Pontificia Universidad Católica de Chile

Master's Thesis

Dimension of sets and normality of numbers with respect to Cantor series expansions

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Para Luciano

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Contents

Chapter 1

Introduction

Let $Q = (q_n)_{n\geq 1}$ be a sequence of natural numbers such that $q_n \geq 2$ for every $n \in \mathbb{N}$. A *Cantor series* determined by Q is a series of the form

$$\sum_{n\geq 1} \frac{\varepsilon_n}{q_1 \cdots q_n}, \tag{1.1}$$

where $\varepsilon_n \in \Omega_n := \{0, 1, \ldots, q_n - 1\}$. Since $q_n \geq 2$, the series in (1.1) converges absolutely by comparison with $a_n = 2^{-(n-1)}$.

A natural question that arises is if an arbitrary $x \in [0, 1)$ can be written in the form of (1.1). Indeed, given $x \in [0, 1)$, the digits ε_n of the series can be obtained using the following recursive formulas.

Define $r_0(x) := x$ and set $\varepsilon_1(x) = [q_1 r_0(x)]$, $r_1(x) = \{q_1 r_0(x)\}$, where $[\cdot]$ and $\{\cdot\}$ are the integer and fractional parts respectively. For $n \geq 2$, define

$$\varepsilon_n(x) = [q_n r_{n-1}(x)], \quad r_n(x) = \{q_n r_{n-1}(x)\}. \tag{1.2}$$

If $r_m(x) = 0$ for some m, then $\varepsilon_n(x) = 0$ for $n \geq m + 1$.

We write

$$x = \sum_{n\geq 1} \frac{\varepsilon_n(x)}{q_1 \cdots q_n} =: [\varepsilon_1(x), \varepsilon_2(x), \ldots]_Q.$$

A direct consequence of (1.2) is that every $x \in [0, 1)$ admits at least one representation in the form of (1.1). We call a representation in this form a *Cantor series expansion* of x with respect to Q. The expansion obtained as in (1.2) converges to x, as we will see now.

Define $x_n := [\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x)]_Q$ and $\alpha_n := x_{n+1} - x_n$ for $n \geq 1$. It is clear from the definition of $\varepsilon_n(x)$ that $x - x_n \geq 0$ and $\alpha_n \geq 0$, as can be seen in Figure 1.1. Also,

$$\alpha_n = \frac{\varepsilon_{n+1}(x)}{q_1 \cdots q_{n+1}} \leq \frac{q_{n+1} - 1}{q_1 \cdots q_{n+1}} = \frac{1}{q_1 \cdots q_n} - \frac{1}{q_1 \cdots q_{n+1}}.$$

Then, $|\alpha_n| \leq (q_1 \cdots q_n)^{-1} - (q_1 \cdots q_{n+1})^{-1}$.

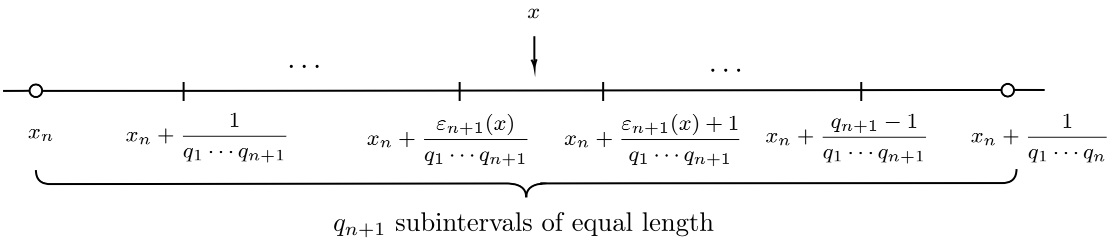


FIGURE 1.1

Using induction on n, suppose that $x - x_{n-1} \leq (q_1 \cdots q_{n-1})^{-1}$. Then,

$$x - x_n = x - x_{n-1} - \alpha_{n-1} \leq x - x_{n-1} + (q_1 \cdots q_n)^{-1} - (q_1 \cdots q_{n-1})^{-1} \leq (q_1 \cdots q_n)^{-1}.$$

Hence,

$$|x - x_n| \leq (q_1 \cdots q_{n-1})^{-1} \leq 2^{-n} \xrightarrow{n} 0.$$

It follows that $x_n \xrightarrow{n} x$.

Now that we have seen that every $x \in [0,1)$ has a Cantor series expansion with respect to Q, the question that follows concerns the uniqueness of this representation. In order to study this, consider $x \in [0,1)$ with two different expansions

$$\sum_{n \geq 1} \frac{\varepsilon_n(x)}{q_1 \cdots q_n} = \sum_{n \geq 1} \frac{\delta_n(x)}{q_1 \cdots q_n}.$$

Let $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0}(x) \neq \delta_{n_0}(x)$ but $\varepsilon_n(x) = \delta_n(x)$ if $n < n_0$. Multiplying both expansions by $q_1 \cdots q_{n_0}$ and subtracting the first $n_0 - 1$ terms we get

$$\varepsilon_{n_0}(x) + \sum_{n > n_0} \frac{\varepsilon_n(x)}{q_{n_0+1} \cdots q_n} = \delta_{n_0}(x) + \sum_{n > n_0} \frac{\delta_n(x)}{q_{n_0+1} \cdots q_n}. \tag{1.3}$$

Without loss of generality, assume $\varepsilon_{n_0}(x) > \delta_{n_0}(x)$. Subtracting in (1.3) we obtain

$$0 < \varepsilon_{n_0}(x) - \delta_{n_0}(x) = \sum_{n > n_0} \frac{\delta_n(x) - \varepsilon_n(x)}{q_{n_0+1} \cdots q_n}. \tag{1.4}$$

The right-hand side of (1.4) is bounded above by

$$\sum_{n>n_0} \frac{\delta_n(x)-\varepsilon_n(x)}{q_{n_0+1}\cdots q_n} \leq \sum_{n>n_0} \frac{q_n-1}{q_{n_0+1}\cdots q_n} = \sum_{n>n_0} \frac{1}{q_{n_0+1}\cdots q_{n-1}} - \frac{1}{q_{n_0+1}\cdots q_n} \leq 1.$$

Since $\varepsilon_{n_0} - \delta_{n_0} \in \mathbb{N}$, we have that $\varepsilon_{n_0} - \delta_{n_0} = 1$. Thus,

$$\sum_{n\geq n_0+1} \frac{\delta_n(x)-\varepsilon_n(x)}{q_{n_0+1}\cdots q_n} = 1,$$

and $\delta_n - \varepsilon_n = q_n - 1$ for every $n \geq n_0+1$. Recall that $\delta_n, \varepsilon_n \in \Omega_n$. Then, $\delta_n(x) = q_n - 1$, $\varepsilon_n(x) = 0$.

The uniqueness follows if we ask $\varepsilon_n(x) \neq q_n - 1$ for all but finitely many n. This is the case of the expansion obtained in (1.2), which we will call *the* Cantor series representation of x with respect to Q. Some examples of representations of certain x with respect to some Q are

- $Q = (n+1)_{n\geq 1}$ and $\{e\} = \sum_{n\geq 1} \frac{1}{(n+1)!} = [1,1,1,\ldots]_Q$.
- $Q = (q_n)_{n\geq 1}$ where

$$q_n = \begin{cases} 2 & \text{if } n \text{ is even.} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

  Hence $\frac{1}{24} = [0,0,0,1,1,1,0,0,0,0,\ldots]_Q = [0,0,0,1,1,0,2,1,2,1,2,\ldots]_Q$. This second representation is not correct if we ask $\varepsilon_n \neq q_n - 1$ for all but finitely many n.
- $Q = (q_n)_{n\geq 1}$ where $q_n = b$ for every n and b is a natural number greater than 1. Then $x = [\varepsilon_1(x), \varepsilon_2(x), \ldots]_Q$ is the expansion in base b of x.

The third example shows that the Cantor series expansion of x with respect to Q is a generalization of the base b expansion.

1.1. Cantor series as a dynamical system

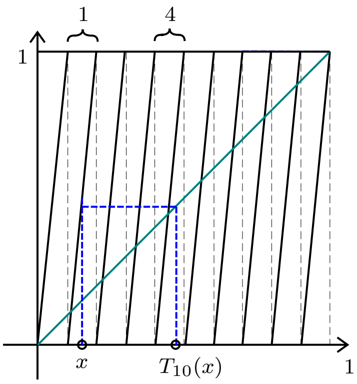
The expansion in base b is a classic example of a dynamical system. If we consider $x \in [0,1)$ we can relate its b-ary expansion, defined in the last example, with a transformation. Let $T_b : [0,1) \longrightarrow [0,1)$ be defined by $x \longmapsto bx \mod 1$. The map T_b acts as a shift on the digits of the expansion in the following sense.

If $x = [\varepsilon_1, \varepsilon_2, \ldots]$ is the b-ary expansion of x, then $T_b([\varepsilon_1, \varepsilon_2, \ldots]) = [\varepsilon_2, \varepsilon_3, \ldots]$.

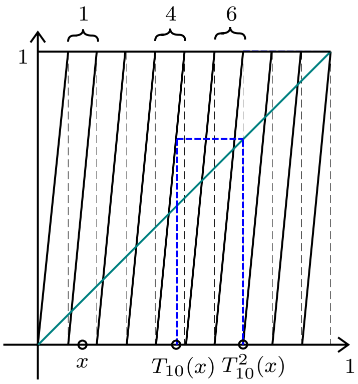
The transformation T_b allows to calculate the digits ε_n of the expansion as follows. Given $x \in [0,1)$, $\varepsilon_n(x) = [b\{T_b^{n-1}(x)\}]$, where $T_b^0(x) = x$. For example, consider the number $x = 0.1469$ where the representation is its decimal expansion. The formula for ε_n gives

- $\varepsilon_1(x) = [10 \cdot \{0.1469\}] = [1.469] = 1$
- $\varepsilon_2(x) = [10 \cdot \{10 \cdot 0.1469\}] = [10 \cdot 0.469] = [4.69] = 4.$
- $\varepsilon_3(x) = [10 \cdot \{100 \cdot 0.1469\}] = 6.$
- $\varepsilon_4(x) = 9$ and $\varepsilon_n(x) = 0$ if $n \geq 5$.

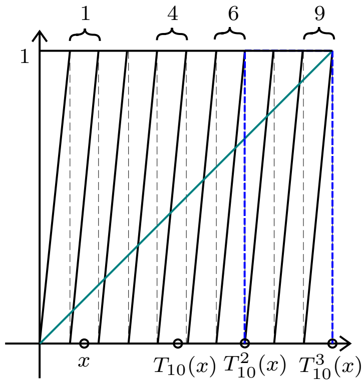
Then, we know the orbit of a point under T_b if and only if we know its expansion in base b. This calculation can be seen graphically in Figure 1.2.



(A) Step 1: $\varepsilon_0(x), \varepsilon_1(x)$.



(B) Step 2: $\varepsilon_3(x)$



(C) Step 3: $\varepsilon_4(x)$.

FIGURE 1.2: Expansion in base $b = 10$ of $x = 0.1469$.

The Lebesgue measure is ergodic for the dynamical system $([0,1), T_b)$ (c.f. [EW11]) Hence, the Birkhoff ergodic theorem (Theorem 3.1.2) is a tool that we can use. In fact, in Chapter 3 this theorem is used to prove the Borel theorem on normal numbers.

In a similar way, we can introduce a dynamical system to study the Cantor series expansion with respect to a given Q.

Consider $x \in [0,1)$ and its Cantor series expansion with respect to Q given by $x = [\varepsilon_1(x), \varepsilon_2(x), \ldots]_Q$. Note that $q_1 x \mod 1 = [\varepsilon_2(x), \varepsilon_3(x), \ldots]_Q$. More generally,

$$q_n q_{n-1} \cdots q_1 x \mod 1 = [\varepsilon_{n+1}(x), \varepsilon_{n+2}(x), \ldots]_Q \tag{1.5}$$

The family of transformations $(g_n)_{n\geq 1}$ defined by

$$g_n : [0,1) \longrightarrow [0,1), \quad x \longmapsto q_n x \mod 1 \tag{1.6}$$

acts as a shift in the expansion of x. The digits of the expansion can be recovered as in the b-ary case. If we set $G_n := g_n \circ g_{n-1} \circ \cdots \circ g_1$, the formula to do this is

$$\varepsilon_n(x) = [q_n\{G_{n-1}(x)\}].$$

For example, consider Q such that $q_1 = 5$, $q_2 = 2$ and $q_3 = 4$. Choose $x = 7/10$. Its Q expansion is calculated in the following way.

