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FIBRED NON-HYPERBOLIC QUADRATIC FAMILIES

by

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Abstract

The aim of this thesis is two-folding. In the first instance, we make a significant progress in the problem of density of hyperbolic components in the context of fibred quadratic polynomial dynamics by proving the existence of robust non-hyperbolic fibred quadratic polynomials. Secondly, we exhibit a more complex class of invariant set distinct than the invariant curves for fibred polynomial dynamics, called *multi-curves*. Moreover, a construction for multi-curves in the lowest interesting dynamics is shown, obtaining not only invariant multi-curves, but also with the property of being *attracting*.

The thesis is organized as follows. In Chapter 2, the fundamentals of fibred dynamics is stated, first in the general context , and then in the particular case for *fibred polynomial dynamics with base an irrational rotation over the unit circle*, where the main results are stated. The theory in this chapter are classical known results, except for Section 2.4, where the local linearization theory for invariant curves is generalized to the case when the invariant curve intersects the critical set (only possible for finitely many points).

In Chapter 3, a series of results are proved in the direction of the main theorem. The Hausdorff continuity of the fibred Julia sets is proved and a new mechanism for non-hyperbolicity is given depending on the existence of at least two invariant curves and a critical path intersecting both. These provide a criterion for the non-hyperbolicity of certain fibred polynomials, and the main theorem can be proved.

Finally in Chapter 4, multi-curves are defined and the classification theory is extended for this new class of invariant sets and the local linearization theory can be applied to them. Specific examples of invariant 2-curves with jumping integer equal 0 and 1 are given, also an attracting one is exhibited, see Appendix B for images.

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Contents

Chapter 1

Introduction

Structural stability and hyperbolicity as one of its most classic mechanisms, has been one of the most important questions in dynamical systems since the beginning of the 20th century. The possibility that most dynamical systems are stable under some kind of perturbation (and in some relevant sense) has guided much of the research in the area ever since. Complex dynamics, with well-defined formulas for the map in question, had a sustained boom in the early 1900s with the works of Fatou and Julia, and thanks, by the way, to the possibility or perspective of understanding them from the apparent simplicity of the formulas involved. We quickly realized that this hope was illusory. Already in the apparently elementary case of quadratic polynomial dynamics, the situation becomes complicated, and questions quickly emerge that keep us with fascinating uncertainties to this day. One of the first questions of this type is the one known today as the Fatou Conjecture in 1920, which establishes that among the dynamics of polynomial maps of degree two, the notion of hyperbolicity is dense. Therefore, the property of structural stability should also be dense in this case. In this work, we do not seek to answer this question but are inspired by it, to study the density (or not) of a notion of hyperbolicity for a family of dynamical systems that, a priori, are close to complex quadratic dynamics.

We will revisit the notion of hyperbolicity for complex skew-product polynomials, which was introduced by Sester in the late 90's [Se1]. In general, a *fibred polynomial over* φ is a map of the form

$$\begin{aligned} P : T \times \mathbb{C} &\longrightarrow T \times \mathbb{C}, \\ (t, z) &\longmapsto (\varphi(t), p_t(z)), \end{aligned}$$

where $\varphi : T \to T$ is a continuous dynamics defined over a compact metric space T (the *base map*) and $p_t : \mathbb{C} \to \mathbb{C}$ is a complex polynomial, that depends continuously on $t \in T$ (see precise definitions in Section 2.1).

The foundational elements of the theory were established mainly in [Jo1, Se1, Se2]. Maps in the form above are also studied in a more general setting with the denomination of *fibred holomorphic dynamics* (see [Po1, Po2]). The particular structure of the dynamics (that is, an independent base map dynamics in the first coordinate followed by a dynamics in a second space, but this time dependent on the first coordinate) gives this type of dynamical system the general name of *skew-product dynamics*. This skew-product structure is used to model physical phenomena whose law varies depending on an external variable (forced systems) and dynamical systems whose iterations are subject, at each time, to small random perturbations (random dynamics), among others (see [St, Jä]). Skew-product dynamics have been the subject of an accelerated study in recent years (see [FaJoJoTa]). In [Vi], Viana proposes the study of fibred quadratic polynomials over an expansive transformation, opening the way to a huge development in the understanding of multidimensional non hyperbolic attractors. In the particular case of fibred polynomials, they can be considered as an intermediate step between the one-dimensional complex polynomial dynamics and higher-dimensional complex dynamics.

In the classic situation (non-fibred setting), it is well known that the quadratic polynomial family exhibits really interesting events. The uniqueness of the critical point is crucial. As a simple but deep consequence, any quadratic polynomial has at most one finite attracting cycle and both cases may occur, that is, there are quadratic polynomials with one finite attracting cycle and quadratic polynomials without finite attracting cycles (recall that the point at infinity is always a super-attracting fixed point for any polynomial). As a counterpart in the fibred situation, M. Ponce in [Po3], proved that there exists a one-parameter family of fibred quadratic polynomials over an irrational rotation of the circle with at least *two attracting invariant curves*.

Although fibred polynomials may have no fixed points nor periodic points (due to the lack of them in the base map), the existence of *invariant curves* allows a local study of these dynamical systems, in the same spirit of the classic local study of the dynamics around a fixed point. This local study is based on the average behavior around the invariant curve, which is defined by the *multiplier* of the curve (Lyapunov exponent). This number also classifies the curve as *attracting, repelling* or *indifferent*. In the case when the base map is an irrational rotation of the circle, in [Po1] Ponce describes a local linearization for attracting (respectively repelling) invariant curves.

Going back to the classic case of the iterations of a complex quadratic polynomial, if the unique critical point is attracted by an attracting cycle, then the polynomial is *hyperbolic* (uniform expansion over the Julia set, see Section 2.2 for a formal definition). This feature defines the connected components (also called *hyperbolic components*) of the interior of the well-known *Mandelbrot set*. Since the mid-1980s, a new impetus has appeared in the study of complex dynamics. The density of the hyperbolic components of the Mandelbrot set has been stated as a famous *conjecture*. This conjecture is closely related to the topological *MLC-conjecture* (Is the Mandelbrot set Locally Connected?). Even though, at the time of writing this work, it is still an open conjecture, there had been at least two breakthroughs: first, all non-infinitely *renormalizable* parameters represent locally-connected parameters for the Mandelbrot set as showed by Yoccoz [Hu], and second, for the real quadratic family, Lyubich, and Graczyk and Swiatek independently showed that hyperbolicity is a dense property [Ly, GrSw]. For (large degree) polynomials in $\mathbb{C}^2$, remarkable results have been obtained. In [Bu], Buzzard shows that structurally stable maps are not dense in the space of large-degree polynomial automorphisms in $\mathbb{C}^2$. In [BuJe], the authors proved the structural stability of hyperbolic polynomial automorphisms in $\mathbb{C}^2$. Recently, Biebler [Bi] has introduced a notion of complex blender to show the persistence of the Newhouse phenomenon, which gives rise to new robust families of polynomials in $\mathbb{C}^3$ presenting notions of non-stability.

In recent years, the question has resurfaced with interesting and novel results, mainly due to Dujardin [Duj1, Duj2] and Taflin [Taf]. In these works the authors show robust non-stability mechanisms for holomorphic endomorphisms in dimension $k \geq 2$ with sufficiently chaotic dynamics at each entry. Regarding polynomial skew-products in $\mathbb{C}^2$, the works by Astorg and Bianchi [AsBi1, AsBi2] provide a complete description of the hyperbolic components and of the bifurcation locus (including situations with non-empty interior). All these results require either the existence of Cantor Julia sets or blenders. Neither of these mechanisms is available in our context, for which the dynamics of the basis is a rotation of the circle.

For further reference in the density conjecture and MLC-conjecture of hyperbolic quadratic polynomials, we refer to the remarkable survey [Be], where the author exposes, in a well-detailed way, how this conjecture is related to other different subjects in complex dynamics, such as topological and quasi-conformal rigidity, No Invariant Line Fields conjecture, etc.

The principal objective of this work is to contribute to a better understanding of the notion of hyperbolicity and the possibility of density for quadratic families in the fibred polynomial context. In particular, we will concentrate on fibred quadratic polynomials over an irrational rotation. Our key result creates a very simple mechanism (*critical connection*, see Definition 10 at Section 3.1.2) that detects the non-hyperbolicity of a fibred polynomial. Indeed, a fibred polynomial over an irrational rotation that admits two attracting invariant curves whose basins of attraction are *connected* by the curve of critical points cannot be hyperbolic (see Theorem 5 at Section 3.1.2). To have a critical connection is a robust property, in the sense that for any fixed irrational α and fixed degree $d \geq 2$, the property of having a critical connection is open on the space of fibred polynomials of degree d over the circle rotation R_α (see Proposition 10 at Section 3.1.3). Since every element of the above-mentioned Ponce's family of fibred quadratic polynomials has a critical connection, as a corollary, we obtain that for any fixed irrational α, the hyperbolic fibred quadratic polynomials over the circle rotation R_α are not dense in the (parameter) space of fibred quadratic polynomials over R_α (see Theorem 6 at Section 3.1.3).

As a secondary objective, we exhibit the existence of simple invariant objects, located at an intermediate point between periodic or invariant curves and the invariant (chaotic) Julia set. In Holomorphic dynamics, the complexity and chaotic nature of the corresponding Julia set make it concentrate the most significant part of the dynamics. Even though this is a complicated invariant set, we can recover it from a much simpler class of invariant set, the *repelling periodic orbits*. It is a classic result that this set is dense in the Julia set for holomorphic dynamics.

It is a classical fact that invariant sets are part of a central key for understanding many features in dynamical systems. Focusing on the (simple) invariant curves, its multiplier (Lyapunov exponent) allows us to determine the local behavior of the fibred dynamics around the invariant curves. more precisely, we know that there exists an open invariant set containing the invariant curve that is attracted to it when the multiplier is less than 1.

When the base map of a fibred dynamics is an irrational rotation over the unit circle, there are no fixed nor periodic cycles. This way, invariant (periodic curves) curves are the natural extension of fixed (periodic) points in the setting of *fibred dynamics*, since constitute the most simple invariant objects herein. The existence of more complex invariant sets (other than the Julia set) arises as a natural question.

A good, and very interesting, candidate for this new kind of invariant objects is what we called *multi-curves* (see Section 4.1 for precise definition), that is simple closed curves that turn many times in the base space direction ($\mathbb{T}^1$). The aim of the last part of this work is to prove the existence of such invariant objects in the low degree setting, that is, on *quadratic fibred polynomials*. Moreover, it would be possible to obtain examples of multi-curves that have a dynamical nature of being attracting.

The main tool to prove the main results is a fine study of the continuity of the fibers of the filled-in Julia set in the particular case of irrational rotation in the base map. We obtain the following result, which presents an interest in itself and that in some way represents an important advance to Sester's initial work on the fibred (filled-in) Julia set structure. We show that for a hyperbolic fibred polynomial over an irrational rotation of the circle, the fibers of the filled-in Julia set vary continuously in the Hausdorff topology. In particular, the Julia set equals the boundary of the filled-in Julia set (see Proposition 9 and Corollary 5 at Section 3.1). It is worth mentioning that in [Se1], Example 4.1, the author presents an example of a hyperbolic fibred polynomial with a discontinuous filled-in Julia set. Hence, the additional hypothesis on the base map is necessary in order to obtain the equality between the boundary of the filled-in Julia set and the Julia set.

In order to exhibit the multi-curve for fibred quadratic polynomials, we construct these invariant objects as *small perturbations* of invariant objects for a *'static' fibred dynamics*, i.e. a particular parametric family of (classic) quadratic polynomials. The perturbations will consist of small closed curve, 'around' special points in the Mandelbrot set.

Chapter 2

Rewriting the Theory

This chapter contains the fundamentals and known results of the fibred dynamics theory. First, we will be presented in a general context, extending the basic notions of non-fibred or classical polynomial dynamics. The particular case that we are interested in studying is when the base mapping is an irrational rotation. At the end of the chapter, the local dynamical theory developed by Ponce is extended, through original new results, to the case of invariant curves intersecting critical points.

2.1 Preliminaries

In this section, we review some of the standard definitions and notations (see [Po2, Se0] for further details). Consider a compact metric space T and $\varphi : T \to T$ a continuous map. For $d \geq 2$, we denote by $\mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$ the set of continuous functions from T to $\mathbb{C}^* \times \mathbb{C}^d$ endowed with the uniform convergence topology.

Given $c \in \mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$, that is, $c(t) = (c_d(t), ..., c_0(t))$ with $c_i \in \mathcal{C}(T, \mathbb{C})$ for each $0 \leqslant i \leqslant d$ and $c_d(t) \neq 0$ for all $t \in T$, we associate to each $t \in T$ a complex polynomial p_t, of degree d,

$$p_t(z) = c_d(t)z^d + c_{d-1}(t)z^{d-1} + \cdots + c_1(t)z + c_0(t). \tag{2.1}$$

We are going to use the notation above to define our object of study, the fibred systems.

Definition 1. *Given $c \in \mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$, we call a* fibred polynomial dynamics over φ *to the map*

$$\begin{aligned} P : T \times \mathbb{C} &\rightarrow T \times \mathbb{C}, \\ (t, z) &\mapsto (\varphi(t), p_t(z)). \end{aligned}$$

where $p_t(z)$ is the polynomial associated to c defined in equation (2.1).

We are interested in studying the dynamics of iterating these objects, that is P^n. We will denote by $P^n = P \circ \cdots \circ P$ the composition of P, n-times, and by p_t^n the second coordinate of P^n, that is to say,

$$p_t^n = p_{\varphi^{n-1}(t)} \circ p_{\varphi^{n-2}(t)} \circ \cdots \circ p_{\varphi(t)} \circ p_t.$$

With this we have $P^n(t, z) = (\varphi^n(t), p_t^n(z))$. Note that the first coordinate of P^n is the actual orbit by φ.

We notice that, by iterating P, the coefficients of the polynomial p_t "vary" on the fiber according to the base mapping φ, which is why we can consider these systems in the context of skew products.

The topological and metric structure of the space $\mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$ endows the structure of space to the set of fibred polynomials.

Definition 2. *The space of fibred polynomials of degree d over φ is denoted by $\mathcal{F}_{d,\varphi}$. The uniform convergence topology on $\mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$ endows $\mathcal{F}_{d,\varphi}$ with a natural topology.*

Although the coefficients of the polynomial vary, the dynamics near infinity are still attracted to the point at infinity, so the following definitions have a natural extension to the fibred case.

Definition 3. *Let $P(t, z) = (\varphi(t), p_t(z))$ be a fibred polynomial dynamics in $\mathcal{F}_{d,\varphi}$. We define the* filled-in Julia set *of P, as:*

$$\mathcal{K} = \{(t, z) \in T \times \mathbb{C} : \sup_{n \in \mathbb{N}} |p_t^n(z)| < +\infty\},$$

and its fiber on $t \in T$ by

$$\mathcal{K}_t = \{z \in \mathbb{C} : (t, z) \in \mathcal{K}\}.$$

For each $t \in T$, let $\mathcal{J}_t = \partial\mathcal{K}_t$ denotes the topological boundary of the fiber on t of the filled-in Julia set. Finally, we call

$$\mathcal{J} = \overline{\bigcup_{t \in T} \{t\} \times \mathcal{J}_t},$$

the Julia set *of the fibred polynomial P.*

Some basic properties of these sets, extended from the non-fibred case, are mentioned in the following result.

Proposition 1 (see Sester [Se1]). *Let $\varphi : T \to T$ continuous, $c \in \mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$ and $P : T \times \mathbb{C} \to T \times \mathbb{C}$ the associated polynomial. Then:*

1. *There exists $R = R(P) > 0$ such that, for all $t \in T$, $\mathcal{K}_t$ is contained in the disc with center 0 and radius $R(P)$.*

2. *$\mathcal{K}$ is compact and $\mathcal{K}_t$ is a "full set" for all $t \in T$.*

3. *If $\varphi : T \to T$ is surjective, then $\mathcal{K}$ is completely invariant.*

In the non-fibred polynomial case (the classical case), it is well known that there is a strong relation between the Julia set and the filled-in Julia set. In fact, in that case, the Julia set coincides with the boundary of the filled-in Julia set. As mentioned in the introduction, this is not necessarily the case in the fibred setting, not even in the hyperbolic case (see [Se1, Example 4.1]). Even so, by definition, we have the inclusion

$$\mathcal{J} \subseteq \partial\mathcal{K}. \tag{2.2}$$

The question about equality in the above inclusion arises. In this work, we will obtain equality in the case of hyperbolic fibred polynomials over an irrational rotation on the circle, see Corollary 5 in the next chapter.

A final topological property of the fiber-filled Julia sets, which follows from the dynamics of the polynomial P, is directly related to the set of critical points, which is defined as:

$$\Omega = \{(t, z) \in T \times \mathbb{C} : P_t'(z) = 0\}$$

and the corresponding part of the fiber:

$$\Omega_t = \{z : (t, z) \in \Omega\}$$

Proposition 2 ([Se1], Corollary 2.6)**.** *Let us fix $c \in \mathcal{C}_d(T)$, then $\mathcal{K}_t$ is connected for all $t \in T$ if and only if $\Omega \subset \mathcal{K}$.*

It is worth mentioning that this property follows from the construction of Green's function of fibred polynomial P, which will not be of interest to us, but you can consult its definition and properties in Chapter 1 of [Se0].

The above result is the generalization, in the fibred setting, of a result of Fatou and Julia which gives the equivalence between the connectivity of the filled Julia set and the fact that no critical point escapes.

2.1.1 Semi-continuity of $\mathcal{K}_t$ and $\mathcal{J}_t$

Part of the results of this work is to study, under what conditions equality is obtained in equation (2.2). A necessary condition for this equality is the Hausdorff continuity of the $\mathcal{K}_t$ fibers.

This section deals with the properties of semi-continuity by analyzing these properties with respect to the base point t of the sets $\mathcal{K}_t$ and $\mathcal{J}_t$. We will designate by $\text{Comp}^*(\mathbb{C})$ the set of non-empty compact subsets of $\mathbb{C}$ provided with the Hausdorff distance. The proposition below is a characterization of upper semi-continuity for a family of compacts of $\mathbb{C}$. A proof of this criterion could be found in [Se0, Proposition 1.43].

Proposition 3.

1. *The map* $t \mapsto K_t$ *of* T *in* $Comp^*(\mathbb{C})$ *is upper semi-continuous.*

2. *The map* $t \mapsto \mathcal{J}_t$ *of* T *in* $Comp^*(\mathbb{C})$ *is lower semi-continuous.*

2.2 Fatou set by chains

Unlike the classical case, fibred polynomials can have attracting cycles, in fact, if the base mapping φ has no cycles, the fibred polynomial has no cycles at all. From this, it follows that a description of the normality set is not as straightforward as in the classic case. In the works of Sester and Jonson, we can find a characterization of the complement of the Julia set by fibers, J_t, in terms of normality, the equivalent of the Fatou set of the polynomial over the fiber t.

Attracting cycles, or cycles in general, are important objects from the dynamical point of view, which will allow us to give a first description of what the Fatou set by fibers should be. It is possible to find a good counterpart in the fibred setting in the *stable set by chains.* To define the stable set by chains of a fibred polynomial P, it is necessary to review the concept of *pseudo-orbit.*

Definition 4. *Let* $\epsilon > 0$ *and* $(t, z) \in T \times \mathbb{C}$*. We call an* ϵ*-pseudo-orbit of* (t, z) *to any sequence* $\{s_n\}_{n\in\mathbb{N}} = \{(\varphi^n(t), z_n)\}_{n\in\mathbb{N}} \subset T \times \mathbb{C}$ *such that:*

- $z_0 = z$

- $|z_{n+1} - p_{\varphi^n(t)}(z_n)| < \epsilon$ *for all* $n \in \mathbb{N}$.

Note that the first coordinate of a pseudo-orbit consists of an actual orbit of the base map, which is consistent with the definition of the iterates of the polynomial itself.

In this way, we define the *stable sets by chains.* For $\epsilon > 0$, we define the set $\mathcal{U}(\epsilon)$ as the pairs $(t, z) \in T \times \mathbb{C}$ such that every ϵ-pseudo-orbit of (t, z) is bounded. Analogously, we define the set $\mathcal{U}^\infty(\epsilon)$ as the pairs $(t, z) \in T \times \mathbb{C}$ for which every ϵ-pseudo-orbit of (t, z) is contained in the complement of $\mathcal{K}$, which is equivalent that they tend to infinity.

Now, consider the following sets:

$$\mathcal{U}_t(\epsilon) = \{z \in \mathbb{C} : (t, z) \in \mathcal{U}(\epsilon)\},$$

$$\mathcal{U}_t^{\infty}(\epsilon) = \{z \in \mathbb{C} : (t, z) \in \mathcal{U}^{\infty}(\epsilon)\},$$

$$\mathcal{U} = \bigcup_{\epsilon > 0} \mathcal{U}(\epsilon) \text{ and } \mathcal{U}_t = \bigcup_{\epsilon > 0} \mathcal{U}_t(\epsilon).$$

Observation. $\mathcal{U}_t(\varepsilon) \subset \mathcal{U}_t(\varepsilon')$ if $\varepsilon' > \varepsilon$.

The following result summarizes the basic properties of the set of ϵ-pseudo-orbits.

Proposition 4 (see Sester [Se1], Section 3)**.**

1. *$\mathcal{U}_t$ is an open set contained in $\text{int}(\mathcal{K}_t)$.*

2. *$\mathcal{U}$ is invariant by P.*

3. *If z and z' belongs to the same component of $\mathcal{U}_t$, then $|p_t^n(z) - p_t^n(z')| \to 0$, when $n \to \infty$.*

4. *The connected components of $\mathcal{U}_t$ are connected components of $\text{int}(\mathcal{K}_t)$.*

We will now focus on the description of the connected components of $\mathcal{U}_t$. Let V be one of them and set $V^n = P_t^n(V)$. Recall that Ω_t is the set of critical points of P_t. In the same way as the immediate basins of attracting periodic orbits always contain critical points (the well-known Fatou Lemma), Sester showed a proposition where $P_t^n(V)$ mach $\Omega_{\varphi^n(t)}$ for an infinity of integers n. It is worth mentioning that properties 4 and 5 of the proposition above are a consequence of this result. Furthermore, this relationship will be piece key in the study of hyperbolicity.

