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Cohomological equation for isometries of Gromov hyperbolic spaces

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Chapter 1

Introduction

1.1 History and Motivation

A dynamical system is a map $T : \Omega \to \Omega$ where Ω or T have an interesting property. It could be a metric, topological or measurable space. In this case our space Ω will be a compact metric space; T will also have properties described later. Given a map $A : \Omega \to \mathcal{G}$, where $\mathcal{G}$ is a topological group, one of the main problems of dynamical systems is to determine when this map A, called cocycle, is conjugated to another cocycle taking values in a small subgroup $\mathcal{H} \to \mathcal{G}$. That means that there is a map $B : \Omega \to \mathcal{G}$ such that the conjugated map

$$C(\omega) = B(T\omega)^{-1} A(\omega) B(\omega) \tag{1.1}$$

takes values in $\mathcal{H}$. One very studied case is when $\mathcal{H}$ is the identity. In this case the equation looks like

$$A(\omega) = B(T\omega) B(\omega)^{-1} \tag{1.2}$$

This equation will be our main object of study along this work.

The first result related to this subject is due to Livsic (see [Liv72]) who proved that there is a solution for the equation in the case $A : \mathbb{C} \to \mathbb{C}$ is Hölder, satisfies a certain condition over the periodic orbits and T is topology transitive satisfying a shadowing type property. (See chapter 3 for further details). The Livšic problem consist in solve equation 1.2 for a $A : \mathcal{G} \to \mathcal{G}$ under those assumptions. There are several types of generalizations varying the group, the cocycle or the base T.

Another famous result regarding this problem is due to Kalinin (see [Kal08]) who solved Livšic problem for cocycles taking values on the group $\mathbf{GL}(n, \mathbb{R})$. In his work one can see the deep relation between the existence of the solution and the estimation of Lyapunov exponents. Inspired by Kalinin's result, there has been a lot of interest in resolve the problem for diffeomorphism of manifolds. Kocsard and Potrie solved the problem for low dimension manifolds (see [KP14]). Recently Avila, Kocsard and Liu, solved this problem for arbitrary dimension (see [AKL17]).

Another result related can be found in [GG14]. There, the Livšic problem is solved for cocycles taking values on a Banach ring.

Our work is in the framework of A taking values on the group $\textsc{Isom}(H)$ where H is a non-positive curved space. We deal particularly when H is a Gromov hyperbolic space. Related works, when H is a Busemann space or a $\mathrm{CAT}(\kappa)$ type space, are [BN15] and [CNP11].

We proceed now to give a brief explanation on how we approach to the problem. One main tool we will use is the fact that there is a solution for the equation 1.2 if and only if for every point (ω_0, h_0) exist a map $s_{(\omega_0,h_0)} : \Omega \to H$ such that

$$A(\omega)s_{(\omega_0,h_0)}(\omega) = s_{(\omega_0,h_0)}(T\omega) \tag{1.3}$$

You can think this maps as curves on the product $\Omega \times H$. To prove that claim, we pass through on intermediate step defining the so-called skew-product

$$F(\omega, h) = (T\omega, A(\omega)h) \tag{1.4}$$

We will prove that the existence of the solution, or equivalent the existence of the curves, is the same the skew product is conjugated to the map $G(\omega, h) = (T\omega, h)$.

Passing to the geometric considerations, roughly speaking, a δ (geodesic) hyperbolic space is a space where the triangles are thin. Typical examples are the classic hyperbolic space $\mathbb{H}^n$ and $\mathbb{R}$-trees. There are a lot of equivalences of δ-hyperbolicity under certain conditions. The most general one is given by Gromov in [Gro87]. Given a metric space X the Gromov product is defined by

$$(x, y)_p = \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$$

If every quadruple of points $x, y, z, p \in X$ satisfies

$$(x, y)_p \geq \min\{(x, z)_p, (y, z)_p\} - \delta$$

we will say that X is a δ (not necessarily the same as before) hyperbolic space. One can see it isn't needed the space to be geodesic when working with this last definition. Nevertheless, through this work we will only deal with the geodesic case. Even more, we only deal with uniquely geodesic spaces.

We consider next the Gromov boundary of the space H. It can be defined in therms of geodesic, as the equivalence relation $\gamma_1 \sim \gamma_2$ if and only if $d(\gamma_1(t), \gamma_2(t))$ is uniformly bounded. It isn't the only way of define it, we will indicate the other ones in chapter 4.

Given an isometry of H, say A, there is a natural extension A^* to the boundary ∂H due to isometries send geodesic into geodesics. Given all this setup we wonder if

> Does the existence of a solution of 1.2 for the extended action A^* (as an action of ∂H) implies the existence of a solution of 1.2 for A?

Although we couldn't give an absolute answer to the question, we manage to affirmative answer it for some particular cases, specifically in the case of $\mathbb{H}^n$ and the Cayley graph of the free group in two symbols. Our main tool use the fact that geodesics images are isometric to segments of $\mathbb{R}$ and get a solution here using the Livšic theorem. In order to do that, we construct a family of geodesics γ_ω, $\omega \in \Omega$ and one crucial point of the construction is determine a Hölder parametrization for each geodesic, that is, there exists constants $C > 0$, $\alpha \in (0, 1]$ such that

$$d(\gamma_{\omega_1}(t), \gamma_{\omega_2}(t)) \leq Cd(\omega_1, \omega_2)^\alpha$$

The constants may depend on t. In two cases we work around and find a way to get that type of parametrization, but we won't be able to do it in the general case.

1.2 About the chapters

In the second chapter we talk about the equivalences of the existence of a solution for 1.2 and the existence of the maps with the property 1.3 and some regularity issues. As we mention, these maps will be fundamental to obtain our result. Third chapter is a revisit to the Livšic theorem, its proof and examples. In chapter four we consider the geometric context; we talk about Gromov Hyperbolic spaces, the multiply ways of define them, their boundary and examples. We highlight the result 4.7 which will be very helpful along this work. Chapter five contains the results on the problem regarding the main question. Finally in the last chapter, we discuss about what we can expect on the problem and we review concepts of coarse geometry.

Chapter 2

Problem Equivalences

In this chapter we present two problems that are equivalent to finding a solution of the equation 1.2. They are just reformulations, but give us a graphic or geometric way of looking at the problem. One is construct the so called skew product for (T, A) and the other one consist in the existence of the invariant maps (we will detail what means invariant).

Later we talk about the regularity of those construction which is a crucial issue for us, because in order to use the Livsic theorem we need the objects to be Hölder.

2.1 Invariant curves

Let $T\colon \Omega \to \Omega$ and $A\colon \Omega \to G$ where G is a group acting on H, and both Ω, H are metric spaces. The first map will be called the dynamic or base, and A the cocycle.

Theorem 2.1. *The next three propositions are equivalent*

- ***a.*** *The equation*
$$A(\omega) = B(T\omega)B(\omega)^{-1}$$
has a solution.That is, there is a continuous map $B : \Omega \to \mathrm{HOM}(H)$ *satisfying the equation above.*

- ***b.*** *The skew product* $F(\omega, h) = (T\omega, A(\omega)h)$*, is conjugated, through a map that preserve the first coordinate, to* $\boldsymbol{I} = (T\omega, h)$*. That means there exists a contoinuous* $G : \Omega \times H \to \Omega \times H$ *given by* $(\omega, h) \mapsto (\omega, b(\omega, h))$*, where* $b(\omega, \cdot)$ *belongs to* $\mathrm{HOM}(H)$*, and such that* $F = G \circ \boldsymbol{I} \circ G^{-1}$*.*

- ***c.*** *For each* (ω_0, h_0) *there exists a continuous* $s_{(\omega_0,h_0)} \colon \Omega \to H$ *and such that* $s_{(\omega_0,h_0)}(\omega_0) = h_0$ *and* $A(\omega)s_{(\omega_0,h_0)}(\omega) = s_{(\omega_0,h_0)}(T\omega)$

Remark. We will refer to the maps defined in c as *curves*, because the graph $(\omega, s(\omega))$ can be thought as that.

Proof. First, let's prove *a* implies *b*. Define $G(\omega, h) = (\omega, B(\omega)^{-1}h)$ and see that

$$G \circ F \circ G^{-1}(\omega, h) = (T\omega, B(T\omega)^{-1}A(\omega)B(\omega)h) = (T\omega, h)$$

So G is the conjugation map that we are looking for. Now, let's see b implies a. Under this assumption, there is $b \colon \Omega \times H \to H$ such that $G(\omega, h) = (\omega, b(\omega, h))$ conjugate F to $\boldsymbol{I}$. In the second coordinate, this means

$$b(T\omega, A(\omega)b^{-1}(\omega, h)) = h \tag{2.1}$$

So, it's enough to define $B(\omega)h = b^{-1}(\omega, h)$. The maps $b(\omega, h)$ and $B^{-1}(\omega)h$ are essentially the same, but it's more familiar to work with b because is can be seen as a coordinate change. See 2.1.

It's important to note that $b^{-1}(\omega, h)$ is a notation abuse, since the domain of b^{-1} is H. But with ω fixed we have a map $b_\omega \colon H \to H$. that has inverse. So every time we write down $b^{-1}(\omega, h)$ we are thinking on that map.

To prove b implies c, we will use the fact we already have curves when the map A act as the identity. These curves are constants passing through every height h. Using the conjugation map G over those curves, we can construct $s_{(\omega_0,h_0)}(\omega) = b^{-1}(\omega, b(\omega_0, h_0))$. Let's verify this definition works. First it's clear $s_{(\omega_0,h_0)}(\omega) = b^{-1}(\omega, b(\omega_0, h_0)) = h_0$. Second, note that

$$b(T\omega, A(\omega)b^{-1}(\omega, h)) = h \iff A(\omega)b^{-1}(\omega, h) = b^{-1}(T\omega, h)$$

So under the assumption that the equation 2.1 holds, we have

$$A(\omega)s_{(\omega_0,h_0)} = A(\omega)b^{-1}(\omega, b(\omega_0, h_0)) = b^{-1}(T\omega, b(\omega_0, h_0)) = s_{(\omega_0,h_0)}(T\omega)$$

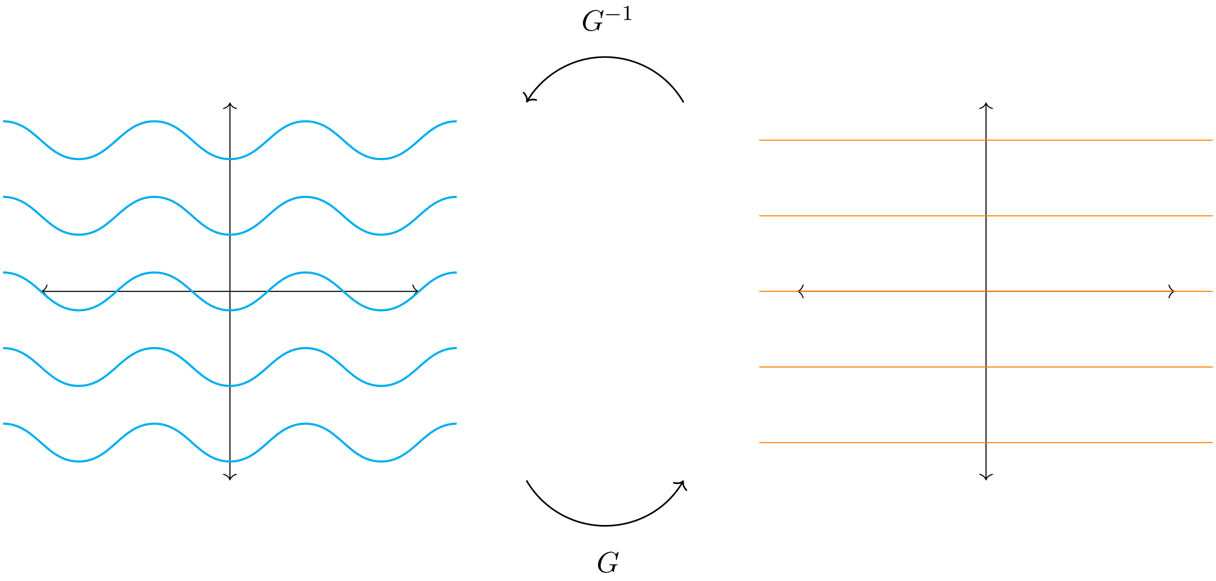


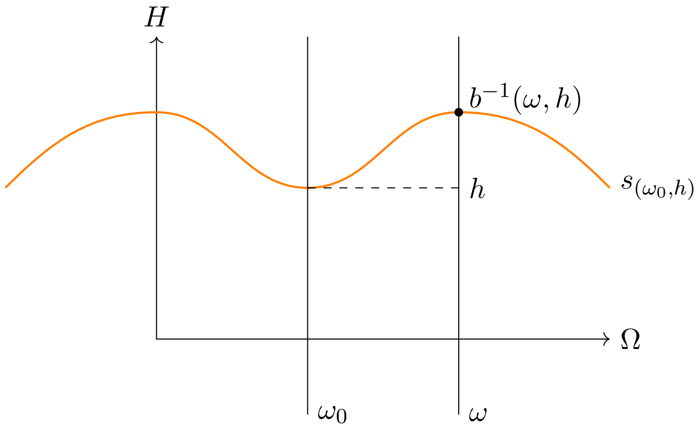
FIGURE 2.1: Coordinate Change

Now, to prove *c* implies *b*, fix an ω_0. The curves passing through this fiber will help us to define the function $b(\omega, h)$. Given a point (ω, h), go forward or back until find the fiber ω_0, i.e, stop at the point (ω_0, h). Now, follow the curve $s_{(\omega_0,h)}$ (the curve passing through (ω_0, h)) until you go back the fiber ω (see figure). Define this point of intersection as $b^{-1}(\omega, h)$, that is $b^{-1}(\omega, h) := s_{(\omega_0,h)}(\omega)$. To define $b(\omega, h)$ simply do the inverse process. Given a point (ω, g), follow the curve $s_{(\omega,g)}$ until the fiber ω_0 and define the intersection point $s_{(\omega,g)}(\omega_0)$ as $b(\omega, h)$. Graphically it's clear that these definitions are inverse one of each other.