- $\varepsilon_1\left(\frac{7}{10}\right) = \left[5 \cdot \frac{7}{10}\right] = 3.$
- $\varepsilon_2\left(\frac{7}{10}\right) = \left[2\left\{5 \cdot \frac{7}{10}\right\}\right] = \left[2 \cdot \frac{1}{2}\right] = 1.$
- $\varepsilon_3\left(\frac{7}{10}\right) = \left[4\left\{2 \cdot 5 \cdot \frac{7}{10}\right\}\right] = 0.$
- $\varepsilon_n(x) = 0$ for every $n \geq 4$,

a calculation illustrated in Figure 1.3. Hence, its representation as a Cantor series with respect to Q is

$$\frac{7}{10} = \frac{3}{5} + \frac{1}{2 \cdot 5} = [3, 1, 0, 0, \ldots]_Q.$$

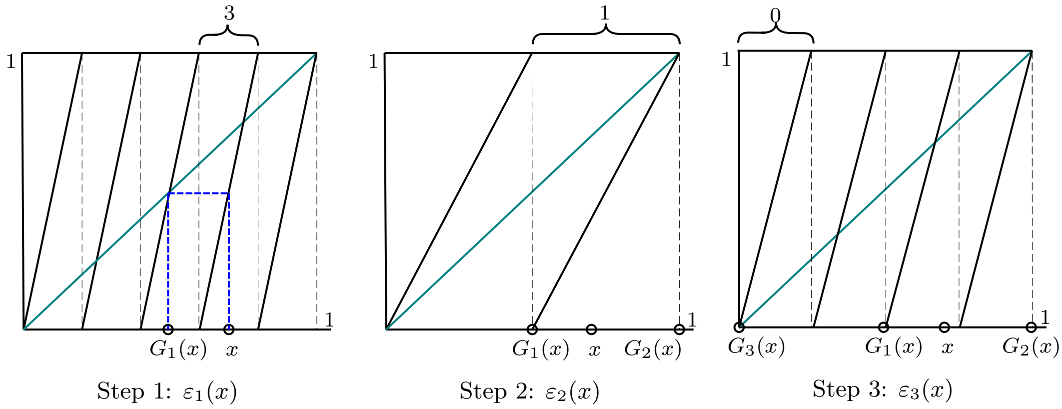


FIGURE 1.3: Cantor series expansion with respect to Q of $x = 7/10$

Then, if we want to know the digits of the Cantor series expansion of a given x, we need to understand the orbit of x under the dynamical system $([0,1), (G_n)_n)$,

$$\{x, G_1(x), G_2(x), \ldots, G_n(x), \ldots\}.$$

However, we need to apply a different function if we want to shift a digit in a specific position. This essential difference with the b-ary case changes completely how to approach the problems concerned with Cantor series expansion. Its associated dynamical system $([0,1),(g_n)_n)$ is non-autonomous, and its theory is not fully developed.

Chapter 2

Hausdorff dimension of sets defined in terms of Cantor series

2.1. Hausdorff dimension

2.1.1. Definition of the Hausdorff dimension

Let $\mathbb{T}$ be the compact space $\mathbb{R}/\mathbb{Z}$. Henceforth, $\mathbb{T}$ will be identified with the interval $[0, 1)$, E will denote an arbitrary subset of $[0, 1)$, μ a positive Radon measure on $[0, 1)$, and $\mathcal{F}$ a family of half-open intervals of $[0, 1)$.

Definition 2.1.1. Let $\varepsilon > 0$ and $E \subseteq [0, 1)$. An $(\varepsilon, \mathcal{F}, \mu)$-cover of E is a countable collection of elements $\{F_n\}_n \subseteq \mathcal{F}$ such that

$$E \subseteq \bigcup_n F_n, \quad \mu(F_n) < \varepsilon \;\; \forall n \in \mathbb{N}.$$

For every $\alpha > 0$, we define

$$\mathcal{H}^{\alpha}_{(\varepsilon,\mathcal{F},\mu)}(E) := \inf \left\{ \sum_n \mu(F_n)^{\alpha} : \{F_n\}_n \text{ is an } (\varepsilon, \mathcal{F}, \mu)\text{-cover of } E \right\}. \tag{2.1}$$

In this definition, the sequence of sets $\{\sum_n \mu(F_n)^{\alpha} : \{F_n\}_n \text{ is an } (\varepsilon, \mathcal{F}, \mu)\text{-cover of } E\}$ is increasing in ε. Hence, $\mathcal{H}^{\alpha}_{(\varepsilon,\mathcal{F},\mu)}(E)$ is decreasing in ε and consequently $\lim_{\varepsilon\to 0} \mathcal{H}^{\alpha}_{(\varepsilon,\mathcal{F},\mu)}(E)$ exists and could be $+\infty$. We denote this limit by $\mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E)$ and we call this quantity the *α-dimensional Hausdorff outer measure* of E.

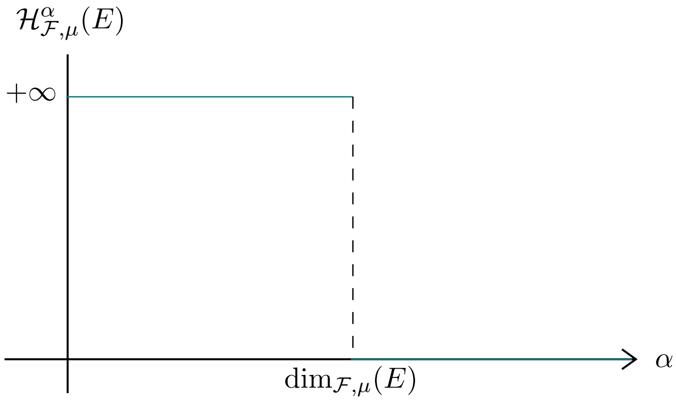


FIGURE 2.1: Hausdorff outer measure and Hausdorff dimension.

It is clear from (2.1) that if α is such that $\mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E) < \infty$, then $\mathcal{H}^{\beta}_{\mathcal{F},\mu}(E) = 0$ for every $\beta > \alpha$ and that $\mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E)$ is decreasing in α. So, there exists a unique value of the parameter α in which the outer measure jumps from $+\infty$ to 0. This observation allows to make the following crucial definition.

Definition 2.1.2. Let $E \subseteq [0,1)$. The $(\mathcal{F},\mu)$-Hausdorff dimension of E is the non-negative number

$$\dim_{\mathcal{F},\mu}(E) := \inf\{\alpha > 0 : \mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E) = 0\} = \sup\{\alpha > 0 : \mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E) = \infty\}. \tag{2.2}$$

If $\alpha = \dim_{\mathcal{F},\mu}(E)$, then $\mathcal{H}^{\alpha}_{\mathcal{F},\mu}(E)$ can be finite or infinite.

When $\mathcal{F}$ is the family of all half-open intervals of $[0,1)$ and μ is the Lebesgue measure, Definition 2.2 coincides with the definition of Hausdorff dimension. In that case, we will omit the subscripts.

Changing the set of possible covers of a set could also vary its Hausdorff dimension. Hence, a natural question would be

Which assumptions about the family $\mathcal{F}$ should be made to obtain the same Hausdorff dimension as when we consider the family of all half-open intervals of $[0,1)$?

Definition 2.1.3. Let $\mathcal{F}$ be a family of right-open intervals of $[0,1)$. The class $\mathcal{F}$ *allows the calculation of the Hausdorff dimension* if

$$\dim_{\mathcal{F}}(E) = \dim(E) \text{ for every } E \subseteq [0,1).$$

2.1.2. Assumptions on $\mathcal{F}$

In this section we will give conditions on the family $\mathcal{F}$ in order to give a partial answer to the previous question.

From now on, $\mathcal{F}$ will satisfy the following hypotheses:

(H_1) $\mathcal{F} = \bigcup_{n\geq 0} \mathcal{F}_n$, where $\mathcal{F}_n$ is a finite partition of $[0,1)$.

(H_2) $\mathcal{F}_{n+1}$ is a strict refinement of $\mathcal{F}_n$, namely, for every $I \in \mathcal{F}_{n+1}$ there exists $J \in \mathcal{F}_n$ such that $I \subsetneq J$. The set J is called the father of I and is denoted by $p(I)$.[1]

Since our aim is to capture the local shape of the sets, we also assume that

(H_3) For every $x \in [0,1)$ and $\varepsilon > 0$ there exists $I \in \mathcal{F}$ such that $x \in I$ and $|I| < \varepsilon$,

where $|I|$ denotes the Lebesgue measure of the half-open interval I.

Finally, a fourth assumption is made to control the speed at which the length of the elements of $\mathcal{F}$ decrease in each step. Define the function $k : \mathcal{F} \to \overline{\mathbb{R}}$ by

$$I \longmapsto \sup\left\{\frac{|I|}{|J|} : J \in \bigcup_{n\geq 1} \mathcal{F}_n,\ p(J) = I\right\}.$$

(H_4) For every $\alpha > 0$, $|I|^\alpha k(I) \longrightarrow 0$ whenever $|I| \longrightarrow 0$.

2.1.3. Calculation of the Hausdorff dimension

The objective of this section is to prove the following theorem of Peyrière (cf. [Pey77]).

Theorem 2.1.1. If $\mathcal{F}$ satisfies the hypotheses H_1, H_2, H_3 and H_4, then $\mathcal{F}$ allows the calculation of the Hausdorff dimension.

Its proof requires some auxiliary results.

Lemma 2.1.1. If $\mathcal{F}$ satisfies H_1, H_2 and H_3, then $\sup\{|I| : I \in \mathcal{F}_n\} \xrightarrow{n} 0$.

Proof. We proceed by contradiction. If that is not the case, there exists $\eta > 0$ and $\{I_n\}_n \subseteq \mathcal{F}$ a sequence of intervals such that $I_n \in \mathcal{F}_{N_n}$ with $N_n \xrightarrow{n} \infty$ and $|I_n| > \eta$.

[1]The function is denoted by p because its original name is *père* in [Pey77]

Order $\{I_n\}$ by inclusion. Then it must exist an infinite chain, because if not, there are infinite disjoint intervals I such that $|I| > \eta$, contradicting $|[0,1)| = 1$. Thus, we can construct a sequence $\{J_n\}_{n\geq 0} \subseteq \mathcal{F}$ such that $J_n \in \mathcal{F}_n$, $J_{n+1} \subsetneq J_n$ and $|J_n| > \eta$ for every $n \geq 0$.

Let $x \in \bigcap_{n\geq 0} J_n$, which is non-empty because its measure is greater or equal to η (since $\mathcal{F}_{n+1}$ is a refinement of $\mathcal{F}_n$). This x cannot be covered by arbitrarily small intervals, contradicting (H_3). ■

Lemma 2.1.2. Let $\mathcal{F}$ satisfy $(H_1), (H_2)$ and (H_3). Then, for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$I \in \mathcal{F} \text{ and } |I| < \eta \Longrightarrow |p(I)| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. By Lemma 2.1.1, there exists $n_0 \in \mathbb{N}$ such that $\sup\{|I| : I \in \mathcal{F}_{n_0}\} < \varepsilon$.

Consider the set A of the intervals in $\bigcup_{n\leq n_0} \mathcal{F}_n$ such that their lengths are less than ε, ordered by inclusion.

The maximal elements of A, say $J_1, J_2, \ldots, J_N$, form a finite partition of $[0,1)$. Set $\eta = \inf\{|J_k| : k \in \{1,2,\ldots,N\}\}$. We claim that η is the one we were looking for. In fact, if $I \in \mathcal{F}$ satisfies $|I| < \eta$, then there exists $k \in \{1,2,\ldots,N\}$ such that $I \subsetneq J_k$. By (H_2) we have that $p(I) \subseteq J_k$, and therefore $|p(I)| \leq |J_k| < \varepsilon$. ■

Lemma 2.1.3. Let I be an arbitrary right-open interval of $[0,1)$ such that $|I| < \inf\{|J| : J \in \mathcal{F}_0\}$. Then, one of the following cases occurs:

(1) There exists $J \in \mathcal{F}$ such that $J \subseteq I \subseteq p(J)$,

(2) There exist $J_1, J_2 \in \mathcal{F}$ disjoint sets such that $J_1 \cup J_2 \subseteq I \subseteq p(J_1) \cup p(J_2)$.

Proof. For $J \in \mathcal{F}$, define $g(J) \in \mathbb{N}$ as the unique number such that $J \in \mathcal{F}_{g(J)}$.

Set $A := \{J \in \mathcal{F} : J \subseteq I\}$, which is non-empty by (H_3), ordered by inclusion. We choose one of its maximal elements, say J_1, such that g is the least possible. If $p(J_1) \supseteq I$, the first case occurs.

