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A GENERALIZATION OF THE ZERO-ONE LAW FOR PLANAR RANDOM WALKS IN FINITE RANK AND UNIFORMLY ELLIPTIC ENVIRONMENTS

POR

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>
> — *Albert Camus*, "Retour à Tipasa".

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Contents

Chapter 1

Abstract

> "Begin at the beginning" the King said, very gravely, "and go on till you come to the end: then stop."
>
> — *Lewis Carroll*, "Alice in Wonderland".

The theory of "Random Walk in Random Environment" (henceforward, abbreviated as RWRE) has played a role as a new mathematical model (which started being studied in the 70, see [11]). This theory describes several phenomena in physics and biology, quoting [4],

"Random walk in random environment is a simple but powerful model for a variety of phenomena including homogenization in disordered materials [7], DNA chain replication [1], crystal growth [10] and turbulent behavior in fluids [8]".

Briefly speaking, a RWRE is a stochastic process, which describes a path that is made up of a sequence of random steps on some space endowed with randomness as well. Hence, the randomness is not only on the walk itself, but on the medium where the random process moves through. Nonetheless, there are several open question for seemingly naive problems, even Sznitman, Zeitouni and Zerner have said that some of them are embarrassing, see [9] and [11]. This thesis tries to give a partial

answer to one of the mentioned embarrassing problems.

Sznitman & Zerner demonstrated Kalikow's theorem (see [9], page 1854). Rougly speaking, this theorem says that given a line $\mathscr{L}$ in $\mathbb{Z}^d$ with $d \geq 2$, a random walk in an i.i.d. environment moves in a *rough sense* along the line $\mathscr{L}$ with probability 0 or 1. Therefore, it seems natural to ask if the same conclusion holds if we have a ray starting at zero rather than a line. This was proposed as an open question in [9] and answered two years later by Zerner & Merkl in [13], assuming $d = 2$. Furthermore, that answer was improved by Zerner in [14]. The response was "*it depends*." If we assume that the environments are planar and i.i.d., then the response is affirmative, which we call *the zero-one law*. However, if we assume that the environments are planar, stationary and totally ergodic, the response is negative. This was a major step toward a complete answer of an unsolved problem. Regardless of the previous fact, the problem remains unsolved for arbitrary dimension or for other kind of environments. Indeed, for dimension equal to 2 there are a *jungle* of types of environments between the i.i.d. environments and totally and ergodic environments, where the former proposed problem is unsolved, which leads to this thesis. In this thesis, we prove that even after introducing certain dependence in the environments, the zero-one law still holds, although we still assume that the dimension equal to 2.[1]

This thesis aims to be self-contained. Therefore, the thesis is organized as follows: Firstly, chapter 2 contains the essential concepts to understand the proof of Theorem 4. Secondly, chapter 3 contains, just like the name suggests, all the results obtained in order to prove the main result of this thesis, which is a generalization of the zero-one law for planar random walks in finite rank and uniformly elliptic environments.

[1] The problem remains unsolved for dimension greater than or equal to 3. In fact, during the proof of Theorem 4, we point out where the hyphotesis $d = 2$ is used.

Chapter 2

Introduction

No volvió a pensar en ella, ni en ninguna otra, después de que entró al taller con la taza humeante, y encendió la luz para contar los pescaditos de oro que guardaba en un tarro de lata. Había diecisiete. Desde que decidió no venderlos, seguía fabricando dos pescaditos al día, y cuando completaba veinticinco volvía a fundirlos en el crisol para empezar a hacerlos de nuevo.

— *Gabriel García Marquez*, "Cien años de soledad".

In this chapter we expose the general framework to be used in this thesis. Namely, the present chapter is designed to be a summary of the essential aspects of the RWRE theory. Hence, the basic definitions are presented and some results of the theory are set without proving them. Having said this, it is important to stress the fact that this chapter is based strongly on [2]. For further information about the theory of RWRE, see [11].

This thesis aims to be self-contained, however in order to understand this chapter, the reader must have an elementary knowledge of probability theory, two good books about the subject are [3] and [6].

2.1 Basic notions

Our workspace is $\mathbb{Z}^d$ for $d \in \mathbb{Z}^+$, which is endowed with the norm $\|\cdot\|_1$. This space provides the medium where the random walk moves through. Having said this, we start defining the concept of *environment*, which is an important ingredient so as to model the movement of a particle through a random medium.

Definition 1 (Environment and environment space)**.** *Let $\mathscr{P}$ be the set of all the $2d-$random vectors, i.e.,*

$$\mathscr{P} := \{p = (p(e))_{e \in U} : U := \{\pm \hat{e}_k\}_{k=1}^d, p \in [0,1]^U, \|p\|_1 = 1\},^1$$

where $\{\hat{e}_k\}_{k=1}^d$ is the canonical basis in $\mathbb{Z}^d$. The ***environment space*** *is said to be the space $\Omega := \mathscr{P}^{\mathbb{Z}^d}$. For obvious reasons, a d-dimensional* ***environment*** *is defined as any element in the former space. Namely , an environment $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$ is a vector, in which each one of its coordinates is a vector in $\mathscr{P}$. By an abuse of notation, if we are interested on some coordinate of the vector $\omega(x)$, say the e-th coordinate, we denote that number as $\omega(x,e)$.*

This definition allows us to define a random walk moving in a given environment ω.

Definition 2 (Random walk in an environment ω)**.** *Given $d \in \mathbb{Z}^+$. Let ω be a $d-$dimensional*

[1] This set provides the admissible transition probabilities for a particle moving from one point to one of the nearest neighbors.

environment, i.e., $\omega \in \Omega := \mathscr{P}^{\mathbb{Z}^d}$ *and let* $\mathscr{G}$ *be the* $\sigma-$*algebra on* $(\mathbb{Z}^d)^{\mathbb{N}}$ *defined by the cylinder functions. Given a point* $x \in \mathbb{Z}^d$, ***the random walk in the environment*** ω *is defined as the Markov chain* $\{X_n\}_{n\in\mathbb{N}}$ *on* $\mathbb{Z}^d$ *whose law* $P_{x,\omega}$ *on* $(\mathbb{Z}^d, \mathscr{G})$ *is determined by the following relations*

$$P_{x,\omega}[X_0 = x] = 1$$

$$P_{x,\omega}[X_{n+1} = y + e | X_n = y] = \begin{cases} \omega(y,e), \textit{ if } e \in U \textit{ and } P_{x,\omega}[X_n = y] > 0. \\ 0, \textit{ otherwise.} \end{cases}$$

Besides, $P_{x,\omega}$ *is called the* ***quenched law*** *or* ***probability*** *of the random walk in the random environment* $\{X_n\}$.

So far we have defined the random walk in a given environment ω with its intrinsic law, the quenched probability. Now we introduce another probability measure defined not on $(\mathbb{Z}^d, \mathscr{G})$, but on the environment space Ω. Ω is endowed with the product topology, which enables us to construct the measurable space $(\Omega, \mathscr{B}(\Omega))$, where $\mathscr{B}(\Omega)$ is the Borel $\sigma-$algebra of Ω.

Definition 3 (Law of the environment)**.** *Let* $\mathbb{P}$ *be a probability measure defined on* $(\Omega, \mathscr{B}(\Omega))$. $\mathbb{P}$ *is called the* ***law of the environment***.

Having already defined two different probabilities, we proceed to mix them up to form a new probability, which we will call *annealed law* or *probability* . But, before stating this concept, we need to observe that for each $x \in \mathbb{Z}^d$,

$$G \in \mathscr{G} \longmapsto P_{x,\omega}[G]$$

is a function $\mathscr{B}(\Omega)-$measurable, which follows by Dynkin's theorem. Now we are able to define the new law.