The invariance of the curves give us the condition that we are looking for

$$A(\omega)b^{-1}(\omega, h) = A(\omega)s_{(\omega_0,h)}(\omega) = s_{(\omega_0,h)}(T\omega) = b^{-1}(T\omega, h)$$

Which as we saw it's equivalent to the equation 2.1, and therefore this b is the correct map.



■

Remark. It's interesting to note $b(\omega_0, h) = s_{(\omega_0,h)}(\omega_0) = h_0$ hence is the identity for that fixed point. Also note that b depends on the choice of ω_0 and therefore isn't unique. This is a natural consequence, because the co-homological equation doesn't have a unique solution either. If B is a solution of 1.2 and f an element of G, i.e map from H in itself, $B \circ f$ is another solution. Indeed,

$$B(T\omega)f(B(\omega)f)^{-1} = B(T\omega)B(\omega)^{-1}$$

Along the proof of 2.1 we didn't talk about the regularity and continuity of all the maps involved. We would like, for example, the curves $s_{(\omega_0,h_0)}(\omega)$ will be continuous as a function, but maybe we also want to be continuous in the sense that two curves are close if the points were they pass through are close. As the curves s and b are defined one in function of the other, we will require the map b to be continuous in some sense, and also if we ask for b to be continuous in the first coordinate, we will need to $B(\cdot)$ to be also continuous. We will formalize all this ideas in the next section.

2.2 Regularity of the curves

Theorem 2.2. *Using the notation of 2.1*

1. *For every $\varepsilon > 0$ there is $\delta \leq 0$ such that if $|h_1 - h_2| \leq \delta$ then*

$$\left|s_{(\omega_1,h_1)}(\omega) - s_{(\omega_2,h_2)}(\omega)\right| < \varepsilon$$

for every $\omega_1, \omega_2, \omega$, implies the maps $b(\omega, \cdot)\colon H \to H$ belongs to $\mathrm{Hom}(H)$.

2. *For every $\varepsilon > 0$ there is $\delta \leq 0$ such that if $|\omega_1 - \omega_2| \leq \delta$ then*

$$\sup_{h\in H}\left|s_{(\omega_1,h)}(\omega) - s_{(\omega_2,h)}(\omega)\right| < \varepsilon$$

implies the map $b\colon \Omega \to H^\Omega$ is continuous in the uniform convergence topology.

3. *The curves are Lipschitz in the formal variable, that is,there is a $K \in \mathbb{R}$*

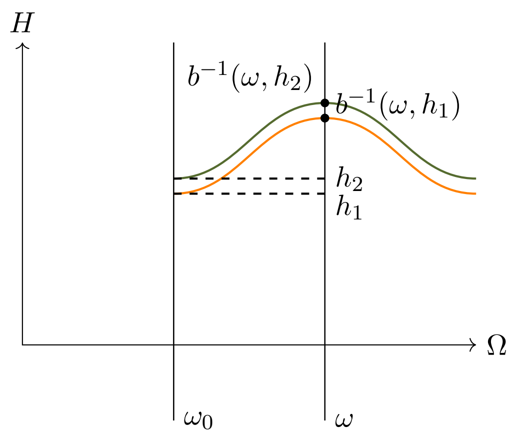
$$\left|s_{(\omega,h)}(\omega_1) - s_{(\omega,h)}(\omega_2)\right| < K|\omega_1 - \omega_2|$$

implies the map $b^{-1} : \Omega \to H^\Omega$ is continuous in the uniform convergence topology.

Proof. The proofs are pretty much write the definitions constructed in 2.1.

We have already proved $b(\omega, \cdot)$ has an inverse. The fact both of them are continuous can be easily deduced from condition one and the identities

$$|b(\omega, h_1) - b(\omega, h_2)| = |s_{(\omega,h_1)}(\omega_0) - s_{(\omega,h_2)}(\omega_0)|$$
$$|b^{-1}(\omega, h_1) - b^{-1}(\omega, h_2)| = |s_{(\omega_0,h_1)}(\omega) - s_{(\omega_0,h_2)}(\omega)|$$



For the second condition we have

$$|b(\omega_1, h) - b(\omega_2, h)| = |s_{(\omega_1,h)}(\omega_0) - s_{(\omega_2,h)}(\omega_0)|$$

So if ω_1 and ω_2 are close, $|b(\omega_1, h) - b(\omega_2, h)|$ are small, not depending on h. Therefore, the map b is continuous in the said topology. The proof of the last statement is pretty much the same. Just write down the definition of b^{-1} given in 2.1 and check the condition. ■

Recall a map $f\colon X \to Y$ between metric spaces is Hölder if there is constants $C > 0$, $\alpha \in (0, 1]$ such that

$$d(f(x), f(y)) \leq Cd(x, y)^{\alpha}$$

For $B : \Omega \to G$ and G acting on H we say that is Hölder if for every compact $K \subset H$ there exists constants

The next proposition established what we need for the curves in order to apply the Livsic theorem in the following sections.

Proposition 2.3. *The solution of 1.2, B, is Hölder if and only if the curves defined in 2.1 are Hölder maps.*

Proof. Following the construction of the proof of 2.1, one can deduce that the curve passing through the point (ω_0, h) satisfies

$$s_{(\omega_0,h)}(\omega) = B^{-1}(\omega)B(\omega_0)h_0$$

So you can see that B acts Hölder on h_0 if and only if the map $s_{(\omega_0,h_0)} : \Omega \to H$ is Hölder for every h.

■

Finally we discuss the nature of the solution B.

Lemma 2.4. $\text{ISOM}(H)$ *is a closed subset of* $\text{HOM}(H)$ *in the point-wise topology.*

Proof. The proof is a simple computation. If i_n is a sequence of isometries converging to $j \in \text{HOM}(H)$ and $h_1, h_2 \in H$.

$$|j(h_1) - j(h_2)| = \lim_n |i_n(h_1) - i_n(h_2)| = |h_1 - h_2|$$

■

Proposition 2.5. *Let be $A : \Omega \to \textsc{Isom}(H)$ and suppose the cohomological equation*

$$A(\omega) = B(T\omega)B^{-1}(\omega)$$

has a solution $B : \Omega \to \textsc{Hom}(H)$. Then B is in fact a cocycle of isometries, i.e, $B : \Omega \to \textsc{Isom}(H)$.

Proof. Given ω_0 without loss generality suppose that $B(\omega_0)$ is the identity. Note this implies

$$\begin{gathered} B(T\omega_0) = A(\omega_0) \\ B(T^2\omega_0) = A(T\omega_0)B(T\omega_0) = A(T\omega_0)A(\omega_0) \\ \vdots \end{gathered}$$

And so on until the n-step where we have

$$B(T^n\omega) = \prod_{i=1}^{n} A(T^i\omega_0)$$

Hence $B(T^n\omega_0)$ is an isometry for every n. As the extension to the whole space exists by assumption and using the previous lemma, B is a cocycle of isometries.

∎

Chapter 3

Livsic Theorem

This chapter is mainly based on [NP11].

In this chapter we talk about the Livsic theorem which is probably the first result ever proved respect to cohomolgy equations for reals (or complex) numbers.

Given the cohomoly equation

$$A(x) = B(Tx)B(x)^{-1}$$

The simplest obstruction for the existence of the solution B is the *Periodic orbit Obstruction*: if $p \in X$ and $n \in \mathbb{N}$ satisfy $T^n p = p$,

$$A(n,p) = \prod_{i=0}^{n-1} A(T^i x) = \prod_{i=0}^{n-1} B(T^{i+1}x)B(T^i x)^{-1} = B(T^n p)B(p)^{-1} = e_{\mathcal{G}}. \tag{3.1}$$

The *Livšic problem* consists in determining whether the condition (3.1) is not only necessary but also sufficient for A being a coboundary. Livsic prove the result when $\mathcal{G} = \mathbb{R}, \mathbb{C}$ (in fact, for any abelian group), A is Hölder continuous and T is a topologically transitive hyperbolic homeomorphism.

The key property of hyperbolic homeomorphism used in the proof, is the (exponential) closing property. Let $T : X \to X$ be a homeomorphism and let x, y be points of X. We say that the orbit segments $x, Tx, \dots, T^k x$ and $y, Ty, \dots, T^k y$ are *exponentially δ-close with exponent* $\lambda > 0$ if for every $j = 0, \dots, k$,

$$dist_X(T^j x, T^j y) \leq \delta e^{-\lambda \min\{j, k-j\}}.$$

We say that T satisfies the *closing property* if there exist $c, \lambda, \delta_0 > 0$ such that for every $x \in X$ and $k \in \mathbb{N}$ so that $dist_X(x, T^k x) < \delta_0$, there exists a point $p \in X$ with $T^k p = p$ so that letting

$\delta := c\, dist_X(x, T^k x)$, the orbit segments $x, Tx, \ldots, T^k x$ and $p, Tp, \ldots, T^k p$ are exponentially δ-close with exponent λ and there exists a point $y \in X$ such that for every $j = 0, \ldots, k$,

$$dist_X(T^j p, T^j y) \le \delta e^{-\lambda j} \quad \text{and} \quad dist_X(T^j y, T^j x) \le \delta e^{-\lambda(n-j)}.$$

Important examples of maps satisfying the closing property are hyperbolic diffeomorphisms of compact manifolds (See [Fie02] or [ST10] for references) as full shifts and linear Anosov maps.

Theorem 3.1 ([Liv72]). *Let $T : X \to X$ be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $\psi : X \to \mathbb{R}$ be an $\alpha-$Hölder-continuous function for which the condition (3.1) holds, that is, for every point $p \in X$ and $k \ge 1$ such that $T^k p = p$, one has $\sum_{j=0}^{k-1} \psi(T^j p) = 0$. Then there exists an $\alpha-$Hölder-continuous function $\phi : X \to \mathbb{R}$ that is a solution of the cohomological equation*

$$\phi \circ T - \phi = \psi.$$

Proof. Let $x_0 \in X$ be such that $\overline{\{T^n x_0\}_{n\in\mathbb{N}}} = X$. We define ϕ by letting $\phi(x_0) := 0$ and $\phi(T^n x_0) := \sum_{j=0}^{n-1} \psi(T^j x_0)$. We next check that ϕ is $\alpha-$Hölder-continuous on $\{T^n x_0\}_{n\in\mathbb{N}}$. Let $n > m$. There are two cases to consider:

- Assume that $dist_X(T^m x_0, T^n x_0) < \delta_0$. Then there exists a point $p \in X$ satisfying $T^{n-m} p = p$ and such that for every $j = 0, \ldots, n-m$,

$$dist_X(T^j(T^m x_0), T^j p) \le c\, dist_X(T^n x_0, T^m x_0) e^{-\lambda \min\{j, n-m-j\}}.$$

  This yields

$$
\begin{aligned}
|\phi(T^n x_0) - \phi(T^m x_0)| &= \left|\sum_{j=0}^{n-m-1} \psi(T^{m+j} x_0)\right| \\
&= \left|\sum_{j=0}^{n-m-1} \big(\psi(T^{m+j} x_0) - \psi(T^j p)\big) + \sum_{j=0}^{n-m-1} \psi(T^j p)\right| \\
&\leq \sum_{j=0}^{n-m-1} \big|\psi(T^{m+j} x_0) - \psi(T^j p)\big| \\
&\leq \sum_{j=0}^{n-m-1} [\psi]_\alpha \, dist_X(T^{m+j} x_0, T^j p)^\alpha \\
&\leq \sum_{j=0}^{n-m-1} c^\alpha [\psi]_\alpha \, dist_X(T^n x_0, T^m x_0)^\alpha \, e^{-\lambda\alpha \min\{j, n-m-j\}} \\
&\leq \frac{2\, c^\alpha [\psi]_\alpha}{1 - e^{-\lambda\alpha}} \, dist_X(T^n x_0, T^m x_0)^\alpha .
\end{aligned}
$$

■

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Chapter 4

Geometry

This section is based [BH11] chapter III.

4.1 Delta Hyperbolic spaces

The δ-hyperbolicity can be defined in several ways depending on what characteristics has the space that one is dealing. The most general one is 4.3 given by Gromov in [Gro87] that doesn't require any additional assumptions over the space. Although, We start given a more particular, but intuitive, definition involving geodesic triangles.