If $I \not\subseteq p(J_1)$, since J_1 is a maximal element of A, $p(J_1) \not\subseteq I$. Observe that $I \cap p(J_1) \supseteq J_1$, and in particular, $I \cap p(J_1) \neq \varnothing$. Since I and $p(J_1)$ are intervals, I must contain exactly one extreme of $p(J_1)$. Consider $B := \{J \in \mathcal{F} : J \subseteq I \cap$

$p(J_1)^c$ such that J is adjacent to $p(J_1)\}$ ordered by inclusion, and choose one of its maximal elements, say J_2. Then

- $p(J_2) \not\subseteq p(J_1)$, because if this is not the case, $J_2 \subseteq p(J_2) \subseteq p(J_1)$, contradicting the fact that J_2 is an element of B.
- $p(J_1) \not\subseteq p(J_2)$, because if this is not the case, $p(J_1) \subsetneq p(J_2)$ and therefore $g(J_1) \geq g(J_2) + 1 > g(J_2)$, contradicting the minimality of $g(J_1)$, since $J_2 \in A$.

Since $\mathcal{F}$ is a family of partitions, this implies that $p(J_1) \cap p(J_2) = \varnothing$ and, consequently, $J_1 \cap J_2 = \varnothing$. Also, $J_1 \cup J_2 \subseteq I$. Clearly, $p(J_1) \cap I \subseteq p(J_1)$ and $p(J_1)^c \cap I \subseteq p(J_2)$ because J_2 is maximal. It follows that $I \subseteq p(J_1) \cup p(J_2)$. ■

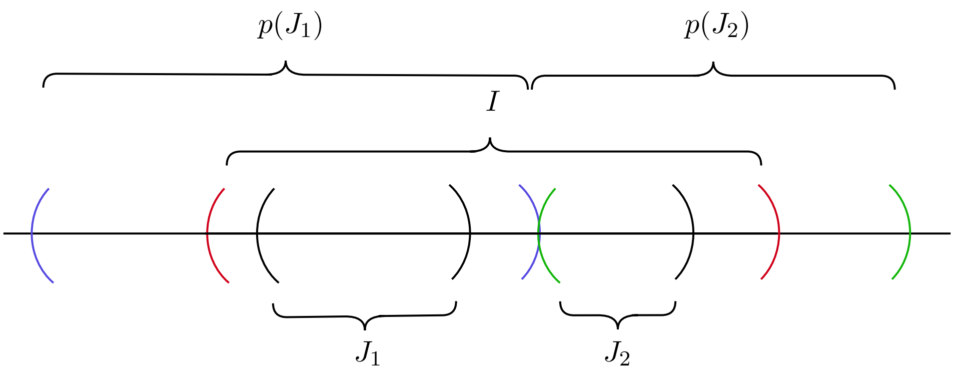


FIGURE 2.2: Lemma 2.1.3, second case.

Hypothesis (H_4) has not been used until now. This is the essential hypothesis in the proof of Theorem 2.1.1, which we present now. This proof can be found in [Pey77].

Proof (of Theorem 2.1.1). Let $\mathcal{F}$ be a family satisfying $(H_1), (H_2), (H_3)$ and (H_4). Since $\mathcal{F}$ is a sub-family of all right-open intervals of $[0, 1)$, we have $\mathcal{H}^\alpha_\varepsilon(E) \leq \mathcal{H}^\alpha_{(\varepsilon,\mathcal{F})}(E)$ for every $E \subset [0, 1)$. Making $\varepsilon \to 0$, $\mathcal{H}^\alpha(E) \leq \mathcal{H}^\alpha_{\mathcal{F}}(E)$ and, consequently, $\dim(E) \leq \dim_{\mathcal{F}}(E)$.

For the remaining inequality, lets fix $E \subseteq [0, 1)$ and $\alpha > 0$ such that $\dim(E) < \alpha$. We will prove that for every $\beta > 0$, $\dim_{\mathcal{F}}(E) \leq \alpha(1 + \beta)$.

Let $\beta > 0$ and define $\varphi_\beta(\varepsilon) := \sup\{|J|^\beta k(J) : J \in \mathcal{F}, |J| < \varepsilon\}$, which is well-defined by (H_3), and satisfies $\varphi_\beta(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ by Lemma 2.1.1 and (H_4).

Let $\varepsilon > 0$ and choose η as in Lemma 2.1.2, also satisfying $\eta < \inf\{|J| : J \in \mathcal{F}_0\}$. Consider an η-cover $\mathcal{T}$ of E by right-open intervals. We can divide $\mathcal{T}$ in two families of sets by Lemma 2.1.3:

- $\mathcal{T}_1 := \{I \in \mathcal{T} : I$ satisfies (1) of Lemma 2.1.3$\}$,
- $\mathcal{T}_2 := \{I \in \mathcal{T} : I$ satisfies (2) of Lemma 2.1.3$\}$.

For every $I \in \mathcal{T}_1$, we choose $J^I \in \mathcal{F}$ satisfying condition (1) in Lemma 2.1.3 and for every $I \in \mathcal{T}_2$ we consider $J_1^I, J_2^I \in \mathcal{F}$ satisfying condition (2) in the same lemma. By Lemma 2.1.2, $\{J^I\}_{I\in\mathcal{T}_1} \cup \{J_1^I\}_{I\in\mathcal{T}_2} \cup \{J_2^I\}_{I\in\mathcal{T}_2}$ is an $(\varepsilon, \mathcal{F})$-cover of E.

Hence,

$$\sum_{I\in\mathcal{T}_1} |p(J^I)|^{\alpha(1+\beta)} \leq \sum_{I\in\mathcal{T}_1} |I|^\alpha \left(\frac{|p(J^I)|}{|J^I|}\right)^\alpha |p(J^I)|^{\alpha\beta} \leq \sum_{I\in\mathcal{T}_1} k(J^I)^\alpha |I|^\alpha |p(J^I)|^{\alpha\beta} \leq$$
$$\leq \sum_{I\in\mathcal{T}_1} \varphi_\beta(\varepsilon)^\alpha |I|^\alpha,$$

where the first inequality follows from $|I|/|J^I| \geq 1$, the second from the definition of k, and the third from the definiton of $\varphi_\beta(\varepsilon)$.

The inequalities

$$\sum_{I\in\mathcal{T}_2} |p(J_1^I)|^{\alpha(1+\beta)} \leq \sum_{I\in\mathcal{T}_2} \varphi_\beta(\varepsilon)^\alpha |I|^\alpha, \qquad \sum_{I\in\mathcal{T}_2} |p(J_2^I)|^{\alpha(1+\beta)} \leq \sum_{I\in\mathcal{T}_2} \varphi_\beta(\varepsilon)^\alpha |I|^\alpha$$

are proven analogously. Then,

$$\mathcal{H}^{\alpha(1+\beta)}_{(\varepsilon,\mathcal{F})}(E) \leq 2\varphi_\beta(\varepsilon)^\alpha \mathcal{H}^\alpha_\eta(E) \xrightarrow{\varepsilon\to 0} 0.$$

Recalling that $\alpha > \dim(E)$ and $\varphi_\beta(\varepsilon)^\alpha \xrightarrow{\varepsilon\to 0} 0$, it follows that $\mathcal{H}^\alpha_\eta(E) \xrightarrow{\varepsilon\to 0} \mathcal{H}^\alpha(E) = 0$. In consequence, considering the inequality, $\dim_{\mathcal{F}}(E) \leq \alpha(1+\beta)$.

To prove $\dim_{\mathcal{F}}(E) = \dim(E)$, suppose that $\dim(E) < \dim_{\mathcal{F}}(E)$. Then there exists $\alpha > 0$ such that $\dim(E) < \alpha < \dim_{\mathcal{F}}(E)$. By the previous argument, $\dim_{\mathcal{F}}(E) \leq \alpha(1+\beta)$ for every $\beta > 0$ and therefore $\dim_{\mathcal{F}}(E) \leq \alpha$, a contradiction. ■

2.2. Cantor series and Hausdorff dimension

Let $Q = (q_n)_{n\geq 1} \subseteq \mathbb{N}$ be a sequence such that $q_n \geq 2$ for every $n \in \mathbb{N}$. The Cantor series expansion determined by Q defines a natural family of finite partitions of $[0, 1)$ consisting of right-open intervals. In order to compute the Hausdorff dimension of certain sets related to this expansion, it would be easier if we restrict our definition of Hausdorff dimension to this particular family. To obtain the same dimension that we obtain using all right-open intervals of $[0,1)$, we will make some assumptions on Q that will fulfill the requirements in Theorem 2.1.1 proved previously.

2.2.1. Natural partitions and hypotheses

Let $Q = (q_n)_n \subseteq \mathbb{N}$ such that $q_n \geq 2$ for every $n \in \mathbb{N}$, conditions that will be assumed throughout the chapter. Consider the family $\mathcal{F}$ of partitions by right-open intervals given by

$$\mathcal{F}_0 := \{[0,1)\}, \quad \mathcal{F}_n := \left\{ \left[\frac{k}{q_1 \cdots q_n}, \frac{k+1}{q_1 \cdots q_n} \right) \right\}_{0 \leq k \leq q_1 \cdots q_n - 1} \quad \text{for } n \geq 1.$$

The family $\mathcal{F}$ satisfies (H_1). Condition (H_2) is fulfilled because $q_n \geq 2$. Since $q_1 \ldots q_n \geq 2^n$, and for any interval $I \in \mathcal{F}_n$ we have $|I| = (q_1 \cdots q_n)^{-1} \leq 2^{-n} \xrightarrow{n} 0$, condition (H_3) also holds. If $I \in \mathcal{F}_n$ with $n \geq 1$, then $p(I) \in \mathcal{F}_{n-1}$. Therefore, $|p(I)| = (q_1 \cdots q_{n-1})^{-1}$ if $n \geq 2$ and $|p(I)| = 1$ if $n = 1$. In any case, $|p(I)|/|I| = q_n$ and $k(I) = q_{n+1}$. Thus, is sufficient to assume that

$$|I|^\alpha k(I) = \frac{q_{n+1}}{(q_1 \cdots q_n)^\alpha} \xrightarrow{n} 0$$

for every $\alpha > 0$ if we want (H_4) to hold.

Under these assumptions, $\mathcal{F}$ allows the calculation of the Hausdorff dimension in virtue of Theorem 2.1.1.

2.2.2. Setting and auxiliary results

We will identify $[0,1)$ with a symbolic space. Consider the set $\Omega_n := \{0, 1, \ldots, q_n - 1\}$. If we define

$$\Omega = \prod_{n \geq 1} \Omega_n,$$

an element $\varepsilon = (\varepsilon_n)_{n\geq 1} \in \Omega$ can be identified with $x = [\varepsilon_1, \varepsilon_2, \ldots]_Q \in [0,1)$. The cylinders of rank n in Ω are then identified with the elements of $\mathcal{F}_n$ defined before.

The main idea to prove the results in this section is to construct measures on Ω, calculate the Hausdorff dimension for those measures and deduce from that results in $[0,1)$. The following classical results will be used to achieve such an aim.

Theorem 2.2.1 (Kolmogorov's strong law of large numbers)**.** Let $\{X_n\}_n$ be a sequence of independent random variables such that $\mathbb{E}(X_n), \mathbb{V}(X_n) < \infty$ for every $n \in \mathbb{N}$.

$$\text{If } \sum_{n\geq 1} \frac{\mathbb{V}(X_n)}{n^2} < \infty, \text{ then } \frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_i) \xrightarrow{n} 0 \text{ almost surely.}$$

The proof can be found in [SS93]. The result that will allow us to translate our calculations into $[0, 1)$ is due to Billingsley and can be found in [Bil61].

Let $I_n(x)$ denote the unique element of $\mathcal{F}_n$ containing the point x.

Theorem 2.2.2. Let μ, m be probability measures on $[0, 1)$, $\mathcal{F}$ a family of sets satisfying $(H_1), (H_2)$ and (H_3). If given $B \subseteq [0, 1)$ there exists a $\delta > 0$ satisfying

$$B \subseteq \left\{x \in [0, 1) : \liminf_{n} \frac{\log \mu(I_n(x))}{\log m(I_n(x))} \geq \delta\right\},$$

then $\dim_{\mathcal{F},m}(B) \geq \delta \dim_{\mathcal{F},\mu}(B)$.

2.2.3. Mean of digits

First, consider the expansion in base b. Eggleston in [Egg49] proved that for $0 < \ell < 1$ the Hausdorff dimension ρ of

$$\left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{\varepsilon_j(x)}{b} = \ell\right\},$$

the set of $x \in [0, 1]$ such that their mean of digits is ℓ, satisfies the equation $b^\rho = (1 + r + r^2 + \ldots + r^{b-1})r^{-b\ell}$, where r is the real root between 0 and 1 of the polynomial $p(x) = (b - 1 - b\ell)x^{b-1} + (b - 2 - b\ell)x^{b-2} + \cdots + (1 - b\ell)x - b\ell$. In particular, the value $\ell = (b - 1)/2$ is the only one that produces a set of Hausdorff dimension 1, and it corresponds to the value given by calculation using Birkhoff's ergodic theorem.

