HOW TO APPROXIMATE THE HEAT EQUATION WITH NEUMANN BOUNDARY CONDITIONS BY NONLOCAL DIFFUSION PROBLEMS

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Abstract. We present a model for nonlocal diffusion with Neumann boundary conditions in a bounded smooth domain prescribing the flux through the boundary. We study the limit of this family of nonlocal diffusion operators when a rescaling parameter related to the kernel of the nonlocal operator goes to zero. We prove that the solutions of this family of problems converge to a solution of the heat equation with Neumann boundary conditions.

1. Introduction

The purpose of this article is to show that the solutions of the usual Neumann boundary value problem for the heat equation can be approximated by solutions of a sequence of nonlocal "Neumann" boundary value problems.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function with $\int_{\mathbb{R}^N} J(z)\, dz = 1$. Assume also that J is strictly positive in $B(0,d)$ and vanishes in $\mathbb{R}^N \setminus B(0,d)$. Nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\, dy - u(x,t), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [10], if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\, dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\, dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1). For recent references on nonlocal diffusion see, [1], [2], [3], [4], [5], [6], [9], [10], [12], [13], [14] and references therein.

Given a bounded, connected and smooth domain Ω, one of the most common boundary conditions that has been imposed in the literature to the heat equation, $u_t = \Delta u$, is the *Neumann boundary condition*, $\partial u/\partial \eta(x,t) = g(x,t)$, $x \in \partial\Omega$, which leads to the following

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classical problem,

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = g & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

In this article we propose a nonlocal "Neumann" boundary value problem, namely

$$u_t(x,t) = \int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} G(x,x-y)g(y,t)\,dy, \tag{1.3}$$

where $G(x,\xi)$ is smooth and compactly supported in ξ uniformly in x.

In this model the first integral takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position x from or to other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The last term takes into account the prescribed flux of individuals that enter or leave the domain.

The nonlocal Neumann model (1.3) and the Neumann problem for the heat equation (1.2) share many properties. For example, a comparison principle holds for both equations when G is nonnegative and the asymptotic behavior of their solutions as $t \to \infty$ is similar, see [8].

Existence and uniqueness of solutions of (1.3) with general G is proved by a fixed point argument in Section 2. Also, a comparison principle when $G \geq 0$ is proved in that section.

Our main goal is to show that the Neumann problem for the heat equation (1.2), can be approximated by suitable nonlocal Neumann problems (1.3).

More precisely, for given J and G we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \qquad G_\varepsilon(x,\xi) = C_1 \frac{1}{\varepsilon^N} G\left(x, \frac{\xi}{\varepsilon}\right) \tag{1.4}$$

with

$$C_1^{-1} = \frac{1}{2}\int_{B(0,d)} J(z) z_N^2\,dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Then, we consider the solution $u^\varepsilon(x,t)$ to

$$\begin{cases} u_t^\varepsilon(x,t) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega J_\varepsilon(x-y)(u^\varepsilon(y,t)-u^\varepsilon(x,t))\,dy \\ & \qquad\qquad + \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^N\setminus\Omega} G_\varepsilon(x,x-y)g(y,t)\,dy, \\ u^\varepsilon(x,0) &= u_0(x). \end{cases} \tag{1.5}$$

We prove in this paper that

$$u^\varepsilon \to u$$

in different topologies according to two different choices of the kernel G.

Let us give an heuristic idea in one space dimension, with $\Omega = (0,1)$, of why the scaling involved in (1.4) is the correct one. We assume that

$$\int_1^\infty G(1, 1-y)\,dy = -\int_{-\infty}^0 G(0,-y)\,dy = \int_0^1 J(y)\,y\,dy$$

and, as stated above, $G(x,\cdot)$ has compact support independent of x. In this case (1.5) reads

$$\begin{aligned} u_t(x,t) = \ & \frac{1}{\varepsilon^2}\int_0^1 J_\varepsilon(x-y)\big(u(y,t)-u(x,t)\big)dy + \frac{1}{\varepsilon}\int_{-\infty}^0 G_\varepsilon(x, x-y)\, g(y,t)\,dy \\ & + \frac{1}{\varepsilon}\int_1^{+\infty} G_\varepsilon(x,x-y)\,g(y,t)\,dy := \mathcal{A}_\varepsilon u(x,t). \end{aligned}$$

If $x \in (0,1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $\mathcal{A}_\varepsilon u$ in (1.5) becomes

$$\mathcal{A}_\varepsilon u(x) = \frac{1}{\varepsilon^2}\int_0^1 J_\varepsilon(x-y)\,(u(y)-u(x))\,dy \approx u_{xx}(x)$$

and if $x = 0$ and ε small,

$$\mathcal{A}_\varepsilon u(0) = \frac{1}{\varepsilon^2}\int_0^1 J_\varepsilon(-y)\,(u(y)-u(0))\,dy + \frac{1}{\varepsilon}\int_{-\infty}^0 G_\varepsilon(0,-y)\,g(y)\,dy \approx \frac{C_2}{\varepsilon}(u_x(0)-g(0)).$$

Analogously, $\mathcal{A}_\varepsilon u(1) \approx (C_2/\varepsilon)(-u_x(1)+g(1))$. However, the proofs of our results are much more involved than simple Taylor expansions due to the fact that for each $\varepsilon > 0$ there are points $x \in \Omega$ for which the ball in which integration takes place, $B(x, d\varepsilon)$, is not contained in Ω. Moreover, when working in several space dimensions, one has to take into account the geometry of the domain.

Our first result deals with homogeneous boundary conditions, this is, $g \equiv 0$.

Theorem 1.1. *Assume $g \equiv 0$. Let Ω be a bounded $C^{2+\alpha}$ domain for some $0 < \alpha < 1$. Let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution to* (1.2) *and let u^ε be the solution to* (1.5) *with J_ε as above. Then,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$$

as $\varepsilon \to 0$.