We further denote

$$\Lambda(V) = \{n \geq 0 \mid \Omega_{\varphi^n(t)} \cap V^n \neq \emptyset\}$$

Proposition 5 (see [Se1], Proposition 3.2)**.** *The set $\Lambda(V)$ is infinite and if*

$$\Lambda(V) = \{n_0 < n_1 < \cdots < n_k < n_{k+1} < \dots\},$$

then there exists an integer $N(V)$ such that $n_{k+1} - n_k \leq N(V)$.

Hyperbolicity and the Critical Set

One of the main results in [Se1] is the generalization to the fibred setting of the distinct characterizations of the hyperbolic parameters in terms of the *critical set*, as we have defined in the previous section:

$$\Omega = \{(t, z) \in T \times \mathbb{C} : p_t'(z) = 0\},$$

and *the post-critical set*,

$$\mathcal{P}\Omega = \bigcup_{n \geq 0} P^n(\Omega).$$

We denote by Ω_t the fibre of Ω in $t \in T$.

A fibred polynomial is called *hyperbolic* if it is uniformly expansive on its Julia set $\mathcal{J}$, i.e., there exist $A > 0$ and $\lambda > 1$ such that $|(p_t^n)'(z)| \geqslant A\lambda^n$ for all $(t, z) \in \mathcal{J}$. This definition is independent of the nature of the polynomial. Some properties of a hyperbolic polynomial in the non-fibred setting have their generalization in the following results.

In the context of general fibred dynamics, that is, for an arbitrary compact metric space T, $\varphi : T \to T$ any continuous map, hyperbolicity brings interesting properties to basic sets, for the family of Julia sets by fibers we have:

Proposition 6 (see Sester [Se1], Proposition 4.1)**.** *If P is hyperbolic, then the mapping $t \mapsto \mathcal{J}_t$ is continuous for the Hausdorff distance of compact sets of $\mathbb{C}$.*

On the other hand, analogous to the non-fibered case, it is possible to determine a characterization of the fibered systems depending on the dynamics of their critical set, see [Se1].

Theorem 1 (see Sester [Se1], Theorem 4.2)**.**

The following statements are equivalent:

1. *P is hyperbolic;*

2. *there exists $\epsilon_0 > 0$ such that $\Omega \subset \mathcal{U}(\epsilon_0) \cup \mathcal{U}^\infty(\epsilon_0)$;*

3. *there exists a family of open sets* $\{V_t\}_{t\in T}$ *and* ϵ_0 *such that*

$$\mathcal{J}_t \subset \mathbb{C}\backslash V_t, \quad \Omega_t \subset V_t$$

$$P_t(V_t) \subset V_{\varphi(t)} \quad \textit{and} \quad d(P_t(V_t), \partial V_{\varphi(t)}) \geqslant \epsilon_0;$$

4. $\overline{\mathcal{P}\Omega} \cap \mathcal{J} = \emptyset$.

Finally, as was announced at the beginning of the section, hyperbolicity allows us to give a description of the Fatou set, the normality/stable set by fibers, through the concept of pseudo orbit.

Corollary 1 (see Sester [Se1], Corollary 4.3)**.** *If* P *is hyperbolic, then for all* $t \in T$, $\mathrm{int}(\mathcal{K}_t) = \mathcal{U}_t$ *and* $\mathcal{J}_t$ *has zero Lebesgue measure.*

We close the general setting by considering the following. Analogous to the classical case, we can consider the Moduli space of degree d fibred polynomials, $\mathcal{F}_{d,\varphi}$, that is, the parameter space reduced under conformal conjugation with fibred linear applications. The following result is a summary of Propositions 1.1.1, 1.1.2, and 1.1.3 of [Se0].

We denote $C_d(T) = \mathcal{C}(T, \mathbb{S}^1 \times \mathbb{C}^{d-1})$ with the conventions that if $c \in \mathcal{C}_d(T)$ then

$$c(t) = (c_d(t), c_{d-2}(t), \ldots, c_0(t)), \qquad \text{and} \qquad |c_d(t)| = 1.$$

Also, if $u \in \mathcal{C}(T, \mathbb{T}^1)$ we denote by $[u]$ its homotopy class and $H = [T : \mathbb{T}^1]$ the group of homotopy classes. The map φ defines a homomorphism φ^* from H in H by

$$\varphi^*([u]) = [u \circ \varphi].$$

Proposition 7. *If* $c \in \mathcal{C}(T, \mathbb{C}^* \times \mathbb{C}^d)$ *then there exists* $u \in \mathcal{C}(T, \mathbb{R}^*_+)$ *and* $v \in \mathcal{C}(T, \mathbb{C})$ *such that the map* $(t, z) \mapsto (t, u(t)z + v(t))$ *conjugates* P_c *to* $P_{c'}$ *with* $c' \in \mathcal{C}_d(T)$*. Moreover, if* $c \in \mathcal{C}_d(T)$ *satisfies the homotopic hypothesis that there exists* u_0 *such that* $[c'_d]\varphi^*([u_0])[u_0^2]^{-1} = 1$, *then* $P_{c'}$ *will be conformally conjugated to a fiber polynomial* $P_{c''}$ *of the form*

$$P_{c'',t}(z) = z^d + c''_{d-2}(t)z^{d-2} + \cdots + c''_1(t)z + c''_0(t).$$

Due to the nature of Chapter 4 (multi-curves in quadratic polynomials), in such chapter, we will give a proof/construction of this proposition for the quadratic case ($d = 2$), and the homotopic hypothesis will be satisfied trivially.

2.3 Fibred polynomials over irrational rotations

From this point on, we will concentrate on the case of fibred polynomials over an irrational rotation of the circle. That is, we take $T = \mathbb{T}^1 := \mathbb{R}/2\pi\mathbb{Z}$, and for $\theta \in \mathbb{T}^1, \varphi(\theta) = \mathcal{R}_\alpha(\theta) := \theta + \alpha \mod 2\pi$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

From here on, we want to restrict ourselves to this case. Let's start by saying that we are interested in the study of the case in which $T = \mathbb{T}^1 := \mathbb{R}/2\pi\mathbb{Z}$. And let's give an explanation of why we restrict ourselves to irrational rotations.

We could consider fibred holomorphic dynamics, where the transformation on the basis is a minimal homeomorphism of the circle preserving the orientation (a $\mathcal{C}^\infty$ (analytic) diffeomorphism in the $\mathcal{C}^\infty$ (analytic) case). But this greater generality is only apparent because a conjugation on the basis (under the appropriate arithmetic hypothesis in the $\mathcal{C}^\infty$ or analytical case) brings us back to the case considered above. Indeed, let r be a diffeomorphism of the circle preserving the orientation, and let h be the linearization given by the Theorems of Herman or Yoccoz depending on the case ($\mathcal{C}^\infty$, $\mathcal{C}^w$, see [Her1], [Yoc1], [Yoc2]). The change in coordinates given by $H(\theta, z) = (h(\theta), z)$ therefore brings us back to the case $r(\theta) = \theta + \alpha$. This way, by considering the irrational rotations, we are covering (up to conjugation) every $\mathcal{C}^\infty$ (analytic) case.

2.4 Invariant curves

In a given dynamical system, it is possible to find many types of invariant objects, such as fixed and periodic points, a minimal complicated attractor, and the support of an invariant measure, among others. Through these objects, we can understand many features of the system.

The Fatou and Julia sets, form a dichotomy in the Riemann sphere for global complex dynamics. Moreover, it is known that the Julia set concentrates the most significant part of the (chaotic) dynamics. Although in general, this is a complicated invariant set, in non-fibred holomorphic dynamics we can focus on a simpler invariant set, the repelling periodic orbits. It is a classical result, from Fatou and Julia, that those objects are dense in the Julia set.

Since irrational rotations of the circle are minimal, it follows that these fibred polynomials do not contain fixed nor periodic points. Arises the natural question of the existence of minimal invariant objects, distinct from the Julia set, for fibred polynomial dynamics. Given the nature of the base space of fibred polynomial dynamics, it is well expected that a minimal invariant object possesses the same topological structure of the base space: a closed curve.

As part of his doctoral thesis [Po2], M. Ponce proved that a natural extension for fixed and periodic points are *invariant curves*, that is, continuous functions $u : \mathbb{T}^1 \to \mathbb{C}$ satisfying the equation

$$P(\theta, u(\theta)) = (\theta + \alpha, u(\theta + \alpha)). \tag{2.3}$$

Remark 1. *It is important to note that, unlike the non-fibred case, the existence of invariant curves for fibred dynamics is a cohomological problem instead of an algebraic one.*

2.4.1 Invariant curves with critical points

Part of the novelties presented in this work is the extension of local dynamical theory developed by Ponce for invariant curves over which the polynomial is injective, to invariant curves with possible finite intersections with the critical set.

Analogous to the classical case, there is a complex conformal invariant number with a considerable amount of information from the local dynamics of an invariant curve.

Let us remember that in classical dynamics, once the fixed or periodic points are located, the value of their derivative provides us with local dynamic information about the system. In the fibered case, the panorama is not very different, although the definition of a multiplier is not so direct. Here, the invariant object is a non-countable set of points, a curve. One way to extend this conformal invariant is through the appropriate average of the derivatives of the function. More precisely, let us consider the following definition:

Definition 5. *Let $u : \mathbb{T}^1 \to \mathbb{C}$ be an invariant curve for a fibred polynomial P over an irrational rotation. Provided that $\theta \to \log|p'_\theta(u(\theta))|$ is a $L^1(\mathbb{T}^1)$ function, we call* multiplier of the curve *to the positive number*

$$\kappa(u) = \exp\Big(\int_{\mathbb{T}^1} \log|p'_\theta(u(\theta))| d\theta\Big).$$

When the multiplier $\kappa(u) < 1$ we say that the curve is attracting*. If $\kappa(u) > 1$ we call it* repulsor*, and in the case $\kappa(u) = 1$ the curve is called* indifferent*.*

Remark 2. *Linearization results by Ponce in* [Po1] *follow once we notice that, the compactity of $\mathbb{T}^1$, together with the injectivity of the polynomial over the invariant curve ($p'_\theta(u(\theta)) \neq 0$), imply that the map $\theta \to \log|p'_\theta(u(\theta))|$ is L^1.*

When the invariant curve is also a critical curve, it is possible to extend the definition by making $\kappa(u) = 0$, note that this is the case for the constant curve $z \equiv \infty$. The integrability condition of the invariant curve allows non-empty intersections, between the invariant curves and the critical set, only on finite sets.

As mentioned above, the complex multiplier, under certain conditions, provides us with information about the local dynamics, in fact, in the attracting case for example, we can find an entire neighborhood around the attracting cycle that is "attracted" to such cycle. This property can be extended for invariant attracting curves in the fibred case. Even more, if P is injective on the invariant curve, it is possible to obtain a linearization around the curve (see [Po1], Proposition 3.1). In the next paragraphs, we perform an analogous result when $\log|p'_\theta(u(\theta))|$ is just a $L^1(\mathbb{T}^1)$ function (in order to allow critical points on the invariant curve).

We start by considering a conformal normalization for fibred quadratic polynomials in such a way that the invariant curve is a simple section in the space $\mathbb{T}^1 \times \mathbb{C}$. This could be considered as a first step to the *moduli space*.

Lemma 1. *Let P be a fibred polynomial over an irrational rotation, and u an attracting invariant curve. Then there exists a continuous change of coordinates $H(\theta, z) = (\theta, a(\theta)z+$*

$b(\theta))$ *such that* $H^{-1} \circ P \circ H$ *has the form*

$$(\theta, z) \longmapsto (\theta + \alpha, \rho_1(\theta)z + z^2 q_\theta(z)),$$

where $q_\theta(z)$ *is a polynomial on* z *that depends continuously on* θ. *Moreover, there exists* $c < 1$ *so that we can choose* ρ_1 *in such a way that* $\sup_{\theta\in\mathbb{T}^1} |\rho_1(\theta)| < c$.

Proof. A first conjugacy in the form $H(\theta, z) = (\theta, z + u(\theta))$ allows to assume that P has the form

$$(\theta, z) \longmapsto (\theta + \alpha, \rho_1(\theta)z + z^2 q_\theta(z)).$$

We look for a conjugacy in the form $H(\theta, z) = (\theta, e^{v(\theta)}z)$ with a continuous $v : \mathbb{T}^1 \to \mathbb{R}$ in order to obtain the uniform upper bound on ρ_1. For an integer N we define the continuous function $\gamma_N(\theta) = \max\{\log|\rho_1(\theta)|, -N\}$. Since $\log|\rho_1(\theta)|$ is $L^1(\mathbb{T}^1)$, we have $\int_{\mathbb{T}^1} \log|\rho_1(\theta)|d\theta = \lim_{N\to\infty} \int_{\mathbb{T}^1} \gamma_N(\theta)d\theta$. We can find an integer $N > 0$ and a trigonometric polynomial $l : \mathbb{T}^1 \to \mathbb{R}$ such that

1. $\int_{\mathbb{T}^1} \gamma_N(\theta)d\theta < 0$,
2. $\int_{\mathbb{T}^1} l(\theta)d\theta = \int_{\mathbb{T}^1} \gamma_N(\theta)d\theta$,
3. $e^{\gamma_N - l} < e^{-\int \gamma_N}$.

Since l is a trigonometric polynomial and α is irrational, we choose v as a (continuous) solution to the cohomological equation

$$v(\theta + \alpha) - v(\theta) = l(\theta) - \int_{\mathbb{T}^1} \gamma_N(\theta)d\theta.$$

By performing the conjugacy of P by H we obtain the linear term $\frac{e^{v(\theta)}}{e^{v(\theta+\alpha)}}\rho_1(\theta)$, and we can estimate:

$$\begin{aligned} \frac{e^{v(\theta)}}{e^{v(\theta+\alpha)}} e^{\log|\rho_1(\theta)|} &= \frac{e^{v(\theta)}}{e^{v\theta+\alpha}} e^{l(\theta)} \frac{e^{\log|\rho_1(\theta)|}}{e^{l(\theta)}}, \\ &= e^{\int \gamma_N} e^{\log|\rho_1(\theta)| - \gamma_N(\theta)} e^{\gamma_N(\theta) - l(\theta)}, \\ &< 1. \end{aligned}$$

□

From the proof of the above result, it follows that with suitable modifications, we have an analogous result for a repulsor invariant curve.

Lemma 2. *Let P be a fibred polynomial over an irrational rotation, and u a repulsor invariant curve. Then there exists a continuous change of coordinates $H(\theta, z) = (\theta, a(\theta)z + b(\theta))$ such that $H^{-1} \circ P \circ H$ has the form*

$$(\theta, z) \longmapsto (\theta + \alpha, \rho_1(\theta)z + z^2 q_\theta(z)),$$

where $q_\theta(z)$ is a polynomial on z that depends continuously on θ. Moreover, there exists $c > 1$ so that we can choose ρ_1 in such a way that $\sup_{\theta \in \mathbb{T}^1} |\rho_1(\theta)| > c$.

Lemma 1 allows us to find a neighborhood around an invariant attracting curve that is invariant by the polynomial P and that is "attracted" under the dynamics towards the invariant curve, as described by the following result:

Lemma 3. *Let P be a fibred polynomial over an irrational rotation, and u an attracting invariant curve. Then there exists an open set $\mathcal{T} \subset \mathbb{T}^1 \times \mathbb{C}$ containing the curve, that is, $(\theta, u(\theta)) \in \mathcal{T}$ for every $\theta \in \mathbb{T}^1$, and such that for every $(\theta, z) \in \mathcal{T}$ we have $|p_\theta^n(z) - u(\theta + n\alpha)| \to 0$ as $n \to \infty$. In particular $u \subset int(\mathcal{K})$ and $u(\theta) \in int(\mathcal{K}_\theta)$ for every $\theta \in \mathbb{T}^1$.*

Proof. We assume that P has the form as in the previous lemma. Hence

$$p_\theta(z) = z(\rho_1(\theta) + z(q_\theta(z))).$$

Let $M = \sup_{\theta \in \mathbb{T}^1, |z|<1} |q_\theta(z)|$. Hence for $|z| < M^{-1}\frac{(1-c)}{2}$ one knows that

$$|p_\theta(z)| < \frac{1+c}{2}|z|, \tag{2.4}$$

where $c < 1$ is the constant given by Lemma 1.

We obtain $\mathcal{T}$ as $H\left(\mathbb{T}^1 \times B(0, M^{-1}\frac{(1-c)}{2})\right)$, where H is given in Lemma 1. □

Condition (2.4) in the proof of the above lemma tells us, roughly speaking, that the local dynamics around the invariant curve is *contracting*. *Contracting/stability* theory around invariant curves, was developed by Ponce in [Po2, Chapter 2]. We recover here the fundamentals that will be key in the proof of (in)stability results.

Definition 6. *We say that a fibred polynomial P over an irrational rotation has an* invariant open tube *if there is an open set $\mathcal{T} \subset \mathbb{T}^1 \times \mathbb{C}$ satisfying:*

i) $P(\mathcal{T}) \subset \mathcal{T}$.

ii) The fiber $\mathcal{T}_\theta$ over the point $\theta \in \mathbb{T}^1$ is a topological disc for all $\theta \in \mathbb{T}^1$.

We also say that the invariant curve $u : \mathbb{T}^1 \to \mathbb{C}$ is a stable curve *if there is an open invariant tube that contains it, that is, $u(\theta) \in \mathcal{T}_\theta$ for all $\theta \in \mathbb{T}^1$.*

This way, we can re-formulate the above result.

Corollary 2. *Let P be a fibred polynomial over an irrational rotation, and u an attracting invariant curve. Then u is a stable curve.*

The open invariant tube constructed in the proof above is a generalization of the immediate basin of attraction for the non-fibred case. In this way, we extend the definition of a basin of attraction in the fibred case as follows:

Definition 7. *Let P be a fibred polynomial over an irrational rotation, and u an attracting invariant curve. We define the* attracting basin *of the curve u, and denote it by $\mathcal{A}(u)$, the set of points $(\theta, z) \in \mathbb{T}^1 \times \mathbb{C}$ for which*

$$\lim_{n\to\infty} |p_\theta^n(z) - u(\theta + n\alpha)| = 0. \tag{2.5}$$

In the case when u is normalized so that $u = \{z \equiv 0\}$ (which is always possible by Lemma 1), the condition (2.5) can be rewritten as

$$\lim_{n\to\infty} |p_\theta^n(z)| = 0. \tag{2.6}$$

Note that if u is an *invariant attracting curve*, and $\mathcal{T}$ is its corresponding invariant tube, then

$$\mathcal{A}(u) = \bigcup_{n\geq 0} P^{-n}(\mathcal{T}).$$

This way, as a straightforward corollary of Lemma 3 we obtain the following.

Corollary 3. *If $u : \mathbb{T}^1 \to \mathbb{C}$ is an attracting invariant curve, then the attracting basin $\mathcal{A}(u) \subset \mathbb{T}^1 \times \mathbb{C}$ is an open set.*

The following is a trivial example. Consider the *escape set* of a fibred polynomial

$$I(P) = \mathcal{K}^c = \{(\theta, z) \in \mathbb{T}^1 \times \mathbb{C} : |(P_\theta^n(z)| \to \infty\}.$$

In this case, we write $\mathcal{A}(\infty) = I(P)$. By the compacity of $\mathcal{K}$, it follows that $\mathcal{A}(\infty)$ is an (unbounded) open set. It is not difficult to see that the invariant curve $\{z \equiv \infty\}$ is (super)-attracting.

Next, for completeness in the invariant curves' theory, we present a reciprocal result to Corollary 2, that is, if $\mathcal{T}$ is an invariant tube for a fibred dynamics P, then $\mathcal{T}$ contains an invariant stable curve for P.

Let $P(\theta, z) = (\mathcal{R}_\alpha(\theta), f_\theta(z))$ be a fibred holomorphic dynamics defined in a neighborhood of $\mathbb{T}^1 \times \overline{\mathbb{D}}$ satisfying:

(i) $P(\mathbb{T}^1 \times \overline{\mathbb{D}}) \subset \mathbb{T}^1 \times \overline{\mathbb{D}}$;

(ii) there exists a point $\tilde{\theta} \in \mathbb{T}^1$ such that $f_{\tilde{\theta}}(\overline{\mathbb{D}}) \subset \mathbb{D}$.

It follows that the former hypothesis is equivalent to the fact that $f_{\tilde{\theta}} : \mathbb{D} \to \mathbb{D}$ is a contraction in the Poincaré metric of the unit disk $\mathbb{D}$. We say that P verifies *strong contraction hypothesis in a point.* Note that if $P(\mathbb{T}^1 \times \overline{\mathbb{D}}) \subset \mathbb{T}^1 \times \mathbb{D}$, as in Lemma 3, then P is a global (strong) contraction.

Lemma 4 (Lemma 2.10, [Po2])**.** *Let $F(\theta, z) = (\mathcal{R}_\alpha(\theta), f_\theta(z))$ be a* fhd *defined in a neighborhood of $\mathbb{T}^1 \times \overline{\mathbb{D}}$ which satisfies the strong contraction hypothesis at a point, then for all $\theta \in \mathbb{T}^1$ the diameter (measured in the Poincaré metric) of the set $K_\theta^n = f_{\theta - n\alpha}^n(\mathbb{D})$ tends* *to zero uniformly in θ as n tends to infinity. More precisely, there are constants C,$c > 0$ such that*

$$diam_{\mathbb{D}}(K_\theta^n) \leqslant Cc^n$$

for all $n > 0$.

We will need the following fibred analogous to the Schwarz-Pick Lemma. Let $V \subset \mathbb{C}$ be a simply connected region containing the origin, and let $h : \mathbb{D} \to V$ be a biholomorphism with $h(0) = 0$ and $h'(0) > 0$, this further conditions make the biholomorphism h unique. Then $h'(0)$ is called the *conformal radius* of V. The following result is a general version of the Schwartz-Pick Lemma (see [K] for reference).

