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Characterization of Kollár Surfaces

por

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Introduction

Throughout this work the base field will be $\mathbb{C}$. In 2008 Janos Kollár introduced in [Ko08] the following family of hypersurfaces. Let $n \geq 3$ be an integer, and let $a_1, \dots, a_n$ be positive integers such that there is no $(a_i, a_{i+2}, \dots, a_{i+n-2}) = (1, \dots, 1)$ when n is even. The indices are and will be taken modulo n. For every $1 \leq i \leq n$, we define the positive integers

$$W_i := \sum_{j=1}^{n} (-1)^{j-1} \prod_{l=i+j}^{i+n-1} a_l \quad \text{and} \quad D := \prod_{l=1}^{n} a_l + (-1)^{n-1}.$$

For example, for $n = 4$ we have

$$W_i = a_{i+1}a_{i+2}a_{i+3} - a_{i+2}a_{i+3} + a_{i+3} - 1 \quad \text{and} \quad D = a_1a_2a_3a_4 - 1.$$

We also define

$$w^* := gcd(W_1, \dots, W_n).$$

Then $w^* = gcd(W_i, W_{i+1}) = gcd(W_i, D)$ since $a_iW_i + W_{i+1} = D$ for all i. More details on these numbers will be given in Chapter 2.

Set

$$w_i := \frac{W_i}{w^*} \quad \text{and} \quad d := \frac{D}{w^*}.$$

Definition. The *Kollár hypersurface* [Ko08] of type $(a_1, \dots, a_n)$ is

$$X(a_1, \dots, a_n) := (x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_n^{a_n}x_1 = 0) \subset \mathbb{P}(w_1, \dots, w_n).$$

We can define the map $\psi \colon X(a_1, \dots, a_n) \dashrightarrow (y_1 + \cdots + y_n = 0) \subset \mathbb{P}^{n-1}$ given by the linear system $|x_1^{a_1}x_2, x_2^{a_2}x_3, \dots, x_n^{a_n}x_1|$. Let $0 < \mu_i < w^*$ be such that $\mu_i \equiv (-1)^{i+1} \prod_{l=i+1}^{i+n-1} a_l \mod w^*$. Consider the normal projective variety Y' defined as the w^*-th root cover of $(y_1 + \cdots + y_n = 0) \subset \mathbb{P}^{n-1}$ totally branched along $(y_1^{\mu_1} \cdots y_n^{\mu_n} = 0)$, as defined in Section 1.6. Then by studying ψ we can prove that $X(a_1, \dots, a_n)$ and Y' are birational (Corollary 2.6).

The main focus of this thesis is $n = 4$. In this case we can describe precisely the birational map between X and Y'. We dedicate Section 2.1 to give a geometrical description of the map, summarized as follows.

Theorem. *There is a configuration Γ of 6 rational curves in $X(a_1, a_2, a_3, a_4)$ such that if $\hat{X} \to X$ is a log resolution of (X, Γ), then $\hat{X} \to X \overset{\psi}{\dashrightarrow} \mathbb{P}^2$ is a morphism which factors through $Y' \to \mathbb{P}^2$ via a birational morphism $\hat{X} \to Y'$.*

On the other hand, in Section 2.3 we prove that for every m-th root cover of $\mathbb{P}^2$ totally branched along four lines in general position, there are infinitely many Kollár surfaces with $w^* = m$ birational to it. Therefore we can obtain birational-invariant information from Kollár surfaces via the study of m-th root covers of $\mathbb{P}^2$, and vice versa.

Kollár surfaces are related to a conjecture posed by Kollár in the same article, regarding the number of certain type of singularities on $\mathbb{Q}$-homology projective planes.

Definition. A normal projective surface is called a *$\mathbb{Q}$-homology projective plane* ($\mathbb{Q}$HPP) if it has the same Betti numbers as $\mathbb{P}^2$.

Conjecture ([Ko08], Conjecture 30)**.** Let S be a $\mathbb{Q}$HPP with quotient singularities. If $S^0 := S \setminus S_{sing}$ is simply connected, then S has at most 3 singular points.

The purpose of Kollár surfaces was to give examples of $\mathbb{Q}$HPP with ample canonical class. This occurs when $w^* = 1$, after contracting the rational curves $(x_1 = x_3 = 0)$ and $(x_2 = x_4 = 0)$ in $X(a_1, a_2, a_3, a_4)$ when possible. This contraction gives a $\mathbb{Q}$HPP with two cyclic quotient singularities. Even more, when $a_i \geq 4$ for all i, then the canonical class is ample.

Hwang and Keum in a series of articles have proved the conjecture in almost all the cases, narrowing it to prove it when S is rational, K_S is ample, and it has at worst cyclic singularities. They also proved that the surface can have at most 4 singularities and in [HK12] they construct examples of rational $\mathbb{Q}$HPP with ample canonical class and with one, two and three cyclic singularities. In particular, some of their examples with two singularities have the same singularities as the Kollár surfaces with $w^* = 1$. In Section 2.2 we prove the following result.

Theorem. *Kollár surfaces with $w^* = 1$ are Hwang-Keum surfaces.*

In Section 2.3 we give formulas for invariants of Kollár surfaces via the invariants of Y' when $w^* > 1$. We pay special attention to the geometric genus, which depends on classical Dedekind sums, defined in Section 1.7. In Chapter 3 we proceed to classify Kollár surfaces in terms of their geometric genus.

First we prove that for every nonnegative integer m there is a Kollár surface with $p_g = m$, and that for a given positive integer m there is a positive integer N such that if $w^* > N$ and $p_g > 0$, then $p_g > m$. The rest of the sections of Chapter 3 are devoted to prove the following.

Theorem. *For $w^* > 1$, we have that*

(a) $p_g = 0$ if and only if the Kollár surface is rational. This happens when $a_i \equiv 1$ or $a_i a_{i+1} \equiv -1$ modulo w^* for some i.

(b) $p_g = 1$ if and only if the Kollár surface is birational to a $K3$ surface. We classify this situation in 8 cases.

(c) There are families of Kollár surfaces with Kodaira dimension 1 and 2. Even more, for $w^* >> 0$, the smooth minimal model S of a generic Kollár surface is of general type with $K_S^2/e(S) \to 1$, where K_S is the canonical class, and $e(S)$ is the topological Euler characteristic.

Even more, we give explicit families of Kollár surfaces with Kodaira dimension 1 elliptic fibrations, and Kodaira dimension 2 surfaces of general type, both for w^* arbitrarily large.

Chapter 1

Preliminaries

Throughout this work, we will assume that the reader is familiar with the contents of [Hart77]. If needed, some results of it will be mentioned explicitely. In this chapter we will list the definitions and results that we will use. Through Section 1.1 to 1.4 we will introduce Weighted Projective Spaces and properties that will be useful when studying Kollár surfaces. Section 1.5 describes cyclic quotient singularities, their minimal resolution and their connection with Hirzebruch-Jung continued fractions. In Section 1.6 we define n-th root covers of a variety, with special interest in n-th root covers of surfaces. Finally in Section 1.7 we study Dedekind sums and show some results that will be essential for Chapter 3.

1.1 Weighted projective spaces

Weighted projective spaces are a generalization of the usual projective space. They are singular varieties, but they are useful in the sense that we can study certain singular subvarieties as they were nonsingular varieties. We just go through this theory, to then study Kollár surfaces as hypersurfaces of certain 3-dimensional weighted projective spaces. Even though we will work over $\mathbb{C}$, most of the results still hold for an arbitrary field $k = \overline{k}$, having certain restrictions when $\mathrm{char}(k) = p > 0$. The following results can be found in [Dolg82] and [Ian00].

Definition 1.1. Let $Q = \{q_0, \ldots, q_n\}$ be positive integers, and define $S(q_0, \ldots, q_n) = S(Q)$ as the graded polynomial ring $\mathbb{C}[X_0, \ldots, X_n]$, graded by $\deg x_i = q_i$. The *weighted projective space*

$\mathbb{P}(q_0, \ldots, q_n)$ is defined by

$$\mathbb{P}(q_0, \ldots, q_n) = \mathbb{P}(Q) := \operatorname{Proj} S(Q).$$

Geometrically we can see this space as follows: let $\mathbb{A}^{n+1}$ the affine space and $X_0, \ldots, X_n$ its coordinates. Define the action of $\mathbb{C}^*$ as

$$\lambda \cdot (X_0, \ldots, X_n) = (\lambda^{q_0} X_0, \ldots, \lambda^{q_n} X_n).$$

Then $\mathbb{P}(Q) = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{C}^*$ (see [Dolg82, 1.2.1]).

Let $\mathbb{Z}_Q = \mathbb{Z}/q_0\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_n\mathbb{Z}$, and let $\zeta \cdot (Z_0 \colon \ldots \colon Z_n) = (\zeta_0 Z_0 \colon \ldots \colon \zeta_n Z_n)$ be the action of the group on $\mathbb{P}^n$, where $\zeta = (\zeta_0, \ldots, \zeta_n) \in \mathbb{Z}_Q$, and ζ_i is a primitive i-th root of 1. The ring of invariants $(\mathbb{C}[Z_0, \ldots, Z_n])^{\mathbb{Z}_Q} = \mathbb{C}[Z_0^{q_0}, \ldots, Z_n^{q_n}]$ is isomorphic to $S(Q)$ as graded rings by $X_i = Z_i^{q_i}$.

Definition 1.2. Let $\mathbb{Z}_Q$ act on $\mathbb{P}^n$ as mentioned before. Then the *weighted projective space* $\mathbb{P}(Q)$ is the quotient space $\mathbb{P}^n/\mathbb{Z}_Q$.

Corollary 1.3. *The intersection number of n hypersurfaces in $\mathbb{P}(Q)$ of degree $d_1, \ldots, d_n$ respectively, is $\prod_{i=1}^n d_i / \prod_{i=0}^n q_i$.*

Proof. Let $p \colon \mathbb{P}^n \to \mathbb{P}(Q)$ the quotient map of Definition 1.2, and let H_i be the hypersurface of degree d_i. As p is a finite and surjective morphism, it is flat. Then $p^*(H_1 \cdots H_n) = (p^*H_1) \cdots (p^*H_n)$, and we obtain

$$(\deg p)(H_1 \cdots H_n) = p^*(H_1 \cdots H_n) = (p^*H_1) \cdots (p^*H_n).$$

As p^*H_i is a degree d_i hypersurface of $\mathbb{P}^n$, then $(p^*H_1) \cdots (p^*H_n) = \prod_{i=1}^n d_i$. On the other hand, $\deg p$ is the order of the group $\mathbb{Z}_Q$, which is $\prod_{i=0}^n q_i$ (cf. [Ful98, Example 8.3.12]). $\square$

Therefore we have three equivalent definitions for a weighted projective space: as the Proj of a graded polynomial ring, as the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$, and as the quotient of the usual projective space $\mathbb{P}^n$. Each of them will be useful to study these spaces and the behaviour of subvarieties of them.

The following two properties allows us to choose the weights $q_0, \ldots, q_n$ in a convenient way.

Proposition 1.4. $\mathbb{P}(q_0, \ldots, q_n) \simeq \mathbb{P}(dq_0, \ldots, dq_n)$.

Proof. ([EGA] Proposition 2.4.7(i)) Let S be our graded algebra $\mathbb{C}[x_0, \ldots, x_n]$ with $\deg(x_i) = q_i$, and define $S^{(d)} = \bigoplus_{n=0}^{\infty} S_{nd}$. We will show that the map $\varphi : \operatorname{Proj} S \to \operatorname{Proj} S^{(d)}$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a set bijection. Let $\mathfrak{p}' \in \operatorname{Proj} S^{(d)}$ be a prime ideal and let $\mathfrak{p}_{nd} = \mathfrak{p}' \cap S_{nd}$. For each $n > 0$ such that $d \nmid n$, we define $\mathfrak{p}_n$ as the set of $x \in S_n$ such that $x^d \in \mathfrak{p}_{nd}$. This set is a subgroup of S_n because $\mathfrak{p}'$ is a prime ideal, so we can find an unique prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap S^{(d)} = \mathfrak{p}'$. We have that $V_+(f) = V_+(f^d)$, given by $V_+(fg) = V_+(f) \cup V_+(g)$, then the bijection defined above gives an homeomorphism between $\operatorname{Proj}(S)$ and $\operatorname{Proj}(S^{(d)})$. Finally, there is a canonical correspondence between $S_{(f)}$ and $S_{(f^d)}$ (see [EGA], Lemma 2.2.2). Hence we have an isomorphism of sheaves, therefore we have an isomorphism of schemes between

$$\mathbb{P}(q_0, \ldots, q_n) = \operatorname{Proj} S \simeq \operatorname{Proj} S^{(d)} = \mathbb{P}(dq_0, \ldots, dq_n).$$

□

Proposition 1.5. *Let* $q_0, \ldots, q_n$ *be positive integers, with* $\gcd(q_0, \ldots, q_n) = 1$ *and* $\gcd(q_1, \ldots, q_n) = d$. *Then*

$$\mathbb{P}(q_0, q_1, \ldots, q_n) \simeq \mathbb{P}(q_0, q_1/d, \ldots, q_n/d).$$

Proof. Let $S' = \bigoplus_{n=0}^{\infty} S_{nd}$. From Proposition 1.4 we have that

$$\operatorname{Proj} S(q_0, \ldots, q_n) \simeq \operatorname{Proj} S'.$$

Suppose that $x_0^{a_0} \cdots x_n^{a_n}$ is a monomial of degree md, where $m \in \mathbb{Z}_{\geq 0}$. Then

$$a_0 q_0 + \cdots + a_n q_n = md$$

so $d \mid a_0 q_0$. As $d \nmid q_0$ because $\gcd(q_0, \ldots, q_n) = 1$, then $d \mid a_0$. Hence x_0 only appears in S' as x_0^d, so $S' = \mathbb{C}[x_0^d, \ldots, x_n] \simeq S(dq_0, q_1, \ldots, q_n)$. Then, using again Proposition 1.4, we obtain

$$\operatorname{Proj} S' \simeq \operatorname{Proj} S(dq_0, q_1, \ldots, q_n) \simeq \operatorname{Proj} S(q_0, q_1/d, \ldots, q_n/d)$$

□

Corollary 1.6. *Given* $\mathbb{P}(q_0, \ldots, q_n)$ *there exists* $\mathbb{P}(q_0', \ldots, q_n')$ *such that*

$$\mathbb{P}(q_0, \ldots, q_n) \simeq \mathbb{P}(q_0', \ldots, q_n')$$

with

$$\gcd(q'_0, \ldots, \widehat{q'_i}, \ldots, q'_n) = 1, \qquad \textit{for all } i$$

where $(q'_0, \ldots, \widehat{q'_i}, \ldots, q'_n)$ *is the list of weights with the element* q'_i *ommited.*

Corollary 1.7. *For every positive integers* a, b, $\mathbb{P}(a, b) \simeq \mathbb{P}^1$.

Proof. Using Proposition 1.5, $\mathbb{P}(a, b) \simeq \mathbb{P}(1, b) \simeq \mathbb{P}(1, 1) = \mathbb{P}^1$. □

Definition 1.8. The space $\mathbb{P}(q_0, \ldots, q_n)$ is *well formed* if it satisfies the properties of Corollary 1.6.

We would like to know the behavior of $\mathcal{O}_{\mathbb{P}}(n)$ under this isomorphism. To do so we have the following proposition .

Proposition 1.9 ([Del75],Prop. 1.3)**.** *Let*

$$\begin{aligned} d_i &= \gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) \\ c_i &= lcm\ (d_0, \ldots, d_{i-1}, d_{i+1} m \ldots, d_n) \\ c &= lcm\ (d_0, \ldots, d_n) \end{aligned}$$

Then $\mathbb{P} = \mathbb{P}(q_0 \ldots, q_n) \simeq \mathbb{P}(q_0/c_0, \ldots, q_n/c_n) = \mathbb{P}'$, *and* $\mathbb{P}'$ *is a well formed weighted projective space.*

Proof. The proof follows from Proposition 1.4 and Proposition 1.5. □

Proposition 1.10 ([Dolg82],Remarks 1.3.2)**.** *The isomorphism of Proposition 1.9 induces an isomorphism of sheaves*

$$\mathcal{O}_{\mathbb{P}}(n) \simeq \mathcal{O}_{\mathbb{P}'}\left(\left(n - \sum_{i=0}^{n} b_i(n) q_i\right) / c\right),$$

where $b_i(n)$ *is uniquely determined by the equality*

$$n = b_i(n) q_i + r_i(n) d_i, \quad 0 \leq b_i(n) < d_i.$$

Now we will study the singularities of $\mathbb{P}(Q)$. The type of singularities that appear on weighted projective spaces are cyclic quotient singularities.

Definition 1.11. A cyclic quotient singularity is a germ at the origin of the quotient of $\mathbb{C}^n$ by the action

$$(z_1, \ldots, z_n) \mapsto (\zeta_m^{b_1} z_1, \ldots, \zeta_m^{b_n} z_n),$$

where ζ_m is a primitive m-th root of 1, and the b_i are positive integers relatively prime to m. It is denoted by $\frac{1}{m}(b_1, \ldots, b_n)$.

Definition 1.12. Let G be a finite group of linear automorphisms of a finite-dimensional vector space V over $\mathbb{C}$. An element $g \in G$ is a *pseudoreflection* if there exists an element $e_g \in V$ and $f_g \in V\check{}$, the dual vector space, such that

$$g(x) = x + f_g(x)e_g, \quad \text{for every } x \in V.$$

Example 1.13. Recall the action of $\mathbb{Z}_Q$ on $S(1, \ldots, 1)$ as mentioned in Definition 1.2. The generators of $\mathbb{Z}_Q$ act on the vector space of degree 1 elements of $S_1(1, \ldots, 1)$ by the formula

$$(Z_0, \ldots, Z_i, \ldots, Z_n) \mapsto (Z_0, \ldots, \zeta_i Z_i, \ldots, Z_n) = (Z_0, \ldots, Z_n) + (\zeta_i - 1)Z_i V_i,$$

with V_i the i-th unit vector. Therefore they are pseudoreflections.

To study the singularities of $\mathbb{P}(Q)$ we will use the following algebraic lemma.

Lemma 1.14 ([Bo68], ch. V5, Thm. 4)**.** *Let G be a finite group acting on a vector space V over $\mathbb{C}$, B the symmetric algebra of V and $A = B^G$ the subalgebra of G-invariant elements. Then the following are equivalent:*

(i) G is generated by pseudoreflections.

(ii) A is a polynomial $\mathbb{C}$-algebra.

(iii) $V/G \simeq \mathrm{Spec}\, A \simeq V$

Theorem 1.15.

(a) The space $\mathbb{P}(Q)$ is a normal irreducible projective algebraic variety.

(b) All singularities of $\mathbb{P}(Q)$ are cyclic quotient singularities.

Proof. Consider the definition $\mathbb{P}(Q) = \mathbb{P}^n/\mathbb{Z}_Q$.

(a) This follows from the fact that all those properties are preserved under the action of a finite group.

(b) Consider the open coverings $\mathbb{P}(Q) = \bigcup_{i=0}^n D_+(X_i)$ and $\mathbb{P}^n = \bigcup_{i=0}^n D_+(Z_i)$, where $D_+(f)$ is the set of points x such that $f(x) \neq 0$. Notice that $D_+(Z_i)$ is invariant under the action of $\mathbb{Z}_Q$, so

$$D_+(X_i) \simeq D_+(Z_i)/\mathbb{Z}_Q.$$

Without loss of generality, assume $i = 0$. Then $D_+(Z_0) = \operatorname{Spec}\mathbb{C}\left[\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right] \simeq \mathbb{A}^n$. Write $Z_Q = \mathbb{Z}_{\hat{q_0}} \times \mathbb{Z}/q_0\mathbb{Z}$, where $\mathbb{Z}_{\hat{q_0}} = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_n\mathbb{Z}$. Example 1.13 tells us that the generators of $\mathbb{Z}_{\hat{q_0}}$ are pseudoreflections when acting on $D_+(Z_0)$. Therefore, by Lemma 1.14, $D_+(Z_0)/\mathbb{Z}_{\hat{q_0}} \simeq \mathbb{A}^n$, thus

$$D_+(X_0) \simeq D_+(Z_0)/\mathbb{Z}/q_0\mathbb{Z} \simeq \mathbb{A}^n/\mathbb{Z}/q_0\mathbb{Z}.$$

□

The following proposition characterize in a more precise way the singular locus of a weighted projective space.

Proposition 1.16. *If $\mathbb{P} = \mathbb{P}(Q)$ is a well formed weighted projective space, then*

$$(x_0 : \dots : x_n) \in \mathbb{P}_{sing} \Leftrightarrow \gcd\{q_j : x_j \neq 0\} > 1.$$

Proof. ([DD85], Proposition 7) Let $X = \{x \in \mathbb{P} : \gcd\{q_j : x_j \neq 0\} > 1\}$. It is clear that X is a closed set, so let $\mathbb{P}_0$ be the open set $\mathbb{P} \setminus X$. We have that $\mathbb{C}^*$ acts freely on $U_0 = p^{-1}(\mathbb{P}_0)$, with $p : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}$, so $\mathbb{P}_0 \subset \mathbb{P}_{\mathrm{reg}}$ the set of nonsingular points of $\mathbb{P}$.

To prove the other inclusion, let q be a common multiple of the weights q_i and let $a_i = q/q_i$. Define the weighted projective space $\mathbb{P}' = \mathbb{P}(q_0, \dots, q_n, 1)$ and consider the hypersurface

$$V = (x_0^{a_0} + \cdots + x_n^{a_n} + t^q = 0).$$

V is a quasismooth hypersurface (see Definition 1.32) and hence it is normal and irreducible. If we take the covering $f : V \to \mathbb{P}$, $(x, t) \mapsto x$, then the minimal branching set of f is $X \cup H_0$, where $H_0 = (x_0^{a_0} + \cdots + x_n^{a_n} = 0) \subset \mathbb{P}$.

The result now follows from [DD85, Corollary 3], which says that the branching set has codimension 1 at any nonsingular point, but $\dim_x X < n-1$ for any $x \in X$ because $\mathbb{P}$ is well formed. Therefore $X \subset \mathbb{P}_{\text{sing}}$. □

The well formed condition tells us that $\mathbb{P}_{\text{sing}} \subset \bigcup_i \{X_i = 0\}$. Even more, let

$$p_i = [0 : \ldots : 0 : \underset{i\text{-th}}{1} : 0 : \ldots : 0].$$

We already saw that p_i is a singularity of type $\frac{1}{q_i}(q_0, \ldots, \widehat{q_i}, \ldots, q_n)$. For $\overline{p_i p_j}$, the 1-dimensional line passing through p_i and p_j, each point P has an analytic neighborhood which is analytically isomorphic to $(0, Q) \in \mathbb{A}^1 \times Y$, with $Q \in Y$ a singularity of type $\frac{1}{\gcd(q_i, q_j)}(q_0, \ldots, \widehat{q_i}, \ldots, \widehat{q_j}, \ldots, q_n)$. The analogous result holds for higher dimension hyperplanes.

1.2 Cohomology of $\mathcal{O}_{\mathbb{P}(Q)}(m)$

From now on we assume that $\mathbb{P}(Q)$ is a well formed projective space.

Recall that $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is the sheaf associated to the $S(Q)$-module $\widetilde{S(Q)(m)}$ on $\mathbb{P}(Q)$. Given an homogeneus $f \in S(Q)$, define $S(Q)(m)_{(f)}$ the group of elements of degree 0 in the localization $S(Q)(m)_f$, i.e.

$$S(Q)(m)_{(f)} = \{\frac{g}{f^d} \mid \deg(g) = d \deg(f)\}.$$

We have a natural homomorphism

$$S(Q)_m \to S(Q)(m)_{(f)}, \quad f \mapsto \frac{f}{1},$$

which induces a k-linear map called the Serre homomorphism

$$h_m \colon S(Q)_m \to H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)).$$

Theorem 1.17.

(a) For any $m \in \mathbb{Z}$, the homomorphism $h_m \colon S(Q)_m \to H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m))$ is bijective.