Note that this result holds for every G since $g \equiv 0$, and that the assumed regularity in u is guaranteed if $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and $\partial u_0/\partial\eta = 0$. See, for instance, [11].

We will prove Theorem 1.1 by constructing adequate super and subsolutions and then using comparison arguments to get bounds for the difference $u^\varepsilon - u$.

Now we will make explicit the functions G we will deal with in the case $g \neq 0$.

To define the first one let us introduce some notation. As before, let Ω be a bounded $C^{2+\alpha}$ domain. For $x \in \Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) < d\varepsilon\}$ and ε small enough we write

$x = \bar{x} - s\,d\,\eta(\bar{x})$ where $\bar{x}$ is the orthogonal projection of x on $\partial\Omega$, $0 < s < \varepsilon$ and $\eta(\bar{x})$ is the unit exterior normal to Ω at $\bar{x}$. Under these assumptions we define

$$G_1(x,\xi) = -J(\xi)\,\eta(\bar{x})\cdot\xi \quad \text{for } x \in \Omega_\varepsilon. \tag{1.6}$$

Notice that the last integral in (1.5) only involves points $x \in \Omega_\varepsilon$ since when $y \notin \Omega$, $x - y \in supp\, J_\varepsilon$ implies that $x \in \Omega_\varepsilon$. Hence the above definition makes sense for ε small.

For this choice of the kernel, $G = G_1$, we have the following result.

Theorem 1.2. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,(1+\alpha)/2}(\overline{(\mathbb{R}^N \setminus \Omega)} \times [0,T])$, $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ the solution to* (1.2)*, for some $0 < \alpha < 1$. Let J as before and $G(x,\xi) = G_1(x,\xi)$, where G_1 is defined by* (1.6)*. Let u^ε be the solution to* (1.5)*. Then,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\Omega)} \to 0$$

as $\varepsilon \to 0$.

Observe that G_1 may fail to be nonnegative and hence a comparison principle may not hold. However, in this case our proof of convergence to the solution of the heat equation does not rely on comparison arguments for (1.3). If we want a nonnegative kernel G, in order to have a comparison principle, we can modify $(G_1)_\varepsilon$ by taking

$$(\tilde{G}_1)_\varepsilon(x,\xi) = (G_1)_\varepsilon(x,\xi) + \kappa\varepsilon J_\varepsilon(\xi) = \frac{1}{\varepsilon} J_\varepsilon(\xi)\left(-\eta(\bar{x})\cdot\xi + \kappa\varepsilon^2\right)$$

instead.

Note that for $x \in \overline{\Omega}$ and $y \in \mathbb{R}^N \setminus \Omega$, $(\tilde{G}_1)_\varepsilon(x,x-y) = \frac{1}{\varepsilon} J_\varepsilon(x-y)\,(-\eta(\bar{x})\cdot(x-y) + \kappa\varepsilon^2)$ is nonnegative for ε small if we choose the constant κ as a bound for the curvature of $\partial\Omega$, since $|x-y| \le d\,\varepsilon$. As will be seen in Remark 4.1, Theorem 1.2 remains valid with $(G_1)_\varepsilon$ replaced by $(\tilde{G}_1)_\varepsilon$.

Finally, the other "Neumann" kernel we propose is

$$G(x,\xi) = G_2(x,\xi) = C_2 J(\xi),$$

where C_2 is such that

$$\int_0^d \int_{\{z_N > s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds = 0. \tag{1.7}$$

This choice of G is natural since we are considering a flux with a jumping probability that is a scalar multiple of the same jumping probability that moves things in the interior of the domain, J.

Several properties of solutions to (1.3) have been recently investigated in [8] in the case $G = G_2$ for different choices of g.

For the case of G_2 we can still prove convergence but in a weaker sense.

Theorem 1.3. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,(1+\alpha)/2}(\overline{(\mathbb{R}^N \setminus \Omega)} \times [0,T])$, $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ the solution to* (1.2)*, for some $0 < \alpha < 1$. Let J as before and*

$G(x,\xi) = G_2(x,\xi) = C_2 J(\xi)$, where C_2 is defined by (1.7)*. Let u^ε be the solution to* (1.5)*. Then, for each $t \in [0,T]$*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad * - weakly\ in\ L^\infty(\Omega)$$

as $\varepsilon \to 0$.

The rest of the paper is organized as follows: in Section 2 we prove existence, uniqueness and a comparison principle for our nonlocal equation. In Section 3 we prove the uniform convergence when $g = 0$. In Section 4 we deal with the case $G = G_1$ and finally in Section 5 we prove our result when $G = G_2$.

2. Existence and uniqueness

In this section we deal with existence and uniqueness of solutions of (1.3). Our result is valid in a general L^1 setting.

Theorem 2.1. *Let Ω be a bounded domain. Let $J \in L^1(\mathbb{R}^N)$ and $G \in L^\infty(\Omega \times \mathbb{R}^N)$. For every $u_0 \in L^1(\Omega)$ and $g \in L^\infty_{loc}([0,\infty); L^1(\mathbb{R}^N \setminus \Omega))$ there exists a unique solution u of* (1.3) *such that $u \in C([0,\infty); L^1(\Omega))$ and $u(x,0) = u_0(x)$.*

As in [7] and [8], existence and uniqueness will be a consequence of Banach's fixed point theorem. We follow closely the ideas of those works in our proof, so we will only outline the main arguments. Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = C([0,t_0]; L^1(\Omega))$$

with the norm

$$|||w||| = \max_{0 \le t \le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $T_{u_0,g} : X_{t_0} \to X_{t_0}$ defined by

$$\begin{aligned} T_{u_0,g}(w)(x,t) = u_0(x) \quad &+ \int_0^t \int_\Omega J(x-y)\,(w(y,s) - w(x,s))\,dy\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N \setminus \Omega} G(x,x-y) g(y,t)\,dy\,ds. \end{aligned} \tag{2.1}$$

The following lemma is the main ingredient in the proof of existence.