Lemma 5. *Let $W > 1$ be a constant and D a topological disk whose conformal radius verifies*

$$W^{-1} < R(D) < W.$$

Then there exists a constant $C = C(W)$ such that

$$C^{-1}|z| < d_D(z, 0) < C|z|,$$

for $d_D(z, 0) < 1$. Let D, d' be topological disks. If $g : D \to D'$ is a holomorphic function, then

(i) g is a contraction in the strong sense in the induced metrics d_D, $d_{D'}$, i.e., for all $z_1, z_2 \in D$ we have

$$g^*(d_{D'}) \leq d_D,$$

$$\mathrm{dist}_{D'}(g(z_1), g(z_2)) \leq \mathrm{dist}_D(z_1, z_2),$$

where dist_D, and $\mathrm{dist}_{D'}$ are the distances induced by the hyperbolic metrics d_D and $d_{D'}$, respectively.

(ii) If f maps D compactly to D', that is if we have the inclusion $\overline{g(D)} \subset D'$, then g is a strict contraction in the induced metrics, in other words, there exists a positive constant $c < 1$, such that for all $z_1, z_2 \in D$ we have that g is a c-contraction.

The following is a detailed complete proof of a result from Ponce's PhD. Thesis [Po2]:

Proposition 8 (Proposition 2.11, [Po2])**.** *If* $F(\theta, z) = (\mathcal{R}_\alpha(\theta), f_\theta(z))$ *is a* fhd *defined in a neighborhood of* $\mathbb{T}^1 \times \overline{\mathbb{D}}$ *which satisfies the strong contraction hypothesis in a point, then there exists a continuous and invariant curve* $u : \mathbb{T}^1 \to \mathbb{D}$*. In addition, this invariant curve is attracting and attracts the whole set* $\mathbb{T}^1 \times \overline{\mathbb{D}}$ *in the future.*

Proof. Let $\widetilde{\theta} \in \mathbb{T}$ be the point where F satisfies the strong contraction hypothesis. From the proof of Lemma 4, we know there exists an open interval I from $\widetilde{\theta}$ such that f_θ is a $\widetilde{c}$-contraction, with $\widetilde{c} \in (0, 1)$, for all $\theta \in I$.
Given the definition of $K_\theta^n = f_{\theta - n\alpha}^n(\overline{\mathbb{D}})$, it follows that

$$K_\theta^{n+1} \subset K_\theta^n$$

So we have a nested compact intersection of non-empty compact sets. Then

$$\bigcap_{n \geqslant 0} K_\theta^n = \{p_\theta\}$$

for some point p_θ in the θ fiber.
With the above, we defined the curve $u : \mathbb{T}^1 \to \mathbb{D}$ given by $u(\theta) := p_\theta$ the intersection point of the family $\{K_\theta^n\}_{n \geqslant 0}$.

As $0 \in K_\theta^0 = \overline{\mathbb{D}}$ for all θ, we can see that u is in fact the uniform limit of the sequence of continuous curves defined by the iterates of the preimages of the zero section $\{z \equiv 0\}_{\theta \in \mathbb{T}}$:

$$u_n(\theta) = f_{\theta - n\alpha}^n(0).$$

Then u is continuous.

Moreover, from the definition of K_θ^n, we have:

$$f_\theta(K_\theta^n) = K_{\theta+\alpha}^{n+1}$$

taking the limit on both sides, we have

$$f_\theta(u(\theta)) = u(\theta+\alpha)$$

so that u is, in fact, invariant.

Let's show that u is an attracting curve. For each $\theta \in \mathbb{T}^1$, let A_θ be a neighborhood of $z_\theta = u(\theta)$, from the invariance of u, we have that

$$f_\theta : A_\theta \to f_\theta(A_\theta) \subset A_{\theta+\alpha}$$

is holomorphic. If $h_\theta : A_\theta \to \mathbb{D}$ is a conformal Riemann map, we get the following commutative diagram:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "g_\theta"] \arrow[d, "h_\theta^{-1}"'] & \mathbb{D} \arrow[d, "h_{\theta+\alpha}^{-1}"] \\
A_\theta \arrow[r, "f_\theta"] & A_{\theta+\alpha}
\end{tikzcd}$$

With $g_\theta : \mathbb{D} \to \mathbb{D}$ given by $g := h_{\theta+\alpha} \circ f_\theta \circ h_\theta^{-1}$. From the Schwarz-Pick Lemma 5 we observe that:

$$\mathrm{dist}_{\mathbb{D}}(g_\theta(h_\theta(\epsilon_1)), g_\theta(h_\theta(\epsilon_2)) \leqslant \mathrm{dist}_{\mathbb{D}}(h_\theta(\epsilon_1), h_\theta(\epsilon_2))) = \mathrm{dist}_{A_\theta}(\epsilon_1, \epsilon_2)$$

But

$$\begin{aligned}
\mathrm{dist}_{\mathbb{D}}(g_\theta(h_\theta(\epsilon_1)), g_\theta(h_\theta(\epsilon_2)) &= \mathrm{dist}_{\mathbb{D}}(h_{\theta+\alpha}(f_\theta(\epsilon_1)), h_{\theta+\alpha}(f_\theta(\epsilon_2)) \\
&= \mathrm{dist}_{A_{\theta+\alpha}}(f_\theta(\epsilon_1), f_\theta(\epsilon_2))
\end{aligned}$$

So that

$$\mathrm{dist}_{A_{\theta+\alpha}}(f_\theta(\epsilon_1), f_\theta(\epsilon_2)) \leqslant \mathrm{dist}_{A_\theta}(\epsilon_1, \epsilon_2)$$

i.e. $f_\theta : A_\theta \to A_{\theta+\alpha}$ it is a semi-contraction. But if $\theta \in I$, we know that f_θ is a strong contraction.

Given that $I = (\tilde{\theta} - \epsilon_1, \tilde{\theta} + \epsilon_2)$ for $\epsilon_1, \epsilon_2 > 0$ by the minimal ergodicity of $\mathcal{R}_\alpha(\theta) = \theta + \alpha$ there exists $n^* > 0$, such that $\mathcal{R}_\alpha^{n^*}(I) = \mathbb{T}^1$.

For all $\theta \in \mathbb{T}^1$, we have:

$$|f^n_{\theta-n\alpha}(z_1) - f^n_{\theta-n\alpha}(z_2)|_\theta = |f_{\theta-n\alpha}(f^{n-1}_{\theta-(n-1)\alpha}(z_1)) - f_{\theta-n\alpha}(f^{n-1}_{\theta-(n-1)\alpha}(z_2))|_\theta$$

Hence, if $\theta - n_0\alpha \in I$, for some $n_0 \in \mathbb{N}$, then

$$\begin{aligned} |f^{n_0}_{\theta-n_0\alpha}(z_1) - f^{n_0}_{\theta-n_0\alpha}(z_2)| &\leqslant \tilde{c}|f^{n_0-1}_{\theta-(n_0-1)\alpha}(z_1) - f^{n_0-1}_{\theta-(n_0-1)\alpha}(z_2)|_{\theta-\alpha} \\ &\leqslant \tilde{c}|z_1 - z_2|_{\theta-n_0\alpha} \end{aligned}$$

this way

$$\mathrm{diam} f^{n_0}_{\theta-n_0\alpha}(A_{\theta-n_0\alpha}) \leqslant \tilde{c}\ \mathrm{diam} A_{\theta-n_0\alpha}, \quad \tilde{c} \in (0,1)$$

For every $\theta \in \mathbb{T}^1$, let $\{n_k\}_{k\geqslant 0}$ the sub-sequence given by $\theta - n_k\alpha \in I$, from the semi-contraction of f_θ we have

$$\mathrm{diam} f^n_{\theta-n\alpha}(A_{\theta-n\alpha}) \leqslant \mathrm{diam}(f^{n_k}_{\theta-n_k\alpha}(A_{\theta-n_k\alpha})) \leqslant \tilde{c}^k \mathrm{diam}(A_{\theta-n_k\alpha})$$

we can choose I in such a way that $\mathrm{diam}(A_\theta) \leqslant 1$ for all $\theta \in I$, we conclude that

$$\begin{aligned} &|\partial_z f^{n_k}_\theta(p_\theta)| \leqslant \tilde{c}^k \\ &|\partial f^n_\theta(u(\theta))| \leqslant |\partial_z f^{n_k}_\theta(u(\theta))| \leqslant \tilde{c}^k \to 0, \quad \tilde{c} \in (0,1) \end{aligned}$$

Now, we consider the Birkhoff sums:

$$\lim_n \frac{1}{n} \log |\partial_z f^n_\theta(u(\theta))| \leqslant \lim_k \frac{1}{k} \log |\partial_z f^{n_k}_\theta(u(\theta))| < 0$$

it follows that

$$\kappa(u) = \int_{\mathbb{T}^1} \log |\partial_z f_\theta u(\theta)| d\theta < 0$$

i.e. $u(\theta)$ is an attracting invariant curve. $\square$

Chapter 3

Hyperbolic Polynomials over irrational rotations

Hyperbolicity is a strong property for dynamical systems in general. Broadly speaking, in a hyperbolic dynamical system "almost every orbit" has a well-defined asymptotic behavior: either tends to an attracting invariant object or "escape" uniformly to the border of the space where the system is defined.

As was mentioned in the previous chapter, we are especially interested in fibred holomorphic dynamics with base an irrational rotation over the unitary circle. In this chapter, we will study some nice properties these systems hold in the hyperbolic case. What will lead us, under contradiction, to the non-density of these types of systems.

3.1 The map $\theta \mapsto \mathcal{K}_\theta$ is continuous

As was mentioned earlier, [Se1, Example 4.1] shows that, although hyperbolicity of a polynomial is enough condition for the Hausdorff continuous of the Julia set by fiber, $\mathcal{J}_\theta$, it is not enough to have Hausdorff continuity of the filled-in Julia sets, by fiber $\mathcal{K}_\theta$. Recall that $\mathcal{J}_\theta = \partial\mathcal{K}_\theta$.

Let's analyze in detail the example provided by Sester.

Example. Consider the compact subset $T = \{0\} \cup \{\frac{1}{n}\}_{n\in\mathbb{N}^*} \cup \{2\} \subset \mathbb{R}$ and let $\varphi : T \to T$

be the continuous map given by

$$\varphi(t) = \begin{cases} 0 & t = 0, \\ \dfrac{1}{n-1} & t = \dfrac{1}{n},\ n > 1, \\ 2 & t \in \{1, 2\}. \end{cases}$$

Now, for each $t \in T\backslash\{2\}$, we set $c(t) = 0$ and $c(2) = -3$. We define the fibred polynomial $P : T \times \mathbb{C} \to T \times \mathbb{C}$ by the formula

$$P(t, z) = (\varphi,\ p_t(z) = z^2 + c(t)).$$

We note first that, 0 and 2 are fixed points for the map φ, hence the orbits starting at the fibers $t = 0$ and $t = 2$ stay in their respective fiber.

On one hand, $\varphi(0) = 0$ implies that $p_0(z) = z^2$ and then $\mathcal{K}_0 = \overline{B}(0, 1)$, hence $\mathcal{J}_0 = \mathbb{S}^1 = \partial\overline{B}(0, 1) = \partial\mathcal{K}_0$.

On the other hand, $\varphi(2) = -3$ implies that $p_2(z) = z^2 - 3$, this way, $\mathcal{K}_2 = \mathcal{J}_2$ is a Cantor set, and this implies that $\mathrm{int}(\mathcal{K}_2) = \emptyset$.

For $t \in T\backslash\{0, 2\}$, the orbits of φ goes away from $t = 0$, until they *land* in the fiber $t = 2$. The same happens with the dynamics of P fiber by fiber. It is not difficult to notice that $\mathcal{K}_1 = \mathcal{J}_1 = p_0^{-1}(\mathcal{J}_2)$, and for $n > 1$,

$$\mathcal{K}_{1/n} = \mathcal{J}_{1/n} = p_0^{-1}(\mathcal{J}_{\frac{1}{n-1}}).$$

It follows that

$$\partial\mathcal{K}\backslash\mathcal{J} = \{0\} \times B(0, 1).$$

This means that there is no continuity of the mapping $t \mapsto \mathcal{K}_t$ at $t = 0$. Discreteness elsewhere makes the map trivially continuous.

Remark 3. *The value $c(2) = -3$ is totally arbitrary, in fact, any value $c \notin \mathcal{M}_2$ outside the Mandelbrot set will work for this example. So, this is a robust family for which the map $t \mapsto \mathcal{K}_t$ is not continuous.*

Two important aspects of this example are highlighted: the base space T is not a continuum and for every $t \in T$ with $t \neq 0$, $\mathrm{int}(\mathcal{K}_t) = \emptyset$.

The first issue is well covered when $T = \mathbb{T}^1$ is the unit circle. The former will be necessary to be covered since, as often happens when dealing with Hausdorff convergence of compact sets, it is easier to deal with their complements, which are open sets. That is why this condition will be assumed in our results.

Analogous to the Hausdorff convergence for compact sets, it is possible to define a convergence for open sets in the Riemann sphere (see [MT, p. 222] for further references).

Definition 8. *Let O and O_n, $n \in \mathbb{N}$, be open sets in the Riemann sphere $\widehat{\mathbb{C}}$. We say that O_n converge to O in the sense of Carathèodory if the following conditions hold:*

(1) For each compact subset $I \subset O$, there is $N > 0$ such that $I \subset O_n$ for every $n \geq N$.

(2) If an open connected set V is contained in O_m for infinitely many $m \in \mathbb{N}$, then $V \subset O$.

The following result allows us to analyze the continuity of the set, indistinctly as collections of compact sets (Hausdorff), or as collections of open sets (Carathèodory) through their complements.

Lemma 6. *The closed sets K_n converge to K in the Hausdorff metric if and only if the complements K_n^c of K_n, converge to the complement K^c of K in the sense of Carathèodory.*

Now, we are in a position to prove the first property of a hyperbolic fibred dynamics with base an irrational rotation over the unit circle. Recall that the unit circle $\mathbb{T}^1$ is a continuum.

Lemma 7. *Let P be a hyperbolic fibred polynomial over an irrational rotation, such that for every $\theta \in \mathbb{T}^1$, $\mathrm{int}(\mathcal{K}_\theta) \neq \emptyset$. Then the map $\theta \mapsto \mathrm{int}(\mathcal{K}_\theta)$ is continuous in the sense of Carathèodory.*

Proof. Since P is hyperbolic, recall that, from Corollary 1, $\mathrm{int}(\mathcal{K}_\theta) = \mathcal{U}_\theta$. Suppose that $\theta \mapsto \mathcal{U}_\theta$ is discontinuous at some $\theta_0 \in \mathbb{T}^1$. Then, one of the two conditions in the definition of convergence in the sense of Carathèodory does not hold, that is: or (1) There exists a compact set $I \subset \mathcal{U}_{\theta_0}$ such that for every $\delta > 0$ there exists $\theta' \in \mathbb{T}^1$ with $|\theta' - \theta_0| < \delta$ and $I \not\subset \mathcal{U}_{\theta'}$. Or (2), there exists an open set V such that for every $\delta > 0$, there exists $\theta' \in \mathbb{T}^1$ with $|\theta' - \theta_0| < \delta$ such that $V \subset \mathcal{U}_{\theta'}$ but $V \not\subset \mathcal{U}_{\theta_0}$.

From point (1) we have the following: we know that P is uniformly continuous over the compact set $\mathbb{T}^1 \times \overline{B(0, R^{d^2})}$, where $R = R(P)$ is the escape rate of the polynomial P. Let $\varepsilon > 0$, set $\delta' = \delta'(\varepsilon) > 0$ such that $|P_\theta(z) - P_{\theta_0}(z)| < \varepsilon/2$ if $(\theta, z) \in \mathbb{T}^1 \times \overline{B(0, R^{d^2})}$, and $|\theta - \theta_0| < \delta'$. For such a δ' let $\theta' \in \mathbb{T}^1$ such that $I \cap \mathcal{U}^c_{\theta'} \neq \emptyset$ and $|\theta' - \theta_0| < \delta'$ and take z_0 in such intersection. Note that $z_0 \in \mathcal{U}_{\theta_0}$ since $I \subset \mathcal{U}_{\theta_0}$, so $z \in \mathcal{U}_{\theta_0}(\varepsilon')$ for some $\varepsilon' > 0$. From the definition of $\mathcal{U}_\theta(\varepsilon)$, we can take $\varepsilon' > \varepsilon$.

Since $z_0 \in \mathcal{U}^c_{\theta'}$, there exists an $\varepsilon/2-$pseudo orbit $\{(\mathcal{R}^n_\alpha(\theta'), x_n)\}$ of (θ', z_0) that is unbounded. Let $N > 0$ be the minimal n such that $R < |x_n| < R^{d^2}$. Now, we construct an $\varepsilon'-$pseudo orbit $\{(\mathcal{R}^n_\alpha(\theta_0), z_n)\}$ of (θ_0, z_0) as follows: for $1 \leq n \leq N$, $z_n = x_n$, and for $n > N$, $z_n = P^{n-N}_{\mathcal{R}^N_\alpha(\theta_0)}(z_N)$ the actual orbit of the point $(\mathcal{R}^N_\alpha(\theta_0), z_N)$. We claim that $\{(\mathcal{R}^n_\alpha(\theta_0), z_n)\}$ is an $\varepsilon-$pseudo orbit of (θ_0, z_0): note that we only have to check the condition for $1 \leq n \leq N$ since $\{z_n\}$ is an orbit for $n \geq N$. So, for $1 \leq n \leq N - 1$ we have

$$
\begin{aligned}
|z_{n+1} - P_{\mathcal{R}^n_\alpha(\theta_0)}(z_n)| &= |z_{n+1} - P_{\mathcal{R}^n_\alpha(\theta')}(z_n) + P_{\mathcal{R}^n_\alpha(\theta')}(z_n) - P_{\mathcal{R}^n_\alpha(\theta_0)}(z_n)| \\
&\leq |z_{n+1} - P_{\mathcal{R}^n_\alpha(\theta')}(z_n)| + |P_{\mathcal{R}^n_\alpha(\theta')}(z_n) - P_{\mathcal{R}^n_\alpha(\theta_0)}(z_n)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon < \varepsilon'.
\end{aligned}
$$

The first $\varepsilon/2$ comes from the fact that $z_n = x_n$ is part of an $\varepsilon/2-$pseudo orbit of (θ', z_0), and the second one is obtained from the uniform continuity with $|\mathcal{R}^n_\alpha(\theta') - \mathcal{R}^n_\alpha(\theta_0)| = |\theta' - \theta_0|$.

Then $\{(\mathcal{R}^n_\alpha(\theta_0), z_n)\}$ is an $\varepsilon'-$pseudo orbit of (θ_0, z_0), but $|z_n| \to \infty$ since $z_N \notin \mathcal{K}(P)$ which contradicts that $z_0 \in \mathcal{U}_{\theta_0}(\epsilon)$.

For (2) we make an analogous construction of an unbounded $\varepsilon-$pseudo orbit of a point (θ', z_0) with $V \subset \mathcal{U}_{\theta'}, z_0 \in V \cap \mathcal{U}^c_{\theta_0} \neq \emptyset$ and $z_0 \in \mathcal{U}_{\theta'}(\epsilon)$. $\square$

We have then the first important result of fibred hyperbolic polynomials $P : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ over the unit circle. This result marks a difference with respect to the counter-example given by Sester (example above).

Proposition 9. *Let P be a hyperbolic fibred polynomial over an irrational rotation, such that* $\text{int}(\mathcal{K}_\theta) \neq \emptyset$ *for every $\theta \in \mathbb{T}^1$. Then the map $\theta \mapsto \mathcal{K}_\theta$ is Hausdorff continuous.*

Proof. By the hyperbolicity of P we have that $\theta \mapsto \mathcal{J}_\theta$ is Hausdorff continuous, by Lemma 6, $\theta \mapsto \mathcal{J}_\theta^C$ is continuous in the sense of Carathèodory, but

$$\mathcal{J}_\theta^C = \text{int}(\mathcal{K}_\theta) \sqcup \mathcal{A}_\theta(\infty),$$

since $\theta \mapsto \text{int}(\mathcal{K}_\theta)$ is continuous, the above equation implies that $\theta \mapsto \mathcal{A}_\theta(\infty)$ is also continuous in the sense of Carathèodory, again, by Lemma 6 the map $\theta \mapsto \mathcal{A}_\theta(\infty)^C$ is Hausdorff continuous, but $\mathcal{A}_\theta(\infty)^C = \mathcal{K}_\theta$. $\square$

We close this section with the following consequences of the above result. These corollaries give us a *picture* of the above results by showing that if a hyperbolic polynomial is fibred over $\mathbb{T}^1$ by an irrational rotation, then the *boundary* of the filled-in Julia set, it is a *nice boundary* fiber by fiber.

Corollary 4. *Let P be a hyperbolic fibred polynomial over an irrational rotation, such that int($\mathcal{K}_\theta$) $\neq \emptyset$ for every $\theta \in \mathbb{T}^1$. If $(\theta, z) \in \mathcal{K}$ is such that $z \in$* $\text{int}(\mathcal{K}_\theta)$, *then* $(\theta, z) \in \text{int}(\mathcal{K})$.