(b) $H^n(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) \simeq S(Q)_{-m-\sum q_i}$.

(c) For $0 < i < n$ and all $m \in \mathbb{Z}$, $H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) = 0$.

Proof. In [Dolg82] this result is proved using local cohomology (see [Dolg82, §1.4]). In this case we refer to the proof found in [Ke97, Thm. 2.1], which uses Čech cohomology and is analogous to the proof of [Hart77, III, §5, Thm. 5.1]. Let $\mathcal{F} := \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}(Q)}(m)$. Since cohomology commutes with infinite direct sums on a noetherian topological space, the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(m)$. Therefore we compute the cohomology of $\mathcal{F}$, keeping track of the grading by m.

As $D_+(X_i)$ is an open affine subset of $\mathbb{P}(Q)$, we can compute the Čech cohomology for the covering $\mathcal{U} = \{D_+(X_i)\}_{i=0}^n$.

Notice that $D_+(X_{i_1}) \cap \cdots \cap D(X_{i_k}) = D_+(X_{i_1} \cdots X_{i_k})$ and that

$$\mathcal{F}(D_+(X_{i_1} \cdots X_{i_k})) = \bigoplus_{m\in\mathbb{Z}} S(Q)(m)_{(T_{i_1}\cdots T_{i_k})} = S(Q)_{T_{i_1}\cdots T_{i_k}},$$

and furthermore, the grading of $\mathcal{F}$ is the natural grading of $S(Q)_{T_{i_1}\cdots T_{i_k}}$ under this isomorphism.

The Čech complex of $\mathcal{F}$ is given by

$$C^\bullet(\mathcal{U}, \mathcal{F})\colon \prod S(Q)_{X_{i_0}} \to \prod S(Q)_{X_{i_0}X_{i_1}} \to \cdots \to S(Q)_{X_0\cdots X_n},$$

and all the modules have a natural grading compatible with the grading of $\mathcal{F}$.

Then $H^0(\mathbb{P}(Q), \mathcal{F})$ is the kernel of the first map. This corresponds to the intersection $\bigcap_{i=0}^n S(Q)_{X_i}$ inside $S(Q)_{X_0\cdots X_n}$, which is $S(Q)$ (cf. [Hart77, II, §5, Prop. 5.13]). This proves (a).

For (b) we have that $H^r(\mathbb{P}(Q), \mathcal{F})$ is the cokernel of the last map

$$\prod_k S_{X_0\cdots\hat{X}_k\cdots X_n} \to S(Q)_{X_0\cdots X_n}.$$

We note that $S(Q)_{X_0\cdots X_n}$ can be considered as the free $S(Q)$-module generated by the elements $X_0^{l_0}\cdots X_n^{l_n}$, where $l_j \in \mathbb{Z}$. The image of the previous map is the free submodule generated by $X_0^{l_0}\cdots X_n^{l_n}$, where at least one $l_j \geq 0$. Therefore $H^r(\mathbb{P}(Q), \mathcal{F})$ is a $\mathbb{C}$-vector space with basis consisting of the monomials

$$\{X_0^{l_0}\cdots X_n^{l_n} \mid l_i < 0 \text{ for each } i\}.$$

Thus $H^r(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m))$ is generated by those monomials such that $\sum l_i q_i = m$, The number of these monomials is equivalent to the number of monomials the set

$$\{X_0^{t_0}\cdots X_n^{t_n} \mid t_i \geq 0 \text{ for each } i \text{ and } \sum t_i q_i = -(m + \sum q_i)\}$$

which is exactly $\dim_{\mathbb{C}} S(Q)_{-m-\sum q_i}$. This proves (b).

For (c), we will use induction on n. If $n = 1$, then there is nothing to prove, so let $n > 1$. If we localize the complex $C^{\bullet}(\mathcal{U}, \mathcal{F})$ with respect to X_n, as graded $S(Q)$-modules, we get the Čech complex for the sheaf $\mathcal{F}_{D_+(X_n)}$ on the space $D_+(X_n)$ with respect to the open affine covering $\{D_+(X_n) \cap D_+(X_i)\}_{i=0}^{n}$. This complex gives the cohomology of $\mathcal{F}|_{D_+(X_n)}$ on $D_+(X_n)$, which is 0 for $i > 0$, because $D_+(X_n)$ is affine. Since localization is an exact functor, we have that $H^i(\mathbb{P}(Q), \mathcal{F})_{X_n} = 0$ for $i > 0$. Therefore every element of $H^i(\mathbb{P}(Q), \mathcal{F})$, for $i > 0$ is annihilated by some power of X_n.

Now we will prove that for $0 < i < n$, multiplication by X_n induces a bijective map of $H^i(\mathbb{P}(Q), \mathcal{F})$ into itself, which implies that this vector space is 0.

Consider the exact sequence of graded $S(Q)$-modules

$$0 \to S(Q)(-q_n) \xrightarrow{X_n} S(Q) \to S(Q)/(X_n) \to 0,$$

which induces the exact sequence of sheaves on $\mathbb{P}(Q)$

$$0 \to \mathcal{O}_{\mathbb{P}(Q)}(-q_n) \xrightarrow{X_n} \mathcal{O}_{\mathbb{P}(Q)} \to \mathcal{O}_H \to 0,$$

where $H = (X_n = 0) \simeq \mathbb{P}(q_0, \dots, q_{n-1})$. Twisting by all $m \in \mathbb{Z}$ and taking direct sum we have

$$0 \to \mathcal{F}(-q_n) \xrightarrow{X_n} \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{F}_H = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_H(m)$. Hence we have the long exact sequence of cohomology

$$\cdots \to H^i(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^i(\mathbb{P}(Q), \mathcal{F}) \to H^i(\mathbb{P}(Q), \mathcal{F}_H) \to \cdots .$$

Considered as graded $S(Q)$-modules, $H^i(\mathbb{P}(Q), \mathcal{F}(-q_n))$ is $H^i(\mathbb{P}(Q), \mathcal{F})$ shifted by $-q_n$, and

$$H^i(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^i(\mathbb{P}(Q), \mathcal{F})$$

is multiplication by X_n.

By the induction hypothesis, $H^i(\mathbb{P}(Q), \mathcal{F}_H) = H^i(H, \mathcal{F}_H) = 0$ for $0 < i < n-1$, so $H^i(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^i(\mathbb{P}(Q), \mathcal{F})$ is bijective for $1 < i < n-1$. Therefore we have to check the injectivity of $H^1(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^1(\mathbb{P}(Q), \mathcal{F})$ and the surjectivity of $H^{r-1}(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^{r-1}(\mathbb{P}(Q), \mathcal{F})$. This is equivalent to prove that

$$0 \to H^0(\mathbb{P}(Q), \mathcal{F}(-q_n)) \xrightarrow{X_n} H^0(\mathbb{P}(Q), \mathcal{F}) \to H^0(\mathbb{P}(Q), \mathcal{F}_H) \to 0$$

and

$$0 \to H^{n-1}(\mathbb{P}(Q), \mathcal{F}_H) \xrightarrow{\delta} H^n(\mathbb{P}(Q), \mathcal{F}(-q_n)) \xrightarrow{X_n} H^n(\mathbb{P}(Q), \mathcal{F}) \to 0$$

are exact sequences of sheaves. The first one is a consequence of part (a), since $H^0(\mathbb{P}(Q), \mathcal{F}_H)$ is $S(Q)/(X_n)$. For the second one, it is enoguh to prove that δ is injective.

To prove this, recall from part (b) that $H^n(\mathbb{P}(Q), \mathcal{F})$, and therefore $H^n(\mathbb{P}(Q), \mathcal{F}(-q_n))$, is the vector space generated by the negative monomials in $X_0, \ldots, X_n$. Therefore the kernel of multiplication by X_n are the monomials $X_0^{l_0} \cdots X_{n-1}^{l_{n-1}} X_n^{-1}$, so δ is division by X_n. Since $H^{n-1}(\mathbb{P}(Q), \mathcal{F}_H)$ is the vector space generated by the negative monomials on $X_0, \ldots, X_{n-1}$, δ is injective.

Hence $X_n \colon H^i(\mathbb{P}(Q), \mathcal{F}(-q_n)) \to H^i(\mathbb{P}(Q), \mathcal{F})$ is bijective for $0 < i < n$, which concludes the proof of (c). □

Definition 1.18. Let $S = \bigoplus_{r \geq 0} S_r$ a graded $\mathbb{C}$-algebra. The *Poincaré series* $P_S(t)$ is defined by

$$P_S(t) = \sum_{r=0}^{\infty} (\dim_{\mathbb{C}} S_r) t^r.$$

Proposition 1.19. *The Poincaré series of $S(Q)$ is*

$$P_{S(Q)}(t) = \prod_{i=0}^{n} \frac{1}{1 - t^{q_i}}.$$

Proof. Note that the Poincaré series for the polynomial ring with one variable is

$$1 + X + X^2 + \cdots = \frac{1}{1 - X}.$$

If we take the product of these expressions on each variable we obtain

$$\prod_{i=0}^{n} \frac{1}{1 - X_i} = \sum X_0^{l_0} \cdots X_n^{l_n}.$$

The RHS corresponds to the list of every monomial in $\mathbb{C}[X_0, \ldots, X_n]$ counted once each. If we replace $X_i = t^{q_i}$ in this formal expression, we will have on the RHS as many t^r as monomials of degree r were.

Therefore

$$\prod_{i=0}^{n} \frac{1}{1 - t^{q_i}} = \sum_{r=0}^{\infty} (\dim_{\mathbb{C}} S(Q)_r) t^r.$$

□

Putting together Proposition 1.19 and Theorem 1.17 we obtain the following result.

Corollary 1.20. *Let a_m be the integers determined by the identity*

$$\prod_{i=0}^{n} \frac{1}{1-t^{q_i}} = \sum_{m=0}^{\infty} a_m t^m.$$

Then

$$\dim_{\mathbb{C}} H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) = \begin{cases} a_m & \text{if } i = 0 \\ 0 & \text{if } 0 < i < n \\ a_{-m-\sum q_i} & \text{if } i = n \end{cases}.$$

Finally, we will discuss some pathologies of $\mathbb{P}(Q)$. If $\mathbb{P}(Q) = \mathbb{P}^n$ the following properties hold (see [Hart77, II,§5, Prop. 5.12] and [Hart77, II,§7, Example 7.6.1]).

(i) $\mathcal{O}_{\mathbb{P}^n}(m)$ is an invertible sheaf.

(ii) $\mathcal{O}_{\mathbb{P}^n}(m)$ is ample for $m > 0$.

(iii) The multiplication homomorphism $S(m_1) \otimes S(m_2) \to S(m+n)$ induces an isomorphism $\mathcal{O}_{\mathbb{P}^n}(m_1) \otimes \mathcal{O}_{\mathbb{P}^n}(m_2) \to \mathcal{O}_{\mathbb{P}^n}(m_1 + m_2)$, where $S = S(1, \dots, 1)$.

(iv) For any grade S-module M and $m \in \mathbb{Z}$, $\tilde{M}(m) \simeq \tilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m)$.

None of these are valid for a general $\mathbb{P}(Q)$.

Counterexamples

(i) Let $Q = \{1, 1, 2\}$ and consider the sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1)$. The restriction of this sheaf to the open set $D_+(X_2)$ is given by the $S(Q)_{(X_2)}$-module

$$S(Q)(1)_{(X_2)} = \{\frac{f}{X_2^k} \mid f \in S(Q)_{2k+1}\}.$$

We can see that $S(Q)(1)_{(X_2)} = S(Q)_{(X_2)} X_0 + S(Q)_{(X_2)} X_1$, so it is not a free $S(Q)_{(X_2)}$-module of rank 1.

(ii) Let $Q = \{q_0, q_1\}$, $\gcd(q_0, q_1) = 1$ and $q_i \geq 2$ for some i. All sheaves $\mathcal{O}_{\mathbb{P}(Q)}(m)$ are invertible. By Proposition 1.5, we have that $\mathbb{P}(Q) \simeq \mathbb{P}^1$, and so an invertible sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is isomorphic to some $\mathcal{O}_{\mathbb{P}^1}(b_m)$. Even more, if $\Gamma(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) \neq 0$, then $b_m = \dim_{\mathbb{C}} \Gamma(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(n)) - 1$. Hence $\mathcal{O}(m)$ is ample if and only if $\dim_{\mathbb{C}} \Gamma(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) \geq 2$. But if $0 < m < \min\{q_0, q_1\}$, then by Theorem 1.17 we have that $\Gamma(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) < 2$.

(iii) Let $Q = \{q_0, q_1\}$ with $q_1 = q_0 + 1$ and $q_0 \geq 3$. Then $b_{q_0} = b_{q_0+q_1+1} = 0$ and $b_{q_1+1} < 0$. But

$$\mathcal{O}_{\mathbb{P}(Q)}(q_0) \otimes \mathcal{O}_{\mathbb{P}(Q)}(q_1 + 1) \simeq \mathcal{O}_{\mathbb{P}^1}(b_{q_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(b_{q_1+1}) \simeq \mathcal{O}_{\mathbb{P}^1}(b_{q_0} + b_{q_1+1})$$

and

$$\mathcal{O}_{\mathbb{P}(Q)}(q_0 + q_1 + 1) \simeq \mathcal{O}_{\mathbb{P}^1}(b_{q_0+q_1+1}).$$

(iv) Take $M = S(Q)(m)$. Then (iii) gives a counterexample to property (iv).

1.3 Differentials

In the following section, we describe the sheaf of differentials of a weighted projective space. These results are in [Dolg82, Section 2.1 and Section 2.2].

Definition 1.21. Let $\Omega^1_{S(Q)}$ be the *module of relative differential forms of* $S(Q)$ *over* $\mathbb{C}$. This is a free $S(Q)$-module with basis $\{dX_0, \ldots, dX_n\}$.

Definition 1.22. $\Omega^i_{S(Q)} = \bigwedge^i(\Omega^1_{S(Q)})$. We define $\Omega^0_{S(Q)} = S(Q)$.

We have that $\Omega^i_{S(Q)}$ is a free $S(Q)$-module with basis $\{dX_{s_1} \wedge \cdots \wedge dX_{s_i} \mid 0 \leq s_1 \leq \cdots \leq s_i \leq n\}$. We give $\Omega^i_{S(Q)}$ a graduation by the condition

$$\deg(dX_{s_1} \wedge \cdots \wedge dX_{s_i}) = q_{s_1} + \cdots + q_{s_i}.$$

Then we have an isomorphism of graded $S(Q)$-modules

$$\Omega^i_{S(Q)} \simeq \bigoplus_{0 \leq s_1 \leq \cdots \leq s_i \leq n} S(Q)(-q_{s_1} - \ldots - q_{s_i}),$$

with $f dX_{s_1} \wedge \cdots \wedge dX_{s_i} \mapsto f$.

In particular, $\Omega^{n+1}_{S(Q)} \simeq S(Q)(-\sum q_i)$.

We have from the definition of a $\mathbb{C}$-derivation that for $f \in S(Q)$

$$df = \sum_{i=0}^{n} \frac{\partial f}{\partial X_i} dX_i,$$

and this map d extends to the exterior derivation

$$d \colon \Omega^i_{S(Q)} \to \Omega^{i+1}_{S(Q)},$$

which is uniquely determined by

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^i \omega \wedge d\omega',\ \omega \in \Omega^i_{S(Q)},\ \omega' \in \Omega^j_{S(Q)}$$

and

$$d(d(w)) = 0,\ \omega \in \Omega^i_{S(Q)}.$$

Lemma 1.23 (Euler formula)**.** *If $f \in S(Q)_m$, then*

$$mf = \sum_{i=0}^{n} \frac{\partial f}{\partial X_i} q_i X_i.$$

Proof. Because of the linearity of both RHS and LHS, it is enough to check the assertion only for monomials $X_0^{s_0} \cdots X_n^{s_n}$, which is easy to verify. □

Define the homomorphism of graded $S(Q)$-module

$$\triangle \colon \Omega^i_{S(Q)} \to \Omega^{i-1}_{S(Q)},$$

given by

$$dX_{s_1} \wedge \cdots \wedge dX_{s_i} \mapsto \sum_{k=1}^{i} q_{s_k} X_{s_k} dX_{s_1} \wedge \cdots \wedge \widehat{dX_{s_k}} \wedge \cdots \wedge dX_{s_i}.$$

This homomorphism has the following properties.

Lemma 1.24. *(a)* $\triangle^2 = 0$*;*

(b) $\triangle(\omega \wedge \omega') = \triangle(\omega) \wedge \omega' + (-1)^i \omega \wedge \triangle(\omega')$, $\omega \in \Omega^i_{S(Q)}$, $\omega' \in \Omega^j_{S(Q)}$*;*

(c) $\triangle(df) = mf$, $f \in S(Q)_m$*;*

(d) $\triangle(d\omega) + d(\triangle(\omega)) = n\omega$, $\omega \in (\Omega^i_{S(Q)})_n$.

Proof. (a) is easy to check.

For (b) it is enough to check it only for $\omega = dX_{s_1} \wedge \cdots \wedge dX_{s_i}$ and $\omega' = dX_{s'_1} \wedge \cdots \wedge dX_{s'_j}$, and the result follows from the definition of $\triangle$.

(c) is a corollary of Lemma 1.23.

To prove (d) it is enough to consider $\omega = f dX_{s_1} \wedge \cdots \wedge dX_{s_i}$, with $f \in S(Q)_l$. Then

$$\begin{aligned}\triangle(d\omega) &= \triangle(df \wedge dX_{s_1} \wedge \cdots \wedge dX_{s_i}) \overset{\text{(b)}}{=} \triangle(df) \wedge dX_{s_1} \wedge \cdots \wedge dX_{s_i} - df \wedge \triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i}) \\ &\overset{(c)}{=} l f dX_{s_1} \wedge \cdots \wedge dX_{s_i} - df \wedge \triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i})\end{aligned}$$

and

$$\begin{aligned}d(\triangle\omega) &= d(f\triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i})) = df \wedge \triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i}) + fd(\triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i})) \\ &= df \wedge \triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i}) + fd\left(\sum_{k=1}^{i} q_{s_k} X_{s_k} dX_{s_1} \wedge \cdots \wedge \widehat{dX_{s_k}} \wedge \cdots \wedge dX_{s_i}\right) \\ &= df \wedge \triangle(dX_{s_1} \wedge \cdots \wedge dX_{s_i}) + \left(\sum_{k=1}^{i} q_{s_k}\right) f dX_{s_1} \wedge \cdots \wedge dX_{s_i}.\end{aligned}$$

Therefore $\triangle(d\omega) + d(\triangle\omega) = (l + \sum_{k=1}^{i} q_{s_k})\omega = n\omega$. □

The complex

$$0 \to \Omega^{n+1}_{S(Q)} \overset{\triangle}{\to} \Omega^{n}_{S(Q)} \overset{\triangle}{\to} \cdots \to \Omega^{1}_{S(Q)} \to S(Q) \to S(Q)/(q_0X_0, \ldots, q_nX_n) \to 0$$

is the Koszul complex for the regular sequence $(q_0X_0, \ldots, q_nX_n)$, and therefore it is exact (see [Ma70, Thm. 43, p. 135]).

Definition 1.25. Define $\overline{\Omega}^{i}_{S(Q)} = \ker(\triangle\colon \Omega^{i}_{S(Q)} \to \Omega^{i-1}_{S(Q)}) = \operatorname{Im}(\triangle\colon \Omega^{i+1}_{S(Q)} \to \Omega^{i}_{S(Q)})$, with the induced grading.

We have short exact sequences of graded $S(Q)$-modules

$$0 \to \overline{\Omega}^{i}_{S(Q)}(m) \overset{\triangle}{\to} \Omega^{i}_{S(Q)}(m) \overset{\triangle}{\to} \overline{\Omega}^{i-1}_{S(Q)}(m) \to 0,\ i \geq 1,\ m \in \mathbb{Z}.$$

Definition 1.26. Denote by $\Omega^{i}_{\mathbb{P}(Q)}(m)$ the sheaf on $\mathbb{P}(Q)$ associated to the graded $S(Q)$-module $\overline{\Omega}^{i}_{S(Q)}(m)$, for $i = 0, 1, \ldots, n$.

Because the functor $M \to \tilde{M}$ is exact, then the exact sequence above induces an exact sequence of sheaves

$$0 \to \Omega^{i}_{\mathbb{P}(Q)}(m) \to \tilde{\Omega}^{i}_{S(Q)}(m) \to \Omega^{i-1}_{\mathbb{P}(Q)}(m) \to 0,\ i \geq 1,\ m \in \mathbb{Z}.$$

Notice that $\Omega^{n+1}_{\mathbb{P}(Q)}(m) = 0$, thus

$$\Omega^{n}_{\mathbb{P}(Q)}(m) \simeq \tilde{\Omega}^{n+1}_{S(Q)}(m) \simeq \tilde{S(Q)}(n - \sum q_i) = \mathcal{O}_{\mathbb{P}(Q)}(n - \sum q_i).$$

The goal of the following three propositions is to give a justification of the use of $\Omega^i_{\mathbb{P}(Q)}$ as a good substitute for the sheaf of differential forms $\Omega^i_{\mathbb{P}^n}$ on the usual projective space $\mathbb{P}^n$. Their proofs can be found in [Dolg82, Section 2.2].

Proposition 1.27 ([Dolg82],2.2.1)**.** *For $Q = \{1, \dots, 1\}$, the sheaf $\Omega^i_{\mathbb{P}(Q)}(m)$ coincides with the twisted sheaf of differential i-forms $\Omega^i_{\mathbb{P}^n}(m)$ on the usual projective space $\mathbb{P}^n$.*

Proof. Let $U = \mathbb{A}^{n+1} \setminus \{0\}$, and let $S = S(1, \dots, 1)$, and consider the projection $p\colon U \to \mathbb{P}^n$. We have the exact sequence of sheaves of differentials

$$0 \to p^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_U \to \Omega^1_{U/\mathbb{P}^n} \to 0. \tag{1.1}$$

This sequence induces the exact sequences (see [Hart77, II, Ex. 5.16(d)])

$$0 \to p^*\Omega^i_{\mathbb{P}^n} \to \Omega^i_U \to \Omega^1_{U/\mathbb{P}^n} \otimes p^*\Omega^{i-1}_{\mathbb{P}^n} \to 0.$$

The homomorphism $\triangle\colon \Omega^1_S \to S$ given by $\sum f_i dX_i \mapsto \sum f_i X_i$, restricted to U induces a surjective morphism of sheaves

$$\triangle\colon \Omega^1_U \to \mathcal{O}_U.$$

We have that

$$\triangle\left(d\left(\frac{X_i}{X_j}\right)\right) = \triangle\left(\frac{X_j dX_i - X_i dX_j}{X_j^2}\right) = 0,$$

so $\triangle(p^*\Omega^1_{\mathbb{P}^n}) = 0$. Hence, by the exact sequence 1.1, we have that $\triangle$ induces a surjective morphism of sheaves

$$\tilde{\triangle}\colon \Omega^1_{U/\mathbb{P}^n} \to \mathcal{O}_U.$$

As p is smooth morphism, $\Omega^1_{U/\mathbb{P}^n}$ is invertible, so $\tilde{\triangle}$ is an isomorphism.

Therefore we have the exact sequences

$$0 \to p^*\Omega^i_{\mathbb{P}^n} \to \Omega^i_U \to p^*\Omega^{i-1}_{\mathbb{P}^n} \to 0.$$

By taking p_*, and using that $p_*p^*\mathcal{F} \simeq p_*\mathcal{O}_U \otimes \mathcal{F}$ and $p_*\mathcal{O} \simeq \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(m)$, we obatin the exact sequences

$$0 \to \bigoplus_{m\in\mathbb{Z}} \Omega^i_{\mathbb{P}^n}(m) \to \bigoplus_{m\in\mathbb{Z}} \tilde{\Omega}^i_S(m) \to \bigoplus_{m\in\mathbb{Z}} \Omega^{i-1}_{\mathbb{P}^n}(m) \to 0,$$

and so we obtain the same exact sequences of Definition 1.26. $\square$

To prove the next proposition we will use the following algebraic fact.