Lemma 2.1. *Let J and G as in Theorem 2.1. Let $g,\ h \in L^\infty((0,t_0); L^1(\mathbb{R}^N \setminus \Omega))$ and $u_0,\ v_0 \in L^1(\Omega)$. There exists a constant C depending only on Ω, J and G such that for $w, z \in X_{t_0}$,*

$$|||T_{u_0,g}(w) - T_{v_0,h}(z)||| \le \|u_0 - v_0\|_{L^1} + C t_0 \left(|||w - z||| + \|g - h\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N \setminus \Omega))} \right). \tag{2.2}$$

Proof. We have

$$\int_\Omega |T_{u_0,g}(w)(x,t) - T_{v_0,h}(z)(x,t)|\,dx \le \int_\Omega |u_0(x) - v_0(x)|\,dx$$
$$+ \int_\Omega \left| \int_0^t \int_\Omega J\,(x-y)\, \Big[(w(y,s) - z(y,s)) - (w(x,s) - z(x,s)) \Big]\, dy\, ds \right| dx$$
$$+ \int_\Omega \int_0^t \int_{\mathbb{R}^N \setminus \Omega} |G(x, x-y)||g(y,s) - h(y,s)|\, dy\, ds\, dx.$$

Therefore, we obtain (2.2). □

Proof of Theorem 2.1. Let $T = T_{u_0,g}$. We check first that T maps X_{t_0} into X_{t_0}. From (2.1) we see that for $0 \le t_1 < t_2 \le t_0$,

$$\|T(w)(t_2) - T(w)(t_1)\|_{L^1(\Omega)} \le A \int_{t_1}^{t_2} \int_\Omega |w(y,s)|\, dy\, ds + B \int_{t_1}^{t_2} \int_{\mathbb{R}^N \setminus \Omega} |g(y,s)|\, dy\, ds.$$

On the other hand, again from (2.1)

$$\|T(w)(t) - u_0\|_{L^1(\Omega)} \le Ct \big\{ |||w||| + \|g\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N \setminus \Omega))} \big\}.$$

These two estimates give that $T(w) \in C([0,t_0]; L^1(\Omega))$. Hence T maps X_{t_0} into X_{t_0}.

Choose t_0 such that $Ct_0 < 1$. From Lemma 2.1 we get that T is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial datum $u(x,t_0) \in L^1(\Omega)$ and obtain a solution in $[0, 2\,t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

Our next aim is to prove a comparison principle for (1.3) when $J,\ G \ge 0$. To this end we define what we understand by sub and supersolutions.

Definition 2.1. *A function $u \in C([0,T); L^1((\Omega))$ is a supersolution of* (1.3) *if $u(x,0) \ge u_0(x)$ and*

$$u_t(x,t) \ge \int_\Omega J(x-y)\big(u(y,t) - u(x,t)\big)\, dy + \int_{\mathbb{R}^N \setminus \Omega} G(x, x-y) g(y,t)\, dy.$$

Subsolutions are defined analogously by reversing the inequalities.

Lemma 2.2. *Let $J,\ G \ge 0$, $u_0 \ge 0$ and $g \ge 0$. If $u \in C(\overline{\Omega} \times [0,T])$ is a supersolution to* (1.3)*, then $u \ge 0$.*

Proof. Assume that $u(x,t)$ is negative somewhere. Let $v(x,t) = u(x,t) + \varepsilon t$ with ε so small such that v is still negative somewhere. Then, if we take (x_0,t_0) a point where v

attains its negative minimum, there holds that $t_0 > 0$ and

$$v_t(x_0, t_0) = u_t(x_0, t_0) + \varepsilon > \int_\Omega J(x-y)(u(y,t_0) - u(x_0,t_0))\,dy$$
$$= \int_\Omega J(x-y)(v(y,t_0) - v(x_0,t_0))\,dy \geq 0$$

which is a contradiction. Thus, $u \geq 0$. ☐

Corollary 2.1. *Let J, $G \geq 0$ and bounded. Let u_0 and v_0 in $L^1(\Omega)$ with $u_0 \geq v_0$ and g, $h \in L^\infty((0,T); L^1(\mathbb{R}^N \setminus \Omega))$ with $g \geq h$. Let u be a solution of* (1.3) *with initial condition u_0 and flux g and v be a solution of* (1.3) *with initial condition v_0 and flux h. Then,*

$$u \geq v \qquad a.e.$$

Proof. Let $w = u - v$. Then, w is a supersolution with initial datum $u_0 - v_0 \geq 0$ and boundary datum $g - h \geq 0$. Using the continuity of solutions with respect to the initial and Neumann data (Lemma 2.1) and the fact that $J \in L^\infty(\mathbb{R}^N)$, $G \in L^\infty(\Omega \times \mathbb{R}^N)$ we may assume that u, $v \in C(\overline{\Omega} \times [0,T])$. By Lemma 2.2 we obtain that $w = u - v \geq 0$. So the corollary is proved. ☐

Corollary 2.2. *Let J, $G \geq 0$ and bounded. Let $u \in C(\overline{\Omega} \times [0,T])$ (resp. v) be a supersolution (resp. subsolution) of* (1.3)*. Then, $u \geq v$.*

Proof. It follows the lines of the proof of the previous corollary. ☐

3. Uniform convergence in the case $g \equiv 0$

In order to prove Theorem 1.1 we set $w^\varepsilon = u^\varepsilon - u$ and let $\tilde{u}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of u to $\mathbb{R}^N \times [0,T]$. We define

$$L_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x-y)\big(v(y,t) - v(x,t)\big)dy$$

and

$$\tilde{L}_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J_\varepsilon(x-y)\big(v(y,t) - v(x,t)\big)dy.$$