When we consider the Cantor series expansion with respect to Q with the additional hypothesis $q_n \to \infty$, the change is radical. Every value of $\ell \in (0, 1)$ produces a set of dimension 1, so we obtain a continuous and disjoint family of subsets of $[0, 1]$ with full Hausdorff dimension.

The aim of this subsection is to prove the aforementioned theorem, due to Peyrière (cf. [Pey77]), which can be stated as follows.

Theorem 2.2.3. Let $Q = (q_n)_n$ be such that $q_n \xrightarrow{n} \infty$. Then, for every $\ell \in (0,1)$,

$$\dim\left(\left\{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{\varepsilon_j(x)}{q_j} = \ell\right\}\right) = 1. \tag{2.3}$$

Proof. In first place, we will see the case when $\ell \in (0, 1/2]$. Define a measure p_n on Ω_n whenever $2\ell q_n > 1$ as follows:

$$p_n(\{j\}) = \begin{cases} 1/[2\ell q_n] & \text{if } 0 \le j < [2\ell q_n], \\ 0 & \text{if } [2\ell q_n] \le j < q_n. \end{cases}$$

We can assume that $2\ell q_n > 1$ for every n since the first terms of Q do not affect the limit in (2.3). Endow Ω with the probability $p = \bigotimes_{n\ge1} p_n$ and consider the independent random variables $\varepsilon_j : \Omega \to \mathbb{R}$, $j \in \mathbb{N}$, giving the j-th term of ε.

The expected values and variances

$$\mathbb{E}\left(\frac{\varepsilon_j}{q_j}\right) = \frac{1}{q_j}\int_\Omega \varepsilon_j\, dp = \frac{1}{q_j}\sum_{k=0}^{q_j-1} k \cdot p_{q_j}(\{k\}) = \frac{[2\ell q_j]-1}{2q_j},$$
$$\mathbb{V}\left(\frac{\varepsilon_j}{q_j}\right) = \frac{1}{q_j}(\mathbb{E}(\varepsilon_j^2) - \mathbb{E}(\varepsilon_j)^2) = \frac{1}{q_j}\sum_{k=0}^{q_j-1} k^2 \cdot p_{q_j}(\{k\}) - \left(\frac{[2\ell q_j]-1}{2q_j}\right)^2 = \frac{[2\ell q_j]^2-1}{12q_j^2}.$$

are finite. Also

$$\sum_{n\ge1} \frac{\mathbb{V}(\varepsilon_n/q_n)}{n^2} = \sum_{n\ge1} \frac{[2\ell q_n]^2-1}{12q_n^2}\cdot\frac{1}{n^2} \le C\sum_{n\ge1}\frac{1}{n^2} < \infty,$$

since $([2\ell q_n]^2 - 1)/12q_n^2 \xrightarrow{n} \ell^2/3$.

By Theorem 2.2.1,

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\varepsilon_j}{q_j} - \mathbb{E}\left(\frac{\varepsilon_j}{q_j}\right) \xrightarrow{n} 0 \quad p\text{-almost surely.}$$

In order to apply theorem 2.2.2, we need to prove that

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\frac{\varepsilon_j}{q_j}\right) = \frac{1}{n}\sum_{j=1}^{n}\frac{[2\ell q_j]-1}{2q_j} \xrightarrow{n} \ell,$$

but this is a consequence of the Stolz-Cesàro theorem (c.f. [PS98]), since $([2\ell q_n]-1)/2q_n \xrightarrow{n} \ell$. We conclude that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\varepsilon_j}{q_j} \xrightarrow{n} \ell \quad p\text{-almost surely.}$$

To finish the proof, we need to prove an auxiliary result.

Lemma 2.2.1. If A is a Borel set such that $p(A)>0$, then $\dim(A)=1$.

Proof. First note that $p(\{x : \varepsilon_j > [2\ell q_j] \text{ for some } j \in \mathbb{N}\}) = 0$. Then, $\log p(I_n(x)) = -\sum_{j=1}^{n}\log[2\ell q_j]$ for every n p-almost everywhere. Thus,

$$\frac{\log p(I_n(x))}{\log|I_n(x)|} = \frac{\sum_{j=1}^{n}\log[2\ell q_j]}{-\sum_{j=1}^{n}\log q_j} \xrightarrow{n} 1 \quad p\text{-almost everywhere,}$$

in virtue of Stolz-Cesàro theorem. Therefore,

$$p\left(\left\{x : \lim_{n\to\infty}\frac{\log p(I_n(x))}{\log|I_n(x)|}=1\right\}\right)=1.$$

Set $B := \left\{x : \lim_{n\to\infty}\frac{\log p(I_n(x))}{\log|I_n(x)|}=1\right\}$. If A is a Borel set such that $p(A)>0$, then $1 \geq p(A\cap B) = p(A) > 0$. In consequence, $\dim_{\mathcal{F},p}(A\cap B)=1$. Since $A\cap B \subseteq B$, Theorem 2.2.2 gives $1=\dim_{\mathcal{F},p}(A\cap B)=\dim_{\mathcal{F}}(A\cap B)=\dim(A\cap B)$, because $\mathcal{F}$ allows the calculation of the Hausdorff dimension. Hence, $\dim(A)\geq 1$. Since $A\subseteq[0,1)$, we have that $\dim(A)\leq 1$, which concludes the proof. ■

Now, note that

$$p\left(\left\{x : \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{\varepsilon_j(x)}{q_j}=\ell\right\}\right)=1>0,$$

and by Lemma 2.2.1 its Hausdorff dimension is 1.

For the case $\ell\in(1/2,1)$, consider the measure on Ω_n defined by

$$p_n(\{j\}) = \begin{cases} 0 & \text{if } 0\leq j<[(2\ell-1)q_n], \\ \dfrac{1}{q_n-[(2\ell-1)q_n]} & \text{if } [(2\ell-1)q_n]\leq j<q_n. \end{cases}$$

and the procedure is analogous. ■

2.2.4. Dimension of sets determined by the frequency of digits

In this section we will study the Hausdorff dimension of sets of points for which every digit appears with a given frequency in the Cantor series expansion. To achieve this purpose, we need to make some assumptions.

2.2.4.1. Hypotheses on Q

Define the sets $F_n = \{1, \dots, n\}$ and $E_\nu = \{j \in \mathbb{N} : q_j = \nu\}$ for $\nu \geq 2$. The cardinality of the intersection $F_n \cap E_\nu$ is the number of digits in $\{q_1, \dots, q_n\}$ equal to ν. We will make the following assumptions.

First, we need that every digit in Q occurs with a well defined frequency:

(h_1) There exists a sequence $(d_\nu)_{\nu \geq 2} \subseteq [0, \infty)$ such that for every ν,

$$\lim_{n \to \infty} \frac{\operatorname{card}(F_n \cap E_\nu)}{n} = d_\nu.$$

And some conditions that will allow the use of limit theorems:

(h_2) There exists $c > 0$ such that for every ν,

$$\sup_{n \geq 1} \frac{\operatorname{card}(F_n \cap E_\nu)}{n} \leq c d_\nu.$$

(h_3) The series

$$\sum_{\nu \geq 2} d_\nu (\log \nu)^2$$

is convergent.

Lemma 2.2.2. Under the assumptions $(h_1), (h_2)$ and (h_3) we have

(a) $\sum_{\nu \geq 2} d_\nu = 1$,

(b) $\sum_{j \in E_\nu} \dfrac{1}{j^2} \leq 2 c d_\nu$,

(c) $\lim_{n \to \infty} \dfrac{1}{n} \log(q_1 \cdots q_n) = \sum_{\nu \geq 2} d_\nu \log \nu$.

Proof. Note that

$$\sum_{\nu=2}^{k} d_\nu = \lim_{n\to\infty} \sum_{\nu=2}^{k} \frac{\operatorname{card}(F_n \cap E_\nu)}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{card}\left(F_n \cap \bigcup_{\nu=2}^{k} E_\nu\right) \le 1.$$

Hence $\sum_{\nu\ge 2} d_\nu \le 1$. Also, $\sum_{\nu\ge 2} \dfrac{\operatorname{card}(F_n \cap E_\nu)}{n} = 1$. Then, by (h_2) we have that $\dfrac{\operatorname{card}(F_n \cap E_\nu)}{n} \le c d_\nu \in L^1_\nu(\mathbb{N})$ (the space of integrable functions $f : \mathbb{N} \to \mathbb{R}$ with ν as the variable). In virtue of the dominated convergence theorem

$$\sum_{\nu\ge 2} d_\nu = 1 = \lim_{n\to\infty} \sum_{\nu\ge 2} \frac{\operatorname{card}(F_n \cap E_\nu)}{n},$$

which proves (a).

To prove (b), consider the sequences defined by

$$g_j = \begin{cases} 0 & \text{if } j = 1, \\ \operatorname{card}(F_{j-1} \cap E_\nu) & \text{if } j \ge 2, \end{cases} \qquad \text{and} \qquad f_j = \frac{1}{j^2} \text{ for } j \ge 1.$$

Noting that $g_{j+1} - g_j = 1$ if and only if $q_j = \nu$ and it is 0 otherwise a summation by parts gives

$$\begin{aligned}
\sum_{j\in E_\nu} \frac{1}{j^2} &= \sum_{j\ge 1} f_j(g_{j+1} - g_j) = \lim_{n\to\infty} \sum_{j=1}^{n} f_j(g_{j+1} - g_j) \\
&= \lim_{n\to\infty} \left(f_n g_{n+1} - f_1 g_1 - \sum_{j=2}^{n} g_j(f_{j-1} - f_j) \right) \\
&= \lim_{n\to\infty} \left(\frac{\operatorname{card}(F_n \cap E_\nu)}{n^2} - \sum_{j=2}^{n} g_j(f_j - f_{j-1}) \right) = \lim_{n\to\infty} \sum_{j=2}^{n} g_j(f_{j-1} - f_j) \\
&\le c d_\nu \lim_{n\to\infty} \sum_{j=2}^{n} \left(\frac{1}{j-1} - \frac{j-1}{j^2} \right) = \frac{\pi^2}{6} c d_\nu \le 2 c d_\nu.
\end{aligned}$$

For (c), observe that $\sum_{\nu\ge 2} \operatorname{card}(F_n \cap E_\nu) \log \nu = \log(q_1 \dots q_n)$. Therefore, for every $\nu \ge 2$.

$$\frac{\log(q_1 \cdots q_n)}{n} = \sum_{\nu\ge 2} \frac{\operatorname{card}(F_n \cap E_\nu) \log \nu}{n}.$$

Since

$$\frac{\operatorname{card}(F_n \cap E_\nu)}{n} \log \nu \le c d_\nu \log \nu \le c d_\nu (\log \nu)^2 \in L^1_\nu(\mathbb{N}),$$

by (h_3), the dominated convergence theorem gives

$$\lim_{n\to\infty}\frac{\log(q_1\dots q_n)}{n}=\sum_{\nu\geq 2}d_\nu\log\nu.$$

■

2.2.4.2. Proof of Theorem 2.2.4.

First, we need the following result.

Lemma 2.2.3. The following holds:

(a) For every $n\in\mathbb{N}$,

$$\sup\left\{\sum_{j=1}^{n}p_j\log\frac{1}{p_j}:p_j\geq 0,\sum_{j=1}^{n}p_j=1\right\}=\log n. \tag{2.4}$$

(b) For every $\nu\geq 2$,

$$\sup\left\{\sum_{j=1}^{n}p_j\left(\log\frac{1}{p_j}-\sum_{k=1}^{n}p_k\log\frac{1}{p_k}\right)^2:p_k\geq 0,\sum_{k=1}^{n}p_k=1\right\}\leq(\log n)^2. \tag{2.5}$$

Proof. (a) is a direct consequence of Jensen's inequality. For (b), consider the random variable $Y(x)=\log(1/x)$. Thus

$$\sum_{j=1}^{n}p_j\left(\log\frac{1}{p_j}-\sum_{k=1}^{n}p_k\log\frac{1}{p_k}\right)^2=\mathbb{E}(Y-\mathbb{E}(Y))^2\leq\mathbb{E}(Y^2)=\sum_{j=1}^{n}p_j\log\left(\frac{1}{p_j}\right)^2.$$

The function $x\log^2(1/x)$ is concave, so Jensen's inequality gives $\mathbb{E}(Y^2)\leq(\log n)^2$. ■

Finally, we can state the main theorem of the subsection, proved by Peyrière in [Pey77].

