Angle-Bounded condition for FitzHugh-Nagumo equations in cardiac electrophysiology

Thesis submitted in partial fulfillment for the requirements of the Master's degree in Mathematics in the Faculty of Mathematics at Pontificia Universidad Católica de Chile.

Author:

Joaquín Oyarzún Quintupurrai

Supervisors:

Duvan Henao Manrique & Manuel Sánchez Uribe

Committee:

Duvan Henao Manrique, Daniel Hurtado Sepúlveda & Manuel Sánchez Uribe

August, 2023
Santiago, Chile

To my parents: Sandra and José,
thank you for your unconditional support and for always believing in me.

Agradecimientos/Acknowledgement

Resumir en pocas líneas esta travesía iniciada en 2021 resulta ser un gran desafío. Definitivamente no fue un camino sencillo y estuvo compuesto de muchos momentos luminosos y oscuros. No obstante, los frutos plasmados en esta tesis encuentran sus raíces en el apoyo inquebrantable de las personas excepcionales que compartieron este recorrido conmigo.

Agradezco de todo corazón el esfuerzo y apoyo incondicional de mis padres: Sandra y José. El camino recorrido hasta este magíster no fue fácil: ha estado lleno de muchos sacrificios. Sin embargo, en esos momentos más oscuros ustedes siempre estuvieron a mi lado. Gracias por creer siempre en mí e inculcarme el valor de la perserverancia y el cariño por querer aprender. Estaré eternamente agradecido con ustedes.

Deseo expresar mi más profundo agradecimiento al profesor Duvan Henao por su inmensa paciencia, excelente disposición y compromiso excepcional. Gracias por guiarme en este proceso e introducirme en el maravilloso mundo del cálculo de variaciones. Realmente ha sido una experiencia muy grata y enriquecedora que jamás olvidaré. No hubiese conseguido los resultados de este trabajo sin su valiosa colaboración y apoyo incondicional. De la misma manera, agradezco a los profesores Manuel Sánchez y Daniel Hurtado. A Manuel, por su vasta paciencia, gran dedicación y vital apoyo al momento de llevar a cabo la parte numérica de la tesis. A Daniel, por sus valiosos comentarios retroalimentativos y por aceptar generosamente ser parte del comité revisor de esta tesis.

No hubiese podido llegar al magíster sin los aprendizajes que adquirí durante mi paso por el colegio y en la licenciatura. No puedo dejar de darle las gracias a mis profesores del San Mateo: Gleny Acevedo y Leopoldo Wagner, por introducirme en este fantastico mundo de las matemáticas. Tampoco puedo dejar de mencionar a los profesores que conocí durante la licenciatura y que me impulsaron a seguir especializandome en matemáticas: Robert Auffarth y Gonzalo Robledo. Agradezco profundamente la disposición, compromiso y vocación pedagógica del profesor Robledo, que me motivó a continuar estos estudios de postgrado.

Deseo agradecer también el incondicional apoyo de mis compañeros y amigos. Son ustedes los que me impulsaron a seguir adelante en este complejo, pero maravilloso mundo de las matemáticas. Gracias totales al Gonza (*el tío*) y Cristhian *(guagualongo)* por toda la ayuda que me brindaron en los momentos más oscuros de la tesis. Agradezco a mis compañer@s matemátic@s en esta travesía: David *(tío Dave)*, Xavi, Tamara, Marcos, John Alex, Jp, Pao, Fitz, Hali, Jaime y Nico. No puedo dejar de agradecerles también a mis amigos de toda una vida que siempre han estado conmigo: Tami, Ricky, Sánchez, Papo, Javi y Juanisimo.

Finalmente, agradezco a la Facultad de Matemáticas por aceptarme en el programa de magíster. Gracias por creer en mí y entregarme las competencias necesarias para poder desarrollarme como un futuro investigador/docente.

Contents

1 Introduction **4**

2 Mathematical Preliminaries **8**

3 Continuous-Discrete Problem **13**
3.1 Continuous Problem 14
3.1.1 FitzHugh-Nagumo as Evolution Equation 14
3.1.2 Weak Formulation 15
3.1.3 Existence of Solutions 18
3.2 Discrete Problem 20

4 Error Estimates and Angle-Bounded Condition **24**
4.1 Error Estimates 25
4.2 Main Result 26
4.3 Angle-Bounded Condition 28
4.4 Maximal Monotone Operators 36
4.5 Proof of Theorem 4.3. 40
4.6 Consequences of Theorem 4.3. 50

5 Numerical Simulations **52**
5.1 Numerical Method 53
5.1.1 Finite Element Method and Geometry of the Problem 53
5.1.2 Temporal Discretization: Forward-Euler Time Step 56
5.1.3 Parameters for Simulations and CFL Condition 57
5.2 Simulation and Conclusions 58

6 Conclusions **62**

Appendix **68**

Bibliography **69**

Chapter 1

Introduction

Consider the bounded set $\Omega \subset \mathbb{R}^n$. The boundary of Ω is denoted by $\partial\Omega$ and $\mathbf{n}$ represents the outward-pointing normal vector on $\partial\Omega$. Let $T > 0$ be the final time. We define the space-time domain as $\Omega_T := \Omega \times (0, T)$ and the boundary of space-time domain as $\partial\Omega_T := \partial\Omega \times (0, T)$. We introduce the space:

$$\mathcal{S}(\Omega) := \{u_1 \in H^1(\Omega) \,:\, u_1 = 0 \text{ in } \partial\Omega_\phi\},$$

such that $\partial\Omega_\phi$ is the Dirichlet part of the boundary $\partial\Omega$. Let $\phi(x,t) = \phi = \overline{\phi} + u_1 \in L^2(\Omega)$ be the membrane potential, where $\overline{\phi}$ is defined in $\overline{\Omega}$ (i.e, $\overline{\phi}$ can be extended to a $C^\infty(\Omega)$ function), $u_1 \in \mathcal{S}(\Omega)$ and $r(x,t) = r \in L^2(\Omega)$ be the plasma membrane recovery current.

In this thesis we will work with the FitzHugh-Nagumo equations in cardiac electrophysiology (based in the Hodgkin-Huxley model, winner of the Nobel Prize in Physiology or Medicine, 1963), defined as the following system of equations:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - D\Delta\phi - c_1\phi(\phi-\alpha)(1-\phi) + c_2 r = 0, \quad (x,t) \in \Omega_T \\[2ex] \dfrac{\partial r}{\partial t} - b(\phi - dr) = 0, \quad (x,t) \in \Omega_T \\[2ex] \phi = \overline{\phi}, \quad \text{in } \partial\Omega_\phi \times (0,T) \\[2ex] D\nabla\phi \cdot \mathbf{n} = 0, \quad \text{in } \partial\Omega_q \times (0,T) \\[2ex] \phi(x,0) = \phi^0(x), \quad x \in \Omega, \\[2ex] r(x,0) = r^0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

In (1.1), $D > 0$ is the diffusion rate (depending of the membrane potential ϕ), $\alpha \in \mathbb{R}$ and $c_1, c_2, b, d \in \mathbb{R}^+$. Note that FitzHugh-Nagumo equations are a coupled system of a partial parabolic differential equation and an ordinary differential equation with boundary and initials conditions.

The boundary $\partial\Omega$ satisfies the conditions:

$$\begin{aligned} \partial\Omega_\phi \cap \partial\Omega_q &= \emptyset, \\ \overline{\partial\Omega_\phi} \cup \overline{\partial\Omega_q} &= \partial\Omega, \end{aligned}$$

where $\partial\Omega_\phi$, $\partial\Omega_q$ are relatively open and smooth in $\partial\Omega$. We will emphasize that $\partial\Omega_q$ is the Neumann part of boundary $\partial\Omega$ and $\partial\Omega_\phi$ is the Dirichlet part of boundary $\partial\Omega$. As an example, in [1] a simulation is performed with the following values for the parameters:

Parameter	Value	Description
α	0.08	Normalized threshold potential
c_1	0.175	Excitation rate constant
c_2	0.03	Excitation decay constant
b	0.011	Recovery rate constant
d	0.55	Recovery decay constant

Table I: Parameter values for a single-cell example.

Let's define $F'_{\text{ion}}(\phi) := -c_1\phi(\phi - \alpha)(1 - \phi)$ as the ionic current and consider the membrane potential $\phi = \overline{\phi} + u_1$, where $u_1 = \phi - \overline{\phi} = 0 \in \mathcal{S}(\Omega)$ on the boundary $\partial\Omega_\phi$. Moreover, if we introduce $c_r := \frac{c_2}{b}$ as the relative weight of c_2 (which represents the transfer coefficient from r to ϕ in the partial differential equation (PDE) $\frac{\partial\phi}{\partial t} - D\Delta\phi = c_1\phi(\phi - \alpha)(1 - \phi) - c_2 r$) and b (the transfer coefficient from ϕ to r in the ordinary differential equation (ODE) $\frac{\partial r}{\partial t} = b\phi - bdr$), we can rewrite Equation (1.1) as follows:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D\Delta u_1 + F'_{\text{ion}}(\overline{\phi} + u_1) + c_2 r + L u_1 = L u_1 - \dfrac{\partial\overline{\phi}}{\partial t} + D\Delta\overline{\phi}, \\ c_r \dfrac{\partial r}{\partial t} + c_2 d r - c_2 u_1 + L r = L r + c_2\overline{\phi}, \\ D\nabla u_1 \cdot \mathbf{n} = -D\nabla\overline{\phi} \cdot \mathbf{n}, \\ u_1(x, 0) = \phi_0(x) - \overline{\phi}(x) = u_1^0(x), \\ r(x, 0) = r^0(x). \end{cases} \tag{1.2}$$

where $L > 0$.

In Chapter 3 it will be shown that FitzHugh-Nagumo equations given in (1.2) can be expressed as the following evolution equation:

$$u'(t) + \mathcal{F}(u(t)) = g(t), \quad t \in (0, T) \tag{1.3}$$

where $\mathcal{F}$ is called evolution operator, $g(t)$ is a function and $u(t)$ is the solution of (1.3). If we consider the time discretization of the equation $u'(t)+\mathcal{F}(u(t)) = g(t)$ in $(0, T)$ using a Backward-Euler finite difference scheme, we will obtain the semi-discrete equation $\delta U_n + \mathcal{F}(U_n) = G_n$, which can be described as follow:

$$U' + \mathcal{F}(\overline{U}) = \overline{G}, \tag{1.4}$$

with step-time $\tau_n = t_n - t_{n-1}$, $\tau = \max\limits_{1\leq n\leq N} \tau_n$ and U is the numerical solution of (1.4), where $\mathcal{P}$ is a partition of the time interval $[0, T]$:

$$\mathcal{P} = \{0 = t_0 < t_1 < ... < t_{N-1} < t_N = T\}.$$

The advantage of considering (1.2) instead (1.1) lies in the fact that the coefficients of r (in the PDE) and ϕ (in the ODE) have the same magnitude, preserving the opposite sign that they had in (1.1). This implies that the antisymmetric part in the evolution operator $\mathcal{F}$ is well-balanced and the mathematical analysis of Backward-Euler scheme in time that we will present in this thesis can be carried out successfully.

The main purpose of this thesis is to show that the solution $U(t)$ of discrete problem (1.4) converges to weak solution $u(t)$ of continuous problem (1.3) when $\tau \to 0$ (in $\|\|_{L^\infty(\Omega)}$ norm). To do this, we must introduce the error estimate $E_{\mathcal{H}}$ (defined in Chapter 4). This will be allow us to validate the Backward-Euler finite difference scheme in time for (1.1). These analytical results and conclusions will be complemented with a numerical simulation.

In fact, we use a Finite Element Method (FEM) in the space Ω and Forward-Euler finite difference scheme in time for a generalized FitzHugh-Nagumo equations. In Chapter 5, the simulations are performed with fixed initial conditions and parameters (as in Table I).

The angle-bounded condition (defined in Chapter 4) will be extremely useful in the development of this thesis, which are studied in more details by Haim Brézis in [2] and [3]. These concepts also allow us to definite a variational structure for the FitzHugh-Nagumo equations.

Being able to provide a specific variational structure for FitzHugh-Nagumo equations is the concrete achievement of this thesis.

The remainder of this work is organized as follows:

- Chapter 2: Mathematical Preliminares. In this chapter we will introduce the spaces $L^1(\Omega)$, $H^1(\Omega)$, $\mathcal{S}(\Omega)$ and the key results that will allow us to demonstrate the convergence from discrete to continuous time.

- Chapter 3: Continuous-Discrete Problem. In this chapter we are going to define a continuous problem (writing the FitzHugh-Nagumo equations as an abstract evolution equation) and the discrete problem (time discretization of continuous problem using a Backward-Euler finite difference scheme). Additionally, we will ensure the existence of solutions for these problems (discrete and continuous) utilizing certain properties satisfied by the evolution operator $\mathcal{F}$.

- Chapter 4: Error Estimates. In this chapter we will define the error estimates, which will make it possible to show that the solution $U(t)$ of discrete problem (1.4) converges to weak solution $u(t)$ of continuous problem (1.3) when $\tau \to 0$. To achieve this purpose, we will introduce the concept of angle-bounded operators and provide a concise overview of the theory of maximal monotone operators.

- Chapter 5: Numerical Simulations. We are going to provide a numerical simulation for the FitzHugh-Nagumo equations using a Finite Element Method (FEM) in space and Forward-Euler finite difference scheme in time with the purpose to visualizing and interpreting the behavior of the membrane potential under fixed initial conditions in a tridimensional domain.

- Chapter 6: We are going to define a concrete variational structure for the FitzHugh-Nagumo equations using the error estimates and the angle-bounded condition.

- Appendix #1: Electrophysiological Glossary. We provide the biological and electrophysiological concepts used throughout this thesis (plasma membrane, membrane potential, ionic current and plasma membrane recovery current).

- Appendix #2: Code (in NGSolve) for numerical simulation.

Chapter 2

Mathematical Preliminaries

Let Ω be a bounded subset of $\mathbb{R}^n$. The main purpose of this chapter is to introduce three important function spaces: $L^2(\Omega)$, $H^1(\Omega)$, and $\mathcal{S}(\Omega)$. These spaces will be used extensively throughout this thesis and they play a crucial role in demonstrating the convergence from discrete to continuous time. More details about these spaces can be found in [9]

Firstly, we introduce the space $L^2(\Omega)$ (which consists of square-integrable functions on Ω). Next, we discuss the Sobolev space $H^1(\Omega)$ (consisting in functions on Ω whose first-order weak derivatives are square-integrable). Finally, we define the space $\mathcal{S}(\Omega)$ (which consists of the functions $u_1 \in H^1(\Omega)$ such that they have zero Dirichlet boundary condition).