Proof. Let $z \in \text{int}(\mathcal{K}_\theta)$. From condition (1) of continuity in the sense of Carathèodory, there exists $\delta > 0$ such that if $|\theta - \theta'| < \delta$ then $z \in \text{int}(\mathcal{K}_{\theta'})$. For $|\theta - \theta'| \leq \delta/2$, let $r = r(\theta') > 0$ be such that $B_r(z) \subset \text{int}(\mathcal{K}_{\theta'})$, from the compacity of $C = \{|\theta - \theta'| \leq \delta/2\}$, it follows that

$$r' = \inf_{\theta' \in C} r(\theta') > 0,$$

thus

$$B_{r'}((\theta, z)) \subset \bigcup_{\theta' \in C} \{\theta'\} \times \mathcal{K}_{\theta'}.$$

Hence, $d((\theta, z), \mathcal{K}^C) > 0$ and $(\theta, z) \in \operatorname{int}(\mathcal{K})$. □

Corollary 5. *Let P be a hyperbolic fibred polynomial over an irrational rotation, such that int$(\mathcal{K}_\theta) \neq \emptyset$ for every $\theta \in \mathbb{T}^1$. We have*

$$\partial\mathcal{K} = \mathcal{J} \qquad \textit{and} \qquad \operatorname{int}(\mathcal{K}) = \bigcup_{\theta \in \mathbb{T}^1} \{\theta\} \times \operatorname{int}(\mathcal{K}_\theta).$$

Proof. In general, we have the inclusion $\mathcal{J} \subset \partial\mathcal{K}$. Suppose that $\partial\mathcal{K} \setminus \mathcal{J} \neq \emptyset$ and take $(\theta, z) \in \partial\mathcal{K} \setminus \mathcal{J}$. Since P is hyperbolic we know that $\mathcal{J} = \bigcup_{\theta \in \mathbb{T}^1} \{\theta\} \times \mathcal{J}_\theta$. Hence, $z \notin \mathcal{J}_\theta = \partial\mathcal{K}_\theta$ and $z \in \operatorname{int}(\mathcal{K}_\theta)$. By Corollary 4 it follows that $(\theta, z) \in \operatorname{int}(\mathcal{K})$ which contradicts that $(\theta, z) \in \partial\mathcal{K}$. □

3.1.1 Recent result on (in)stability

Fibred dynamical systems can be considered as being in an *intermediate* dimension. They are not complex, but they mix both real dynamics and complex dynamics, the space where they live is a 3-real dimensional space. Given the importance of hyperbolicity among general dynamical systems, the question of their density always appeared. Recently, this question has been studied for higher dimensional complex polynomial dynamics. In this section, we extend a comment on these results and how the techniques used do not apply directly to our context.

In (complex) higher dimension, stability/bifurcation theory is based on the so-called "small Julia set" $\mathcal{J}^*$, which by definition is the support of the measure of maximal entropy. Thanks to the work of Bianchi, Berteloot, and Dupont [BeBiDu], we now know that the $\mathcal{J}^*$-stability is the correct generalization of the Mañé-Sad-Sullivan-Lyubich theory in higher dimension, and they leave the existence of persistent bifurcations as an open problem.

From a classical work of Newhouse [New1] persistent bifurcations can be obtained for invertible dynamics in dimension 2. On the other hand, in sufficiently high degree polynomial automorphism of $\mathbb{C}^2$, persistent homoclinic tangencies exist. There is not, however, a direct relation between these former results and the $\mathcal{J}^*$-(in)stability.

Simultaneously and independently, Dujardin [Duj2], Bianchi and Taflin [BT1], and later Taflin [Taf] by itself, address the open problem of density.

Let $\mathcal{H}_d(\mathbb{P}^k)$ denote the space of holomorphic mappings of $\mathbb{P}^k$ of degree d.

Theorem 2 (Dujardin, [Duj2]). *The bifurcation locus has a non-empty interior in the space $\mathcal{H}_d(\mathbb{P}^k)$ for every $k \geq 2$ and $d \geq 2$.*

Dujardin developed two mechanisms to prove this theorem: one topological and one geometric fractal. In the first mechanism, a topological manifold contained in $\mathcal{J}^*$ is constructed in such a way that it intersects the post-critical set, the intersection is proved to be non-empty by *homological reasons*. For the second mechanism, a Canton-type fractal set is constructed and perturbed, achieving bifurcation persistency. Such Cantor sets are called *blenders*.

Bianchi and Taflin, on the other hand, took advantage of the dynamical richness of the *elementary Desboves family* of an endomorphism of $\mathbb{P}^K$. One of their main properties is that the small and large Julia sets coincide.

Theorem 3 (Bianchi and Taflin [BT1]). *The family of endomorphisms of $\mathbb{P}^k$ given by*

$$f_\lambda = [-x(x^3 + 2z^3) : y(z^3 - x^2 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3)]$$

with $\lambda \in \mathbb{C}^*$ satisfies the following properties:

(1) the Julia set of f_λ depends continuously on λ, for the Hausdorff topology;

(2) the bifurcation locus coincides with $\mathbb{C}^*$.

The idea to prove point (2) is the show that *Misiurewicz parameters* are dense in the parameter space.

Definition 9. *A parameter $\lambda_0 \in \mathbb{C}^*$ is called a* Misiurewicz parameter *if there exist a neighborhood $N_{\lambda_0} \subset \mathbb{C}^*$ of λ_0 and a holomorphic map $\sigma : N_{\lambda_0} \to \mathbb{P}^2$ such that*

(1) for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ is a repelling periodic point;

(2) $\sigma(\lambda_0)$ is in the Julia set of F_{λ_0};

(3) there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0)) \in f^{n_0}(C_f)$;

(4) $\sigma(N_{\lambda_0}) \nsubseteq f^{n_0}(C_f)$.

Here C_f *denotes the critical set of* f.

Finally, using the concept of *blenders*, Taflin reduces the construction of Dujardin, obtaining similar results. Let $\mathcal{P}_d$ denote the space of degree d polynomials on $\mathbb{C}$, and $\mathrm{Bif}(\mathcal{P}_d)$ its bifurcation locus.

Theorem 4 (Taflin [Taf])**.** *Let* $d \geq 2$. *If* p *and* q *are two elements of* $\mathcal{P}_d$ *such that* $p \in \mathrm{Bif}(\mathcal{P}_d)$ *then the map* $(p, q) \in \mathcal{H}_d(\mathbb{P}^2)$ *can be approximated both by polynomial skew products of the form* $(z, w) \mapsto (\tilde{p}(z, w), q(w))$ *having an iterate with a blender of repelling type and by others having an iterate with a blender of saddle type.*

Corollary 6. *The bifurcation locus of the family* $\mathcal{P}_d \times \mathcal{P}_d$ *is contained in the closure of the interior of the bifurcation locus in* $\mathcal{H}_d(\mathbb{P}^2)$.

The first thing that we must highlight about these results is that the Bianchi and Taflin example is of degree greater than 2 and is not a polynomial dynamics, but rational. For the other two results, the difference with fibred dynamics is very significant.

Firstly, both constructions are based on *skew products* or mapping products of $\mathbb{C}^2$. So, what would be the fibred part of the system is, in fact, holomorphic not only continuous.

Moreover, the base polynomial is already chaotic enough: the Julia set is a Cantor set. Each coefficient of the polynomials are complex constant values, while in the fibred case are *continuous curves.* Hence, via the analyticity of the global parameter space, it is possible to generate *Holomorphic Motions* which proves the bifurcation persistency.

These notable differences make this novel work within the area.

3.1.2 Critical connections prevent hyperbolicity

In this section, we present and prove the main result of this work. As has been mentioned before, hyperbolicity is a significant property among dynamical systems, and on some occasions it turns out that this property is dense in the respective space. On this occasion, we give a negative answer to the natural question: are the fibred hyperbolic systems dense? We will see that when the system is given by a fibred polynomial with base an irrational rotation, non-hyperbolic polynomials are "fat" enough to avoid density of hyperbolicity.

Unlike the mechanisms described in the above section, the mechanism we will use to prove that the *bifurcation locus* of fibred polynomials is open, is a simple mechanism.

The next definition is inspired by the example provided by Ponce [Po3], where a (real) family of fibred quadratic polynomials admitting two invariant attracting curves is presented (see Section 3.1.3 below for details). Given that the polynomials are fibred over the unit circle via an irrational rotation, the critical set consists of continuous curves.

In that example, the curve of critical points intersects both invariant attracting curves. Instead, in the next definition, we will only require that the set of critical points "connects" the basins of attraction of the curves. Recall that $P : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ is a fibred polynomial with base an irrational rotation over the unit circle $\mathbb{T}^1$, and if $u : \mathbb{T}^1 \to \mathbb{C}$ is an invariant curve for P, then $\mathcal{A}(u)$ denotes the basin of attraction of the curve u, that is,

$$\mathcal{A}(u) = \{(\theta, z) \in \mathbb{T}^1 \times \mathbb{C} : \lim_{n\to\infty} |p_\theta^n(z) - u(\mathcal{R}_\alpha^n(\theta))| = 0\}.$$

Definition 10. *We say that a fibred polynomial P has a* critical connection *when P admits two distinct invariant attracting curves $u_1, u_2 : \mathbb{T}^1 \to \mathbb{C}$ and there exists a connected component $\Omega_0 \subset \Omega$ of the critical set of P such that $\mathcal{A}(u_1) \cap \Omega_0 \neq \emptyset$ and $\mathcal{A}(u_2) \cap \Omega_0 \neq \emptyset$.*

In classical holomorphic dynamics, Fatou proved that there exists a direct relation between the set of critical points (which is always a discrete set) and the different stable components of the system. As part of his doctoral work [Se0], O. Sester established a similar relation for the fibred case in the general context. Recall that, in the fibred case, the stable set (by chains) is defined in terms of ε-pseudo orbit.

We rewrite the definitions of such sets for a better understanding of the proofs of the results below.

For $\varepsilon > 0$, we denote by $\mathcal{U}(\varepsilon)$ the set of points in $\mathbb{T}^1 \times \mathbb{C}$ whose ε-pseudo orbits are bounded, and then consider the following sets

$$\mathcal{U}_\theta(\varepsilon) = \{z \in \mathbb{C} : (\theta, z) \in \mathcal{U}(\varepsilon)\},$$

$$\mathcal{U} = \bigcup_{\varepsilon>0} \mathcal{U}(\varepsilon), \qquad \text{and} \qquad \bigcup_{\varepsilon>0} \mathcal{U}_\theta(\varepsilon).$$

The first thing we want to do is to find a sufficient condition to avoid hyperbolicity. The next result says that *critical connection* is the right candidate.

Theorem 5. *Let $P : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ be a fibred polynomial over an irrational rotation. If P has a critical connection then P is non-hyperbolic.*

Proof. Suppose that P is hyperbolic. We know that invariant-attracting curves are contained in open invariant tubes. Also, the basins of attraction of each invariant attracting curve are open subsets of $\mathbb{T}^1 \times \mathbb{C}$.

First of all, from the fact that Ω intersect both attracting invariant curves and that $\mathcal{J} = \partial\mathcal{K}$ since P is hyperbolic, it follows that $\Omega \subset \text{int}(\mathcal{K})$, because if $\Omega \cap \mathcal{A}(\infty) \neq \emptyset$ then $\Omega \cap \partial\mathcal{K} \neq \emptyset$.

Without loss of generality, we assume that Ω is a simple close curve in $\mathbb{T}^1 \times \mathbb{C}$. We use the following notation. For $z \in \mathcal{U}_\theta$, we denote by $V_{\theta,z} \subset \mathcal{U}_\theta$ the connected component of $\mathcal{U}_\theta$ containing point z. After a suitable change of coordinates, we may assume that one of the invariant curves is $u_1 = \{z \equiv 0\}_{\theta\in\mathbb{T}^1}$ and $\Omega_0 \cap \mathcal{A}_0(0) \neq \emptyset$. We have that $(\theta, z) \in \mathcal{A}(0)$, the basin of attraction of u_1, if and only if $|P_\theta^n(z)| \to 0$, as $n \to \infty$.

Let $w_{\theta,0} = w_0 \in \Omega_0 \cap \mathcal{A}_0(0)$, i.e., $(0, w_0) \in \mathcal{A}(0)$. We know that Ω is a continuous curve, $\mathcal{A}(0)$ is open, and $\Omega \cap \mathcal{A}(u_2) \neq \emptyset$, thus there exists $\theta_0 > 0$ such that $\Omega_{\theta_0} \cap \mathcal{A}_{\theta_0}(0) = \emptyset$, but $\Omega_\theta \cap \mathcal{A}_\theta(0) \neq \emptyset$ for $0 \leq \theta < \theta_0$. We denote by $w_\theta \in \Omega_\theta$ the critical point that is in the same segment of Ω than w_0, $0 \leq \theta \leq \theta_0$.

We claim that $(\theta_0, w_{\theta_0}) \notin \mathcal{A}(u_2) \cup \mathcal{A}(\infty)$ since both sets are open in $\mathbb{T}^1 \times \mathbb{C}$. Recall we are assuming P is hyperbolic. From (4) of Theorem 1 we have $\overline{\mathcal{P}\Omega} \cap \mathcal{J} = \emptyset$. Given that $(\mathcal{J})_\theta = \mathcal{J}_\theta$, there exists $\varepsilon > 0$ and $k > 0$ such that, for $n_k \in \Lambda(V_{\theta_0,w_{\theta_0}})$,

$$d(P_{\theta_0}^{n_k}(w_{\theta_0}), \mathcal{J}_{\theta_k}) \geq \epsilon > 0.$$

Now, $(\theta_0, w_{\theta_0}) \notin \mathcal{A}(0)$, implies $P_{\theta_0}^{n_k}(w_{\theta_0}) \notin \mathcal{A}_{\theta_{n_k}}(0)$. Let $W_k := \overline{\mathcal{A}_{\theta_k}(0)}$, then

$$d(P_{\theta_0}^{n_k}(w_{\theta_0}), W_k) \geq \varepsilon > 0,$$

since $\partial \mathcal{A}_\theta(0) \subset \mathcal{J}_\theta$ for every θ by hyperbolicity.

For $r > 0$ and a compact subset $B \subset \mathbb{C}$, we denote by

$$\Delta_r(B) = \{z \in \mathbb{C} : d(z, B) < r\},$$

an r-neighborhood of the compact set B. From the Hausdorff continuity of the map $\theta \mapsto \mathcal{J}_\theta$, there exists $\delta_1 > 0$ such that if $|\theta - \theta_{n_k}| < \delta_1$, then

$$\mathcal{J}_\theta \subset \Delta_{\varepsilon/4}(\mathcal{J}_{\theta_{n_k}}).$$

It is clear that P^N is a continuous function for every $N \geq 1$. Let $\delta_2 > 0$ be such that $|(\theta, z) - (\theta_0, w_{\theta_0})| < \delta_2$, implies

$$|P_\theta^{n_k}(z) - P_{\theta_0}^{n_k}(w_{\theta_0}(w_{\theta_0}))| < \varepsilon/2.$$

Take $\delta = \min(\delta_1, \delta_2)$. If $0 < \theta_0 - \theta < \delta$, then

$$|(\theta, w_\theta) - (\theta_0, w_{\theta_0})| < \delta \qquad \Rightarrow \qquad |P_\theta^{n_k}(w_\theta) - P_{\theta_0}^{n_k}(w_{\theta_0})| < \varepsilon/4,$$

and

$$d(P_{\theta_0}^{n_k}(w_{\theta_0}), W_k) \geq \varepsilon > 0.$$

Therefore,

$$d(P_\theta^{n_k}(w_\theta), W_k) > 0,$$

which is a contradiction since $(\theta, w_\theta) \in \mathcal{A}_\theta(0)$. Hence, we conclude that P cannot be hyperbolic and the proof of the Theorem 5 is complete. $\square$

3.1.3 Hyperbolic fibred quadratic polynomials are not dense

Critical connections are robust

Once we have a model for non-hyperbolicity, the critical connection, we need to know how fat this condition is among the space of fibred polynomials. Let α be a fixed irrational and $d \geq 2$. We consider the space $\mathcal{F}_{d,\mathcal{R}_\alpha}$ of fibred polynomials of degree d over the irrational rotation $\mathcal{R}_\alpha$.

Definition 11. *We say that a property $\mathcal{H}$ is robust whenever the set of fibred polynomials in $\mathcal{F}_{d,\mathcal{R}_\alpha}$ that verify the property $\mathcal{H}$ is open in $\mathcal{F}_{d,\mathcal{R}_\alpha}$.*

The following lemmas relate continuity condition to the critical connection.

Lemma 8. *Let P be a fibred polynomial in $\mathcal{F}_{d,\mathcal{R}_\alpha}$ and u be an attracting invariant curve for P. Then there exists an open neighborhood $\mathcal{V} \subset \mathcal{F}_{d,\mathcal{R}_\alpha}$ of P such that for every $Q \in \mathcal{V}$ there exists an attracting invariant curve v of Q. Moreover, the map $Q \mapsto v$ is continuous at P, u.*

Proof. The inequality (2.4) at the proof of Lemma 3 and Corollary 2, allows claiming that there exists an open invariant tube $\mathcal{T}$, i.e., such that every fiber $\mathcal{T}_\theta$ is a disc $B(u(\theta), r)$, for some $r > 0$ and such that $P(\theta, \mathcal{T}_\theta) \subset B(u(\theta + \alpha), \lambda r)$ for some $\lambda < 1$. As is pointed out in Section 2.4, from the stability theory for invariant curves in Chapter 2, page 20 of [Po2], we deduce that $\mathcal{T}$ is an open tube that is (strongly) contracted by P. But, from its definition, this is an open property in $\mathcal{F}_{d,\mathcal{R}_\alpha}$, that is, for Q close enough to P, the map q_θ is a strong contraction in the hyperbolic metric at every fiber $\mathcal{T}_\theta$. Following the ideas in the proof of Proposition 8, we can deduce that for every $\theta \in \mathbb{T}^1$ the set

$$\bigcap_{n\in\mathbb{N}} q^n_{\theta-n\alpha}(\mathcal{T}_{\theta-n\alpha}) \tag{3.1}$$

consists in exactly one point $v(\theta)$, and hence it produces an invariant curve $\theta \mapsto v(\theta)$. Since the convergence is uniform, the curve is continuous and attracts exponentially every point of $\mathcal{T}$. From this, similar to the conclusion in Proposition 8, we deduce that $\kappa(v) < 1$ and that v depends continuously on Q. $\square$

Lemma 9. *Let P be a fibred polynomial in $\mathcal{F}_{d,\mathcal{R}_\alpha}$ and Ω_0 be a connected component of the critical set Ω of P. Then there exists an open neighborhood $\mathcal{V} \subset \mathcal{F}_{d,\mathcal{R}_\alpha}$ of P such that for every $Q \in \mathcal{V}$ there exists a connected component Ω_0^Q of the critical set Ω^Q of Q such that $Q \mapsto \Omega_0^Q$ is continuous in the Hausdorff topology at P, Ω_0.*

Proof. The fibers of the critical set of any element of $\mathcal{F}_{d,\mathcal{R}_\alpha}$ are algebraic roots of a $d-1$ degree polynomial, which varies continuously with respect to the fiber polynomial. □

Proposition 10. *The property of admitting a critical connection is robust.*

Proof. Let $P \in \mathcal{F}_{d,\mathcal{R}_\alpha}$ having a critical connection. Since basins of attraction of attracting invariant curves are open, together with the fact the immediate basin of attraction of invariant curves contains a set whose size is uniform in a neighborhood of P, Lemma 8 and Lemma 9 allows us to conclude. □

Ponce's examples are (robustly) non-hyperbolic

In the quadratic non-fibred case, to have a finite attracting cycle implies hyperbolicity. In [Po3] the author proposes that, for every irrational α, for each $0 < \epsilon < 1$, the fibred quadratic polynomial

$$P(\theta, z) = (\theta + \alpha, z(1 + a(\theta)(z-1))), \tag{3.2}$$

possesses two attracting invariant curves, where $a(\theta) = \cos\theta + i(1-\epsilon)\sin\theta$. Note that $a(\theta)$ parametrizes an ellipse having mayor axis equals to the segment $[-1, 1] \subset \mathbb{R}$, and minor axis equals to $i\,[-(1-\epsilon), (1-\epsilon)] \subset i\mathbb{R}$.

By a direct verification, we can see that $u_- = \{z \equiv 0\}_{\theta\in\mathbb{T}^1}$ and $u_+ = \{z \equiv 1\}_{\theta\in\mathbb{T}^1}$ are invariant curves of the fibred quadratic polynomial P, with multipliers

$$\kappa(u_\pm) = \exp\Big(\int_{\mathbb{T}^1} \log|1 \pm a(\theta)|d\theta\Big) < 1, \tag{3.3}$$

that is, u_- and u_+ are attracting invariant curves. Moreover, the critical set of P is given by

$$\Omega(P) = \left\{ \left(\theta, \frac{a(\theta) - 1}{2a(\theta)} \right) \right\}_{\theta \in \mathbb{T}^1},$$

which is a simple closed curve. Since $(0, 0), (\pi, 1) \in \Omega$, P has a critical connection, and thus, it is non-hyperbolic. This fact, together with Proposition 10 allows obtaining the next result.

Theorem 6. *For any irrational α, the set of hyperbolic polynomials is non-dense in the space $\mathcal{F}_{2,\mathcal{R}_\alpha}$ of fibred quadratic polynomials over the circle rotation $\mathcal{R}_\alpha$.*

Chapter 4

Invariant Multi-curves

As was shown in the works by Ponce (see [Po2, Po1]), the primer objects that describe the local behavior of a fibred dynamics, are curves, invariant curves. It is natural to wonder if there exist more complex invariant objects, containing dynamical information from the system.

In this chapter, we describe a more general class of objects that can be found in *fibred polynomial dynamics* with base an irrational rotation. Moreover, the strategy in this chapter is to construct invariant multi-curves

4.1 Multi-curves

Let $n \in \mathbb{N}$, and $\tilde{\gamma} : [0, n] \to \mathbb{C}$ be a continuous function holding the following conditions:

- $\tilde{\gamma}(t) \neq \tilde{\gamma}(t + p)$, for $t \in (0, n)$ and $p \in \{1, 2, ..., n-1\}$; and
- $\tilde{\gamma}(n) = \tilde{\gamma}(0)$.

The function $\tilde{\gamma}$ induces a simple, closed, continuous curve in the fibred space $\mathbb{T}^1 \times \mathbb{C}$ given by

$$\begin{array}{rccc} \gamma : & [0, n] & \to & \mathbb{T}^1 \times \mathbb{C} \\ & \theta & \mapsto & (\langle \theta \rangle, \tilde{\gamma}), \end{array}$$

where $\langle \cdot \rangle$ denotes the fractional part of θ. In other words, γ is a simple and closed curve in $\mathbb{T}^1 \times \mathbb{C}$ turning n-times in the direction of the base space $\mathbb{T}^1$. We call a n**-curve,** a curve $\gamma : [0,n] \to \mathbb{T}^1 \times \mathbb{C}$ induced in this way. In general, a subset $\Gamma \subset \mathbb{T}^1 \times \mathbb{C}$ is called a $(p,n)-$ *curve* or **multi-curve** if Γ consists of p components, each of which is a n-curve.