Given a metric space H recall a geodesic is an isometric map $\gamma : [a, b] \to H$. The space H is called geodesic if every pair of points can be joined by a geodesic; if it is unique we called the space unique geodesic. A geodesic triangle (or simply a triangle) is a set consisting of the union of three geodesic that join three points x, y, z between each other. Points are called vertices and the geodesic that join two vertices, x, y, denoted by $[x, y]$, is called a side of the triangle.

Definition 4.1. Let $\delta > 0$. A geodesic triangle $\triangle ABC$ is said to be δ*-slim* if each of its sides is contained in a δ neighborhood of the union of the other two sides. That is to say, for every point $p \in [A, B]$ there exists $q \in [B, C] \cup [A, C]$ such that $d(p, q) \leq \delta$. A space H is said to be δ-hyperbolic if every triangle is δ-slim.

The first and obvious example of these spaces are trees (undirected graphs in which two vertices are joined by one path) which are 0-hyperbolic. Classic hyperbolic space $\mathbb{H}^n$, as expected, is also $\delta_{\mathbb{H}}$-hyperbolic, and, as a consequence, CAT(κ) spaces are δ-hyperbolic too for $\kappa < 0$. Later it will be show what values $\delta_{\mathbb{H}}$ can take . The usual euclidean plane $\mathbb{E}^2$ isn't δ-hyperbolic neither $\mathbb{E}^n$, but the real line it is. More examples will be reviewed later, using the Gromov definition for δ-hyperbolic spaces.

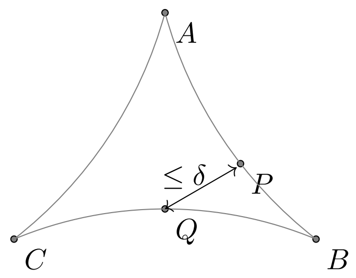


FIGURE 4.1: Slim Triangle

Moving on, we now introduce an equivalence to 4.1. One can think the δ as a measure of how far are the triangle to be a *tripod*. A tripod is a graph with three vertices of valence one and a forth with valence three. Indeed, for each geodesic triangle we can construct a natural tripod. This can be done collapsing the triangle respect to the in-center. Given a triangle Δ with vertex x, y, z the triangle inequality tell us there exists unique three numbers a, b, c such that $d(x, y) = a + b$, $d(x, z) = a + c$ and $d(y, z) = b + c$. Consider the tripod $T = T(a, b, c)$ that has one central vertex O and vertex v_x, v_y, v_z such that $d(v_x, O) = a$, $d(v_y, O) = b$ and $d(v_z, O) = c$. There is a natural isometry $\chi : \Delta \to T$ that maps the vertex to the vertex. A nice way to measure the thickness of the triangle is measuring the diameter of $\chi^{-1}(O)$, called in-size. In the case the in-circle of the triangle exists, $\chi^{-1}(O)$ consists in three points where the in-circle and the triangle intersects. The more short is the in-size, more thin is the triangle.

Proposition 4.2. *A proper geodesic metric space X is δ-hyperbolic if and only there is a δ' such that every triangle $\triangle$, insize $\triangle \leq \delta_2$.*

Proof. See [BH11] III. 1.17. ■

This is a very useful tool. For example, you can prove easily that $\mathbb{H}$ is a δ-hyperbolic space using the proposition 4.2. As triangle areas are bounded by π in $\mathbb{H}$, the in-circles have bounded areas too, so the diameters of them are bounded. You can use a similar argument to show euclidean plane is not hyperbolic: as the in-circles are not bounded there is no such δ'.

Note that numbers a, b, c (the lengths of the tripod sides) satisfies the equations

$$
\begin{aligned}
2a &= d(x,y) + d(x,z) - d(y,z) \\
2b &= d(y,z) + d(x,y) - d(x,z) \\
2c &= d(x,z) + d(y,z) - d(x,y)
\end{aligned}
$$

This give us a clue about how you can extend the hyperbolic notion to spaces which are not geodesic. Indeed, Gromov did it and we review now that definition.

Definition 4.3. Let X a metric space. Given three points $x, y, p \in X$ the *Gromov Product* is defined as

$$(x,y)_p = \frac{1}{2}(d(x,p) + d(y,p) - d(x,y))$$

As we said, one way of understand this definition is to see it as the distance of the in-center of the triangle with vertex p, x, y, to the sides $[p,x]$ or $[p,y]$ and therefore this measure how thick the triangle is. The more large is the Gromov product less thick is the triangle. In the hyperbolic plane it can be illustrated clear how this work. As expected one can reformulate the δ-hyperbolic condition in therms of the Gromov product.

Proposition 4.4. *A geodesic metric space X is δ-hyperbolic if and only there is a δ'' such that every quadruple of points u, x, y, z*

$$(x,y)_u \leq \min\{(x,z)_u, (y,z)_u\} - \delta'' \tag{4.1}$$

Proof. See [BH11] III. 1.22. ∎

Remark. We can define the δ-hyperbolicty for any metric space X as one where the condition 4.1 holds for every quadruple of points.

Remark. The equation 4.1 is equivalent to

$$d(x,u) + d(z,y) \leq \max\{d(x,y) + d(u,z), d(x,z) + d(y,u)\} + 2\delta'' \tag{4.2}$$

For all points x, y, z, u. It's known as the 4-point condition.

Indeed, δ, δ' and δ'' can be different. In the case of the hyperbolic plane $\mathbb{H}$ In [NS14] it's proved that the optimal δ'' is equal to $\log 2$. On other hand, consider the ideal triangle with vertices on the points $(1,0)$, $(-1,0)$ and infinity. We can find the δ for the neighborhood halving the distance between i and $i+2$ which is given by

$$\delta = \tanh^{-1}\left(\frac{(i+2)-i}{(i+2)+i}\right) = \log(1+\sqrt{2})$$

We will show and detail next a lot of interesting examples which were given by Gromov in [Gro87].

Example 4.1.1. Any metric space with finite diameter D is D-Gromov Hyperolic. Indeed,

$$\min\{(x,z)_u,(y,z)_u\}-(x,y)_u \leq \min\{d(y,z),d(x,z)\} \leq D$$

Not that, for example, any bounded subset of the euclidean plane is δ-hyperbolic. This illustrate δ-hyperbolic spaces doesn't need to be spaces with *real* negative curvature as someone can expect due to the previous examples.

Example 4.1.2. Let X be any metric space with metric d and define a new metric

$$d_l(x,y)=\log(1+d(x,y))$$

(X,d') is $2\log 2$-hyperbolic.

Set

$$\begin{aligned} a &= d(x,y); & a_l &= d_l(x,y) \\ b &= d(x,z); & b_l &= d_l(x,z) \\ c &= d(y,z); & c_l &= d_l(y,z) \end{aligned}$$

Suppose $b=\max\{b,c\}$ by the triangle inequality $a+1 \leq 2b+2$ which implies $a_l \leq b_l+\log 2$. Hence

$$a_l \leq \max\{b_l,c_l\}+\log 2 \tag{4.3}$$

Consider the same inequality for x,z,t and sum it with 4.3 to get

$$\begin{aligned} d(x,y)+d(z,t) &\leq \max\{d(x,z),d(y,z)\}+\max\{d(x,t),d(z,t)\}+2\log 2 \\ &= \max\{d(x,z)+d(y,z),d(x,t)+d(z,t)\}+2\log 2 \end{aligned}$$

Example 4.1.3. Let X_0 be a metric space and let $f\colon \mathbb{R}\to\mathbb{R}$ be a positive monotone increasing function satisfying $f(t+1)\geq \lambda f(t)$ for some fixed $\lambda>1$. Consider a metric D on $X=X_0\times\mathbb{R}$ such that the embedding $t\mapsto(x_0,t)$ is isometric and

$$D[(x,t),(y,t)] \leq f(t)d(x,y)$$

It can be prove (X,D) is a hyperbolic space.

4.2 Gromov Boundary

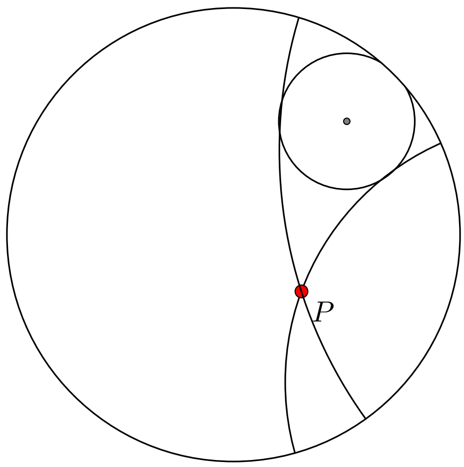
We proceed to define the boundary of a geodesic δ-hyperbolic space

Definition 4.5. Fix a point O of the space H. We will say two geodesics $\gamma_1, \gamma_2 : [0, \infty] \to H$, passing through O, are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) \leq K$. The *Gromov Boundary* is defined as the equivalence classes among all geodesics of the space, namely,

$$\partial(H, p) = \{[\gamma] : \gamma(0) = O, \gamma \text{ is a geodesic in } H\}$$

It can be prove the boundary doesn't depend, up to an homeomorphism, on the point chosen.

The gromov boundary of trees are Cantor sets. The boundary of $\mathbb{H}^n$ is a sphere.



However, this definition won't be as useful as we want for our purposes, so we will define the boundary in another way that is more convenient for us.

Definition 4.6. Two geodesics rays $\gamma_1, \gamma_2 : \mathbb{R} \to H$ are equivalents if $\sup_{t \in \mathbb{R}} d(\gamma_1(t), \gamma_2(t)) \leq K$. As before, this boundary ∂H, is defined as the set of all the equivalence classes.

Why is this boundary more convenient? Because we can think every geodesic ray has two *ends*, $\gamma(\infty)$ and $\gamma(-\infty)$. Although, both boundaries are the same up to an homeomorphism.

The next property is fundamental for our proposes.

Proposition 4.7. *The boundary ∂H is visible, that is to say, given two points $x, y \in \partial H$, there exists a geodesic γ such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$.*

Proof. See [BH11] III.3.2. ■

Remark. The geodesic ray γ, of course doesn't need to be unique, but if the space H is unique geodesic, γ is unique.

Our next aim is to give a topology and metric for the boundary.

4.3 Topology and metric

We start defining a topology, but this one doesn't give us much information about the geometry of the space. Later, we define a more convenient using the Gromov product.

Definition 4.8. A generalized geodesic ray is a geodesic ray $\gamma : \mathbb{R} \to H$ or a path that is a geodesic until a certain point and then constant.

Definition 4.9. Let H be a proper geodesic hyperbolic space and a base-point p. We say that a sequence x_n converges to a point $x \in H \cup \partial H$, if there are generalized geodesic rays such that $\gamma_n(0) = p$ and $\gamma_n(\infty) = x_n$ if every subsequence of (γ_n) contains a subsequence that converges, in the compact-uniform topology, to a generalized ray γ with $\gamma(\infty) = x$.

Next we define a boundary for an arbitrary hyperbolic space with the help of the Gromov product.

Let be X an hyperbolic space and a base point $p \in X$. We say that a sequence (x_i) converges to infinity if $(x_i, x_j)_p$ when $i, j \to \infty$. Two such sequences $x_i, y_j \in X$ are related if $(x_i, y_j) \to \infty$. This defines an equivalence relation.

i. It's reflexive by definition.

ii. It's symmetric by the symmetry of $(\cdot, \cdot)_p$

iii. If $x_i \sim y_i$ and $y_i \sim z_i$, hyperbolicty give us

$$(x_i, z_i)_p \geq \min\{(x_i, y_i)_p, (y_i, z_i)_p\} - \delta$$

Definition 4.10. The boundary $\partial_s X$ is the set of equivalences classes of sequences (x_i) which converges to infinity.

Proposition 4.11. *Let H be a proper geodesic hyperbolic space. The map $x_i \to \lim_i x_i$ is a bijection among $\partial_s H$ and ∂H*

We need a lemma

Lemma 4.12. *Let H be a proper geodesic hyperbolic space. A sequence $(x_i) \in H$ converges to a point in ∂H if and only if converges to infinity.*

Proof. Suppose $\lim_i x_i = x \in \partial H$. The definition of converges implies there are generalized geodesic rays c_{n_k} ending on a subsequence x_{n_k} such that

$$d(c_{n_i}(t), c_{n_j}(t)) \leq \varepsilon$$

uniform for every compact $K \subset \mathbb{R}$ for a sufficient large pair of indexes i, j. So for a suitable choice of K and i, j, $d(x_i, x_j)$ is bounded implying immediately that $(x_i, x_j)_p \to \infty$.