Definition 4 (Annealed or averaged probability)**.** *We consider the measurable space* $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathscr{B}(\Omega) \otimes \mathscr{G})$ *and given* $x \in \mathbb{Z}^d$, *we can define on the previous measurable* *space the semi-direct product $\mathbb{P}_{x,\mathbb{P}}$ of $\mathbb{P}$ and $P_{x,\omega}$, which is characterized by the formula*

$$P_{x,\mathbb{P}}[F \times G] = \int_F P_{x,\omega}[G]\mathbb{P}(\mathrm{d}\omega), \qquad F \in \mathscr{B}(\Omega), G \in \mathscr{G}.$$

The ***annealed*** *or* ***averaged probability*** *of the random walk in the random environment $\mathbb{P}_x$ is the marginal law of $P_{x,\mathbb{P}}$ on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathscr{G})$.*

2.2 Background of this thesis

Now we focus on the necessary background for understanding the main result proved in this thesis, which is a generalization of the zero-one law.

2.2.1 Different kind of environment spaces

We define a complete list of all the necessary environments for comprehending this thesis.

Definition 5 (Ellipticity and uniform ellipticity)**.** *The environment space Ω is said to be*

- ***elliptic*** *if*

$$\mathbb{P}\left[\min_{e \in U} \omega(x,e) > 0\right] = 1, \qquad \forall x \in \mathbb{Z}^d.$$

- ***uniformly elliptic*** *(abbreviated as u.e.) if there exists a constant $\kappa > 0$ such that*

$$\mathbb{P}\left[\min_{e \in U} \omega(x,e) \geq \kappa\right] = 1, \qquad \forall x \in \mathbb{Z}^d.$$

It should be stressed that throughout chapter 3 we assume that the environment space is uniformly elliptic.

Definition 6 (IID)**.** *We say that the environment space Ω is* ***independent and identically distributed****, abbreviated as iid, if the coordinate maps on the product space Ω are independent and identically distributed under $\mathbb{P}$.*

Definition 7 (Translation defined on the environment space). *Given $y \in \mathbb{Z}^d$, t_y is said to be a **translation** on Ω if*

$$t_y\omega(x,e) = \omega(x+y,e), \quad \forall x \in \mathbb{Z}^d \, \forall e \in U.$$

Definition 8 (Stationary). *The environment space Ω is said to be **stationary** if*

$$\mathbb{P}[A] = \mathbb{P}[t_x A], \quad \forall A \in \mathscr{B}(\Omega) \, \forall x \in \mathbb{Z}^d$$

Now, we are able to define the quality of being *totally ergodic*.

Definition 9 (Totally ergodic). *We say that the environment space Ω is **totally ergodic** if the family of transformations $\{t_x\}_{x\in\mathbb{Z}^d}$ is an ergodic family acting on $(\Omega, \mathscr{B}(\Omega), \mathbb{P})$.*

The last kind of environment and one of our assumptions throughout chapter 3 (together with the hypothesis of being uniformly elliptic) is the quality of being *finite rank*.

Definition 10 (Finite rank). *The environment space Ω is said to be **finite rank** if there exists a constant $R \in \mathbb{N}$ such that $\{\omega(x)\}_{x\in A}$ and $\{\omega(x)\}_{x\in B}$ are independent whenever $A,B \subseteq \mathbb{Z}^2$ and $d(A,B) > R$. Furthermore, the constant R is called the **rank** of the environment space.*

2.2.2 Results

Now we are able to discuss some results proved by Zerner & Merkl in [13] as well as Sznitman & Zerner in [9]. In addition, we can state the main result provided by this thesis and we can see how our result is related to Zerner & Merkl's work and Sznitman & Zerner's work.

In 1999, Sznitman & Zerner (see [9]) demonstrated the Kalikow's zero-one law for iid and uniformly elliptic environments. Moreover, the theorem was improved by Zerner & Merkl in [13] for elliptic environments. This theorem can be stated as

Theorem 1 (Kalikow's zero-one law)**.** *Assume $d \geq 2$. Suppose the environment space $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ is elliptic and iid. Then*

$$\mathbb{P}_0\left[\left\{\lim_{n\to\infty}\langle X_n, l\rangle = \infty\right\} \cup \left\{\lim_{n\to\infty}\langle X_n, -l\rangle = \infty\right\}\right] \in \{0,1\}. \tag{2.1}$$

Roughly speaking, Theorem 1 says that given a line $\mathscr{L}$ in $\mathbb{Z}^d$ with $d \geq 2$, a random walk moving in a *suitable* environment moves in a *rough sense* along the line $\mathscr{L}$ with an annealed probability 0 or 1. Hence, one natural question which arises from Kalikow's zero-one law is the following:

Question 1. *Instead of considering a line $\mathscr{L}$, let us consider a ray $\mathscr{R}$ starting at zero. Is it true that the same result holds?*

If we assume that the dimension is 2, the answer is yes and is Theorem 2, but before establishing so, we proceed to translate and formalize this question into a mathematical setting. We follow the formalization given in [14].
Given $d \in \mathbb{Z}^+$ and $l \in \mathbb{S}^{d-1}$, we define the event A_l as

$$A_l := \left\{\lim_{n\to\infty}\langle X_n, l\rangle = \infty\right\}.$$

The previous event is what we denominate the event that "**the walk tends in a rough sense towards the direction** l".

Now we are able to state the original zero-one law, which can be found in [14].

Theorem 2 (Zero-one law)**.** *Assume $d = 2$. Let $l \in \mathbb{S}^1$ and let $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ be i.i.d. and elliptic under $\mathbb{P}$. Then*

$$\mathbb{P}_0[A_l] \in \{0,1\}.$$

Whereas, the above result is false if we assume another kind of environment. In [14], Zerner showed a counterexample of the result in theorem 2 for stationary and totally ergodic environments.

Theorem 3. *Assume $d = 2$. There exists a stationary and totally ergodic environment $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ such that*

$$\mathbb{P}_0\left[\lim_{n\to\infty}\frac{\langle X_n, \hat{e}_1\rangle}{n} \geq \frac{1}{2}\right] > 0, \qquad \mathbb{P}_0\left[\lim_{n\to\infty}\frac{\langle X_n, -\hat{e}_1\rangle}{n} \geq \frac{1}{2}\right] > 0.$$

In particular,

$$0 < \mathbb{P}_0\left[A_{\hat{e}_1}\right] < 1.$$

Keeping in mind the above results, we show in this thesis a version of the zero-one law for finite rank environment, which is a weaker condition than being independent. Nevertheless, we were unable to keep the the ellipticity condition and we use uniformly elliptic environments.

Theorem 4 (Maturana, R. (2018))**.** *Assume $d = 2$. Let $l \in \mathbb{S}^1$ and let $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ be finite rank R and uniformly elliptic under $\mathbb{P}$. Then*

$$\mathbb{P}_0[A_l] \in \{0,1\}.$$

Chapter 3

A generalization of the zero-one law

Each new discovery furnishes a step which leads on the complete truth.

— *Arthur Conan Doyle,* "The Adventure of the engineer's thumb".

This chapter is devoted to prove a generalization of the zero-one law for planar random walks in finite rank and uniformly elliptic environments, which gives the title to this thesis. So as to accomplish this task, we need to demonstrate two previous theorems. Since it is an arduous task and we want to expose the thesis as clearly as possible, this chapter is divided into two sections. Section 3.1 is focused on proving a generalization of a theorem presented in [5]. This theorem is the cornerstone for the generalization of Kalikow's zero-one law, which was proved in [9] for iid environments. The main feature of the first section is the fact that it is developed for an arbitrary dimension greater than or equal to two. Afterwards, in section 3.2, we focus on demonstrating the generalization of the zero-one law, the proof of this theorem relies strongly on Kalikow's zero-one law.

3.1 A generalization of the Kalikow's zero-one law

As already stressed above, in this chapter we present two theorems for arbitrary dimension greater than or equal to two. The first theorem has interest in its own right. Roughly speaking, the following theorem says that given any line $\mathscr{L}$, the event that the random walk hits the previous line i.o. has annealed probability zero or one. Additionally, another important feature of the mentioned theorem is the fact that it is the main ingredient in the proof of Theorem 6. Besides, a version of this theorem can be found in [5] for iid environments with a "certain" ellipticity condition; namely,

$$\mathbb{P}_0[\min\{\omega(\cdot,\cdot-e),\omega(\cdot,\cdot+e)\}>0]=1,$$

which clearly holds for u.e. environments.