Recalling that $\mathbb{T}^1 := \mathbb{R}/2\pi\mathbb{Z} \cong [0,1)$, a n-curve $\gamma : [0,n] \to \mathbb{T}^1 \times \mathbb{C}$, viewed as a continuous function $\gamma : [0,n] \to [0,1] \times \mathbb{C}$ (with the corresponding identification), may be viewed as a concatenated list of n curves $\gamma_0,\ \gamma_1, ..., \gamma_{n-1} : [0,1] \to [0,1] \times \mathbb{C}$ that satisfy $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 0, 1, ..., n-1$ and $\gamma_n = \gamma_0$. Each curve may be defined as

$$\gamma_i(\theta) = \tilde{\gamma}(i + \theta),\ i = 0, 1, ..., n-1. \tag{4.1}$$

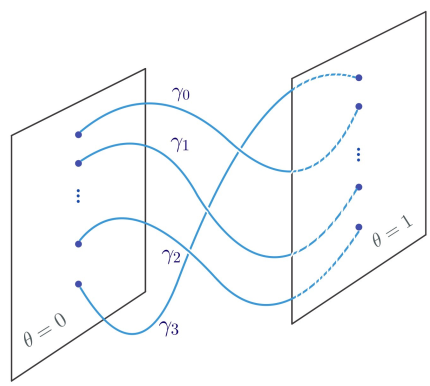


Figure 4.1: Displaying a 4-curve in $\mathbb{T}^1 \times \mathbb{C}$

Now, if we 'extend' the $\tilde{\gamma} : [0,n] \to \mathbb{C}$ over the fibred space $[0,n] \times \mathbb{C}$ in the most natural way,

$$\begin{array}{rccc} \tilde{\gamma} : & [0,n] & \to & [0,n] \times \mathbb{C} \\ & \theta & \mapsto & (\theta, \tilde{\gamma}), \end{array}$$

we recover a simple closed curve by identifying $[0,n) \cong \mathbb{R}/(2\pi n\mathbb{Z})$, since $\tilde{\gamma}(n) = \tilde{\gamma}(0)$.

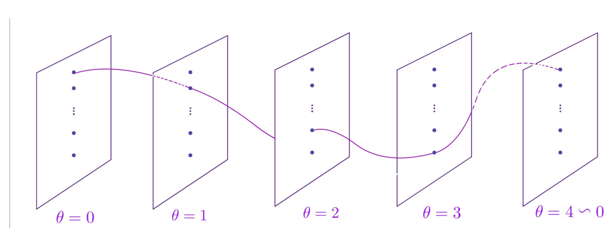


Figure 4.2: "Unfolding" process for the 4-curve

This new extended curve may be thought as an *'unfolding'* of the n-curve γ. We can reparametrize the curve $\tilde{\gamma}$ to be defined in the unit interval, and hence the unit circle.

Definition 12. *Let $\gamma = (\gamma_0\ \gamma_1\ \dots\ \gamma_{n-1})$ be a n-curve induced by a closed continuous function $\tilde{\gamma} : [0, n] \to \mathbb{C}$, the curve $\hat{\gamma} : [0, 1] \to [0, 1] \times \mathbb{C}$ given by*

$$\hat{\gamma}(\theta) = \gamma_{\lfloor n\theta \rfloor}(n\theta - \lfloor n\theta \rfloor) = \gamma_{\lfloor n\theta \rfloor}(\langle n\theta \rangle), \tag{4.2}$$

is called the ***unfolding curve*** *of γ. Here $\lfloor \cdot \rfloor$ denotes the* floor *function.*

From the properties of a n-curve, we have.

Proposition 11. *Let $\gamma = (\gamma_0\ \gamma_1\ \dots\ \gamma_{n-1})$ be a n-curve and $\hat{\gamma}$ be its unfolding curve as described above. Then, the function*

$$\begin{array}{rccc} \hat{\Gamma} : & \mathbb{T}^1 & \to & \mathbb{T}^1 \times \mathbb{C} \\ & \theta & \mapsto & (\theta, \hat{\gamma}(\theta)) \end{array} \tag{4.3}$$

is a well-defined, closed, continuous curve.

Take $k \in \{1, 2, \dots, n-1\}$ and define $\hat{\gamma}_k : [0, n] \to \mathbb{C}$ as a k-shift of the n-components of $\hat{\gamma}$, that is, $\hat{\gamma}_k = (\gamma_k\ \gamma_{1+k}\ \dots\ \gamma_{n-1+k})$. It follows that $\hat{\gamma}_0, \hat{\gamma}_1, \dots, \hat{\gamma}_{n-1}$ are pairwise disjoint curves in the fibred space $[0, n] \times \mathbb{C}$, that *induce the same n-curve* γ (as a topological object in $\mathbb{T}^1 \times \mathbb{C}$). In other words, each n-curve possesses n unfolding curves (as topological objects) by reparametrization.

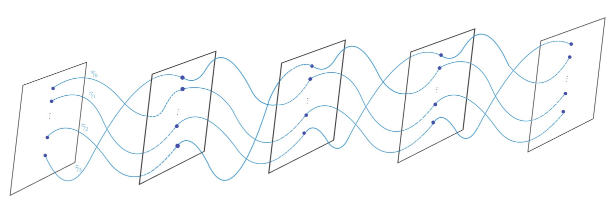


Figure 4.3: 4 *unfoldings* for a 4-curve

Remark 4. *When we talk of a n-curve as a dynamical object, it will be clear why we consider only integer translation for reparametrization.*

4.1.1 Invariant multi-curves

Consider the fibred holomorphic dynamics $F : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ and let $\gamma = (\gamma_0\ \gamma_1\ ...\ \gamma_{n-1})$ be a n-curve in $\mathbb{T}^1 \times \mathbb{C}$ such that $F(\theta, \gamma(\theta)) \in \gamma$, i.e. the restriction of F to the n-curve γ is (at least) an endomorphism. For each $\theta \in \mathbb{T}^1$, the fiber of γ over θ consists of n distinct points, and then Equation (2.3) makes no sense as an invariant notion. Although $\gamma = (\gamma_0\ \gamma_1\ ...\ \gamma_{n-1})$ is invariant under F as a subset of $\mathbb{T}^1 \times \mathbb{C}$, γ may be 'dynamically jumping' along the set of curves $(\gamma_0\ \gamma_1\ ...\ \gamma_{n-1})$

For $n \geq 1$, $\tau \in \{0, 1, 2, ..., n-1\}$, and the fibred dynamics $F : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$, define the following fibred mappings

$$\begin{array}{rccc} \hat{F}_\tau : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & \left(\theta + \dfrac{\alpha + \tau}{n}, f_{\langle n\theta \rangle}(z)\right) \end{array}$$

and

$$\begin{array}{rccc} \Pi : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (n\theta, z) \end{array}$$

A direct calculation gives

$$\begin{aligned} \Pi \circ \hat{F}_\tau(\theta, z) &= \Pi\left(\theta + \frac{\alpha + \tau}{n}, f_{\langle n\theta \rangle}(z)\right) \\ &= (n\theta + \alpha + \tau, f_{\langle n\theta \rangle}(z)) \\ &= (n\theta + \alpha, f_{\langle n\theta \rangle}(z)), \end{aligned}$$

and

$$\begin{aligned} F \circ \Pi &= F(n\theta, z) \\ &= (n\theta + \alpha, f_{\langle n\theta \rangle}(z)), \end{aligned}$$

obtaining the following commutative diagram.

$$\begin{tikzcd} \mathbb{T}^1 \times \mathbb{C} \arrow[r, "\hat{F}_\tau"] \arrow[d, "\Pi"'] & \mathbb{T}^1 \times \mathbb{C} \arrow[d, "\Pi"] \\ \mathbb{T}^1 \times \mathbb{C} \arrow[r, "F"'] & \mathbb{T}^1 \times \mathbb{C}. \end{tikzcd} \tag{4.4}$$

Notice that if we divide the interval $[0, 1]$ into n sub-intervals of length $1/n$, the integer τ in the fibred dynamics $\hat{F}_\tau$, represents a jump among these intervals. We then consider the following definition.

Definition 13. *A n-curve (or multi-curve) $\gamma = (\gamma_0\ \gamma_1\ \dots\ \gamma_{n-1})$ is called a* ***dynamically invariant curve*** *(or* ***invariant multi-curve****) for the fibred dynamics $F : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ if for each unfolding $\hat{\Gamma} : \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{C}$ as defined in (4.2), there exists an integer $\tau \in \{0, 1, \dots, n-1\}$ such that $\hat{\Gamma}$ is an invariant curve for the (lifted) fibred dynamics $\hat{F}_\tau : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$.*

The above commutative diagram makes this definition well-defined. More over, it is clear that $\hat{F}_\tau\big|_{\hat{\Gamma}}$ is a homeomorphism if and only if $F\big|_\gamma$ is a homeomorphism. Finally, τ determine how $F\big|_\gamma$ dynamically jumps among the curves $\gamma = (\gamma_0\ \gamma_1\ \dots\ \gamma_{n-1})$, this way, τ is called the **jumping integer** for $\gamma = (\gamma_0\ \gamma_1\ \dots\ \gamma_{n-1})$.

4.1.2 Dynamically invariant multi-curves exist.

In this short subsection, we exhibit a couple of examples of multi-curves for fibred dynamics. The examples are extreme opposite in the sense that the former is a trivial construction of several multi-curves for a fibred dynamics in the unit circle (with rotation as base map), while the further is a forced construction of a fibred polynomial dynamics based on a given *topological multi-curve.*

Example 1. Let $n \in \mathbb{N}$, $\alpha \in \mathbb{T}^1$, and $F : \mathbb{T}^1 \times \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{T}^1$ be a fibred dynamics on the unit circle over a rotation by α in the unit circle itself, $F(x, y) = (x + \alpha, y + \dfrac{\alpha}{n})$. Then the curve $\gamma : [0, n] \to \mathbb{T}^1$ defined by

$$t \mapsto \gamma(t) = \frac{t}{n},$$

induce an invariant n-curve for F. In fact, the phase space $\mathbb{T}^1 \times \mathbb{T}^1$ is foliated by invariant copies of this invariant n-curve.

Example 2. As is mentioned in Appendix A, the Interpolation Lagrange polynomial is a useful tool to construct *invariant multi-curves*. Let $\gamma = (\gamma_0\ \gamma_1\ ...\ \gamma_{n-1})$ be a n-curve, $\alpha \in \mathbb{R}$, and $\tau \in \{0, 1, ..., n-1\}$. For every $\theta \in \mathbb{T}^1$, let p_θ be the $n-1$ degree Lagrange interpolation polynomial taking the points $\{\gamma_0(\theta), \gamma_1(\theta), ..., \gamma_{n-1}(\theta)\}$ to the points $\{\gamma_{0+\tau}(\theta+\alpha), \gamma_{1+\tau}(\theta+\alpha), ..., \gamma_{n-1+\tau}(\theta+\alpha)\}$ sending point $\gamma_i(\theta)$ to the point $\gamma_{i+\tau}(\theta+\alpha)$, where $i+\tau$ is taken $(\mod n)$. Then, the fibred polynomial

$$\begin{array}{rccc} P: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\theta+\alpha, p_\theta(z)) \end{array}$$

is continuous and leaves γ (dynamically) invariant with jumping integer equal to τ. A similar construction can be made to get a fibred higher degree polynomial dynamics ($(n+p-1)$ degree) leaving invariant a prescribed (p, n)-curve.

4.1.3 Dynamical nature of multi-curves

One wonders if it is possible to determine a (locally) dynamical nature of an invariant multi-curves as for simple invariant curves. This will be possible since the invariance of the multi-curve is defined through an (unfolding) invariant curve.

The fibred multiplier can then be extended for invariant multi-curves in the following way.

Definition 14. *Suppose that γ is a (dynamically) invariant n-curve for the fibred polynomial $P(\theta, z) = (\theta+\alpha, p_\theta(\theta))$, if $\theta \mapsto \log|\partial_z \hat{p}_\theta(\hat{\gamma}(\theta))| \in L^1(\mathbb{T}^1)$ then the fibred multiplier of γ is defined as*

$$\kappa_f(\gamma) := \kappa(\hat{\gamma}) = \exp\left(\int_{\mathbb{T}^1} \log|\partial_z \hat{p}_\theta(\hat{\gamma}(\theta))| d\theta\right), \tag{4.5}$$

where $\hat{P}(\theta, z) = \left(\theta + \dfrac{\tau+\alpha}{n}, \hat{p}_\theta(z)\right)$ and $\hat{\gamma}$ are the lifted fpd *and its (unfolding) invariant curve associated.*

Remark 5. *Note that the multiplier of an invariant multi-curve is simply the multiplier of its corresponding unfolding invariant curve, this way, we can extend the dynamical nature of the unfolding invariant curve to the invariant multi-curve. So it makes sense to talk of an* **attracting, repulsor or indifferent invariant multi-curve** *if $\kappa_f(\gamma)$ is minor, greater or equal to 1, respectively.*

Remark 6. *In Example 2, by increasing the degree of p_θ, we can impose extra mild conditions on the complex derivative $\partial_z P$ at points of the multi-curve γ, so that γ yields into an attracting invariant n-curve.*

Remark 5 allows extending the local theory for multi-curves from Section 2.4. The following results follow directly from results in Section 2.4 through the commutative diagram (4.4).

Lemma 10 (The attracting case)**.** *Let P be a fibred polynomial dynamics over an irrational rotation, and γ be an attracting invariant multi-curve. Then there exists a continuous change of coordinates $H(\theta, z) = (\theta, a(\theta)z + b(\theta))$ such that γ is still an attracting invariant multi-curve for the conjugated fibred polynomial dynamics $Q = H^{-1} \circ P \circ H$. Moreover, if $Q(\theta, z) = (\theta + \alpha, q_\theta(z))$, then there exists $c < 1$ such that*

$$\sup_{\theta \in \mathbb{T}^1} |\partial_z q_\theta(\gamma(\theta))| < c.$$

Lemma 11 (The repulsor case)**.** *Let P be a fibred polynomial dynamics over an irrational rotation, and γ be a repulsor invariant multi-curve. Then there exists a continuous change of coordinates $H(\theta, z) = (\theta, a(\theta)z + b(\theta))$ such that γ is still a repulsor invariant multi-curve for the conjugated fibred polynomial dynamics $Q = H^{-1} \circ P \circ H$. Moreover, if $Q(\theta, z) = (\theta + \alpha, q_\theta(z))$, then there exists $c > 1$ such that*

$$\sup_{\theta \in \mathbb{T}^1} |\partial_z q_\theta(\gamma(\theta))| > c.$$

Also, the basin of attraction is well-defined in the attracting case.

Lemma 12. *Let P be a fibred polynomial dynamics over an irrational rotation, and let γ be an attracting invariant* n*-curve. Then there exists an open set $\mathcal{T} \subset \mathbb{T}^1 \times \mathbb{C}$ containing* *the multi-curve γ, and such that every point in $\mathcal{T}$ is attracted to γ, i.e., $z_\theta \in \gamma_\theta$ then*

$$\mathrm{dist}(P^n(\theta, z_\theta), \gamma) \to 0, \text{ as } n \to \infty.$$

Moreover, for every $\theta \in \mathbb{T}^1$, the fiber $\mathcal{T}_\theta$ consists of n*-components each of which contains a point of γ_θ.*

The open set $\mathcal{T}$ defined in the above lemma, may be thought as a (neighborhood) **multi-tube** around the multi-curve γ. This allows us to formally define its basin of attraction.

Definition 15. *Let P be a fibred polynomial dynamics over an irrational rotation, and let γ be an attracting invariant multi-curve. If $\mathcal{O}^+(\theta, z)$ defines the forward orbit under P of a point (θ, z), then*

$$\mathcal{A}(\gamma) = \{(\theta, z) : \mathrm{dist}(\mathcal{O}^+(\theta, z), \gamma) \to 0,\ n \to \infty\}$$

*is called **the basin of attraction of the multi-curve** γ.*

Analogous to the simply invariant case, we have that

$$\mathcal{A}(\gamma) = \bigcup_{n \geq 0} P^{-n}(\mathcal{T}),$$

where $\mathcal{T}$ is the invariant multi-tube defined in Lemma 12.

Corollary 7. *If γ is an attracting invariant multi-curve, then the basing of attraction of γ, $\mathcal{A}(\gamma)$, is an open subset of $\mathbb{T}^1 \times \mathbb{C}$.*

4.2 Invariant multi-curves in the quadratic case

4.2.1 Quadratic polynomials

The final aim of this section is to exhibit multi-curves in the lowest grade for polynomials where interesting dynamics appear: the quadratic case (we recall trivial Example 1 is of degree one). Consider the family $\mathcal{F}_\alpha$ of **canonical** fibred quadratic polynomials

$$
\begin{aligned}
P_{\mathcal{C}}^{\alpha}: \quad \mathbb{T}^1 \times \mathbb{C} &\rightarrow \mathbb{T}^1 \times \mathbb{C} \\
(\theta, z) &\mapsto (\theta + \alpha, z^2 + \mathcal{C}(\theta)),
\end{aligned}
\tag{4.6}
$$

where $\alpha \in \mathbb{T}^1$ is the irrational rotation angle and $\mathcal{C} : \mathbb{T}^1 \rightarrow \mathbb{C}$ is a continuous function that may be thought as a parameter. A wider family of quadratic polynomial dynamics has been widely studied by Sester in [Se1], here the author defines the corresponding *principal cardioid of the fibred Mandelbrot set*. We will be only focusing on those quadratic polynomials with a good normalization as stated in Proposition 7.

We will find invariant n-curves by choosing a parameter $\mathcal{C} : \mathbb{T}^1 \rightarrow \mathbb{C}$ that *wanders* through some special places around the classical Mandelbrot set.

Analogous to the non-fibred case, under a mild condition on the quadratic coefficient, every quadratic polynomial can be normalized to the form in (4.6). To make the constructions of the multi-curves clearer, we add the proof of the following result.

Lemma 13 ([Se1]). *Given $\alpha \in \mathbb{T}^1$, and three continuous functions $A : \mathbb{T}^1 \rightarrow \mathbb{C}^*$, $B : \mathbb{T}^1 \rightarrow \mathbb{C}$, and $C : \mathbb{T}^1 \rightarrow \mathbb{C}$, with $\mathrm{wind}(A(\mathbb{T}^1), 0) = 0$, consider the fibred quadratic polynomial*

$$
\begin{aligned}
F: \quad \mathbb{T}^1 \times \mathbb{C} &\rightarrow \mathbb{T}^1 \times \mathbb{C} \\
(\theta, z) &\mapsto (\theta + \alpha, A(\theta)z^2 + B(\theta)z + C(\theta)).
\end{aligned}
$$

Then, there exists a continuous fibred transformation, affine at each fiber,

$$
\begin{aligned}
W: \quad \mathbb{T}^1 \times \mathbb{C} &\rightarrow \mathbb{T}^1 \times \mathbb{C} \\
(\theta, z) &\mapsto (\theta, u_1(\theta)z + u_2(\theta))
\end{aligned}
$$

and a continuous parameter $\mathcal{C} : \mathbb{T}^1 \rightarrow \mathbb{C}$ such that

$$
W \circ F \circ W^{-1}(\theta, z) = (\theta + \alpha, z^2 + \mathcal{C}(\theta)),
$$

for every $\theta \in \mathbb{T}^1$.

Proof. The transformation W will be obtained as a composition of two affine transformations. First, by eliminating the linear part of the polynomial F, and secondly, by normalizing the coefficient of the quadratic term.

Let $W_B(\theta, z) = (\theta, z + w(\theta))$ be a continuous map. calculating, we have

$$\begin{aligned} W_B^{-1} \circ F \circ W_B(\theta, z) = (\theta + \alpha, A(\theta)z^2 + \{2A(\theta)w(\theta) + B(\theta)\}z + \\ + \{C(\theta) + A(\theta)w(\theta)^2 + B(\theta)w(\theta) - w(\theta + \alpha)\}). \end{aligned}$$

By taking

$$w(\theta) = -\frac{B(\theta)}{2A(\theta)},$$

the linear coefficient of F vanishes. So, after conjugacy, we may assume that $F(\theta, z) = A(\theta)z^2 + C(\theta))$. Consider the continuous map $W_A(\theta) = (\theta, u(\theta)z)$, after a careful calculation, we have

$$W_A^{-1} \circ F \circ W_A(\theta, z) = \left(\theta + \alpha, \frac{A(\theta)u(\theta)^2}{u(\theta + \alpha)} z^2 + \frac{C(\theta)}{u(\theta + \alpha)}\right).$$

We want that the quadratic term above has a coefficient equal to 1. Putting $\hat{u}(\theta) = u(\theta)A(\theta)$, the equation to solve changes to

$$\frac{A(\theta + \alpha)}{A(\theta)} \frac{\hat{u}(\theta)^2}{\hat{u}(\theta + \alpha)} = 1. \tag{4.7}$$

Set

$$\phi(\theta) = -\log\left(\frac{A(\theta + \alpha)}{A(\theta)}\right),$$

note that ϕ is well-defined since the topological degree of $\dfrac{A(\theta + \alpha)}{A(\theta)}$ with respect to the origin is zero. If $\hat{v} = \log \hat{u}$, Equation (4.7) yields

$$\begin{aligned} 2\hat{v}(\theta) &= \phi(\theta) + \hat{v}(\theta + \alpha) \\ \hat{v}(\theta) &= \frac{1}{2}\phi(\theta) + \frac{1}{4}\phi(\theta + \alpha) + \frac{1}{8}\phi(\theta + 2\alpha) + ... \end{aligned}$$

This series is uniformly convergent in $\mathbb{T}^1$ and defines a continuous function, and then a continuous function $u : \mathbb{T}^1 \to \mathbb{C}^*$ as required.