Then

$$w_t^\varepsilon = L_\varepsilon(u^\varepsilon) - \Delta u + \frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} G_\varepsilon(x, x-y) g(y,t)\,dy$$
$$= L_\varepsilon(w^\varepsilon) + \tilde{L}_\varepsilon(\tilde{u}) - \Delta u + \frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} G_\varepsilon(x, x-y) g(y,t)\,dy$$
$$- \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t) - \tilde{u}(x,t)\big)\,dy.$$

Or

$$w_t^\varepsilon - L_\varepsilon(w^\varepsilon) = F_\varepsilon(x,t),$$

where, noting that $\Delta u = \Delta \tilde{u}$ in Ω,

$$F_\varepsilon(x,t) = \tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u} + \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} G_\varepsilon(x,x-y)g(y,t)\,dy \\ - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t)-\tilde{u}(x,t)\big)\,dy.$$

Our main task in order to prove the uniform convergence result is to get bounds on F_ε.

First, we observe that it is well known that by the choice of C_1, the fact that J is radially symmetric and $\tilde{u} \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$, we have that

$$\sup_{t\in[0,T]} \|\tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u}\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha)\,. \tag{3.1}$$

In fact,

$$\frac{C_1}{\varepsilon^{N+2}}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)(\tilde{u}(y,t)-\tilde{u}(x,t))\;dy - \Delta\tilde{u}(x,t)$$

becomes, under the change variables $z = (x-y)/\varepsilon$,

$$\frac{C_1}{\varepsilon^2}\int_{\mathbb{R}^N} J\,(z)\,(\tilde{u}(x-\varepsilon z,t)-\tilde{u}(x,t))\,dy - \Delta\tilde{u}(x,t)$$

and hence (3.1) follows by a simple Taylor expansion.

Next, we will estimate the last integral in F_ε. We remark that the next lemma is valid for any smooth function, not only for a solution to the heat equation.

Lemma 3.1. *If θ is a $C^{2+\alpha,1+\alpha/2}$ function on $\mathbb{R}^N \times [0,T]$ and $\dfrac{\partial\theta}{\partial\eta} = h$ on $\partial\Omega$, then for $x \in \Omega_\varepsilon = \{z \in \Omega \,|\, \mathrm{dist}\,(z,\partial\Omega) < d\varepsilon\}$ and ε small,*

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\theta(y,t)-\theta(x,t)\big)\,dy = \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}h(\bar{x},t)\,dy \\ + \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]dy + O(\varepsilon^\alpha),$$

where $\bar{x}$ is the orthogonal projection of x on the boundary of Ω so that $\|\bar{x}-y\| \le 2d\varepsilon$.

Proof. Since $\theta \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$ we have

$$\theta(y,t)-\theta(x,t) = \theta(y,t)-\theta(\bar{x},t) - \big(\theta(x,t)-\theta(\bar{x},t)\big) \\ = \nabla\theta(\bar{x},t)\cdot(y-x) + \sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\big[(y-\bar{x})^\beta - (x-\bar{x})^\beta\big] \\ + O(||\bar{x}-x||^{2+\alpha}) + O(||\bar{x}-y||^{2+\alpha}).$$

Therefore,

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\theta(y,t)-\theta(x,t)\big)\,dy = \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\nabla\theta(\bar{x},t)\cdot\frac{(y-x)}{\varepsilon}\,dy$$
$$+\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta-\big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]\,dy+O(\varepsilon^\alpha).$$

Fix $x\in\Omega_\varepsilon$. Let us take a new coordinate system such that $\eta(\bar{x})=e_N$. Since $\dfrac{\partial\theta}{\partial\eta}=h$ on $\partial\Omega$, we get

$$\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\nabla\theta(\bar{x},t)\cdot\frac{(y-x)}{\varepsilon}\,dy$$
$$=\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}h(\bar{x},t)\,dy+\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\,dy.$$

We will estimate this last integral. Since Ω is a $C^{2+\alpha}$ domain we can chose vectors e_1, e_2, ..., e_{N-1} so that there exists $\kappa>0$ and constants $f_i(\bar{x})$ such that

$$B_{2d\varepsilon}(\bar{x})\cap\left\{y_N-\big(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\big)>\kappa\varepsilon^{2+\alpha}\right\}\subset\mathbb{R}^N\setminus\Omega,$$
$$B_{2d\varepsilon}(\bar{x})\cap\left\{y_N-\big(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\big)<-\kappa\varepsilon^{2+\alpha}\right\}\subset\Omega.$$

Therefore

$$\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy$$
$$=\int_{(\mathbb{R}^N\setminus\Omega)\cap\ \left|y_N-\left(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\right)\right|\le\kappa\varepsilon^{2+\alpha}} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy$$
$$+\int_{y_N-\left(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\right)>\kappa\varepsilon^{2+\alpha}} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy$$
$$=I_1+I_2.$$

If we take $z=(y-x)/\varepsilon$ as a new variable, recalling that $\bar{x}_N-x_N=\varepsilon s$, we obtain

$$|I_1|\le C_1\sum_{i=1}^{N-1}|\theta_{x_i}(\bar{x},t)|\int_{\left|z_N-\left(s+\varepsilon\sum_{i=1}^{N-1}f_i(\bar{x})(z_i)^2\right)\right|\le\kappa\varepsilon^{1+\alpha}} J(z)|z_i|\,dz\le C\,\kappa\,\varepsilon^{1+\alpha}.$$

On the other hand,

$$I_2 = C_1 \sum_{i=1}^{N-1} \theta_{x_i}(\bar{x},t) \int_{z_N - \left(s+\varepsilon \sum_{i=1}^{N-1} f_i(\bar{x})(z_i)^2\right) > \kappa\varepsilon^{1+\alpha}} J(z)\, z_i\, dz.$$

Fix $1 \le i \le N-1$. Then, since J is radially symmetric, $J(z)\, z_i$ is an odd function of the variable z_i and, since the set $\left\{ z_N - \left(s + \varepsilon \sum_{i=1}^{N-1} f_i(\bar{x})(z_i)^2\right) > \kappa\varepsilon^{1+\alpha}\right\}$ is symmetric in that variable we get

$$I_2 = 0.$$

Collecting the previous estimates the lemma is proved. □

We will also need the following inequality.