Theorem 5 (Maturana, R. (2018)). *Assume $d\geq 2$. Suppose the environment $\{\omega(x,e)\}_{x\in\mathbb{Z}^d,e\in U}$ is uniformly elliptic and has finite rank R. Then, the annealed probability that the process hits the hyperplane $\mathscr{H}=\{a\in\mathbb{Z}^d:\langle a,l\rangle=0\}$ i.o. is zero or one for any $l\in\mathbb{S}^{d-1}$.*

But before giving a proof to Theorem 5, we need to prove the following lemma, which is going to be quite useful.

Lemma 1. *Assume $d\geq 2$. Suppose the environment $\{\omega(x,e)\}_{x\in\mathbb{Z}^d,e\in U}$ is uniformly elliptic and has finite rank R. Let $l\in\mathbb{S}^{d-1}$ and assume*

$$\mathbb{P}_0[\exists N\in\mathbb{N}\text{ such that }\langle X_n,l\rangle>0\;\forall n>N]>0.$$

Hence

$$\mathbb{P}_0\left[\{\langle X_n,l\rangle>0\;\forall n\in\mathbb{Z}^+\}\right]>0$$

Proof of Lemma 1. Set

$$D:=\{\langle X_n,l\rangle>0\;\forall n\in\mathbb{Z}^+\}.$$

Assume, for the sake of contradiction, $\mathbb{P}_0(D) = 0$. Hence $P_{0,\omega}(D) = 0$ $\mathbb{P}-$a.s. Given $v \in \mathbb{Z}^d$, let $D^v := \{\langle X_n, l\rangle > \langle v, l\rangle \ \forall n \in \mathbb{Z}^+\}$. Using the invariance under translations of the annealed probability, we have $\mathbb{P}_v(D^v) = \mathbb{P}_0(D) = 0$. Therefore, $P_{v,\omega}(D^v) = 0$ $\mathbb{P}-$a.s. Let

$$\Omega' := \{\omega \in \Omega : P_{v,\omega}(D^v) = 0 \ \forall v \in \mathbb{Z}^d\}$$

$$= \bigcap_{v \in \mathbb{Z}^d} \{\omega \in \Omega : P_{v,\omega}(D^v) = 0\}.$$

It follows easily that $\mathbb{P}(\Omega') = 1$.

Given $\omega \in \Omega'$, by the Markov property, we have that

$$\mathbb{P}_{0,\omega}[\langle X_n, l\rangle > \langle v, l\rangle \ \forall n > N | X_N = v] = 0, N \in \mathbb{N}.$$

As a consequence,

$$\mathbb{P}_{0,\omega}[\langle X_n, l\rangle \le \langle X_N, l\rangle \text{ for some } n > N] = 1, N \in \mathbb{N}.$$

It follows that

$$\mathbb{P}_{0,\omega}[\langle X_n, l\rangle \le 0 \text{ i.o.}] = 1.$$

From the former equation and the datum that Ω' has probability 1, it follows at once

$$\mathbb{P}_0[\langle X_n, l\rangle \le 0 \text{ i.o.}] = \mathbb{E}[\mathbb{P}_{0,\omega}[\langle X_n, l\rangle \le 0 \text{ i.o.}]] = 1.$$

This fact contradicts our assumption that

$$\mathbb{P}_0[\exists N \in \mathbb{N} \text{ such that } \langle X_n, l\rangle > 0 \ \forall n > N] > 0.$$

In consequence, $\mathbb{P}_0(D) > 0$, which is the desired result.

$\square$

Proof of Theorem 5. Define A, B and C as the following events:

$$\begin{aligned} A &:= \{X_n \text{ hits the hyperplane } \mathscr{H} \text{ i.o.}\} \\ B &:= \{\exists N \in \mathbb{N} \text{ such that } \langle X_n, l\rangle > 0 \, \forall n > N\} \\ C &:= \{\exists N \in \mathbb{N} \text{ such that } \langle X_n, l\rangle < 0 \, \forall n > N\} \end{aligned}$$

Firstly, assume that $\mathbb{P}_0(B) = \mathbb{P}_0(C) = 0$, hence $\mathbb{P}_0(A) = 1$, which is what we claim. Secondly, assume that either $\mathbb{P}_0(B) > 0$ or $\mathbb{P}_0(C) > 0$. Suppose without loss of generality, $\mathbb{P}_0(B) > 0$. We claim that $\mathbb{P}_0(A) = 0$.

For $N \in \mathbb{Z}^+$ and $v \in \mathbb{Z}^d$ with $\|v\|_1 \leq N$. We define the following events

$$G := \left\{ \overline{\lim_{n\to\infty}} \langle X_n, l\rangle = \infty \right\} \qquad \& \quad G_{N,v} = \{\langle X_n, l\rangle > \langle X_N, l\rangle \, \forall n > N, X_N = v\} \tag{3.1}$$

We proceed to compute the conditional probability of $G_{N,v}$ given $X_N = v$, X_{N-1}, ... , X_1.

$$\begin{aligned} &\mathbb{P}_0[\langle X_n, l\rangle > \langle v, l\rangle : \forall n > N | X_1, X_2, \ldots, X_N = v] \\ &= \frac{\mathbb{E}[P_{0,\omega}[\langle X_n, l\rangle > \langle v, l\rangle : \forall n > N, X_N = v, \ldots, X_2, X_1]}{\mathbb{P}_0[X_1, X_2, \ldots, X_N = v]}. \end{aligned}$$

We have just used both the definition of conditional probability and the definition of annealed probability.

Set $A = \mathbb{P}_0[X_1, X_2, \ldots, X_N = v]^{-1}$. Let L be the least natural number u such that

$$\begin{cases} \|(v + u\hat{e}_1)\|_1 > R + N, \, \forall j \in \{1, 2, \cdots, N\} : \text{ if } l \neq \pm e_2, \\ \|(v + u\hat{e}_2)\|_1 > R + N, \, \forall j \in \{1, 2, \cdots, N\} : \text{ otherwise} \end{cases}$$

Observe that the constant R is the rank of the environments and since the random walk moves from one point to the nearest neighbor at each step of time, then the walk moves inside the square $[-N, N]^2$. So the following set $\{a \in \mathbb{Z}^d : \langle a, l\rangle \geq \langle v + L\hat{e}_1\rangle\}$

is independent of the former square. The same happens with $\{a \in \mathbb{Z}^d : \langle a, l\rangle \geq \langle v + L\hat{e}_2\rangle\}$.

Suppose $l \neq \hat{e}_2$. From the previous computation, it follows that

$$\begin{aligned}
&\mathbb{P}_0[\langle X_n, l\rangle > \langle v, l\rangle : \forall n > N | X_1, X_2, \ldots, X_N = v] \\
&\geq A\mathbb{E}[P_{0,\omega}[\langle X_n, l\rangle > \langle v + L\hat{e}_1, l\rangle : \forall n > N + L, X_{N+L} = v + L\hat{e}_1, X_{L-1} = v + (L-1)\hat{e}_1 \\
&\quad , \ldots, X_{N+1} = v + \hat{e}_1, X_N = v, \ldots, X_2, X_1] \\
&= A\mathbb{E}[P_{0,\omega}[X_N = v, \ldots, X_2, X_1] P_{0,\omega}[X_{N+1} = v + \hat{e}_1 | X_N = v, \ldots, X_2, X_1] \cdot \\
&\quad \ldots \cdot P_{0,\omega}[X_{N+L-1} = v + (L-1)\hat{e}_1 | X_{N+L-2} = v + (L-2)\hat{e}_1, \\
&\quad \ldots, X_{N+1} = v + \hat{e}_1, X_N = v, \ldots, X_2, X_1] \cdot \\
&\quad P_{0,\omega}[X_{N+L} = v + L\hat{e}_1 | X_{N+L-1} = v + (L-1)\hat{e}_1, \\
&\quad \ldots, X_N = v, \ldots, X_2, X_1] \cdot \\
&\quad P_{0,\omega}[\langle X_n, l\rangle > \langle v + L\hat{e}_1, l\rangle : \forall n > N + L | X_{N+L} = v + L\hat{e}_1, \\
&\quad X_{N+L-1} = v + (L-1)\hat{e}_1, \ldots, X_N = v, \ldots, X_2, X_1]]
\end{aligned}$$