Defining $W = W_B \circ W_A$, we have the desired normalization. $\square$

Remark 7. *After a (very) careful calculation, we have that*

$$u(\theta) = [A(\theta)]^{-1} \cdot \prod_{j=0}^{\infty} \left[\frac{A(\theta + j\alpha)}{A(\theta + (j+1)\alpha)} \right]^{\frac{1}{2^{j+1}}}$$

$$\frac{1}{u(\theta+\alpha)} = A(\theta+\alpha) \cdot \prod_{j=1}^{\infty} \left[\frac{A(\theta + (j+1)\alpha)}{A(\theta + j\alpha)} \right]^{\frac{1}{2^{j+1}}}.$$

Hence, if $\alpha > 0$ is sufficiently small, then the products in the above relations are very close to 1, so $u(\theta)$ is very close to $[A(\theta)]^{-1}$ and $[u(\theta+\alpha)]^{-1}$ is very close to $A(\theta+\alpha)$.

4.3 Invariant 2-curves for small perturbation of a static quadratic dynamics

The 'static' fibred polynomial

Let $P(\theta, z) = (R_\alpha(\theta), p_\theta(z))$ be a fibred polynomial dynamics, where $p_\theta(z)$ is a degree d polynomial and $\alpha = 0$, in other words, $P(\theta, z)$ may be viewed as a continuous parametrized family of polynomial dynamics. We refer to this case as the *static fibred polynomial* case.

Suppose that $\mathcal{Z} \subset \mathbb{T}^1 \times \mathbb{C}$ is a connected component of the continuous solution to the *fixed-points* equation:

$$p_\theta(z(\theta)) = z(\theta).$$

Since $\alpha = 0$, it follows that $P\big|_{\mathcal{Z}}$ is a homeomorphism. Suppose also that $\mathcal{Z}$ is a n-curve, then it is easy to see that $\mathcal{Z}$ is an *invariant multi-curve* according to Definition 4.3.

The strategy of this chapter is to construct invariant multi-curves through the curves generated by the set of fixed points of the static fibred polynomial. We will consider suitable parametric curves (small circle with the classical parabolic parameter $c_0 = 1/4$ in its interior). Then, after a post-composition with a Lagrange Interpolation Polynomial, adding the fibred nature with the irrational rotation as described in Example2, we will maintain the invariance of the multi-curve for a fibred polynomial dynamics which will be still quadratic since the Lagrange Interpolation polynomial will be linear. Finally, via

Lemma 13 we will come back to the canonical form of the fibred quadratic polynomial.

4.3.1 Fixed Points of the quadratic polynomial

Consider the canonical form of a quadratic polynomial

$$q_c(z) = z^2 + c,\ z \in \mathbb{C}^*,$$

it is well known that the fixed points of q_c are given by

$$z_1(c) = \frac{1}{2} + \sqrt{\frac{1}{4} - c}, \qquad \& \qquad z_2(c) = \frac{1}{2} - \sqrt{\frac{1}{4} - c} \tag{4.8}$$

If $c = 1/4$, q_c possesses one, and only one fixed point; otherwise, there are always two distinct fixed points.

For fixed $\varepsilon > 0$ sufficiently small, let $\mathcal{C} : \mathbb{T}^1 \to \mathbb{C}$ be the continuous function given by

$$\mathcal{C}(\theta) = 1/4 - \varepsilon^2 e^{2\pi i\theta},$$

that is, $\mathcal{C} : \mathbb{T}^1 \to \mathbb{C}$ is a simple continuous loop around the parabolic parameter $c = 1/4$. If we keep tracking the fixed points $z_0(\mathcal{C}(\theta)), z_1(\mathcal{C}(\theta))$ when θ goes from 0 to 1 (in $\mathbb{T}^1$), we see that this tour to the loop of parameters gives rise to a transposition of the fixed points, as we will see in the next lines.

Now, if we substitute the form of $c = \mathcal{C}(\theta) = 1/4 - \varepsilon^2 e^{2\pi i\theta}$ in the solutions (4.8) we have

$$z_1(\theta) := z_1(\mathcal{C}(\theta)) = \frac{1}{2} + \varepsilon e^{\pi i\theta}, \qquad \& \qquad z_2(\theta) := z_2(\mathcal{C}(\theta)) = \frac{1}{2} - \varepsilon e^{\pi i\theta},$$

and if we take $-1 = e^{\pi i}$, then

$$z_1(\theta) = \frac{1}{2} + \varepsilon e^{\pi i\theta}, \qquad \& \qquad z_2(\theta) = \frac{1}{2} + \varepsilon e^{\pi i(\theta+1)}.$$

So, let $\tilde{\gamma} : [0, 2] \to \mathbb{C}$ be the continuous function defined as

$$\tilde{\gamma}(\theta) = 1/2 + \varepsilon e^{\pi i\theta}.$$

It is clear that $\tilde{\gamma}(\theta) \neq \tilde{\gamma}(\theta + p)$, for $\theta \in (0, 2)$ and $p \in \{0, 1\}$, and that $\tilde{\gamma}(2) = \tilde{\gamma}(0)$. It follows that $\tilde{\gamma}$ induces the 2-curve $\gamma = (\gamma_1 \ \gamma_2)$,

$$\begin{array}{rccc} \gamma : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\langle \theta \rangle, \tilde{\gamma}(\theta)) \end{array}$$

where $\gamma_0 = z_1$ and $\gamma_1 = z_2$.

Remark 8. *As was mentioned above, this γ is our candidate for an invariant 2-curve for a static quadratic dynamics.*

Consider the static fibred quadratic dynamics, that is, with $\alpha = 0$,

$$\begin{array}{rccc} Q : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\theta, q_{\mathcal{C}(\theta)}(z)). \end{array} \tag{4.9}$$

.

Proposition 12. *The 2-curve $\gamma = (z_1 \ z_2)$ is invariant for the static fibred polynomial $Q(\theta, z) = (\theta, q_{\mathcal{C}(\theta)}(z))$.*

Proof. Since $\gamma = (\gamma_0 \ \gamma_1) = (z_1 \ z_2)$, we have that $Q\big|_\gamma$ is a homeomorphism, further given that $z_0(\theta)$ and $z_1(\theta)$ are fixed points of $q_{\mathcal{C}(\theta)}$, we can take $\tau = 0$ for the *lifted fhd*

$$\begin{array}{rccc} \widehat{Q} : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\theta, q_{\mathcal{C}(\langle 2\theta \rangle)}(z)), \end{array}$$

with $\Pi(\theta, z) = (2\theta, z)$. Rest to prove that each *unfolding* curve $\widehat{\gamma}_{0,1} : [0, 1] \to \mathbb{C}$ of the 2-curve γ is a (simple) invariant curve for $\widehat{Q}$, which follows from the next result.

Lemma 14. *Let $\widehat{\gamma}_i$, $i = 0, 1$, be the unfolding curves of the 2-curve $\gamma = (z_1 \ z_2)$. Then $\widehat{\gamma}_i$ is an invariant curve for $\widehat{Q}$ (here $\alpha = 0$).*

Proof. We want to show that

$$q_{\mathcal{C}(\langle 2\theta \rangle)}(\widehat{\gamma}_i(\theta)) = \widehat{\gamma}_i(\theta).$$

This follows directly from Equation (4.2). Since

$$\widehat{\gamma}_i(\theta) = z_{\lfloor i+2\theta \rfloor}(\langle 2\theta \rangle),$$

we have,

$$q_{\mathcal{C}(\langle 2\theta \rangle)}(\widehat{\gamma}_i(\theta)) = q_{\mathcal{C}(\langle 2\theta \rangle)}(z_{\lfloor i+2\theta \rfloor}(\langle 2\theta \rangle)) = z_{\lfloor i+2\theta \rfloor}(\langle 2\theta \rangle) = \widehat{\gamma}_i(\theta).$$

□

This finishes the proof of the proposition. □

4.3.2 From Static to Fibred. The Post-Composition

For $\alpha > 0$, we want to "transform" $\widehat{Q}$ in such a way that z_1 and z_2 still form an invariant 2-curve for a fibred quadratic polynomial. For each $\theta \in \mathbb{T}^1$, consider the two pairs of points $(\widehat{\gamma}_0(\theta), \widehat{\gamma}_1(\theta))$ and $(\widehat{\gamma}_0(\theta + \frac{\alpha}{2}), \widehat{\gamma}_1(\theta + \frac{\alpha}{2}))$, and define the linear fibred map

$$\begin{array}{rccc} \tilde{L}_\alpha : & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, \zeta) & \mapsto & (\theta, \tilde{\ell}_\theta(\zeta)), \end{array}$$

where $\tilde{\ell}_\theta$ is given by the Lagrange interpolation polynomial (affine, since $n = 2$) between the pairs of points considered above. Then,

$$\tilde{\ell}_\theta(\widehat{\gamma}_0(\theta)) = \widehat{\gamma}_0(\theta + \frac{\alpha}{2}), \qquad \text{and} \qquad \tilde{\ell}_\theta(\widehat{\gamma}_1(\theta)) = \widehat{\gamma}_1(\theta + \frac{\alpha}{2}).$$

Explicitly, we have

$$\tilde{\ell}_\theta(z) = z + (\widehat{\gamma}_0(\theta + \frac{\alpha}{2}) - \widehat{\gamma}_0(\theta)) \left(\frac{z - \widehat{\gamma}_1(\theta)}{\widehat{\gamma}_0(\theta) - \widehat{\gamma}_1(\theta)} \right) + (\widehat{\gamma}_1(\theta + \frac{\alpha}{2}) - \widehat{\gamma}_1(\theta)) \left(\frac{z - \widehat{\gamma}_0(\theta)}{\widehat{\gamma}_1(\theta) - \widehat{\gamma}_0(\theta)} \right)$$

Proposition 13. *For $\alpha > 0$, and $\tilde{\ell}_\theta$ and $q_{\mathcal{C}(\theta)}$ as above, define the fibred quadratic polynomial*

$$\widetilde{Q} = \widetilde{L}_\alpha \circ \widehat{Q},$$

that is,

$$\begin{array}{rccc} \widetilde{Q}: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, \zeta) & \mapsto & (\theta + \frac{\alpha}{2}, \tilde{\ell}_\theta \circ q_{\mathcal{C}(\langle 2\theta \rangle)}(\zeta)). \end{array}$$

Then the unfolding *curve* $\widehat{\gamma}$ *is an invariant curve for* $\widetilde{Q}$.

Proof. The result follows once again by a direct calculation.

$$\begin{aligned} \tilde{\ell}_\theta \circ q_{\mathcal{C}(\langle 2\theta \rangle)}(\widehat{\gamma}_0(\theta)) &= \tilde{\ell}_\theta(\widehat{\gamma}_0(\theta)) \\ &= \widehat{\gamma}_0(\theta + \frac{\alpha}{2}), \end{aligned}$$

and

$$\begin{aligned} \tilde{\ell}_\theta \circ q_{\mathcal{C}(\langle 2\theta \rangle)}(\widehat{\gamma}_1(\theta)) &= \tilde{\ell}_\theta(\widehat{\gamma}_1(\theta)) \\ &= \widehat{\gamma}_1(\theta + \frac{\alpha}{2}), \end{aligned}$$

where the first equality follows since both curves are invariants by $q_{\mathcal{C}(\langle 2\theta \rangle)}$ with $\alpha = 0$, and the second one follows by the definition of $\tilde{\ell}_\theta$. □

Proposition 14. *The* 2*-curve* $\gamma = (z_0 \ z_1)$ *is an* indifferent *invariant multi-curve, with jumping integer* $\tau = 0$*, of the fibred quadratic polynomial*

$$F(\theta, z) = (\theta + \alpha, \tilde{\ell}_{\theta/2}(q_{\mathcal{C}(\langle \theta \rangle)}(z))). \tag{4.10}$$

Proof. Recall that $\widetilde{Q}$ *lives in the lifting world* of Q. Now, we want to 'project' these curves to the world of Q as a single 2-curve of a fibred quadratic polynomial. By using the same factor $\Pi(\theta, z)$ as in the case of $\alpha = 0$ (here $\tau = 0$), we have

$$\begin{aligned} \Pi \circ \widetilde{Q}(\theta, \zeta) &= \Pi \circ \widetilde{L}_\alpha \circ \widehat{Q}(\theta, \zeta) \\ &= \Pi \circ \widetilde{L}_\alpha(\theta + \frac{\alpha}{2}, q_{\mathcal{C}(\langle 2\theta \rangle)}(\zeta)) \\ &= \Pi(\theta + \frac{\alpha}{2}, \tilde{\ell}_\theta(q_{\mathcal{C}(\langle 2\theta \rangle)}(\zeta))) \\ &= (2\theta + \alpha, \tilde{\ell}_\theta(q_{\mathcal{C}(\langle 2\theta \rangle)}(\zeta))). \end{aligned}$$

Since the fibred quadratic polynomial $F : \mathbb{T}^1 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ is given by,

$$F(\theta, z) = (\theta + \alpha, \tilde{\ell}_{\frac{\theta}{2}} \circ q_{\mathcal{C}(\langle\theta\rangle)}(z)),$$

by checking compositions, we obtain

$$\begin{aligned} F \circ \Pi(\theta, z) &= F(2\theta, z) \\ &= (2\theta + \alpha, \tilde{\ell}_{\theta}(q_{\mathcal{C}(\langle 2\theta\rangle)}(z))), \end{aligned}$$

hence

$$\Pi \circ \widetilde{Q} = F \circ \Pi,$$

so F and $\widetilde{Q}$ are semi-conjugated. Since $\widehat{\gamma}_0$ and $\widehat{\gamma}_1$ are invariants for $\widetilde{Q}$, it follows that their projection forms a 2-curve for F.

In order to see the dynamical nature of the invariant curve, we have to calculate its multiplier. For this, a careful calculation will prove that

$$\tilde{\ell}_{\frac{\theta}{2}} \circ q_{\mathcal{C}(\theta)}(z) = \frac{\delta(\theta) + \delta_1(\theta) - \delta_2(\theta)}{\delta(\theta)} z^2 + \frac{\mathcal{C}(\theta)(\delta(\theta) + \delta_1(\theta) - \delta_2(\theta)) + \delta_2(\theta) z_1(\theta) - \delta_1(\theta) z_2(\theta)}{\delta(\theta)}$$

where

$$\delta(\theta) = z_1(\theta) - z_2(\theta) = 2\varepsilon e^{\pi i\theta},$$

$$\delta_1(\theta) = z_1(\theta + \alpha) - z_1(\theta) = \varepsilon e^{\pi i\theta}(e^{\pi i\alpha} - 1),$$

and

$$\delta_2(\theta) = z_2(\theta + \alpha) - z_2(\theta) = \varepsilon e^{\pi i\theta}(1 - e^{\pi i\alpha}).$$

A simple calculation gives that

$$A(\theta) = \frac{\delta(\theta) + \delta_1(\theta) - \delta_2(\theta)}{\delta(\theta)} = e^{\pi i\alpha}$$

is a constant coefficient with respect to the coordinate θ. And

$$\begin{aligned} C(\theta) &= \frac{\mathcal{C}(\theta)(\delta(\theta)+\delta_1(\theta)-\delta_2(\theta))+\delta_2(\theta)z_1(\theta)-\delta_1(\theta)z_2(\theta)}{\delta(\theta)} \\ &= e^{\pi i\alpha}\left(\frac{1}{4}-\varepsilon^2 e^{2\pi i\theta}\right)+\frac{1}{2}(1-e^{\pi i\alpha}). \end{aligned}$$

Applying Lemma 13 with these coefficients (note that $B(\theta)=0$) we have the corresponding normalized fibred quadratic polynomial

$$P(\theta,z)=\left(\theta+\alpha, z^2+e^{2\pi i\alpha}\left(\frac{1}{4}-\varepsilon^2 e^{2\pi i\theta}\right)+\frac{e^{\pi i\alpha}}{2}(1-e^{\pi i\alpha})\right). \tag{4.11}$$

With $\gamma=(\gamma_0\ \gamma_1)$ given by

$$\gamma_{0,1}(\theta)=e^{\pi i\alpha}z_{1,2}(\theta)=e^{\pi i\alpha}\left(\frac{1}{2}\pm\varepsilon e^{\pi i\theta}\right)$$

as its invariant 2-curve.

Note that the canonical parameter in Equation (4.11) is given by

$$\mathcal{C}_\varepsilon(\theta)=e^{2\pi i\alpha}\left(\frac{1}{4}-\varepsilon^2 e^{2\pi i\theta}\right)+\frac{e^{\pi i\alpha}}{2}(1-e^{\pi i\alpha}). \tag{4.12}$$

The following lemma is direct by small perturbations.

Lemma 15. *For suitable $\varepsilon>0$ and $\alpha>0$ sufficiently small, the curve $\mathcal{C}\varepsilon(\theta)$ defined in (4.12) of the above quadratic polynomial, is a simple closed curve whose winding number with respect to $c_0=1/4$ is 1.* □

Applying the definition of the *fibred multiplier for multi-curves*, we have to calculate the corresponding *unfolding* curve in the lifted *fpd*:

$$\tilde{P}(\theta,z)=\left(\theta+\frac{\alpha}{2}, z^2+e^{2\pi i\alpha}\left(\frac{1}{4}-\varepsilon^2 e^{2\pi i(2\theta)}\right)+\frac{e^{\pi i\alpha}}{2}(1-e^{\pi i\alpha})\right),$$

with invariant curve given by

$$\tilde{\gamma}(\theta)=e^{\pi i\alpha}\left(\frac{1}{2}+\varepsilon e^{2\pi i\theta}\right),$$

this way,

$$\begin{aligned}
\log(\kappa(\gamma)) &= \log(\kappa(\tilde{\gamma})) \\
&= \int_{\mathbb{T}^1} \log|2\tilde{\gamma}(\theta)|d\theta \\
&= \int_0^1 \log|e^{\pi i\alpha}(1+2\varepsilon e^{2\pi i\theta})|d\theta. \\
&= \int_0^1 \log|e^{\pi i\alpha}|d\theta + \int_0^1 \log|1+2\varepsilon e^{2\pi i\theta}|d\theta
\end{aligned}$$

Taking $f(z) = 1 \pm z$, it follows from the *Jensen's formula* applied over $\{|z| = 2\varepsilon\}$, that

$$\kappa(\gamma) = 1.$$

In other words, γ is an *indifferent (dynamically) invariant multi-curve.* $\square$

Fibred combinatory ($\tau = 1$)

In the above construction, by obtaining the invariance of the curve in the lifting, dynamically we stay over the same 'part' of the 2-curve. We recall that a n-curve may have defined a combinatorics "over the fibred".

For the case of the 2-curve, there are only two possible combinatorics.

- The dynamics stay in the same part of the curve ($\tau = 0$).
- The dynamics do "jumps" between the two parts ($\tau = 1$).

It is clear that $\tau = 0$ in the above construction. For $\tau = 1$, the Lagrange interpolation polynomial (affine) $\tilde{\ell}$, may be defined by the pairs

$$(\widehat{\gamma}_0(\theta), \widehat{\gamma}_1(\theta)) \qquad \text{and} \qquad (\widehat{\gamma}_1(\theta + \frac{\alpha}{2}), \widehat{\gamma}_0(\theta + \frac{\alpha}{2})).$$

However, the idea of taking $\alpha > 0$ sufficiently small is that the Lagrangian interpolation polynomial $\tilde{l}$ is very close to the identity so that the composed fibered polynomial $P \circ \tilde{l}$ is in fact a perturbation of the static dynamics.

There is another way to obtain a 2-invariant curve with "jumping integer" $\tau = 1$. Let's consider the parameterized curve, $\epsilon > 0$ small,

$$C(\theta) = -\frac{3}{4} - \epsilon^2 e^{2\pi i\theta}$$

That is, $C : \mathbb{T}^1 \to \mathbb{C}$ is a simple closed curve around the parameter, which is a parameter with parabolic multiplicity equal to 2.

Given the static quadratic polynomial

$$P(\theta, z) = (\theta, z^2 + C(\theta)),$$

we have that the sets (curves) of periodic points of period 2 are given by:

$$z_1(\theta) = -\frac{1}{2} + \epsilon e^{\pi i\theta} \qquad \text{and} \qquad z_2(\theta) = -\frac{1}{2} - \epsilon e^{\pi i\theta}$$

with

$$p_\theta(z_1(\theta)) = z_2(\theta) \qquad \text{and} \qquad p_\theta(z_2(\theta)) = z_1(\theta).$$

In other words, the dynamics (in each iteration) "jumps" between the two curves z_1 and z_2.

Similar to the previous case, the curve $\tilde{\gamma} : [0, 2] \to \mathbb{C}$ given by

$$\tilde{\gamma}(\theta) = -\frac{1}{2} + \epsilon e^{\pi i\theta}$$

induces the 2-curve $\gamma = (z_1, z_2)$. Furthermore $\tilde{\gamma}$, is an invariant curve for the static quadratic polynomial

$$(\theta, z) \mapsto (\theta, z^2 + C(\langle 2\theta \rangle)).$$

For sufficiently small $\alpha > 0$ we take the (linear) Lagrange Interpolation polynomial that sends $z_1(\theta)$ to $z_1(\theta + \frac{\alpha}{2})$ and $z_2(\theta)$ to $z_2(\theta + \frac{\alpha}{2})$, we have the following analogous result for $\tau = 1$.

Lemma 16. *For sufficiently small $\alpha > 0$ and $\tau = 1$ define the fibred quadratic polynomial.*

$$\begin{array}{rccl} \tilde{P}: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\theta + \frac{\alpha+\tau}{2}, z^2 + C(\langle 2\theta \rangle)), \end{array}$$

Then the curve (unfolding) $\tilde{\gamma}$ is invariant for $\tilde{P}$.

Proposition 15. *The 2-curve $\gamma = (z_0\ z_1)$ is an invariant multi-curve of* fibred quadratic polynomial

$$F(\theta, z) = (\theta + \alpha + \tau, \tilde{\ell}_{\theta/2}(q_{\mathcal{C}(\langle\theta\rangle)}(z))), \tag{4.13}$$

where $\tau = 1$ is the jumping integer for the fqp.