Lemma 1.28 ([Dolg82], 2.2.2)**.** *Let G be a finite group acting on a vector space V over $\mathbb{C}$, B the symmetric algebra of V and $A = B^G$ the subalgebra of G-invariant elements. Assume that G is generated by pseudoreflections. Then the canonical homomorphism*

$$\Omega^i_{A/\mathbb{C}} \to (\Omega^i_{B/\mathbb{C}})^G$$

is an isomorphism of A-modules.

Proposition 1.29. *Let $\pi\colon \mathbb{P}^n \to \mathbb{P}(Q) = \mathbb{P}^n/\mathbb{Z}_Q$. Then*

$$\Omega^i_{\mathbb{P}(Q)} \simeq \pi^G_*(\Omega^i_{\mathbb{P}^n}),$$

where $G = \mathbb{Z}_Q$ and π^G_* is the invariant direct image $\pi^G_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))^G$.

Proof. The action of G on $\mathbb{P}^n$ is induced by an action on $S = S(1,\dots,1)$, and as seen in Example 1.13 it is generated by pseudoreflections. Then by Lemma 1.28 we have an isomorphism of $S(Q)$-modules

$$\Omega^i_{S(Q)} \simeq (\Omega^i_S)^G,$$

which induces an isomorphism of sheaves

$$\tilde{\Omega}^i_{S(Q)} \simeq \pi^G_*(\tilde{\Omega}^i_S).$$

From Proposition 1.27 we have the exact sequence

$$0 \to \Omega^i_{\mathbb{P}^n} \to \tilde{\Omega}^i_S \to \Omega^{i-1}_{\mathbb{P}^n} \to 0.$$

Because π is affine, we have that $R^1\pi_*(\Omega^i_{\mathbb{P}^n}) = 0$, and we have that the functor $(\cdot)^G$ is exact. Therefore we have the exact sequence of sheaves

$$0 \to \pi^G_*(\Omega^i_{\mathbb{P}^n}) \to \tilde{\Omega}^i_{S(Q)} \to \pi^G_*(\Omega^{i-1}_{\mathbb{P}^n}) \to 0,$$

and since $\pi^G_*(\mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}(Q)}$, the result follows by using the five lemma and induction on i. ☐

Proposition 1.30. *Let $j\colon W \to \mathbb{P}(Q)$ be the open immersion of the nonsingular locus of $\mathbb{P}(Q)$. Then*

$$\Omega^i_{\mathbb{P}(Q)} = j_*(\Omega^i_W).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(W) & \xrightarrow{j'} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow[j]{} & \mathbb{P}(Q) \end{array}$$

where $\pi' = \pi|_{\pi^{-1}(W)}$, and j' is the natural immersion. As W is nonsingular, the action of Z_Q on $\pi^{-1}(W)$ is generated by pseudoreflections. Then, by Lemma 1.28 we have

$$\Omega^i_W \simeq \pi'^G_*(\Omega^i_{\pi^{-1}(W)}).$$

Since $\mathbb{P}(Q)$ is normal, then $\mathbb{P}(Q) - W$, and hence $\mathbb{P}^n - \pi^{-1}(W)$, has codimension ≥ 2. Because $\mathbb{P}^n$ is smooth, we have

$$j'_*(\Omega^i_{\pi^{-1}(W)}) \simeq \Omega^i_{\mathbb{P}^n}.$$

Finally we obtain that

$$j_*(\Omega^i_W) \simeq j_*(\pi'^G_*(\Omega^i_{\pi^{-1}(W)})) \simeq \pi^G_*(j'_*(\Omega^i_{\pi^{-1}(W)})) \simeq \pi^G_*(\Omega^i_{\mathbb{P}^n}) \simeq \Omega^i_{\mathbb{P}(Q)}.$$

□

1.4 Hypersurfaces of weighted projective spaces

We are now interested in studying properties of closed subvarieties of codimension 1 in our weighted projective spaces. Most of these results still hold for complete intersection of higher codimension, but for the purpose of this document they are not relevant.

Definition 1.31. Let X be a closed subvariety in $\mathbb{P}$ a weighted projective space, and

$$q : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}$$

the quotient map. The *punctured affine cone* C^*_X over X is given by $C^*_X = q^{-1}(X)$. The *affine cone* C_X over X is $\overline{C^*_X} = C^*_X \cup \{0\}$ in $\mathbb{A}^{n+1}$.

Definition 1.32. A closed subvariety $X \subset \mathbb{P}$ of dimension m is *quasismooth* if C^*_X is nonsingular of dimension $m+1$ outside the vertex 0.

Proposition 1.33. C_X^* *has no isolated singularities.*

Proof. If $P \in C_X^*$ is singular, then all the fiber $q^{-1}(P)$ will be singular. $\square$

Another property that will be useful when proving some general properties for hypersufaces is the notion of well formed.

Definition 1.34. A closed subvariety $X \subset \mathbb{P}$ of codimension m is *well formed* if $\mathbb{P}$ is well formed (definition 1.8) and X contains no codimension $m+1$ singular stratum of $\mathbb{P}$.

For example, if X is a well formed surface in a 3 dimensional weighted projective space $\mathbb{P}$, then X does not contain any dimension 1 subvariety of $\mathbb{P}_{\text{Sing}}$.

Proposition 1.35. *Let X be a quasismooth and well formed hypersurface of degree d. Then the dualizing sheaf $\omega_X = \mathcal{O}_X(K_X)$ of X is isomorphic to $\mathcal{O}_X(d - \sum q_i)$.*

Proof. See [Dolg82, Thm. 3.3.4] and [Ian00, 6.14]. $\square$

1.5 Cyclic singularities on surfaces

Previously, in Definition 1.11, we defined a cyclic quotient singularity. Now we focus in the 2-dimensional case. The goal of this section is to introduce cyclic singularities and certain combinatorial numbers that arise from them. More details can be found in [BHPV, Ch. III, §5], [R03] or [Is14, §7.4].

Recall that 2-dimensional cyclic singularities correspond locally to the quotient of $\mathbb{C}^2$ by the action $(x, y) \mapsto (\zeta_m^a x, \zeta_m^b y)$, where ζ_m is a primitive m-th root of 1, and a, b are integers relatively prime to m.

Let $0 < q < m$ be such $aq - b \equiv 0 \pmod{m}$. Then $\frac{1}{m}(a, b) = \frac{1}{m}(1, q)$, by considering ζ'_m, another primitive m-th root of 1, such that $\zeta'^a_m = \zeta_m$.

The minimal resolution of these singularities is closely related to the Hirzebruch-Jung continued fraction of a rational number.

Proposition 1.36. *A rational number $\frac{m}{q}$, with $m > q$, is uniquely expanded by using a finite number of integers $b_1, \ldots, b_s$, all of them greater than or equal to 2, as follows.*

$$\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}}.$$

Proof. Define $\beta_1 := q$ and take b_1 be the positive integer such that $m = b_1\beta_1 - \beta_2$, with $0 \leq \beta_2 < \beta_1$. Because $m > q$, then $b_1 \geq 2$. Analogously, decompose:

$$\begin{aligned} \beta_1 &= b_2\beta_2 - \beta_3, \qquad (b_2 \geq 2, 0 \leq \beta_3 < \beta_2) \\ \beta_2 &= b_3\beta_3 - \beta_4, \qquad (b_3 \geq 2, 0 \leq \beta_4 < \beta_3) \\ &\vdots \end{aligned}$$

As q_i are integers, then there exists an integer $s > 0$ such that $\beta_{s+1} = 0$. Therefore

$$\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}}.$$

To prove the uniqueness, supose that

$$\frac{m}{q} = b'_1 - \cfrac{1}{b'_2 - \cfrac{1}{\ddots - \frac{1}{b'_r}}}.$$

Notice that

$$b'_2 - \cfrac{1}{\ddots - \frac{1}{b'_r}} > 1.$$

Hence

$$m = b'_1 q - \cfrac{q}{b'_2 - \cfrac{1}{\ddots - \frac{1}{b'_r}}} = b'_1\beta'_1 - \beta'_2,$$

where $0 < \beta'_2 \leq \beta'_1 = \beta_1 = q$. Therefore $b'_1 = b_1$ and $\beta'_2 = \beta_2$. In the same way it follows that $b'_i = b_i$ and $r = s$. □

Definition 1.37. Let $\frac{m}{q}$ be a rational number, with $m > q$ and $\gcd(m, q) = 1$, and expand it as in Proposition 1.36:

$$\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}}.$$

This expansion is called the *Hirzebruch-Jung continued fraction* and it is denoted by

$$\frac{m}{q} = [b_1, \ldots, b_s].$$

We denote the order of the fraction by $|[b_1, \ldots, b_s]| := m$.

As in the proof of Proposition 1.36 this continued fraction defines the sequence of integers

$$0 = \beta_{s+1} < 1 = \beta_s < \ldots < q = \beta_1 < m = \beta_0,$$

where $\beta_{i+1} = b_i\beta_i - \beta_{i-1}$. Therefore, $\frac{\beta_{i-1}}{\beta_i} = [b_i, \ldots, b_s]$. Partial fractions $\frac{\alpha_i}{\gamma_i} = [b_1, \ldots, b_{i-1}]$ are computed through the sequences

$$0 = \alpha_0 < 1 = \alpha_1 < \ldots < q^{-1} = \alpha_s < m = \alpha_{s+1},$$

where $\alpha_{i+1} = b_i\alpha_i - \alpha_{i-1}$ (q^{-1} is the integer such that $0 < q^{-1} < m$ and $qq^{-1} \equiv 1(\text{mod } m)$), and $\gamma_0 = -1$, $\gamma_1 = 0$, $\gamma_{i+1} = b_i\gamma_i - \gamma_{i-1}$. We have $\alpha_{i+1}\gamma_i - \alpha_i\gamma_{i+1} = -1$, $\beta_i = q\alpha_i - m\gamma_i$, and $\frac{m}{q^{-1}} = [b_s, \ldots, b_1]$.

These numbers are important because they will appear in the minimal resolution of $S = \mathbb{C}^2/\mathbb{Z}/m\mathbb{Z}$.

Lemma 1.38. *The affine coordinate ring of S is $\mathbb{C}[x^iy^j]$, where $i + qj \equiv 0(\text{mod } m)$ and $0 \leq i \leq m$, $0 \leq j \leq m$.*

Proof. The affine coordinate ring of S corresponds to $\mathbb{C}[x, y]^{\mathbb{Z}/m\mathbb{Z}}$, which is generated by the monomial x^iy^j such that $\zeta_m^{i+qj} = 1$. □

Theorem 1.39 ([R03], Theorem 3.2). *Let $S = \mathbb{C}^2/\mathbb{Z}/m\mathbb{Z}$ be a cyclic singularity of type $\frac{1}{m}(a, b)$, and let $\frac{1}{m}(a, b) = \frac{1}{m}(1, q)$. Let N be the lattice $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{m}(1, q)$, and*

$$M = \{(r, s) : r + qs \equiv 0 \bmod m\} \subset \mathbb{Z}^2$$

the dual lattice of invariant monomials under the action $(x, y) \mapsto (\zeta_m x, \zeta_m^q y)$ with ζ_m an m-th primitive root of unity.

Let $\frac{m}{q} = [b_1, \ldots, b_s]$ and let $z_0, z_1, \ldots, z_{s+1}$ vectors in N defined as

$$z_i = \frac{1}{m}(\alpha_i, \beta_i).$$

Then for each $i = 0, \dots, s$, let u_i, v_i be monomials forming the dual basis of M to z_i, z_{i+1}; that is, $u_i = (\beta_i, -\alpha_i); v_i = (-\beta_{i+1}, \alpha_{i+1})$.

Then S has a resolution of singularities $\tilde{S} \to S$ constructed as follows:

$$\tilde{S} = U_0 \cup U_1 \cup \cdots \cup U_s,$$

where $U_i \simeq \mathbb{C}^2$ with coordinates u_i, v_i.

The glueing $U_i \cup U_{i+1}$ and the morphism $\tilde{S} \to S$ are both determined by the definition of u_i, v_i and they consist of

$$U_i \setminus (v_i = 0) \xrightarrow{\simeq} U_{i+1} \setminus (u_{i+1} = 0) \quad \textit{given by} \quad u_{i+1} = v_i^{-1},\ v_{i+1} = u_i v_i^{b_i}.$$

It follows from the definition of the numbers α_i and β_i that $u_0 = x^m$ and $v_s = y^m$, and they satisfy the relations

$$x^m = u_i^{\alpha_{i+1}} v_i^{\alpha_i} \quad \textit{and} \quad y^m = u_i^{\beta_{i+1}} v_i^{\beta_i}.$$

Notice that the closed subset $E = (u_0 = v_1 = 0) \cup (u_1 = v_2 = 0) \cup \cdots \cup (u_{s-1} = v_s = 0)$ is isomorphic to $\mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1$. Even more, by looking at the image of the open set $(U_0 \cap U_1) \setminus E = (u_0 v_0 \neq 0)$ under the glueing we obtain that $\tilde{S} = (u_0 v_0 \neq 0) \cup E \cup (v_0 = 0) \cup (u_s = 0)$.

To see that this is a resolution of S, consider Y the affine variety with affine coordinate ring

$$\mathbb{C}[x^m, x^{m-q}y, y^m] \simeq \mathbb{C}[x_1, x_2, x_3]/(x_1^{m-q} x_3 - x_2^m).$$

The ring $\mathbb{C}[x^i y^j]$ that appears in Lemma 1.38 is integral over $\mathbb{C}[x^m, x^{m-q}y, y^m]$, and it is integrally closed because it is the ring of invariants of a integrally closed domain, under the action of finite group of automorphism. Therefore, S is the normalization of Y. Now notice that u_0, v_s and $u_0 v_0$ are regular functions in $\tilde{S}$. To see this it is enough to show that they are written as $u_i^a v_i^b$, for $a, b \geq 0$, at every U_i. For u_0 and v_s follows from $u_0 = u_i^{\alpha_{i+1}} v_i^{\alpha_i}$ and $v_s = u_i^{\beta_{i+1}} v_i^{\beta_i}$, and it easy to check that $u_0 v_0$ is also regular. We have that $v_s = u_0^q v_0^n$, so we can define the morphism $\Phi \colon \tilde{S} \longrightarrow \mathbb{C}^3$ given by $(u_i, v_i) \mapsto (u_0, u_0 v_0, u_0^q v_0^n)$. Thus Im $\Phi = Y$, so Φ factors

$$\tilde{S} \xrightarrow{\sigma} S \xrightarrow{\phi} Y.$$

Even more, the morphism Φ gives an isomorphism between $(u_0 v_0 \neq 0)$ and $(x_1 x_2 x_3 \neq 0)$, and that restricted to $Y \setminus \{0\}$, Φ is finite.

As σ is isomorphic outside the singular point, then $\tilde{S}$ is a resolution of the singularity S. The exceptional divisor of σ is $E = E_1 \cup E_2 \cup \cdots \cup E_s$ and one can compute the self-intersection of them and obtain that $E_i^2 = -b_i$ (see [Is14, Thm. 7.4.16]). Because $b_i \geq 2$ for all i, $\tilde{S}$ is the minimal resolution of S. Figure 1.1 shows the exceptional curves $E_i = \mathbb{P}^1$ of σ, for $1 \leq i \leq s$, and the strict transforms E_0 and E_{s+1} of $(y = 0)$ and $(x = 0)$ respectively.

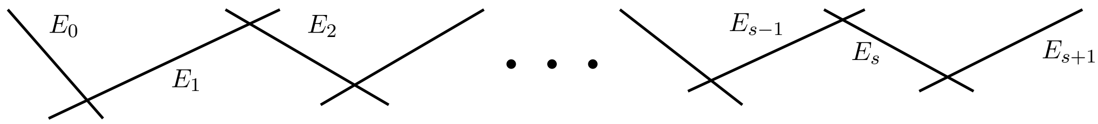


Figure 1.1: Exceptional divisors over $\frac{1}{m}(1, q)$, E_0 and E_{s+1}

Finally, we have the following pull-back formulas (see [BHPV, Ch. III, §5])

$$\sigma^*\big((y=0)\big) = \sum_{i=0}^{s+1} \frac{\beta_i}{m} E_i, \quad \text{and} \quad \sigma^*\big((x=0)\big) = \sum_{i=0}^{s+1} \frac{\alpha_i}{m} E_i. \tag{1.2}$$

Even more, we have that $K_{\tilde{S}} \equiv \sigma^*(K_S) + \Delta$, where Δ is a $\mathbb{Q}$-divisor supported on the exceptional divisor, say $\Delta = \sum_{i=1}^{s} \Delta_i E_i$, with $\Delta_i \in \mathbb{Q}$. To find the coefficients Δ_i we use the adjuction formula for E_i, and Cramer's rule we obtain that

$$\Delta_i = -1 + \frac{\alpha_i + \beta_i}{m}.$$

1.6 n-th root covers

One of the main result of this thesis is to prove that Kollár surfaces are birational to n-th root covers of the projective plane totally branched over four lines in general position. In this section we follow [EV92, §3] to show how to construct this n-th root covers.

Let X be a smooth projective variety of dimension m.

Definition 1.40. An effective divisor $D = \sum D_i$ on X is a *simple normal crossing divisor* (*SNC divisor*) if D is reduced, each component D_i is smooth, and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type $(z_1 \cdots z_k = 0)$, with $k \leq n$. We say that a divisor $E = \sum \mu_i D_i$ has *simple normal crossing support* if the reduced divisor $\sum D_i$ is a SNC divisor.

Let $D = \sum \mu_i D_i \neq 0$ be an effective SNC divisor on X. Assume that there is a positive integer n and a line bundle $\mathcal{L}$ such that $\mathcal{L}^n \simeq \mathcal{O}_X(D)$.

Let s be a section of $\mathcal{O}_X(D)$ such that its divisor of zeros is equal to D. The dual of this section $s^\vee : \mathcal{L}^{-n} \to \mathcal{O}_X$ defines a $\mathcal{O}_X$-algebra structure on

$$\mathcal{A} := \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}.$$

The multiplication is the multiplication $\mathcal{L}^{-i} \times \mathcal{L}^{-j} \to \mathcal{L}^{-i-j}$ composed with $s^\vee \colon \mathcal{L}^{-i-j} \to \mathcal{L}^{-i-j+n}$ if $i + j \geq n$.

Let $Y_0 := \mathbf{Spec}\ \mathcal{A} \xrightarrow{f_1} X$ as defined in [Hart77, Exercise II 5.17]. This variety may not be normal, so we consider the normalization $Y' \to Y_0$ and let $f_2 \colon Y' \to X$ the composition of f_1 with the normalization. Let $\lfloor x \rfloor$ be the greatest integer that is less than or equal to x. Following [EV92] define the line bundles

$$\mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_X\left(-\sum \left\lfloor \frac{\mu_j i}{n} \right\rfloor D_j\right),$$

for $0 \leq i \leq n-1$, and

$$\mathcal{A}' := \bigoplus_{i=0}^{n-1} \mathcal{L}^{(i)^{-1}}.$$

Proposition 1.41. *$\mathcal{A}' = f_{2*}\mathcal{O}_{Y'}$ or equivalently $Y' =$ **Spec** $\mathcal{A}'$, and the cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on Y' and on $g_*\mathcal{O}_{Y'}$. Furthermore, we have that $Y'/\mathbb{Z}/n\mathbb{Z} = X$.*

Proof. See [EV92, Claim 3.10] and [EV92, Corollary 3.11]. ☐

Proposition 1.42. *If we change the multiplicities μ_i to ν_i such that $\nu_i \equiv \mu_i \pmod{n}$ for all i, then the corresponding variety $\overline{Y}'$ is isomorphic to Y'. Even more, if b is a positive integer such that $\gcd(b, n) = 1$, then if we change the multiplicities μ_i to ν_i such that $\nu_i \equiv b\mu_i \pmod{n}$ for all i, then the corresponding variety $\overline{Y}'$ is isomorphic to Y'.*

Proof. First, let $D' = \sum \nu_i D_i$ with $\mu_i = \nu_i + c_i n$, and define $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_X(-\sum c_i D_i)$. Then $\mathcal{L}'^n \simeq \mathcal{O}_X(D')$. Even more, we have that $\mathcal{L}'^{(i)} = \mathcal{L}^{(i)}$ which define an isomorphism between the $\mathcal{O}_X$-algebras $\bigoplus_{i=0}^{n-1} \mathcal{L}^{(i)^{-1}}$ and $\bigoplus_{i=0}^{n-1} \mathcal{L}'^{(i)^{-1}}$.

For the second case, let $D'' = \sum \nu_i D_i$ with $b\mu_i = \nu_i + c_i n$, and define $\mathcal{L}'' = \mathcal{L}^b \otimes \mathcal{O}_X(-\sum c_i D_i)$. This definition also induces an isomorphism between the respective $\mathcal{O}_X$-algebras. ☐

Finally, we consider $f_3 \colon Y \to X$ be f_2 composed with a minimal resolution of singularities of Y'. In the case of surfaces, the minimal resolution is unique.

If we restrict to the case when X is a surface, as D only have nodes we can compute the minimal resolution as follows. Let $0 < \mu_i, \mu_j < n$ be the multiplicities of D_i and D_j respectively. Assume that D_i and D_j do intersect. Then over a point on Y' we have an open neighborhood isomorphic to the normalization of $\mathrm{Spec}\,(\mathbb{C}[x,y,z]/(z^p - x^{\mu_i}y^{\mu_j}))$. Then in [BHPV, III, §5] it is proven that this normalization is isomorphic to the normalization of $\mathrm{Spec}\,(\mathbb{C}[x,y,z]/(z^n - xy^{n-q}))$, where $\mu_i q + \mu_j \equiv 0 (\mathrm{mod}\ n)$. Therefore, as seen in Section 1.5, the resolution locally corresponds to the resolution of the singularity $1/n(1,q)$.

1.7 Dedekind sums

Definition 1.43. Let a, b, n integers, with $\gcd(a,n) = \gcd(b,n) = 1$. The *generalized Dedekind sum* $s(a,b;n)$ is defined by

$$s(a,b;n) = \sum_{i=0}^{n-1} \left(\left(\frac{ai}{n}\right)\right)\left(\left(\frac{bi}{n}\right)\right),$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

($\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x).

These Dedekind sums appear in many different contexts, including Number Theory, Algebraic Geometry and Topology (see [HiZa74, Section 5]. For example Dedekind sums and Hirzebruch-Jung continued fractions relate as (see e.g. [Ba77], [Urz10, Example 3.5])

$$12s(a,b;n) = \frac{q + q^{-1}}{n} + \sum_{i=1}^{l(a,b;n)} (e_i - 3),$$

where q is the integer such that $0 < q < n$ and $a + qb \equiv 0 \mod n$, and $l(a,b;n)$ is the length of the Hirzebruch-Jung continued fraction of n/q, i.e. $l(a,b;n) = s$ where $n/q = [b_1, \ldots, b_s]$.

Even though these Dedekind sums are still not completely understood, they have certain properties that will be useful to study the geometric genus of Kollar surfaces.

Proposition 1.44.

(1) $s(a,b;n) = s(b,a;n)$*;*

(2) $s(a,b;n) = s(a+tn, b+sn;n)$*, with* $t,s \in \mathbb{Z}$*;*

(3) $s(-a,b;n) = s(a,-b;n) = -s(a,b;n)$*;*

(4) $s(a,b;n) = s(ac,bc;n)$ *for all* c *coprime with* n*;*

(5) $s(a,b;n) = s(1,ba^{-1};n)$*, with* $a^{-1}a \equiv 1 \pmod{n}$*;*

(6) $s(1,a;n) = s(1,a^{-1};n)$*.*

Proof. (1) and (2) follow inmediately from the definition. Notice that $((x))$ is an odd function, which implies (3). For (4) see [HiZa74, p. 96]. (5) and (6) are consequences of (1), (2) and (4). □

The main tool to compute explicitily Dedekind sums is the following *Reciprocity law.*

Theorem 1.45 (Reciprocity law)**.** *If* a, n *are relatively prime integers, then*

$$s(1,a;n) + s(1,n;a) = \frac{1}{12}\left(\frac{n}{a} + \frac{1}{na} + \frac{a}{n}\right) - \frac{1}{4}.$$

Proof. See [HiZa74, Ch. II, §5, Thm. 1] □

Finally, we prove the following bounds for Dedekind sums.