Now we use the fact that the random walk in the environment ω is a Markov chain and the fact that the environment is u.e. to obtain

$$\begin{aligned}
&\mathbb{P}_0[\langle X_n, l\rangle > \langle v, l\rangle : \forall n > N | X_1, X_2, \ldots, X_N = v] \geq \\
&\quad A\mathbb{E}[P_{0,\omega}[X_N = v, \ldots, X_2, X_1] \cdot \\
&\quad P_{0,\omega}[X_{N+1} = v + \hat{e}_1 | X_N = v] \cdot \\
&\quad \ldots \cdot P_{0,\omega}[X_{N+L-1} = v + (L-1)\hat{e}_1 | X_{N+L-2} = v + (L-2)\hat{e}_1] \cdot \\
&\quad P_{0,\omega}[X_{N+L} = v + L\hat{e}_1 | X_{N+L-1} = v + (L-1)\hat{e}_1] \cdot \\
&\quad P_{0,\omega}[\langle X_n, l\rangle > \langle v + L\hat{e}_1, l\rangle : \forall n > N + L | X_{N+L} = v + L\hat{e}_1] \\
&\geq A\kappa^L \mathbb{E}[P_{0,\omega}[X_N = v, \ldots, X_2, X_1] \cdot \\
&\quad P_{0,\omega}[\langle X_n, l\rangle > \langle v + L\hat{e}_1, l\rangle : \forall n > N + L | X_{N+L} = v + L\hat{e}_1]]
\end{aligned}$$

Note that $\{X_N = v,\dots,X_2,X_1\}$ is independent of the set $Y := \{z \in \mathbb{Z}^d : \langle z,l\rangle > \langle v + L\hat{e}_1,l\rangle\}$ by the choice of L. Then,

$$\mathbb{P}[\langle X_n,l\rangle > \langle v,l\rangle : \forall n > N | X_1,X_2,\dots,X_N = v] \geq \kappa^L \mathbb{P}_{v+L\hat{e}_1}(D^{v+L\hat{e}_1}) = \kappa^L \mathbb{P}_0[D] > 0, \tag{3.2}$$

where in the last step we use the Lemma (1).
The same result is true if we suppose $l = \pm\hat{e}_2$. We just need to change $\hat{e}_1$ by $\hat{e}_2$ in the former computation.
Using inequality (3.2), we infer that on G there are infinitely many N for which

$$\mathbb{P}_0[\langle X_n,l\rangle > \langle X_N,l\rangle \;\forall n > N | X_1,\dots,X_N] \geq \kappa^L \mathbb{P}_0[D]$$

and $\langle X_N,l\rangle > 0$. Let A^c the complement of A. Hence, on G, we infer that

$$\overline{\lim_{n\to\infty}} \mathbb{P}_0[A^c | X_1,\cdots,X_N] \geq \kappa^L \mathbb{P}_0[D] > 0.$$

On the other hand, by the martingale convergence theorem,

$$\lim_{n\to\infty} \mathbb{P}_0[A^c | X_1,\cdots,X_N] = 0$$

on A a.a. In consequence,

$$\mathbb{P}_0[A \cap G] = 0. \tag{3.3}$$

Besides, we claim that

$$\mathbb{P}_0[A \cap G^c] = 0. \tag{3.4}$$

Because, if $l \neq \pm\hat{e}_2$, then $\mathbb{P}_0[\pi_x(X_{n+N}) - \pi_x(X_N) = n | X_1,X_2,\dots,X_N] \geq \kappa^n$. Otherwise, $\mathbb{P}_0[\pi_y(X_{n+N}) - \pi_y(X_n) = n | X_1,X_2,\dots,X_N] \geq \kappa^n$. From this argument follows either $\pi_x(X_n) \geq R$ or $\pi_y(X_n) \geq R$ for infinitely many n on almost all of A for all $R \in \mathbb{Z}^+$.

Using equations (3.3) and (3.4), we assert that

$$\mathbb{P}_0[A] = 0,$$

which was the desired result. □

Having proved the previous theorem we are able to generalize Kalikow's zero-one law for finite rank and uniformly elliptic environments. The proof follows the same lines as in [9]. However, so as to keep the fact that this thesis is self-contained, we paraphrase the proof in the previously mentioned paper.

Theorem 6 (Kalikow's zero-one law for finite rank and uniformly elliptic environments)**.** *Assume $d \geq 2$. Suppose the environment $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ is uniformly elliptic and has finite rank R. Then*

$$\mathbb{P}_0[A_l \cup A_{-l}] \in \{0,1\}. \tag{3.5}$$

Proof of Theorem 6. We start defining the events B_l and C_l as

$$\begin{aligned} B_l &:= A_l \cup A_{-l} \\ C_l &:= \{\langle X_n, l\rangle \text{ remains of constant sign for large } n\} = A^c \end{aligned}$$

(see the definition of A given at the beginning of the proof of theorem 5). From the above definition, there are two immediate consequences. The first one is

$$B_l \subseteq C_l. \tag{3.6}$$

The second one is

$$\mathbb{P}_0[C_l] \in \{0,1\}, \tag{3.7}$$

this follows at once from theorem 5.

Using (3.6) and the result (3.7), we deduce that

$$\mathbb{P}_0[C_l] = 0 \Rightarrow \mathbb{P}_0[B_l] = 0.$$

Therefore, the only case that remains is when $\mathbb{P}_0[C_l] = 1$. Henceforward, we assume this condition.

We proceed to demonstrate the following lemma, which is going to help us to prove the result (3.5).

Lemma 2. *For* $M > 0$,

$$\{\langle X_n, l\rangle \in [0,M] \; i.o.\} \subseteq \{\langle X_n, l\rangle < 0 \; i.o.\} \quad \mathbb{P}_0\text{-}a.s. \tag{3.8}$$

Grosso modo, for $d = 2$, Lemma 2 asserts that the event that the walk reaches i.o. the slab $\{a \in \mathbb{Z}^d : 0 \le \langle a, l\rangle \le M\}$. Then a.s. in average the walker reaches i.o. the half-plane $\{a \in \mathbb{Z}^d : \langle a, l\rangle < 0\}$.

Proof of Lemma 2. Using the condition of u.e. of the environment, we are able to choose N large enough and $c > 0$ such that

$$P_{x,\omega}\left[H_{\{a\in\mathbb{Z}^d:\langle a,l\rangle<0\}} \le N\right] \ge c, \text{ for } \omega \in \Omega \text{ and } x \in \{a \in \mathbb{Z}^d : 0 \le \langle a, l\rangle \le M\} \tag{3.9}$$

In other words, roughly speaking, for a large enough period of time N, the walker starting at the point x in the environment ω reaches the set $\{a \in \mathbb{Z}^d : \langle a, l\rangle < 0\}$ in at most N steps with a positive (quenched) probability.

Hence, we construct recursively the successive return times to the set $\{a \in \mathbb{Z}^d : 0 \le$

$\langle a,l\rangle \leq M\}$.

$$V_0 := 0$$
$$V_1 := H_{\{a\in\mathbb{Z}^d:0\leq\langle a,l\rangle\leq M\}} \leq \infty$$
$$V_{k+1} := V_1\circ\theta_{V_k+N} + V_k + N \leq \infty \text{ for } k\in\mathbb{Z}^+$$

Given $k\in\mathbb{Z}^+$, we define the events G_k and H_k such that

$$\mathbb{1}_{G_k} := \mathbb{1}_{\{V_k<\infty\}} \qquad \mathbb{1}_{H_k} := \begin{cases} \mathbb{1}_{\{H_{\{a\in\mathbb{Z}^2:\langle a,l\rangle<0\}}\}}\circ\theta_{V_k}, \text{ if } V_k<\infty \\ 0, \text{ otherwise.}\end{cases}$$

From the definition, it is quite clear that $G_k\in\mathscr{F}_{V_k}$ and $H_k\in\mathscr{F}_{V_{k+1}}$.
Using inequality (3.9), the strong Markov property and $\mathbb{P}-$integration, we infer that

$$\mathbb{P}_0[H_k|\mathscr{F}_{V_k}] = \mathbb{E}[P_{0,\omega}[H_k|\mathscr{F}_{V_k}]] = \mathbb{E}[P_{X_{V_k},\omega}[H_{\{a\in\mathbb{Z}^2:\langle a,l\rangle<0\}}\leq N|\mathscr{F}_{V_k}]] \geq c\,\mathbb{1}_{G_k}, \quad k\geq 1. \tag{3.10}$$

We proceed to quote the following version of the Borel-Cantelli's lemma. This lemma can be found in [3].