4.3.3 Attracting 2-curves

In non-fibred dynamics, indifferent (parabolic) periodic points are part of the *bifurcation locus* of the system. Applied to the fibred dynamics, we can expect this to be the case as well.

In this section, we consider a slight perturbation in the above construction to obtain *attracting invariant 2-curves*, and then the possibility to linearize the local dynamics around it.

Let $\epsilon > 0$, $x_0 > 0$ such that

$$0 < x_0 < \frac{1}{4} \quad \text{and} \quad \frac{\epsilon^2}{\frac{1}{4} - x_0} < 1 \tag{4.14}$$

and consider the parametric curve $C : \mathbb{T}^1 \to \mathbb{C}$ given by,

$$C(\theta) = \frac{1}{4} - \epsilon^2 x_0 - \epsilon^2 e^{2\pi i\theta} \tag{4.15}$$

Proposition 16. *Let $\alpha \in (0, 1)\backslash\mathbb{Q}$ small enough. For ϵ and x as in (4.14), there exists a fibred quadratic polynomial in canonical form*

$$(\theta, z) \mapsto (\theta + \alpha, z^2 + \mathcal{C}(\theta))$$

containing a 2-invariant attracting curve.

Proof. This demonstration will be done using the *small perturbation* procedure described in Section 4.3.2. We define our perturbation on the static model

$$P(\theta, z) = (\theta, z^2 + C(\theta)),$$

where $C(\theta)$ is given by (4.15). Following the construction in Proposition 14, we have that the curves

$$z_1(\theta) = \frac{1}{2} + \epsilon e^{\pi i \theta}\sqrt{1 + x_0 e^{-2\pi i \theta}}$$

and

$$z_2(\theta) = \frac{1}{2} - \epsilon e^{\pi i \theta}\sqrt{1 + x_0 e^{-2\pi i \theta}},$$

they form a set of fixed points and their concatenation $\gamma : [0, 2] \to \mathbb{C}$

$$\gamma(\theta) = 1/2 + \varepsilon e^{\pi i \theta}\sqrt{1 + x_0 e^{-2\pi i \theta}},$$

induces an invariant 2-curve for the static quadratic polynomial $P(\theta, z)$.

In order to determine the dynamical nature of this 2-curve, we have to calculate the multiplier of the (unfolding) invariant curve of the lifted system $\tilde{P}(\theta, z)$.

For this, note that, if $\tilde{z}(\theta)$ is the (unfolding) invariant curve for the lifted fibred polynomial, then $\tilde{z}(\theta)\big|_{[0,1/2]} = z_1(2\theta)$ and $\tilde{z}(\theta)\big|_{[1/2,1]} = z_2(2\theta)$, and hence

$$\begin{aligned}
\log(\kappa(\tilde{z}(\theta))) &= \int_{\mathbb{T}}^{1} \log|2\tilde{z}(\theta)|d\theta \\
&= \int_0^{1/2} \log|2\tilde{z}(\theta)|d\theta + \int_{1/2}^{1} \log|2\tilde{z}(\theta)|d\theta \\
&= \int_0^{1/2} \log|2z_1(2\theta)|d\theta + \int_{1/2}^{1} \log|2z_2(2\theta)|d\theta \\
&= \int_0^{1/2} \log|4z_1(2\theta)z_2(2\theta)|d\theta \\
&= \int_0^{1/2} \log|4C(2\theta)|d\theta \\
&= \frac{1}{2}\int_0^{1} \log|4C(\theta)|d\theta.
\end{aligned}$$

So, for the case $\alpha = 0$ it is enough to calculate the integral

$$2\log(\kappa(\gamma)) = \int_0^1 \log|4C(\theta)|d\theta.$$

From the form in 4.15 and the conditions on ε and x_0, we have

$$\begin{aligned}
2\log(\kappa(\gamma)) &= \int_0^1 \log|4C_T(\theta)|d\theta \\
&= \int_0^1 \log|4(1/4 - \varepsilon^2 e^{2\pi i\theta} - x_0)|d\theta \\
&= \int_0^1 \log|4(1/4 - x_0)(1 - \frac{\varepsilon^2}{1/4 - x_0}e^{2\pi i\theta})|d\theta \\
&= \int_0^1 \log|4(1/4 - x_0)|d\theta + \int_0^1 \log|1 - \frac{\varepsilon^2}{1/4 - x_0}e^{2\pi i\theta}|d\theta,
\end{aligned}$$

so, if

$$4(1/4 - x_0) < 1 \qquad \text{and} \qquad \frac{\varepsilon^2}{1/4 - x_0} < 1,$$

then

$$\kappa(\gamma) < 1.$$

Hence, the curve is attracting for the static fibred polynomial. But we are interested in the non-static fibred case $\alpha > 0$, that is, after applying Lagrange interpolation and normalizing to the canonical form.

A direct calculus shows that the fibred quadratic polynomial, in its canonical form obtained this way, is given by

$$Q(\theta, z) = \left(\theta + \alpha, z^2 + \frac{C(\theta)}{u(\theta+\alpha)}\right),$$

where $u(\theta)$ is the coefficient function described in Remark 7. Here, we have

$$A(\theta) = \frac{e^{\pi i \alpha}\sqrt{1 + x_0 e^{-2\pi i(\theta+\alpha)}}}{\sqrt{1 + x_0 e^{-2\pi i \theta}}},$$

with the invariant 2-curve given by

$$\gamma = \frac{1}{u(\theta)} \cdot (z_1 \; z_2),$$

and corresponding *unfolding*

$$\tilde{\gamma}(\theta) = \frac{\gamma(\theta)}{u(\theta)}.$$

So, the multiplier of the (unfolding) invariant 2-curve is

$$\begin{aligned}\kappa(\tilde{\gamma}) &= \exp\left(\int \log\left|\frac{2\gamma(\theta)}{u(\theta)}\right| d\theta\right) \\ &= \exp\left(\int \log|\gamma(\theta)| d\theta - \int \log\left|\frac{u(\theta)}{2}\right| d\theta\right).\end{aligned}$$

Hence, for $\tilde{\gamma}$ (and then γ) be attracting, is enough that

$$\int \log\left|\frac{u(\theta)}{2}\right| d\theta = 0,$$

which is equivalent, from the form of $u(\theta)$ to the condition

$$\int \log\left|\frac{A(\theta)}{2}\right| d\theta = 0 \tag{4.16}$$

But, $A(\theta) = \dfrac{e^{\pi i\alpha}\sqrt{1+x_0 e^{-2\pi i(\theta+\alpha)}}}{\sqrt{1+xe^{-2\pi i\theta}}}$, so (4.16) reduces to

$$\int \log|\sqrt{1+x_0e^{-2\pi i\theta}}|d\theta = \int \log|\sqrt{1+x_0e^{-2\pi i(\theta+\alpha)}}|d\theta = 0,$$

which follows by noticing that each integral above is the real part of the integral

$$\int_{|z|=r} f(z)dz,$$

where $f(z) = \sqrt{1+z}$ and $r = x_0$ in the former and $f(z) = \sqrt{1+e^{\pi i\alpha}z}$ and $r = x_0$ in the former. We conclude that

$$\kappa(\tilde{\gamma}) = \kappa(\gamma) < 0.$$

This way, the 2-curve $\gamma = \dfrac{1}{u(\theta)} \cdot (z_1\ z_2)$ is an *attracting invariant* multi-curve for the fibred quadratic polynomial

$$P(\theta, z) = \left(\theta+\alpha, z^2 + \mathcal{C}(\theta)\right),$$

with $\mathcal{C}(\theta) = C(\theta)/u(\theta+\alpha)$. $\square$

Remark 9. *Conditions (4.14), also imply that the parametric curve of the perturbed fibred quadratic polynomial $P(\theta, z)$ has topological degree one with respect to $c_0 = 1/4$.*

4.4 Searching for 3-curves

There are two well-known parametrizations for quadratic dynamics.

$$z \mapsto P_c(z) = z^2 + c,\ c \in \mathbb{C} \qquad \text{and} \qquad z \mapsto Q_\lambda(z) = \lambda z + z^2,\ \lambda \in \mathbb{C}.$$

The former generates the picture of the famous Mandelbrot set, defined as

$$\mathcal{M}_c = \{c \in \mathbb{C} : \{P_c^n(c)\}_{n\in\mathbb{N}} \text{ is bounded}\}$$

This parametrization is based on the behavior of the only critical value $z_c = c$. On the other hand, the further parametrization is based on the dynamical nature of the two fixed

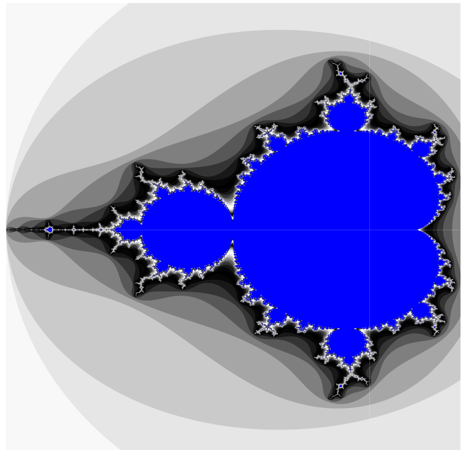


Figure 4.4: Mandelbrot Set

points of the system, In particular, $\lambda \in \mathbb{D}$ corresponds to a quadratic dynamics with an attracting fixed point (for $\lambda = 0$, $z = 0$ is a super-attracting fixed point). The parameter space can be defined as

$$\Lambda_\lambda = \{\lambda \in \mathbb{C} : \{Q_\lambda^n(-\lambda/2)\}_{n \in \mathbb{N}} \text{ is bounded}\}.$$

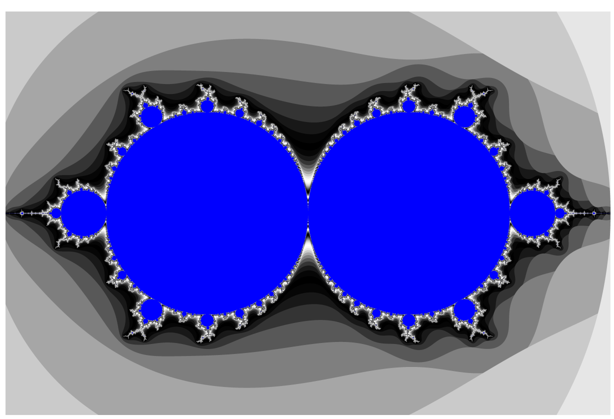


Figure 4.5: The Lambda space Λ

From the definition, it follows that

$$\mathcal{P}ar := \{\lambda = e^{2\pi i \theta} : \theta \in \mathbb{Q}\},$$

is the set of parameters with a parabolic fixed point. We have a natural correspondence (2 to 1) between the sets Λ_λ and $\mathcal{M}_c$, given by the conjugation by $T_{\lambda/2}(z) = z + \lambda/2$ ($z_\lambda = -\lambda/2$ corresponds to the critical point of Q_λ). The corresponding quadratic function is

$$P_\lambda(z) = z^2 + \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right).$$

Note that $\lambda = 1$ corresponds to the critical value $c = 1/4$ and $\lambda = -1$ to $c = -3/4$ as expected. It follows that the map

$$\lambda \mapsto \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right),$$

is a correspondence (2-1) between the Lambda space and the Mandelbrot set.

One of the parabolic fixed points with 3-petals, in the Lambda space, is given by the parameter $\lambda_0 = e^{\frac{2\pi i}{3}}$, then the quadratic polynomial

$$p_{\lambda_0}(z) = z^2 + \frac{\lambda_0}{2}\left(1 - \frac{\lambda_0}{2}\right)$$

has a parabolic fixed point with 3-petals. So, λ_0 is a center candidate for the fibred (family

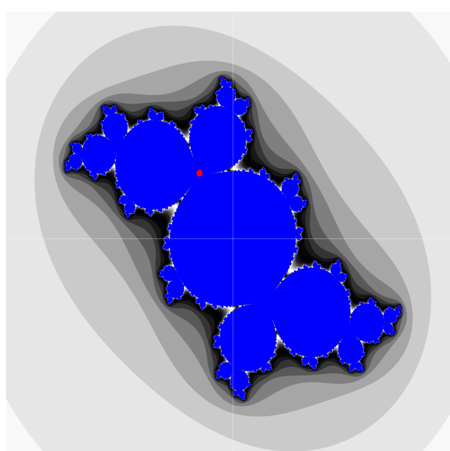


Figure 4.6: Filled-in Julia set for p_{λ_0}. The **red** point corresponds to the parabolic fixed point. It can be appreciate the 3-petals around it.

of) quadratic polynomial with $\alpha = 0$.

For $\alpha = 0$, consider the fibred quadratic polynomial

$$\begin{array}{rccc} P: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (z, p_{\lambda_0}(z) + \varepsilon^2 e^{2\pi i \theta}), \end{array}$$

with $\varepsilon > 0$ sufficiently small ($\varepsilon \sim 1/100$). Here are some images of the *filled Julia set,* corresponding to some of the θ's values.

Conjecture 1. *By "tracking" the corresponding 3-periodic points generated by the perturbation of the parabolic fixed point at* λ_0*, we have an invariant 3-curve.*

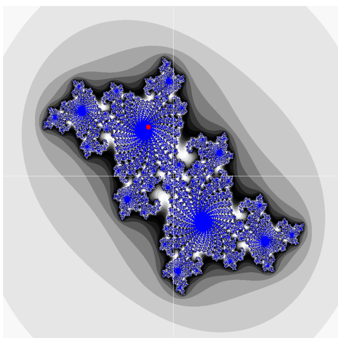
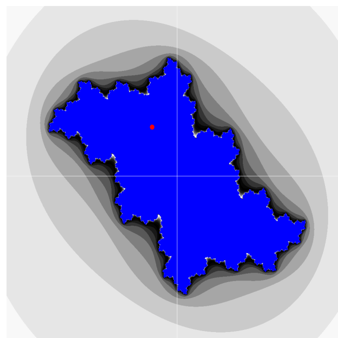
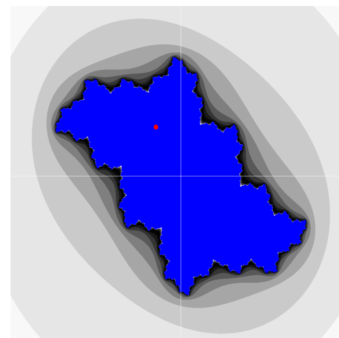
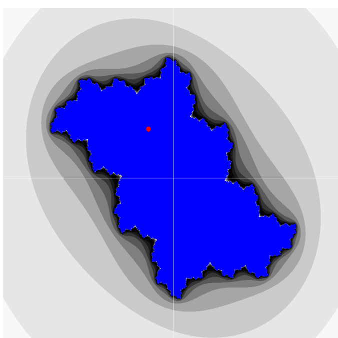
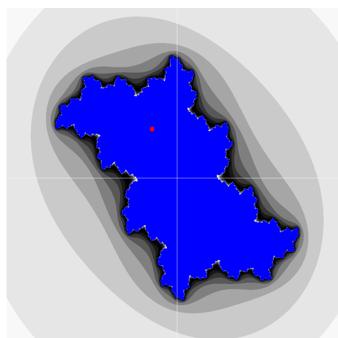
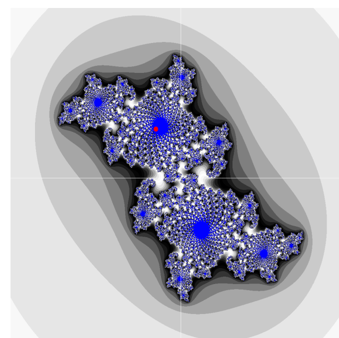
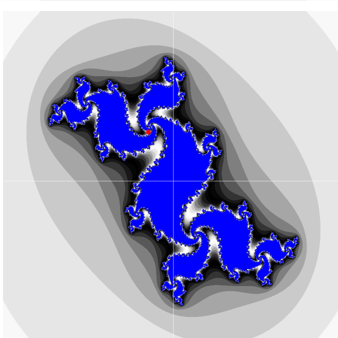
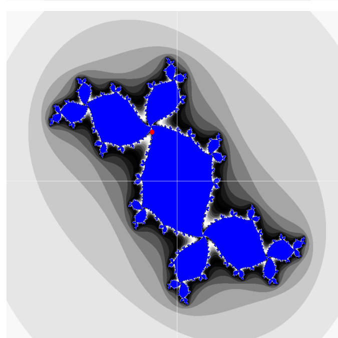
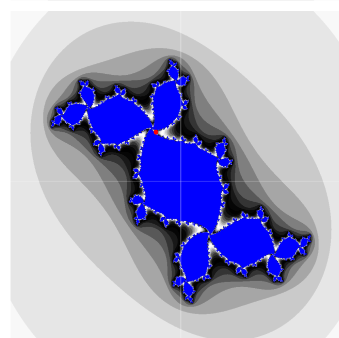
a	b	c
		
		
		

Table 4.1: The table shows the Filled-in Julia sets (in **blue** for the static polynomial $(\theta, z) \mapsto (\theta, z^2 + C(\theta))$, where $C(\theta)$ is a simple loop around the parabolic parameter λ_0. In all of the images the **red** point corresponds to the parabolic fixed point of multiplicity 28 of $z \mapsto z^2 + c_0$. From left to right and from upper to lower, we have the fiber for the values of $\theta = 0.0$ to $\theta = 0.8$.

4.4.1 The rational quadratic 3-curve

Conjecture 1 in the previous section has two basic obstacles: there is no closed formula to find the 3-period points of the quadratic polynomial, and if we could know the 3-cycle, the Lagrange Interpolation polynomial is no longer *linear*, now is quadratic, so the polynomial obtained with it is now of degree 4.

Nevertheless, it is possible to maintain the degree when we construct the 3-curve, but there is a price to pay. The fibred dynamics is now *rational*. It is well known that the Möbius transformations are 3-*transitive*: that is, given two set of points (z_1, z_2, z_3) and $(w_1, w_2.w_3)$, there exists a Möbius transformation mapping z_i to w_i.

Let $\gamma : \mathbb{T}^1 \to \mathbb{C}^*$ be a simple (and small) loop around the parabolic parameter λ_0 mentioned before. Consider the "fibred" quadratic polynomial given by

$$\begin{array}{rccc} P: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (\theta, z^2 + \gamma(\theta)). \end{array}$$

The polynomial $p_\theta(z) = z^2 + \gamma(\theta)$ may be considered as a loop-perturbation of the polynomial $z \mapsto z^2 + \lambda_0$ (*parabolic implosion*). In this sense, for every $\theta \in \mathbb{T}^1$, p_θ has a 3-cycle $(\gamma_0(\theta), \gamma_1(\theta), \gamma_2(\theta))$. By the continuity of γ, this 3-cycle moves continuously on $\mathbb{T}^1 \times \mathbb{C}$.

Now, for $\alpha > 0$ sufficiently small, consider the two 3-tuples $(\gamma_0(\theta), \gamma_1(\theta), \gamma_2(\theta))$ and $(\gamma_0(\theta + \alpha), \gamma_1(\theta + \alpha), \gamma_2(\theta + \alpha))$, for each $\theta \in \mathbb{T}^1$. Now, for each $\theta \in \mathbb{T}^1$, let $M : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the Möbius transformation that maps $(\gamma_0(\theta), \gamma_1(\theta), \gamma_2(\theta))$ into $(\gamma_0(\theta+\alpha), \gamma_1(\theta+\alpha), \gamma_2(\theta+\alpha))$ (here, $\widehat{\mathbb{C}}$ denotes the Riemann sphere). The following result follows from a direct calculation.

Proposition 17. *Given the fibred rational quadratic dynamics,*

$$\begin{array}{rccc} Q: & \mathbb{T}^1 \times \mathbb{C} & \to & \mathbb{T}^1 \times \mathbb{C} \\ & (\theta, z) & \mapsto & (z + \alpha, M(z^2 + \gamma(\theta))), \end{array}$$

for a suitable loop γ with index number 1 with respect to the parabolic parameter λ_0, then the 3-cycle $(\gamma_0(\theta), \gamma_1(\theta), \gamma_2(\theta))$ is an invariant 3-curve for Q.

Appendix A

Lagrange interpolation

The most (simple) useful tool to construct invariant multi-curves will be the **Lagrange interpolation polynomial**.

Let $w_1, w_2, ..., w_n$ be distinct points in $\mathbb{C}$ and $y_1, y_2, ..., y_n$ be distinct points in $\mathbb{C}$. The idea is to construct a polynomial $s(z)$ such that $s(w_i) = y_i$ for each $i = 1, ..., n$. Furthermore, we want that $s(z) - z$ tends to zero uniformly over compact sets when $w_i - y_i \to 0$. Take $\delta_i = y_i - w_i$ and $\tilde{s}(z)$ the Lagrange interpolation polynomial taking values δ_i in $z = w_i$, that is

$$\tilde{s}(z) = \sum_{i=0}^{n} \delta_i \left(\prod_{j \neq i} \frac{z - w_j}{w_i - w_j} \right).$$

Then, the polynomial $s(z) = z + \tilde{s}(z)$ verifies our requirements. We call $s(z)$ the **interpolation polynomial closed to identity** related with points $\{w_i, y_i\}_{1 \leq i \leq n}$. From the definition is clear that $s(z)$ is a degree $n-1$ polynomial.

Remark 10. *If $n = 2$, then $s(z)$ is an affine mapping.*

Appendix B

Julia sets through the lens of fibered dynamics: A computational exploration

We would like to include some graphical examples of the dynamical planes (Julia and filled-in Julia sets) in the context of *fibred dynamics*.

B.1 Non-hyperbolic model

The Ponce's family,

$$P^{\varepsilon}(\theta, z) = (\theta + \alpha, z(1 + a(\theta)(z - 1))),$$

where $a(\theta) = \cos(\theta) + i(1 - \varepsilon)\sin(\theta)$, marks a significant difference between the classic quadratic dynamics and the fibred quadratic dynamics by exhibiting fibred quadratic polynomials with two distinct attracting invariant curves, a phenomenon that cannot happen in the classic quadratic case.

Moreover, it was this family which inspired the mechanism to obtain the non-hyperbolicity in the fibred quadratic context.