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. We define the $L^2(\Omega)$ space as follow:*

$$L^2(\Omega) := \left\{ \varphi : \Omega \to \mathbb{R} \,:\, \varphi \text{ is measurable and } \int_\Omega |\varphi|^2 \, d\mu < \infty \right\}.$$

Definition 2.2. *From the space $L^2(\Omega)$, we can define the Sobolev space $H^1(\Omega)$ as follow:*

$$\begin{aligned} H^1(\Omega) := \Bigg\{ \varphi \in L^2(\Omega) \,:\, & \text{ For all } i = 1, 2, ..., n, \text{ exists } \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega) \text{ such that} \\ & \int_\Omega \varphi \frac{\partial \psi}{\partial x_i} \, d\mu = -\int_\Omega \frac{\partial \varphi}{\partial x_i} \psi \, d\mu \quad \forall\, \psi \in C_0^\infty(\Omega) \Bigg\}. \end{aligned}$$

Remark. $C_0^\infty(\Omega)$ denotes the set of functions that are C^∞ whose support is contained on a compact set Ω. The spaces $L^2(\Omega)$ and $H^1(\Omega)$ are Hilbert spaces equipped with the following inner products:

$$\begin{aligned} \langle f, g \rangle_{L^2(\Omega)} &= \int_\Omega f g \, d\mu, \quad \forall\, f, g \in L^2(\Omega), \\ \langle f, g \rangle_{H^1(\Omega)} &= \int_\Omega f g \, d\mu + \int_\Omega \nabla f \cdot \nabla g \, d\mu, \quad \forall\, f, g \in H^1(\Omega). \end{aligned}$$

Definition 2.3. *From the Sobolev Space $H^1(\Omega)$, we define the space $\mathcal{S}(\Omega)$ as follow:*

$$\mathcal{S}(\Omega) := \{u_1 \in H^1(\Omega) \,:\, u_1 = 0 \text{ in } \partial\Omega_\phi\},$$

where $\partial\Omega_\phi$ is the Dirichlet part of the boundary $\partial\Omega$.

Remark. The space $\mathcal{S}(\Omega)$ is a vectorial subspace of $L^2(\Omega)$. By the trace-zero function theorem (see [8], Theorem 2, Page 259), we can conclude that $\mathcal{S}(\Omega)$ is a closed subset of $L^2(\Omega)$. Moreover, since $L^2(\Omega)$ is a Banach space it follows that $\mathcal{S}(\Omega)$ is a complete vector subspace of $L^2(\Omega)$. This allow us to conclude that $\mathcal{S}(\Omega)$ is a Banach space with the subspace norm.

Throughout the thesis, we will work with the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ equipped with the inner product:

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \langle f_1, g_1 \rangle_{L^2(\Omega)} + \langle f_2, g_2 \rangle_{L^2(\Omega)},$$

for all $f_1, f_2, g_1, g_2 \in L^2(\Omega)$.

Theorem 2.4. (Poincaré Inequality) *Let $1 \leq p < \infty$ and Ω be a bounded and connected subset of $\mathbb{R}^n$. Then, exists a constant $C > 0$ (who depends only from Ω and p) such that:*

$$\|\varphi\|_{L^p(\Omega)} \leq C \|\nabla\varphi\|_{L^p(\Omega)}, \quad \forall\, \varphi \in H^1(\Omega).$$

Proof. See [8], Theorem 1, Page 275.

Throughout this section, we will use V to denote a Banach space with and norm $\|\,|_V$.

Definition 2.5. *Let V be a Banach space. We will say that $\Lambda : V \to \mathbb{R}$ is a linear continuous functional if:*

- $\Lambda(\varphi_1 + \varphi_2) = \Lambda(\varphi_1) + \Lambda(\varphi_2), \quad \forall\, \varphi_1, \varphi_2 \in V.$
- $\Lambda(k\varphi) = k\Lambda(\varphi), \quad \forall\, \varphi \in V.$
- $\{\varphi_n\}_{n\in\mathbb{N}}$ *is a sequence of functions in V such that* $\lim\limits_{n\to+\infty} \|\varphi_n - \varphi\|_V = 0$ *then:*

$$\lim_{n\to+\infty} \Lambda(\varphi_n) = \Lambda(\varphi).$$

From Definition 2.5, we can define the dual space of Banach space. For this, consider the following definition:

Definition 2.6: *Let V a Banach Space. We define the dual space of V as follow:*

$$V^{'} := \{\Lambda : V \to \mathbb{R} \, : \, \Lambda \text{ is a linear continuous functional}\}.$$

The dual space V' is equipped with the norm:

$$\|\Lambda\|_{V'} = \sup_{\|\varphi\|_V \leq 1, \, \varphi \in V} |\Lambda(\varphi)|.$$

Remark. The dual space of $L^p(\Omega)$ is the space $L^q(\Omega)$ where p, q satisfies $q = \dfrac{p}{p-1}$ (with $p \neq 1$).

Remark. $(L^2(\Omega))' = L^2(\Omega)$

Definition 2.7. *Let V be a Banach space. We will said that $\Lambda : V \to \mathbb{R}$ is lower semi-continuous on V if it satisfies the two equivalent conditions:*

- *For all $a \in \mathbb{R}$, the set $\mathcal{D} := \{u \in V \, : \, \Lambda(u) \leq a\}$ is closed.*
- *For all $\overline{u} \in V$, we have $\liminf_{u \to \overline{u}} \Lambda(u) \geq \Lambda(\overline{u})$.*

The space $H^1(\Omega)$ can be further generalized to a more extensive class of Sobolev spaces known as $W^{1,p}(\Omega)$. This generalization encompasses the functions whose weak derivatives up to order one belongs to $L^p(\Omega)$. To provide a formal definition, consider the following:

Definition 2.8. *Let $1 \leq p < \infty$. From the space:*

$$L^p(\Omega) := \left\{ \varphi : \Omega \to \mathbb{R} \, : \, \varphi \textit{ is measurable and } \int_\Omega |\varphi|^p \, d\mu < \infty \right\},$$

we can define the Sobolev space $W^{1,p}(\Omega)$ as follow:

$$W^{1,p}(\Omega) := \Bigg\{ \varphi \in L^p(\Omega) \, : \, \text{For all } i = 1, 2, ..., n, \text{ exists } \frac{\partial \varphi}{\partial x_i} \in L^p(\Omega) \text{ such that} \\ \int_\Omega \varphi \frac{\partial \psi}{\partial x_i} \, d\mu = - \int_\Omega \frac{\partial \varphi}{\partial x_i} \psi \, d\mu \quad \forall \, \psi \in C_0^\infty(\Omega) \Bigg\}.$$

Remark. Clearly $W^{1,2}(\Omega) = H^1(\Omega)$.

Now it's time to introduce one of the fundamental theorems in Functional Analysis:

Theorem 2.9. (Hahn-Banach Theorem): *Let $p : E \to \mathbb{R}$ be a function satisfying:*

$$p(\lambda x) = \lambda p(x), \quad \forall\, x \in E,\ \forall\, \lambda \in \mathbb{R},$$
$$p(x + y) \le p(x) + p(y), \quad \forall\, x, y \in E.$$

Let $G \subset E$ be a linear subspace (where E is a vectorial space) and let $g : G \to \mathbb{R}$ be a linear functional such that $g(x) \le p(x)$, for all $x \in G$. Then, there exists a linear functional f defined on all E that extends g. Indeed, $g(x) = f(x)$, for all $x \in G$ such that $f(x) \le p(x)$, for all $x \in E$.

Proof. See [9], Theorem 1.1, Page 16.

Theorem 2.10. (Riesz Representation Theorem) *Let $< p < \infty$ and $\varphi \in (L^p(\Omega))'$. Then there exists a unique function $u \in (L^p(\Omega))'$ such that:*

$$\langle \varphi, f \rangle_{(L^p(\Omega))'} = \int u f \, d\mu, \quad \forall\ f \in L^p(\Omega).$$

Proof. See [9], Theorem 4.11, Page 97.

Definition 2.11. *Let $\mathcal{H}$ be a Hilbert space and consider the function $h : [0, T] \to \mathcal{H}$. If $\mathcal{P}$ is a partition of time interval $[0, T]$ given by:*

$$\mathcal{P} = \{0 = t_0 < t_1 < ... < t_{N-1} < t_N = T\}.$$

We define the total variation of h in $[0, T]$ as follow:

$$Var(h) := \sup_{\{t_i\} \in \mathcal{P}} \left\{ \sum_{j=1}^{n} |h(t_j) - h(t_{j-1})| \right\}.$$

Definition 2.12. *Let $\mathcal{H}$ be a Hilbert space and consider the function $h : [0, T] \to \mathcal{H}$. We will say that g is a bounded variation function (denoted by $g \in BV((0, T), \mathcal{H})$) if for all $t_0 \in [0, T)$, there exists the right limit:*

$$\lim_{t \to t_0^+} g(t).$$

Now, we define the key results that will allow us to demonstrate the convergence from discrete to continuous time:

Theorem 2.13. *Let's consider the measurable functions $a, b, c, d : (0, T) \to [0, +\infty[$ and a^2 be an absolutely continuous function on $[0, T]$. If the differential inequality is satisfied almost everywhere in $(0, T)$ for all $\ell \in \mathbb{R}$:*

$$\frac{d}{dt}(a^2(t)) + b^2(t) + 2\ell a^2(t) \le c^2(t) + 2d(t)a(t).$$

Then

$$\max\left(\max_{t\in[0,T]} e^{\ell t}a(t), \left(\int_0^T e^{2\ell s}b^2(s)\,ds\right)^{\frac{1}{2}}\right) \le \left(a^2(0)+\int_0^T e^{2\ell t}c^2(t)\,dt\right)^{\frac{1}{2}} + \int_0^T e^{\ell t}d(t)\,dt.$$

Proof. See [6], Lemma 3.7, Page 551.

Theorem 2.14. *Let $\{a_n\}_{n=0}^N$ and $\{b_n, c_n, d_n\}_{n=0}^N$ be nonneagitve numbers. Consider the given coefficients $\{\mu_n\}_{n=1}^N$ such that $-1 < \mu_n \le 0$ and:*

$$\begin{aligned}
\mu &:= \min_{1\le n\le N} \mu_n,\\
s_n &:= \sum_{k=1}^{n} \mu_k,\\
\overline{s_n} &:= \frac{s_n + s_{n-1}}{2},\\
s_0 &:= 0.
\end{aligned}$$

If $2a_n(a_n - a_{n-1}) + b_n^2 + 2\mu_n a_n^2 \le c_n^2 + 2a_n d_n$ for all $1 \le n \le N$, then:

$$\max\left(\max_{1\le n\le N} e^{\frac{s_n}{1+\mu}}a_n, \left(\sum_{n=1}^N e^{\frac{2\overline{s_n}}{1+\mu}}b_n^2\right)^{\frac{1}{2}}\right) \le \left(a_0^2 + \sum_{n=1}^N e^{2\overline{s_n}}c_n^2\right)^{\frac{1}{2}} + \sqrt{2}\sum_{n=1}^N e^{s_{n-1}}d_n.$$

Proof. See [6], Lemma 4.12, Page 569.

Chapter 3

Continuous-Discrete Problem

Let us recall the definitions of the variables involved in the FitzHugh-Nagumo equations:

- The membrane potential (refers to electrical potential difference that exists between the interior and exterior of a plasma membrane) is denoted as:

  $$\phi(x,t) = \phi = \overline{\phi} + u_1 \in L^2(\Omega),$$

  where $\overline{\phi}$ is defined over the domain $\overline{\Omega}$ and u_1 belongs to the space $\mathcal{S}(\Omega)$. Throughout this thesis, we will be working with the following relation:

  $$u_1 = \phi - \overline{\phi} \in \mathcal{S}(\Omega).$$

- The plasma membrane recovery current (refers to the electrical current that flows across the cell membrane during this restoration phase) $r(x,t) = r \in L^2(\Omega)$.

In Chapter 1 we introduce the FitzHugh-Nagumo equations, which constitute a coupled system consisting of a parabolic partial differential equation (PDE) and an ordinary differential equation (ODE). Indeed, the equations that we will work with throughout this thesis are given by:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D\Delta u_1 + F'_{\text{ion}}(\overline{\phi} + u_1) + c_2 r + L u_1 = L u_1 - \dfrac{\partial \overline{\phi}}{\partial t} + D\Delta\overline{\phi}, \\[2ex] \dfrac{\partial r}{\partial t} + c_2 d r - c_2 u_1 + L r = L r + c_2 \overline{\phi}, \\[2ex] D\nabla u_1 \cdot \mathbf{n} = -D\nabla\overline{\phi} \cdot \mathbf{n}, \\[2ex] u_1(x,0) = u_1^0(x), \\[2ex] r(x,0) = r^0(x). \end{cases} \tag{3.1}$$

where $L > 0$. In order to modify the first equation (PDE) and the second equation (ODE) in (1.2), we introduced the terms Lu_1 and Lr on both sides respectively and set c_r equals to 1.

The purpose of this chapter is to introduce a continuous and discrete problem associated to the FitzHugh-Nagumo equations in (3.1). To do this, we will rewrite the FitzHugh-Nagumo equations as an evolution equation. This formulation allows us to describe the system behavior over continuous time. In parallel, we introduce the discrete problem, which involves the time discretization of the evolution equation using a Backward-Euler finite difference scheme. This discretization method allows us to approximate the continuous dynamics of the system by considering a sequence of discrete time step, facilitating numerical analysis.

To accomplish this, we will define an evolution operator $\mathcal{F}$ that will play a crucial role in establishing a variational structure for FitzHugh-Nagumo. Additionally, we will review specific theoretical results that guarantee the existence of solutions for the evolution equation and the discrete problem associated with (3.1).

3.1 Continuous Problem

3.1.1 FitzHugh-Nagumo as Evolution Equation

An evolution equation is a mathematical formulation that describes the temporal evolution of a dynamical system in space. It captures how the system behavior changes over time, taking into account how the variable evolves and how it interacts with its surrounding environment.

To define the evolution equation associated with the system (3.1), we need to introduce definitions and preliminary results:

Definition 3.1. *Let's consider the application $u : (0,T) \to \mathcal{S}(\Omega) \times L^2(\Omega)$ given by:*

$$u(t) := \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} (u_1(t))(x) \\ (r(t))(x) \end{pmatrix}.$$

Remark. Note immediately that $(u(t))(x)$ describes the temporal evolution of variables u_1 and r, where the spatial component $x \in \Omega$ is fixed.