Theorem 2.2.4. Let Q be a sequence satisfying the hypotheses $(h_1),(h_2)$ and (h_3). Let $\{p_{\nu,k}\}_{\nu\geq 2,0\leq k\leq\nu-1}\subseteq[0,\infty)$ be such that for every $\nu\geq 2$ we have

$$\sum_{0\leq k\leq\nu-1}p_{\nu,k}=1.$$

Consider the set

$$A := \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{card}(\{j \in F_n \cap E_\nu : \varepsilon_j = k\}) = d_\nu p_{\nu,k} \ \forall \nu \geq 2, 0 \leq k < \nu \right\}.$$

Then

$$\dim(A) = \frac{-\sum_{\nu\geq 2} d_\nu \sum_{k=0}^{\nu-1} p_{\nu,k} \log p_{\nu,k}}{\sum_{\nu\geq 2} d_\nu \log \nu}.$$

To simplify notation, set $h_\nu := -\sum_{k=0}^{\nu-1} p_{\nu,k} \log p_{\nu,k}$ and $\lambda := \dfrac{-\sum_{\nu\geq 2} d_\nu h_\nu}{\sum_{\nu\geq 2} d_\nu \log \nu}$. Thus, we want to prove $\dim(A) = \lambda$.

Notice that all the conditions imposed on the sequence Q are satisfied when $q_n = b$ for a fixed natural number $b \geq 2$. In this particular case, λ is a quotient between the entropy of $([0,1], T_b)$ and its Lyapunov exponent.

In the non-autonomous case, when Q is not constant, the concepts of entropy or Lyapunov exponent are not defined, but maybe we should expect that the numerator and the denominator of λ could be the appropriate generalization of those concepts in this more general setting.

Proof. First, we prove that $\dim(A) \geq \lambda$.

Endow Ω_{q_j} with the probability measure $\mu_j(\{k\}) = p_{q_j,k}$ and Ω with the product probability measure $\mu = \bigotimes_{j\geq 1} \mu_j$. Fix $\nu \geq 2$ and $0 \leq k \leq \nu - 1$. If we consider the independent random variables

$$X_j(\varepsilon) = \begin{cases} 1 & \text{if } q_j = \nu, \varepsilon_j = k, \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$\mathbb{E}(X_j) = \mathbb{1}_{E_\nu}(j) p_{\nu,k} < \infty, \quad \mathbb{V}(X_j) = \mathbb{1}_{E_\nu}(j)(p_{\nu,k} - p_{\nu,k}^2) < \infty, \quad \sum_{j\geq 1} \frac{\mathbb{V}(X_j)}{j^2} \leq \sum_{j\geq 1} \frac{1}{j^2} < \infty,$$

where $\mathbb{1}_X$ denotes the indicator function of the set X.

In virtue of Theorem 2.2.1,

$$\frac{1}{n} \sum_{j=1}^{n} X_j - p_{\nu,k} \mathbb{1}_{E_\nu}(j) \xrightarrow{n} 0 \quad \mu\text{-almost surely,}$$

or equivalently, by (h_1),

$$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow{n} \lim_{n\to\infty} p_{\nu,k}\frac{\mathrm{card}(F_n\cap E_\nu)}{n} = p_{\nu,k}d_\nu \quad \mu\text{-almost surely.} \tag{2.6}$$

Observe that

$$\frac{1}{n}\sum_{j=1}^{n} X_j = \frac{\mathrm{card}(\{j\in F_n\cap E_\nu : \varepsilon_j = k\})}{n},$$

so we have proved that $\mu(A)=1$ by (2.6).

Now, define the independent random variables $Y_j = -\log p_{q_j,\varepsilon_j}$. We have that

$$\mathbb{E}(Y_j) = h_{q_j} \le \log q_j < \infty, \quad \mathbb{V}(Y_j) \le K(\log q_j)^2 < \infty$$

by Lemma 2.2.3. In virtue of Lemma 2.2.2 and (2.5),

$$\sum_{j\ge 1}\frac{\mathbb{V}(Y_j)}{j^2} \le K\sum_{j\ge 1}\frac{(\log q_j)^2}{j^2} = K\sum_{\nu\ge 2}\sum_{j\ge 1}\mathbb{1}_{E_\nu}(j)\frac{(\log\nu)^2}{j^2} \le 2Kc\sum_{\nu\ge 2} d_\nu(\log\nu)^2 < \infty,$$

where the finiteness follows from (h_3). Again, by Theorem 2.2.1

$$\frac{1}{n}\sum_{j=1}^{n} Y_j - \mathbb{E}(Y_j) \xrightarrow{n} 0 \quad \mu\text{-almost surely.}$$

Since $\lim_{n\to\infty}\dfrac{\log(q_1\cdots q_n)}{n}$ exists by Lemma 2.2.2,

$$\frac{\sum_{j=1}^{n} Y_j - \mathbb{E}(Y_j)}{\log(q_1\cdots q_n)} \xrightarrow{n} 0 \quad \mu\text{-almost surely.}$$

If we rewrite this limit we obtain

$$\frac{\log\mu(I_n(x))}{\log|I_n(x)|} - \frac{h_{q_1}+\cdots+h_{q_n}}{\log(q_1\cdots q_n)} \xrightarrow{n} 0 \quad \mu\text{-almost surely.}$$

Since

$$\frac{1}{n}\sum_{j=1}^{n} h_{q_j} = \sum_{\nu\ge 2}\frac{\mathrm{card}(F_n\cap E_\nu)h_\nu}{n}$$

and, by (h_2), each term of this series is bounded by $cd_\nu h_\nu \leq cd\nu \log \nu \in L^1_\nu(\mathbb{N})$ by formula (2.4) and (h_3), the dominated convergence theorem gives

$$\frac{\sum_{j=1}^n h_{q_j}}{\log(q_1 \cdots q_n)} = \frac{\frac{1}{n}\sum_{j=1}^n h_{q_j}}{\frac{1}{n}\log(q_1 \cdots q_n)} \xrightarrow{n\to\infty} \frac{\sum_{\nu\geq 2} d_\nu h_\nu}{\sum_{\nu\geq 2} d_\nu \log \nu} = \lambda. \tag{2.7}$$

Therefore

$$\frac{\log \mu(I_n(x))}{\log |I_n(x)|} \xrightarrow{n\to\infty} \lambda \quad \mu\text{-almost surely.}$$

Define

$$B = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \geq \lambda \right\}.$$

The limit in (2.7) implies that $\mu(B) = 1$, thus $\mu(A \cap B) = 1$, and consequently $\dim_{\mathcal{F},\mu}(A \cap B) = 1$. Since $A \cap B \subseteq B$, by Theorem 2.2.2 and the fact that $\mathcal{F}$ allows the calculation of the Hausdorff dimension,

$$\dim(A \cap B) = \dim_{\mathcal{F}}(A \cap B) \geq \lambda \dim_{\mathcal{F},\mu}(A \cap B) = \lambda.$$

It follows that $\dim(A) \geq \lambda$ because $A \supseteq A \cap B$.

It remains to show that $\dim(A) \leq \lambda$.

For every $N \geq 1$ define the sets

$$A_N := \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{(\mathrm{card}\{j \in F_n \cap E_\nu : \varepsilon_j = k\})}{n} = d_\nu p_{\nu,k}, \right.$$
$$\left. \nu = 1, \ldots, N \text{ and } 0 \leq k \leq \nu - 1 \right\}.$$

Clearly $A \subseteq A_N$ for every N. We will prove that $\dim(A_N) \xrightarrow{N\to\infty} \lambda$.

Define the measure μ_j^N on Ω_{q_j} as follows:

$$\mu_j^N(\{k\}) = \begin{cases} p_{q_j,k} & \text{if } q_j \leq N, \\ \dfrac{1}{q_j} & \text{if } q_j > N, \end{cases}$$

and endow Ω with the product measure $\mu^N = \bigotimes_{j\geq 1} \mu_j^N$. We can see that $\dim_{\mathcal{F},\mu^N}(A_N) = 1$ just like we did to prove $\dim_{\mathcal{F},\mu}(A) = 1$.

Let $x \in A_N$ and set $E_{\nu,k} := \{j \in E_\nu : \varepsilon_j = k\}$. By definition, $\lim_{n\to\infty} \dfrac{\operatorname{card}(F_n \cap E_{\nu,k})}{n} = p_{\nu,k}$ whenever $\nu \leq N$. Thus,

$$-\frac{1}{n}\log \mu_N(I_n(X)) = -\sum_{\nu=2}^{N}\sum_{k=0}^{\nu-1} \frac{\operatorname{card}(F_n \cap E_{\nu,k})\log p_{\nu,k}}{n} + \sum_{\nu>N} \frac{\operatorname{card}(F_n \cap E_\nu)\log \nu}{n},$$

and taking limit when $n \longrightarrow \infty$,

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu_N(I_n(x)) = \sum_{\nu=2}^{N} p_{\nu,k}\log p_{\nu,k} + \sum_{\nu>N} d_\nu \log \nu = \sum_{\nu=2}^{N} d_\nu h_\nu + \sum_{\nu>N} d_\nu \log \nu.$$

Then,

$$\lim_{n\to\infty} \frac{-\frac{1}{n}\log \mu_N(I_n(x))}{\frac{1}{n}\log |I_n(x)|} = \frac{\sum_{\nu=2}^{N} d_\nu h_\nu + \sum_{\nu>N} d_\nu \log \nu}{\sum_{\nu\geq 2} d_\nu \log \nu} =: C(N) \xrightarrow{N\to\infty} \lambda$$

since $\sum_{\nu\geq 2} d_\nu \log \nu$ converges by (h_3). Hence, in virtue of Theorem 2.2.2 and $\mathcal{F}$ allowing the calculation of the Hausdorff dimension,

$$1 = \dim_{\mathcal{F},\mu^N}(A_N) = C(N)^{-1}\dim_{\mathcal{F}}(A_N) = C(N)^{-1}\dim(A_N).$$

Therefore, $\dim(A_N) = C(N)$ for every N, and since $A \subset A_N$, $\dim(A) \leq \dim(A_N) \xrightarrow{N} \lambda$. This concludes the proof. ■

Chapter 3

Normal numbers with respect to the Cantor series expansion

In this chapter, we will review the definition of a normal number for two different representations of real numbers: base b and continued fractions expansions. An essential tool to prove that almost every $x \in (0,1)$ in each setting is normal will be Birkhoff's ergodic theorem. In order to mimic the definitions for the base b expansion, we will extend the normality notion for the Cantor series representations in three different and non-equivalent ways. We shall focus in just one of them and go on to prove that, under certain assumptions on the sequence which determines the series, Lebesgue almost every x is normal. This is an analogous result to Borel's theorem for normal numbers.

3.1. Normality of classic expansions

3.1.1. Base b expansion

Definition 3.1.1. Let $b \geq 2$ be an integer. For every $x \in [0,1)$ we define the *b-ary expansion of x* as

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{b^n} = [\varepsilon_1(x), \varepsilon_2(x), \ldots]_b,$$

where $\varepsilon_n(x) \in \{0, 1, \ldots, b-1\}$.

This expansion is essentially unique. Every $x \in [0,1)$ can be written uniquely as an infinite series of this type and the numbers which also have a finite expansion are the rationals, which have zero Lebesgue measure.

In this setting, there are three well known and equivalent definitions of normality that we shall discuss, and all of them are important because each one will give rise to a possible definition of normality for a Cantor series expansion. In order to define these generalizations, we need to review some concepts first.

Definition 3.1.2. Given an integer $k \geq 1$, *a block B of length k* is an ordered k-tuple $B = (b_1, b_2, \ldots, b_k) \in \{0, 1, \ldots, b-1\}^k$.

We say that *a block B of length k occurs at x with starting position n* in the $b-$ary expansion of $x \in [0,1)$ if the equality $(\varepsilon_n(x), \varepsilon_{n+1}(x), \ldots, \varepsilon_{n+k-1}(x)) = B$ holds.

Denote by $N_n^b(B, x)$ the number of times that B occurs at x with starting position no greater than n. In other words,

$$N_n^b(B, x) = \operatorname{card}(\{1 \leq j \leq n : (\varepsilon_j(x), \varepsilon_{j+1}(x), \ldots, \varepsilon_{j+k-1}(x)) = B\}).$$

Now we can state the first definition of normality for the base b expansion.

Definition 3.1.3. Let $x \in [0,1)$, and $b \geq 2$ be a integer. We say that *x is normal in base b* if

$$\lim_{n\to\infty} \frac{N_n^b(B, x)}{n} = b^{-k} \tag{3.1}$$

for every $k \geq 1$ and every block B of length k.