The Table B.1 shows a sequel of images of fibres for one member of the family P^{ε} for

a	b	c

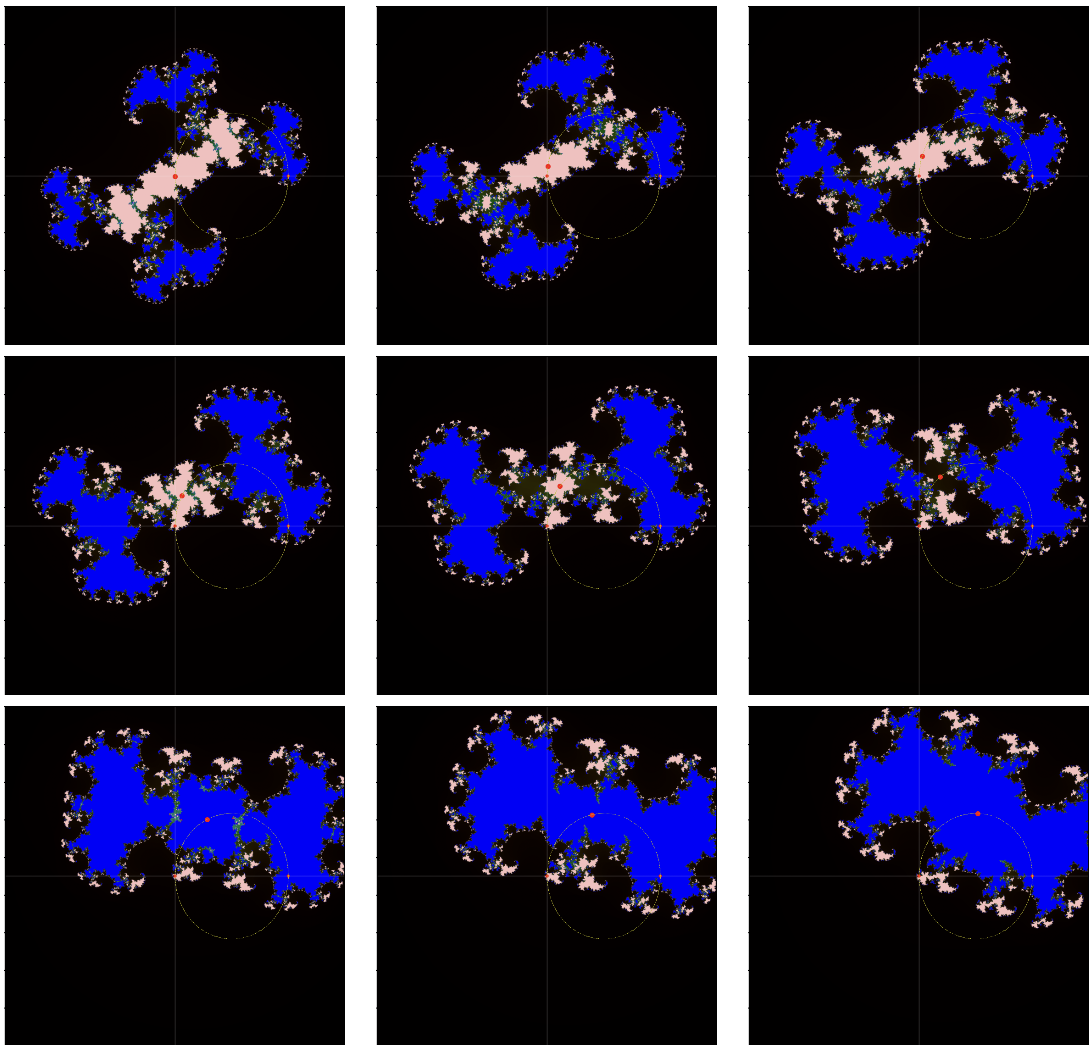


Table B.1: The table shows some filled-in Julia sets (blue and pink) of a member of the family P^{ε}. In **blue** we have the basin of attraction of the attracting invariant curve $\{z \equiv 1\}$ (red point), while the **pink** zone correspond to the basin of attraction of the attracting invariant curve $\{z \equiv 0\}$(red point). The curve in **yellow** denotes the critical set, and the red point on it is the critical point of the corresponding fiber. From left to right and upper to lower, we have the corresponding fibers to $t = 0$ to $t = 0.4$.

different values of θ. Independent of $0 < \varepsilon < 1$, the invariant curves $\{z \equiv 0\}_{\theta \in \mathbb{T}}$ and $\{z \equiv 0\}_{\theta \in \mathbb{T}}$ are attracting in the sense defined in Section 2.4.

In the table, we have on each fiber: the invariant curves (points in red) $z = 0$ and $z = 1$, the critical set (the curve projected to the fiber in yellow)) and the corresponding critical point on the fiber. In the sequel of images, we can appreciate how the critical point 'pass' from the basin of attraction of the invariant curve $\{z \equiv 0\}$ (the pink zone) to the basin of attraction of the invariant curve $\{z \equiv 1\}$.

From the continuity of the critical curve, we can say that both basin of attraction are *connected via the critical set*. If the given polynomial P^{ε} were hyperbolic, the boundary of the Filled-in Julia set (the union of the pink and blue zones) coincides with the Julia set, and then the critical curve should intersects the Julia set, which contradicts the Sester's condition form hyperbolicity.

The reasoning does not depends on the degree of the polynomial, for the contradiction holds we only need that there exists at least two basins of attraction and that a path of the critical curve *connects* them.

B.2 Invariant multi-curves

The idea of this section is to show (numerically) how the different loops around parabolic fixed points in the Mandelbrot set, degenerate into multi-curves via *small perturbations,* as described in Section 4.3.2.

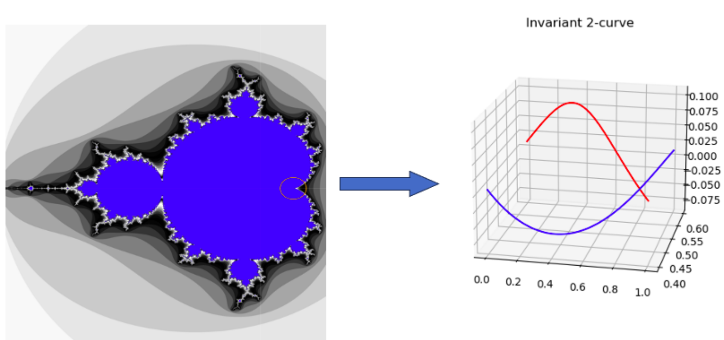


Figure B.1: The 2-curve from *small perturbation*. Left: we appreciate a small loop around the point $c_0 = 1/4$ in the Mandelbrot set. Right: the 2-curve induces by $\gamma(\theta) = 1/2 + \varepsilon e^{\pi i \theta}$, the set of fixed points.

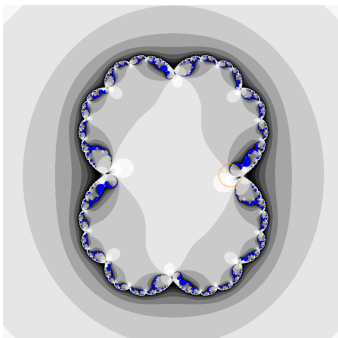
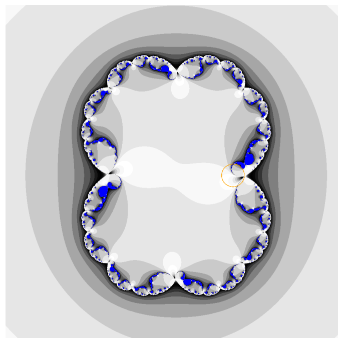
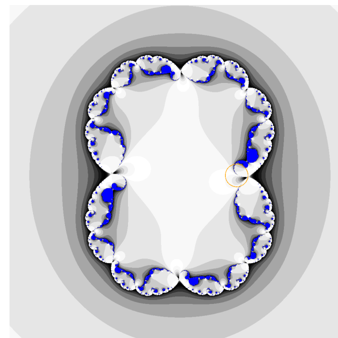
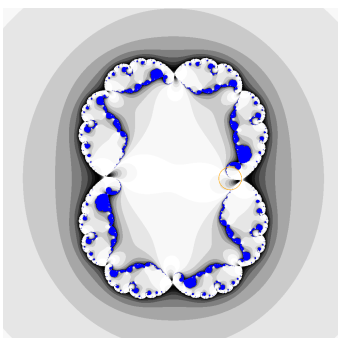
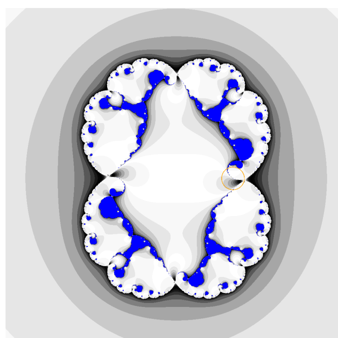
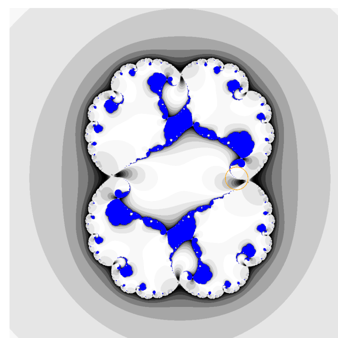
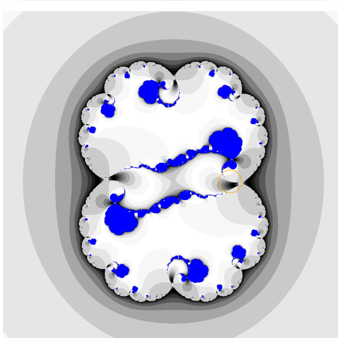
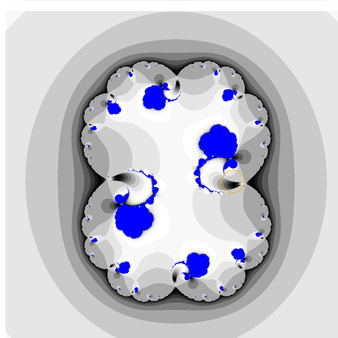
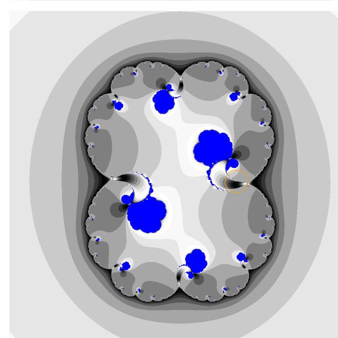
a	b	c
		
		
		

Table B.2: The table shows some filled-in Julia sets (blue) for the unperturbed fibred system $(\theta, z) \mapsto (\theta + \alpha, z^2 + C(\theta))$. From left to right and upper to lower, we have the corresponding fibres from $t = 0$ to $t = 0.08$.

For the first table of images, we consider a simple loop around the parabolic parameter $c_0 = 1/4$, as shown in Figure B.1. As was prove in Section 4.3.2, this curve generates an invariant 2-curve for the static quadratic polynomial $(\theta, z) \mapsto (\theta, z^2 + C(\theta))$.

Table B.2 shows some Filled-in Julia sets for distinct values of fiber θ close to 0 for the fibred polynomial $(\theta, z) \mapsto (\theta + \alpha, z^2 + C(\theta))$. In this case, there is no invariant 2-curve.

In Table B.3, the Filled-in Julia sets (in blue again) are shown for the **attracting invariant 2-curve** obtained after applied Lagrange interpolation and normalizing to the form $(\theta, z) \mapsto (z + \alpha, z\ 2 + \mathcal{C}(\theta))$ (recall $\alpha > 0$ is taken small enough). Since this is a very small perturbation of the parabolic parameter $c_0 = 1/4$, it is notorious the similarity of the Filled-in Julia sets to the cauliflower describe in the classic dynamics.

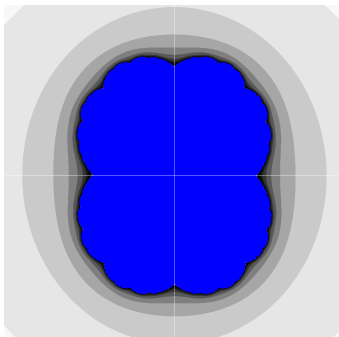
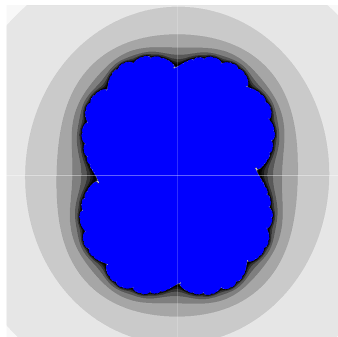
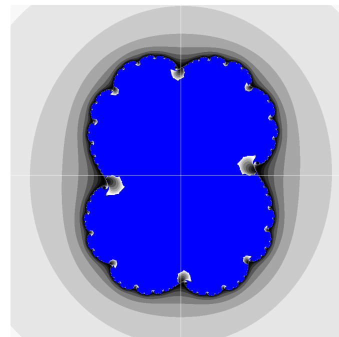
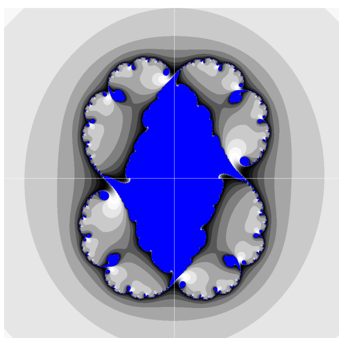
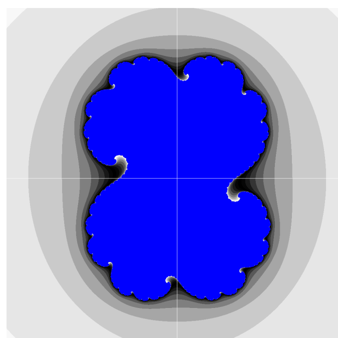
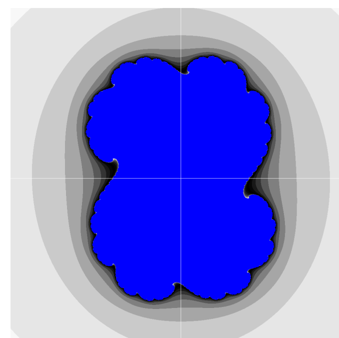
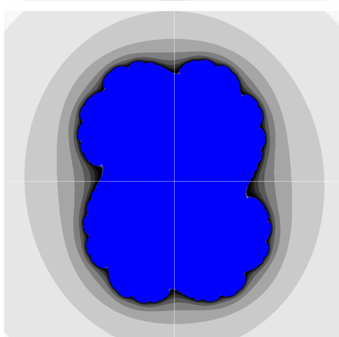
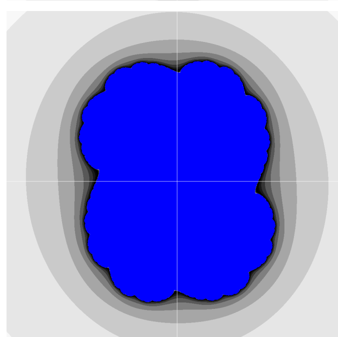
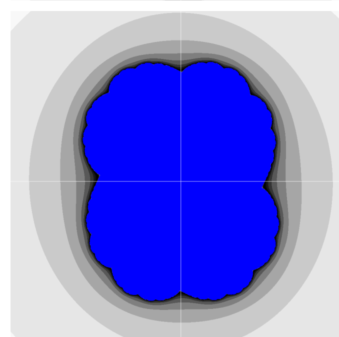
a	b	c
		
		
		

Table B.3: The table shows some filled-in Julia sets (blue) for the normalized fibred system $(\theta, z) \mapsto (\theta + \alpha, z^2 + \mathcal{C}(\theta))$ obtained via small perturbation with an attracting invariant 2-curve. From left to right and upper to lower, we have the corresponding fibres from $t = 0$ to $t = 0.8$.

Jumping integer $\tau = 1$

Finally, we consider a simple loop around the parabolic parameter of multiplicity 2 $c_0 = -3/4$. Given the polynomial $z \mapsto z^2 + c$, we know that the 2-period points are given by

$$z = -\frac{1}{2} \pm \sqrt{-\frac{3}{4} - c}.$$

Hence, the loop generates a set (curve) of 2-period points

$$z_{1,2}(\theta) = -\frac{1}{2} \pm \varepsilon + e^{\pi i \theta},$$

which in turn, after applying Lagrange interpolation to them in the fiber θ to the fiber $\theta + \alpha$, (α small enough), into an **invariant 2-curve**, see Figure B.2. In Table B.4, we

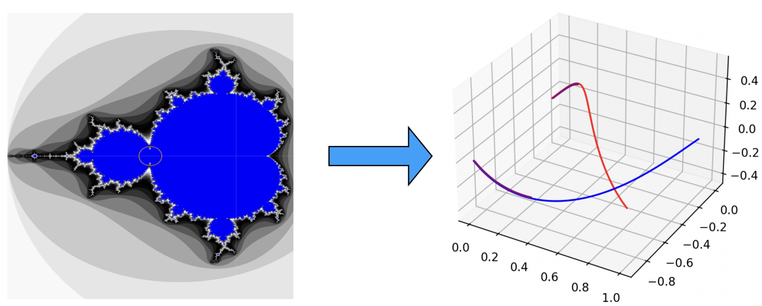


Figure B.2: The 2-curve from *small perturbation*. Left: we appreciate a small loop around the point $c_0 = -3/4$ in the Mandelbrot set. Right: the 2-curve induced by $\gamma(\theta) = -1/2 + \varepsilon e^{\pi i \theta}$, the set of 2-period points.

visualize the fibers of Filled-in Julia sets (in blue) for different values of θ's. Since the points $z_{1,2}$ for a cycle, with the Lagrange interpolation we generate a *jump* from one to other curve, meaning that the **jumping integer** is equal $\tau = 1$. Analogous to the loop around $c_0 = 1/4$, we have similarity with the Filled-in Julia set described by the parabolic of multiplicity 2 fixed point.

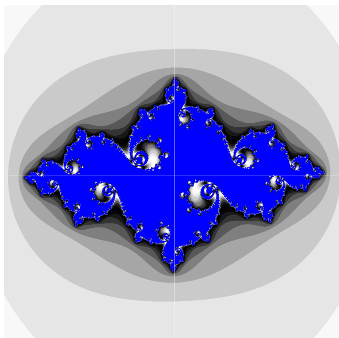
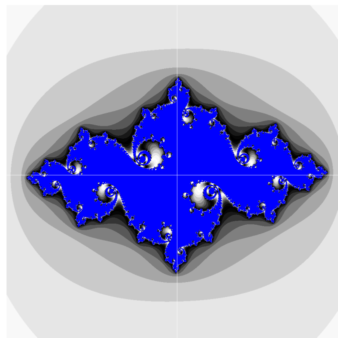
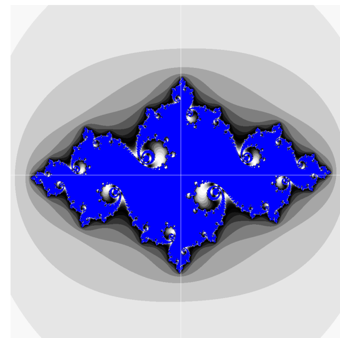
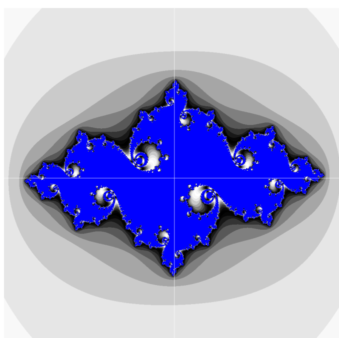
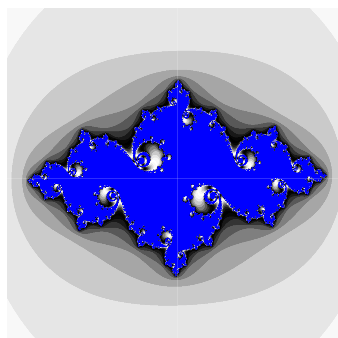
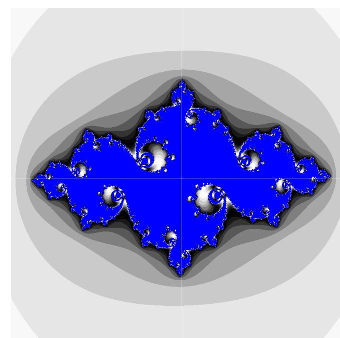
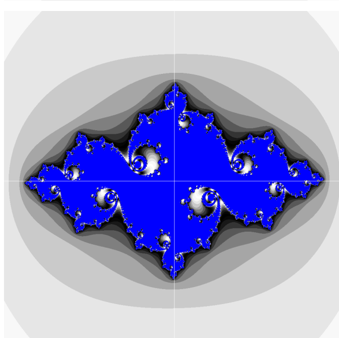
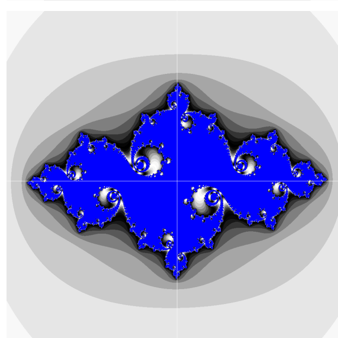
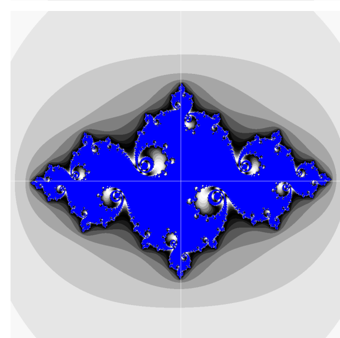
a	b	c
		
		
		

Table B.4: The table shows some filled-in Julia sets (**blue**) for the normalized fibred system $(\theta, z) \mapsto (\theta + \alpha, z^2 + \mathcal{C}(\theta))$ obtained via small perturbation with an invariant 2-curve and *jumping integer* equal 1. From left to right and upper to lower, we have the corresponding fibres from $t = 0$ to $t = 0.8$.

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