For other side, consider the generalized geodesic rays γ_n start starts in a fix point p and ends in the points x_n. As geodesics are equicontinuous, the Àrzela Ascoli theorem ensures there is a subsequence such that converging uniformly in compacts to a generalized ray γ. Suppose that $\gamma(\infty) = x$ doesn't belong to ∂H. We would have

$$|(x_i, x_j)_p - (x_j, x)_p| \leq d(x_i, x)$$

and $(x_j, x)_p$ and $d(x_i, x)$ would be finite, contradicting the assumption. ∎

Proof. (*Of 4.11*) By the previous lemma the map $x_i \to \lim_i x_i$ is a well-defined injective map. The surjectivity follows from the fact that given a geodesic ray γ, $x_n = \gamma(n)$ is a sequence converging to infinity such that $x_n \to \gamma(\infty)$. ∎

Now we extend $(\cdot, \cdot)_p$ to the boundary. We could hope that the extension can be defined simply as

$$(x, y)_p = \lim_{i,j \to \infty} (x_i, y_j)_p \tag{4.4}$$

But it isn't possible in general. The limit could not exist or even depend on the representative sequences. ([BH11] 3.16) is a example of why in general the limit 4.4 doesn't exist. Nevertheless, there is a large list of spaces where 4.4 does holds, and in that list we can find familiar examples, like the traditional hyperbolic space and trees. In section 4.5 we aboard those type of spaces more in-depth.

We need so give another definition of the Gromov product for points in the boundary, for arbitrary metric spaces.

Definition 4.13. Let X be a δ-hyperbolic space and p a base point. Define the Gromov product on ∂X by

$$(x, y)_p = \sup \liminf_{i,j \to \infty} (x_i, y_j)_p$$

where the supremum is taken over all sequences $x_i, y_j \in H$ such that $\lim_i x_i = x$ and $\lim_j y_j = y$.

Recall that given a triangle with vertices x, y, z, the more large is $(x, y)_z$ more thin is the triangle (see figure). With this in mind, we define now a topology for the boundary.

Definition 4.14. A sequence $\xi_n \in \partial H$ converges to a point $\xi \in \partial H$ if

$$(\xi_n, \xi)_p \to \infty$$

Proposition 4.15. *The definition 4.14 doesn't depend on the base point p.*

Proof. Given two points in the boundary ξ, ζ for all sequences x_i, y_j such that $\lim_i x_i = \xi$ and $\lim_j y_j = \zeta$ we have

$$(\xi, \zeta)_p - 2\delta \leq \liminf_{i,j} (x_i, y_j)_p \leq (\xi, \zeta)_p \tag{4.5}$$

as stated in [BH11] 3.17(5). On the other hand, if $q \in H$ and using the triangle inequality, it follows that

$$|(x_i, y_j)_p - (x_i, y_j)_q| \leq d(p, q) \tag{4.6}$$

From these two inequalities we can deduce the result. If $(\xi, \xi_n) \to \infty$, the limit inferior of the p-products of representative sequences go to infinity due to 4.5. From 4.6 the limit inferior of the q-products of the sequences go to infinity. Finally, using again 4.5 the products $(\xi, \xi_n)_q$ go to infinity. Thus $(\xi_n, \xi)_q \to \infty$ if and only if $(\xi_n, \xi)_p \to \infty$.

■

The next step is to construct a metric for the boundary. We doing following [GdLH90] section 7.3 or alternatively [BH11] section 3, from 3.19 onwards. Define

$$\varrho_\varepsilon(x, y) = e^{-\varepsilon(x,y)_p}$$

This map will help us to define a metric. In fact, it has two conditions of a metric As the Gromov product satisfies

- **i.** $(x, y)_p = (y, x)_p$
- **ii.** $(x, y)_p = \infty \iff x = y$
- **iii.** $(x, z)_p \geq \min{(x, y)_p, (y, z)_p} - \delta$ for every $x, y, z \in H \cup \partial H$

$\varrho_\varepsilon(x,y)$ satisfies

i. $\varrho_\varepsilon(x,y) = \varrho_\varepsilon(y,x)$

ii. $\varrho_\varepsilon(x,y) = 0 \Longleftrightarrow x = y$

iii. $\varrho_\varepsilon(x,z) \leq (1+\varepsilon')\max\{\varrho_\varepsilon(x,y), \varrho_\varepsilon(z,y)\}$ for every $x, y, z \in H \cup \partial H$.

Where $\varepsilon' = e^{\varepsilon\delta} - 1$. But one can see ϱ_ε doesn't always satisfies the triangle inequality. When it does, the space is called strong hyperbolic and that kind of spaces will be studied in later sections. Although, we can circumvent this situation and construct a metric d_ε, and a very friendly one, because it turns to be equivalent to ϱ_ε in the sense there is constants c_1 and c_2 such that

$$c_1\varrho_\varepsilon(x,y) \leq d_\varepsilon(x,y) \leq c_2\varrho_\varepsilon(x,y) \tag{4.7}$$

Such types of metrics are knowns or called visual metrics. The construct it's a standard technique. A chain between two points is a finite sequence $x = x_1, x_2, \ldots, x_n = y$ with $x_i \in \partial H$ and no bound on n. Denote $\mathcal{C}_{x,y}$ the set of all the chains joining x and y and for $c \in \mathcal{C}_{x,y}$ define

$$\varrho_\varepsilon(c) = \sum_{1\leq i\leq n} \varrho_\varepsilon(x_i, x_{i-1})$$

Finally define the metric as

$$d_\varepsilon(x,y) = \inf\{\varrho_\varepsilon(c) : c \in \mathcal{C}_{x,y}\}$$

Proposition 4.16. *If $\varepsilon' < \sqrt{2} - 1$ then d_ε is a visual metric on the boundary of H and*

$$(1-2\varepsilon')\varrho_\varepsilon(x,y) \leq d_\varepsilon(x,y) \leq \varrho_\varepsilon(x,y)$$

Proof. See [BH11] or [GdLH90]. ■

Remark. The metric does have dependence on the base point p. See [V05] for an in-depth analysis.

Continuing with this section, we show examples of boundaries of some classic and iconic spaces.

Example 4.3.1. *Poincaré Disk Model.* Let be $\mathbb{D}$ the unit disk and two points in there, z_1 and z_2. Let be $c \colon \mathbb{R} \to \mathbb{H}$ the unique geodesic ray passing through z_1, z_2 with $c(\infty) = p$, $c(-\infty) = q$.

Denote $|\cdot|$ as the euclidean distance and suppose $|pz_1| > |qz_1|$, $|qz_2| > |pz_2|$. The hyperbolic distance between z_1 and z_2 can be computed as

$$d(z_1, z_2) = \log \frac{|pz_1||qz_2|}{|qz_1||pz_2|} \tag{4.8}$$

We will calculate the Gromov product for any two points x, x' on the boundary ∂H. Let z be a point in the geodesic ray that joins x with the origin o and call y the other end. Let z and y' the analogous for X'. Finally call p and q the end points of the geodesic joining P and Q. Using 4.8

$$\begin{aligned}
2(z, z')_o &= d(z, o) + d(z', o) - d(z, z') \\
&= \log \frac{|zy||ox|}{|zx||oy|} + \log \frac{|z'y'||ox'|}{|z'x'||oy'|} - \log \frac{|zq||z'p|}{|zp||z'q|} \\
&= \log \frac{|zy||zy'||zp||z'q|}{|zx||zx'||zq||z'p|}
\end{aligned}$$

Let $z \to x$, $z' \to x$. This also implies $p \to x$, $q \to x'$. It is not clear that $(z, z')_o \to (x, x')_o$ but we will see in 4.5 it's actually true in the case of the hyperbolic plane. Hence

$$2(x, x')_o = \log \frac{|xy||x'y'|}{|xx'|^2} \implies (x, x')_o = \log \frac{2}{|xx'|}$$

One can shows easily that $|xx'| = 2\sin(\theta/2)$, where $\theta = \angle xox'$, due to the triangle with that vertices is isosceles. So finally

$$(x, x')_o = \log \frac{1}{\sin(\theta/2)}$$

Later, we will see in 4.5 $d(x, x') = e^{-(x,x')_o} = \sin(\theta/2)$ is actually a metric for the boundary. This metric makes the boundary ∂H homeomorphic to $\mathbb{S}^1$.

Example 4.3.2. *Cayley Graph of free group on two symbols.* In this example we will show the boundary of the Cayley graph of $\mathbb{F}_2$ is a Cantor like-set, understood as a perfect , compact and totally disconnected set.

First we compute the Gromov product for two vertex points. Remember vertex of the graph are represented as word in two letters, and the distance between two vertex x, y is the number of letters of the word $x^{-1}y$. If o denote the center of the graph, i.e, the identity, we have

$$\begin{aligned}
d(x, o) &= \#\text{letters of } x \\
d(y, o) &= \#\text{letters of } y \\
d(x, y) &= \#\text{letters of } x^{-1}y
\end{aligned}$$

Therefore $(x, y)_o$ is exactly the number of letters x and y have in common until the first difference.

We can represent the points in the boundary are infinites words. Now denote $\partial\mathbb{F}_2$ and we pick one infinite word, say x. It's easy to see if you cut the first n letters of x and let the rest random, we have a sequence $x_n \in K$ converging to x. Hence K have no isolated points. To verify K is compact it's enough to check sequential compact due to K is metrizable. Let $x_1^{j_1}x_2^{j_2}\dots x_n^{j_n}\dots$ be a sequence with $x_i^{j_i} \in a, b$. Using a classic diagonal argument we can construct a convergence subsequence. There must be infinites elements starting at a if not at b. Of those elements, there must be again infinite starting at one of the two elements. Iterating this process you obtain a the convergence subsequence and hence K is compact. Finally to see K is totally disconnected for every set $U \subset K$ define

$$U_a = \{k \in U : k \text{ start with } a\}$$
$$U_b = \{k \in U : k \text{ start with } b\}$$

It's clear that $U_a \cap U_b = \emptyset$ and $U_a \cup U_b = U$. Both are open in U because the distance between an element starting with a and one starting with b is never small.

We end this section with a proposition related to the extended action of the isometries to the boundaries.

Proposition 4.17. *Let H be a hyperbolic space and $g\colon H \to H$ an isometry. The extended action $g^* : \partial H \to \partial H$ is a homeomorphism on the topology defined in 4.14 and hence for every visual metric.*

Proof. If $\xi_n \to \xi$ in the boundary, x_i, y_j^n are representative sequences and $gq = p$ Applying g in 4.5 we obtain

$$(g\xi, g\xi_n)_p - 2\delta \leq \liminf_{i,j}(x_i, y_j^n)_q \leq (g\xi, g\xi_n)_p$$

As $(\xi, \xi_n)_p \to \infty$, by 4.15 $(\xi, \xi_n)_q \to \infty$, hence the q-product of the sequences goes to infinity too and so does $(g\xi, g\xi_n)_p$ ■

4.4 Horofunctions

Horofunctions or Busemann functions have their origin the non-Euclidean geometry and they have been largely studied on spaces of non-positive curvature and Gromov spaces too. They are intimately related with the boundary; furthermore, you can define another boundary using these functions which is homeomorphic, if the space is proper, to the one defined above with geodesics.

Given a geodesic ray γ starting at p the Busemann function or horofunction $b : H \to \mathbb{R}$ for γ is defined by

$$b_\gamma(h) = \lim_{n\to\infty} d(\gamma(n), h) - n$$

Existence of the limit is ensured by the triangle inequality and monotonicity. Note that if $\gamma_(\infty) = \gamma'(\infty)$ then $|b_\gamma(t) - b_{\gamma'}(t)|$ is bounded and if $\gamma_(\infty) = \gamma'(\infty)$, $|b_\gamma(t) - b_{\gamma'}(t)|$ is unbounded, so you can clearly see how this functions can define a boundary similar to the geodesic one. If we modify the base-point, the horofunction varies by a constant, so they're a coarse object in that regard. For the explicit equivalence see [HW97] where is proved that a Gromov space which is also Busemann, the horoboundary and the geodesic boundary are homeomorphic.

As we already have a boundary established, this definition is equivalent to take a base-point p and a point $\xi \in \partial H$ (H proper) and

$$b_{p,\xi}(h) = \lim_{n\to\infty} d(x_n, h) - d(x_n, p)$$

where x_n is a succession in H such that $x_n \to \xi$.

In a similar way you can extended this concept to a large variety of metric spaces and define a boundary. See [KL06] for the explicit construction.

Pre-images $b_{p,\xi}^{-1}(t)$ are called horospheres and play an important role in general and for us. They received that name because in certain spaces, as $\mathbb{H}^n$, they looks like a sphere. In [Pap05] you have a large list of properties and characteristics of horospheres.

Next we show a proposition we will need in later chapters

Proposition 4.18. *Let H be a δ-hyperbolic proper geodesic space. Let be $\xi \in \partial H$ and γ a geodesic ray ending with one ending on ξ. Then, for every $p \in H$ and $t \in \mathbb{R}$, γ intersect once and just once $b_{p,\xi}^{-1}(t)$.*

First, for the existence, note $b_{p,\xi}$ satisfies $|b_{p,\xi}(h) - b_{p,\xi}(g)| \leq d(h, g)$ thus horofunctions are 1-Lipschitz and so are continuous. Suppose $\gamma(\infty) = \xi$ and $\gamma(-\infty) = \zeta$. As $b_{p,\xi}(p) = 0$, there are points $\gamma(s)$ such that $b_{p,\xi}(\gamma(s)) < t$. Analogously, as $\gamma(-\infty) = \zeta \neq \xi$ there are points such that $b_{p,\xi}(\gamma(s)) > t$. Hence, as γ_ω and $b_{p,\xi}$ are continuous, there exists a point such that $b_{p,\xi}(\gamma(s_0)) = t$.