Remark. Using Definition 3.1, we can express the initial conditions of the FitzHugh-Nagumo equation (3.1) as:

$$u(0) := u^0(x) = \begin{pmatrix} u_1^0(x) \\ r^0(x) \end{pmatrix}.$$

Definition 3.2. *We define the evolution operator of the FitzHugh-Nagumo equations $\mathcal{F} : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ as:*

$$\mathcal{F}(u(t)) := \begin{pmatrix} -D\Delta u_1 + F'_{ion}(\overline{\phi} + u_1) + c_2 r + Lu_1 \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix}.$$

where $L > 0$ and $D(\mathcal{F}) := \left\{ \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} \in \mathcal{S}(\Omega) \times L^2(\Omega) \;:\; -\Delta u_1(t) \in L^2(\Omega) \right\}$ is the proper domain of $\mathcal{F}$.

Definition 3.3. *We define the function* $g : (0, T) \to L^2(\Omega) \times L^2(\Omega)$ *as follow:*

$$g(t) := \begin{pmatrix} Lu_1 - \frac{d\overline{\phi}}{dt} + D\Delta\overline{\phi} \\ Lr + c_2\overline{\phi} \end{pmatrix},$$

where $g \in L^1((0, T), L^2(\Omega) \times L^2(\Omega))$.

We must emphasize that we are only studying the temporal evolution of the equations. In this manner (using the Definitions 3.1, 3.2 and 3.3), we can rewrite the FitzHugh-Nagumo equations given in (3.1) as a Cauchy problem. Indeed:

Definition 3.4. *We define the evolution equation associated to FitzHugh-Nagumo equations (3.1) by the expression:*

$$\begin{cases} u'(t) + \mathcal{F}(u(t)) = g(t), \quad t \in (0, T) \\ \\ u(0) = u^0. \end{cases} \tag{3.2}$$

Remark. Since we are only studying the temporal evolution of the FitzHugh-Nagumo equations, the partial derivative is total. In this way, we will consider the notation:

$$(u'(t))(x) = \begin{pmatrix} (u_1'(t))(x) \\ (r'(t))(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u_1}{\partial t} \\ \dfrac{\partial r}{\partial t} \end{pmatrix}$$

.

3.1.2 Weak Formulation

In the last subsection, we have defined the evolution equation (3.2), which describes the temporal evolution of the FitzHugh-Nagumo equations. The next question that we need to address is ensuring the existence of solutions for (3.2). In fact, this is possible. But before that, we need to determine the weak formulation of the equation (3.1). To do this, consider the test function:

$$\widehat{u}(t) = \begin{pmatrix} \widehat{u_1}(t) \\ \widehat{r}(t) \end{pmatrix} \in L^2(\Omega) \times L^2(\Omega).$$

If we multiply the equation (3.1) by the test function $\widehat{u}$ and integrate over the domain Ω, we obtain the following expresion:

$$\int_\Omega \left(\frac{\partial u_1}{\partial t} - D\Delta u_1 + F'_{\text{ion}}(\overline{\phi} + u_1) + c_2 r + L u_1 - L u_1 + \frac{\partial \overline{\phi}}{\partial t} - D\Delta\overline{\phi} \right) \widehat{u_1} = 0,$$

$$\int_\Omega \left(\frac{\partial r}{\partial t} + c_2 d r - c_2 u_1 - c_2 \overline{\phi} + L r - L r \right) \widehat{r} = 0.$$

Using integration by parts:

$$\int_\Omega \frac{\partial u_1}{\partial t}\widehat{u_1} - \int_{\partial\Omega} \widehat{u_1} \cdot D\frac{\partial u_1}{\partial \mathbf{n}} + \int_\Omega D\nabla u_1 \cdot \nabla \widehat{u_1} + \int_\Omega F'_{\text{ion}}(\overline{\phi} + u_1)\widehat{u_1} + \int_\Omega c_2 r \widehat{u_1}$$
$$+ \int_\Omega L u_1 \widehat{u_1} - \int_\Omega L u_1 \widehat{u_1} + \int_\Omega \frac{\partial \overline{\phi}}{\partial t}\widehat{u_1} - \int_{\partial\Omega} \widehat{u_1} \cdot D\frac{\partial \overline{\phi}}{\partial \mathbf{n}} + \int_\Omega D\nabla\overline{\phi} \cdot \nabla\widehat{u_1} = 0,$$

$$\int_\Omega \frac{\partial r}{\partial t}\widehat{r} + \int_\Omega c_2 d r \widehat{r} - \int_\Omega c_2 u_1 \widehat{r} - \int_\Omega c_2 \overline{\phi} \widehat{r} + \int_\Omega L r \widehat{r} - \int_\Omega L r \widehat{r} = 0.$$

Since we have a zero Neumann boundary condition in a part of the boundary $\partial\Omega_q$ given by $D\nabla u_1 \cdot \mathbf{n} = -D\nabla\overline{\phi} \cdot \mathbf{n}$, we can conclude:

$$\int_{\partial\Omega} \widehat{u_1} \cdot D\frac{\partial u_1}{\partial \mathbf{n}} = -\int_{\partial\Omega} \widehat{u_1} \cdot D\frac{\partial \overline{\phi}}{\partial \mathbf{n}}.$$

In this way, we obtain the following weak formulation for the FitzHugh-Nagumo equations (3.1):

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial u_1}{\partial t}\widehat{u_1} + \int_\Omega D\nabla u_1 \cdot \nabla\widehat{u_1} + \int_\Omega F'_{\text{ion}}(\overline{\phi} + u_1)\widehat{u_1} + \int_\Omega c_2 r\widehat{u_1} + \int_\Omega L u_1 \widehat{u_1} = \int_\Omega L u_1 \widehat{u_1} \\ \displaystyle -\int_\Omega \frac{\partial \overline{\phi}}{\partial t}\widehat{u_1} - \int_\Omega D\nabla\overline{\phi} \cdot \nabla\widehat{u_1}, \\ \\ \displaystyle\int_\Omega \frac{\partial r}{\partial t}\widehat{r} + \int_\Omega c_2 d r\widehat{r} - \int_\Omega c_2 u_1 \widehat{r} + \int_\Omega L r\widehat{r} = \int_\Omega L r \widehat{r} + \int_\Omega c_2 \overline{\phi}\widehat{r}. \end{cases} \tag{3.3}$$

From (3.3), we can define (for reasons that will be justified in Chapter 4) for all $\widehat{u} \in \mathcal{S}(\Omega) \times L^2(\Omega)$:

- Bilinear forms:

$$\begin{aligned}\mathfrak{b}(u,\widehat{u}) &:= \int_\Omega [u_1\widehat{u_1} + r\widehat{r}]\,,\\ \mathfrak{a}(u,\widehat{u}) &:= \int_\Omega [c_2(r\widehat{u_1} - u_1\widehat{r}) + (c_2 d + L)r\widehat{r} + D\nabla u_1\cdot\nabla\widehat{u_1}]\,.\end{aligned}$$

- Functionals:

$$\begin{aligned}\langle\mathcal{L}(t),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} &:= \int_\Omega \left[Lu_1\widehat{u_1} + Lr\widehat{r} + c_2\overline{\phi}\widehat{r} - \frac{\partial\overline{\phi}}{\partial t}\widehat{u_1} - D\nabla\overline{\phi}\cdot\nabla\widehat{u_1}\right],\\ \left\langle\mathcal{L}^{(0)},\widehat{u}\right\rangle_{L^2(\Omega)\times L^2(\Omega)} &:= \int_\Omega \left[u_1^0\widehat{u_1} + r^0\widehat{r}\right].\end{aligned}$$

- Non-linear term: $\langle\mathcal{F}_2(u),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} := \displaystyle\int_\Omega \left[F'_{\text{ion}}(\overline{\phi}+u_1)\widehat{u_1} + Lu_1\widehat{u_1}\right].$

In this way, we can express (3.3) as follow:

$$\begin{cases}\mathfrak{b}(u',\widehat{u}) + \mathfrak{a}(u,\widehat{u}) + \langle\mathcal{F}_2(u),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} = \langle\mathcal{L}(t),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)}\,, & t\in(0,T)\\ \mathfrak{b}(u(0),\widehat{u}) = \left\langle\mathcal{L}^{(0)},\widehat{u}\right\rangle_{L^2(\Omega)\times L^2(\Omega)}\,.\end{cases} \tag{3.4}$$

Let $\mathcal{V} := \mathcal{S}(\Omega)\times L^2(\Omega)$. Because $\mathfrak{a},\mathfrak{b} : \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ are positive, continuous and weakly-coercive bilinear forms, then $\mathfrak{a},\mathfrak{b}$ can be associated to linear operators $\mathcal{F}_1, \text{Id} : D(\mathcal{F})\to L^2(\Omega)\times L^2(\Omega)$ (respectively) such that:

$$\begin{aligned}\langle\mathcal{F}_1(w),v\rangle_{L^2(\Omega)\times L^2(\Omega)} &= \mathfrak{a}(w,v),\\ \langle\text{Id}(w),v\rangle_{L^2(\Omega)\times L^2(\Omega)} &= \mathfrak{b}(w,v),\end{aligned}$$

for all $v\in\mathcal{S}(\Omega)\times L^2(\Omega)$. Thus, we can rewrite (3.4) as follow:

$$\begin{cases}\langle u',\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle\mathcal{F}_1(u),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle\mathcal{F}_2(u),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} = \langle\mathcal{L}(t),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)}\,, & t\in(0,T)\\ \langle u(0),\widehat{u}\rangle_{L^2(\Omega)\times L^2(\Omega)} = \left\langle\mathcal{L}^{(0)},\widehat{u}\right\rangle_{L^2(\Omega)\times L^2(\Omega)}\,.\end{cases} \tag{3.5}$$

The expression (3.4) motivates us to decompose the evolution operator $\mathcal{F}$ into the sum of two operators: $\mathcal{F}_1$ and $\mathcal{F}_2$. Indeed, such decomposition will be crucial when studying the angle-bounded condition for operators. Indeed, consider the following definition:

Definition 3.5. *We define the operators* $\mathcal{F}_1, \mathcal{F}_2 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ *as follow:*

$$\mathcal{F}_1(u(t)) := \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix},$$
$$\mathcal{F}_2(u(t)) := \begin{pmatrix} F'_{ion}(\overline{\phi} + u_1) + Lu_1 \\ 0 \end{pmatrix}.$$

Furthermore, $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *satisfies* $\mathcal{F}(u(t)) = (\mathcal{F}_1 + \mathcal{F}_2)(u(t)) = \mathcal{F}_1(u(t)) + \mathcal{F}_2(u(t))$*, for all* $u(t) \in D(\mathcal{F}) = \left\{ \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} \in \mathcal{S}(\Omega) \times L^2(\Omega) \,:\, -\Delta u_1(t) \in L^2(\Omega) \right\}$.

Utilizing (3.5), we can express the FitzHugh-Nagumo equations in the weak form, which can be written compactly as:

$$\begin{cases} \langle u'(t) + \mathcal{F}(u(t)), \widehat{u}(t) \rangle_{L^2(\Omega)\times L^2(\Omega)} = \langle \mathcal{L}(t), \widehat{u}(t) \rangle_{L^2(\Omega)\times L^2(\Omega)}, \quad t \in (0,T) \\ \\ \langle u(0), \widehat{u}(t) \rangle_{L^2(\Omega)\times L^2(\Omega)} = \left\langle \mathcal{L}^{(0)}, \widehat{u}(t) \right\rangle_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{3.6}$$

3.1.3 Existence of Solutions

In subsection 3.1.1, we defined an evolution equation associated to the FitzHugh-Nagumo system (3.1). Our next mission is to ensure the existence of solutions for the evolution equation (3.2) in order to demonstrate the robustness of the Backward-Euler method (i.e the solution of the discrete problem converge to the continuous-time solution). To do this, we need to consider the following definitions:

Definition 3.6. *We say that* $u \in C^0((0,T), L^2(\Omega) \times L^2(\Omega))$ *is a strong solution of (3.2) if* u *is locally absolutely continuous in* $(0,T)$ *and satisfies (3.4) almost everywhere for* $t \in (0,T)$.

Definition 3.7. *We say that* $u \in C^0((0,T), L^2(\Omega) \times L^2(\Omega))$ *is a weak solution of (3.2) if* u *can be uniformly approximated by a sequence* $\{u_k\}_{k\in\mathbb{N}}$ *of strong solutions that solve (3.2) with respect to a family of data* $\{g_k\}_{k\in\mathbb{N}}$ *approaching* g *in* $L^1((0,T), L^2(\Omega) \times L^2(\Omega))$ *and* $u_k(0) \to u_0$ *in* $L^2(\Omega) \times L^2(\Omega)$.

Proposition 3.8. *The evolution equation (3.2) admits a unique strong solution.*

Proof. Let's remember that:

$$u(0) := u^0(x) = \begin{pmatrix} u_1^0(x) \\ r^0(x) \end{pmatrix}.$$

Clearly, $u_1^0(x) \in \mathcal{S}(\Omega)$ and $r^0(x) \in L^2(\Omega)$. This implies that $u^0 \in \mathcal{S}(\Omega) \times L^2(\Omega)$.

In the other side, the function $g : (0,T) \to L^2(\Omega) \times L^2(\Omega)$ given by:

$$g(t) = \begin{pmatrix} Lu_1 - \frac{d\overline{\phi}}{dt} + D\Delta\overline{\phi} \\ Lr + c_2\overline{\phi} \end{pmatrix},$$

is bounded total variation (i.e, $g \in BV((0,T), L^2(\Omega) \times L^2(\Omega))$), because exists the right limit for all $t_0 \in [0,T)$:

$$\lim_{t\to t_0^+} g(t) = \begin{pmatrix} Lu_1(t_0) - \frac{d\overline{\phi}}{dt}(t_0) + D\Delta\overline{\phi}(t_0) \\ Lr(t_0) + c_2\overline{\phi}(t_0) \end{pmatrix}.$$

Indeed, $\overline{\phi}$ is a smooth function that does not depend on $x \in \Omega$. By [6] (see Section 2.1, Basic Existence Results and Energy Solutions), we can ensure the existence and uniqueness of strong solution for the evolution equation (3.2) associated to FitzHugh-Nagumo system.