Roughly speaking, a number x is normal if every block of a given length occurs at x with the same frequency.

The second way to define a normal number for this expansion is related directly with the distribution of the orbit of x under an appropriate mapping.

Definition 3.1.4. Given a sequence $X = (x_n)_n \subset [0,1]$ and a subinterval $J \subset [0,1]$ we define $A_n(X, J)$ as the number of times X enters in J, namely,

$$A_n(X, J) = \operatorname{card}(\{1 \leq j \leq n : x_j \in J\}).$$

We say that X *is uniformly distributed modulo 1* (u.d. mod 1) if for every pair of real numbers a, b such that $0 \leq a < b \leq 1$,

$$\lim_{n\to\infty} \frac{A_n(X,[a,b])}{n} = b - a. \tag{3.2}$$

The map $T_b : x \mapsto bx \mod 1$ is closely related to the base b expansion because it acts as a shift on the digits, namely, if $x = [\varepsilon_1, \ldots, \varepsilon_n, \ldots]_b$, then $bx \mod 1 = [\varepsilon_2, \ldots, \varepsilon_n, \ldots]_b$. So, if we consider the sequence $(T_b^n x)_{n\geq 0}$, we can define a distribution of digits of x.

Definition 3.1.5. Let $x \in [0,1)$ be given. We say that x *is normal in base* b if the sequence $(T_b^n x)_{n\geq 0}$ is u.d. mod 1.

It is easy to see that this definition implies (3.1). Note that a given block $B = (b_1, b_2, \ldots, b_k)$ occurs in x at position n if and only if

$$T_b^{n-1} x \in I_B := \left[\frac{b_1}{b} + \frac{b_2}{b^2} + \cdots + \frac{b_k}{b^k}, \frac{b_1}{b} + \frac{b_2}{b^2} + \cdots + \frac{b_k+1}{b^k}\right). \tag{3.3}$$

If $(T_b^n x)_{n\geq 0}$ is u.d. mod 1, then

$$\frac{N_n^b(B,x)}{n} = \frac{A_n((T_b^j)_j, I_B)}{n} \xrightarrow{n\to\infty} \lambda(I_B) = b^{-k},$$

where λ denotes the Lebesgue measure. Since B and k were arbitrary, the result follows. For the converse, see [KN74].

Finally, we give a third definition of a normal number in this setting.

Definition 3.1.6. Let $x \in [0,1)$. We say that x *is normal in base* b if for every two blocks B_1, B_2 of the same length,

$$\lim_{n\to\infty} \frac{N_n^b(B_1,x)}{N_n^b(B_2,x)} = 1.$$

Now we provide a proof of the equivalence of Definition 3.1.6 and Definition 3.1.3. Let $x \in [0,1)$ be a normal number as in Definiton 3.1.3. Let B_1, B_2 be two blocks of some length k. Then,

$$\lim_{n\to\infty} \frac{N_n^b(B_1,x)}{N_n^b(B_2,x)} = \lim_{n\to\infty} \frac{N_n^b(B_1,x)/n}{N_n^b(B_2,x)/n} = 1.$$

For the other direction, fix k and note that there exist b^k different blocks of this length, $B_1, \ldots, B_{b^k}$. Let B be of length k. For every $1 \leq i \leq b^k$,

$$\lim_{n\to\infty} \frac{N_n^b(B_i, x)}{N_n^b(B, x)} = 1,$$

and therefore,

$$\lim_{n\to\infty} \frac{n}{N_n^b(B,x)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{b^k} N_n^b(B_i, x)}{N_n^b(B,x)} = \sum_{i=1}^{b^k} \lim_{n\to\infty} \frac{N_n^b(B_i,x)}{N_n^b(B,x)} = b^k.$$

Given that we already know the equivalence of these three definitions of normality, we are able to prove the Borel classical result.

Theorem 3.1.1. Lebesgue almost every $x \in [0, 1)$ is normal in base b.

Proof. Consider the map $T_b : [0, 1) \to [0, 1)$, $x \mapsto bx \mod 1$. It is a well known result that $([0, 1), \mathcal{B}, \lambda, T_b)$ is an ergodic dynamical system (cf. [Has17]). This fact allows us to invoke Birkhoff's ergodic theorem.

Theorem 3.1.2. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system. If $f \in L^1_\mu$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \int_X f \, d\mu$$

for μ-almost every $x \in X$.

Fix $k \geq 1$. Let $B = (b_1, \ldots, b_k)$ be a block of length k, and consider I_B as defined in (3.3). Let $f = \mathbb{1}_{I_B} \in \mathscr{L}^1_\mu$ be the characteristic function of I_B. Hence, for almost every $x \in [0, 1)$,

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T_b^j(x) = \int_{[0,1)} f \, d\lambda = \lambda(I_B) = b^{-k}.$$

Since k and B are arbitrary, and there are only finitely many choices of B and countable many choices of k, the result follows. ∎

The original proof of this theorem relies heavily in results of probability theory and it is more difficult than the one presented here. The ergodic theorem turns out to be a powerful method to conclude the same result.

3.1.2. Continued fraction expansion

In the same spirit of Definition 3.1.5 for the base b expansion, we can define normal numbers with respect to the continued fractions expansion. Our interest in this case relies in the fact that the possible digits appearing in the expansion are infinite, just like in the case of Cantor series expansions.

Definition 3.1.7. Let $x \in (0,1)$. A *continued fraction expansion of x* is a expansion of the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \tag{3.4}$$

where $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$.

It is a well-known result that every $x \in (0,1)$ can be written in the form (3.4) (c.f. [EW11]). We will denote

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}} = [a_1(x), a_2(x), a_3(x), \ldots].$$

This representation of x is known as *the continued fraction expansion of x*.

If we consider the Gauss map $T : (0,1] \to (0,1]$ defined by $x \mapsto \{1/x\}$, where $\{\cdot\}$ denotes the fractional part, that mapping is closely related to the continued fraction expansion. As an example of this connection, T acts as a shift in the digits of the continued fraction expansion, namely, if $x = [a_1, a_2, a_3, \ldots]$, then $Tx = [a_2, a_3, \ldots]$.

The measure-preserving system $((0,1], \mathcal{B}, \mu, T)$, where

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x}\, d\lambda \text{ whenever } A \in \mathcal{B},$$

is also ergodic (cf. [EW11]).

Given this probability measure μ (or any other probability measure ν on $(0,1]$), we can make the following definition in the spirit of the Lebesgue case of (3.2).

Definition 3.1.8. Let $(x_n)_n \subset \mathbb{R}$ be a sequence. We say that $(x_n)_n$ *is uniformly distributed modulo 1 with respect to* ν if for every $a, b \in \mathbb{R}$ such that $0 \leq a < b \leq 1$,

$$\lim_{n\to\infty} \frac{A_n((x_j)_j, (a,b])}{n} = \nu((a,b]).$$

Because of the shift behavior of T with respect to the continued fraction expansion, an appropriate definition of a normal number in this setting should be the following.

Definition 3.1.9. Let $x \in (0,1]$. We say that x *is normal with respect to the continued fraction expansion* if the sequence $(T^n x)_n$ is uniformly distributed modulo 1 with respect to μ.

Normality of Lebesgue almost every x then follows as a direct application of Birkhoff's ergodic theorem and the equivalence of μ with the Lebesgue measure, as we will see now.

Theorem 3.1.3. Lebesgue almost every $x \in (0,1]$ is normal with respect to the continued fraction expansion.

Proof. Let $(a,b] \subset (0,1]$ be a non-empty interval and $x \in (0,1]$. We can write

$$A_n((T^j x)_j, (a,b]) = \operatorname{card}(\{1 \leq j \leq n : T^{j-1}x \in (a,b]\}) = \sum_{j=0}^{n-1} \mathbb{1}_{(a,b]} \circ T^j(x),$$

hence

$$\lim_{n\to\infty} \frac{A_n((x_j)_j, (a,b])}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(a,b]} \circ T^j(x) = \int_{(0,1]} \mathbb{1}_{(a,b]}(x)\, d\mu = \mu((a,b])$$

for μ-almost every $x \in (0,1]$ because $\mathbb{1}_{(a,b]}$ is integrable. Since $\mu \ll \lambda$, the conclusion is also valid for λ-almost every $x \in (0,1]$. ■

3.1.3. Expected properties of normal numbers with respect to Cantor series expansions

One question appears naturally during the development of the theory of normal numbers for b-ary expansions.

- Is it possible to define a normal number in the Cantor series setting? Is this definition a generalization of the b-ary notion?

We will see in the next section that the answers to these questions are affirmative, and in three different ways. Each one extends one of the previous equivalent definitions of a normal number.

Comparing the cases of different series expansions, such as the previous analyzed and some others (cf. [Man10]), we can find the following similarities between them:

- The set of normal numbers has full Lebesgue measure.
- A condition on the frequency of occurrence of blocks is equivalent to a condition on the distribution of the orbit of a map that acts as a shift in the expansion.

The first question has a partial answer, restricted to some conditions on the sequence which defines the Cantor series expansion. The second question is not affirmative in general, as we will see later.

3.2. Definitions of a normal number

In this section we will provide three different definition of a normal number, and a stronger version of each one will be presented. The non-equivalence of them will be highlighted with some examples at the end of the section.

Henceforth, the sequence $Q = (q_n)_{n\geq 1}$ always satisfies $q_n \geq 2$ for all $n \in \mathbb{N}$.

3.2.1. Q-normal numbers

The definition we will develop now is the analogous of Definition 3.1.3. It consists in the frequency of appearance of blocks of digits in this new setting.

Definition 3.2.1. Let $k \in \mathbb{N}$. A *block B of length k* is a k-tuple $B = (b_1, \ldots, b_k) \in (\mathbb{N} \cup \{0\})^k$.

Note that we are changing the set of possible entries for a block because the digits in the Cantor series expansion determined by an arbitrary Q can be unbounded, contrary to what happens in the case of base b expansions.

Definition 3.2.2. Let $x \in [0, 1)$ and B a block of length k. We say that B *occurs in x at position j* if $(\varepsilon_j(x), \ldots, \varepsilon_{j+k-1}(x)) = B$. The number of times B occurs in x at position no greater than n is defined by

$$N_n^Q(B.x) := \mathrm{card}(\{1 \leq j \leq n : (\varepsilon_j(x), \ldots, \varepsilon_{j+k-1}(x)) = B\}).$$

We will denote this quantity also as $N_n(B, x)$ when the sequence Q is fixed.

Note that the quotient in (3.1) can be written as

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{nb^{-k}} = 1.$$

The denominator in the quotient represents the sum of the frequencies that should have a block of length k if its starting position goes from 1 to n. Since each frequency should be b^{-k}, the sum gives nb^{-k}. In the Cantor series expansion determined by Q, the number of blocks of length k starting at position j depends on Q, so we need to change the denominator by an analogous quantity.

Definition 3.2.3. Let Q be a sequence and $k \in \mathbb{N}$. Q is said *k-divergent* if

$$Q_n^k := \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}$$

satisfies $\lim_{n\to\infty} Q_n^k = +\infty$. If this condition holds for every k, we say that Q is *fully divergent*. If Q is not k-divergent, we say that Q is k-convergent.

The number Q_n^k is the substitute of nb^{-k} because the number of blocks of length k starting at position j is $q_j q_{j+1} \cdots , q_{j+k-1}$ given that $\varepsilon_i \in \{0, 1, \ldots, q_i - 1\}$.

We are finally able to define a normal number in this context.

Definition 3.2.4. Let Q be a sequence and $k \in \mathbb{N}$. A number $x \in [0, 1)$ is said *Q-normal of order k* if for every block B of length k,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^k} = 1$$

If this is holds for every k, we say that x is *Q-normal*.

If we restrict the possible entries for a block and consider $q_n = b$ for all $n \in \mathbb{N}$, we recover the definition of a normal number in base b.

3.2.2. Q-distribution normal numbers

In this section we will extend the definition of a normal number following the guidelines of Definition 3.1.5. In the b-ary expansion case, the normality of $x \in [0,1)$ is stated in terms of the distribution of the orbit of x under a map that acts as a shift in that expansion.

In the Cantor series expansion, there exists not one but a sequence of functions shifting the digits depending on Q and on how many digits we want to shift. The sequence of maps $\{G_n(x) : x \mapsto q_n q_{n-1} \cdots q_1 x \mod 1\}_n$ is such that, given $x \in [0,1)$ with expansion $x = [\varepsilon_1(x), \varepsilon_2(x), \ldots]_Q$, it happens that $G_n(x) = [\varepsilon_{n+1}(x), \varepsilon_{n+2}, \ldots]_Q$. Then, the following definition generalizes Definition 3.1.5.