Lemma 1.46. *Let* $0 < a < n$ *be relatively prime. Then*

(1) $s(1,1;n) > 2s(1,a;n)$ *if* $a \not\equiv 1$*;*

(2) $s(1,1;n) > 3s(1,a;n)$ *if* $a \not\equiv 1, 2, 2^{-1}$*;*

(3) $s(1,1;n) > 4s(1,a;n)$ *if* $a \not\equiv 1, 2, 2^{-1}, 3, 3^{-1}$*.*

Proof. First of all, using the Reciprocity law we have that $s(1,1;n) = (n-1)(n-2)/12n$ and

$$\begin{aligned} 2s(1,2;n) &= \frac{n^2-6n+5}{12n} < s(1,1;n) \\ 3s(1,3;n) &\leq \frac{n^2-7n+10}{12n} < s(1,1;n) \\ 4s(1,4;n) &\leq \frac{n^2-6n+17}{12n} < s(1,1;n) \end{aligned}$$

with $\gcd(n,2)=1$, $\gcd(n,3)=1$ and $\gcd(n,4)=1$ respectively. In [Girs16, Thm.1], the author describes how Dedekind sums $s(1,m;n)$ grow for a fixed m, given a positive integer k. To do so, Girstmair divides the numbers $1 \leq m \leq n-1$ as ordinary and not ordinary, and proves that if m is ordinary, then $s(1,m;n) \leq \dfrac{n}{12(k+1)} + O(1)$, and if m is not ordinary then there exists $d \in \{1,\dots,2k+1\}$ and $c \in \{0,1,\dots,d\}$, $\gcd(c,d)=1$, such that $s(1,m;n) = \dfrac{n}{12dq} + O(1)$, where $q = md - nc$.

First assume that $k=2$. Notice that $\frac{s(1,1;n)}{2} = \frac{n}{24} + O(1)$. If m is ordinary, then $s(1,m;n) \leq \dfrac{n}{36} + O(1)$, and if m is not ordinary and $dq \geq 3$, then $s(m,n) \leq \dfrac{n}{36} + O(1)$. Therefore, we have to find a bound for the three $O(1)$ involved, and find an N such that if $n > N$, then $s(1,1;n)/2 > s(1,m;n)$ for ordinary numbers and nonordinary numbers with $qd \geq 3$. The procedure to do so is shown by Girstmair in [Girs16, Thm. 2], and for the case $k=2$ such N is 132. The nonordinary numbers with $qd \leq 2$ correspond to $m \equiv 1, 2, 2^{-1}$, but the first case was ruled out in the proposition, and the inequality for 2 and 2^{-1} was shown at the beginning of the proof. Therefore, we have (1) for $n > 132$, and using a computer we can check that it holds true for every $n \leq 132$.

For $k=3$ and $k=4$ we obtain similar results, with $N=320$ and $N=630$ respectively. The cases with $qd \leq 3$ and $qd \leq 4$ are the ones ruled out in the proposition, and using a computer we can check that (2) and (3) are true for $n \leq 320$ and $n \leq 630$. □

Corollary 1.47.

(1) $2s(1,1;n) - 2s(1,2;n) + s(1,4;n) - s(1,3;n) + s(1,2\cdot 3^{-1};n) - s(1,4\cdot 3^{-1};n) > 0$ *for all* $n > 5$*;*

(2) $2s(1,1;n) - s(1,2;n) - s(1,3;n) - s(1,4;n) + s(1,6;n) - s(1,2\cdot 3^{-1};n) + s(1,4\cdot 3^{-1};n) > 0$ *for all* $n > 7$*;*

(3) $2s(1,1;n) - s(1,2;n) - s(1,3;n) - s(1,5;n) + s(1,6;n) + s(1,2\cdot 5^{-1};n) - s(1,6\cdot 5^{-1};n) > 0$ *for all* $n > 7$.

Proof. Using the inequalities from Lemma 1.46 we see that to prove (1) it is enough to prove that $\frac{2}{3}s(1,1;n) + s(1,4;n) + s(1,2\cdot 3^{-1};n) - s(1,4\cdot 3^{-1};n) > 0$. On the other hand, we have that $s(1,4;n) > 0$ if $n \notin \{7,13,19,25,31\}$, that $s(1,-2\cdot 3^{-1};n) < s(1,1;n)/3$ if $n \notin \{5,7\}$ and $s(1,4\cdot 3^{-1};n) < s(1,1;n)/3$ if $n \neq 5$. Therefore, if n is not one of those cases, then the inequality holds. We check the remaining cases and find that (1) is false only if $n = 5$. We repeat the same argument and prove that we have to check the cases when $n \in \{7,11,13,19,25,31\}$ for (2), and when $n \in \{7,13,19,31\}$ for (3). Both cases give us that (2) or (3) are false only if $n = 7$. $\square$

Chapter 2

Kollár hypersurfaces

As in the introduction, let $n \geq 3$ be an integer, and let $a_1, \ldots, a_n$ be positive integers such that there is no $(a_i, a_{i+2}, \ldots, a_{i+n-2}) = (1, \ldots, 1)$ when n is even. The indices are and will be taken modulo n. For every $1 \leq i \leq n$, we define the positive integers

$$W_i := \sum_{j=1}^{n} (-1)^{j-1} \prod_{l=i+j}^{i+n-1} a_l \quad \text{and} \quad D := \prod_{l=1}^{n} a_l + (-1)^{n-1}.$$

For example, for $n = 4$ we have

$$W_i = a_{i+1}a_{i+2}a_{i+3} - a_{i+2}a_{i+3} + a_{i+3} - 1 \quad \text{and} \quad D = a_1a_2a_3a_4 - 1.$$

The numbers W_i and D come as a solution of the following system of equations.

Proposition 2.1. *The system*

$$a_i x_i + x_{i+1} = 1 \;\; ; \quad i = 1, \ldots, n$$

has a unique solution given by

$$x_i = \frac{W_i}{D}.$$

Proof. The matrix associated to the system is

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & 1 & \ddots & \cdots & \vdots \\ 0 & 0 & a_3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & a_{n-1} & 1 \\ 1 & 0 & \cdots & \cdots & 0 & a_n \end{pmatrix}$$

and its determinar is exactly D. Now the result follows from Cramer's rule. □

Using Proposition 2.1 we have for $i = 1, \dots, n$

$$a_i W_i + W_{i+1} = D. \tag{2.1}$$

Define $w^* := \gcd(W_1, W_2, W_3, W_4)$. From Equation (2.1) we have that $w^* = \gcd(W_i, D) = \gcd(W_i, W_{i+1})$ for all i. Set

$$w_i := \frac{W_i}{w^*} \quad \text{and} \quad d := \frac{D}{w^*}.$$

Notice that $\gcd(w_i, w_{i+1}) = \gcd(w_i, d) = 1$, and that $\gcd(a_i, w^*) = 1$.

Definition 2.2. The *Kollár hypersurface* of type $(a_1, \dots, a_n)$ is

$$X(a_1, \dots, a_n) := (x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_n^{a_n} x_1 = 0) \subset \mathbb{P}(w_1, \dots, w_n)$$

Kollár proves in [Ko08, Thm.39] the following.

Theorem 2.3.

(1) The weighted projective space $\mathbb{P}(w_1, \dots, w_n)$ is well formed, and its singular set has dimension $\leq [n/2] - 1$.

(2) The hypersurface $X(a_1, \dots, a_n)$ is quasi-smooth, and $\mathbb{P}(w_1, \dots, w_n) \backslash X(a_1, \dots, a_n)$ is smooth.

(3) If $w^* = 1$, then $X(a_1, \dots, a_n)$ is birational to $\mathbb{P}^{n-2}$.

To prove (3) above, Kollár uses the linear system $|x_1^{a_1} x_2, x_2^{a_2} x_3, \dots, x_n^{a_n} x_1|$. In general, this linear system defines a rational map

$$\psi \colon \mathbb{P}(w_1, \dots, w_n) \dashrightarrow \mathbb{P}^{n-1}_{y_1, \dots, y_n}$$

given by $y_i = x_i^{a_i} x_{i+1}$.

Proposition 2.4. *The rational map ψ defines the field extension*

$$\mathbb{C}(y_1/y_n, \ldots, y_{n-1}/y_n) \subset \mathbb{C}(y_1/y_n, \ldots, y_{n-1}/y_n)[z]/(z^{w^*} - f/y_n^{W_1})$$

where $z = x_1^d/y_n^{w_1}$ and $f = y_1^{a_2a_3\cdots a_n} y_2^{-a_3\cdots a_n} y_3^{a_4\cdots a_n} \cdots y_{n-1}^{(-1)^{n-2}a_n} y_n^{(-1)^{n-1}}$.

Proof. At the affine cover level, the field extension induced by ψ is

$$\mathbb{C}(y_1, \ldots, y_n) \subset \mathbb{C}(y_1, \ldots, y_n)[x_1]/(x_1^D - f)$$

where the other variables $x_2, \ldots, x_n$ can be written using $y_1, \ldots, y_n, x_1$ as

$$\begin{aligned} x_2 &= \frac{y_1}{x_1^{a_1}} \\ x_3 &= \frac{x_1^{a_1a_2} y_2}{y_1} \\ x_4 &= \frac{y_1^{a_2a_3} y_3}{x_1^{a_1a_2a_3} y_2^{a_3}} \\ &\vdots \; . \end{aligned}$$

The action of $\mathbb{C}^*$ compatible with the map is: Given $\lambda \in \mathbb{C}^*$, $y_i \mapsto \lambda^d y_i$ and $x_i \mapsto \lambda^{w_i} x_i$. Then the rational map ψ is determined by

$$\big(\mathbb{C}(y_1, \ldots, y_n)\big)^{\mathbb{C}^*} \subset \big(\mathbb{C}(y_1, \ldots, y_n)[x_1]/(x_1^D - f)\big)^{\mathbb{C}^*}.$$

Notice that $\big(\mathbb{C}(y_1, \ldots, y_n)\big)^{\mathbb{C}^*} = \mathbb{C}(y_1/y_n, \ldots, y_{n-1}/y_n)$, and that $z = x_1^d/y_n^{w_1}$ is a $\mathbb{C}^*$-invariant element such that $z^{w^*} - f/y_n^{W_1} = 0$. Since geometrically the map ψ has degree w^*, then

$$\big(\mathbb{C}(y_1, \ldots, y_n)[x_1]/(x_1^D - f)\big)^{\mathbb{C}^*} = \mathbb{C}(y_1/y_n, \ldots, y_{n-1}/y_n)[z]/(z^{w^*} - f/y_n^{W_1}).$$

□

Corollary 2.5. *The corresponding restriction map*

$$\psi|_X \colon X(a_1, \ldots, a_n) \dashrightarrow \mathbb{P}^{n-2} = \{y_1 + \ldots + y_n = 0\}$$

is cyclic of degree w^* totally branch along $(y_1 \cdots y_n = 0) \subset \mathbb{P}^{n-2}$.

In this way, we can write down another normal projective model Y' of $X(a_1, \ldots, a_n)$ using a w^*-th root cover as described in Section 1.6.

As in the introduction, let $0 < \mu_i < w^*$ be such that

$$\mu_i \equiv (-1)^{i+1} \prod_{l=i+1}^{i+n-1} a_l \pmod{w^*}.$$

In $\mathbb{P}^{n-2} = \{y_1 + \ldots + y_n = 0\}$, we write $L_i := \{y_i = 0\}$, and so

$$\mathcal{O}_{\mathbb{P}^{n-2}}(t)^{\otimes w^*} \simeq \mathcal{O}_{\mathbb{P}^{n-2}}(\mu_1 L_1 + \ldots + \mu_n L_n),$$

where $tw^* = \sum_{i=1}^n \mu_i$. Then

$$Y_0 := \mathrm{Spec}_{\mathbb{P}^{n-2}} \Big(\bigoplus_{i=0}^{w^*-1} \mathcal{O}_{\mathbb{P}^{n-2}}(-ti) \Big) \to \mathbb{P}^{n-2}$$

is the cyclic cover given by $z^{w^*} - f/y_n^{W_1}$ above. We want to consider the normalization of Y_0. As in 1.6, we define the line bundles $\mathcal{L}^{(i)}$ on $\mathbb{P}^{n-2}$ as

$$\mathcal{L}^{(i)} := \mathcal{O}_{\mathbb{P}^{n-2}}(ti) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}\Big(- \sum_{j=1}^{n} \Big[\frac{\mu_j \, i}{w^*} \Big] L_j \Big)$$

for $i \in \{0, 1, ..., w^* - 1\}$. Then, the normalization of Y_0 is $Y' := \mathrm{Spec}_{\mathbb{P}^{n-2}} \Big(\bigoplus_{i=0}^{w^*-1} {\mathcal{L}^{(i)}}^{-1} \Big)$. Notice that $\gcd(\mu_i, w^*) = 1$, and so this cyclic morphism is totally branch at the L_i's.

Corollary 2.6. *There is a birational map* $X(a_1, \ldots, a_n) \dashrightarrow Y'$.

In the next section we describe explicitly this birational map for $n = 4$.

2.1 Explicit birational map for Kollár surfaces

From now on we concentrate in the case of Kollár surfaces, where $n = 4$. Let $X(a_1, a_2, a_3, a_4)$ be a Kollár surface. Let

$$p_1 = (1:0:0:0), \; p_2 = (0:1:0:0), \; p_3 = (0:0:1:0), \; p_4 = (0:0:0:1).$$

Proposition 2.7. *The surface* $X(a_1, a_2, a_3, a_4)$ *is normal, and it has only singularities of type* $\frac{1}{w_i}(w_{i+2}, w_{i+3})$ *at the points* p_i *when* $gcd(w_i, w_{i+2}) = 1$*, and of type* $\frac{1}{t_i}(t_{i+2}, w_{i+3})$ *when* $gcd(w_i, w_{i+2}) = h > 1$*, where* $w_j = ht_j$.

Proof. Here we follow the idea in [Ian00, §10.1]. Without loss of generality, it is enough to check the singularity at p_1. Consider the affine cone $C_X \subset \mathbb{C}^4$ of $X(a_1, a_2, a_3, a_4)$ (see Definition 1.31) and the corresponding action of $\mathbb{C}^*$ given by

$$\lambda \in \mathbb{C}^*, \qquad \lambda \cdot (x_1, x_2, x_3, x_4) = (\lambda^{w_1} x_1, \lambda^{w_2} x_2, \lambda^{w_3} x_3, \lambda^{w_4} x_4).$$

Then to study the singularities around p_1, we check how the action behaves when we restrict to $(x_1 = 1)$. Notice that, when $x_1 \neq 0$,

$$\frac{\partial}{\partial x_2}(x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_4 + x_4^{a_4} x_1) = x_1^{a_1} + a_2 x_2^{a_2 - 1} x_3 \neq 0,$$

so locally, by the Implicit Function Theorem, we can write x_2 as a function of x_3 and x_4, which become local parameters. Then the action of $\mathbb{C}^*$ restricted to $(x_1 = 1)$ is

$$\zeta_1 \cdot (1, x_2, x_3, x_4) = (1, \zeta_1^{w_2} x_2, \zeta_1^{w_3} x_3, \zeta_1^{w_4} x_4),$$

where ζ_1 is a w_1-th primitive root of 1. Therefore, after taking the quotient, the singularity is a cyclic singularity of type $\frac{1}{w_1}(w_3, w_4)$, if $gcd(w_i, w_{i+2}) = 1$. If $gcd(w_i, w_{i+2}) = h > 1$, then there are elements which fix the axis $(x_3 = 0)$, so they are quasi-reflections. We eliminate them by dividing $w_i = ht_i$ and $w_{i+2} = ht_{i+2}$ by h, obtaining that the singularity is $\frac{1}{t_i}(t_{i+2}, w_{i+3})$. □

Assume that $a_i \geq 2$ for all i [1]. We have the following key configuration of curves on $X(a_1, a_2, a_3, a_4)$:

$$\begin{aligned}
C_1 &:= (x_1 = x_3 = 0) \\
C_2 &:= (x_2 = x_4 = 0) \\
\Gamma_{1,2} &:= (x_3 = x_4^{a_4} + x_1^{a_1 - 1} x_2 = 0) \\
\Gamma_{2,3} &:= (x_4 = x_1^{a_1} + x_2^{a_2 - 1} x_3 = 0) \\
\Gamma_{3,4} &:= (x_1 = x_2^{a_2} + x_3^{a_3 - 1} x_4 = 0) \\
\Gamma_{4,1} &:= (x_2 = x_3^{a_3} + x_4^{a_4 - 1} x_1 = 0)
\end{aligned}$$

Proposition 2.8. *The curves C_1, C_2 are smooth and rational. The curve $\Gamma_{i,j}$ is rational, and it may only have a unibranch singularity at p_j.*

Proof. The curves C_1, C_2 are isomorphic to $\mathbb{P}^1$ by Corollary 1.7. To prove the assertion about $\Gamma_{i,j}$, it is enough to do it for $\Gamma_{2,3}$. Notice that this curve lives in $(x_4 = 0) = \mathbb{P}(w_1, w_2, w_3)$, and that it is possibly singular only at $(0 : 0 : 1)$. Let us consider the $\mathbb{Z}/w_1 \oplus \mathbb{Z}/w_2 \oplus \mathbb{Z}/w_3$ quotient map

$$\mathbb{P}^2 \to \mathbb{P}(w_1, w_2, w_3)$$

[1] This is to have the key configuration of curves as shown. By Theorem 2.28, Kollár surfaces with $a_i = 1$ are birationally included in our analysis. Also, check Corollary 2.27 when $w^* = 1$.

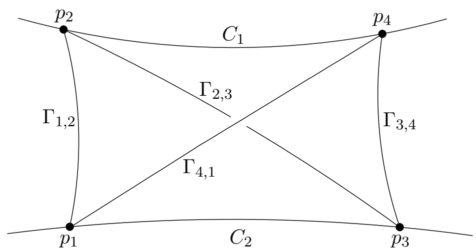


Figure 2.1: Key configuration of curves on a Kollár surface.

given by $(x : y : z) \mapsto (x^{w_1} : y^{w_2} : z^{w_3})$. Then the preimage of $\Gamma_{2,3}$ is

$$\Gamma'_{2,3} = (x^{w_1 a_1} + y^{w_2(a_2-1)} z^{w_3} = 0),$$

and so $\Gamma_{2,3}$ is rational since all irreducible components (branches at $(0 : 0 : 1)$) of $\Gamma'_{2,3}$ are rational curves.

To see that $\Gamma_{2,3}$ is unibranch at $(0 : 0 : 1)$, we will show that the (possible) branches of $\Gamma'_{2,3}$ form one orbit under the $\mathbb{Z}/w_1 \oplus \mathbb{Z}/w_2 \oplus \mathbb{Z}/w_3$ action. We take the canonical affine chart at $(0 : 0 : 1)$, where $\Gamma'_{2,3} = (x^{w_1 a_1} + y^{w_2(a_2-1)} = 0)$. We consider the action of $\mathbb{Z}/w_3$ given by $(x, y) \mapsto (\zeta_3^k x, \zeta_3^k y)$ where $k \in \mathbb{Z}$ and $\zeta_3 = e^{\frac{2\pi i}{w_3}}$. Notice that $\gcd(w_2, w_1) = 1$ and $\gcd(w_2, a_1) = 1$ by definition, and so we write $a_2 - 1 = rb$ and $w_1 a_1 = ra$ where $\gcd(a, b) = 1$, to factor in branches

$$x^{w_1 a_1} + y^{w_2(a_2-1)} = \prod_{c=0}^{r-1} (y^{w_2 b} - \zeta_{2r}^{2c+1} x^a)$$

where $\zeta_{2r} = e^{\frac{\pi i}{r}}$. Then we take $y^{w_2 b} - \zeta_{2r} x^a$ and apply $(x, y) \mapsto (\zeta_3^k x, \zeta_3^k y)$ to obtain the branch $y^{w_2 b} - \zeta_{2r} \zeta_3^{k(a - w_2 b)} x^a$, but $a - w_2 b = \frac{w_3}{r}$, and so it goes to $y^{w_2 b} - \zeta_{2r}^{2k+1} x^a$. Therefore branches form one orbit, and the curve $\Gamma_{2,3}$ is unibranch at $(0 : 0 : 1)$. □

Proposition 2.9. *Assume that $a_i > w^*$ for some i. Then $\Gamma_{i+2,i+3}$ is nonsingular.*

Proof. We take $a_1 > w^*$ to prove that $\Gamma_{3,4}$ is nonsingular. For this we will compute the arithmetic genus of $\Gamma_{3,4}$. Let $\mathbb{P} = \mathbb{P}(w_2, w_3, w_4)$, and consider the exact sequence of sheaves $0 \to \mathcal{O}_{\mathbb{P}}(-a_2 w_2) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\Gamma_{3,4}} \to 0$. From it we have that $\chi(\mathcal{O}_{\Gamma_{3,4}}) = \chi(\mathcal{O}_{\mathbb{P}}) - \chi(\mathcal{O}_{\mathbb{P}}(-a_2 w_2))$. If $\gcd(w_2, w_4) = 1$, then by Proposition 1.35 we have that $\chi(\mathcal{O}_{\mathbb{P}}) - \chi(\mathcal{O}_{\mathbb{P}}(-a_2 w_2)) = 1 - h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a_2 w_2 - w_2 - w_3 - w_4))$. Therefore

$$p_a(\Gamma_{3,4}) = 1 - \chi(\mathcal{O}_{\Gamma_{3,4}}) = h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a_2 w_2 - w_2 - w_3 - w_4)),$$

so we have to compute the number of nonnegative integer solutions of the equation $w_2x+w_3y+w_4z = a_2w_2 - w_2 - w_3 - w_4$. As $a_2w_2 + w_3 = a_3w_3 + w_4$, then our equation can be written as

$$w_2(x + a_2z) + w_3(y + (1 - a_3)z) = (a_3 - 2)w_3 - w_2$$

and its solutions are

$$x = -1 - tw_3 - a_2z \quad , \quad y = a_3 - 2 + tw_2 + (a_3 - 1)z \quad , \quad z = z \tag{2.2}$$

If x, y and z are nonnegative, then $t < 0$, so we will change the sign of t and assume that $t > 0$. Then from Equations (2.2) we obtain that

$$a_2z \leq tw_3 - 1$$

and $(a_3 - 1)z \geq tw_2 - a_3 + 2$. Hence we have that

$$\frac{tw_3 - 1}{a_2} \geq z \geq \frac{tw_2 + 2 - a_3}{a_3 - 1} \tag{2.3}$$

Replacing with $w_2 = \frac{1}{w^*}(a_3a_4a_1 - a_4a_1 + a_1 - 1)$ and $w_3 = \frac{1}{w^*}(a_4a_1a_2 - a_1a_2 + a_2 - 1)$ we obtain

$$ta_4a_1 - t(a_1 - 1) - \frac{t + w^*}{a_2} \geq w^*z \geq ta_4a_1 - w^* + \frac{t(a_1 - 1) + w^*}{a_3 - 1}.$$

Because $a_1 > w^*$ and $t \geq 1$, then $t(a_1 - 1) \geq w^*$, so $ta_4a_1 - w^* \geq ta_4a_1 - t(a_1 - 1)$. We have that both $\frac{t+w^*}{a_2}$ and $\frac{t(a_1-1)+w^*}{a_3-1}$ are positive, therefore the RHS of the system (2.3) is greater than the LHS, so the system has no solution. Hence the arithmetic genus of $\Gamma_{3,4}$ is zero and therefore nonsingular.