□

Remark. Note that:

$$u(t) = \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} = \underbrace{g(t)}_{\in L^2(\Omega)\times L^2(\Omega)} - \underbrace{\mathcal{F}(u(t))}_{\in L^2(\Omega)\times L^2(\Omega)}.$$

Because $L^2(\Omega) \times L^2(\Omega)$ is a Hilbert space, we can conclude that $u(t) \in L^2(\Omega) \times L^2(\Omega)$. Also, $u(t)$ is a Lipschitz continuous function. Because the derivative of $u(t)$ exists, it follow that:

$$u'(0) = g(0) - \mathcal{F}(u(0)),$$

where $\lim_{t\to 0^+} g(t) = g(0)$.

Proposition 3.9. *The evolution equation (3.2) admits a unique weak solution.*

Proof. Proposition 3.8 guarantees the existence and uniqueness of strong solutions for the evolution equation (3.2) associated with the FitzHugh-Nagumo system (3.1). Given $f \in L^1((0,T), L^2(\Omega) \times L^2(\Omega))$ (see Definition 3.3) and because u_0 can be extended to $\overline{D(\mathcal{F})}$, in virtue of [6] (Section 2.1, Basic Existence Results and Energy Solutions) we can ensure the existence and uniqueness of weak solutions.

□

Remark. Based on Propositions 3.8 and 3.9, we have established the existence and uniqueness of weak/strong solutions for the evolution equation (3.2) associated to the FitzHugh-Nagumo system (3.1).

Remark. Using the weak formulation (3.5) for the FitzHugh-Nagumo equations (3.1), Theorem 1 in [10] (page 59) also guarantees the existence and uniqueness of (strong) solutions $u(t)$ for the evolution equation (3.2).

Definition 3.10. *The continuous problem involves to obtaining the weak solution for the evolution equation (3.2) associated to the FitzHugh-Nagumo system (3.1).*

However, our objective is not to determine a specific weak solution for (3.2). We are interested in demonstrating the convergence of the solution from discrete problem to the solution of the continuous problem. In order to accomplish this, we will define the discrete problem based on the evolution equation (3.2).

3.2 Discrete Problem

Once we have defined the continuous problem (which consists of obtaining a weak solution to the evolution equation associated with the FitzHugh-Nagumo system (3.1)), we will focus on obtaining a discrete problem. To do this, we need to perform the temporal discretization of equation (3.1) using Backward-Euler method defined as follow:

Definition 3.11. (Backward-Euler scheme in time) *Consider the Cauchy problem:*

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ \\ y(t_0) = y_0, \end{cases}$$

The function $f(t, x)$ and the initial data $y(t_0) = y_0$ are known. The Backward-Euler finite difference scheme in time gives a sequence $\{y_n\}_{n=0}^{N}$ such that y_n approximates the solution $y(t_0 + nh)$ where h is the step-time, $1 \leq n \leq N$ and:

$$\frac{y_n - y_{n-1}}{h} = f(t_n, y_n).$$

Remark. The information at time $t = t_n$ is assumed to be known.

Because we need to discretize the evolution equation (3.2), let us consider a partition $\mathcal{P}$ of time interval $[0, T]$ given by:

$$\mathcal{P} = \{0 = t_0 < t_1 < ... < t_{N-1} < t_N = T\},$$

with step-time $\tau_n = t_n - t_{n-1}$, $L\tau_n < 1$ (for robustness of numerical method) and $\tau := \max \tau_n$, for all $1 \leq n \leq N$. For the evolution equation (3.2), we know the operator $\mathcal{F}$ (see Definition 3.2), the function g (see Definition 3.3), and the initial condition $u(0) = u^0$.

Our goal is to approximate the temporal derivative in (3.2). Using a Backward-Euler finite difference scheme in time given in Definition 3.16 for the evolution equation (3.2), we will obtain the discrete equation:

$$\begin{cases} \dfrac{U_n - U_{n-1}}{\tau_n} + \mathcal{F}(U_n) = G_n, \\ \\ U(0) = U^0, \end{cases} \tag{3.7}$$

where $\{U_n\}_{n=0}^N \in D(\mathcal{F})$ is a sequence whose first term $U(0) = U^0$ is given (by the initial conditions) and the other ones are recursively defined for $1 \leq n \leq N$. Furthermore $U_n \approx u(t_n)$ and U is the piecewise linear interpolant of the values $\{U_n\}_{n=0}^N$ on the grid $\mathcal{P}$. Also, $\{G_n\}_{n=0}^N \in D(\mathcal{F})$ is a sequence that approximates $g(t)$: $G_n(t_n) \approx g(t_n)$.

From the semi-discrete equation, consider the following definition:

Definition 3.12. *For any sequence $\{U_n\}_{n=0}^N$ we define in the interval $(t_{n-1}, t_n]$:*

- *The piecewise upper constant function: $\overline{U}(t) := U_n$.*
- *The piecewise lower constant function: $\underline{U}(t) := U_{n-1}$.*
- *The piecewise linear interpolation $U(t) := (1 - \lambda(t))U_n + \lambda(t)U_{n-1}$, where:*

$$\lambda(t) := \frac{t_n - t}{\tau_n}, \quad \forall\ 1 \leq n \leq N.$$

Definition 3.13. *We define the discrete derivative for the sequence $\{U_n\}_{n=0}^N$ as:*

$$\delta U_n := \frac{U_n - U_{n-1}}{\tau_n}, \quad 1 \leq n \leq N.$$

Definition 3.14. *We define the second discrete derivative for the sequence $\{U_n\}_{n=0}^N$ as follow:*

$$\delta^2 U_n := \frac{\delta U_n - \delta U_{n-1}}{\tau_n}, \quad 1 \leq n \leq N$$

Using the Definitions 3.12 and 3.13, we can obtain the following result:

Proposition 3.15. *For all $t \in (t_{n-1}, t_n]$, we have $\delta U_n = U'(t)$.*

Proof. For all $t_j \in (t_{n-1}, t_n]$, it is verified:

$$\begin{aligned}
U'(t_j) &= \lim_{t\to t_j} \frac{U(t) - U(t_j)}{t - t_j}, \\
&= \lim_{t\to t_j} \frac{[(1-\lambda(t))U_n + \lambda(t)U_{n-1}] - [(1-\lambda(t_j))U_n + \lambda(t_j)U_{n-1}]}{t - t_j}, \\
&= \lim_{t\to t_j} \frac{U_n - \lambda(t)U_n + \lambda(t)U_{n-1} - U_n + \lambda(t_j)U_n - \lambda(t_j)U_{n-1}}{t - t_j}, \\
&= \lim_{t\to t_j} \frac{U_n[\lambda(t_j) - \lambda(t)] - U_{n-1}[\lambda(t_j) - \lambda(t)]}{t - t_j}, \\
&= \lim_{t\to t_j} \frac{(U_n - U_{n-1})(\lambda(t_j) - \lambda(t))}{t - t_j}, \\
&= -(U_n - U_{n-1} \lim_{t\to t_j} \frac{\lambda(t_j) - \lambda(t)}{t - t_j}, \\
&= -(U_n - U_{n-1})\lambda'(t_j), \\
&= \frac{U_n - U_{n-1}}{\tau_n}, \\
&= \delta U_n.
\end{aligned}$$

□

Remark. We have used the fact $\lambda'(t) = -\dfrac{1}{\tau_n}$ for all $t \in (t_{n-1}, t_n]$, Definitions 3.12 and 3.13.

Using Proposition 3.15, Definitions 3.12 and 3.13, we can rewrite the semi-discrete equation (3.7) more compactly. Indeed, consider the following definition:

Definition 3.16. *We define the discrete equation associated to FitzHugh-Nagumo evolution equation (3.2) by the expression:*

$$\begin{cases} U'(t) + \mathcal{F}(\overline{U}(t)) = \overline{G}(t), \quad t \in [t_{n-1}, t_n) \\ U(0) = U^0, \end{cases} \tag{3.8}$$

Remark. The existence of (strong/weak) solution for the continuous problem (corresponding to the evolution equation) guarantees the existence of solution for the discrete equation (3.8), which corresponds to the temporal discretization of equation (3.2) using the Backward-Euler scheme in time. Moreover, this solution is unique (guaranteed in [NSV00], page 549). Taking this into account, we can define the discrete problem as follows:

Definition 3.17. *The discrete problem consists in obtaining a discrete solution of the equation (3.8) associated to FitzHugh-Nagumo system.*

If we consider the definitions of the continuous problem and discrete problem, we can introduce an error estimate (see in Chapter 4, Section 4.1). This will allow us to demonstrate that the solution $U(t)$ of discrete problem (3.8) converges to weak solution $u(t)$ of continuous problem (3.2) when $\tau \to 0$ in $\|\|_{L^\infty(\Omega)}$ norm.

Now, we will focus on defining some concepts and properties associated with the discrete problem. These results are explicitly presented in the development of Theorem 4.3 (the cornerstone of this thesis).

Definition 3.18. *Let's consider the discrete solution $\{U_n\}_{n=0}^N$ of the discrete problem (3.8). We define the discrete estimators for all $1 \leq n \leq N$ as follow:*

$$\widetilde{\mathcal{D}_n} := \tau_n \left\langle \delta G_n - \delta^2 U_n, \delta U_n \right\rangle_{L^2(\Omega)\times L^2(\Omega)}.$$

Remark. Let us recall that $u'(0) = g(0) - \mathcal{F}(u(0))$. We are inquiring about the existence of a numerical expression for δU_0, because the definition of discrete derivative (see Definition 3.13) is only applicable for $1 \leq n \leq N$. In fact, drawing from Section 3 in [6] we can obtain:

$$\delta U_0 = g(0) - \mathcal{F}(u(0)).$$

Chapter 4

Error Estimates and Angle-Bounded Condition

The main purpose of this chapter is to introduce the error estimates between the weak solution $u(t)$ of the continuous problem (3.2) and the discrete solution $U(t)$ of the problem (3.8). The error estimates allows us to conclude that $U(t)$ converges to $u(t)$ in $L^{\infty}(\Omega)$ norm as $\tau \to 0$, where $\tau := \max_{1 \leq n \leq N} \tau_n$ and $\tau_n = t_n - t_{n-1}$ (associated to a partition $\mathcal{P}$ of $[0, T]$).

This result is the cornerstone of this thesis and will bring us very interesting consequences. Indeed:

- We can validate the Backward Euler numerical method when approximating the solutions of the continuous problem (which correspond to the evolution equation associated with the FitzHugh-Nagumo system). In this way, we will have all the advantages of working with a numerical problem (efficiency, flexibility, experimental validation).
- We will be able to give a specific variational structure to the FitzHugh-Nagumo equations, where our evolution operator $\mathcal{F}$ will play a very important role.

However, in order to establish the convergence of solutions from discrete to continuous time, it is necessary to introduce the concept of angle-bounded operators. The notion of angle-boundness for operators was originally introduced by Haim Brézis and Felix Browder in [3] and it plays a fundamental role in establishing a well-defined variational structure for the FitzHugh-Nagumo equations. Furthermore, we will provide an overview of the theory of maximal monotone operators.

4.1 Error Estimates

In Chapter 3, a continuous problem was introduced with the objective of obtaining a weak solution for the following evolution equation associated with the FitzHugh-Nagumo system (3.1):

$$\begin{cases} u'(t) + \mathcal{F}(u(t)) = g(t), \quad t \in (0,T) \\ \\ u(0) = u^0, \end{cases} \tag{4.1}$$

where $\mathcal{F} : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is the evolution operator associated to FitzHugh-Nagumo equations given by:

$$\mathcal{F}(u(t)) = \begin{pmatrix} -D\Delta u_1 + F'_{\text{ion}}(\overline{\phi} + u_1) + c_2 r + L u_1 \\ c_2 d r - c_2 u_1 \end{pmatrix}.$$

with $L > 0$ and $D(\mathcal{F}) = \left\{ \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} \in \mathcal{S}(\Omega) \times L^2(\Omega) \,:\, -\Delta u_1(t) \in L^2(\Omega) \right\}$.

Proposition 3.9 ensured the existence (and uniqueness) of weak solutions for (4.1). This motivated us to introduce a continuous problem, which involved obtaining the temporal discretization of the evolution equation (4.1) using the Backward-Euler scheme in time. By employing a brief theory of discrete derivatives, we obtained the expression:

$$\begin{cases} U'(t) + \mathcal{F}(\overline{U}(t)) = \overline{G}(t), \quad t \in [t_{n-1}, t_n) \\ \\ U(0) = U^0, \end{cases} \tag{4.2}$$

We will focus on estimating the difference between the solutions of the continuous and discrete problems. To carry out this task, we will define an estimation error. Consider the following definition:

Definition 4.1. *Let $u(t)$ be the weak solution of continuous problem (4.1) and $U(t)$ be the discrete solution of problem (4.2). We define the error estimates $E_{\mathcal{H}}$ as follow:*

$$E_{\mathcal{H}} := \max_{t \in [0,T]} |u(t) - U(t)|.$$

4.2 Main Result

We follow [6] in order to estimate the error estimates $E_{\mathcal{H}}$. The idea is to subtract the differential equations satisfied independently by the continuous-in-time solution and the discrete-in-time solution (that exists). The resulting equation for the error is then tested against the error itself, obtaining a differential inequality for the error which yields an uniform-in-time error estimate. More precisely, subtracting (4.1) from (4.2) we obtain:

$$(u(t) - U(t))' + \mathcal{F}(u(t)) - \mathcal{F}(\overline{U}(t)) = g(t) - \overline{G}(t).$$

Taking the inner product with $u(t) - U(t)$, we find:

$$\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 = \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle g(t) - \overline{G}(t), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)}.$$

Using now that:

$$u(t) - U(t) = (\lambda(t) - 1)(\overline{U}(t) - u(t)) + \lambda(t)(u(t) - \underline{U}(t)),$$

we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 &= (\lambda(t) - 1)\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t\overline{U}(t) - u(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle g(t) - \overline{G}(t), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} \\ &\quad + \lambda(t)\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)}\end{aligned}$$

If $\mathcal{F}$ is monotone (see Definition 4.13) which is a customary assumption in evolution problems, then the term with prefactor $(\lambda - 1)$ is negative and:

$$\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 \leq \lambda\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle g(t) - \overline{G}(t), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)}.$$

If $\mathcal{F}$ was the gradient of a convex potential Γ, then we could write:

$$\begin{aligned}\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} &= \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} \\ &\quad + \underbrace{\langle \mathcal{F}(\overline{U}(t)), u(t) - \overline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \Gamma(\overline{U}(t)) - \Gamma(u(t))}_{\leq 0} \\ &\quad + \underbrace{\langle \mathcal{F}(u(t)), \underline{U}(t) - u(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \Gamma(u(t)) - \Gamma(\underline{U}(t))}_{\leq 0} \\ &\quad + \underbrace{\langle \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \Gamma(\underline{U}(t)) - \Gamma(\overline{U}(t))}_{\leq 0}.\end{aligned}$$

This leads to the differential inequality:

$$\frac{1}{2}\frac{d}{dt}|u(t)-U(t)|^2 \leq \lambda\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle g(t) - \overline{G}(t), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)}.$$

Note that the difference between this and the differential inequality above is that $\mathcal{F}$ is being compared -in the sense of monotone operators- between only two functions $\overline{U}$ and $\underline{U}$, whereas in the former stage of the differential inequality three actors were in scene: $\overline{U}$, u, and $\underline{U}$.