Definition 3.2.5. Let $Q = (q_n)_n$ be a sequence. A number $x \in [0,1)$ is Q-*distribution normal* if the sequence $(G_n(x))_{n\geq 0}$ is u.d. mod 1.

3.2.3. Q-ratio normal numbers

The notion of normality given by Definition 3.1.6 can be easily extended to this general setting as follows.

Definition 3.2.6. Let Q be a sequence. We say that x is Q-*ratio normal* if given two arbitrary blocks B_1, B_2 of the same length, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B_1, x)}{N_n^Q(B_2, x)} = 1.$$

It is clear that Q-normal implies Q-ratio normal.

3.2.4. Stronger definitions of normality

Definitions 3.2.4, 3.2.5 and 3.2.6 can be stated in stronger versions, as it is done in [Man10]. In order to prove the main result of this chapter, we will require the following stronger version of Definition 3.2.4.

Definition 3.2.7. Let Q be a sequence, $x \in [0,1)$, B a block of length k, and integers $n \in \mathbb{N}, p \in [1,k]$. We define $N_{n,p}^Q(B,x)$ as the number of times that B occurs in x with starting position of the form $jk + p$, $0 \leq j < n/k$. That is,

$$N^Q_{n,p}(B, x) = \operatorname{card}(\{0 \leq j < n/k : (\varepsilon_{jk+p}, \ldots, \varepsilon_{jk+p+k-1}) = B\}).$$

The theorem that we will prove relies heavily in certain estimates of $N^Q_n(B, x)$, which are proved using appropriate random variables. The main problem with $N_n(B, x)$ is that it counts non-necessarily disjointed positions in which the block B occurs. If we consider disjoint position, we can obtain independence of these random variables. Independence is a crucial hypothesis in the law of the iterated logarithms, the theorem that will be used to prove the main estimate. Hence, introducing p in the definition of $N^Q_{n,p}(B, x)$ solves the non-disjointedness problem.

In a similar spirit of the discussion of how to generalize the quotient for the definition of a Q-normal number in Definition 3.2.3, we need to replace the denominator to make it count the total number of blocks considered in $N_{n,p}(B, x)$.

Definition 3.2.8. Let Q be a sequence. We define

$$Q^k_{n,p} := \sum_{j=0}^{\rho(n,k)} \frac{1}{q_{jk+p} q_{jk+p+1} \cdots q_{jk+p+k-1}},$$

where $\rho(n, k) = \max\{i \in \mathbb{Z} : i < n/k\}$ is the maximum possible value of j considered in Definition 3.2.7. If $\lim_{n\to\infty} Q^k_{n,p} = +\infty$ for every p, we say that Q is *strongly k-divergent*. If that holds for every $k \in \mathbb{N}$, we say that Q is *strongly fully divergent*.

We need to define the strong version of k-divergence because it can happen that the limit $\lim_{n\to\infty} Q^k_{n,p} < +\infty$ with Q k-divergent, as it can be seen in [Man11]. We are interested in divergence of this quantity because we want that a block occurs infinitely many times in the expansion of a normal number. So, if the numerator in the definition of Q-normal is unbounded, then the denominator should be too.

The definition of strong Q-normality can be stated as follows.

Definition 3.2.9. Let Q be a sequence, $x \in [0, 1)$ and $k \in \mathbb{N}$. We say that x is *strongly Q-normal of order k* if for every block B of length k and every integer $p \in [1, k]$ it is true that

$$\lim_{n\to\infty} \frac{N^Q_{n,p}(B, x)}{Q^k_{n,p}} = 1.$$

If x is strongly Q-normal of order k for every $k \in \mathbb{N}$, we say that x is *strongly Q-normal*.

Now we can enunciate the main result that will be proved in the last section of this chapter.

Theorem 3.2.1. Let Q be a sequence such that $q_n \xrightarrow{n} \infty$. Then Lebesgue almost every $x \in [0, 1)$ is Q-normal if and only if Q is fully divergent.

It is also possible to define stronger versions of Q-distribution normality and Q-ratio normality as follows.

Definition 3.2.10. Let Q be a sequence. We say that $x \in [0, 1)$ is *strongly Q-distribution normal* if for every $k \in \mathbb{N}$ and integer $p \in [1, k]$ the sequence

$$(q_1 \cdots q_{nk+p}\, x \mod 1)_{n \geq 0}$$

is u.d. mod 1.

This definition is stronger than Q-distribution normality and it has its analogous result to Theorem 3.2.1 and can be found in [Man10].

Theorem 3.2.2. Let Q be a sequence. Then Lebesgue almost every $x \in [0, 1)$ is strongly Q-distribution normal.

Finally, we can define a strong Q-ratio normal number.

Definition 3.2.11. Let Q be a sequence. We say that $x \in [0, 1)$ is *strongly Q-ratio normal of order k* if for every two blocks B_1, B_2 of length k and every integer $p \in [1, k]$

$$\lim_{n \to \infty} \frac{N^Q_{n,p}(B_1, x)}{N^Q_{n,p}(B_2, x)} = 1.$$

If x is strongly Q-ratio normal of order k for every $k \in \mathbb{N}$, we say that x is *strongly Q-ratio normal.*

In this case, there is no analogous result to Theorem 3.2.1. A weaker result can be found in [Man11]. This result states that if $q_n \xrightarrow{n} \infty$, then there exists a Q-ratio normal number. A more interesting result (see [Man11]), of topological nature, states that if $q_n \xrightarrow{n} \infty$, then the set of Q-ratio normal numbers is of the first category and dense in $[0, 1)$.

3.2.5. Non-equivalence of definitions

In the first section of this chapter we remarked that the three notions of a normal number in base b were equivalent. Interestingly, in the Cantor series expansion this is, in general, not true. To demonstrate this fact, we will mention some illustrative examples.

Example 3.2.1. Consider $Q = (2, 3, 3, 4, 4, 4, 5, 5, 5, 5, \ldots)$ and

$$x = [1, 1, 2, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots]_Q.$$

Since $(\varepsilon_1(x)/q_1, \ldots, \varepsilon_n(x)/q_n, \ldots)$ is uniformly distributed modulo 1 (cf. [Mur08]) and this condition is equivalent to Q-distribution normality whenever $q_n \xrightarrow{n} \infty$ (cf. [Man10]), x is Q-distribution normal.

Note that the block (0), of length 1, never occurs in x. In conclusion, x is not Q-normal of order 1 and therefore not Q-normal.

Example 3.2.2. Consider $Q = (2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots)$ and $x = [0, 1, 0, 1, 2, 0, 1, 2, 3, \ldots]_Q$. Then x is Q-distribution normal and Q-normal of order 1 (cf. [Man10]).

Example 3.2.3. Consider $Q = (4, 4, 6, 6, 6, 8, 8, 8, \ldots)$ and $x = [0, 1, 0, 1, 2, 0, 1, 2, 3, \ldots]_Q$. If we set $Q' = (2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots)$ and $y = [0, 1, 0, 1, 2, 0, 1, 2, 3, \ldots]_{Q'}$, the previous example gives

$$\lim_{n\to\infty} \frac{N_n^{Q'}(B, y)}{\sum_{k=1}^{n} (q'_k)^{-1}} = 1$$

for every block B of length 1. Since $N_n^{Q'}(B, y) = N_n^{Q}(B, x)$ and $q_n = 2q'_n$, we get

$$\lim_{n\to\infty} \frac{N_n^{Q}(B, x)}{\sum_{k=1}^{n} (q_k)^{-1}} = 2,$$

and x is not Q-normal of order 1. Also, if we consider $(\varepsilon_1(x)/q_1, \ldots, \varepsilon_n(x)/q_n, \ldots) = (0, 1/4, 0, 1/6, 2/6, \ldots)$, this sequence is not dense in $[0, 1)$ since its terms are always less than $1/2$. We conclude that x is not Q-distribution normal.

However, since $N_n^{Q'}(B, x) = N_n^{Q'}(B, y)$ for every block B of arbitrary length, the Q-ratio normality of x follows as an immediate consequence of the Q'-normality of y.

An example of a sequence Q and x that is Q-normal but not Q-distribution normal is constructed in [Man10]. Some techniques for the construction of Q-normal, Q-distribution normal and Q-ratio normal numbers can also be found in this work.

3.3. Measure of Q-normal numbers

The aim of this section is to prove the following result.

Theorem 3.3.1. Let Q be a sequence such that $q_n \xrightarrow{n} \infty$. Then almost every $x \in [0,1)$ is Q-normal if and only if Q is fully divergent.

First of all, we start discussing the condition $q_n \xrightarrow{n} \infty$. It is related to the admissibility of a block B, namely, when it can occur infinitely many times. Recall that $B \in \mathbb{N}_0^k$ for some k, and their entries can be as large as we want. If b is the maximum value of its entries, the condition $q_n \to \infty$ guarantees that B can occur at position n for every n large enough. For a more concrete example of this condition, see [Rén55]. However, this condition can be relaxed to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{q_j} = 0,$$

which is equivalent to $q_n \xrightarrow{n} \infty$ for n in a subset of $\mathbb{N}$ whose complement has density 0 (cf. [Man10]).

As said in the Introduction, the dynamical system associated to a general Cantor series expansion is non-autonomous. Hence, Birkhoff's ergodic theorem (Theorem 3.1.2) cannot be used in this setting. The main tool that we will use to obtain Theorem 3.3.1 is the law of iterated logarithms, a classical probabilistic result. Then, it is necessary to introduce some concepts of probability theory.

Recall that for a random variable X in $(X, \mathcal{B}, \mu)$, $\mathbb{E}(X) = \int_X X \, d\mu$ denotes the expected value of X. Also, $\mathbb{V}(X)$ denotes the variance of X, namely, $\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. We will also use $\mathbb{P}(X = j)$ to indicate the probability of $X = j$.

Henceforth, Q is a fixed sequence. Every super index denoting the dependence on Q will be suppressed to soften the notation.

If we write $x = [\varepsilon_1(x), \varepsilon_2(x), \ldots]$, then ε_n is a random variable in the probability space $([0,1), \mathcal{B}, \mathbb{P})$ for all n, where $\mathbb{P}$ will denote the Lebesgue measure on $[0,1)$.

It is clear that for $j \in \mathbb{N}_0$

$$\mathbb{P}(\varepsilon_n(x) = j) = \begin{cases} \dfrac{1}{q_n} & 0 \leq j \leq q_n - 1, \\ 0 & j \geq q_n. \end{cases}$$

Our first claim concerns the independence of these random variables.

Lemma 3.3.1. The random variables $\varepsilon_1(x), \varepsilon_2(x), \ldots$ are independent.

Proof. Let $n \in \mathbb{N}$ and $0 \leq \eta_j \leq q_j - 1$ for $j \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned}
&\mathbb{P}(\varepsilon_1(x) = \eta_1, \ldots, \varepsilon_n(x) = \eta_n) = \\
&= \lambda\left(\left\{x \in [0,1) : x \in \left[\frac{\eta_1}{q_1} + \frac{\eta_2}{q_1 q_2} + \cdots + \frac{\eta_n}{q_1 \cdots q_n}, \frac{\eta_1}{q_1} + \frac{\eta_2}{q_1 q_2} + \cdots + \frac{\eta_n + 1}{q_1 \cdots q_n}\right)\right\}\right) = \\
&= \frac{1}{q_1 \cdots q_n} = \frac{1}{q_1} \cdots \frac{1}{q_n} = \prod_{j=1}^{n} \mathbb{P}(\varepsilon_j(x) = \eta_j).
\end{aligned}$$

Since n is arbitrary, the independence follows. ■

Definition 3.3.1. Let $b \in \mathbb{N}_0$ and $n \in \mathbb{N}$. We define the random variable

$$r_{b,n}(x) := \begin{cases} 1 & \varepsilon_n(x) = b, \\ 0 & \varepsilon_n(x) \neq b. \end{cases}$$

This is a random variable because $r_{b,n}(x) = \mathbb{1}_{\{\varepsilon_n = b\}}$. As a direct consequence of Lemma 3.3.1, we obtain the following.

Lemma 3.3.2. Let $b \in \mathbb{N}_0$. The random variables $r_{b,1}, r_{b,2}, \ldots, r_{b,n}, \ldots$ are independent.

Now we are able to define the occurrence of a block B in terms of random variables.