If $\gcd(w_2, w_4) = h > 1$, then $p_a(\Gamma_{3,4}) = h^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-a_2w_2))$. To compute it, we first have to consider the well formed weighted projective plane $\mathbb{P}' = \mathbb{P}(t_2, w_3, t_4) \simeq \mathbb{P}$, where $t_2 = w_2/h$ and $t_4 = w_4/h$, and following 1.10, we have that $\mathcal{O}_{\mathbb{P}}(-a_2w_2) \simeq \mathcal{O}_{\mathbb{P}'}(-a_2t_2)$. Then $p_a(\Gamma_{3,4}) = h^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(a_2t_2 - t_2 - w_3 - t_4))$, which is equivalent to the number of nonnegative integer solutions of the equation

$$t_2x + w_3y + t_4z = a_2t_2 - t_2 - w_3 - t_4.$$

The general solution of this equation is

$$x = -1 - tw_3 - a_2z \quad , \quad y = \frac{a_3 - 1}{h} - 1 + t_2t + \frac{a_3 - 1}{h}z \quad , \quad z = z,$$

with $t \in \mathbb{Z}$. Then $t < 0$, and changing the sign of t as above, we have that the arithmetic genus is equal to the number of solutions of the system

$$a_1a_4t - t(a_1 - 1) - \frac{t + w^*}{a_2} \geq w^*z \geq a_1a_4t - w^* + \frac{hw^* + (a_1 - 1)t}{a_3 - 1},$$

but again, as $a_i > w^*$, the RHS is greater than the LHS, so the arithmetic genus is 0. □

Proposition 2.10. *The map ψ is defined precisely in $X(a_1,a_2,a_3,a_4)\setminus\{p_1,p_2,p_3,p_4\}$, and it contracts*

$$\psi(C_1\setminus\{p_2,p_4\})=(0:1:0:-1)\qquad \psi(C_2\setminus\{p_1,p_3\})=(1:0:-1:0)$$

$$\psi(\Gamma_{1,2}\setminus\{p_1,p_2\})=(-1:0:0:1)\qquad \psi(\Gamma_{2,3}\setminus\{p_2,p_3\})=(1:-1:0:0)$$

$$\psi(\Gamma_{3,4}\setminus\{p_3,p_4\})=(0:1:-1:0)\qquad \psi(\Gamma_{4,1}\setminus\{p_4,p_1\})=(0:0:1:-1)$$

Proof. We have that $\psi|_{\Gamma_{1,2}\setminus\{p_1,p_2\}}=(x_1^{a_1-1}x_2:0:0:x_4^{a_4})$, and because $x_1^{a_1-1}x_2=-x_4^{a_4}$ over $\Gamma_{1,2}$, then $\psi|_{\Gamma_{1,2}\setminus\{p_1,p_2\}}=(-1:0:0:1)$. This gives the result for all curves $\Gamma_{i,i+1}$.

For C_1, let $x_4=1$ and $x_2=b\neq 0$. Then the equation of the surface with these restrictions is

$$bx_1^{a_1}+b^{a_2}x_3+x_3^{a_3}+x_1=x_1(1+bx_1^{a_1-1})+x_3(b^{a_2}+x_3^{a_3-1})=0.$$

The map is $\psi(x_1:b:x_3:1)=(bx_1^{a_1}:b^{a_2}x_3:x_3^{a_3}:x_1)$. We multiply every coordinate by $(1+bx_1^{a_1-1})$, and use the relation $x_1(1+bx_1^{a_1-1})=-x_3(b^{a_2}+x_3^{a_3-1})$, to write down $\psi(x_1:b:x_3:1)$ as

$$(bx_1^{a_1}(1+bx_1^{a_1-1}):b^{a_2}x_3(1+bx_1^{a_1-1}):x_3^{a_3}(1+bx_1^{a_1-1}):x_1(1+bx_1^{a_1-1}))=$$

$$(-x_3bx_1^{a_1-1}(b^{a_2}+x_3^{a_3-1}):b^{a_2}x_3(1+bx_1^{a_1-1}):x_3^{a_3}(1+bx_1^{a_1-1}):-x_3(b^{a_2}+x_3^{a_3-1}))$$
$$=(-bx_1^{a_1-1}(b^{a_2}+x_3^{a_3-1}):b^{a_2}(1+bx_1^{a_1-1}):x_3^{a_3-1}(1+bx_1^{a_1-1}):-(b^{a_2}+x_3^{a_3-1})).$$

Hence $\psi(0:b:0:1)=(0:b^{a_2}:0:-b^{a_2})=(0:1:0:-1)$. A similar argument works for C_2. □

Remark 2.11. By Theorem 2.28, we know that any $X(a_1,a_2,a_3,a_4)$ has a birational model $X(a_1',a_2',a_3',a_4')$ with $\gcd(w_i',w_{i+2}')=1$. **From now on, we assume that**

$$\gcd(w_1,w_3)=\gcd(w_2,w_4)=1.$$

Now we want to study the behavior of ψ on a resolution of the singularities in $X(a_1,a_2,a_3,a_4)$. To do so, we need to write this map in terms of local coordinates in the resolution, which are described in the following theorem.

The main theorem of this section is the following.

Theorem 2.12. *Let $\sigma\colon \tilde{X} \to X(a_1, a_2, a_3, a_4)$ be the minimal resolution, and let*

$$\hat{X} \xrightarrow{\varphi} \tilde{X} \xrightarrow{\sigma} X(a_1, a_2, a_3, a_4)$$

be the minimal log resolution of X together with the key configuration of curves. Then $\psi \circ \sigma \circ \varphi$ is a morphism.

To prove Theorem 2.12 we have to compute the strict transform of the curves $\Gamma_{i,i+1}$ on $\tilde{X}$. As in Section 1.5, let $E_{i,j}$ be the components of the exceptional divisor over the point p_i, let $\frac{1}{w_i}(w_{i+2}, w_{i+3}) = \frac{1}{w_i}(1, q_i)$, and let $\alpha_{i,j}$, $\beta_{i,j}$ and $\gamma_{i,j}$ the integers defined for the continued fraction of $\frac{w_i}{q_i}$. Recall from the proof of Proposition 2.7 that x_{i+2} and x_{i+3} are toric local coordinates at p_i, so we have that $E_{i,0}$ and E_{i,s_i+1} are the strict transform of $(x_{i+3} = 0)$ and $(x_{i+2} = 0)$ at the open set $(x_i \neq 0)$. This means that $E_{1,0} = E_{3,0}$ and $E_{2,0} = E_{4,0}$ and correspond to the strict transform of C_2 and C_1 respectively. On the other hand, E_{i,s_i+1} corresponds to the strict transform of the curve $\Gamma_{i,i+1}$. Then it remains to compute the strict transform of $\Gamma_{i,i+1}$ around the point p_{i+1}, and without loss of generality, we will compute the strict transform $\Gamma_{3,4}$ at the point p_4. As all the results will hold locally for $\Gamma_{3,4}$, we can modify the following proofs for every $\Gamma_{i,i+1}$.

Proposition 2.13. *Let $U_{4,j}$ the open sets of the resolution of $\frac{1}{w_4}(1, q_4)$ as defined in Theorem 1.39. Then the local equation of the strict transform of the curve $\Gamma_{3,4}$ restricted to the open set $U_{4,j}$ is*

$$\Gamma'_{34} = \begin{cases} 1 + u_j^{((a_3-1)\beta_{4,j+1} - a_2\alpha_{4,j+1})/w_4} v_j^{((a_3-1)\beta_{4,j} - a_2\alpha_{4,j})/w_4} = 0 \\ u_j^{(a_2\alpha_{4,j+1} - (a_3-1)\beta_{4,j+1})/w_4} v_j^{(a_2\alpha_{4,j} - (a_3-1)\beta_{4,j})/w_4} + 1 = 0 \\ u_j^{(a_2\alpha_{4,j+1} - (a_3-1)\beta_{4,j+1})/w_4} + v_j^{((a_3-1)\beta_{4,j} - a_2\alpha_{4,j})/w_4} = 0 \end{cases},$$

if

$$a_2\alpha_{4,j} - (a_3 - 1)\beta_{4,j} < a_2\alpha_{4,j+1} - (a_3 - 1)\beta_{4,j+1} \leq 0$$
$$0 \leq a_2\alpha_{4,j} - (a_3 - 1)\beta_{4,j} < a_2\alpha_{4,j+1} - (a_3 - 1)\beta_{4,j+1},$$
$$a_2\alpha_{4,j} - (a_3 - 1)\beta_{4,j} \leq 0 \leq a_2\alpha_{4,j+1} - (a_3 - 1)\beta_{4,j+1},$$

respectively.

Proof. We can assume that $x_4 = 1$ and $x_1 = 0$, so we must study the curve $(x_2^{a_2} + x_3^{a_3-1} = 0) \subset (x_4 \neq 0) \subset \mathbb{P}(w_2, w_3, w_4)$. By Theorem 1.39, to find the total transform of $\Gamma_{3,4}$ in U_i we replace x_2 and x_3 with $u_i^{\alpha_{4,i+1}/w_4} v_i^{\alpha_{4,i}/w_4}$ and $u_i^{\beta_{4,i+1}/w_4} v_i^{\beta_{4,i}/w_4}$ respectively, where u_i and v_i are the local

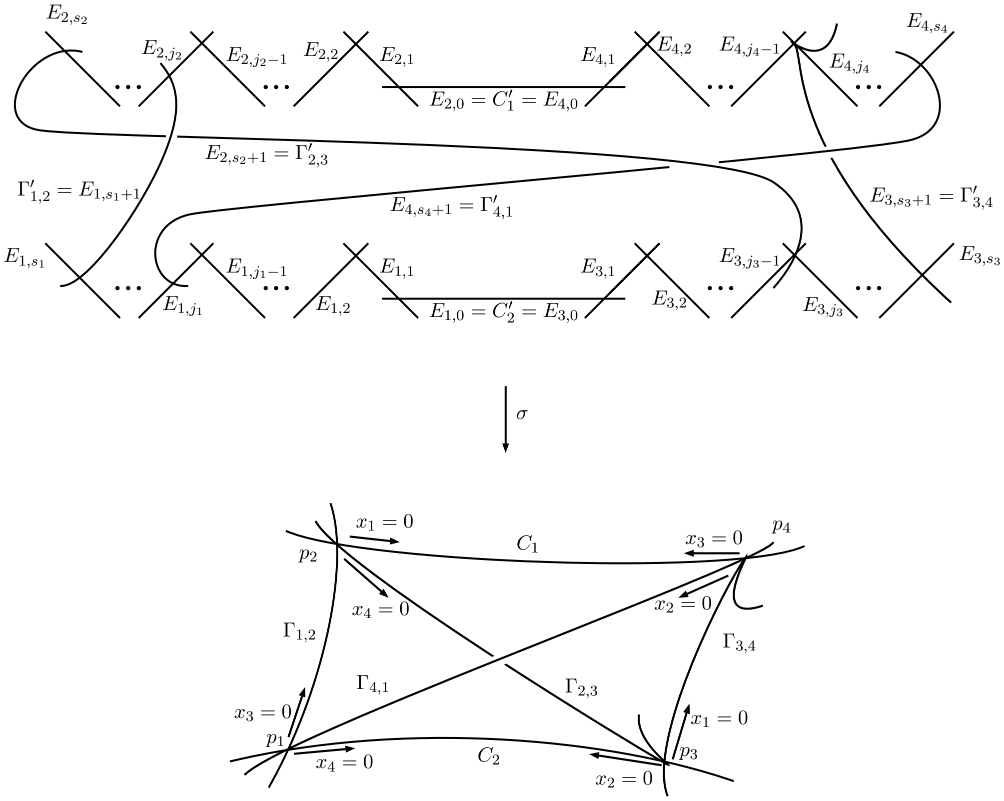


Figure 2.2: Key configuration of curves on $X(a_1, a_2, a_3, a_4)$ and the curve configuration of the minimal resolution $\tilde{X}$.

coordinates of U_i, and so the total transform is

$$(u_i^{\alpha_{4,i+1}/w_4} v_i^{\alpha_{4,i}/w_4})^{a_2} + (u_i^{\beta_{4,i+1}/w_4} v_i^{\beta_{4,i}/w_4})^{a_3-1} = 0.$$

Recall that $\alpha_{4,i} < \alpha_{4,i+1}$ and $\beta_{4,i+1} < \beta_{4,i}$, so

$$a_2\alpha_{4,i} - (a_3 - 1)\beta_{4,i} < a_2\alpha_{4,i+1} - (a_3 - 1)\beta_{4,i+1}.$$

Thus if $a_2\alpha_{4,i} - (a_3 - 1)\beta_{4,i} < a_2\alpha_{4,i+1} - (a_3 - 1)\beta_{4,i+1} \leq 0$, we factor out $(u_i^{\alpha_{4,i+1}/w_4} v_i^{\alpha_{4,i}/w_4})^{a_2}$. If $0 \leq a_2\alpha_{4,i} - (a_3 - 1)\beta_{4,i} < a_2\alpha_{4,i+1} - (a_3 - 1)\beta_{4,i+1}$, we factor out $(u_i^{\beta_{4,i+1}/w_4} v_i^{\beta_{4,i}/w_4})^{a_3-1}$. If $a_2\alpha_{4,i} - (a_3 - 1)\beta_{4,i} \leq 0 \leq a_2\alpha_{4,i+1} - (a_3 - 1)\beta_{4,i+1}$, we factor out $u_i^{((a_3-1)\beta_{4,i+1})/w_4}$ and $v_i^{a_2\alpha_{4,i}/w_4}$, obtaining what we wanted to prove. □

Notice that $\Gamma'_{3,4}$ intersects the exceptional divisor in U_i if and only if

$$a_2\alpha_{4,i} - (a_3 - 1)\beta_{4,i} \leq 0 \leq a_2\alpha_{4,i+1} - (a_3 - 1)\beta_{4,i+1}.$$

If $a_2\alpha_{4,i} - (a_3-1)\beta_{4,i} < 0 < a_2\alpha_{4,i+1} - (a_3-1)\beta_{4,i+1}$, then the curve intersects two components of the exceptional divisor, and if $a_2\alpha_{4,i} - (a_3-1)\beta_{4,i} = 0$ or $a_2\alpha_{4,i+1} - (a_3-1)\beta_{4,i+1} = 0$, then it intersects only one component.

Proposition 2.14. *Let us say that $\Gamma'_{3,4}$ intersects the exceptional divisor over p_4 at the components $E_{4,j}$ and $E_{4,j+1}$ with multiplicity m_j and m_{j+1} respectively (possibly $m_{j+1} = 0$). Then $a_3 - 1 = \alpha_{4,j}m_j + \alpha_{4,j+1}m_{j+1}$ and $a_2 = \beta_{4,j}m_j + \beta_{4,j+1}m_{j+1}$.*

Proof. Let H be the restriction to $X(a_1,a_2,a_3,a_4)$ of a generator of the class group of $\mathbb{P}(w_1,w_2,w_3,w_4)$. We have that

$$w_1H \cdot w_2H = \frac{w_1w_2(a_3w_3+w_4)}{w_1w_2w_3w_4} = \frac{1}{w_3} + \frac{a_3}{w_4}.$$

On the other hand, $w_1H \cdot w_2H = \sigma^*(w_1H)\cdot\sigma^*(w_2H)$, where $\sigma^*(w_1H) = \sigma^*(\Gamma_{3,4}+C_1)$, and $\sigma^*(w_2H) = \sigma^*(\Gamma_{4,1}+C_2)$. Because the pull-back of a divisor has intersection zero with any component of the exceptional divisor, and using the pull-back formulas in (1.2) we have that

$$\begin{aligned}
\sigma^*(w_1H)\cdot\sigma^*(w_2H) &= (\Gamma'_{3,4}+C'_1)\cdot(\sum_{i=0}^{s_3+1}\frac{\beta_{3,i}}{w_3}E_{3,i} + \sum_{i=0}^{s_4+1}\frac{\alpha_{4,i}}{w_4}E_{4,i}) \\
&= \Gamma'_{3,4}\cdot\sum_{i=0}^{s_3+1}\frac{\beta_{3,i}}{w_3}E_{3,i} + C'_1\cdot\sum_{i=0}^{s_4+1}\frac{\alpha_{4,i}}{w_4}E_{4,i} + \Gamma'_{3,4}\cdot\sum_{i=0}^{s_4+1}\frac{\alpha_{4,i}}{w_4}E_{4,i} \\
&= \frac{1}{w_3} + \frac{1}{w_4} + \sum_{i=0}^{s_4+1}\frac{\alpha_{4,i}}{w_4}\Gamma'_{3,4}\cdot E_{4,i}.
\end{aligned}$$

Then $a_3 - 1 = \alpha_{4,j}\Gamma'_{3,4}\cdot E_{4,j} + \alpha_{4,j+1}\Gamma'_{3,4}\cdot E_{4,j+1} = \alpha_{4,j}m_j + \alpha_{4,j+1}m_{j+1}$. To simplify the computation of the second equality, we will restrict to the plane $\mathbb{P}(w_2,w_3,w_4)$, with L a generator of the class group. We can do this because at the point p_4 the singularity is the same as the one at the point $(0:0:1) \in \mathbb{P}(w_2,w_3,w_4)$, so locally σ does not change.

Then $w_3L \cdot a_2w_2L = \frac{a_2w_2w_3}{w_2w_3w_4} = \frac{a_2}{w_4}$ and also

$$\sigma^*(w_3L)\cdot\sigma^*(a_2w_2L) = \Gamma'_{3,4}\cdot\sum_{i=0}^{s_4+1}\frac{\beta_{4,i}}{w_4}E_{4,i},$$

where $\sigma^*(w_3L) = \sigma^*(C_1)$ and $\sigma^*(a_2w_2L) = \sigma^*(\Gamma_{3,4})$. Then $a_2 = \beta_{4,j}m_j + \beta_{4,j+1}m_{j+1}$. □

Corollary 2.15. *If $\Gamma'_{3,4}$ intersects the exceptional divisor in one component, then it does it transversally.*

Proof. Recall that in the open subset $U_{4,i}$, the exponents of the variables u_i and v_i of the strict transform of $\Gamma_{3,4}$ are $\pm(a_2\alpha_{4,i+1} - (a_3-1)\beta_{4,i+1})/w_4$ and $\pm(a_2\alpha_{4,i} - (a_3-1)\beta_{4,i})/w_4$.

Suppose that $\Gamma'_{3,4}$ intersects E_j with multiplicity m_j. Then, using Proposition 2.14, we have that for all i

$$\frac{a_2\alpha_{4,i} - (a_3-1)\beta_{4,i}}{w_4} = m_j\frac{\beta_{4,j}\alpha_{4,i} - \alpha_{4,j}\beta_{4,i}}{w_4},$$

but the singularity at p_4 was unibranch, so it is locally irreducible. Therefore the exponents on the resolution must be relatively prime. Thus $m_j = 1$. □

Theorem 2.16. *The curve $\Gamma'_{3,4}$ intersects the exceptional divisor in one component if and only if $\psi \circ \sigma$ is defined on the whole exceptional divisor over p_4.*

Proof. The equation of our surface is $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0$, so locally at p_4 our surface is $(x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3} + x_1 = 0)$. Then analytically the power series expansion of x_1 in terms of x_2 and x_3 is

$$x_1 = -x_2^{a_2}x_3 - x_3^{a_3} + (\text{higher order terms in } x_2 \text{ and } x_3).$$

Therefore, at the open set U_i

$$\begin{aligned}\sigma^*(x_1) &= -(u_i^{\alpha_{4,i+1}/w_4}v_i^{\alpha_{4,i}/w_4})^{a_2}(u_i^{\beta_{4,i+1}/w_4}v_i^{\beta_{4,i}/w_4}) - (u_i^{\beta_{4,i+1}/w_4}v_i^{\beta_{4,i}/w_4})^{a_3}\\ &\quad +(\text{higher order terms}).\end{aligned}$$

and so

$$\begin{aligned}\psi\circ\sigma|_{U_i} = ((^*) : u_i^{(a_2\alpha_{4,i+1}+\beta_{4,i+1})/w_4}v_i^{(a_2\alpha_{4,i}+\beta_{4,i+1})/w_4} : u_i^{a_3\beta_{4,i+1}/w_4}v_i^{a_3\beta_{4,i}/w_4} :\\ -u_i^{(a_2\alpha_{4,i+1}+\beta_{4,i+1})/w_4}v_i^{(a_2\alpha_{4,i}+\beta_{4,i+1})/w_4} - u_i^{a_3\beta_{4,i+1}/w_4}v_i^{a_3\beta_{4,i}/w_4} + (^*)),\end{aligned}$$

where $(^*)$ are terms in u_i and v_i of degree higher than $(a_2\alpha_{4,i+1} + \beta_{4,i+1} + a_2\alpha_{4,i} + \beta_{4,i+1})/w_4$ and $(a_3\beta_{4,i+1} + a_3\beta_{4,i})/w_4$.

Assume now that u_i and v_i are both nonzero. If $a_2\alpha_{4,i} - (a_3-1)\beta_{4,i} < a_2\alpha_{4,i+1} - (a_3-1)\beta_{4,i+1} < 0$, then we can factor out

$$(u_i^{\alpha_{4,i+1}/w_4}v_i^{\alpha_{4,i}/w_4})^{a_2}(u_i^{\beta_{4,i+1}/w_4}v_i^{\beta_{4,i}/w_4})$$

from $\psi\circ\sigma$ to obtain

$$\psi\circ\sigma|_{U_i} = ((^*) : 1 : u_i^{(a_2\alpha_{4,i+1}-(a_3-1)\beta_{4,i+1})/w_4}v_i^{(a_2\alpha_{4,i}-(a_3-1)\beta_{4,i})/w_4} : -1 + (^*))$$

Then $(\psi\circ\sigma|_{U_i})(u_i,0) = (\psi\circ\sigma|_{U_i})(0,v_i) = (0:1:0:-1)$. Repeating the same procedure for $0 < a_2\alpha_{4,i} - (a_3-1)\beta_{4,i} < a_2\alpha_{4,i+1} - (a_3-1)\beta_{4,i+1}$, we obtain that restricted to that open set U_i,

$$(\psi\circ\sigma|_{U_i})(u_i,0) = (\psi\circ\sigma|_{U_i})(0,v_i) = (0:0:1:-1).$$

Now we are left with the case $a_2\alpha_{4,i}-(a_3-1)\beta_{4,i} \le 0 \le a_2\alpha_{4,i+1}-(a_3-1)\beta_{4,i+1}$. Suppose first that the curve $\Gamma'_{3,4}$ intersect transversally the exceptional divisor, so we know that there is some j such that $a_2\alpha_{4,j} - (a_3-1)\beta_{4,j} = 0$, and by Corollary 2.15, $a_2\alpha_{4,j+1} - (a_3-1)\beta_{4,j+1} = 1$, and $a_2\alpha_{4,j-1} - (a_3-1)\beta_{4,j-1} = -1$. Then in U_{j-1} we can still factor out

$$(u_i^{\alpha_{4,i+1}/w_4} v_i^{\alpha_{4,i}/w_4})^{a_2}(u_i^{\beta_{4,i+1}/w_4} v_i^{\beta_{4,i}/w_4}),$$

so assuming that u_{j-1} and v_{j-1} are not zero, the maps looks like

$$\psi\circ\sigma|_{U_{j-1}} = ((*):1:v_{j-1}:-1-v_{j-1}+(*)).$$

Therefore $(\psi\circ\sigma|_{U_{j-1}})(u_{j-1},0) = (0:1:0:-1)$ and $(\psi\circ\sigma|_{U_{j-1}})(0,v_{j-1}) = (0:1:v_{j-1}:-1-v_{j-1})$. Doing the same for U_j we find that $(\psi\circ\sigma|_{U_j})(0,v_j) = (0:0:1:-1)$ and $(\psi\circ\sigma|_{U_j})(u_j,0) = (0:u_j:1:-u_j-1)$. Then we see that $\psi\circ\sigma(\bigcup_{i=0}^{j-1} E_{4,i}) = (0:1:0:-1)$, $\psi\circ\sigma(\bigcup_{i=j+1}^{s_4+1} E_{4,i}) = (0:0:1:-1)$. Notice that v_{j-1} and u_j are the coordinates of the charts of $E_j \simeq \mathbb{P}^1$ and that

$$(\psi\circ\sigma|_{U_{j-1}})(0,v_{j-1}) = (0:1:v_{j-1}:-1-v_{j-1})$$

and

$$(\psi\circ\sigma|_{U_j})(u_j,0) = (0:u_j:1:-u_j-1).$$

So $\psi\circ\sigma$ is an isomorphism from E_j onto the line $(y_1=0)\subset(y_1+y_2+y_3+y_4=0)\subset\mathbb{P}^3_{y_1,y_2,y_3,y_4}$. Therefore $\psi\circ\sigma$ is defined at the exceptional divisor over p_4, and it is totally branch over the line $L_1 = (y_1=0)\subset(y_1+y_2+y_3+y_4=0)$.