From here it is possible to use Gronwall-type arguments (albeit more sophisticated) to obtain an a-posteriori error estimates. Now, what follows works all the same even if instead of the previous differential inequality we would have obtained:

$$\frac{1}{2}\frac{d}{dt}|u(t)-U(t)|^2 \leq \lambda\gamma^2\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle g(t) - \overline{G}(t), u(t) - U(t)\rangle_{L^2(\Omega)\times L^2(\Omega)},$$

for some γ (which can be greater than 1). This leads to the consider the following property (see Definition 4.2), which gives what seems to be the right intermediate class of evolution operators between monotone operators and those that are gradients of a convex potential.

For practical purposes, we will use $\mathcal{H}$ to denote a separable Hilbert space with scalar product $\langle , \rangle_{\mathcal{H}}$.

Definition 4.2. *An operator $\mathcal{G} : D(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$ is said to be angle-bounded (or more specifically: γ^2-angle-bounded) if there exists a positive constant γ such that:*

$$\langle \mathcal{F}(v) - \mathcal{F}(w), w - z\rangle_{\mathcal{H}} \leq \gamma^2 \langle \mathcal{F}(v) - \mathcal{F}(z), v - z\rangle_{\mathcal{H}},$$

for all $v, w, z \in D(\mathcal{G})$.

For the purpose of proving Theorem 4.3, we will decompose the function $g : (0,T) \to L^2(\Omega) \times L^2(\Omega)$ given by Definition 3.3. In fact:

$$g(t) = \begin{pmatrix} Lu_1 - \frac{d\overline{\phi}}{dt} + D\Delta\overline{\phi} \\ Lr + c_2\overline{\phi} \end{pmatrix} = L\begin{pmatrix} u_1 \\ r \end{pmatrix} + \begin{pmatrix} -\frac{d\overline{\phi}}{dt} + D\Delta\overline{\phi} \\ c_2\overline{\phi} \end{pmatrix} = Lu(t) + f(t),$$

where $f : (0,T) \to L^2(\Omega) \times L^2(\Omega)$ is given by $f(t) = \begin{pmatrix} -\frac{d\overline{\phi}}{dt} + D\Delta\overline{\phi} \\ c_2\overline{\phi} \end{pmatrix}$.

Under this consideration, the sequence $\{G_n\}_{n=0}^{N} \in D(\mathcal{F})$ that approximates $g(t)$ can be expressed as follows:

$$G_n(t) = \overline{G}(t) = L\overline{U}(t) + \overline{F}(t),$$

where $F_n(t) = \overline{F}(t)$ is a suitable approximation of $f(t)$. The following theorem is the cornerstone of this thesis:

Theorem 4.3. *Let $u(t)$ be the weak solution of (4.1) and $\{U_n\}_{n=0}^N$ be the solution of discrete problem (4.2), with $U_0 \in D(\mathcal{F})$ and $L\tau_n < 1$, for all $1 \leq n \leq N$. If $\mathcal{F}$ is maximal monotone and γ^2 - angle bounded operator, then:*

$$E_{\mathcal{H}} \leq e^{LT} \left[\left(|u^0 - U^0|^2 + \gamma^2 \sum_{n=1}^{N} \tau_n^2 \widetilde{\mathcal{D}_n} \right)^{\frac{1}{2}} + \frac{L}{2} \sum_{n=1}^{N} \tau_n^2 \, |\delta U_n| + \left\| f - \overline{F} \right\|_{L^1((0,T),\mathcal{H})} \right],$$

where $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$. If $f \in BV((0,T),\mathcal{H})$ and we fix $U_0 := u_0$, then we obtain:

$$E_{\mathcal{H}} \leq \tau e^{LT} \left(\widetilde{C} \, |\delta U_0| + (1 + \widetilde{C}\sqrt{2}) \, Var(f) \right),$$

where $\widetilde{C} := 2e^{\frac{L\tau(1+LT)}{1-L\tau}} \left(\gamma + \frac{LT}{2} \right)$ and $\widetilde{\mathcal{D}_n} := \tau_n \left\langle \delta G_n - \delta^2 U_n, \delta U_n \right\rangle_{\mathcal{H}}$, for all $1 \leq n \leq N$.

Unfortunately, we still lack the necessary tools to provide the proof of Theorem 4.3. It will be necessary to work on and develop the theory of angle-bounded operators. For this, consider the results of section 4.3.

4.3 Angle-Bounded Condition

Throughout this section, we will use $\mathcal{H}$ to denote a separable Hilbert space with scalar product $\langle , \rangle_{\mathcal{H}}$.

Recall that an operator $\mathcal{G} : D(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$ is said to be γ^2-angle-bounded, if exists a positive constant γ such that:

$$\langle \mathcal{F}(v) - \mathcal{F}(w), w - z \rangle_{\mathcal{H}} \leq \gamma^2 \, \langle \mathcal{F}(v) - \mathcal{F}(z), v - z \rangle_{\mathcal{H}},$$

for all $v, w, z \in D(\mathcal{G})$. The angle-bounded condition represents an intermediate class of evolution operators, bridging the gap between monotone operators and operators that are gradients of a convex potential.

One of the most important properties of the angle-bounded condition is additivity. In fact, the sum of two angle-bounded operators is itself angle-bounded. We will formalize this result in Proposition 4.4, which plays a key role in determining a specific variational structure for the FitzHugh-Nagumo equations.

Proposition 4.4. (Additivity) *Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 : D(\mathcal{G}) \to \mathcal{H}$ be operators such that $\mathcal{G}_1$ is γ_1^2-angle-bounded and $\mathcal{G}$ is γ_2^2-angle-bounded. Then $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 : D(\mathcal{G}) \to \mathcal{H}$ is γ^2-angle-bounded with $\gamma^2 = \max\{\gamma_1^2, \gamma_2^2\}$.*

Proof. By the algebraic properties of an operator, we know that:

$$\mathcal{G}(x) = (\mathcal{G}_1 + \mathcal{G}_2)(x) = \mathcal{G}_1(x) + \mathcal{G}_2(x),$$

for all $x \in D(\mathcal{G})$ (domain of operators $\mathcal{G}_1, \mathcal{G}_2$). Let's consider $v, w, z \in D(\mathcal{G})$. Using the properties of scalar product, sum of operators and the hypotheses that $\mathcal{G}_1$ and $\mathcal{G}_2$ are γ_1^2-angle-bounded and γ_2^2-angle-bounded respectively, we obtain:

$$\begin{aligned}
\langle \mathcal{G}(v) - \mathcal{G}(w), w - z\rangle_{\mathcal{H}} &= \langle (\mathcal{G}_1 + \mathcal{G}_2)(v) - (\mathcal{G}_1 + \mathcal{G}_2)(w), w - z\rangle_{\mathcal{H}}, \\
&= \langle \mathcal{G}_1(v) + \mathcal{G}_2(v) - \mathcal{G}_1(w) - \mathcal{G}_2(w), w - z\rangle_{\mathcal{H}}, \\
&= \langle (\mathcal{G}_1(v) - \mathcal{G}_2(w)) + (\mathcal{G}_1(v) - \mathcal{G}_2(w)), w - z\rangle_{\mathcal{H}}, \\
&= \langle \mathcal{G}_1(v) - \mathcal{G}_2(w), w - z\rangle + \langle \mathcal{G}_1(v) - \mathcal{G}_2(w), w - z\rangle_{\mathcal{H}}, \\
&\leq \gamma_1^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle + \gamma_2^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}}, \\
&\leq \gamma^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle + \gamma^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}},
\end{aligned}$$

where $\gamma^2 := \max\{\gamma_1^2, \gamma_2^2\}$. This implies:

$$\begin{aligned}
\gamma^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}} + \gamma^2 \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}} &= \gamma^2 \left[\langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}} + \langle \mathcal{G}_1(v) - \mathcal{G}_2(z), v - z\rangle_{\mathcal{H}}\right], \\
&= \gamma^2 \langle (\mathcal{G}_1(v) - \mathcal{G}_2(z)) + (\mathcal{G}_1(v) - \mathcal{G}_2(z)), v - z\rangle_{\mathcal{H}}, \\
&= \gamma^2 \langle (\mathcal{G}_1(v) + \mathcal{G}_2(v)) - (\mathcal{G}_1(z) + \mathcal{G}_2(z)), v - z\rangle_{\mathcal{H}}, \\
&= \gamma^2 \langle (\mathcal{G}_1 + \mathcal{G}_2)(v) - (\mathcal{G}_1 + \mathcal{G}_2(z)), v - z\rangle_{\mathcal{H}}, \\
&= \gamma^2 \langle \mathcal{G}(v) - \mathcal{G}(z), v - z\rangle_{\mathcal{H}}.
\end{aligned}$$

Therefore:

$$\langle \mathcal{G}(v) - \mathcal{G}(w), w - z\rangle_{\mathcal{H}} \leq \gamma^2 \langle \mathcal{G}(v) - \mathcal{G}(z), v - z\rangle_{\mathcal{H}}.$$

Using Definition 4.2, we can conclude that $\mathcal{G}$ is γ^2-angle-bounded.

□

Now, we will study the linear operators (defined below), since they are of great interest to us:

Definition 4.5. *An operator $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$ is lineal if it satisfies the following properties:*

$$\begin{cases} \mathcal{G}(v+w) & = \ \mathcal{G}(v) + \mathcal{G}(w), \\ \mathcal{G}(kv) & = \ k\mathcal{G}(v), \end{cases}$$

for all $v, w \in D(\mathcal{G})$ and for all $k \in \mathbb{R}$.

Let's remember that in Subsection 3.1.2 (Definition 3.5), we decomposed our evolution operator $\mathcal{F}$ as the sum of two operators: $\mathcal{F}_1$ and $\mathcal{F}_2$. Since the bilinear form $\mathfrak{a}(v, w)$ is positive, continuous, and weakly coercive; then it is satisfied:

$$\langle \mathcal{F}_1(w), v \rangle_{L^2(\Omega)\times L^2(\Omega)} = \mathfrak{a}(w, v), \tag{4.3}$$

for all $v \in \mathcal{S}(\Omega) \times L^2(\Omega)$. Now, let's note that our operator $\mathcal{F}_1$ has the following property: ¡it is linear! In fact, consider the following result:

Proposition 4.7. *The operator $\mathcal{F}_1 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ given by:*

$$\mathcal{F}_1(u(t)) = \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix},$$

is linear.

Proof. We know that the Laplacian is a linear operator. Let's consider:

$$v(t) = \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix}, \quad w(t) = \begin{pmatrix} \widehat{u_1}(t) \\ \widehat{r}(t) \end{pmatrix}.$$

Then:

$$\begin{aligned} \mathcal{F}_1(v(t) + w(t)) &= \begin{pmatrix} -D\Delta(u_1 + \widehat{u_1}) + c_2(r + \widehat{r}) \\ (c_2 d + L)(r + \widehat{r}) - c_2(u_1 + \widehat{u_1}) \end{pmatrix}, \\ &= \begin{pmatrix} -D\Delta u_1 - D\Delta\widehat{u_1} + c_2 r + c_2\widehat{r} \\ (c_2 d + L)r + (c_2 d + L)\widehat{r} - c_2 u_1 - c_2\widehat{u_1}) \end{pmatrix}, \\ &= \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix} + \begin{pmatrix} -D\Delta\widehat{u_1} + c_2\widehat{r} \\ (c_2 d + L)\widehat{r} - c_2\widehat{u} \end{pmatrix}, \\ &= \mathcal{F}_1(v(t)) + \mathcal{F}_2(w(t)). \end{aligned}$$

Finally, let $k \in \mathbb{R}$. Then:

$$\begin{aligned}
\mathcal{F}_1(kv(t)) &= \begin{pmatrix} -D\Delta k u_1 + c_2 k r \\ (c_2 d + L)kr - c_2 k u_1 \end{pmatrix}, \\
&= \begin{pmatrix} -kD\Delta u_1 + kc_2 r \\ k(c_2 d + L)r - kc_2 u_1 \end{pmatrix}, \\
&= k \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix}, \\
&= k\mathcal{F}_1(v(t)).
\end{aligned}$$

By Definition 4.5, we can conclude that $\mathcal{F}_1$ is a linear operator.

□

Remark. It is evident that the operator $\mathcal{F}_2$ is non-linear due to the presence of the ionic current $F'_{\text{ion}}(\phi)$, which is a third-degree polynomial.

The fact that $\mathcal{F}_1$ is a linear operator brings significant advantages when demonstrating that it is indeed an angle-bounded operator. Thanks to the theory of angle-bounded operators developed by Haim Brézis and Felix Browder in [3], we have the following result for the angle-bounded condition in linear operators:

Proposition 4.7. *Let $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$ be a linear operator. Then the following propositions are equivalent for all $v, w \in D(\mathcal{G})$:*

(a) *$\mathcal{G}$ is γ^2-angle-bounded.*

(d) $|\langle \mathcal{G}(v), w\rangle - \langle \mathcal{G}(w), v\rangle| \leq 2\mu\sqrt{\langle \mathcal{G}(v), v\rangle}\sqrt{\langle \mathcal{G}(w), w\rangle}$, *for a suitable $\mu \geq 0$.*

Proof. See [3], Proposition 1, Page 124.

Remark. From Proposition 4.7, $\gamma^2 = \dfrac{\mu^2 + 1}{4}$.

In virtue of (4.3), we can study the angle-bounded property for the operator $\mathcal{F}_1$ in terms of the bilinear form $\mathfrak{a}(v, w)$. Indeed, we have the following result for $\mathcal{F}_1$:

Corollary 4.8. *The operator $\mathcal{F}_1 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is γ^2 - angle bounded if only if:*

$$|\mathfrak{a}(v, w) - \mathfrak{a}(w, v)| \leq 2\mu\sqrt{\mathfrak{a}(v, v)}\sqrt{\mathfrak{a}(w, w)},$$

for all $v, w \in D(\mathcal{F})$ and suitable $\mu \geq 0$.