Definition 3.3.2. Let $k \in \mathbb{N}$ and B be a block of length k. For $i \in \mathbb{N}_0$ and $p \in \mathbb{N} \cap [1, k]$, we define $\varepsilon_{ik+p,k}$ as the block of k digits of x starting at position $ik + p$, that is, $\varepsilon_{ik+p,k}(x) = (\varepsilon_{ik+p}(x), \ldots, \varepsilon_{ik+p+k-1}(x))$. We also define

$$r_{B,i,p}(x) = \begin{cases} 1 & \varepsilon_{ik+p,k}(x) = B, \\ 0 & \varepsilon_{ik+p,k}(x) \neq B. \end{cases}$$

Note that if $B = (b_1, \ldots, b_k)$, then $r_{B,i,p} = r_{b_1,ik+p} r_{b_2,i+p+1} \cdots r_{b_k,ik+p+k-1}$. As an immediate consequence of the disjointedness of the positions considered in Definition 3.3.2 for k, p fixed and $i \in \mathbb{N}_0$ and Lemma 3.3.2, we obtain the following result.

Corollary 3.3.1. Let B be a block of length k and an integer $p \in [1, k]$. Then $r_{B,0,p}, r_{B,1,p}, \ldots$ are independent random variables.

A consequence of this corollary is the following lemma.

Lemma 3.3.3. Let $k \in \mathbb{N}$. If B is a block of length k, then

$$\mathbb{E}(r_{B,i,p}(x)) = \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}},$$
$$\mathbb{V}(r_{B,i,p}(x)) = \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2.$$

Proof. If $B = (b_1, \ldots, b_k)$, then by the independence

$$\mathbb{E}(r_{B,i,p}) = \mathbb{E}(r_{b_1,ik+p}\cdots r_{b_2,ik+p+k-1}) = \prod_{j=1}^{k}\mathbb{E}(r_{b_j,ik+p+j-1}) = \prod_{j=0}^{k-1}\frac{1}{q_{ik+p+j}}.$$

Now, for the variance, note that

$$\mathbb{V}(r_{B,i,p}) = \mathbb{E}(r_{B,i,p}^2) - \mathbb{E}(r_{B,i,p})^2 = \mathbb{E}(r_{B,i,p}) - \mathbb{E}(r_{B,i,p})^2 = \prod_{j=0}^{k-1}\frac{1}{q_{ik+p+j}} - \left(\prod_{j=0}^{k-1}\frac{1}{q_{ik+p+j}}\right)^2.$$

■

Since we have calculated all the probabilistic quantities that we will require in the demonstration, we can state our main tool. This is the law of iterated logarithms.

Theorem 3.3.2. Let $\xi_1,\ \xi_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}(\xi_k) = 0, \mathbb{E}(\xi_k^2) = \sigma_k^2$. Assume that $\lim_{n\to\infty} s_n = \infty$ and

$$\mathbb{P}\left(|\xi_n| \leq \frac{s_n}{(\log\log s_n)^{3/2}}\right) = 1,$$

where $s_n^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. Then

$$\mathbb{P}\left(\varlimsup_{n\to\infty}\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\sqrt{2s_n^2\log\log s_n^2}} = 1\right) = 1.$$

This theorem can be found in [Rév68]. It can be rewritten as

$$\xi_1 + \xi_2 + \cdots + \xi_n = O(\sqrt{s_n^2\log\log s_n^2}) \tag{3.5}$$

for almost every x.

Definition 3.3.3. Let $k \in \mathbb{N}, p \in \mathbb{N} \cap [1, k]$ and B a block of length k. For $i \in \mathbb{N}_0$ we define

$$F_i^k = \mathbb{E}(r_{B,i,p}(x)), \quad V_i^k = \mathbb{V}(r_{B,i,p}(x)), \quad t_{n,p}^k = \sum_{i=0}^{\rho(n,k)} V_i^k.$$

Recall that $\rho(n, k) = \max\{i \in \mathbb{Z} : i < n/k\}$.

Lemma 3.3.4. Let $n, k \in \mathbb{N}$, B a block of length k and $p \in \mathbb{N} \cap [1, k]$. Then

$$\frac{1}{2} Q_{n,p}^k \leq t_{n,p}^k < Q_{n,p}^k. \tag{3.6}$$

Proof. Note that $Q_{n,p}^k = \sum_{i=0}^{\rho(n,k)} F_i^k$ in virtue of Lemma 3.3.3. Moreover, given $n, k \in \mathbb{N}$, B a block of length k and $p \in \mathbb{N} \cap [1, k]$, we can see that

$$\begin{aligned} t_{n,p}^k &= \sum_{i=0}^{\rho(n,k)} V_i^k = \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} \right)^2 < \\ &< \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} = \sum_{i=0}^{\rho(n,k)} F_i^k = Q_{n,p}^k, \end{aligned}$$

and

$$\begin{aligned} t_{n,p}^k &= \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} \right)^2 \geq \\ &\geq \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} - \frac{2^{-k}}{q_{ik+p} \cdots q_{ik+p+k-1}} \geq \\ &\geq \frac{1}{2} \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p} \cdots q_{ik+p+k-1}} = \frac{1}{2} Q_{n,p}^k. \end{aligned}$$

■

The next step is to prove an estimate closely related to strong Q-normality.

Lemma 3.3.5. Let $k \in \mathbb{N}$ and B a block of length k. Then, for almost every $x \in [0, 1)$,

$$N_{n,p}(B, x) = Q_{n,p}^k + O\left(\sqrt{Q_{n,p}^k \log \log Q_{n,p}^k} \right).$$

Proof. Fix $p \in \mathbb{N} \cap [1, k]$. We will consider two cases.

- $\lim_{n \to \infty} Q_{n,p}^k < +\infty$.

By the definition of $r_{B,i,p}$,

$$\lim_{n\to\infty} Q_{n,p}^k = \lim_{n\to\infty} \sum_{i=0}^{\rho(n,k)} \mathbb{E}(r_{B,i,p}) = \lim_{n\to\infty} \sum_{i=0}^{\rho(n,k)} \mathbb{P}(r_{B,i,p} = 1) < +\infty.$$

So, by the Borel-Cantelli lemma, $\mathbb{P}(r_{B,i,p} = 1 \text{ i.o.}) = 0$. Thus, $\mathbb{P}(r_{B,i,p} = 0 \text{ a.a.}) = 1$. and therefore $\lim_{n\to\infty} N_{n,p}(B,x) < +\infty$. The conclusion follows immediately.

- $\lim_{n\to\infty} Q_{n,p}^k = +\infty$. Because of (3.6),

$$\lim_{n\to\infty} t_{n,p}^k \geq \lim_{n\to\infty} \frac{1}{2} Q_{n,p}^k = +\infty. \tag{3.7}$$

Also, $N_{n,p}(B,x) = \sum_{i=0}^{\rho(n,k)} r_{B,i,p}(x)$. If we take $\xi_i = r_{B,i,p} - \mathbb{E}(r_{B,i,p})$, we have $\mathbb{E}(\xi_i) = 0$, $\mathbb{E}(\xi_i^2) = \mathbb{V}(r_{B,i,p}) = V_i^k$. Note that $s_n^2 = \sum_{i=0}^{\rho(n,k)} V_i^k = t_{n,p}^k \xrightarrow{n} +\infty$. Since $t_{n,p}^k \geq 0$ and $t_{n,p}^k \xrightarrow{n} \infty$,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (t_{n,p}^k)^{1/2} = +\infty.$$

Moreover, $|\xi_i| \leq 1$ for all i and $x(\log\log x)^{-3/2} \geq 1$ for $x > 0$, so we also have

$$\mathbb{P}\left(|\xi_n| \leq \frac{s_n}{(\log\log s_n)^{3/2}}\right) = 1.$$

By equation (3.5),

$$N_{n,p}(B,x) - Q_{n,p}^k = \sum_{i=0}^{\rho(n,k)} r_{B,i,p} - F_i^k = \sum_{i=0}^{\rho(n,k)} \xi_i = O\left(\sqrt{t_{n,p}^k \log\log t_{n,p}^k}\right)$$

for almost every x. Since $t_{n,p}^k < Q_{n,p}^k$ by (3.6), it follows that

$$N_{n,p}(B,x) - Q_{n,p}^k = O\left(\sqrt{Q_{n,p}^k \log\log Q_{n,p}^k}\right) \text{ for almost every } x.$$

■

Note that this estimate would imply the Q-normality of order k of x if the index p did not appear. However, these estimates will allow us to deduce analogous asymptotics for $N_n(B,x)$. The next step is to prove that $\sum_p N_{n,p} \sim N_n$ and $\sum_p Q_{n,p}^k \sim Q_n^k$.

From now on, the big and little O notations will be replaced by O_n and o_n respectively, to indicate that the limit involved is taken in n while k remains fixed.

Lemma 3.3.6. Let Q such that $q_n \xrightarrow{n} \infty$, $n, k \in \mathbb{N}$ such that $p \in \mathbb{N} \cap [1, k]$. Then,

$$\sum_{p=1}^{k} \left(Q_{n,p}^k + O_n \left(\sqrt{Q_{n,p}^k \log\log Q_{n,p}^k} \right) \right) = Q_n^k + O_n \left(\sqrt{Q_n^{(k)} \log\log Q_n^{(k)}} \right).$$

Proof. Note that $\sum_{p=1}^{k} Q_{n,p}^k \leq Q_n^k + \left(Q_{\lceil n/k \rceil k}^k - Q_{\lfloor n/k \rfloor k}^k \right)$ and

$$Q_{\lceil n/k \rceil k}^k - Q_{\lfloor n/k \rfloor k}^k \leq \sum_{p=1}^{k} \frac{1}{q_{\rho(n,k)k+p}} = o_n(1)$$

since $q_n \to \infty$. Thus, $\sum_{p=1}^{k} Q_{n,p}^k = Q_n^k + o_n(1)$. Also,

$$\sum_{p=1}^{k} \sqrt{Q_{n,p}^k \log\log Q_{n,p}^k} \leq k \sqrt{\left(\sum_{p=1}^{k} Q_{n,p}^k \right) \left(\sum_{p=1}^{k} \log\log Q_{n,p}^k \right)} = O_n \left(\sqrt{Q_n^k \log\log Q_n^k} \right).$$

∎

In a similar way, $\sum_{p=1}^{k} N_{n,p}(B,x) \leq N_n(B,x) + (N_{\lceil n/k \rceil k}(B,x) - N_{\lfloor n/k \rfloor k}(B,x)) \leq N_n(B,x) + k = N_n(B,x) + O_n(1)$.

By Lemma 3.3.5,

$$N_n(B,x) = \sum_{p=1}^{k} \left(Q_{n,p}^k + O_n \left(\sqrt{Q_{n,p}^k \log\log Q_{n,p}^k} \right) \right) + O_n(1)$$

for almost every x. Now, Lemma 3.3.6 allow us to conclude the following estimate.

Lemma 3.3.7. Let Q be a sequence such that $q_n \xrightarrow{n} \infty$, $k \in \mathbb{N}$ and B be a block of length k. Then almost every $x \in [0, 1)$ satisfies

$$N_n(B,x) = Q_n^k + O_n \left(\sqrt{Q_n^k \log\log Q_n^k} \right).$$

This is the final estimate we need to prove one implication in Theorem 3.3.1. It will be obtained as a corollary of the next result.

Theorem 3.3.3. Let Q be a sequence such that $q_n \xrightarrow{n} \infty$. Then almost every $x \in [0, 1)$ is Q-normal of order k if and only if Q is k-divergent.

Proof. First assume Q is k-divergent. Let B be a block of length k. By Lemma 3.3.7 almost every $x \in [0,1)$ satisfies

$$\lim_{n\to\infty} \frac{N_n(B,x)}{Q_n^k} = \lim_{n\to\infty} \frac{Q_n^k + O_n\left(\sqrt{Q_n^k \log\log Q_n^k}\right)}{Q_n^k} = 1 + \lim_{n\to\infty} O_n\left(\sqrt{\frac{\log\log Q_n^k}{Q_n^k}}\right) = 1.$$

Now, we will prove the reciprocal and assume that Q is k-convergent. Our aim is to prove that there exists a set with positive measure where a block B of length k does not occur. Let $B := (0,0,\dots,0)$ of length k and consider $A := \{x \in [0,1) : B \text{ does not occur in } x\}$.

We have

$$\lambda(A) = \prod_{n\geq 1}\left(1 - \frac{1}{q_n \cdots q_{n+k-1}}\right). \tag{3.8}$$

Since

$$\sum_{n\geq 1} \frac{1}{q_n \cdots q_{n+k-1}} < \infty,$$

and $0 < (q_n \cdots q_{n+k-1})^{-1} < 1$ for all $n \in \mathbb{N}$, the product (3.8) converges to a positive number. ∎

As a corollary of the previous theorem and the countably many choices of k we obtain the desired result.

Corollary 3.3.2. Let Q be a sequence such that $q_n \xrightarrow{n} \infty$. Then almost every $x \in [0,1)$ is Q-normal if and only if Q is fully divergent.

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