Now, if $\Gamma'_{3,4}$ does not intersect transversally the exceptional divisor, then $a_2\alpha_{4,i} - (a_3-1)\beta_{4,i} \neq 0$ for all i, so we will have some j such that

$$a_2\alpha_{4,j} - (a_3-1)\beta_{4,j} < 0 < a_2\alpha_{4,j+1} - (a_3-1)\beta_{4,j+1},$$

and we will not be able to define the map on the open set U_j. This because we can factor out $u_j^{a_3\beta_{4,j+1}} v_j^{a_2\alpha_{4,j}+\beta_{4,j}}$ from $\psi\circ\sigma|_{U_j}$, so the map will be

$$\begin{aligned}\psi\circ\sigma|_{U_j} &= ((*):u_j^{(a_2\alpha_{4,j+1}-(a_3-1)\beta_{4,j+1})/w_4} : v_j^{((a_3-1)\beta_{4,j}-a_2\alpha_{4,j})/w_4} : \\ &\quad -u_j^{(a_2\alpha_{4,j+1}-(a_3-1)\beta_{4,j+1})/w_4} - v_j^{((a_3-1)\beta_{4,j}-a_2\alpha_{4,j})/w_4} + (*))\end{aligned}$$

Then if $v_j \neq 0$, $(\psi \circ \sigma|_{U_j})(0, v_j) = (0 : 0 : 1 : -1)$, and if $u_j \neq 0$, we have $(\psi \circ \sigma|_{U_j})(u_j, 0) = (0 : 1 : 0 : -1)$, and so it is not well-defined when $u_j = v_j = 0$. □

Proposition 2.17. *Assume that $\Gamma'_{3,4}$ does not intersect transversally the exceptional divisor, so it intersect it at the point $(0,0)$ of some affine open set U_j. Let $\varphi_1 \colon X_1 \to \tilde{X}$ be the blowup over that point, let $E_{4,j}^{(1)}$ the new component of the exceptional divisor, and let $u_j, v'_{j,1}$ and $u'_{j,1}, v_j$ be the affine coordinates of $U_j^{(1,1)}$ and $U_j^{(1,2)}$, the two affine charts over U_j. Then they satisfy the relation $x_2^{w_4} = u_j^{\alpha_{4,j}+\alpha_{4,j+1}} v'^{\alpha_{4,j}}_{j,1} = u'^{\alpha_{4,j+1}}_{j,1} v_j^{\alpha_{4,j}+\alpha_{4,j+1}}$ and $x_3^{w_4} = u_j^{\beta_{4,j}+\beta_{4,j+1}} v'^{\beta_{4,j}}_{j,1} = u'^{\beta_{4,j+1}}_{j,1} v_j^{\beta_{4,j}+\beta_{4,j+1}}$.*

Proof. This follows from the fact that the resolution was constructed as a toric variety, and the blowup of an affine variety defined by vectors v_1 and v_2, is the variety associated to the fan generated by the vectors v_1, $v_1 + v_2$ and v_2. □

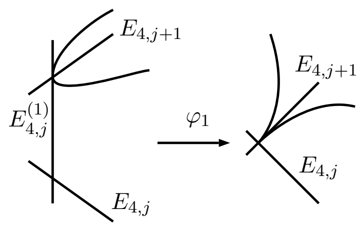


Figure 2.3: An example of the situation in Proposition 2.17.

Notice that if $a_2\alpha_{4,j} - (a_3 - 1)\beta_{4,j} < 0 < a_2\alpha_{4,j+1} - (a_3 - 1)\beta_{4,j+1}$, then

$$a_2\alpha_{4,j} - (a_3 - 1)\beta_{4,j} < a_2(\alpha_{4,j} + \alpha_{4,j+1}) - (a_3 - 1)(\beta_{4,j} + \beta_{4,j+1})$$

and

$$a_2(\alpha_{4,j} + \alpha_{4,j+1}) - (a_3 - 1)(\beta_{4,j} + \beta_{4,j+1}) < a_2\alpha_{4,j+1} - (a_3 - 1)\beta_{4,j+1},$$

so we can use Proposition 2.13 to see that the strict transform of $\Gamma'_{3,4}$ in the blowup intersects at most two components of the exceptional divisor, and that the singularity of the curve is "better". Therefore the map $\psi \circ \sigma \circ \varphi_1$ is defined in one of the charts $U_j^{(1,i)}$, and if $a_2(\alpha_{4,j} + \alpha_{4,j+1}) - (a_3 - 1)(\beta_{4,j} + \beta_{4,j+1}) = 0$, then it is defined in all the exceptional divisor on X_1 over p_4.

Proof of Theorem 2.12. If all the curves $\Gamma'_{i,i+1}$ intersect transversally the exceptional divisor on $\tilde{X}$, then the result follows from Theorem 2.16. If not, then consider the log resolution $\varphi \colon \hat{X} \to X$

of all the curves $\Gamma'_{i,i+1}$. Proposition 2.17 shows that the relations of the new local coordinates are compatible with the previous ones, and as the strict transform of the curves $\Gamma'_{i,i+1}$ intersect transversally the exceptional divisor, we can use the proof of Theorem 2.16 to show that the composition $\psi \circ \sigma \circ \varphi$ is defined over $\hat{X}$. □

Corollary 2.18. *The morphisms $\psi \circ \sigma \circ \varphi\colon \hat{X} \to \mathbb{P}^2$ and the w^*-th root cover $Y' \to \mathbb{P}^2$ factor through a birational morphism $\hat{X} \to Y'$ which contracts precisely six chains of smooth rational curves in*

$$(\sigma \circ \varphi)^*(C_1 + C_2 + \Gamma_{1,2} + \Gamma_{2,3} + \Gamma_{3,4} + \Gamma_{4,1}),$$

each containing one of the proper transforms of $C_1, C_2, \Gamma_{1,2}, \Gamma_{2,3}, \Gamma_{3,4}, \Gamma_{4,1}$, and each contracting to the six cyclic quotient singularities in Y'.

Proof. First, by Theorem 2.12, we note that $\psi \circ \sigma \circ \varphi\colon \hat{X} \to \mathbb{P}^2$ contracts precisely six chains of smooth rational curves in $(\sigma \circ \varphi)^*(C_1 + C_2 + \Gamma_{1,2} + \Gamma_{2,3} + \Gamma_{3,4} + \Gamma_{4,1})$, each containing one of the proper transforms of C_1, C_2, $\Gamma_{1,2}$, $\Gamma_{2,3}$, $\Gamma_{3,4}$, $\Gamma_{4,1}$. This was done locally when we proved definition of the map in Theorem 2.16 at a certain exceptional component over the p_i. Each of these components maps to each of the 4 lines in $\mathbb{P}^2$. Therefore, the birational map $\hat{X} \dashrightarrow Y'$ is defined over these components except possibly over the six singularities of Y'. Because there is a unique minimal resolution for normal two dimensional singularities, the 6 chains of curves in $\hat{X}$ mapping to the 6 nodes of the four lines in $\mathbb{P}^2$ must contract to the 6 singularities of Y'. □

2.2 Kollár surfaces are Hwang-Keum surfaces

We now study the case $w^* = 1$. In this section, we allow $\gcd(w_1, w_3)$ and $\gcd(w_2, w_4)$ to be greater than 1.

In [Ko08, p. 231], it is shown that the curves C_1 and C_2 are extremal rays of the $K_{X(a_1,a_2,a_3,a_4)} + (1-\epsilon)(C_1 + C_2)$ minimal model program if $C_1^2 < 0$ and $C_2^2 < 0$. They are both contractible to quotient singularities. In [HK12] they computed explicitly the type of these singularities.

Theorem 2.19 ([HK12], Theorem 1.1)**.** *The contraction of the curve C_1 forms a singularity of type $\frac{1}{s_1}(w_2, w_4)$, with $s_1 = a_4 w_4 - w_3$, and the contraction of the curve C_2 forms a singularity of* *type* $\frac{1}{s_2}(w_1, w_3)$*, with* $s_2 = a_3w_3 - w_2$*. If* $w^* = 1$*, then their Hirzebruch-Jung continued fractions are*

$$[\underbrace{2,\dots,2}_{a_4-1}, a_3, a_1, \underbrace{2,\dots,2}_{a_2-1}] \quad \textit{and} \quad [\underbrace{2,\dots,2}_{a_3-1}, a_2, a_4, \underbrace{2,\dots,2}_{a_1-1}],$$

respectively.

Proof. For the first part of the theorem we use the unprojection method described in [R00]. Let $F = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1$ be the weighted homogeneous polynomial defining our Kollár surface

$$X = X(a_1, a_2, a_3, a_4) \subset \mathbb{P}(w_1, w_2, w_3, w_4).$$

We have that C_1 and C_2 are disjoint rational curves on X. We write $F = Ax_1 + Bx_3$, where $A = x_1^{a_1-1}x_2 + x_4$ and $B = x_3^{a_3-1}x_4 + x_2^{a_2}$. Therefore we can get an unprojection morphism $X \to X^*$ by introducing the variable

$$y_1 = \frac{A}{x_3} = -\frac{B}{x_1},$$

and

$$X^* = (x_3y_1 = A, x_1y_1 = -B) \subset \mathbb{P}(w_1, w_2, w_3, w_4, s_1),$$

where $s_1 := \deg(y_1) = a_2w_2 - w_1 = a_4w_4 - w_3$. This morphism contracts the curve C_1 to the singular point $(0:0:0:0:1) \in X^*$. Even more, we see that because $\partial f_1, f_2/\partial x_1, x_3 \neq 0$ at $(0,0,0,0,1)$, as in Proposition 2.7 we can say that locally around that point x_2 and x_4 are toric coordinates, so the singular point is of type $\frac{1}{s_1}(w_2, w_4)$.

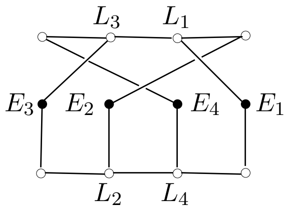
Similarly, we use construct other unprojection morphism contracting C_2 to determine that the singularity obatained is of type $\frac{1}{s_2}(w_1, w_3)$, with $s_2 = a_1w_1 - w_4 = a_3w_3 - w_2$.

To compute the Hirzebruch-Jung continued fraction associated to these singularities it is enough to use certain properties of these continued fractions. The details are in [HK12, Lemma 3.2] and [HK12, Lemma 3.3]. □

Remark 2.20. The statement of Theorem 2.19 was slightly changed to make clear that the first part of it is independent of the value of w^*.

Let $\eta\colon X(a_1, a_2, a_3, a_4) \to X'(a_1, a_2, a_3, a_4)$ be the contraction of C_1 and C_2. In [HK12, §4] they construct several examples of rational $\mathbb{Q}$-homology projective planes with two cyclic singularities. In certain cases the singularities are the same as for $X'(a_1, a_2, a_3, a_4)$ when $w^* = 1$.

The construction of Hwang-Keum is as follows. Let L_1, L_2, L_3, L_4 be four general lines in $\mathbb{P}^2$ and choose four points from the six intersection points, such that every L_i passes through two of them. After blowing up each of these four points twice, we obtain the curve configuration



where • is a (-1)-curve and ∘ is a (-2)-curve. We now blow up r_i times the point $E_i \cap L_i$ to obtain the surface $Z(a_1, a_2, a_3, a_4)$, where $a_i = 2+r_i$. The curve configuration on $Z(a_1, a_2, a_3, a_4)$ is shown in Figure 2.4.

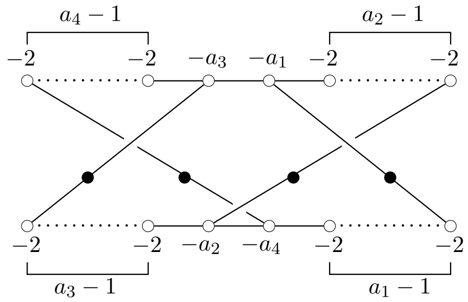


Figure 2.4: Curve configuration over $Z(a_1, a_2, a_3, a_4)$.

Let $T(a_1, a_2, a_3, a_4)$ be the surface obtained by contracting the two chains of rational curves corresponding to the white vertices. Then this surface is a rational $\mathbb{Q}$-homology projective plane with two cyclic singularities. By Theorem 2.19, it has the same singularities as $X'(a_1, a_2, a_3, a_4)$ when $w^* = 1$.

Theorem 2.21. *Let $X(a_1, a_2, a_3, a_4)$ be a Kollár surface with $w^* = 1$, and assume that $a_i \geq 2$ for all i. Then $X'(a_1, a_2, a_3, a_4)$ is the Hwang-Keum surface $T(a_1, a_2, a_3, a_4)$.*

To prove Theorem 2.21 we will show that we can find the same curve configuration of $Z(a_1, a_2, a_3, a_4)$ (Figure 2.4) in $\tilde{X}'$ the minimal resolution of $X'(a_1, a_2, a_3, a_4)$.

First of all, we prove that the rational map ψ is defined in the minimal resolution of X. For this we will use the following proposition.

Proposition 2.22. *Let X be a surface with a cyclic quotient singularity at the point p, and let $C \subset X$ be a curve passing through p. Then C is nonsingular at p if and only if the strict transform of C intersects transversally at one point only one component of the exceptional divisor of the minimal resolution of X.*

Proof. See [GL97]. □

Because $w^* = 1$, by Proposition 2.9 we have that the curves $\Gamma_{i,i+1}$ are smooth, so Proposition 2.22 says that the curves $\Gamma'_{i,i+1}$ intersect transversally the exceptional divisor over p_{i+1}. Hence the minimal resolution of X coincides with the log resolution. If $\gcd(w_1, w_3) = \gcd(w_2, w_4) = 1$, then we already know that the map ψ is defined on the minimal resolution of X. Therefore we only need to check the same assertion when $\gcd(w_1, w_3) > 1$ or $\gcd(w_2, w_4) > 1$.

Proposition 2.23. *The map $\psi \circ \sigma \colon \tilde{X} \to \mathbb{P}^2$ is a morphism.*

Proof. We study the case over the point p_4, with $\gcd(w_2, w_4) = h > 1$. The singularity at p_4 is $1/w_4(w_2, w_3)$ with toric coordinates x_2 and x_3. From Proposition 2.7 we have that $1/w_4(w_2, w_3) \simeq 1/t_4(t_2, w_3)$, with toric coordinates x'_2 and x'_3, and the relation $x'_2 = x_2$ and $x'_3 = x_3^h$. Then from Theorem 1.39 we have $Y = U_1 \cup \cdots U_{s_4}$ in the resolution of p_4, with u_i, v_i the local coordinates in U_i, and the relation $x_2'^{t_4} = u_i^{\alpha_{4,i}} v_i^{\alpha_{4,i+1}}$ and $x_3'^{t_4} = u_i^{\beta_i} v_i^{\beta_{i+1}}$. The curve $\Gamma_{3,4} \subset \mathbb{P}(t_2, w_3, t_4)$, restricted to the open set $(x_4 = 1)$, has equation $x_2'^{a_2} + x_3'^{(a_3-1)/h} = 0$, and we can use Proposition 2.13 to find the equation of the curve in every U_i.

Following the proof of Proposition 2.14, we have that the intersection number

$$\Gamma'_{3,4} \cdot \sum_{i=0}^{s_4+1} \frac{\beta_{4,i}}{t_4} E_{4,i} = \frac{a_2}{t_4},$$

and using the fact that the curve $\Gamma'_{3,4}$ intersects tranversally one component, we have that there exists $\beta_{4,j} = a_2$ and $\alpha_{4,j} = (a_3 - 1)/h$. Therefore

$$\begin{aligned} a_2\alpha_{4,j-1} - \frac{a_3-1}{h}\beta_{4,j-1} &= -1 \\ a_2\alpha_{4,j} - \frac{a_3-1}{h}\beta_{4,j} &= 0 \\ a_2\alpha_{4,j+1} - \frac{a_3-1}{h}\beta_{4,j+1} &= 1 \end{aligned}$$

Hence considering the composition

$$\tilde{X} \xrightarrow{\sigma} \frac{1}{t_4}(t_2, w_3) \xrightarrow{\simeq} \frac{1}{w_4}(w_2, w_3) \overset{\psi}{\dashrightarrow} X(a_1, a_2, a_3, a_4)$$

we have the hypothesis of Theorem 2.16, therefore the map is defined on the whole exceptional divisor. ☐

Proposition 2.24. *The curves* C_1' *and* C_2' *in* $\tilde{X}$ *are* (-1)*-curves. To obtain the chain of curves*

$$K_1 := E_{2,s_2} \cup \cdots \cup E_{2,1} \cup C_1' \cup E_{4,1} \cup \cdots \cup E_{4,s_4}$$

and

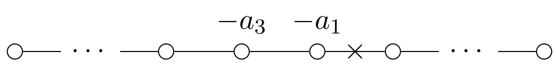
$$K_2 := E_{1,s_1} \cup \cdots \cup E_{1,1} \cup C_2' \cup E_{3,1} \cup \cdots \cup E_{3,s_3}$$

we blowup $\tilde{X}'$ *on the intersection points of the curves with self-intersections* $-a_3$ *and* $-a_1$*, and* $-a_2$ *and* $-a_4$ *respectively.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sigma} & X(a_1, a_2, a_3, a_4) \\ \downarrow & & \downarrow \eta \\ \tilde{X}' & \xrightarrow{\sigma'} & X'(a_1, a_2, a_3, a_4) \end{array}$$

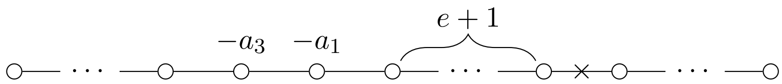
Then, to obtain the chain of curves K_1 we have to blowup on the exceptional divisor over the singularity $\frac{1}{s_1}(w_2, w_4)$. This is because if no blowup were needed, then C_1' would be some of the curves in the exceptional divisor over the singularity $\frac{1}{s_1}(w_2, w_4)$, so we would have that $w_2 \leq a_4 - 1$ or $w_4 \leq a_2 - 1$, which can happen only if one of the a_i is 1. Recall from Theorem 2.19 that the Hirzebruch-Jung continued fraction of the singularity $\frac{1}{s_1}(w_2, w_4)$ is $[2, \ldots, 2, a_3, a_1, 2, \ldots, 2]$. Then we want to show that the blowups needed must be done between the curves with self-intersection $-a_3$ and $-a_1$. For this, we will rule out every other possibility. Suppose first that the blowups are done on the point



then we would obtain that the continued fraction associated to the singularity at p_2 would have an β_i such that

$$\beta_i \geq |[\underbrace{2,\dots,2}_{a_4-1}, a_3, a_1+1]|,$$

but $|[\underbrace{2,\dots,2}_{a_4-1}, a_3, a_1+1]| = w_2 + 2 + a_3a_4 - 2a_4 > w_2$, which is a contradiction. If the blowups are done on the point

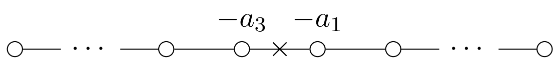


with $e \geq 0$, we would have

$$\beta_i \geq |[\underbrace{2,\dots,2}_{a_4-1}, a_3, a_1, \underbrace{2,\dots,2}_{e}, 3]|,$$

but $|[\underbrace{2,\dots,2}_{a_4-1}, a_3, a_1, \underbrace{2,\dots,2}_{e}, 3]| = (2e+3)w_2 - (2e+1)a_3a_4 - 2a_4 + 1 > w_2$ for all $e \geq 0$.

Therefore, the blowups to obtain the chain of curves K_1 desired have to be done at the point



□

From the proof of Prop. 2.24, we have that the singularity at p_i of the Kollár surface has Hirzebruch-Jung continued fraction

$$[\dots, c_i, \underbrace{2,\dots,2}_{a_{i+2}-1}]$$

with $c_i > 2$. The intersection of $\Gamma'_{i-1,i}$ with the exceptional divisor over p_i is $\beta_{i,j}/w_i = a_{i+2}/w_i$, so the curve $\Gamma'_{i-1,i}$ intersects the exceptional divisor over p_i at the mentioned component with self-intersection $-c_i$. This because $\beta_{i,s_i+1} = 0$ and $\beta_{i,s_i} = 1$, and $\beta_{i,k-1} = b_k\beta_{i,k} - \beta_{i,k+1}$. This implies that $\beta_{i,s_i-(a_2-1)} = a_2 = \beta_j$. Therefore we have the curve configuration shown in Figure 2.5.

Proposition 2.25. *The curves $\Gamma'_{i,i+1}$ are (-1)-curves.*

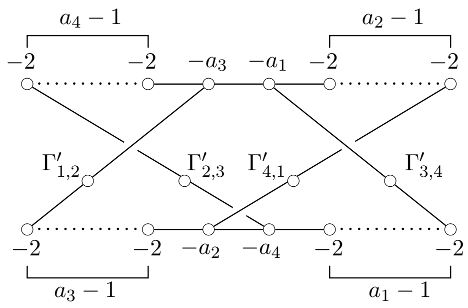


Figure 2.5: Curve configuration on $\tilde{X}'$.

Proof. We have a birational morphism $\psi \circ \sigma \colon \tilde{X} \to \mathbb{P}^2$, so it is a composition of blowups, which contracts (-1)-curves to reach $\mathbb{P}^2$. We start by contracting the curves from the proof of Proposition 2.24 to obtain $\tilde{X}'$ with the curve configuration of Figure 2.5. Recall from Theorem 2.16 that the image of the curves with self-intersection $-a_i$ are the four lines in general position in $\mathbb{P}^2$, so they cannot be contracted. Then, one of the $\Gamma'_{i,i+1}$ is a (-1)-curve, say that it is $\Gamma'_{1,2}$. We contract $\Gamma'_{1,2}$ and the chain of (-2)-curves connected to it, to obtain the diagram in Figure 2.6.

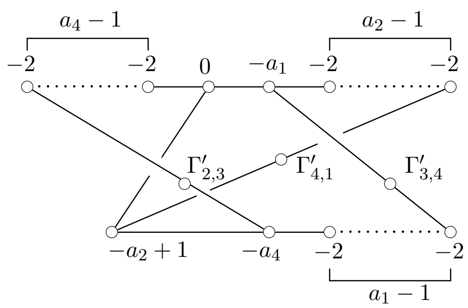


Figure 2.6: Contraction of $\Gamma'_{1,2}$ and the chain of (-2)-curves.

By repeating the procedure, we obtain that all curves $\Gamma'_{i,i+1}$ are (-1)-curves. □

Proof of Theorem 2.21. From Proposition 2.24 and Proposition 2.25, we conclude that $\tilde{X}'$ and $Z(a_1,a_2,a_3,a_4)$ are obtained from the same sequence of blowups of $\mathbb{P}^2$. Therefore

$$\tilde{X}' \simeq Z(a_1,a_2,a_3,a_4)$$

and so $X'(a_1,a_2,a_3,a_4) \simeq T(a_1,a_2,a_3,a_4)$. □

Remark 2.26. Notice that if $w^* \neq 1$, then the surface $T(a_1,a_2,a_3,a_4)$ does not correspond to a Kollár surface, so Kollár surfaces with $w^*=1$ and $a_i \geq 2$ are strictly contained in Hwang-Keum surfaces.

Finally, we check what happens when some $a_i = 1$, say $a_1 = 1$.