Lemma 3.2. *There exist $K > 0$ and $\bar{\varepsilon} > 0$ such that, for $\varepsilon < \bar{\varepsilon}$,*

$$\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy \ge K \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy. \tag{3.2}$$

Proof. Let us put the origin at the point $\bar{x}$ and take a coordinate system such that $\eta(\bar{x}) = e_N$. Then, $x = (0, -\mu)$ with $0 < \mu < d\varepsilon$. Then, arguing as before,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy = \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy \\
&= \int_{\{y_N > \kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy + \int_{\mathbb{R}^N\setminus\Omega\cap\{|y_N|<\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy \\
&\ge \int_{\{y_N > \kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy - C\varepsilon.
\end{aligned}$$

Fix c_1 small such that

$$\frac{1}{2}\int_{\{z_N>0\}} J(z)\, z_N\, dz \ge 2c_1 \int_{\{0<z_N<2c_1\}} J(z)\, dz.$$

We divide our arguments into two cases according to whether $\mu \le c_1\varepsilon$ or $\mu > c_1\varepsilon$.

Case I Assume $\mu \le c_1\varepsilon$. In this case we have,

$$\begin{aligned}
&\int_{\{y_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy = C_1 \int_{\{z_N > \kappa\varepsilon + \frac{\mu}{\varepsilon}\}} J(z)\, z_N\, dz \\
&= C_1 \left(\int_{\{z_N>0\}} J(z)\, z_N\, dz - \int_{\{0<z_N<\kappa\varepsilon+\frac{\mu}{\varepsilon}\}} J(z)\, z_N\, dz \right) \qquad (3.3)\\
&\ge C_1 \left(\int_{\{z_N>0\}} J(z)\, z_N\, dz - 2c_1 \int_{\{0<z_N<2c_1\}} J(z)\, dz \right) \ge \frac{C_1}{2}\int_{\{z_N>0\}} J(z)\, z_N\, dz.
\end{aligned}$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}\,dy - K\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy\\ &\geq C_1\left(\frac{1}{2}\int_{\{z_N>0\}} J(z)\,z_N\,dz - K\right) - C\varepsilon \geq 0,\end{aligned}$$

if ε is small enough and

$$K < \frac{1}{4}\int_{\{z_N>0\}} J(z)\,z_N\,dz.$$

Case II Assume that $\mu \geq c_1\varepsilon$. For y in $\mathbb{R}^N\setminus\Omega\cap B(\bar{x},d\varepsilon)$ we have

$$\frac{y_N}{\varepsilon} \geq -\kappa\varepsilon.$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\,dy - K\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy\\ &\geq (c_1-\kappa\varepsilon)\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy - K\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy\\ &= \big(c_1-\kappa\varepsilon-K\big)\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy \geq 0,\end{aligned}$$

if ε is small and

$$K < \frac{c_1}{2}.$$

This ends the proof of (3.2). □

We now prove Theorem 1.1.

Proof of Theorem 1.1. We will use a comparison argument. First, let us look for a super-solution. Let us pick an auxiliary function v as a solution to

$$\begin{cases} v_t - \Delta v = h(x,t) & \text{in} \quad \Omega\times(0,T),\\ \dfrac{\partial v}{\partial \eta} = g_1(x,t) & \text{on} \quad \partial\Omega\times(0,T),\\ v(x,0) = v_1(x) & \text{in} \quad \Omega. \end{cases}$$

for some smooth functions $h(x,t)\geq 1$, $g_1(x,t)\geq 1$ and $v_1(x)\geq 0$ such that the resulting v has an extension $\tilde{v}$ that belongs to $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N\times[0,T])$, and let M be an upper bound for v in $\bar{\Omega}\times[0,T]$. Then,

$$v_t = L_\varepsilon v + (\Delta v - \tilde{L}_\varepsilon\tilde{v}) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)(\tilde{v}(y,t)-\tilde{v}(x,t))\,dy + h(x,t).$$

Since $\Delta v = \Delta \tilde{v}$ in Ω, we have that v is a solution to

$$\begin{cases} v_t - L_\varepsilon v = H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega, \end{cases}$$

where by (3.1), Lemma 3.1 and the fact that $h \geq 1$,

$$\begin{aligned} H(x,t,\varepsilon) =& (\Delta \tilde{v} - \tilde{L}_\varepsilon \tilde{v}) + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)(\tilde{v}(y,t) - \tilde{v}(x,t))\, dy + h(x,t) \\ \geq& \Big(\frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon} g_1(\bar{x},t)\, dy \\ &+ \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \sum_{|\beta|=2} \frac{D^\beta \tilde{v}}{2}(\bar{x},t) \Big[\big(\frac{(y-\bar{x})}{\varepsilon} \big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon} \big)^\beta \Big]\, dy \Big) + 1 - C\varepsilon^\alpha \\ \geq& \Big(\frac{g_1(\bar{x},t)}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - D_1 \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\, dy \Big) + \frac{1}{2} \end{aligned}$$

for some constant D_1 if ε is small so that $C\varepsilon^\alpha \leq 1/2$.