Proof. This is a immediately consequence of (4.3), because $\langle \mathcal{F}_1(u), v\rangle = \mathfrak{a}(v,w)$ for all $v, w \in D(\mathcal{F})$. Then, by Proposition 4.7 we obtain: $\mathcal{F}_1 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is γ_1^2 - angle bounded if only if:

$$|\mathfrak{a}(v,w) - \mathfrak{a}(w,v)| \leq 2\mu\sqrt{\mathfrak{a}(v,v)}\sqrt{\mathfrak{a}(w,w)},$$

for all $v, w \in D(\mathcal{F})$ and suitable $\mu \geq 0$.

□

Throughout this subsection, we have developed and refined the concept of angle-boundedness for operators. We have demonstrated some basic properties such as additivity and studied the behavior of this angle-bounded property in linear operators. Under these considerations (given the assumptions of Theorem 4.3) we are able to prove that our evolution operator $\mathcal{F}$ is a γ^2-angle bounded operator.

The key to this proof lies in decomposing $\mathcal{F}$ as the sum of two angle-bounded operators using Proposition 4.4. Indeed, it is always good to remember that thanks to Definition 3.5, we have the following decomposition for all $u \in D(\mathcal{F})$:

$$\mathcal{F}(u) = \mathcal{F}_1(u) + \mathcal{F}_2(u),$$

where:

$$\mathcal{F}_1(u(t)) = \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L)r - c_2 u_1 \end{pmatrix}, \quad \mathcal{F}_2(u(t)) = \begin{pmatrix} F'_{\text{ion}}(\overline{\phi} + u_1) + Lu_1 \\ 0 \end{pmatrix},$$

and $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 : D(\mathcal{F}) = \left\{ \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} \in \mathcal{S}(\Omega) \times L^2(\Omega) \,:\, -\Delta u_1(t) \in L^2(\Omega) \right\} \to L^2(\Omega) \times L^2(\Omega).$

Based in the decomposition for $\mathcal{F}$ as the sum of two operators: $\mathcal{F}_1, \mathcal{F}_2$ and in order to make use of Proposition 4.4, we will show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are γ_1^2 and γ_2^2 angle bounded (respectively).

Theorem 4.9. The operator $\mathcal{F}_1 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is γ_1^2 - angle bounded.

Proof. We are going to use the Corollary 4.8. and the identity in (4.3). In fact, we know that $\mathfrak{a}(v,w) = \langle \mathcal{F}_1(v), w\rangle$ for all $v, w \in D(\mathcal{F})$. In this manner, $\mathcal{F}_1$ is γ_1^2-angle-bounded if only if:

$$|\mathfrak{a}(v,w) - \mathfrak{a}(w,v)| \leq 2\mu\sqrt{\mathfrak{a}(v,v)}\sqrt{\mathfrak{a}(w,w)},$$

for all $v, w \in D(\mathcal{F})$ and suitable $\mu \geq 0$. Note that:

$$\mathfrak{a}(v,w) - \mathfrak{a}(w,v) = 2\mathfrak{a}_a(v,w),$$

where $\mathfrak{a}_a$ denotes the anti-symmetric part of the bilinear form $\mathfrak{a}$.

Let's consider $v, w \in D(\mathcal{F})$ such that:

$$v(t) = \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix}, \quad w(t) = \begin{pmatrix} \widehat{u_1}(t) \\ \widehat{r}(t) \end{pmatrix}.$$

In Chapter 3, Subsection 3.1.2, we have defined the bilinear form $\mathfrak{a}$ as follow:

$$\mathfrak{a}(v, w) := \int_\Omega \left[c_2(r\widehat{u_1} - u_1\widehat{r}) + (c_2 d + L)r\widehat{r} + D\nabla u_1 \cdot \nabla \widehat{u_1}\right].$$

Then:

$$\mathfrak{a}_a(v, w) = c_2 \int_\Omega (\widehat{u_1} r - u\widehat{r}).$$

Furthermore:

$$\mathfrak{a}(v, v) := c_2 d \int_\Omega r^2 + D \int |\nabla u_1|^2 = c_2 d \left\| r \right\|^2_{L^2(\Omega)} + D \left\| \nabla u_1 \right\|^2_{L_2(\Omega)}.$$

This implies:

$$\begin{aligned}
|\mathfrak{a}(v, w) - \mathfrak{a}(w, v)|^2 &= 4 \left|\mathfrak{a}_a(v, w)\right|^2, \\
&\leq 4c_2^2 \int_\Omega \left|\widehat{u_1}^2 r^2 + u_1^2 \widehat{r}^2\right| \, d\mu, \\
&\leq 4c_2^2 \int_\Omega \left(\left|\widehat{u_1}^2 r^2\right| + \left|u_1^2 \widehat{r}^2\right|\right) \, d\mu, \\
&= 4c_2^2 \left(\int_\Omega \left|\widehat{u_1}^2 r^2\right| \, d\mu + \int_\Omega \left|u_1^2 \widehat{r}^2\right| \, d\mu\right), \\
&\leq 4c_2 \left(\left\| \widehat{u_1} \right\|^2_{L^2(\Omega)} \left\| r \right\|^2_{L^2(\Omega)} + \left\| u_1 \right\|^2_{L^2(\Omega)} \left\| \widehat{r} \right\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

We have used the Cauchy-Schwarz inequality in the last step. Now, by the Poincare Inequality, exists constants $C_1, C_2 > 0$ such that (Theorem 2.4):

$$\begin{aligned}
\left\| \widehat{u_1} \right\|_{L^2(\Omega)} &\leq C_1 \left\| \nabla \widehat{u_1} \right\|_{L^2(\Omega)}, \\
\left\| u_1 \right\|_{L^2(\Omega)} &\leq C_2 \left\| \nabla u_1 \right\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
|\mathfrak{a}(v, w) - \mathfrak{a}(w, v)|^2 &\leq 4c_2^2 \left(C_1^2 \left\| \nabla \widehat{u_1} \right\|^2_{L^2(\Omega)} \left\| r \right\|^2_{L^2(\Omega)} + C_2^2 \left\| \nabla u_1 \right\|^2_{L^2(\Omega)} \left\| \widehat{r} \right\|^2_{L^2(\Omega)}\right), \\
&\leq 4c_2^2 \left(C_1^2 \left\| \nabla \widehat{u_1} \right\|^2_{L^2(\Omega)} + \left\| \widehat{r} \right\|^2_{L^2(\Omega)}\right) \left(C_2^2 \left\| \nabla u_1 \right\|^2_{L^2(\Omega)} + \left\| r \right\|^2_{L^2(\Omega)}\right), \\
&\leq 4c_2^2 \left(C \left\| \nabla \widehat{u_1} \right\|^2_{L^2(\Omega)} + \left\| \widehat{r} \right\|^2_{L^2(\Omega)}\right) \left(C \left\| \nabla u_1 \right\|^2_{L^2(\Omega)} + \left\| r \right\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Where $C = \max\{C_1^2, C_2^2\}$. Let denote $\beta = \max\{C, 1\}$. Then:

$$\begin{aligned}
|\mathfrak{a}(v,w) - \mathfrak{a}(w,v)|^2 &\le 4c_2^2\beta^2 \left(\|\nabla \widehat{u}_1\|_{L^2(\Omega)}^2 + \|\widehat{r}\|_{L^2(\Omega)}^2\right)\left(\|\nabla u_1\|_{L^2(\Omega)}^2 + \|r\|_{L^2(\Omega)}^2\right), \\
&\le 4\frac{c_2^2\beta^2}{\eta^2}\left(D\,\|\nabla \widehat{u}_1\|_{L^2(\Omega)}^2 + c_2 d\,\|\widehat{r}\|_{L^2(\Omega)}^2\right)\left(D\,\|\nabla u_1\|_{L^2(\Omega)}^2 + c_2 d\,\|r\|_{L^2(\Omega)}^2\right), \\
&\le 4\frac{c_2^2\beta^2}{\eta^2}\mathfrak{a}(v,v)\mathfrak{a}(w,w).
\end{aligned}$$

where $\eta = \min\{D, c_2 d\}$. This implies:

$$0 < |\mathfrak{a}(v,w) - \mathfrak{a}(w,v)|^2 \le 4\frac{c_2^2\beta^2}{\eta^2}\mathfrak{a}(v,v)\mathfrak{a}(w,w).$$

Taking square root in both sides of the last inequality, we obtain:

$$|\mathfrak{a}(v,w) - \mathfrak{a}(w,v)| \le 2\frac{c_2\beta}{\eta}\sqrt{\mathfrak{a}(v,v)}\sqrt{\mathfrak{a}(w,w)}.$$

Let's call $\mu = \dfrac{c_2\beta}{\eta} \ge 0$. In this manner:

$$|\mathfrak{a}(v,w) - \mathfrak{a}(w,v)| \le 2\mu\sqrt{\mathfrak{a}(v,v)}\sqrt{\mathfrak{a}(w,w)}.$$

By Corollary 4.8, we can conclude that $\mathcal{F}_1$ is a γ_1^2-angle-bounded with $\gamma_1^2 = \dfrac{\mu^2+1}{4}$, where $\mu = \dfrac{c_2\beta}{\eta}$.

□

We will focus on working with the angle-bounded condition for our operator $\mathcal{F}_2$. Indeed, $\mathcal{F}_2$ includes the ionic current (a nonlinear term) and a perturbation Lu_1 with $L > 0$. It is the moment that we can justify the reason for adding Lu_1 in equation (3.1). And the reason lies in the fact that the following functional is convex with $L > 0$:

$$I[\phi] := \int_\Omega F_{\text{ion}}(\phi) + \frac{L\phi^2}{2}.$$

And the gradient of a convex potential is always γ^2-angle bounded with $\gamma = 1$. Indeed:

Theorem 4.10. The operator $\mathcal{F}_2 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is γ_2^2 - angle bounded.

Proof. Remember that $\mathcal{F}_2 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is the operator given by:

$$\mathcal{F}_2(u(t)) = \begin{pmatrix} F'_{\text{ion}}(\overline{\phi} + u_1) + Lu_1 \\ 0 \end{pmatrix}.$$

Let $L > 0$ and because $F'_{\text{ion}}(\overline{\phi} + u_1)$ is the derivative of four-degree polynomial in the undetermined u_1 (variable of polynomial), we obtain that $\mathcal{F}_2(u(t))$ is the gradient of a convex potential. In fact, exists a convex potential $\Gamma : D(\mathcal{F}) \subset L^2(\Omega) \times L^2(\Omega) \to (-\infty, +\infty)$ such that $\mathcal{F}_2 = \nabla\Gamma$. This implies for all $v, w, z \in D(\mathcal{F})$:

$$\begin{aligned}\langle \mathcal{F}_2(v) - \mathcal{F}_2(w), w - z\rangle_{L^2(\Omega)} &= \langle \mathcal{F}_2(v) - \mathcal{F}_2(z), v - z\rangle_{L^2(\Omega)} \\ &\quad + \underbrace{\langle \mathcal{F}_2(v), w - v\rangle_{L^2(\Omega)} + \Gamma(v) - \Gamma(w)}_{\leq 0} \\ &\quad + \underbrace{\langle \mathcal{F}_2(w), z - w\rangle_{L^2(\Omega)} + \Gamma(w) - \Gamma(z)}_{\leq 0} \\ &\quad + \underbrace{\langle \mathcal{F}_2(z), v - z\rangle_{L^2(\Omega)} + \Gamma(z) - \Gamma(v)}_{\leq 0}, \\ &\leq \langle \mathcal{F}_2(v) - \mathcal{F}_2(z), v - z\rangle_{L^2(\Omega)}.\end{aligned}$$

We need to justify some calculations in our development. Because Γ is a convex function and satisfied $\mathcal{F}_2 = \nabla\Gamma$, we obtain for all $v, w, z \in D(\mathcal{F})$:

$$\begin{aligned}\Gamma(w) \geq \Gamma(v) + \langle \nabla\Gamma(v), w - v\rangle_{L^2(\Omega)} &\Rightarrow 0 \geq \Gamma(v) - \Gamma(w) + \langle \mathcal{F}_2(v), w - v\rangle_{L^2(\Omega)}, \\ \Gamma(z) \geq \Gamma(w) + \langle \nabla\Gamma(w), z - w\rangle_{L^2(\Omega)} &\Rightarrow 0 \geq \Gamma(w) - \Gamma(z) + \langle \mathcal{F}_2(w), z - w\rangle_{L^2(\Omega)}, \\ \Gamma(v) \geq \Gamma(z) + \langle \nabla\Gamma(z), v - z\rangle_{L^2(\Omega)} &\Rightarrow 0 \geq \Gamma(z) - \Gamma(v) + \langle \mathcal{F}_2(z), v - z\rangle_{L^2(\Omega)}.\end{aligned}$$

In this manner, we can conclude that $\mathcal{F}_2$ is γ_2^2-angle-bounded with $\gamma_2 = 1$.

□

In virtue of the additivity of angle-bounded operators and the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ are γ_1^2 and γ_2^2 angle-bounded operators (respectively), we obtain the following key result of this thesis as an immediate consequence:

Corollary 4.11. The evolution operator $\mathcal{F} : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ associated to FitzHugh-Nagumo equations (3.1) is γ^2 - angle bounded operator.

Proof. By the Theorems 4.9 and 4.10, we know that $\mathcal{F}_1$ and $\mathcal{F}_2$ are γ_1^2 and γ_2^2 - angle bounded operators (respectively). Because $\mathcal{F}(u) = \mathcal{F}_1(u(t)) + \mathcal{F}_2(u(t))$ for all $u(t) \in D(\mathcal{F})$ and using the Proposition 4.4 (sum of two angle-bounded operators is angle-bounded), we can conclude that the operator $\mathcal{F}$ is γ^2 - angle bounded with:

$$\gamma^2 = \max\{\gamma_1^2, \gamma_2^2\} = \max\left\{\frac{\mu^2 + 1}{4}, 1\right\}$$

□

4.4 Maximal Monotone Operators

In this section, we will present a concise overview of maximal monotone operators, as they serve as a important tool for demonstrating the convergence from continuous time to discrete time. However, we must emphasize that we will not delve into the details of this theory. For a more in-depth understanding of maximal monotone operators, we recommend referring to [2], where you can find comprehensive information.

Throughout this section, we will denote by $\mathcal{H}$ a Hilbert space with inner product $\langle , \rangle_{\mathcal{H}}$ and $\mathcal{G} : D(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$ as an operator with a proper domain $D(\mathcal{G})$.