For uniqueness, let s, s' be reals such that $b_{p,\bar{X}}(\gamma_\omega(s)) = b_{p,\bar{X}}(\gamma_\omega(s')) = t$. One can see this implies

$$\lim_{n\to\infty} d(x_n, \gamma_\omega(s)) - d(x_n, \gamma_\omega(s')) = 0$$

We have now to use the fact the geodesic ends in $\bar{X}$ because we can replace x_n with $\gamma_\omega(n)$ and hence

$$\lim_{n\to\infty} d(\gamma_\omega(n), \gamma_\omega(s)) - d(\gamma_\omega(n), \gamma_\omega(s')) = |s - s'|$$

so $s = s'$ which completes the proof.

4.5 Strong Hyperbolicity

As we said, the Gromov product isn't necessarily continuous on the boundary but when it is you can get some interesting results. You can simply ask for that assumption and there are a large category of spaces with such property. However, there is a known condition called Strong Hyperbolicity which implies the continuity and give us a lot more to work and we would like to review in this chapter.

Definition 4.19. A metric space X is said to be strongly hyperbolic if for all points x, y, z and a base point p the following inequality holds

$$\varrho_\varepsilon(x, y) \leq \varrho_\varepsilon(x, z) + \varrho_\varepsilon(y, z)$$

where $\varrho(x, y)$ is the same defined in the section 4.3.

One can think this definition is equivalent to say ϱ is a metric, but note ϱ can be zero only on ∂X so it's just a meta-metric in the whole space. [DSU17] has provided a lot of examples of strong hyperbolic spaces. There, we can find the proof that every CAT(-1) space is strongly hyperbolic.

Theorem 4.20. *Every CAT(-1) space is strongly hyperbolic*

Proof. See [DSU17] ∎

Example 4.5.1. Since every $\mathbb{R}$-tree is CAT(-1), they are strongly hyperbolic too.

Example 4.5.2. The hyperbolic space $\mathbb{H}^n$ is strongly hyperbolic.

As we said before strong hyperbolicity implies the continuity of the Gromov product on the boundary.

Theorem 4.21. *Let X be a strongly hyperbolic space. Then the Gromov product $(\cdot, \cdot)_p$ extends continuously to the boundary ∂X and for every $x, y \in \partial X$ and sequences $x_i \to x$, $y_i \to y$*

$$(x, y)_p = \lim_{i \to \infty} (x_i, y_i)_p$$

And the definition doesn't depend on the representative sequences.

Proof. See [DSU17] ∎

Another worth-mentioning consequence is horofunctions can be rewritten only in therms of the Gromov product as

$$b_{p,\xi}(h) = 2(\xi, p)_h - d(h, p)$$

Proposition 4.22. *If H is a strongly hyperbolic space and $g\colon H \to H$ is an isometry, the extended action $g^* : \partial H \to \partial H$ is a Lipsichtz map.*

Proof. Following [NS14] given a base point p, and $x, y \in H$.

$$-2(gx, gy)_p = d(x, o) - d(x, g^{-1}p) + d(y, o) - d(y, g^{-1}p) - 2(x, y)_p$$

If $x \to \xi$ and $y \to \zeta$ and exponentiating one can deduce

$$\varrho_\varepsilon^2(g\xi, g\zeta) = e^{b_{p,\xi}(g^{-1}p)} e^{b_{p,\zeta}(g^{-1}p)} \varrho_\varepsilon^2(\xi, \zeta)$$

Using the fact $b_{p,\xi}(g^{-1}p) \leq d(p, g^{-1}p)$ (the same goes for the other horofunction) we obtain

$$\varrho_\varepsilon(g\xi, g\zeta) \leq e^{d(p,g^{-1}p)} \varrho_\varepsilon(\xi, \zeta)$$

∎

Chapter 5

Three results on the problem

5.1 First case: Poincaré Disk Model

The first approach will be consider $H = \mathbb{H}$ the hyperbolic space, represented as the Poincaré disk. The main reason whereby the technique use here can't be extended is because of the use of the cross-ratio and its properties. It's a known fact the concept of cross-ratio can't be defined over, for example, hyperbolic spaces of higher dimensions. Let's start with the proof for this case.

Theorem 5.1. *Let $T\colon \Omega \to \Omega$ be a topological transitive map. Let $A\colon \Omega \to \text{ISOM}(\mathbb{D})$ be a continuous map in the point-wise topology. Suppose there exists $B^*\colon \Omega \to \text{HOM}(\mathbb{S}^1)$ continuous such that*

$$A^*(\omega) = B^*(T\omega)(B^*)^{-1}(\omega)$$

Then there exists invariant continuous curves $s_{\omega,h}$ respect to the cocycle A.

Consider the dense orbit given by $\{T^n\omega_0\}_{n\in\mathbb{N}}$. As we said, we will assume there is a solution for the equation in the boundary; by theorem 2.1 there exists a family of curves $s_{(\omega,g)}$ passing through every point of the boundary; we will prove there is also a family of curves passing through every point of the interior H.

Consider the point (ω_0, h_0) and now pick any two geodesics α, β such that $\alpha \cap \beta = h_0$. These two geodesics determine four points in the boundary $A = \alpha(\infty)$, $B = \alpha(-\infty)$, $C = \beta(\infty)$ and $D = \beta(-\infty)$. For every of those points, we have a curve, say $s_{(\omega_0,A)}$, $s_{(\omega_0,B)}$, $s_{(\omega_0,C)}$ and $s_{(\omega_0,D)}$.

We claim for every fiber ω, the curves above determine four different point in the boundary. Assuming using the lemma 4.7,we can join two by two these points by geodesics that intersects. The intersection point will be use to define a the desired curve. Let's start to prove these claims.

!

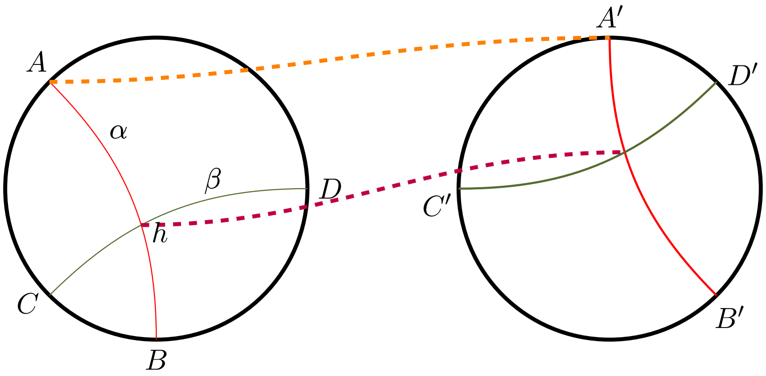


FIGURE 5.1: Circles

Lemma 5.2. *For every ω, the four curves $s_{(\omega_0,A)}$, $s_{(\omega_0,B)}$, $s_{(\omega_0,C)}$ and $s_{(\omega_0,D)}$ determines four different points in the boundary ∂H. Furthermore, the four points stay in the same cycle order for every ω.*

Proof. The cross-ratio of four complex points z_1, z_2, z_3 and z_4 is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$$

It's known fact the cross-ratio is invariant under positive Möbius transformations and it's also known isometries of the complex plane are Möbius transformations.

Set

$$\begin{aligned} z_1 &= s_{(\omega_0,A)}(\omega); & z_1^n &= s_{(\omega_0,A)}(T^n\omega) \\ z_2 &= s_{(\omega_0,B)}(\omega); & z_2^n &= s_{(\omega_0,B)}(T^n\omega) \\ z_3 &= s_{(\omega_0,C)}(\omega); & z_3^n &= s_{(\omega_0,C)}(T^n\omega) \\ z_4 &= s_{(\omega_0,D)}(\omega); & z_4^n &= s_{(\omega_0,D)}(T^n\omega) \end{aligned}$$

Using the invariance of the curves we have $A(T\omega)z_i^n = z_i^{n+1}$ so the points of the curve that belongs to same orbit, are the result of the application of an isometry to the point $s_{(\omega_0,A)}(\omega)$. Combining this with the invariance of the cross-ratio under the we have

$$(z_1, z_2, z_3, z_4) = (z_1^n, z_2^n, z_3^n, z_4^n)$$

Hence, if we have a dense orbit and the curves are continuous the value of the cross ratio can't change in any point ω. So, without loss generality suppose the cross ratio of the initial points different from 1, say $(A, C, B, D) = 1/2$ and suppose for some point ω the order of the points in the boundary changes to (A', B', C', D'). Looking at the definition the cross-ratio the said change would give us that $(A', B', C', D') = 2$, contradicting the fact that we just prove, so they have to stay in the same order. ■

Combining this previous lemma with 4.7. We can ensure there are two geodesics α_ω, β_ω such that for every ω

$$\alpha_\omega(\infty) = s_{(\omega_0,A)}(\omega); \quad \alpha_\omega(-\infty) = s_{(\omega_0,B)}(\omega)$$
$$\beta_\omega(\infty) = s_{(\omega_0,C)}(\omega); \quad \beta_\omega(-\infty) = s_{(\omega_0,D)}(\omega)$$

And they always intersect in one point.

With all this preamble, now we can define $s_{(\omega_0,h_0)}(\omega) := \alpha_\omega \cap \beta_\omega$. We will show this curve has the characteristics that we are looking for, invariance and continuity.

Let's prove the invariance. Recall the invariance of the curve in the boundary

$$A(\omega)s_{(\omega_0,A)}(\omega) = s_{(\omega_0,h_0)}(T\omega)$$

Which implies

$$A(\omega)\alpha_\omega(\infty) = \alpha_{T\omega}(\infty); \quad A(\omega)\alpha_\omega(-\infty) = \alpha_{T\omega}(-\infty)$$

Also it's clear the geodesic $\alpha_{T\omega}$ belongs to the class $\alpha_{T\omega}(\infty)$ (in this particular case is the only one) so we can conclude $A(\omega)\alpha_\omega = \alpha_{T\omega}$ for all $t \in \mathbb{R}$. The same goes to β. It's also clear that the intersection goes to the intersection, so

$$A(\omega)s_{(\omega_0,h_0)}(\omega) = A(\omega)(\alpha_\omega \cap \beta_\omega) = A(\omega)\alpha_\omega \cap A(\omega)\beta_\omega = \alpha_{T\omega} \cap \beta_{T\omega} = s_{(\omega_0,h_0)}(T\omega)$$

It's left to check the continuity of $s_{(\omega_0,h_0)}(\omega)$ which is equivalent to prove given two pair of points in the boundary that are close in pairs, the points defined by the geodesics ending on those points intersected with a third geodesic are close. So set γ, γ', λ with $\gamma(\infty) = X, \gamma'(\infty) = X', \gamma(\infty) = Y, \gamma'(\infty) = Y$. We will show

$$\left.\begin{array}{l} X \to X' \\ Y \to Y' \end{array}\right\} \Longrightarrow \gamma \cap \lambda \to \gamma' \cap \lambda$$

In this particular case, we have a very clear and explicit expression for the boundary distance given by $d(X,X') = \sin(\vartheta)$ with $\vartheta = \angle(X,X')_p$, hence we have immediately $\gamma \to \gamma'$ if both ends get closer. Although it isn't necessarily true $\gamma(t) \to \gamma'(t)$ in this case if $\gamma \cap \lambda \not\to \gamma' \cap \lambda$ that would implies two geodesics intersect twice which is impossible in a unique geodesic space. By 2.1 and 2.2 there is a continuous solution for the cohomological equation for A.

5.2 Second Case: H is strongly hyperbolic and A has a fixed point on the boundary

Moving on to the second case, we will use a different technique also involving the geodesic boundary, but we can't no longer proof the fact that the geodesics intersect in one point, due there isn't a tool to preserve the point configuration like the cross-ratio does. For this instance we will serve us of the Livsic theorem [Liv72] for a translation isometry induced by the original A. Let's start with the proof.

Theorem 5.3. *Let $T\colon \Omega \to \Omega$ be a hyperbolic map. Let H be a strongly hyperbolic metric space and $A\colon \Omega \to \mathrm{ISOM}(H)$ be a continuous map in the point-wise topology. Suppose that there exists $\xi \in \partial H$ such that $A^*(\omega)\xi = \xi$ for every $\omega \in \Omega$ and that there exists $B^*\colon \Omega \to \mathrm{HOM}(\partial H)$ continuous such that*

$$A^*(\omega) = B^*(T\omega)(B^*)^{-1}(\omega)$$

Then for every $(\omega, h) \in \Omega \times H$ there exists invariant continuous curves $s_{\omega,h}$ respect to the cocycle A.

We suppose there is a fixed point for the action of $A(\omega)$ for every ω. It can sound as a very restrictive assumption, but there are cases when you can find such a point. For example, if A acts transitive on the boundary (e.g. rotations of $\mathbb{H}^n$) you can conjugate A by a another co-cycle I and obtain a fixed point for the conjugated cocycle.