Now, observe that Lemma 3.2 implies that for every constant $C_0 > 0$ there exists ε_0 such that,

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - C_0 \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\, dy \geq 0,$$

if $\varepsilon < \varepsilon_0$.

Now, since $g = 0$, by (3.1) and Lemma 3.1 we obtain

$$\begin{aligned} |F_\varepsilon| &\leq C\varepsilon^\alpha + \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \sum_{|\beta|=2} \frac{D^\beta \tilde{u}}{2}(\bar{x},t) \Big[\big(\frac{(y-\bar{x})}{\varepsilon} \big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon} \big)^\beta \Big] dy \\ &\leq C\varepsilon^\alpha + C_2 \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\, dy. \end{aligned}$$

Given $\delta > 0$, let $v_\delta = \delta v$. Then v_δ verifies

$$\begin{cases} (v_\delta)_t - L_\varepsilon v_\delta = \delta H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v_\delta(x,0) = \delta v_1(x) & \text{in} \quad \Omega. \end{cases}$$

By our previous estimates, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for $\varepsilon \leq \varepsilon_0$,

$$|F_\varepsilon| \leq \delta H(x,t,\varepsilon).$$

So, by the comparison principle for any $\varepsilon \leq \varepsilon_0$ it holds that

$$-M\delta \leq -v_\delta \leq w_\varepsilon \leq v_\delta \leq M\delta.$$

Therefore, for every $\delta > 0$,

$$-M\delta \leq \liminf_{\varepsilon \to 0} w_\varepsilon \leq \limsup_{\varepsilon \to 0} w_\varepsilon \leq M\delta.$$

and the theorem is proved. ☐

4. Convergence in L^1 in the case $G = G_1$

First we prove that F_ε goes to zero as ε goes to zero.

Lemma 4.1. *If $G = G_1$ then*

$$F_\varepsilon(x,t) \to 0 \quad \text{in} \quad L^\infty\big([0,T]; L^1(\Omega)\big)$$

as $\varepsilon \to 0$.

Proof. As $G = G_1 = -J(\xi)\,\eta(\bar{x}) \cdot \xi$, for $x \in \Omega_\varepsilon$, by (3.1) and Lemma 3.1,

$$\begin{aligned} F_\varepsilon(x,t) =& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\big(g(y,t) - g(\bar{x},t)\big)\,dy \\ &- \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y) \sum_{|\beta|=2} \frac{D^\beta \tilde{u}}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]\,dy + O(\varepsilon^\alpha). \end{aligned}$$

As g is smooth, we have that F_ε is bounded in Ω_ε. Recalling the fact that $|\Omega_\varepsilon| = O(\varepsilon)$ and $F_\varepsilon(x,t) = O(\varepsilon^\alpha)$ on $\Omega \setminus \Omega_\varepsilon$ we get the convergence result. ☐

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. In the case $G = G_1$ we have proven in Lemma 4.1 that $F_\varepsilon \to 0$ in $L^1(\Omega \times [0,T])$. On the other hand, we have that $w^\varepsilon = u^\varepsilon - u$ is a solution to

$$\begin{aligned} w_t - L_\varepsilon(w) &= F_\varepsilon \\ w(x,0) &= 0. \end{aligned}$$

Let z^ε be a solution to

$$\begin{aligned} z_t - L_\varepsilon(z) &= |F_\varepsilon| \\ z(x,0) &= 0. \end{aligned}$$

Then $-z^\varepsilon$ is a solution to

$$\begin{aligned} z_t - L_\varepsilon(z) &= -|F_\varepsilon| \\ z(x,0) &= 0. \end{aligned}$$

By comparison we have that

$$-z^\varepsilon \le w^\varepsilon \le z^\varepsilon \quad \text{and} \quad z^\varepsilon \ge 0.$$

Integrating the equation for z^ε we get

$$\|z^\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega z^\varepsilon(x,t)\,dx = \int_\Omega \int_0^t |F_\varepsilon(x,s)|\,ds\,dx\,.$$

Applying Lemma 4.1 we get

$$\sup_{t\in[0,T]} \|z^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \to 0$$

as $\varepsilon \to 0$. So the theorem is proved. ☐

Remark 4.1. *Notice that if we consider a kernel G which is a modification of G_1 of the form*

$$G_\varepsilon(x,\xi) = (G_1)_\varepsilon(x,\xi) + A(x,\xi,\varepsilon)$$

with

$$\int_{\mathbb{R}^N\setminus\Omega} |A(x,x-y,\varepsilon)|\,dy \to 0$$

in $L^1(\Omega)$ as $\varepsilon \to 0$, then the conclusion of Theorem 1.2 is still valid. In particular, we can take $A(x,\xi,\varepsilon) = \kappa\varepsilon J_\varepsilon(\xi)$.

5. Weak convergence in L^1 in the case $G = G_2$

First, we prove that in this case F_ε goes to zero as measures.

Lemma 5.1. *If $G = G_2$ then there exists a constant C independent of ε such that*

$$\int_0^T \int_\Omega |F_\varepsilon(x,s)|\,dx\,ds \le C.$$

Moreover,

$$F_\varepsilon(x,t) \rightharpoonup 0 \qquad \text{as measures}$$

as $\varepsilon \to 0$. That is, for any continuous function θ, it holds that

$$\int_0^T \int_\Omega F_\varepsilon(x,t)\theta(x,t)\,dx\,dt \to 0$$

as $\varepsilon \to 0$.