Corollary 2.27. *Let $a_1 = 1$. Then the point p_4 is smooth, and the map ψ is defined in the log resolution $\hat{X}$ of the key curves. The curve $\Gamma_{3,4}$ is smooth, and ψ does not contract C_1. The surface $\hat{X}$ is obtained by doing blowups from $Z(1, a_2, a_3, a_4)$. The curve $C_1 \subset X(1, a_2, a_3, a_4)$ is contractible if and only if $a_3 > a_2$.*

Proof. If $a_1 = 1$, then $w_2 = a_4(a_3-1)$ and $w_4 = a_3-1$. Then by Proposition 2.7 we have that the point p_4 is smooth, and at the point p_2 the singularity is of type $\frac{1}{a_4}(1, a_2a_3a_4 - a_3a_4 + a_4 - 1) = \frac{1}{a_4}(1, a_4 - 1)$. The curve $\Gamma_{1,2}$ intersects transversally the curve C_1 at the point $(0 : -1 : 0 : 1)$, and following the proof of Proposition 2.10 we have that $\psi(0 : 1 : 0 : b) = (b : -1 - b : 0 : 1)$, so the curve ψ does not contract C_1. The curve $\Gamma_{3,4}$ restricted to the weighted projective plane $(x_1 = 0)$ and to the open set $(x_4 \neq 1)$ is $(x_2^{a_2} + x_3 = 0) \subset \mathbb{A}^2$, so it is smooth and to obtain the log resolution $\hat{X}$ is necessary to do a_2 blowups.

Now assume that all the other $a_i \geq 2$. Therefore C_2 is contractible, and by contracting it and all the other (-1)-curves in $\hat{X}$ we obtain the surface $\hat{X}'$ with the curve configuration shown in Figure 2.7. If also $a_2 = 1$, then all the points are smooth but point p_2 with a singularity of

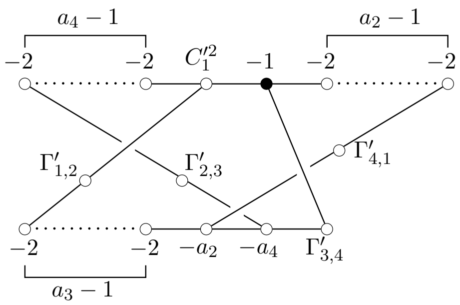


Figure 2.7: Curve configuration on $\hat{X}'$ when $a_1 = 1$.

type $\frac{1}{a_4}(1, a_4 - 1)$, and we obtain the curve configuration on $\hat{X}$ shown in Figure 2.8.

Following the proof of Proposition 2.25 we have that the curves $\Gamma'_{i,i+1}$ are (-1)-curves, $C_1'^2 = -a_3$ and $C_2'^2 = -a_4$. Therefore $\hat{X}' \simeq Z(1, a_2, a_3, a_4)$, and by contracting the (-1)-curve in the top chain along with the (-2)-curves to the right, we obtain that $C_1'^2 = -a_3 + a_2$. Therefore C_1 is contractible if and only if $C_1'^2 < 0$, and this is equivalent to $a_3 > a_2$. ☐

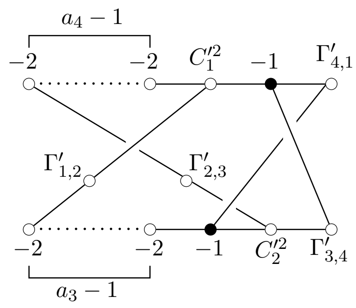


Figure 2.8: Curve configuration on X'_n when $a_1 = 1$ and $a_2 = 1$.

2.3 Kollár surfaces as branch covers of $\mathbb{P}^2$

We now consider the birational model $Y' := \mathrm{Spec}_{\mathbb{P}^2}\left(\bigoplus_{i=0}^{w^*-1} \mathcal{L}^{(i)^{-1}}\right)$ of $X(a_1, a_2, a_3, a_4)$, which was defined at the beginning of this chapter as the w^*-th root cover of $(L_1^{\mu_1} L_2^{\mu_2} L_3^{\mu_3} L_4^{\mu_4} = 0) \subset \mathbb{P}^2$. We recall that $0 < \mu_i < w^*$ are

$$\mu_1 \equiv a_2 a_3 a_4, \quad \mu_2 \equiv -a_3 a_4, \quad \mu_3 \equiv a_4, \quad \mu_4 \equiv -1$$

modulo w^*, and that by definition $\gcd(\mu_i, w^*) = 1$. The lines L_1, L_2, L_3, L_4 form a plane curve with six nodes. We also recall that

$$\mathcal{L}^{(i)} := \mathcal{O}_{\mathbb{P}^2}(ti) \otimes \mathcal{O}_{\mathbb{P}^2}\Big(-\sum_{j=1}^{4} \Big[\frac{\mu_j\ i}{w^*}\Big] L_j\Big)$$

for $i \in \{0, 1, ..., w^* - 1\}$, where $[x]$ is the integer part of x, and $tw^* = \sum_{i=1}^4 \mu_i$. Let Y be the minimal resolution of all singularities in Y'.

Theorem 2.28. *Let $X(a_1, a_2, a_3, a_4)$ be a Kollár surface. Then $X(a_1, a_2, a_3, a_4)$ is birational to*

$$X(a'_1, a'_2, a'_3, a'_4) \subset \mathbb{P}(w'_1, w'_2, w'_3, w'_4)$$

with gcd(w'_1, w'_3) =gcd$(w'_2, w'_4) = 1$, for infinitely many 4-tuples (a'_1, a'_2, a'_3, a'_4).

Proof. By Corollary 2.6, the surface $X(a_1, a_2, a_3, a_4)$ is birational to Y', and so for any $t_i \in \mathbb{Z}_{\geq 0}$ we have that $X(a_1, a_2, a_3, a_4)$ is birational to

$$X(a_1 + t_1 w^*, a_2 + t_2 w^*, a_3 + t_3 w^*, a_4 + t_4 w^*),$$

as soon as w^* =gcd$(W'_1, \ldots, W'_4)$ for the corresponding W'_i. This is because, for a fixed w^*, the isomorphism type of Y' depends only on the multiplicities μ_i modulo w^*. In this way, we must find $t_i \in \mathbb{Z}_{\geq 0}$ such that $\gcd(w'_1, w'_3) = \gcd(w'_2, w'_4) = 1$, and w^* =gcd$(W'_1, \ldots, W'_4)$.

First, choose t_3 such that $\gcd(a_3 + t_3 w^*, 6(a_4 - 1)) = 1$, and let $a_3' := a_3 + t_3 w^*$ and $W_1' := a_2 a_3' a_4 - a_3' a_4 + a_4 - 1 = w_1' w^*$. Next take t_2 such that $\gcd(w_1' + t_2 a_3' a_4, 6(a_4 - 1)) = 1$, and then define $a_2' := a_2 + t_2 w^*$. Now we will choose t_1 such that the final weights $(w_1'', w_2'', w_3'', w_4'')$ satisfy $\gcd(w_1'', w_3'') = \gcd(w_2'', w_4'') = 1$, and $w^* = \gcd(W_1'', \ldots, W_4'')$.

Let $W_2' := a_3' a_4 a_1 - a_4 a_1 + a_1 - 1 = w_2' w^*$, $W_3' := a_4 a_1 a_2' - a_1 a_2' + a_2' - 1 = w_3' w^*$, and $W_4' := a_1 a_2' a_3' - a_2' a_3' + a_3' - 1 = w_4' w^*$, and define

$$W_1'' := w_1'' w^*, \qquad W_2'' := w_2'' w^* = (w_2' + t(a_3' a_4 - a_4 + 1))w^*,$$

$$W_3'' := w_3'' w^* = (w_3' + t(a_4 a_2' - a_2'))w^*, \qquad W_4'' := w_4'' w^* = (w_4' + t a_2' a_3')w^*,$$

where t will be found.

Let $w_1'' = \prod q_{1,j}^{\lambda_{1,j}}$ be its prime factorization. Then we have to find a solution t for $w_4' + t a_2' a_3' \not\equiv 0 \pmod{q_{1,j}}$, $w_3' + t a_2'(a_4 - 1) \not\equiv 0 \pmod{q_{1,j}}$, and $t \not\equiv 0 \pmod{q_{1,j}}$, for all j. This t will exist because we have that $\gcd(a_4 - 1, w_1'') = 1$, and that all $p_{1,j}$ are greater than 3, by the previous choice of t_2 and t_3.

By the Chinese Remainder Theorem, we know that the solutions are of the form $t_1 + r \cdot \prod q_{1,j}$, $r \in \mathbb{Z}$. Hence we have that $\gcd(w_1'', w_3'')$ $=\gcd(w_1'', w_4'') = 1$, for any choice of r. Therefore, considering

$$w_2'' = w_2' + t_1(a_3' a_4 - a_4 + 1) + r \cdot (a_3' a_4 - a_4 + 1) \cdot \prod q_{1,j}$$

and $w_4'' = w_4' + t_1 a_2' a_3' + r \cdot a_2' a_3' \cdot \prod q_{1,j}$, it is enough to find an $r \in \mathbb{Z}_{\geq 0}$ such that $\gcd(w_2'', w_4'') = 1$. Let

$$A := w_2' + t_1(a_3' a_4 - a_4 + 1) \quad B := (a_3' a_4 - a_4 + 1) \cdot \prod q_{1,j}$$

$$C := w_4' + t_1 a_2' a_3' \quad D := a_2' a_3' \cdot \prod q_{1,j}.$$

Notice that $\gcd(A, B) = 1$ by the definition of w_2' and the way t_1 was obtained. Let $AD - BC = q_{2,1}^{\lambda_{2,1}} q_{2,2}^{\lambda_{2,2}} \cdots q_{2,l}^{\lambda_{2,l}}$; $q_{2,j}$ prime number, and let r_1 be a solution of

$$A + Br \not\equiv 0 \pmod{q_{2,j}}. \tag{2.4}$$

Now assume that $\gcd(w_2'', w_4'') = \gcd(A + Br_1, C + Dr_1) > 1$. This means that there is a prime $p \neq q_{2,j}$ for all j, such that it divides both $A + Br$ and $C + Dr$. Then consider the linear transformation $T\colon (\mathbb{Z}/p\mathbb{Z})^2 \to (\mathbb{Z}/p\mathbb{Z})^2$ associated to the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. This matrix maps the vector $(1, r_1)$ to $(0,0)$, so the matrix is singular. But the determinant $AD - BC \neq 0 \pmod p$, which is a contradiction. Therefore $\gcd(A + Br_1, C + Dr_1) = 1$. Let $a_1' := a_1 + (t_1 + r_1 \cdot \prod p_{1,j})w^*$. This gives us that $X(a_1', a_2', a_3', a_4) \subset \mathbb{P}(w_1'', w_2'', w_3'', w_4'')$ is birational to $X(a_1, a_2, a_3, a_4)$, with $\gcd(w_1'', w_3'') = \gcd(w_2'', w_4'')$ and because $\gcd(w_1'', w_4'') = 1$, then $w^* = \gcd(W_1'', \ldots, W_4'')$. Because the equation (2.4) has infinite solutions, then we have infinite 4-tuples $(a_1'', a_2'', a_3'', a_4'')$ that satisfy the result. $\square$

Corollary 2.29. *Let Y' be a n-th root cover of $(L_1^{\mu_1} L_2^{\mu_2} L_3^{\mu_3} L_4^{\mu_4} = 0) \subset \mathbb{P}^2$, with $\gcd(\mu_i, n) = 1$ for all i. Then Y' is birational to a Kollár surface.*

Proof. If we multiply the μ_i by a unit ξ of $\mathbb{Z}/n\mathbb{Z}$, then the n-th root cover does not change (see Proposition 1.42). So we take ξ such that $\xi\mu_4 = -1$. In this way, we have to solve the system $a_2 a_3 a_4 \equiv \xi\mu_1$, $-a_3 a_4 \equiv \xi\mu_2$, $a_4 \equiv \xi\mu_3$, and $a_1 a_2 a_3 a_4 \equiv 1$ modulo n, which has a solution because ξ and the μ_i are units in $\mathbb{Z}/n\mathbb{Z}$. Then, with those a_i we can use Theorem 2.28 to find numbers a_i' such that $X(a_1', a_2', a_3', a_4')$ is a Kollár surface with $w^* = n$, and birational to Y'. $\square$

We want to compute the main numerical invariants of Y. To do so, recall the definitions and results from Section 1.5 and Section 1.7.

Proposition 2.30. *We have that $\pi_1(Y) = 0$, and*

$$p_g(Y) = 2s(1, 1; w^*) + \sum_{i<j} s(\mu_i, \mu_j; w^*)$$

where $s(1, 1; w^*) = \frac{w^*}{12} + \frac{1}{6w^*} - \frac{1}{4}$.

Proof. See [Urz10, Prop.3.2 and Thm.8.5]. $\square$

Remark 2.31. Since the geometric genus $p_g(Y)$ is a nonnegative number, we have $2s(1, 1; w^*) + \sum_{i<j} s(\mu_i, \mu_j; w^*) \geq 0$, which can be rewritten using basic properties of Dedekind sums as

$$p_g(Y) = 2s(1, 1; w^*) - \sum_{i=1}^{4} s(1, a_i; w^*) + s(1, a_1 a_4; w^*) + s(1, a_1 a_2; w^*) \geq 0.$$

We will tell more on this expression in the next section.

Proposition 2.32. *We have that $e(Y) = w^* + 2 + \sum_{i<j} l(\mu_i, \mu_j; w^*)$, and*

$$K_Y^2 = w^* + \frac{4}{w^*} + 4 + \sum_{i<j} (12s(\mu_i, \mu_j; w^*) - l(\mu_i, \mu_j; w^*)).$$

Proof. See [Urz10, Prop. 3.6] and use Noether's formula. ☐

Corollary 2.33. *For $X = X(a_1, a_2, a_3, a_4)$ we have $e(X) = w^* + 4$, $\pi_1(X) = 0$, and $p_g(X) = 2s(1, 1; w^*) - \sum_{i=1}^4 s(1, a_i; w^*) + s(1, a_1a_4; w^*) + s(1, a_1a_2; w^*)$.*

Corollary 2.34. *Let $gcd(w_i, w_{i+2}) = 1$ for all i. Then*

$$12\Big(\sum_{i<j} s(\mu_i, \mu_j; w^*) + \sum_i s(w_{i+2}, w_{i+3}; w_i)\Big) =$$

$$\frac{d(d - \sum_i w_i)^2}{\prod_i w_i} - \sum_i \frac{2}{w_i} - \frac{w^{*2} - 6w^* + 4}{w^*}.$$

Proof. Let $X = X(a_1, a_2, a_3, a_4)$. We are going to compute $p_g(X)$ from X, and then the equality follows from $p_g(X) = p_g(Y)$. Let $\tilde{X} \to X$ be the minimal resolution of singularities. As in the proof of Prop. 3.4 in [Urz10], we have

$$K_{\tilde{X}}^2 - K_X^2 = -12\sum_i s(w_{i+2}, w_{i+3}; w_i) - \sum_i l(w_{i+2}, w_{i+3}; w_i) + \sum_i \frac{2(w_i - 1)}{w_i},$$

and $e(\tilde{X}) - e(X) = \sum_i l(w_{i+2}, w_{i+3}; w_i)$. The conditions $\gcd(w_i, w_{i+2}) = 1$ tell us that X is well formed, so $K_X^2 = \frac{d(d-\sum_i w_i)^2}{\prod_i w_i}$ (Proposition 1.35 and Corollary 1.3) and $e(X) = w^* + 4$, then the formula

$$p_g(X) = \frac{d(d - \sum_{i=1}^4 w_i)^2}{12w_1w_2w_3w_4} - \sum_i s(w_{i+2}, w_{i+3}; w_i) - \frac{1}{6}\sum_i \frac{1}{w_i} + \frac{w^*}{12}$$

is a consequence of the Noether's equality $12\chi(O_{\tilde{X}}) = K_{\tilde{X}}^2 + e(\tilde{X})$. ☐

Chapter 3

Classification of Kollár surfaces

In this section we prove results related to the geometric genus of Kollár surfaces. We will use the results of Section 1.7. Throughout this section, w^* will be greater than 1. All equalities involving $\equiv$ will be modulo w^*, unless stated otherwise. The symbol q^{-1} will denote the inverse of q modulo w^*. To avoid confusions, we will write $\frac{1}{q}$ when it corresponds to a number in $\mathbb{Q}$.

Proposition 3.1. *Any $n \geq 0$ is realizable as the geometric genus of a Kollár surface.*

Proof. We know that $w^* = 1$ implies rational, and so $p_g = 0$. Assume that $n > 0$, and let $w^* = 3n + 1$ and $a_1 \equiv 3^{-1}$, $a_2 \equiv 3$, $a_3 \equiv a_4 \equiv w^* - 1$. This gives the w^*-th root cover Y with $\mu_1 = 3$, $\mu_2 = \mu_3 = \mu_4 = w^* - 1$. The geometric genus of Y is

$$\begin{aligned} p_g(Y) &= 5s(1,1;w^*) - 3s(1,3;w^*) \\ &= 5\left(\frac{w^*}{12} + \frac{1}{6w^*} - \frac{1}{4}\right) - 3\left(\frac{w^*}{36} + \frac{1}{4w^*} + \frac{1}{36w^*} - \frac{1}{18} - \frac{1}{4}\right) \\ &= n. \end{aligned}$$

□

3.1 $p_g = 0$

First we will classify Kollár surfaces with $p_g = 0$. This is equivalent to give a characterization of the n-th root covers of $\mathbb{P}^2$ totally branched along four general lines with $p_g = 0$, which can be done using the bounds for Dedekind sums shown in Section 1.7.

By classification of surfaces we have that $p_g = 0$ occurs in every Kodaiara dimension. It is already known that if two multiplicities μ_i, μ_2 of the w^*-th root cover satisfy $\mu_i + \mu_2 \equiv 0$, then the w^*-th root cover is rational. We will prove the converse.

Theorem 3.2. *Let $X = X(a_1, a_2, a_3, a_4)$ a Kollár surface with $w^* > 1$. Then the following are equivalent*

(a) $p_g(X) = 0$.

(b) $a_i \equiv 1$ *or* $a_i a_{i+1} \equiv -1$ *modulo* w^* *for some* i.

(c) X *is rational.*

Proof of Theorem 3.2. By Corollary 2.33, we have that the geometric genus of $X(a_1, a_2, a_3, a_4)$ is

$$p_g(X) = 2s(1, 1; w^*) - \sum_{i=1}^{4} s(1, a_i; w^*) + s(1, a_1a_4; w^*) + s(1, a_1a_2; w^*)$$

$\underline{(c) \Rightarrow (a):}$ This is trivial.

$\underline{(a) \Rightarrow (b):}$ Assume that $a_i \not\equiv 1$ and $a_i a_{i+1} \not\equiv -1$ for all i. First, if $a_i \not\equiv 2, 2^{-1}$ and $a_i a_{i+1} \not\equiv -2, -2^{-1}$ for all i, then by Lemma 1.46,(2) we have that $p_g > 2s(1, 1; w^*) - \frac{6}{3}s(1, 1; w^*) > 0$. Therefore it is enough to rule out the cases when $a_1 \equiv 2$ or $a_1a_2 \equiv -2^{-1}$. First suppose that $a_1 \equiv 2$, so

$$p_g = 2s(1, 1; w^*) + s(1, 2a_2; w^*) + s(1, 2a_4; w^*) - s(1, 2; w^*) - \sum_{i=2}^{4} s(1, a_i; w^*),$$

and we have to check the cases when we cannot use Lemma 1.46,(3).

If $a_3 \equiv 2$ or $a_3 \equiv 2^{-1}$, then $a_1a_2 \equiv -1$ or $a_4 \equiv 1$ respectively, so they satisfy the hypothesis for $p_g = 0$.

If $a_2 \equiv 2^{-1}$, $2a_2 \equiv -2$, $2a_4 \equiv -2$, $a_4 \equiv 3^{-1}$ or $2a_2 \equiv -3$, then one of the terms is equal to $s(1, 1; w^*)$ or two of the terms cancel, so by Lemma 1.46,(1) we have that $p_g > 0$.

If $a_2 \equiv 2$, $2a_2 \equiv -2^{-1}$ or $2a_4 \equiv -2^{-1}$, then

$$p_g = 2s(1, 1; w^*) - 2s(1, 2; w^*) + s(1, 4; w^*) - s(1, 3; w^*) + s(1, 2 \cdot 3^{-1}; w^*)$$

$$-s(1,4\cdot 3^{-1};w^*)$$

and by Corollary 1.47,(1) $p_g > 0$ when $w^* > 5$. If $w^* = 5$, then it satisfies the conditions for $p_g = 0$.

If $a_2 \equiv 3$ or $2a_4 \equiv -3^{-1}$, then

$$p_g = 2s(1,1;w^*) - s(1,2;w^*) - s(1,3;w^*) - s(1,4;w^*) + s(1,6;w^*)$$

$$-s(1,2\cdot 3^{-1};w^*) + s(1,4\cdot 3^{-1};w^*)$$

and by Corollary 1.47,(2) $p_g > 0$ when $w^* > 7$. If $w^* = 7$, then it satisfies the conditions for $p_g = 0$.

If $a_4 \equiv 3$ or $2a_2 \equiv -3^{-1}$, then

$$p_g = 2s(1,1;w^*) - s(1,2;w^*) - s(1,3;w^*) - s(1,5;w^*) + s(1,6;w^*)$$

$$+s(1,2\cdot 5^{-1};w^*) - s(1,6\cdot 5^{-1};w^*)$$

and by Corollary 1.47,(3) $p_g > 0$ when $w^* > 7$. If $w^* = 7$, then it satisfies the conditions for $p_g = 0$.

These cover all the cases for $a_1 \equiv 2$. Now assume that $a_1 a_2 \equiv -2^{-1}$, so

$$p_g = 2s(1,1;w^*) - s(1,2;w^*) + s(1,a_1a_4;w^*) + s(1,2a_2;w^*) - \sum_{i=2}^{4} s(1,a_i;w^*),$$

and we proceed as the previous case.

If $a_1 a_4 \equiv -2$ or $a_1 a_4 \equiv -2^{-1}$, then $a_1 \equiv 1$ or $a_4 \equiv 1$ respectively, so they satisfy the hypothesis for $p_g = 0$.

If $a_2 \equiv 3^{-1}$ or $a_3 \equiv 3$, then two of the terms in the sum cancel, so by Lemma 1.46,(1) we have that $p_g > 0$.

If $a_4 \equiv 3^{-1}$ or $2a_2 \equiv -3^{-1}$, then

$$p_g = 2s(1,1;w^*) - s(1,2;w^*) - s(1,3;w^*) - s(1,4;w^*) + s(1,6;w^*)$$

$$-s(1,2\cdot 3^{-1};w^*) + s(1,4\cdot 3^{-1};w^*)$$

and by Corollary 1.47,(2) $p_g > 0$ when $w^* > 7$. If $w^* = 7$, then it satisfies the conditions for $p_g = 0$.

If $a_2 \equiv 3$ or $a_3 \equiv 3^{-1}$, then

$$p_g = 2s(1,1;w^*) - s(1,2;w^*) - s(1,3;w^*) - s(1,5;w^*) + s(1,6;w^*)$$
$$+s(1, 2\cdot 5^{-1}; w^*) - s(1, 6\cdot 5^{-1}; w^*)$$

and by Corollary 1.47,(3) $p_g > 0$ when $w^* > 7$. If $w^* = 7$, then it satisfies the conditions for $p_g = 0$.

These cover all the cases for $a_1a_2 \equiv -2^{-1}$.

(b) $\Rightarrow$ *(c)*: Notice that *b*) implies the existence of μ_i and μ_j such that $\mu_i + \mu_j \equiv 0 \pmod{w^*}$. Consider the trivial pencil of lines through $L_i \cap L_j$. Since $\mu_i + \mu_j \equiv 0 \pmod{w^*}$, this pencil defines a pencil of smooth rational curves in Y via pull-back. Therefore Y is rational, and so is X. □

3.2 $p_g = 1$

To classify the Kollár surfaces with $p_g = 1$ we will use the following Lemma, which says that we have to check only a finite number of cases.