Pick a point (ω_0, h_0) and pick a geodesic $\gamma\colon \mathbb{R} \to H$ passing through h_0 with one end in ξ the fixed point . As before this geodesic define two points in the boundary $\xi = \gamma(-\infty), Y = \gamma(\infty)$. Again, considering the action of A extends naturally to the boundary, and assuming the cohomological equation has a solution there, we have invariant curves passing through every point of the boundary. In every fiber, the curves $s_{(\omega_0,\xi)}(\omega)$, $s_{(\omega_0,Y)}(\omega)$ define two points in the boundary X_ω, Y_ω but in this case $X_\omega = \xi$ for every ω. Using the lemma 4.7 there is a geodesic such that $\gamma_\omega(-\infty) = \xi, \gamma_\omega(\infty) = Y_\omega$.

Our aim is construct or induce a map N over $\Omega\times\mathbb{R}$ such that $N(\omega,\cdot)=N_\omega$ acts as an $\mathbb{R}$-isometry for every ω, with all the properties needed to use the livsic theorem. For this purpose, we will use horofunctions defined in the previous chapter.

Consider the fibered set

$$\Gamma=\{(\omega,v):\omega\in\Omega, v\in\gamma_\omega\}$$

where γ_ω satisfies

$$\gamma_\omega(-\infty)=\xi;\quad \gamma_\omega(\infty)=Y_\omega$$

and it's parametrized by

$$\gamma_\omega(t)=b^{-1}_{p,\xi}(t)\cap\gamma_\omega$$

Define the map $N\colon\Omega\times\mathbb{R}\to\Omega$ given by

$$N(\omega,t)=b_{p,\bar{X}}(A(\omega)\gamma_\omega(t))$$

where $b_{p,\bar{X}}$ is there horofunction associated to $p,\bar{X}$ and p is a fixed base-point. This N is our candidate to be the map with the desired properties.

Lemma 5.4. *The induced maps $N(\omega):\mathbb{R}\to\mathbb{R}$ are isometries and act as a translation. There exists $\varphi\colon\Omega\to\mathbb{R}$ such that for every $\omega\in\Omega$ and $t\in\mathbb{R}$,*

$$N(\omega)t=t+\varphi(\omega)$$

Furthermore, $\varphi:\Omega\to\mathbb{R}$ is a Hölder function.

Remark. The map N has and implicit dependence of the points (ω_0,h_0) because it's defined in terms of the geodesics that depends on those point.

Proof. We start checking if it is an isometry

$$|b_{p,\bar{X}}(A(\omega)\gamma_\omega(t)-b_{p,\bar{X}}(A(\omega)\gamma_\omega(s))|=|d(x_n,A(\omega)\gamma_\omega(t))-d(x_n,A(\omega)\gamma_\omega(s))|$$

As before, it's crucial the fact the horofunction and the geodesic have the same initial point $\bar{X}$ because we can take x_n over the geodesic.Hence

$$\begin{aligned}|d(x_n,A(\omega)\gamma_\omega(t))-d(x_n,A(\omega)\gamma_\omega(s))| &= d(A(\omega)\gamma_\omega(t),A(\omega)\gamma_\omega(s))\\ &= d(\gamma_\omega(t),\gamma_\omega(s))\\ &= |s-t|\end{aligned}$$

Continuing with the proof it's well-known there are only two types of isometries in $\mathbb{R}$, translations and reflexions, so it's enough to prove that N never acts reflecting changing the order of the points $t_1 < t_2$ into $t_2 < t_1$. We will check first $A(\omega)$ doesn't change the order.

Up to a translation you can think $t_1 < 0 < t_2$ and now pick $M >$ such that: $|M| \gg t_1, t_2$; $\gamma_\omega(M)$ belongs to the ray starting $\gamma_\omega(t_2)$ going to $\gamma_\omega(\infty)$ and $\gamma_\omega(-M)$ belongs to the ray starting $\gamma_\omega(t_1)$ going to $\gamma_\omega(-\infty)$. It's clear if $A(\omega)$ would change the order of $\gamma_\omega(t_1), \gamma_\omega(t_2)$ then $-M, M$.has to change, hence

$$\lim_{M\to\infty} A(\omega)\gamma_\omega(M) = \gamma_\omega(-\infty)$$

but this contradicts the fact that the ends are fix so $A(\omega)$ has to preserve the order.

It's left to prove the regularity,i.e, the Hölder condition, of the map N. In order to do prove it note that

$$\begin{aligned} b_{p,\xi}(A(\omega)\gamma_\omega(0)) - b_{p,\xi}(A(\omega)p) &= 2(\xi, A(\omega)p)_{A(\omega)\gamma_\omega(0)} - d(A(\omega)\gamma_\omega(0), A(\omega)p) \\ &= 2(\xi, p)_{\gamma_\omega(0)} - d(\gamma_\omega(0), p) \\ &= b_{p,\xi}(\gamma_\omega(0)) \\ &= 0 \end{aligned}$$

So we have

$$\varphi(\omega) = b_{p,\xi}(A(\omega)\gamma_\omega(0)) = b_{p,\xi}(A(\omega)p) \tag{5.1}$$

Hence

$$\begin{aligned} |\varphi(\omega_1) - \varphi(\omega_2)| &= |b_{p,\xi}(A(\omega_1)p) - b_{p,\xi}(A(\omega_2)p)| \\ &= \lim_{x\to\xi} d(A(\omega_1)p, x) - d(A(\omega_2)p, x) \\ &\leq d(A(\omega_1)p, A(\omega_2)p) \\ &\leq Cd(\omega_1, \omega_2)^\alpha \end{aligned}$$

■

The next step is to prove if A satisfies the POO condition, the map φ also does it. As

$$\prod_{i=1}^{n-1} N_p(0) = \sum_{i=1}^{n-1} \varphi(T^j p)$$

we are interested in calculate the iterations of N over an n-periodic orbit and prove that the map $N(\omega)$ acts as the identity over zero, when it's applied $n-1$ times (in fact it will act as the identity over any real). Following the construction, starting with a t_0 we go to H through γ_ω then apply $A(\omega)$ and go back through $b_{p,\bar{X}}$. In the next step, we go to H through $\gamma_{T\omega}$, but note that we return to the same point due to how is defined the geodesic, $\gamma_{T\omega}(s) \in b^{-1}_{p,\bar{X}}(s)$; the invariance, $A(\omega)\gamma_\omega = \gamma_{T\omega}$ (recall is set invariance and not point-wise) , and the fact proved above that horospheres and geodesics with the same base point intersect once and just once. Iterating this process we have

$$\prod_{i=1}^{n-1} N(p)t_0 = b_{p,\bar{X}}\left(\prod_{j=1}^{n-1} A(T^j p)\gamma_p(t_0)\right) = b_{p,\bar{X}}(\gamma_p(t_0)) = t_0$$

It's clear that this is enough to ensure the condition for all $t \in \mathbb{R}$ and in particular for $t = 0$.

Proceeding with the proof, we are ready to use the Livsic theorem for the map φ and hence there is a map $\psi\colon \mathbb{R} \to \mathbb{R}$ Hölder such that

$$\varphi(\omega) = \psi(T\omega) - \psi(\omega)$$

Set r_0 as the point that $\gamma(r_0) = h_0$ (γ is the starting geodesic). Recall our goal is to prove there is an invariant curve passing through (ω_0, h_0). By 2.1 there exists a curve $s_{(\omega_0,r_0)} : \Omega \to \mathbb{R}$ passing through the point $(\omega_0, r_0) \in \Omega \times \mathbb{R}$ with the invariance property

$$s_{(\omega_0,r_0)}(\omega) + \varphi(\omega) = s_{(\omega_0,r_0)}(T\omega)$$

Which is the same that

$$N(\omega)s_{(\omega_0,r_0)}(\omega) = s_{(\omega_0,r_0)}(T\omega) \tag{5.2}$$

Define $s'_{(\omega_0,r_0)}(\omega) = \gamma_\omega(s_{(\omega_0,r_0)}(\omega))$. We claim this is the curve we are looking for, i.e, invariant for the action of $A(\omega)$. We need to prove

$$A(\omega)\gamma_\omega(s_{(\omega_0,r_0)}(\omega)) = \gamma_{T\omega}(s_{(\omega_0,r_0)}(T\omega)) = \gamma_{T\omega} \cap b^{-1}_{\bar{X},p}(s_{(\omega_0,r_0)}(T\omega))$$

Equation 5.2 states

$$b_{p,\bar{X}}(A(\omega)\gamma_\omega(s_{(\omega_0,r_0)}(\omega))) = s_{(\omega_0,r_0)}(T\omega)$$

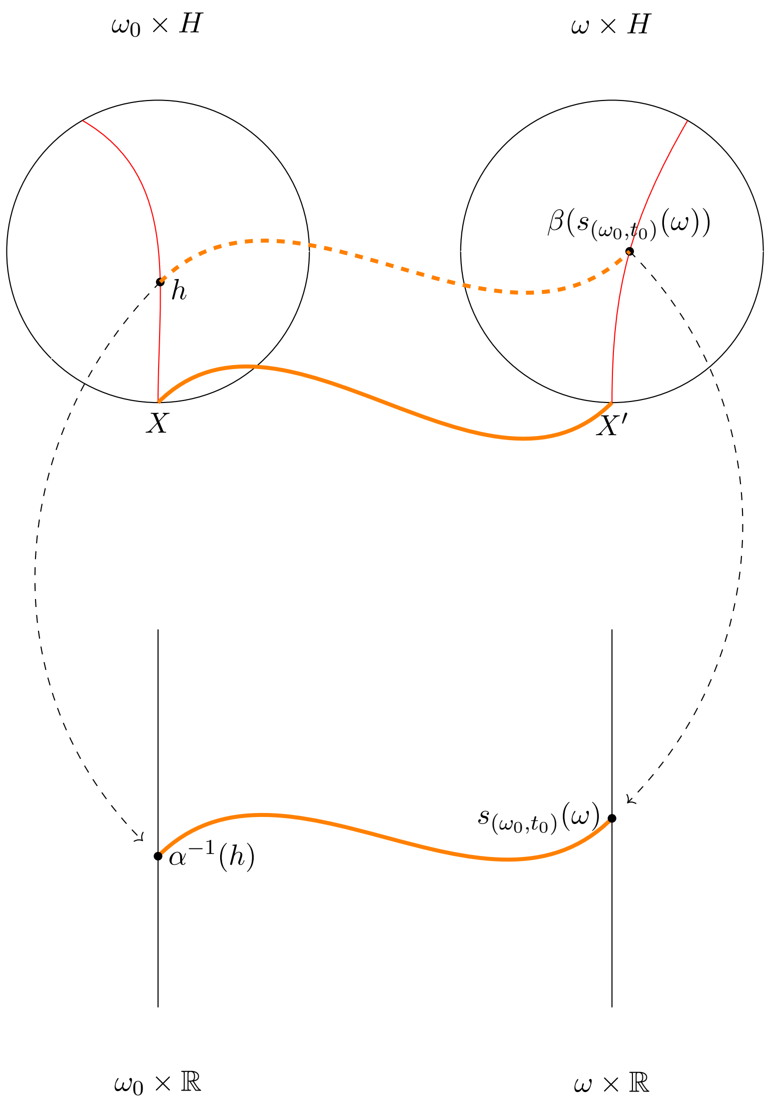
Which proves $A(\omega)\gamma_\omega(s_{(\omega_0,r_0)}(\omega))$ belongs to $b^{-1}_{\bar{X},p}(s_{(\omega_0,r_0)}(T\omega))$. But as before, we also have

$$A(\omega)\gamma_\omega = \gamma_{T\omega}$$

Which proves $b^{-1}_{\bar{X},p}(s_{(\omega_0,r_0)}(T\omega)) \in \gamma_{T\omega}$. Since the intersection is unique the invariance has been proved. The continuity follows from

$$d(\gamma_{\omega_1}(s(\omega_1)), \gamma_{\omega_2}(s(\omega_2))) \leq |s(\omega_1) - s(\omega_2)| + d(\gamma_{\omega_1}(s(\omega_1)), \gamma_{\omega_2}(s(\omega_1)))$$

As the previous section we have $\omega_1 \to \omega_2$ which implies $\gamma_{\omega_1} \to \gamma_{\omega_2}$. Although the convergence isn't point by point, the uniqueness of the intersection with the horosphere ensures $\gamma_{\omega_1}(s(\omega_1)) \to \gamma_{\omega_2}(s(\omega_1))$.



5.3 Third Case: Caley Graph of the free group in two symbols

In this case we manage to obtain a Hölder solution for the Caley graph of $\mathbb{F}_2$. The main reason is being close in the boundary is a lot stronger than other spaces.

Theorem 5.5. *Let be $T\colon \Omega \to \Omega$ with T satisfying all the hypothesis of. Let $A\colon \Omega \to \textsc{Isom}(\mathbb{F}_2)$ be a Hölder cocycle of isometries and A^* its extended action to the boundary of $\mathbb{F}_2$. Assume A^* is continuous and the cohomology equation for A^* has solution. Then there exists a Hölder solution for the cohomology equation for A.*

The proof is very similar to the previous case: we will induce a real-valued Hölder function and then define the invariant curve as the image of the real solution through the geodesic.