Definition 4.12. *Consider the operator $\mathcal{G} : D(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$. We will say that $\mathcal{G}$ is monotone if:*

$$\langle \mathcal{G}(v) - \mathcal{G}(w), v - w \rangle_{\mathcal{H}} \geq 0, \quad \forall\ v, w \in D(\mathcal{G}).$$

An immediate consequence of the definition of angle-bounded operator is related to monotonicity. Indeed, every angle-bounded operator is monotone.

Proposition 4.13. *If $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$ satisfies the angle-bounded condition, then $\mathcal{G}$ is monotone.*

Proof. Taking $w = z$ in Definition 4.2 and using the properties of scalar product, we will obtain:

$$0 = \langle \mathcal{G}(v) - \mathcal{G}(z), 0 \rangle_{\mathcal{H}} \leq \gamma^2 \langle \mathcal{G}(v) - \mathcal{G}(z), v - z \rangle_{\mathcal{H}}.$$

However $\gamma > 0$. This implies $0 \leq \langle \mathcal{G}(v) - \mathcal{G}(z), v - z \rangle_{\mathcal{H}}$. Using the Definition 4.12, we can conclude that $\mathcal{G}$ is monotone.

□

Another immediate consequence of Definition 4.12 is the additivity property for monotone operators:

Proposition 4.14. (Additivity) *Let consider the operators $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 : D(\mathcal{G}) \subset \mathcal{H} \to \mathcal{H}$ with proper domain $D(\mathcal{G})$ such that $\mathcal{G}(u) = \mathcal{G}_1(u) + \mathcal{G}_2(u)$, for all $u \in D(\mathcal{G})$. If $\mathcal{G}_1, \mathcal{G}_2$ are monotone then $\mathcal{G}$ is monotone.*

Proof. Because $\mathcal{G}_1, \mathcal{G}_2$ are monotone, we know that $\forall\ v, w \in D(\mathcal{G})$:

$$\begin{aligned} \langle \mathcal{G}_1(v) - \mathcal{G}_1(w), v - w \rangle_{\mathcal{H}} &\geq 0, \\ \langle \mathcal{G}_2(v) - \mathcal{G}_2(w), v - w \rangle_{\mathcal{H}} &\geq 0. \end{aligned}$$

Using the properties of inner product and sum of operators, we obtain:

$$\begin{aligned} \langle \mathcal{G}_1(v) - \mathcal{G}_1(w), v - w \rangle_{\mathcal{H}} + \langle \mathcal{G}_2(v) - \mathcal{G}_2(w), v - w \rangle_{\mathcal{H}} &= \langle \mathcal{G}_1(v) - \mathcal{G}_1(w) + \mathcal{G}_1(v) - \mathcal{G}_1(w), v - w \rangle_{\mathcal{H}}, \\ &= \langle (\mathcal{G}_1(v) + \mathcal{G}_2(v)) - (\mathcal{G}_1(w) + \mathcal{G}_2(w)), v - w \rangle_{\mathcal{H}}, \\ &= \langle (\mathcal{G}_1 + \mathcal{G}_2)(v) - (\mathcal{G}_1 + \mathcal{G}_2)(w), v - w \rangle_{\mathcal{H}}, \\ &= \langle \mathcal{G}(v) - \mathcal{G}(w), v - w \rangle_{\mathcal{H}}. \end{aligned}$$

But $\langle \mathcal{G}_1(v) - \mathcal{G}_1(w), v - w \rangle_{\mathcal{H}} + \langle \mathcal{G}_2(v) - \mathcal{G}_2(w), v - w \rangle_{\mathcal{H}} \geq 0$. This allow us to conclude that $\langle \mathcal{G}(v) - \mathcal{G}(w), v - w \rangle_{\mathcal{H}} \geq 0$. Thus, we conclude that $\mathcal{G}$ is monotone.

□

Definition 4.15. *Let's consider $w \in \mathcal{H}$ and an operator $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$. We will said that $\mathcal{G}$ is maximal if the equation $w + \varepsilon \mathcal{G}(w) = v$ admits a unique solution for all $\varepsilon > 0$ and for all $v \in \mathcal{H}$.*

Taking into account Definitions 4.12 and 4.15, we can finally introduce the concept of a maximal monotone operator.

Definition 4.16. *An operator $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$ will be called maximal monotone if it is both maximal (in the sense of Definition 4.12) and monotone (in the sense of Definition 4.15) simultaneously.*

Now, we will focus on working with the two aforementioned properties for our operator $\mathcal{F}$ (from Definition 3.2) associated with the FitzHugh-Nagumo equations. Indeed, we will demonstrate that $\mathcal{F}$ is a maximal monotone operator. But first, we need to introduce a key preliminary result.

Theorem 4.17. *Let $\mathcal{G} : D(\mathcal{G}) \to \mathcal{H}$ be a maximal monotone operator and $\Lambda : D(\mathcal{G}) \to (-\infty, +\infty)$ be a convex and lower semi-continuous functional over $D(\mathcal{G})$. Then $\mathcal{G} + \nabla\Lambda$ is maximal monotone.*

Proof. See [2], Proposition 2.17, Page 48.

Remark. Proposition 2.17 of [2] states that the last result (Theorem 4.17) is valid for the sum of a maximal monotone operator and the subdifferential of a convex and lower semicontinuous functional. Since the gradient operator satisfies the properties of subdifferentials (as a particular case), Proposition 2.17 also applies to the sum of a maximal monotone operator and the gradient of a convex and lower semicontinuous functional (as stated in Theorem 4.17).

Theorem 4.18. *The operator $\mathcal{F} : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ associated to FitzHugh-Nagumo (given in Definition 3.2) is maximal monotone.*

Proof. According to Definition 3.5, we know that our evolution operator $\mathcal{F}$ can be decomposed as:

$$\mathcal{F}(u(t)) = \mathcal{F}_1(u(t)) + \mathcal{F}_2(u(t)),$$

for all $u(t) \in D(\mathcal{F})$. We know that $\mathcal{F}_1(u(t))$ is a linear operator and $\mathcal{F}_2(u(t))$ is the gradient of a convex potential (i.e $\mathcal{F}_2 = \nabla\Gamma$). In order to use the Theorem 4.17, we will demonstrate the following:

- $\mathcal{F}_1$ is maximal monotone.
- Γ is a convex and lower semi-continuous functional.

The monotonicity of operator $\mathcal{F}_1$ can be easily deduced from the angle-bounded condition. Indeed, by Theorem 4.9, we obtain that $\mathcal{F}_1$ is angle-bounded. According to Proposition 4.13, we know that every angle-bounded operator is monotone.

For ease of notation, we will consider $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$. Now we will demonstrate that $\mathcal{F}_1$ is maximal. According to Definition 4.15, we need to prove that for any $\varepsilon > 0$ and any $v \in L^2(\Omega) \times L^2(\Omega)$, the equation $w + \epsilon\mathcal{F}_1(w) = v$ admits a unique solution $w \in D(\mathcal{F})$. Indeed, for $z \in \mathcal{S}(\Omega) \times L^2(\Omega)$:

$$\langle w + \epsilon\mathcal{F}_1(w), z\rangle_{\mathcal{H}} = \langle v, z\rangle_{\mathcal{H}}.$$

In virtue of Theorem 2.9 (Hahn-Banach), $\langle \epsilon\mathcal{F}_1(w), z\rangle_{\mathcal{H}}$ becomes a bilinear form over $\mathcal{S}(\Omega) \times L^2(\Omega)$. This is because in $\mathcal{S}(\Omega) \times L^2(\Omega)$, we can define a linear mapping:

$$z \in \mathcal{S}(\Omega) \times L^2(\Omega) \mapsto \mathfrak{c}(v, z).$$

In this manner, $\mathfrak{c}(v, z) = \displaystyle\int \nabla v \cdot \nabla z = \langle \mathcal{F}_1(v), z\rangle_{\mathcal{H}}$. Using the inner product properties, we obtain:

$$\langle w + \epsilon\mathcal{F}_1(w), z\rangle_{\mathcal{H}} = \langle w, z\rangle_{\mathcal{H}} + \epsilon \langle \mathcal{F}_1, z\rangle_{\mathcal{H}}.$$

This implies (using the Cauchy-Schwarz inequality):

$$\begin{aligned}
|\langle w, z\rangle_{\mathcal{H}} + \epsilon \langle \mathcal{F}_1, z\rangle_{\mathcal{H}}| &\leq \|w\|_{\mathcal{H}} \|z\|_{\mathcal{H}} + \varepsilon \|\nabla w\|_{\mathcal{H}} \|\nabla z\|_{\mathcal{H}}, \\
&\leq C_P \|w\|_{\mathcal{H}} \|\nabla z\|_{\mathcal{H}} + \varepsilon \|\nabla w\|_{\mathcal{H}} \|\nabla z\|_{\mathcal{H}}, \\
&= (C_P \|w\|_{\mathcal{H}} + \varepsilon \|\nabla w\|_{\mathcal{H}}) \|\nabla z\|_{\mathcal{H}}, \\
&= (C_P \|w\|_{\mathcal{H}} + \varepsilon \|\nabla w\|_{\mathcal{H}}) \|z\|_{\mathcal{S}(\Omega)\times L^2(\Omega)},
\end{aligned}$$

where C_P is the Poincare constant (see Theorem 2.4). This allows us to conclude that the map:

$$z \mapsto \langle w, z\rangle_{\mathcal{H}} + \epsilon \langle \mathcal{F}_1, z\rangle_{\mathcal{H}},$$

is a element of the dual space of $\mathcal{S}(\Omega) \times L^2(\Omega)$. By the Theorem 2.10 (Riesz), we can ensure the existence and uniqueness of a function $v \in \mathcal{S}(\Omega) \times L^2(\Omega))' = \mathcal{S}(\Omega) \times L^2(\Omega)$ such that:

$$\langle w + \epsilon\mathcal{F}_1(w), z\rangle_{\mathcal{H}} = \langle v, z\rangle_{\mathcal{H}}.$$

Therefore, the equation $w + \epsilon \mathcal{F}_1(w) = v$ admits a unique solution $w \in D(\mathcal{F})$. Thus, $\mathcal{F}_1$ is a maximal operator. In the other side, we know that $\mathcal{F}_2(u(t))$ is the gradient of a convex potential. In this manner, there exists a convex functional $\Gamma : D(\mathcal{F}) \to (-\infty, \infty)$ such that $\nabla\Gamma = \mathcal{F}_2$.

We know that Γ is convex. We need to verify lower semi-continuous. For this, we will use Definition 2.7 and demonstrate that the set $\mathcal{D} := \{u \in D(\mathcal{F}) : \Gamma(u) \leq a\}$ is closed for all $a \in \mathbb{R}$.

Indeed, the ionic current is given by $F'_{\text{ion}}(u_1 + \overline{\phi}) = -c_1(u_1 + \overline{\phi})(u_1 + \overline{\phi} - \alpha)(1 - u_1 - \overline{\phi})$. This allows us to define $\Gamma : D(\mathcal{F}) \to (-\infty, \infty)$ as follow:

$$\Gamma(u) := \frac{c_1}{4}(u_1 + \overline{\phi})^4 - \frac{c_1(\alpha + 1)}{3}(u_1 + \overline{\phi})^3 + \frac{(c_1\alpha + L)}{2}(u_1 + \overline{\phi})^2 + K,$$

where $u = u(t) = \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix}$ and $K \in \mathbb{R}$. Consider a sequence $\{g^n\}_{n\in\mathbb{N}} \in \mathcal{D}$ such that:

$$\lim_{n\to+\infty} g^n = \lim_{n\to+\infty} \begin{pmatrix} u_1^n \\ r^n \end{pmatrix} = \begin{pmatrix} \lim\limits_{n\to+\infty} u_1^n \\ \lim\limits_{n\to+\infty} r^n \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = g.$$

We need to prove that $g \in \mathcal{D}$. Because $\{g^n\}_{n\in\mathbb{N}} \in \mathcal{D}$, we obtain for all $a \in \mathbb{R}$:

$$\frac{c_1}{4}(u_1^n + \overline{\phi})^4 - \frac{c_1(\alpha + 1)}{3}(u_1^n + \overline{\phi})^3 + \frac{(c_1\alpha + L)}{2}(u_1^n + \overline{\phi})^2 + K \leq a.$$

Using the continuity of power and polynomial functions (when we take the limit):

$$\lim_{n\to+\infty}\left[\frac{c_1}{4}(u_1^n + \overline{\phi})^4 - \frac{c_1(\alpha + 1)}{3}(u_1^n + \overline{\phi})^3 + \frac{(c_1\alpha + L)}{2}(u_1^n + \overline{\phi})^2 + K\right] \leq a,$$
$$\left[\frac{c_1}{4}\left(\lim_{n\to+\infty} u_1^n + \overline{\phi}\right)^4 - \frac{c_1\alpha + 1}{3}\left(\lim_{n\to+\infty} u_1^n + \overline{\phi}\right)^3 + \frac{(c_1\alpha + L)}{2}\left(\lim_{n\to+\infty} u_1^n + \overline{\phi}\right)^2 + K\right] \leq a,$$
$$\left[\frac{c_1}{4}(L_1 + \overline{\phi})^4 - \frac{c_1(\alpha + 1)}{3}(L_1 + \overline{\phi})^3 + \frac{(c_1\alpha + L)}{2}(L_1 + \overline{\phi})^2 + K\right] \leq a.$$

The last inequality implies that $g = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \in \mathcal{D}$. Therefore, $\mathcal{D}$ is closed. In this way, Γ is lower semi-continuous in $D(\mathcal{F})$. Using Theorem 4.17, we can conclude that:

$$\mathcal{F}_1 + \nabla\Gamma = \mathcal{F}_1 + \mathcal{F}_2 = \mathcal{F},$$

is maximal monotone.

$\square$

Remark. Because $\mathcal{F}$ is maximal monotone, we can conclude that for every $w \in D(\mathcal{F})$, the (nonempty) set $\mathcal{F}(w)$ is closed and convex set in $L^2(\Omega) \times L^2(\Omega)$.

4.5 Proof of Theorem 4.3.

Finally, we are ready to prove Theorem 4.3, which is the cornerstone of this thesis due to the strong conclusions that can be drawn from this result. To do this, we need to consider the discrete and continuous problems, existence of solutions (for these problems), the error estimates and the conditions satisfied by our evolution operator: $\mathcal{F}$ is maximal monotone and γ^2 angle-bounded.