Proof. As $G = G_2 = C_2 J(\xi)$ and g and $\tilde{u}$ are smooth, taking again the coordinate system of Lemma 3.1, we obtain

$$\begin{aligned}
F_\varepsilon(x,t) =& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(y,t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\
&- \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1} \tilde{u}_{x_i}(\bar{x},t)\frac{(y_i - x_i)}{\varepsilon}\,dy \\
&- \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta \tilde{u}(\bar{x},t)}{2}\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]\,dy + O(\varepsilon^\alpha) \\
=& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\
&- \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1} \tilde{u}_{x_i}(\bar{x},t)\frac{(y_i - x_i)}{\varepsilon}\,dy + O(1)\chi_{\Omega_\varepsilon} + O(\varepsilon^\alpha).
\end{aligned}$$

Let

$$B_\varepsilon(x,t) := \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) - \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\,dy.$$

Proceeding in a similar way as in the proof of Lemma 3.1 we get for ε small,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big)\\
&= g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|y_N-\bar{x}_N|\le\kappa\varepsilon^2\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right)dy\\
&\quad + g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{y_N-\bar{x}_N>0\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right)dy\\
&\quad - g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{0<y_N-\bar{x}_N<\kappa\varepsilon^2\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right)dy\\
&= C_1 g(\bar{x},t)\int_{\{z_N>s\}} J(z)(C_2 - z_N)\,dz + O(\varepsilon)\chi_{\Omega_\varepsilon}.
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\,dy\\
&= \sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{|y_N-\bar{x}_N|\le\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{(y_i-x_i)}{\varepsilon}dy\\
&\quad + \sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{y_N-\bar{x}_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{(y_i-x_i)}{\varepsilon}dy\\
&= C_1\sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{z_N-s>\kappa\varepsilon\}} J(z)z_i dz + O(\varepsilon)\chi_{\Omega_\varepsilon}\\
&= I_2 + O(\varepsilon)\chi_{\Omega_\varepsilon}.
\end{aligned}$$

As in Lemma 3.1 we have $I_2 = 0$. Therefore,

$$B_\varepsilon(x,t) = C_1 g(\bar{x},t)\int_{\{z_N>s\}} J(z)\big(C_2 - z_N\big)\,dz + O(\varepsilon)\chi_{\Omega_\varepsilon}.$$

Now, we observe that B_ε is bounded and supported in Ω_ε. Hence

$$\int_0^t\int_\Omega |F_\varepsilon(x,\tau)|\,dx\,d\tau \le \frac{1}{\varepsilon}\int_0^t\int_{\Omega_\varepsilon} |B_\varepsilon(x,\tau)|\,dx\,d\tau + Ct|\Omega_\varepsilon| + Ct|\Omega|\varepsilon^\alpha \le C.$$

This proves the first assertion of the lemma.

Now, let us write for a point $x \in \Omega_\varepsilon$

$$x = \bar{x} - \mu\eta(\bar{x}) \qquad \text{with } 0 < \mu < d\varepsilon.$$

For ε small and $0 < \mu < d\varepsilon$, let dS_μ be the area element of $\{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) = \mu\}$. Then, $dS_\mu = dS + O(\varepsilon)$, where dS is the area element of $\partial\Omega$.

So that, taking now $\mu = s\varepsilon$ we get for any continuous test function θ,

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T \int_{\Omega_\varepsilon} B_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \\
&\quad = O(\varepsilon) + C_1 \int_0^T \int_{\partial\Omega} g(\bar{x},t)\theta(\bar{x},t) \int_0^d \int_{\{z_N > s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds\,dS\,dt \\
&\quad = O(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

since we have chosen C_2 so that

$$\int_0^d \int_{\{z_N > s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds = 0.$$

Now, with all these estimates, we go back to F_ε. We have

$$F_\varepsilon(x,t) = \frac{1}{\varepsilon}B_\varepsilon(x,t) + O(1)\chi_{\Omega_\varepsilon} + O(\varepsilon^\alpha).$$

Thus, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \to 0 \qquad \text{as } \varepsilon \to 0.$$

Now, if $\sigma(r)$ is the modulus of continuity of θ,

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(x,t)\,dx\,dt = \int_0^T \int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \\
&\quad + \int_0^T \int_{\Omega_\varepsilon} F_\varepsilon(x,t)\big(\theta(x,t) - \theta(\bar{x},t)\big)\,dx\,dt \\
&\le \int_0^T \int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt + C\sigma(\varepsilon)\int_0^T \int_{\Omega_\varepsilon} |F_\varepsilon(x,t)|\,dx\,dt \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Finally, the observation that $F_\varepsilon = O(\varepsilon^\alpha)$ in $\Omega \setminus \Omega_\varepsilon$ gives

$$\int_0^T \int_{\Omega\setminus\Omega_\varepsilon} F_\varepsilon(x,t)\theta(x,t)\,dx\,dt \to 0 \qquad \text{as } \varepsilon \to 0$$

and this ends the proof. □

Now we prove that u^ε is uniformly bounded when $G = G_2$.

Lemma 5.2. *Let $G = G_2$. There exists a constant C independent of ε such that*

$$\|u^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le C.$$

Proof. Again we will use a comparison argument. Let us look for a supersolution. Pick an auxiliary function v as a solution to

$$\begin{cases} v_t - \Delta v = h(x,t) & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial v}{\partial \eta} = g_1(x,t) & \text{on} \quad \partial\Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

for some smooth functions $h(x,t) \geq 1$, $v_1(x) \geq u_0(x)$ and

$$g_1(x,t) \geq \frac{2}{K}(C_2+1) \max_{\partial\Omega\times[0,T]} |g(x,t)| + 1 \qquad (K \text{ as in } (3.2))$$

such that the resulting v has an extension $\tilde{v}$ that belongs to $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$ and let M be an upper bound for v in $\bar{\Omega} \times [0,T]$. As before v is a solution to

$$\begin{cases} v_t - L_\varepsilon v = H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega, \end{cases}$$

where H verifies

$$H(x,t,\varepsilon) \geq \Big(\frac{g_1(\bar{x},t)}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - D_1 \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \Big) + \frac{1}{2}.$$

So that, by Lemma 3.2,

$$H(x,t,\varepsilon) \geq \big(\frac{g_1(\bar{x},t)\, K}{\varepsilon} - D_1 \big) \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy + \frac{1}{2}$$

for $\varepsilon < \bar{\varepsilon}$.