As before we have a geodesic ray γ_ω and end points $\gamma_\omega(\infty) = X_\omega, \gamma_\omega(-\infty) = Y_\omega$ varying continuously on the boundary. We will define next a parametrization for γ_ω.

Given a set C and a point p consider

$$d_{C,p} = \inf_{c \in C} d(c, p)$$

It's a well-known fact if C is closed and convex, there is a point c_p such that $d(c_p, p) = d_{C,p}$. Define now $\gamma_\omega(0) = c_{\gamma_\omega}$. The key point of the proof is we are able to prove $\gamma_\omega(0)$ varies Hölder with ω due to in this case the proximity on the boundary does implies proximity in the interior. In fact we can show the next result

Lemma 5.6. *Let X, X' points on the boundary of $\mathbb{F}_2$, let γ_1 and γ_2 be geodesic rays such that $\gamma_1(\infty) = X$ and $\gamma_2(\infty) = X'$. Suppose also there is a bounded set K such that $\gamma_1(0)$ and $\gamma_2(0)$ belongs to K. Then, there exists $M > 0$ such that if $(X, X')_p > M$, $\gamma_1(0) = \gamma_2(0)$ where the parametrization of the rays is given by $\gamma_i(0) = c_o$.*

Proof. Denote $h_1 = \gamma_{\omega_1}(0)$, $h_2 = \gamma_{\omega_2}(0)$ and suppose $h_1 \neq h_2$. Let η_1 be the geodesic ray which its go from o to h_1 and then continues equal to γ_{ω_1}. Set η_2 analogous for γ_{ω_2}. Let n, m such that $d(\beta_1(n), o) > d(h_1, o)$ and $d(\beta_2(m), o) > d(h_2, o)$. As we saw in 4.3.2 $(\beta_1(n), \beta_2(m))_o$ correspond exactly to how many symbols β_1 and β_2 have in common. Call p the point when . It's also clear that

$$d(o, h_i) = (\beta_1(n), \beta_2(m))_o = (X, X')_o$$

If $(X, X')_o$ is large, i.e, $d(o, p)$ is large, and under the assumption $h_1 \neq h_2$ that could only means p gets pushed further and further in the graph which implies h_1 and h_2 also goes far. But that situation can't happen because $h_1, h_2 \in K$. Hence h_1 has to be equal to h_2. ■

Corollary 5.7. *With the above setup, for every $t \in \mathbb{R}$ there exists $M_t > 0$ such that if $(X, X')_p > M_t$ then $\gamma_1(t) = \gamma_2(t)$*

Proof. Using the previous lemma, suppose (X, X') is sufficiently large to force $h_1 = h_2$. If (X, X') gets even bigger means that the rays have more symbols in common. ■

The rest of the proof is very similar to the previous case. Define the map $M : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$M(\omega, t) = \gamma_{T\omega}^{-1}(A(\omega)\gamma_\omega(t))$$

We will also prove the analogous for the lemma 5.8

Lemma 5.8. *The maps $M(\omega) : \mathbb{R} \to \mathbb{R}$ are isometries and act as a translation. There exists $\psi \colon \Omega \to \mathbb{R}$ such that for every $\omega \in \Omega$ and $t \in \mathbb{R}$,*

$$M(\omega)t = t + \eta(\omega)$$

Furthermore, $\psi : \Omega \to \mathbb{R}$ is a Hölder function.

Proof. The proof is the same except for the part of checking the Hölder regularity. Set

$$z_1 = \gamma_{\omega_1}(0); \quad z_2 = \gamma_{\omega_2}(0); \quad A(\omega_1) = g_1; \quad A(\omega_2) = g_2$$

We have

$$|\gamma_{T\omega_1}^{-1}(g_1 z_1) - \gamma_{T\omega_2}^{-1}(g_2 z_2)| \leq |\gamma_{T\omega_1}^{-1}(g_1 z_1) - \gamma_{T\omega_1}^{-1}(g_2 z_2)| + |\gamma_{T\omega_1}^{-1}(g_2 z_2) - \gamma_{T\omega_2}^{-1}(g_2 z_2)| \tag{5.3}$$

If ω_1 is enough sufficiently close to ω_2, by corollary 5.7, the second therm of 5.3 is equal to zero. For the first we have

$$\begin{aligned} |\gamma_{T\omega_1}^{-1}(g_1 z_1) - \gamma_{T\omega_1}^{-1}(g_2 z_2)| &= d(g_1 z_1, g_2 z_2) \\ &\leq d(g_1 z_1, g_1 z_2) + d(g_1 z_2, g_2 z_2) \\ &= d(z_1, z_2) + d(g_1 z_2, g_2 z_2) \end{aligned}$$

The first therm is zero by the lemma 5.6. For the second therm we use the Hölder regularity of A and get the desired result. ■

Moving forward with the proof we will prove the POO condition for η. The proof is very similar to the previous case when we showed that φ satisfies the POO condition. For the current case is even easier the proof because you go and back only using geodesics.

Let p be an n-periodic point and $t_0 \in \mathbb{R}$. The first loop we end in the point $\gamma_{T\omega}^{-1}(A(\omega)\gamma_\omega(t))$, but then we apply $\gamma_{T\omega}$. Hence

$$\sum_{i=1}^{n-1} \eta(T^j p) = \gamma_p^{-1}\left(\prod_{j=1}^{n-1} A(T^j p)\gamma_p(0)\right) = \gamma_p^{-1}(\gamma_p(0)) = 0$$

Using the Livšic theorem for η and 2.1 there exists an invariant Hölder curve $s_{(\omega_0,r_0)} : \Omega \to \mathbb{R}$ passing through the point $(\omega_0, r_0) \in \Omega \times \mathbb{R}$.The invariance of $s(\omega)$ can be written as

$$\gamma_{T\omega}(s(T\omega)) = A(\omega)\gamma_\omega(s(\omega))$$

which proves the invariance for s'. In contrast with the previous case we can prove the solution is Hölder using the corollary 5.7.

$$\begin{aligned} d(\gamma_{\omega_1}(s(\omega_1)), \gamma_{\omega_2}(s(\omega_2))) &\leq d(\gamma_{\omega_1}(s(\omega_1)), \gamma_{\omega_1}(s(\omega_2))) + d(\gamma_{\omega_1}(s(\omega_2)), \gamma_{\omega_2}(s(\omega_2))) \\ &= |s(\omega_1) - s(\omega_2)| + d(\gamma_{\omega_1}(s(\omega_2)), \gamma_{\omega_2}(s(\omega_2))) \\ &\leq C d(\omega_1, \omega_2)^\alpha \end{aligned}$$

The last inequality is due to s is Hölder and the other therm is zero when ω_1 and ω_2 are sufficiently close by 5.7.

Chapter 6

Final comments and questions

In this final chapter, we will talk about how far you can get with the technique used to resolve the problem, along with the geometric considerations in that regard. Which other spaces (probably with non-positive curvature) will be suitable to aboard the problem? Was Gromov hyperbolic spaces a correct choice?

But first we start giving some remarks on the previous proofs.

6.1 Remarks on the proofs

Can we extend the technique used in the second case of the chapter 5 when there is no fixed point? We can say something about it, but we can't replicate the entire proof. In fact, and as we mention several times, the only question that separate us from solve the problem is whether the parametrization is Hölder or not. Suppose now one can not fix a point for the action on the boundary. The induced isometries would look like

$$N(\omega, t) = b_{p,X_\omega}(A(\omega)\gamma_\omega(t))$$

Recall the parametrization in that case is given by

$$\gamma_\omega(t) = b^{-1}_{p,X_\omega}(t) \cap \gamma_\omega$$

where $X_\omega = \gamma_\omega(\infty)$.

Note that under this setup 5.1 doesn't hold anymore. Denoting the subindex ω_i simply as i, and $\gamma_{\omega_i}(0) = z_i$ and if $\varphi(\omega)$ is defined the same way as case 5.2

$$\begin{aligned}|\varphi(\omega_1) - \varphi(\omega_2)| &= |b_{p,X_1}(A_1 z_1) - b_{p,X_2}(A_2 z_2)| \\ &= |(X_1, p)_q - d(p,q) - ((X_2, p)_r - d(p,r))| \\ &\leq |(X_1, p)_{A_1 z_1} - (X_2, p)_{A_2 z_2}| + |d(A_1 z_1, p) - d(A_2 z_2, p)|\end{aligned}$$

Note that

$$d(A_1 z_1, p) - d(A_2 z_2, p) \leq d(A_1 z_1, A_2 z_2) \leq d(A_1 z_1, A_1 z_2) + d(A_1 z_2, A_2 z_2)$$

As A_1 is an isometry $d(A_1 z_1, A_1 z_2) = d(z_1, z_2)$.

On the other side, the first therm isn't that easy to control. Denoting

$$|(X_1, p)_{A_1 z_1} - (X_2, p)_{A_2 z_2}| \leq |(X_1, p)_{A_1 z_1} - (X_2, p)_{A_1 z_1}| + |(X_2, p)_{A_1 z_1} - (X_2, p)_{A_2 z_2}| \tag{6.1}$$

In general, one has for $x, y, z, w \in H$

$$|(x, y)_z - (x, y)_w| \leq d(z, w) \tag{6.2}$$

$$|(x, y)_w - (x, z)_w| \leq d(x, y) \tag{6.3}$$

Taking limits in 6.2 give us

$$|(X_2, p)_{A_1 z_1} - (X_2, p)_{A_2 z_2}| \leq d(A_1 z_1, A_2 z_2)$$

We would like to apply to first therm, but you can't simply take limit as before, because the distance space doesn't extends to the boundary distance. Even so, we can obtain a similar result that is enough for our purposes

Proposition 6.1. *Given $y, p \in H$ and $\xi, \xi' \in \partial H$, there exists a constant C_y such that*

$$|(\xi, y)_p - (\xi{,}y)_p| \leq C_y d_\varepsilon(\xi, \xi')$$

Proof. Recall in this case we are working under the assumption of strong hyperbolicty, thus we can write the Gromov product as $(\xi, \xi')_p = -\log(d_\varepsilon(\xi, \xi'))$. Hence

$$|(\xi, y)_p - (\xi{,}y)_p| = |\log(d_\varepsilon(\xi', y)) - \log(d_\varepsilon(\xi, y))|$$

In general, there is no constant such that $|\log(t) - \log(s)| \leq C|t-s|$ for all t, s, but as y is fixed, $d_\varepsilon(\xi, y)$, $d_\varepsilon(\xi', y)$ are bounded below and hence there exists a constant C_y such that

$$|\log(d_\varepsilon(\xi', y)) - \log(d_\varepsilon(\xi, y))| \leq C_y|d_\varepsilon(\xi', y) - d_\varepsilon(\xi, y)| \leq C_y d_\varepsilon(\xi, \xi')$$

■

By the previous proposition, there exists a constant C_q such that the first therm of the right side of 6.1 is bounded by $C_q d_\varepsilon(X_1, X_2)$. We have shown that the Hölder condition of $\varphi(\omega)$ depends entirely on the parametrization to be Hölder.

6.2 Coarse Geometry

One can think Gromov hyperbolic spaces as a generalization of the ordinary hyperbolic space $\mathbb{H}^n$, but in reality they aren't in many aspects. Hyperbolic spaces capture the asymptotic or large scale characteristics of $\mathbb{H}^n$, more precisely the *coarse* properties and not the infinitesimal ones. Our problem, therefore, isn't a coarse problem, nevertheless there are very interesting characteristics related to coarse properties. What do we exactly mean with coarse?

Although a coarse structure have its own formal definition, we don't want to go in depth on that. General speaking, the therm coarse is used to refer properties finitely imperturbable. For example, we review the definition of a coarse map.

Definition 6.2. A map $f\colon X \to Y$ between metric spaces is called coarse if

i. For each R there is S such that

$$d(x, x') \leq R \Rightarrow d(f(x), f'(x)) \leq S$$

ii. For each bounded set $B \subset Y$, the inverse image $f^{-1}(B)$ is bounded in X.

We have abused the notation and used the same d for X and Y. Although this definition illustrate how the coarse structure works, we are looking for a more specific definition.

Definition 6.3. A map $f\colon X \to Y$ between metric spaces is called a quasi-isometry if

i. There exists constants $\lambda > 1, \varepsilon > 0$ such that for every pair x, x'

$$\frac{1}{\lambda}d(x,x') - \varepsilon \leq d(f(x), f(x')) \leq \lambda d(x,x') + \varepsilon$$

ii. There exists a constant C such that for each $y \in Y$ there is $x \in X$ such that $d(y, f(x)) < C$

The way one define the Gromov boundary 4.5 is inherently coarse. In fact, we have the following result

Theorem 6.4. *Let X, Y be quasi-isometric hyperbolic proper spaces via $f\colon X \to Y$, then there exists an homeomorphism $\partial f\colon \partial X \to \partial Y$. Furthermore, if $\partial X, \partial Y$ are perfect sets (e.g. 4.3.2), ∂f is Hölder.*

Proof. See [GdLH90] 7.14 ■

Corollary 6.5. *$\mathbb{H}^n$ is not quasi-isometric (so not isometry) $\mathbb{H}^m$ if $n \neq m$.*

Proof. $\mathbb{S}^{n-1}$ is not homeomorphic to $\mathbb{S}^{m-1}$ ■

Despite of that, the δ hyperbolic condition is not a quasi-isometry invariant for arbitrary metric spaces, although it is for geodesic spaces as we will see in the next proposition.