Lemma 3 (Second Borel-Cantelli lemma)**.** *Let $\{\mathscr{F}_n\}_{n\geq0}$ be a filtration with $\mathscr{F}_0 = \{\emptyset,\Omega\}$ and $\{A_n\}_{n\geq1}$ a sequence of events with $A_n\in\mathscr{F}_n$. Then*

$$\{An\ i.o.\} = \left\{\sum_{n\geq1} P[A_n|\mathscr{F}_{n-1}] = \infty\right\} \tag{3.11}$$

We proceed to verify that the hypotheses are hold to apply the second Borel-Cantelli's lemma to our problem.
Firstly, $\{\mathscr{F}_{V_n}\}_{n\geq0}$ is a filtration and $\mathscr{F}_{V_0}$ is trivial, since $V_0 = 0$ (i.e., a deterministic variable). Secondly, setting $A_n = H_n$, it holds that A_n belongs to $\mathscr{F}_{V_n}$. Then we have

equation (3.11), which is

$$\{H_n \text{ i.o.}\} = \left\{ \sum_{n\geq 1} P[H_n | \mathscr{F}_{V_{n-1}}] = \infty \right\} \tag{3.12}$$

in our context.

Using equation (3.12) and inequation (3.10) we can deduce that

$$\sum_{k\geq 1} \mathbb{1}_{H_k} = \infty \text{ on } \left\{ \sum_{k\geq 1} \mathbb{1}_{G_k} = \infty \right\} \qquad \mathbb{P}_0 - a.s.$$

From the above result, it is straightforward that (3.8) is true. □

Making use of Lemma 2, we have the following computation for $M > 0$

$$\mathbb{P}_0[\{\exists N \in \mathbb{N} : \langle X_n, l\rangle > 0\, \forall n > N\} \cap \{\langle X_n, l\rangle \in [-M, M] \text{ i.o.}\}]$$

$$\leq \mathbb{P}_0[\{\exists N \in \mathbb{N} : \langle X_n, l\rangle > 0\, \forall n > N\} \cap \{\exists \tilde{N} \in \mathbb{N} : \langle X_n, l\rangle < 0\, \forall n > \tilde{N}\}] = 0.$$

Replacing l by $-l$ in the Lemma (2) we have that

$$\{\langle X_n, l\rangle \in [-M, 0] \text{ i.o.}\} \subseteq \{\langle X_n, l\rangle > 0 \text{ i.o.}\} \quad \mathbb{P}_0\text{-a.s.}.$$

Then,

$$\mathbb{P}_0[\{\exists N \in \mathbb{N} : \langle X_n, l\rangle < 0\, \forall n > N\} \cap \{\langle X_n, l\rangle \in [-M, M] \text{ i.o.}\}]$$

$$\leq \mathbb{P}_0[\{\exists N \in \mathbb{N} : \langle X_n, l\rangle < 0\, \forall n > N\} \cap \{\exists \tilde{N} \in \mathbb{N} : \langle X_n, l\rangle > 0\, \forall n > \tilde{N}\}] = 0.$$

Hence,

$$\mathbb{P}_0[C_l \cap \{\langle X_n, l\rangle \in [-M, M] \text{ i.o.}\}] = 0,$$

since we have supposed that $\mathbb{P}_0[C_l] = 1$ and M is arbitrary. The previous result claims that the RWRE is not bounded $\mathbb{P}_0 - a.s.$, which means $\mathbb{P}_0[B_l] = 1$. □

Remark 1. C.f. *([12], page 258). In the previous paper there is a theorem proved* *by Zeitouni, which can be applied to the case of finite rank and uniformly elliptic environments. This theorem could replace theorem 6.*

3.2 A generalization of the zero-one law

We have reached the necessary amount of information to prove the main result of this thesis. Besides, we have to say that this proof is based strongly on the proof given by Zerner in [14], therefore there are some arguments from the mentioned paper adapted to our context.

Theorem (Maturana, R. (2018))**.** *Assume $d = 2$. Let $l \in \mathbb{S}^1$ and let $\{\omega(x,e)\}_{x\in\mathbb{Z}^d, e\in U}$ be finite rank R and uniformly elliptic under $\mathbb{P}$. Then*

$$\mathbb{P}_0[A_l] \in \{0,1\}. \tag{3.13}$$

Proof of the main theorem. By Theorem 6

$$\mathbb{P}_0[B_l] = \mathbb{P}_0[A_l \cup A_{-l}] \in \{0,1\}.$$

From the previous argument, it is straightforward that if $\mathbb{P}_0[B_l] = 0$, then $\mathbb{P}_0[A_l] = 0$, which is a desired result.

Henceforth, we assume that $\mathbb{P}_0[B_l] = 1$.

Given $u \in \mathbb{R}$ and $\diamond \in \{<, \leq, >, \geq\}$ we define the following stopping times

$$T_\diamond := \inf\{n \in \mathbb{N} : \langle X_n, l\rangle \diamond u\}.$$

We claim the following useful lemma.

Lemma 4. *Assuming all the previous hypotheses.*

$$0 = \mathbb{P}_0\left[T_{<0} = \infty\right]\mathbb{P}_0\left[T_{>0} = \infty\right]. \tag{3.14}$$

$$\Downarrow$$

$$\mathbb{P}_0[A_l] \in \{0,1\}.$$

Proof of Lemma 4. Suppose equation (3.14) holds, then either $\mathbb{P}_0[T_{<0} = \infty] = 0$ or $\mathbb{P}_0[T_{>0} = \infty] = 0$ if and only if $\mathbb{P}_0-$a.s. $T_{<0} < \infty$ or $\mathbb{P}_0-$a.s. $T_{>0} < \infty$.
Without loss of generality, by analogous arguments, we can assume that $\mathbb{P}_0-$a.s. $T_{<0} < \infty$ holds. Using the translation invariance of the annealed probability, we have that given $x \in \mathbb{Z}^2$

$$1 = \mathbb{P}_x\left[T_{<\langle x,l\rangle} < \infty\right] = \mathbb{E}\left[P_{x,\omega}\left[T_{<\langle x,l\rangle} < \infty\right]\right] \text{ for each } x \in \mathbb{Z}^2.$$

Then, $P_{x,\omega}\left[T_{<\langle x,l\rangle} < \infty\right] = 1$ $\mathbb{P}-$a.s.
Given $x \in \mathbb{Z}^2$, $\Omega_x := \left\{\omega \in \Omega : T_{<\langle x,l\rangle} < \infty\right\}$. Therefore, the event $\Omega' := \bigcap_{x\in\mathbb{Z}^2} \Omega_x$ has probability 1 under $\mathbb{P}$. Since is a countable intersection of events with probability 1. Ergo,

$$P_{x,\omega}[T_{<\langle x,l\rangle} < \infty] = 1 \qquad \mathbb{P} - a.s.$$

Using the strong Markov property, we can infer that $\mathbb{P}_0[A_l] = 0$. Similarly, if $\mathbb{P}_0-$a.s. $T_{>0} < \infty$ holds, then $\mathbb{P}_0[A_{-l}] = 0$, which implies that $\mathbb{P}_0[A_l] = 1$. This follows from the hypothesis that $\mathbb{P}_0[B_l] = 1$ and the fact that A_l and A_{-l} are disjoint sets. □

So, all our proof depends on proving equation (3.14) is true. In order to accomplish that, we note that the restriction that the walker moves from one point to the nearest neighbors implies

$$T_{\geq L} \geq L \qquad\qquad T_{\leq -L} \geq L, \qquad L \in \mathbb{N}. \tag{3.15}$$

Hence, proving equation (3.14) is equivalent to proving the following equation

$$\lim_{L\to\infty} \mathbb{P}_0[T_{\geq L} < T_{<0}]\mathbb{P}_0[T_{\leq -L} < T_{>0}] = 0. \tag{3.16}$$