Let us recall that

$$F_\varepsilon(x,t) = \tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u} + \frac{C_2}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y) g(y,t)\, dy \\ - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t) - \tilde{u}(x,t)\big)\, dy.$$

Then, proceeding once again as in Lemma 3.1 we have,

$$\begin{aligned} |F_\varepsilon(x,t)| &\leq \frac{|g(\bar{x},t)|\, C_2}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy + \frac{|g(\bar{x},t)|}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y) \big|\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\big|\, dy \\ &\quad + C\varepsilon^\alpha + C \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \\ &\leq \Big[\frac{(C_2+1)}{\varepsilon} \max_{\partial\Omega\times[0,T]} |g(x,t)| + C \Big] \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy + C\varepsilon^\alpha \\ &\leq \big(\frac{g_1(\bar{x},t)\, K}{2\varepsilon} + C \big) \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy + C\varepsilon^\alpha \end{aligned}$$

if $\varepsilon < \bar{\varepsilon}$, by our choice of g_1.

Therefore, for every ε small enough, we obtain

$$|F_\varepsilon(x,t)| \le H(x,t,\varepsilon),$$

and, by a comparison argument, we conclude that

$$-M \le -v(x,t) \le u^\varepsilon(x,t) \le v(x,t) \le M,$$

for every $(x,t) \in \overline{\Omega} \times [0,T]$. This ends the proof. □

Finally, we prove our last result, Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1 we have that

$$F_\varepsilon(x,t) \rightharpoonup 0 \quad \text{as measures in} \quad \Omega \times [0,T]$$

as $\varepsilon \to 0$.

Assume first that $\psi \in C_0^{2+\alpha}(\Omega)$ and let $\tilde{\varphi}_\varepsilon$ be the solution to

$$\begin{aligned} w_t - L_\varepsilon w = 0 \\ w(x,0) = \psi(x). \end{aligned}$$

Let $\tilde{\varphi}$ be a solution to

$$\begin{cases} \varphi_t - \Delta\varphi = 0 \\ \dfrac{\partial\varphi}{\partial\eta} = 0 \\ \varphi(x,0) = \psi(x). \end{cases}$$

Then, by Theorem 1.1 we know that $\tilde{\varphi}_\varepsilon \to \tilde{\varphi}$ uniformly in $\Omega \times [0,T]$.

For a fixed $t > 0$ set $\varphi_\varepsilon(x,s) = \tilde{\varphi}_\varepsilon(x,t-s)$. Then φ_ε satisfies

$$\begin{aligned} &\varphi_s + L_\varepsilon\varphi = 0, \qquad \text{for } s < t, \\ &\varphi(x,t) = \psi(x). \end{aligned}$$

Analogously, set $\varphi(x,s) = \tilde{\varphi}(x,t-s)$. Then φ satisfies

$$\begin{cases} \varphi_t + \Delta\varphi = 0 \\ \dfrac{\partial\varphi}{\partial\eta} = 0 \\ \varphi(x,t) = \psi(x). \end{cases}$$

Then, for $w^\varepsilon = u^\varepsilon - u$ we have

$$\int_\Omega w^\varepsilon(x,t)\,\psi(x)\,dx = \int_0^t\int_\Omega \frac{\partial w^\varepsilon}{\partial s}(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds + \int_0^t\int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w^\varepsilon(x,s)\,dx\,ds$$

$$= \int_0^t\int_\Omega L_\varepsilon(w^\varepsilon)(x,s)\varphi_\varepsilon(x,s)\,dx\,ds + \int_0^t\int_\Omega F_\varepsilon(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds$$

$$+ \int_0^t\int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w_\varepsilon(x,s)\,dx\,ds$$

$$= \int_0^t\int_\Omega L_\varepsilon(\varphi_\varepsilon)(x,s)w^\varepsilon(x,s)\,dx\,ds + \int_0^t\int_\Omega F_\varepsilon(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds$$

$$+ \int_0^t\int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w^\varepsilon(x,s)\,dx\,ds$$

$$= \int_0^t\int_\Omega F_\varepsilon(x,s)\varphi_\varepsilon(x,s)\,dx\,ds.$$

Now we observe that, by the Lemma 5.1,

$$\left|\int_0^t\int_\Omega F_\varepsilon(x,s)\varphi_\varepsilon(x,s)\,dx\,ds\right| \le \left|\int_0^t\int_\Omega F_\varepsilon(x,s)\varphi(x,s)\,dx\,ds\right|$$

$$+ \sup_{0<s<t}\|\varphi_\varepsilon(x,s)-\varphi(x,s)\|_{L^\infty(\Omega)}\int_0^t\int_\Omega |F_\varepsilon(x,s)|\,dx\,ds \to 0$$

as $\varepsilon \to 0$. This proves the result when $\psi \in C_0^{2+\alpha}(\Omega)$.

Now we deal with the general case. Let $\psi \in L^1(\Omega)$. Choose $\psi_n \in C_0^{2+\alpha}(\Omega)$ such that $\psi_n \to \psi$ in $L^1(\Omega)$. We have

$$\left|\int_\Omega w^\varepsilon(x,t)\,\psi(x)\,dx\right| \le \left|\int_\Omega w^\varepsilon(x,t)\,\psi_n(x)\,dx\right| + \|\psi_n-\psi\|_{L^1(\Omega)}\|w^\varepsilon\|_{L^\infty(\Omega)}.$$

By Lemma 5.2, $\{w^\varepsilon\}$ is uniformly bounded, and hence the result follows. ☐

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