Theorem 6.6. *Let X, Y be quasi-isometric hyperbolic geodesic spaces via $f\colon X \to Y$. If Y is δ hyperbolic then X is δ' hyperbolic and δ' can be expressed in therms of δ and the quasi-isometry constants.*

Proof. See [BH11] III. 1.9 ■

Corollary 6.7. *Hyperbolicity is an invariant among geodesic metric spaces.*

Proof. Let $f : X \to Y$ be a quasi-isometry. We will construct a quasi-isometry $f' : Y \to X$. We know that for every $y \in Y$ there is a point $x \in X$ such that the image $f(x)$ is C-close to y. Using the axiom of choice we can define $f'(y)$ as one of those x and denote it x_y. The second condition holds immediately. For the first one, we have

$$d(f'(y), f'(y')) \leq \lambda(f(x_y), f(x'_y)) + \varepsilon \leq \lambda(y, y') + 2\lambda C + \varepsilon$$

The lower bound is analogous. ■

The next example illustrate the previous statement doesn't hold for arbitrary metric spaces.

Example 6.2.1. Let $\mathcal{P}$ be the image of the curve $u : [0, \infty) \to \mathbb{R}^2$ given by $t \mapsto (\sqrt{t}, t)$ and let be d the usual euclidean plane metric; $\mathcal{P}$ is not a geodesic space because geodesics are still straight lines. A simple calculation shows that

$$|t - t'| - \frac{1}{2} \leq d(u(t), u(t')) \leq |t - t'| + \frac{1}{2}$$

That means $\mathcal{P}$ is quasi-isometric to the real line which is 0-hyperbolic. But $\mathcal{P}$ isn't hyperbolic for any $\delta \geq 0$. Indeed, set $w = (0, 0)$, $x = (1, 1)$, $y = (2, 4)$ and $z = (t, \sqrt{t})$ and

$$(x, y)_w = a, \quad (x, z)_w = f(t), \quad (y, z)_w = g(t)$$

It's clear that $f(t), g(t) \to \infty$ if $t \to \infty$. Hence there is no finite δ such that

$$a \geq \min\{f(t), g(t)\} - \delta$$

It's the same reason the Euclidean plane isn't δ-hyperbolic, triangles can be arbitrary *large*.

We finish this section with another very important theorem about δ-hyperbolic spaces and quasi-isometries.

Theorem 6.8. *Every δ-hyperbolic space is quasi-isometric to a metric graph that is δ'-hyperbolic.*

Proof. See [BH11]. ■

The previous theorem is very useful to prove coarse properties about δ-hyperbolic spaces, although we can't use apply the theorem for our problem.

6.3 What about CAT(0) spaces?

Finally, in this section we introduce an investigate the initial problem on CAT(0) spaces which seem to be very suitable for that. These spaces also have a geodesic metrizable boundary defined in the same way as in Gromov hyperbolic spaces. In fact, we have more tools to work with in the case of CAT(0) spaces, mainly due to the existence of angles. Along this chapter, we rely on [BH11], majority of definitions and proposition are taken from there.

We start defining a general CAT(κ) space, but for that we need to review some previous concepts

Definition 6.9. The M_κ^n for real number κ is the following metric space

(1) If $\kappa = 0$ then M_0^n is Euclidean space $\mathbb{E}^n$

(2) If $\kappa > 0$ then M_κ^n is obtained from the sphere $\mathbb{S}^n$ multiplying its distance $d_{\mathbb{S}}$ by $1/\sqrt{\kappa}$. The sphere distance is given by

$$\cos d_{\mathbb{S}}(x, y) = (x|y)$$

(3) If $\kappa < 0$ then M_κ^n is obtained from hyperbolic space $\mathbb{H}^n$ multiplying its distance $d_{\mathbb{H}}$ by $1/\sqrt{-\kappa}$. The hyperbolic distance is given by

$$\cos d_{\mathbb{H}}(x, y) = \langle x|y\rangle$$

Where $(x|y)$ is the usual inner product and

$$\langle x|y\rangle = -x_{n+1}y_{n+1} + \sum_{i=1}^{n} x_i y_i$$

Proposition 6.10. *Let κ be a real number and p, q, r three points of a metric space X; if $\kappa > 0$ assume $d(p, q) + d(p, r) + d(q, r) \leq 2D_\kappa$. There are comparison points p', q', r' in M_κ^2 for every κ which are unique up to an isometry.*

Definition 6.11. A metric space X is called a CAT(κ) space if for every geodesic triangle $\triangle$ and points x, y in the geodesic sides of $\triangle$ and comparison points $x, y \in \overline{\triangle}$ in M_κ^2

$$d(x, y) \leq d(\overline{x}, \overline{y})$$

Moving on, we define angles for CAT(κ) spaces. There is a lot of work prior to this proof and definition, defining angles in a general way for metric spaces using comparison triangles, but we omit the details that can be consulted in [BH11].

Proposition 6.12. *Let X be CAT(κ) space and $\gamma : [0, a] \to X$ and $\gamma : [0, a'] \to X$ two geodesics with a common initial point p. Then, the limit*

$$\lim_{t\to 0} 2 \arcsin \frac{1}{2t} d(\gamma(t), \gamma'(t)) \tag{6.4}$$

exists and it will be called the angle ,$\angle_p(\gamma, \gamma')$, between the geodesics.

Now we focus on CAT(0) spaces. In a first approach one can think they are a *flat* space or with zero curvature, but that's not always the case due to a CAT(κ) space is CAT(κ') for every $\kappa' \geq \kappa$ so for example, the hyperbolic space is a CAT(0) space. So in order to study our problem we separate in two cases, the ones that are indeed flat spaces and there ones that aren't.

Definition 6.13. Let X be a complete CAT(0) space. The angle $\angle(\xi, \zeta)$ between $\xi, \zeta \in \partial X$ is defined by

$$\angle(\xi, \zeta) = \sup_{x \in X} \angle_x(\xi, \zeta)$$

This angle defines a metric on the boundary.

Note that if X is the hyperbolic space $\mathbb{H}^n$ (in fact, any complete CAT(-1) space) then $\angle(\xi, \zeta) = \pi$, because there always is a geodesic γ such that $\gamma(\infty) = \xi$ and $\gamma(-\infty) = \zeta$. Thus this metric isn't a good choice for working. There exists another metric for the boundary but it isn't either suitable for our purposes. Regardless, we still define it and study it because give us a lot of information and characteristics about CAT(0) spaces and their boundaries.

Recall that given a metric d the length metric between two points is defined as the infimum of the lengths of all the rectifiable paths (with finite length) joining those points. The length of a path c is

$$l(c) = \sup \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}))$$

Where the supremum is taken over all partitions $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$.

Definition 6.14. The *Tits metric* on ∂X, with X a CAT(0) space, is the length metric associated to the angular metric. We will denote the Tits metric as d_T

Remark. If ξ and ζ are points in ∂X such that there is no rectifiable path joining them in the angular metric, $d_T(\xi, \zeta) = \infty$. Thus, if X is a CAT(-1) space the Tits distance of any two distinct points is infinity. This tell us that the only reasonable distance to work with in CAT(-1) is the one defined previously using the Gromov product.

Next proposition give us a lot information about the Tits metric behavior.

Proposition 6.15. *Let X be a proper CAT(0) space and let ξ and ζ two distinct points of ∂X*

(1) *If $d_T(\xi, \zeta) > \pi$, then there is a geodesic $\gamma \colon \mathbb{R} \to X$ such that $\gamma(\infty) = \xi$ and $\gamma(-\infty) = \zeta$.*

(2) *If there is no geodesic such that $\gamma(\infty) = \xi$ and $\gamma(-\infty) = \zeta$ then $d_T(\xi, \zeta) = \angle(\xi, \zeta)$*

(3) *If $\gamma \colon \mathbb{R} \to X$ is a geodesic, then $d_T(\gamma(-\infty), \gamma(\infty)) \geq \pi$, with equality if and only if $c(\mathbb{R})$ bounds a flat half-plane.*

Proof. See [BH11] II. 9.21. ■

We are ready to aboard our original problem, starting with a proposition about isometries.

Proposition 6.16. *Let X be a proper CAT(0) space and $g : X \to X$ an isometry. Then $g^* : \partial X \to \partial X$ is an isometry for the angular and the Tits metric.*

Proof. If $\xi, \xi' \in \partial X$ and γ, γ' are geodesic rays such that $\gamma(0) = \gamma'(0) = p$, $\gamma(\infty) = \xi$, $\gamma'(\infty) = \xi'$ then

$$\angle(\xi, \xi') = \lim_{t\to\infty} 2\arcsin \frac{1}{2t} d(\gamma(t), \gamma'(t))$$

(See [BH11] II.9.7(4)). Hence

$$\begin{aligned}\angle(g\xi, g\xi') &= \lim_{t\to\infty} 2\arcsin \frac{1}{2t} d(g\gamma(t), g\gamma'(t)) \\ &= \lim_{t\to\infty} 2\arcsin \frac{1}{2t} d(\gamma(t), \gamma'(t)) \\ &= \angle(\xi, \xi')\end{aligned}$$

The fact that g is an isometry for the Tits metric is immediate due to its construction as a length metric. ■

Along the proofs in the previous chapter we made use in every case of the fact that one can join two points of the boundary by a geodesic. But now, as we just saw, one can not ensure it for an arbitrary CAT(0) space. Nevertheless, we can construct the geodesic in certain situations. Suppose x_0 is a point of CAT(0) such that the geodesic γ passing through that point satisfies $d_T(\gamma(-\infty), \gamma(\infty)) > \pi$. Let be (T, A) a dynamic and a co-cycle of isometries as usual. Define $\xi_\omega = A(\omega)\gamma(\infty)$ and $\zeta_\omega = A(\omega)\gamma(-\infty)$; by the previous statement, 6.16, we have

$$d_T(\xi_\omega, \zeta_\omega) > \pi$$

for every $\omega \in \Omega$. Hence, due to 6.15 (1), there exists a geodesic $\gamma_\omega : \mathbb{R} \to X$ such that $\gamma_\omega(\infty) = \xi_\omega$ and $\gamma_\omega(-\infty) = \zeta_\omega$. At this point, the problem is again, find a parametrization of the geodesic that varies Hölder when the points of the boundary varies Hölder.

We finish proposing a solution for the case of $X = \mathbb{E}^2$, a place where everything is quite explicit and simple, so there are no major inconvenient to use our general technique.

Pick a point $x_0 \in X$ and let $\gamma : \mathbb{R} \to X$ the geodesic passing through x_0 and the origin o. By 6.15 (3), $d_T(\gamma(-\infty), \gamma(\infty)) = \pi$. As before, define the points of the boundary $\xi_\omega = A(\omega)\gamma(\infty)$ and $\zeta_\omega = A(\omega)\gamma(-\infty)$ which satisfies $d_T(\xi_\omega, \zeta_\omega) = \pi$. We can no longer apply 6.15 (1), but in this particular case we know that $\partial\mathbb{E}^2 = \mathbb{S}^1$, so there is a geodesic joining γ_ω joining ξ_ω and ζ_ω and as the angle between the points of the boundary is equal to π, the geodesic has to pass through the origin, making that all of them intersect at this point. Thus we have a clear parametrization of all geodesics by setting $\gamma_\omega(0) = o$. Define the same co-cycle of 5.3

$$N_\omega(t) = \gamma_{T\omega}^{-1}(A(\omega)\gamma_\omega(t))$$

Everything that goes for N in 5.3, goes for this N when we assume we have a Hölder (for the Tits metric) solution for the boundary. The maps N_ω are isometries, varying Hölder in ω. As there, applying Livšic and combined with 2.1, we have solution curve $s(\gamma_\omega(t))$. The fact that whether the curve can be Hölder or not depends pretty much on $\gamma_\omega(t)$ to be Hölder in ω for every t.

Let $\xi = \xi_\omega, \xi'_{\omega'}$ be points of the boundary such that

$$d_T(\xi, \xi') \leq Cd(\omega, \omega')^\alpha$$

Without loss generality, suppose $d_T(\xi, \xi') < \pi$ and thus $d_T(\xi, \xi') = \angle(\xi, \xi')$. In general, if $\gamma(0) = \gamma'(0) = o$, $\gamma(\infty) = \xi$ and $\gamma(\infty) = \xi'$ we have

$$\angle(\xi, \xi') = \lim_{t,t' \to \infty} \angle_o(\gamma(t), \gamma'(t'))$$

But in this particular case of the euclidean plane $\angle(\xi, \xi') = \angle_o(\gamma(t), \gamma'(t'))$ for every t, t'. Now let C_t be a circle of radius t centered at o and set $y = \gamma(t) \cap C_t$ and $y' = \gamma'(t) \cap C_t$. We have

$$d(y, y') \leq t\angle_o(\gamma(t), \gamma'(t')) = t\angle(\xi, \xi') \leq tCd(\omega, \omega')^\alpha$$

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