Now, for $L \in \mathbb{N}$ we pick a $z_L \in \mathbb{Z}^2$ such that it fulfills the next three conditions

$$x_L := \langle z_L, l \rangle \geq 2(L+R), \tag{3.17}$$

$$z_L \text{ has a nearest neighbor } w_L \in \mathbb{Z}^2 \text{ with } \langle w_L, l \rangle \leq 2(L+R) \tag{3.18}$$

$$y_L := \langle z_L, l^\perp \rangle \text{ and } \mathbb{P}_0 \left[\langle X_{T_{\geq 2(L+R)}}, l^\perp \rangle \diamond y_L | T_{\geq 2(L+R)} < T_{<0} \right] \leq \frac{1}{2} \quad \diamond \in \{<,>\}, \tag{3.19}$$

where $l^\perp$ is a fixed vector such that $l^\perp \in \mathbb{S}^1$ and $\langle l, l^\perp \rangle = 0$.
Given $L \in \mathbb{N}$, we begin changing the starting point in the second factor of equation (3.16) from 0 to z_L. Hence, after using the translation invariance, we are able to rewrite equation (3.16) as

$$\lim_{L\to\infty} \mathbb{P}_0[T_{\geq L} < T_{<0}] \mathbb{P}_{z_L}[T_{\leq L+2R} < T_{>x_L}] = 0. \tag{3.20}$$

In order to write the product of probabilities in equation (3.20) as a single probability, we create two independent random walks moving in the same environment ω. One of them is starting at 0, the other one is starting at z_L.
Given $\omega \in \Omega$ and $L \in \mathbb{N}$, let $P_{0,z_L,\omega}$ be a probability measure on $(\mathbb{Z}^2)^{\mathbb{N}} \times (\mathbb{Z}^2)^{\mathbb{N}}$ such that the two canonical processes of projections $(X_n^1)_n$ and $(X_n^2)_n$ on this space are independent of each other and have distributions $P_{0,\omega}$ and $P_{z_l,\omega}$, respectively, and denote by $\mathbb{P}_{0,z_L}$ the corresponding annealed measure. Stopping times referring to the walks $(X_n^j)_n$ are going to be marked with an upper index $j \in \{1,2\}$. Therefore, by independence, we have the following identity

$$\lim_{L\to\infty} P_{0,z_L} \left[T^1_{\geq L} < T^1_{<0}, T^2_{\leq L+2R} < T^2_{>x_L} \right] = \lim_{L\to\infty} \mathbb{P}_0[T_{\geq L} < T_{<0}] \mathbb{P}_{z_L}[T_{\leq L+2R} < T_{>x_L}]. \tag{3.21}$$

Consider the walk $(X_n^1)_n$, after crossing the line $\{a \in \mathbb{Z}^2 : \langle a, l \rangle = L\}$, the walk must cross the line $\{a \in \mathbb{Z}^2 : \langle a, l \rangle = 2(L+R)\}$ or $\{a \in \mathbb{Z}^2 : \langle a, l \rangle = 0\}$ a.s., due to Theorem 6. Similarly, the walk $(X_n^2)_n$, after crossing the line $\{a \in \mathbb{Z}^2 : \langle a, l \rangle = L+R\}$, the walk must cross the line $\{a \in \mathbb{Z}^2 : \langle a, l\rangle = 0\}$ or $\{a \in \mathbb{Z}^2 : \langle a, l\rangle = 2(L+R)\}$ a.s. As a consequence, we have the following inequality,

$$\lim_{L\to\infty} P_{0,z_L}\left[T^1_{\geq L} < T^1_{<0}, T^2_{\leq L+2R} < T^2_{>x_L}\right] \tag{3.22}$$

$$\leq \varliminf_{L\to\infty} \mathbb{P}_0\left[T_{\geq L} < T_{<0} < \infty\right] \tag{3.23}$$

$$+\mathbb{P}_{z_L}\left[T_{\leq L+2R} < T_{>x_L} < \infty\right] \tag{3.24}$$

$$+P_{0,z_L}\left[T^1_{\geq 2(L+R)} < T^1_{<0}, T^2_{\leq 0} < T^2_{>x_L}\right]. \tag{3.25}$$

We claim that the inferior limit of (3.23) and (3.24) is zero. This follows easily from the following computation

$$\mathbb{P}_0\left[T_{\geq L} < T_{<0} < \infty\right] \leq \mathbb{P}_0\left[\exists n \geq L : |\langle X_n, l\rangle| \leq 1\right].$$

On the other hand, due to Theorem 6, we infer that

$$\lim_{L\to\infty} \mathbb{P}_0\left[\exists n \geq L : |\langle X_n, l\rangle| \leq 1\right] = 0.$$

Consequently,

$$\varliminf_{L\to\infty} \mathbb{P}_0\left[T_{\geq L} < T_{<0} < \infty\right] = 0. \tag{3.26}$$

Similarly,

$$\varliminf_{L\to\infty} \mathbb{P}_{z_L}\left[T_{\leq L+2R} < T_{>x_L} < \infty\right] = 0. \tag{3.27}$$

Because,

$$\mathbb{P}_{z_L}\left[T_{\leq L+2R} < T_{>x_L} < \infty\right] = \mathbb{P}_0\left[T_{\leq -L} < T_{>0} < \infty\right],$$

by the translation invariance.

By equations (3.21), (3.26) and (3.27), the discussed inequality becomes

$$\mathbb{P}_0\left[T_{<0} = \infty\right]\mathbb{P}_0\left[T_{>0} = \infty\right] \leq \varliminf_{L\to\infty} P_{0,z_L}\left[T^1_{\geq 2(L+R)} < T^1_{<0}, T^2_{\leq 0} < T^2_{>x_L}\right]. \tag{3.28}$$

Consider the event of the right hand side of inequality (3.28). Any random walk starting at zero and crossing the line $\{a \in \mathbb{Z}^2 : \langle a, l\rangle = 2(L+R)\}$ has a tubular neighborhood of width $2R$ restricted to slab $\{z \in \mathbb{Z}^2 : 0 \le \langle z, l\rangle \le 2(L+R)\}$. This tubular neighborhood is the following set:

$$TN^1 := \left\{ z \in \mathbb{Z}^2 : 0 \le \langle z, l\rangle \le 2(L+R), \min_{0 \le n \le T^1_{\ge 2(L+R)}} \left\| z - X^1_n \right\| \le R \right\}.$$

Therefore, for any random walk starting at z_L and crossing the line $\{a \in \mathbb{Z}^2 : \langle a, l\rangle = 0\}$, we have two mutually exclusive options:

1. The second random walk **does not** enter TN^1 at each step of time up to $T^2_{\le 0}$.

2. The second random walk enters TN^1 for some step of time less than or equal to $T^2_{\le 0}$.

Consequently,

$$\{T^1_{\ge 2L+R} < T^1_{<0}, T^2_{\le 0} < T^2_{>x_L}\}$$

$$= N_L \sqcup C_L,$$

where

$$N_L := \left\{ T^1_{\ge 2(L+R)} < T^1_{<0}, T^2_{\le 0} < T^2_{>x_L}, \min\left\{ \left\| X^1_j - X^2_k \right\|_1 : j \le T^1_{\ge 2(L+R)}, k \le T^2_{\le 0} \right\} > R \right\}$$

and

$$C_L := \{\exists x \in \mathbb{Z}^2 : H^1(x) \le T^1_{\ge 2(L+R)} < T^1_{<0}, H^2(x + m\hat{e}_j) \le T^2_{\le 0} < T^2_{>x_L}$$

$$\text{for some } j \in \{1,2\} \text{ and } m \in \{-R, \dots, R\}\}$$

Therefore, inequality (3.28) becomes

$$\mathbb{P}_0[T_{<0} = \infty]\mathbb{P}_0[T_{>0} = \infty] \le \varlimsup_{L\to\infty} P_{0,z_L}[N_L] + \varlimsup_{L\to\infty} P_{0,z_L}[C_L]. \tag{3.29}$$