Lemma 3.3. *Let m be a positive integer. Then there is a positive integer N such that if $w^* > N$ and $p_g \neq 0$, then $p_g > m$.*

Proof. If all a_i, and $-a_1a_2$ and $-a_1a_4$ are not equivalent to $2, 2^{-1}, 3, 3^{-1}$, then by Lemma 1.46,(3) we have that

$$p_g > 2s(1,1;w^*) - \frac{6}{4}s(1,1;w^*) = \frac{1}{2}s(1,1;w^*).$$

Also we note that if we fix two of these values, say for example $a_1 \equiv 2$ and $a_1a_2 \equiv -3$, then the rest of the a_i are completely determined, and they are equivalent to $2, 2^{-1}, 3, 3^{-1}$ only for finitely many w^*. Therefore if we set that two of the a_i, $-a_1a_2$ or $-a_1a_4$ to be equivalent to 3 or 3^{-1}, then for $w^* >> 0$ we have that

$$p_g > 2s(1,1;w^*) - \frac{2}{3}s(1,1;w^*) - s(1,1;w^*) = \frac{1}{3}s(1,1;w^*).$$

If one of the values is 2 or 2^{-1} and the other is 3 or 3^{-1}, then for $w^* >> 0$

$$p_g > 2s(1,1;w^*) - \frac{1}{2}s(1,1;w^*) - \frac{1}{3}s(1,1;w^*) - s(1,1;w^*) = \frac{1}{6}s(1,1;w^*).$$

Both of these cases happen when $w^* > 28$, hence we have to check the case when two of the values are 2 or 2^{-1}. This was done in the proof of Theorem 3.2, and the only relevant case is when p_g is $2s(1,1;w^*)-2s(1,2;w^*)+s(1,4;w^*)-s(1,3;w^*)+s(1,2\cdot 3^{-1};w^*)-s(1,4\cdot 3^{-1};w^*)$. For $w^* >> 0$ we have that

$$p_g > 2s(1,1;w^*) - s(1,1;w^*) - \frac{1}{3}s(1.1;w^*) - \frac{1}{2}s(1,1;w^*) + s(1,4;w^*),$$

and because $s(1,4;w^*) \geq 0$ for $w^* \geq 15$, we have that $p_g > s(1,1;w^*)/6$.

Therefore N is the first integer such that $s(1,1;N) > 6m$. □

In Table 3.1, we show the total transform of the key configuration of curves after successively blowing down several (-1)-curves from the minimal resolution of the indicated surfaces $X(a_1,a_2,a_3,a_4)$.

Theorem 3.4. *Let $X = X(a_1,a_2,a_3,a_4)$ a Kollár surface with $w^* > 1$. Then the following are equivalent*

(a) $p_g(X) = 1$.

(b) X is birational to one of the 8 surfaces in Table 1.

(c) X is birational to a K3 surface.

Proof. *(c) ⇒ (a)*: It is trivial.

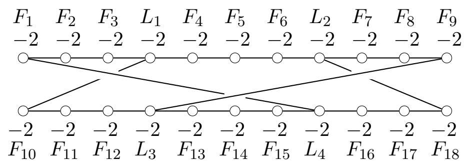
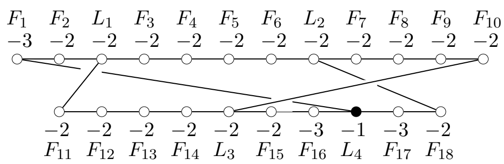
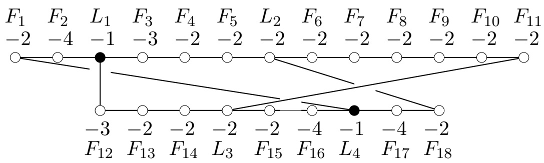
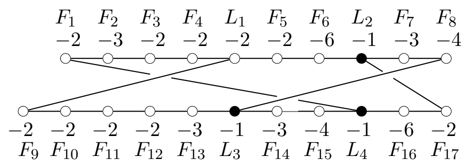
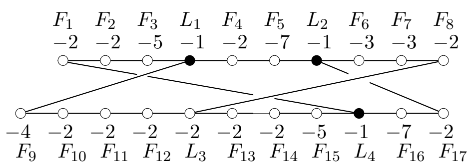
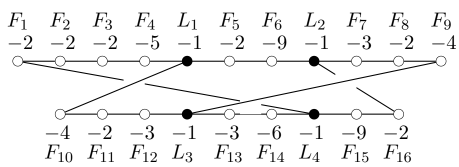
(a) ⇒ (b): To prove this implication, we first use Lemma 3.3 for $m = 1$, which gives us that $N = 75$. We check using a computer all the possible w^*-th root covers for $w^* \leq 75$, and find that there are 8 different cases with $p_g = 1$, which are represented by a Kollár surface in Table 1.

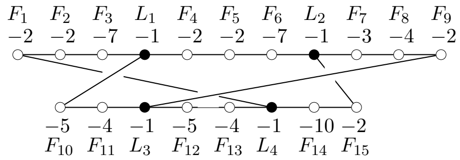
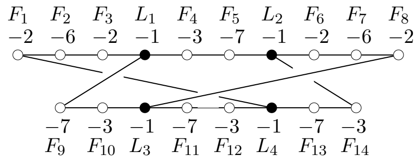
(b) ⇒ (c): We prove this implication by means of the following simple lemma.

Lemma 3.5. *Let S be a smooth projective surface with $p_g = 1$ and $q = 0$. Assume it has an effective connected divisor F with $F^2 = 0$ and $p_a(F) = 1$, and a (-2)-curve C such that $F \cdot C = 1$. Then S is birational to a K3 surface, and F is a fiber of an elliptic fibration $S \to \mathbb{P}^1$, where C is a section.*

Proof. Notice that F has the type of a non-multiple fiber of an elliptic fibration. We want to get such a fibration on S. By the Riemann-Roch inequality and $F \cdot (F - K_S) = 0$, we have $h^0(F) + h^2(F) \geq \chi(\mathcal{O}_S) = 2$. Since in addition $h^2(F) = h^0(K_S - F)$ and $C \cdot (K_S - F) = -1$, we have $h^2(F) = 0$. Therefore, there is a fibration $S \to \mathbb{P}^1$ with general fiber of genus 1 and F is a fiber. Let S' be the relative minimal model of this fibration. By the canonical class formula, $K_S \sim (-2 + \chi(\mathcal{O}_S))F + \sum_i (m_i - 1)G_i + E$ where G_i are the multiple fibers, and E is the exceptional divisor from $S \to S'$. But there is a section C, and so $G_i = 0$ for all i. Then S' has trivial canonical class, and so it is a K3 surface. $\square$

Table 3.1: List for $p_g = 1$

$X(a_1, a_2, a_3, a_4)$	w^*	Total transform of key configuration
$X(7,7,15,15)$	4	
$X(8,9,14,22)$	5	
$X(11,27,10,18)$	7	
$X(17,14,42,18)$	11	
$X(20,21,43,22)$	13	
$X(26,56,39,64)$	17	

$X(29,30,42,32)$	19	
$X(47,51,63,91)$	20	

We now go case by case, showing what the support $\text{supp}(F)$ of F is and its type (using Kodaira's notation), and showing C. Here we are choosing F and C, there are other choices in general.

$w^* = 4$: $\text{supp}(F) = \sum_{i=1}^{6} F_i + L_1 + L_2 + L_4 + F_{16} + F_{17} + F_{18}$, type I_{12}, $C = F_7$.

$w^* = 5$: $\text{supp}(F) = F_1 + F_{16} + F_{17} + L_4$, type IV, $C = F_2$.

$w^* = 7$: $\text{supp}(F) = F_1 + F_{16} + F_{17} + L_4$, type III, $C = F_{15}$.

$w^* = 11$: $\text{supp}(F) = F_6 + L_2 + F_{17} + F_7$, type II, $C = F_5$.

$w^* = 13$: $\text{supp}(F) = F_1 + F_2 + L_4 + L_3 + F_8 + \sum_{i=10}^{15} F_i$, type III^*, $C = F_3$.

$w^* = 17$: $\text{supp}(F) = L_2 + \sum_{i=7}^{9} F_i + F_{12} + L_3 + F_{13} + F_{16}$, type IV, $C = F_{11}$.

$w^* = 19$: $\text{supp}(F) = F_4 + L_1 + F_5 + F_6 + F_7 + L_2 + F_{15}$, type II, $C = F_3$.

$w^* = 20$: $\text{supp}(F) = F_3 + L_1 + F_4 + F_5 + F_6 + L_2 + F_{14}$, type II, $C = F_2$.

$\square$

3.3 $p_g \geq 2$

In this sub-section, we assume that $p_g \geq 2$. We recall that Kollár surfaces are simply-connected. By classification of algebraic surfaces, the Kodaira dimension of the associate surface Y is either 1 or 2. We first present families of explicit examples for each of the two possible Kodaira dimensions, and then we show the general picture for $w^* >> 0$.

Let $g\colon Y' \to \mathbb{P}^2$ be the normal w^*-th root cover branch on $(L_1^{\mu_1} L_2^{\mu_2} L_3^{\mu_3} L_4^{\mu_4} = 0)$, and let $f\colon Y \to \mathbb{P}^2$ be g composed with the minimal resolution of singularities of Y'. Let $p_{i,j} = L_i \cap L_j$ for $i < j$. Let $E_{i,j,k}$ be the k-th exceptional curve over $p_{i,j}$. By [BHPV, Ch. I, Lemma 17.1(iii)]

$$K_{Y'} \equiv g^*\big(-3H + \frac{w^*-1}{w^*}(L_1 + L_2 + L_3 + L_4)\big).$$

Recall that the singularities of Y' are cyclic singularities, and we computed the discrepancies of those singularities at the end of Section 1.5. Then we have

$$K_Y \equiv f^*\big(-3H + \frac{w^*-1}{w^*}(L_1 + L_2 + L_3 + L_4)\big) - \sum_{i<j}\sum_{k} \left(1 - \frac{\alpha_{i,j,k} + \beta_{i,j,k}}{w^*}\right) E_{i,j,k}$$

where H is a line in $\mathbb{P}^2$, and writing $H = (L_1 + L_2 + L_3 + L_4)/4$ we obtain

$$K_Y \equiv \frac{w^*-4}{4}(L_1' + L_2' + L_3' + L_4') + \sum_{i<j}\sum_{k} \left(\frac{\alpha_{i,j,k} + \beta_{i,j,k} - 4}{4}\right) E_{i,j,k},$$

where we are using notation and facts from Section 1.5, and $L_i' \simeq \mathbb{P}^1$ is the (reduced, irreducible) pre-image of L_i.

Now we compute $L_i'^2$. We will consider the case $i = 1$, and for the other curves is analogous. Using the pull-back formulas 1.2 we have

$$f^*(L_1) = w^* L_1' + \sum_{j=1}^{s_{1,2}} \beta_{1,2,j} E_{1,2,j} + \sum_{j=1}^{s_{1,3}} \beta_{1,3,j} E_{1,3,j} \sum_{j=1}^{s_{1,4}} \beta_{1,4,j} E_{1,4,j}.$$

As $f^*(L_1) \cdot (\sum_{j=1}^{s_{1,k}} \beta_{1,k,j} E_{1,k,j}) = 0$ for $k \in \{2,3,4\}$, then

$$0 = w^* \beta_{1,k,1} + (\sum_{j=1}^{s_{1,k}} \beta_{1,k,j} E_{1,k,j})^2,$$

and so

$$w^* L_i^2 = f^*(L_i)^2 = w^{*2} L_1'^2 + w^*(\beta_{1,2,1} + \beta_{1,3,1} + \beta_{1,4,1}).$$

Therefore

$$L_1'^2 = \frac{1}{w^*}(L_1^2 - (\beta_{1,2,1} + \beta_{1,3,1} + \beta_{1,4,1})).$$

Example 3.6. Let b an integer with $b \geq 2$. Consider $w^* = 4(b-1)$, $\mu_1 = \mu_2 = 1$, and $\mu_3 = \mu_4 = 2b-3$. Then, over $p_{1,2}$ and $p_{3,4}$ we have A_{w^*-1} singularities in Y', and over the rest of the $p_{i,j}$ we have $\frac{1}{w^*}(1, 2b-1)$. Notice that $\frac{w^*}{2b-1} = [2, b, 2]$. We have that $L_i'^2 = -2$, and

$$K_Y \equiv \frac{b-2}{2}\Big(2\sum_i L_i' + \sum_k 2(E_{1,2,k} + E_{3,4,k}) + (E_{1,3,k} + E_{1,4,k} + E_{2,3,k} + E_{2,4,k})\Big).$$

Therefore Y is a minimal surface with $K_Y^2 = 0$ and $e(Y) = 3w^* + 12$, and so $p_g(Y) = b-1$. The surface Y is K3 when $b = 2$, and Kodaira dimension 1 when $b > 2$. In fact, one can show that $E_{1,3,2}, E_{1,4,2}, E_{2,3,2}, E_{2,4,2}$ are sections (and $(-b)$-curves) for an elliptic fibration $Y \to \mathbb{P}^1$, and the complement of them in the support above of K_Y give two $I_{w^*}^*$ singular fibers (using Kodaira notation).

Example 3.7. Let $b \geq 1$. Consider $w^* = 28b+1$, $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 4$, and $\mu_4 = 28b-6$. Then, over $p_{i,j}$ we have:

$p_{1,2}$: $\frac{1}{w^*}(1, w^*-2)$, $[2, \ldots, 2, 3]$ with $(14b-1)$ 2's

$p_{1,3}$: $\frac{1}{w^*}(1, 7b)$, $[5, 2, \ldots, 2]$ with $(7b-1)$ 2's

$p_{1,4}$: $\frac{1}{w^*}(1, 7)$, $[4b+1, 2, 2, 2, 2, 2, 2]$

$p_{2,3}$: $\frac{1}{w^*}(1, w^*-2)$, $[2, \ldots, 2, 3]$ with $(14b-1)$ 2's

$p_{2,4}$: $\frac{1}{w^*}(1, 14b+4)$, $[2, 2b+1, 3, 2, 2]$

$p_{3,4}$: $\frac{1}{w^*}(1, 7b+2)$, $[4, b+1, 2, 2, 3]$

One can also compute that $L_1'^2 = L_2'^2 = L_4'^2 = -2$ and $L_3'^2 = -1$. The configuration of all these curves is shown in Figure 3.1.

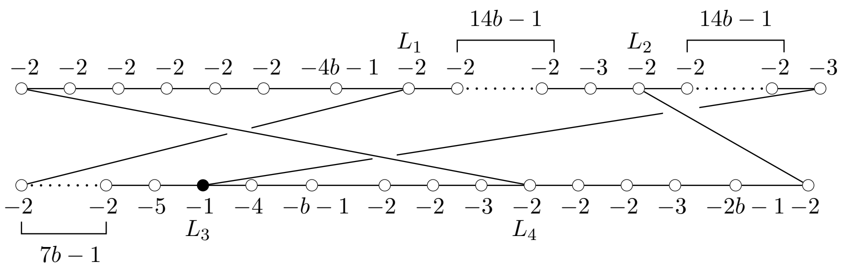


Figure 3.1: Curve configuration of a general type example.

One can verify that $\alpha_{i,j,k} + \beta_{i,j,k} > 4$ for all i, j, k. Therefore, by the formula above, K_Y can be written with positive coefficients supported in the configuration of curves, so that to obtain the minimal model Y'' of Y we only need to contract L_3' since $\frac{w^*-4}{4} > 1$ (and see the figure). We compute using the formulas above: $K_{Y''}^2 = 7(3b-1)$, $e(Y'') = 63b+19$, and $p_g(Y'') = 7b$. In this way, Y'' is of general type for any b.

We now consider prime numbers $w^* >> 0$ and partitions

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = w^*$$

with $0 < \mu_i < w^*$. Let $\mathcal{S}$ be the set of all partitions. Then, as we did before, there are smooth projective surfaces Y constructed as w^*-th root covers $Y \to Y' \to \mathbb{P}^2$, and there are infinitely many Kollár surfaces $X(a_1, a_2, a_3, a_4)$ birational to each Y. Let $X_{\min}$ be a minimal (smooth) model for Y (and so for all $X(a_1, a_2, a_3, a_4)$). The following is based on [Urz10, Urz15].

Theorem 3.8. *There is $\mathcal{S}' \subset \mathcal{S}$ with $\mathcal{S}'/w^* \to 0$ as $w^* >> 0$ such that if $\{\mu_1, \mu_2, \mu_3, \mu_4\} \in \mathcal{S} \backslash \mathcal{S}'$, then X_{min} is a simply-connected surface of general type with $K_{X_{min}}^2 / e(X_{min}) \to 1$ as $w^* >> 0$.*

Proof. By Proposition 2.32, we have $e(Y) = w^* + 2 + \sum_{i<j} l(\mu_i, \mu_j; w^*)$, and

$$K_Y^2 = w^* + \frac{4}{w^*} + 4 + \sum_{i<j} 12 s(\mu_i, \mu_j; w^*) - l(\mu_i, \mu_j; w^*).$$

Notice that by Theorem 4.1 in [Urz15], both $e(Y) >> 0$ and $K_Y^2 >> 0$. In particular Y is of general type by classification of algebraic surfaces. We also note that $K_{Y'}$ is ample since it is numerically $(1 - 4/w^*)$ times the pull-back of the class of a line. Thus, by Theorem 4.3 in [Urz15], the number of potential (-1)-curves to be contracted over w^* tends to zero as w^* approaches infinity, and so $X_{\min}$ satisfies $K_{X_{\min}}^2 / e(X_{\min}) \to 1$ as $w^* >> 0$. ∎

Chapter 4

Open questions for future work

The results of Chapter 2 and Chapter 3 leaves some open questions that can be adressed in a future work.

- The first set of questions are about the birational connection between Kollár surfaces and n-th root covers of $\mathbb{P}^2$. This connection proved to be useful to understand the geometry of Kollár surfaces, and to simplify combinatorial computations for n-th root covers. Even though the birational map was described explicitly, it could be very hard to find a Kollár surface starting from an n-th root cover totally branched along four lines in general position. Then it is interesting to understand if this situation is an isolated case, or we can find conditions for a n-th root cover to have this property. More precisely, if we consider another n-th root cover Z of $\mathbb{P}^2$, now branched along a different divisor, is it be possible to find a 3-dimensional weighted projective space $\mathbb{P}(n_1, n_2, n_3, n_4)$ and a surface $X \subset \mathbb{P}(n_1, n_2, n_3, n_4)$ birational to Z? If so, what are sufficient conditions on Z for this to happen?

- When we described the birational map between a Kollár surface and an n-th root cover of $\mathbb{P}^2$ totally branched along four lines in general position, especially when proving Corollary 2.18, we used strongly the fact that we were working on surfaces. Specifically, we used that every birational map is the composition of blowups and blowdowns, and that the minimal resolution of 2-dimensional singularity is unique. Therefore, the same procedure cannot be applied to higher dimensional Kollár hypersurfaces. Is it possible to describe explicitely the birational map for Kollár hypersurfaces of higher dimension?

- Previous to the work of Hwang and Keum, the only known examples of $\mathbb{Q}$HPP were the ones obtained via Kollár surfaces, and 13 infinite series of examples constructed by Keel and McKernan in [KM99, §19]. In [HK12] the authors construct several examples of rational $\mathbb{Q}$HPP with one, two and three cyclic quotient singularities, and ample canonical class, starting from different curve configurations in $\mathbb{P}^2$. In [HK12, Rem. 4.4] they mention that they can obtain examples of rational $\mathbb{Q}$HPP with the same type of singularities as the one constructed by Keel and McKernan. Is it possible to classify the type of singularities that can appear in a rational $\mathbb{Q}$HPP with only cyclic quotient singularities? If so, can all of them be realized by the examples of Hwang and Keum?

- An interesting topic in algebraic surfaces is to find simply connected surfaces of general type with geometric genus 0, that are different from the few examples known currently. This was one of the motivations to start studying these Kollár surfaces. Finally, it turned that all n-th root covers of $\mathbb{P}^2$ totally branched along four lines in general position with $p_g = 0$ were rational. The next step is to consider n-th root cover of $\mathbb{P}^2$ totally branched along d lines in general position, with $d \geq 5$. Let $\mu_1, \ldots, \mu_d$ the multiplicities asigned to each line in $\mathbb{P}^2$, all of them relatively prime to n, and $\sum \mu_i \equiv 0 \pmod{n}$. Then using [Urz10, Prop. 3.2] the geometric genus is

$$p_g = \frac{(n-1)(3d^2n - 17dn - 2d + 24n)}{24n} + \sum_{i<j} s(\mu_i, \mu_j; n).$$

Recall from Section 1.7 that for all μ_i, μ_j, $s(\mu_i, \mu_j; n) \leq s(1, 1; n) = (n-1)(n-2)/12n$. As d lines in general position have $\binom{d}{2}$ nodes, then

$$\begin{aligned} p_g &\geq \frac{(n-1)(3d^2n - 17dn - 2d + 24n)}{24n} - \frac{d(d-1)(n-1)(n-2)}{24n} \\ &= \frac{(n-1)(d-2)(dn + d - 6n)}{12n}. \end{aligned}$$

Hence if $d \geq 6$, we have that $p_g > 0$. For the case when $d = 5$, notice that

$$p_g = \frac{(n-1)(14n-10)}{24n} + \sum_{i<j} s(\mu_i, \mu_j; n),$$

and that $(n-1)(14n-10)/24n > 7s(1, 1; n)$. As there appears only 10 Dedekind sums in the geometric genus formula when $d = 5$, then $p_g > 0$ for the case $d = 5$. Then we can only find n-th root covers of $\mathbb{P}^2$ totally branched along d lines in general position, with $p_g = 0$ if $d \leq 4$.

Therefore, to look for examples of simply connected surfaces of general type with geometric genus 0 through this method, one should consider n-th root covers o $\mathbb{P}^2$, but now totally branched along lines in special positions.

- In [Al94] Alexeev proved the following result.

  Theorem. *Let $(X, B = \sum b_i B_i)$ be a log canonical pair with coefficients b_i in a set $\mathcal{S} \subset [0,1]$ satisfying descending chain condition (DCC set). Then the set of $(K_X + B)^2$ is also a DCC set.*

  Then it is natural to ask how small could $(K_X + B)^2$ be for a given set $\mathcal{S}$ and $K_X + B$ ample? The most recent achievement in this direction is the result of Alexeev and Liu in [AL16], who proved that if $\mathcal{S} = \emptyset$, then the minimum of $(K_X + B)^2 = K_X^2$ is less or equal than 1/48983. Let

  $$\mathcal{K} := \{K_X^2 \mid X \text{ is a normal surfaces with log canonical singularities and } K_X \text{ is ample}\}$$

  and let $Acc(\mathcal{K})$ be the set of accumulation points of $\mathcal{K}$. As a corollary of Alexeev's theorem, we know that $Acc(\mathcal{K})$ is a DCC set. Regarding this set, Blache proved in [Bl95] that $Acc(\mathcal{K})$ is not empty and proposed the following conjecture.

  Conjecture.

  (a) $\mathbb{N} \subset Acc(\mathcal{K})$;

  (b) The minimum of $Acc(\mathcal{K})$ is 1.

  Notice that by Proposition 2.7 and the computation of the discrepancy at the end of Section 1.6, we have that Kollár surfaces are log canonical. If we restrict to Kollár surfaces X that are well formed (i.e. $\gcd(w_1, w_3) = \gcd(w_2, w_4) = 1$), then it is easy to determine when K_X is ample and to compute K_X^2. Even more, by Theorem 2.28 we can increase arbitrarily the values of a_i, while fixing w^*. By computing the limit of K_X^2 when $a_i \to \infty$, we can prove that $\mathbb{N} \subset Acc(\mathcal{K})$, and that there are infinitely many accumulation points of $\mathcal{K}$ less than 1. The next step for this is to describe what accumulation points of $\mathcal{K}$ can be obtained by sequences of Kollár surfaces, which is equivalent to determine what kind of Kollár surface can be obtained given a_1, a_2, a_3.

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