So far we have not used the hypothesis that the dimension is 2, since if we change the dimension from $d = 2$ to $d \geq 2$ arbitrary, the results up to this point would be the same. Nevertheless, the following argument relies strongly on the fact that the random walk is planar. So as to compute $P_{0,z_L}[N_L]$, we note that the endpoints of the trajectory of the random walks confined to the slab $\{a \in \mathbb{Z}^2 : 0 \leq \langle a, l\rangle \leq 2(L+R)\}$ have a common property, which is $y_L - \langle X^1_{T^1_{\geq 2L}}, l^\perp\rangle$ and $\langle X^2_{T^2_{\leq 0}}, l^\perp\rangle$ share the same sign. Consequently,

$$P_{0,z_L}[N_L] =$$

$$\sum_{s=\pm 1} P_{0,z_L}[T^1_{\geq 2(L+R)} < T^1_{<0}, T^2_{\leq 0} < T^2_{>x_L}, \min\left\{\left\|X^1_j - X^2_k\right\|_1 : j \leq T^1_{\geq 2(L+R)}, k \leq T^2_{\leq 0}\right\} > R$$

$$, s = \mathrm{sign}(y_L - \langle X^1_{T^1_{\geq 2(L+R)}}, l^\perp\rangle) = \mathrm{sign}(\langle X^2_{T^2_{\leq 0}}, l^\perp\rangle)]$$

Denoting $\Pi_{L,s}$, $(s \in \{\pm 1\})$, the set of all the finite nearest-neighbor paths that star at z_L and leave the slab $\{a \in \mathbb{Z}^2 : 0 \leq \langle a, l\rangle \leq 2(L+R)\}$ on the opposite side through a vertex x with sign $\langle x, l^\perp\rangle = s$. Hence,

$$P_{0,z_L}[N_L] =$$

$$\sum_{s=\pm 1} \sum_{\pi \in \Pi_{L,s}} P_{0,z_L}[T^1_{\geq 2(L+R)} < T^1_{<0}, \min\left\{\left\|X^1_j - a\right\|_1 : j \leq T^1_{\geq 2(L+R)}, a \in \pi\right\} > R,$$

$$(X^2_n)_n \text{ follows } \pi, s = \mathrm{sign}(y_L - \langle X^1_{T^1_{\geq 2(L+R)}}, l^\perp\rangle) = \mathrm{sign}(\langle X^2_{T^2_{\leq 0}}, l^\perp\rangle)]$$

Using the fact that the environments are finite rank, we infer that

$$P_{0,z_L}[N_L] \le$$

$$\sum_{s=\pm 1}\sum_{\pi\in\Pi_{L,s}} \mathbb{P}_0[T_{\ge 2(L+R)} < T_{<0}, \min\{\left\|X_j^1 - a\right\|_1$$

$$: j \le T_{\ge 2(L+R)}, a \in \pi\} > R, s = \mathrm{sign}(y_L - \langle X_{T_{\ge 2(L+R)}}, l^\perp\rangle)]$$

$$\cdot \mathbb{P}_{z_L}[(X_n)_n \text{ follows } \pi]$$

$$\le \sum_{s=\pm 1}\sum_{\pi\in\Pi_{L,s}} \mathbb{P}_0[T_{\ge 2(L+R)} < T_{<0}$$

$$, s = \mathrm{sign}(y_L - \langle X_{T_{\ge 2(L+R)}}, l^\perp\rangle)] \cdot \mathbb{P}_{z_L}[(X_n)_n \text{ follows } \pi]$$

$$= \sum_{s=\pm 1} \mathbb{P}_0\left[T_{\ge 2(L+R)} < T_{<0}, s = \mathrm{sign}(y_L - \langle X_{T_{\ge 2(L+R)}}, l^\perp\rangle)\right]$$

$$\cdot \mathbb{P}_{z_L}\left[T_{\le 0} < T_{>x_L}, s = \mathrm{sign}(\langle X_{T_{\le 0}}, l^\perp\rangle)\right]$$

Using the condition (3.19) and the above argument, we deduce that

$$\begin{aligned}
P_{0,z_L}[N_L] \le\ & \frac{1}{2}\mathbb{P}_0\left[T_{\ge 2(L+R)} < T_{<0}\right] \\
& \cdot \sum_{s=\pm 1} \mathbb{P}_{z_L}\left[T_{\le 0} < T_{>x_L}, s = \mathrm{sign}(\langle X_{T_{\le 0}}, l^\perp\rangle)\right] \\
& \le \frac{1}{2}\mathbb{P}_0\left[T_{\ge 2(L+R)} < T_{<0}\right]\mathbb{P}_{z_L}\left[T_{\le 0} < T_{>x_L}\right] \\
& = \frac{1}{2}\mathbb{P}_0\left[T_{\ge 2(L+R)} < T_{<0}\right]\mathbb{P}_0\left[T_{\le -2(L+R)} < T_{>0}\right] \\
& \le \frac{1}{2}\mathbb{P}_0[2(L+R) < T_{<0}]\mathbb{P}_0[2(L+R) < T_{>0}] \\
& \longrightarrow_{L\to\infty} \frac{1}{2}\mathbb{P}_0[T_{<0} = \infty]\mathbb{P}_0[T_{>0} = \infty]
\end{aligned}$$

Using the previous result, inequality (3.29) gets

$$\frac{1}{2}\mathbb{P}_0[T_{<0} = \infty]\mathbb{P}_0[T_{>0} = \infty] \le \overline{\lim_{L\to\infty}} P_{0,z_L}[C_L]. \tag{3.30}$$

Considering the event C_L, we have two options:

1. The first time that the random walk starting at z_L enters TN^1 is within slab $\{a \in \mathbb{Z}^2 : 0 \le \langle a, l\rangle \le (L+R)\}$.

2. The first time that the random walk starting at z_L enters TN^1 is within slab $\{a \in \mathbb{Z}^2 : (L+R) \le \langle a, l\rangle \le 2(L+R)\}$.

After reading the above argument, it seems natural to define the following event

$$C_a^b := \{\exists x \in \mathbb{Z}^2 : a \le \langle x, l\rangle \le b, H^1(x) \le T^1_{\ge 2(L+R)} < T^1_{<0},$$

$$H^2(x + m\hat{e}_j) \le T^2_{\le 0} < T^2_{>x_L} \text{ for some } j \in \{1,2\} \text{ and } m \in \{-R, \dots, R\}\},$$

where $a, b \in \mathbb{R}$ are fixed. Therefore,

$$P_{0,z_L}[C_L] \le P_{0,z_L}\left[C_0^{L+R}\right] + P_{0,z_L}\left[C_{L+R}^{2(L+R)}\right].$$

Due to symmetry and translation invariance, it is sufficient to show for the proof of (3.30) that

$$\lim_{L\to\infty} P_{0,z_L}\left[C_0^{L+R}\right] = 0, \tag{3.31}$$

which is what we are going to demonstrate. Let $\epsilon' > 0$ and $0 < \epsilon < \frac{\epsilon'}{\kappa^R}$ and set $r(x,\omega) := P_{x,\omega}[A_l]$. Therefore,

$$P_{0,z_L}\left[C_0^{L+R}\right] \le \mathbb{P}_0\left[C_{0,1}^{L+R}\right] + \mathbb{P}_{z_L}\left[C_{0,2}^{L+R}\right] + P_{0,z_L}\left[C_{0,3}^{L+R}\right], \tag{3.32}$$

where

$$
\begin{aligned}
C_{0,1}^{L+R} &:= \left\{\exists x \in \mathbb{Z}^2 : 0 \le \langle x, l\rangle \le L+R, H(x) \le T_{\ge(L+R)} < \infty, r(x,\omega) \le \epsilon\right\},\\
C_{0,2}^{L+R} &:= \left\{\exists y \in \mathbb{Z}^2 : 0 \le \langle y, l\rangle \le L, H(y) < \infty, r(y,\omega) \ge \epsilon'\right\},\\
C_{0,3}^{L+R} &:= \{\exists x \in \mathbb{Z}^2 : 0 \le \langle x, l\rangle \le L+R, H^1(x) \le T^1_{\ge(L+R)} < \infty,\\
&\qquad H^2(x+m\hat{e}_j) \le T^2_{\le 0} < \infty \text{ for some } j \in \{1,2\} \text{ and } m \in \{-R,\dots,R\}, r(x,\omega) \ge \epsilon,\\
&\qquad r(x+m\hat{e}_j,\omega) \le \epsilon'\}.
\end{aligned}
$$

So as to prove that $C^{L+R} \to 0$, we are going to prove that each summand on the right-hand side of inequality (3.32) are arbitrarily small as $L \to \infty$. Having said this, let us begin this task. In the first place, in order to bound $\mathbb{P}_0\left[C_{0,1}^{L+R}\right]$, we consider $\sigma := \inf\{n \in \mathbb{N} : r(X_n,\omega) \le \epsilon\}$. It is clear that σ is a stopping time with respect to the filtration $\{\mathscr{F}_n\}_n$, where $\mathscr{F}_n$ is the $\sigma-$algebra generated by $X_0, X_1, \dots, X_n$ and the environment ω. Therefore,

$$\mathbb{P}_0\left[C_{0,1}^{L+R}\right] = \mathbb{P}_0[\sigma \le T_{\ge L+R} < \infty] = \mathbb{E}_0\left[P_{X_\sigma,\omega}[T_{\ge L+R} < \infty], \sigma \le T_{\ge L+R}, \sigma < \infty\right],$$

the last equality is justified by the strong Markov property. Now, by Theorem 6, for all $x \in \mathbb{Z}^2$ and for almost all ω, we have

$$P_{x,\omega}[T_{\ge L+R} < \infty] \downarrow A_l, \text{ as } L \to \infty.$$

After using the dominated convergence theorem, the above facts and the definition of σ, we infer that

$$\lim_{L\to\infty} \mathbb{P}_0\left[C_{0,1}^{L+R}\right] = \mathbb{E}_0\left[P_{X_\sigma,\omega}[A_l], \sigma < \infty\right] \le \epsilon, \tag{3.33}$$

giving a desired bound for the first summand. In the second place, we focus on the second summand, which is $\mathbb{P}_{z_L}\left[C_{0,1}^{L+R}\right]$. Using the translation invariance it follows at once that

$$\mathbb{P}_{z_L}\left[C_{0,1}^{L+R}\right] = \mathbb{P}_0\left[\exists y \in \mathbb{Z}^2 : \langle y, l\rangle \leq -(L+2R), r(y,\omega) \geq \epsilon', H(y) < \infty\right].$$

Now we use the above equation and the assumption that $\mathbb{P}_0[A_l \cup A_{-l}] = 1$, which yield the following inequality

$$\mathbb{P}_{z_L}\left[C_{0,1}^{L+R}\right] \leq \mathbb{P}_0\left[T_{-(L+2R)} < \infty, A_l\right] + \mathbb{P}_0\left[\exists n \in \mathbb{N} : n \geq (L+2R), r(X_n,\omega) \geq \epsilon', A_{-l}\right].$$

One readily verifies that the first summand in the right-hand side of the above inequality tends to zero as $L \to \infty$. The same assertion holds for the second summand, because we claim that

Lemma 5.

$$\lim_{n\to\infty} r(X_n,\omega) = \mathbb{1}_{A_l} \qquad \mathbb{P}_0 - a.s. \tag{3.34}$$

Proof of Lemma 5. Given ω, by the Markov property property, we have that

$$r(X_n,\omega) = P_{0,\omega}[A_l | \mathscr{F}_n] \qquad P_{0,\omega} - \text{a.s.}$$

Therefore, $\{r(X_n,\omega)\}_{n\geq 0}$ is a bounded martingale under the quenched probability $P_{0,\omega}$. Now it is easy to see that the event A_l belongs to $\mathscr{F}_\infty := \cup_{n\geq 0}\mathscr{F}_n$. Hence, by the martingale convergence theorem, we infer that

$$\lim_{n\to\infty} r(X_n,\omega) = \mathbb{1}_{A_l} \qquad P_{0,\omega} - \text{a.s.},$$

since ω was arbitrary, the above argument readily implies equation (3.34). □

We have just proved that the second summand is zero as $L \to \infty$. Finally, the third summand is the unique term that remains. So, by the Markov property, we have the following inequalities

$$
\begin{aligned}
\epsilon' &\geq r(x + m\hat{e}_j, \omega) = P_{x+m\hat{e}_j,\omega}(A_l) \\
&= \sum_{e \in U} \omega(x + m\hat{e}_j, e) P_{x+m\hat{e}_j+e,\omega}(A_l) \\
&\geq \kappa P_{x,(m-1)\hat{e}_j}(A_l) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \text{(inductively)} \\
&\geq \kappa^m P_{x,\omega}(A_l)
\end{aligned}
$$

Therefore, using the previous inequalities and remembering that $0 < \kappa < 1$, we deduce that $r(x,\omega) = P_{x,\omega}(A_l) \leq \frac{\epsilon'}{\kappa^R}$. This results leads to the following inequality

$$
P_{0,z_L}\left[C_{0,3}^{L+R}\right] \leq \mathbb{P}_0\left[\exists x \in \mathbb{Z}^2 : 0 \leq \langle x, l\rangle \leq L + R, H(x) \leq T_{\geq(L+R)} < \infty, r(x,\omega) \leq \frac{\epsilon'}{\kappa^R}\right],
$$

now we apply inequality (3.33) to infer that $P_{0,z_L}\left[C_{0,3}^{L+R}\right] \leq \frac{\epsilon'}{\kappa^R}$.
Since we were able to bound the right-hand side of the inequality (3.32) as $L \to \infty$ we have that

$$
\overline{\lim_{L\to\infty}}\, C_0^{L+R} \leq \epsilon + \frac{\epsilon'}{\kappa^R} \leq \left(1 + \frac{1}{\kappa^R}\right)\epsilon'.
$$

Letting $\epsilon' \downarrow 0$, we conclude

$$
\lim_{L\to\infty} C_0^{L+R} = 0,
$$

which was the desired result. Since equation (3.31) is true, this finishes the proof of Theorem 4. □

Bibliography

[1] Chernov, A. A. (1967). Replication of a multicomponent chain by the "lightning" mechanism. Biophysics, 12(2), 336-341.

[2] Drewitz A., Ramírez A. (2014) Selected Topics in Random Walks in Random Environment. In: Ramírez A., Ben Arous G., Ferrari P., Newman C., Sidoravicius V., Vares M. (eds) Topics in Percolative and Disordered Systems. Springer Proceedings in Mathematics & Statistics, vol 69. Springer, New York, NY.

[3] Durrett, R. (2010). Probability: Theory and Examples. United States of America: Cambridge University Press.

[4] Guerra, E., & Ramírez, A. F. (2017). Asymptotic direction for random walks in mixing random environments. Electronic Journal of Probability, 22.

[5] Kalikow, S. A. (1981). Generalized random walk in a random environment. The Annals of Probability, 753-768.

[6] Klenke, A. (2014). Probability Theory. A Comprehensive Course. London: Springer-Verlag London.

[7] Molchanov, S. (1994). Lectures on random media. In Lectures on probability theory (pp. 242-411). Springer, Berlin, Heidelberg.

[8] Sinai, Y. G. (1983). The limiting behavior of a one-dimensional random walk in a random medium. Theory of Probability & Its Applications, 27(2), 256-268.

[9] Sznitman, A. S., & Zerner, M. (1999). A law of large numbers for random walks in random environment. Annals of probability, 1851-1869.

[10] Temkin, D. E. (1969). The theory of diffusionless crystal growth. Journal of Crystal Growth, 5(3), 193-202.

[11] Zeitouni, O. (2003). Random walks in random environments. eprint arXiv:math/0304374

[12] Zeitouni, O. (2004). Part II: Random walks in random environment. In Lectures on probability theory and statistics (pp. 190-312). Springer, Berlin, Heidelberg.

[13] Zerner, M. P., & Merkl, F. (2001). A zero-one law for planar random walks in random environment. Annals of probability, 1716-1732.

[14] Zerner, M. (2007). The zero-one law for planar random walks in iid random environments revisited. Electronic Communications in Probability, 12, 326-335.