Indeed, let us recall what Theorem 4.3 states:

Let $u(t)$ be the weak solution of (4.1) and $\{U_n\}_{n=0}^N$ be the solution of discrete problem (4.2), with $U_0 \in D(\mathcal{F})$ and $L\tau_n < 1$, for all $1 \leq n \leq N$. If $\mathcal{F}$ is maximal monotone and γ^2 - angle bounded operator, then:

$$E_{\mathcal{H}} \leq e^{LT}\left[\left(\left|u^0 - U^0\right|^2 + \gamma^2 \sum_{n=1}^{N} \tau_n^2 \widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} + \frac{L}{2}\sum_{n=1}^{N} \tau_n^2 \left|\delta U_n\right| + \left\|f - \overline{F}\right\|_{L^1((0,T),\mathcal{H})}\right],$$

where $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$. If $f \in BV((0,T),\mathcal{H})$ and we fix $U_0 := u_0$, then we obtain:

$$E_{\mathcal{H}} \leq \tau e^{LT}\left(\widetilde{C}\left|\delta U_0\right| + (1 + \widetilde{C}\sqrt{2})\, Var(f)\right),$$

where $\widetilde{C} := 2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma + \frac{LT}{2}\right)$ and $\widetilde{\mathcal{D}_n} := \tau_n \left\langle \delta G_n - \delta^2 U_n, \delta U_n \right\rangle_{\mathcal{H}}$, for all $1 \leq n \leq N$.

Proof of Theorem 4.3. In the entire proof of the theorem, we will consider $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$. It is important to emphasize that the condition $L\tau_n < 1$, for every $1 \leq n \leq N$, is imposed for the Backward-Euler method to be robust/efficient. Taking into account the continuous problem (4.1), by Proposition 3.9 we know that the evolution equation associated to the FitzHugh-Nagumo system (3.1) has a unique weak solution $u(t)$. Because $U(t)$ (the piecewise linear interpolation, see Definition 3.12) is a suitable approximation of $u(t_n)$ in $[t_{n-1}, t_n)$, we can ensure the existence of solutions for the discrete problem (4.2). Subtracting the continuous-in-time solution $u(t)$ and the discrete-in-time solution $U(t)$, we obtain the following expression:

$$(u(t) - U(t))' = g(t) - \overline{G}(t) + \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)). \tag{4.4}$$

Taking the inner product in (4.4) with $u(t) - U(t)$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 &= \langle (u(t) - U(t))', u(t) - U(t) \rangle_{\mathcal{H}}\,, \\ &= \langle g(t) - \overline{G}(t) + F(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t) \rangle_{\mathcal{H}}, \\ &= \langle g(t) - \overline{G}(t), u(t) - U(t) \rangle_{\mathcal{H}} + \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t) \rangle_{\mathcal{H}},\end{aligned}$$

by the inner product properties. We will work with the inequality:

$$\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 \leq \langle g(t) - \overline{G}(t), u(t) - U(t) \rangle_{\mathcal{H}} + \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t) \rangle_{\mathcal{H}}. \tag{4.5}$$

Therefore, our mission will be to bound the expressions:

$$\langle g(t)-\overline{G}(t), u(t)-U(t)\rangle_{\mathcal{H}} \text{ and } \langle \mathcal{F}(\overline{U}(t))-\mathcal{F}(u(t)), u(t)-U(t)\rangle_{\mathcal{H}}.$$

We will begin with $\langle g(t)-\overline{G}(t), u(t)-U(t)\rangle_{\mathcal{H}}$. Given that $\overline{\phi}$ is fixed condition (that varies with respect to time), it follows that:

$$g(t)-\overline{G}(t) = L(u(t)-\overline{U}(t)) + f(t) - \overline{F}(t).$$

Therefore, by the Cauchy-Schwarz inequality and the inner product properties:

$$\begin{aligned}
\langle g(t)-\overline{G}(t), u(t)-U(t)\rangle_{\mathcal{H}} &= \langle L(u(t)-\overline{U}(t)) + (f(t)-\overline{F}(t)), u(t)-U(t)\rangle_{\mathcal{H}}, \\
&= L\langle u(t)-\overline{U}(t), u(t)-U(t)\rangle_{\mathcal{H}} + \langle f(t)-\overline{F}(t), u(t)-U(t)\rangle_{\mathcal{H}}, \\
&\leq L\langle (u(t)-U(t)) + (U(t)-\overline{U}(t)), u(t)-U(t)\rangle_{\mathcal{H}} \\
&+ \left|f(t)-\overline{F}(t)\right| \left|u(t)-U(t)\right|, \\
&= L\langle u(t)-U(t), u(t)-U(t)\rangle_{\mathcal{H}} + L\left\langle U(t)-\overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}} \\
&+ \left|f(t)-\overline{F}(t)\right| \left|u(t)-U(t)\right|, \\
&= L\left|u(t)-U(t)\right|^2 + L\left\langle U(t)-\overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}} \\
&+ \left|f(t)-\overline{F}(t)\right| \left|u(t)-U(t)\right|.
\end{aligned}$$

Next, we are going to work with $\left\langle U(t)-\overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}}$. Remember that in Chapter 3 (Definition 3.12), we define the piecewise linear interpolation:

$$U(t) = (1-\lambda(t))U_n + \lambda(t)U_{n-1} = (1-\lambda(t))\overline{U}(t) + \lambda\underline{U}(t),$$

where:

$$\lambda(t) := \frac{t_n - t}{\tau_n}, \quad \forall\, 1 \leq n \leq N.$$

This implies:

$$\begin{aligned}
\left\langle U(t)-\overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}} &= \left\langle (1-\lambda(t))\overline{U}(t) + \lambda\underline{U}(t) - \overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}}, \\
&= \left\langle \lambda\underline{U}(t) - \lambda\overline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}}, \\
&= -\lambda\left\langle \overline{U}(t) - \underline{U}(t), u(t)-U(t)\right\rangle_{\mathcal{H}}, \\
&= -\lambda\left\langle \tau_n\left(\frac{\overline{U}(t)-\underline{U}(t)}{\tau_n}\right), u(t)-U(t)\right\rangle_{\mathcal{H}}, \\
&= -\lambda\tau_n\left\langle U'(t), u(t)-U(t)\right\rangle_{\mathcal{H}}, \\
&\leq -\lambda\tau_n\left|U'(t)\right|\left|u(t)-U(t)\right|, \\
&\leq \lambda\tau_n\left|U'(t)\right|\left|u(t)-U(t)\right|.
\end{aligned}$$

We have used the Cauchy-Schwarz inequality to bounded the inner product $\langle U'(t), u(t)-U(t)\rangle_{\mathcal{H}}$.

Furthermore, in Chapter 3 (Definition 3.13 and Proposition 3.15), we define for all $t \in [t_{n-1}, t_n)$:

$$U'(t) = \frac{\overline{U}(t) - \underline{U}(t)}{\tau_n}.$$

Therefore:

$$\langle g(t) - \overline{G}(t), u(t) - U(t) \rangle_{\mathcal{H}} = L\,|u(t) - U(t)|^2 + L\lambda\tau_n\,|U'(t)|\,|u(t) - U(t)| + \left|f(t) - \overline{F}(t)\right|\,|u(t) - U(t)|\,.$$

Now we are going to estimate $\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t) \rangle_{\mathcal{H}}$. In fact:

$$u(t) - U(t) = (1 - \lambda(t))(u(t) - \overline{U}(t)) + \lambda(t)(u(t) - \underline{U}(t)).$$

Therefore:

$$\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - U(t) \rangle_{\mathcal{H}} = \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), (1 - \lambda(t))(u(t) - \overline{U}(t)) + \lambda(t)(u(t) - \underline{U}(t)) \rangle_{\mathcal{H}}.$$

By the inner product properties, we know that:

$$\begin{aligned}
\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), (1 - \lambda(t))(u(t) - \overline{U}(t)) \rangle_{\mathcal{H}} &= (1 - \lambda(t))\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \overline{U}(t) \rangle_{\mathcal{H}}, \\
\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), \lambda(t)(u(t) - \underline{U}(t)) \rangle_{\mathcal{H}} &= \lambda(t)\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t) \rangle_{\mathcal{H}}.
\end{aligned}$$

Note that:

$$(1 - \lambda(t))\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \overline{U}(t) \rangle_{\mathcal{H}} = (\lambda(t) - 1)\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), \overline{U}(t) - u(t) \rangle_{\mathcal{H}}.$$

Because $\mathcal{F}$ is a maximal monotone operator (Theorem 4.18), we know that $\mathcal{F}$ is not empty and $\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), \overline{U}(t) - u(t) \rangle_{\mathcal{H}} \geq 0$. But $\lambda(t) - 1 \leq 0$. This implies:

$$(1 - \lambda(t))\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \overline{U}(t) \rangle_{\mathcal{H}} \leq 0.$$

Since $\mathcal{F}$ is γ^2 - angle bounded (Corollary 4.11), we can estimate $\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t) \rangle_{\mathcal{H}}$. Note that we have three actors in scene: $\overline{U}(t), u(t)$ and $\underline{U}(t)$. Then, by angle-bounded condition:

$$\lambda(t)\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t) \rangle_{\mathcal{H}} \leq \gamma^2\lambda\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t) \rangle_{\mathcal{H}}.$$

Next, our objective is to express $\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t) \rangle_{\mathcal{H}}$ in terms of discrete estimators $\widetilde{\mathcal{D}_n}$. Because $U'(t) = \delta U_n$ (Proposition 3.15, for $t \in [t_{n-1}, t_n)$) and using the equation (4.2), we obtain:

$$\begin{aligned}
\mathcal{F}(\overline{U}(t)) &= L\overline{G}(t) - \delta U_n = LG_n(t) - \delta U_n, \\
\mathcal{F}(\underline{U}(t)) &= L\underline{G}(t) - \delta U_{n-1} = LG_{n-1}(t) - \delta U_{n-1}.
\end{aligned}$$

Therefore:

$$\mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)) = L(U_n(t) - U_{n-1}(t)) + F_n(t) - F_{n-1}(t) - \delta U_n + \delta U_{n-1}.$$

This implies:

$$\begin{aligned}\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{\mathcal{H}} &= \langle L(U_n(t) - U_{n-1}(t)) + (F_n(t) - F_{n-1}(t)) - (\delta U_n - \delta U_{n-1}), \overline{U}(t) - \underline{U}(t)\rangle_{\mathcal{H}},\\ &= \langle L\tau_n \delta U_n + \tau_n \delta F_n - \tau_n \delta^2 U_n, \tau_n \delta U_n\rangle_{\mathcal{H}},\\ &= \tau_n^2 \langle (L\delta U_n + \delta F_n) - \delta^2 U_n, \delta U_n\rangle_{\mathcal{H}},\\ &= \tau_n^2 \langle \delta G_n - \delta^2 U_n, \delta U_n\rangle_{\mathcal{H}},\\ &= \tau_n \widetilde{\mathcal{D}_n}.\end{aligned}$$

In this way:

$$\begin{aligned}\langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(u(t)), u(t) - \underline{U}(t)\rangle_{\mathcal{H}} &\leq \gamma^2 \lambda \langle \mathcal{F}(\overline{U}(t)) - \mathcal{F}(\underline{U}(t)), \overline{U}(t) - \underline{U}(t)\rangle_{\mathcal{H}},\\ &\leq \gamma^2 \lambda \tau_n \widetilde{\mathcal{D}_n}.\end{aligned}$$

Finally, the inequality (4.5) can be expressed as follow:

$$\frac{1}{2}\frac{d}{dt}|u(t) - U(t)|^2 \leq L\,|u(t) - U(t)|^2 + \left[L\lambda\tau_n\,|U'(t)| + \left|f(t) - \overline{F}(t)\right|\right] |u(t) - U(t)| + \gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n}.$$

Rearranging:

$$\frac{d}{dt}|u(t) - U(t)|^2 - 2L\,|u(t) - U(t)|^2 \leq 2\gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n} + 2\left[L\lambda\tau_n\,|U'(t)| + \left|f(t) - \overline{F}(t)\right|\right] |u(t) - U(t)|. \tag{4.6}$$

Now we are going to use the Theorem 2.13 in the inequality (4.6). In fact, let's consider $a(t) := |u(t) - U(t)|$, $b(t) := 0$, $\ell := -L$, $c^2(t) := 2\gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n}$ and $d(t) := L\lambda\tau_n\,|U'(t)| + \left|f(t) - \overline{F}(t)\right|$. By Theorem 2.13, we have:

$$\begin{aligned}\max_{t\in[0,T]} e^{-Lt}|u(t) - U(t)| \leq &\left(|u(0) - U(0)|^2 + \int_0^T e^{-2Lt} 2\gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n}\,dt\right)^{\frac{1}{2}}\\ &+ \int_0^T e^{-Lt} L\lambda\tau_n\,|U'(t)|\,dt + \int_0^T e^{-Lt}\left|f(t) - \overline{F}(t)\right|\,dt\end{aligned} \tag{4.7}$$

Our goal is to use the Proposition 3.15. To do this, we are going to decompose the integrals:

$$\int_0^T e^{-2Lt} 2\gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n}\,dt,\ \int_0^T e^{-Lt} L\lambda\tau_n\,|U'(t)|\,dt, \text{ and } \int_0^T e^{-Lt}\left|f(t) - \overline{F}(t)\right|\,dt.$$

as the sum of integrals in the interval $[t_{n-1}, t_n)$ for all $1 \leq n \leq N$.

In fact:

$$
\begin{aligned}
0 \le \int_0^T e^{-2Lt} 2\gamma^2 \lambda \tau_n \widetilde{\mathcal{D}_n}\, dt &= 2\gamma^2 \sum_{n=1}^N \int_{t_{n-1}}^{t_n} e^{-2Lt} \lambda \tau_n \widetilde{\mathcal{D}_n}\, dt, \\
&\le 2\gamma^2 \sum_{n=1}^N \widetilde{\mathcal{D}_n} \int_{t_{n-1}}^{t_n} e^{-2Lt_{n-1}} \lambda \tau_n\, dt, \\
&= 2\gamma^2 \sum_{n=1}^N e^{-2Lt_{n-1}} \widetilde{\mathcal{D}_n} \int_{t_{n-1}}^{t_n} (t_n - t)\, dt, \\
&= 2\gamma^2 \sum_{n=1}^N e^{-2Lt_{n-1}} \widetilde{\mathcal{D}_n} \frac{(t_n - t_{n-1})^2}{2}, \\
&= \gamma^2 \sum_{n=1}^N e^{-2Lt_{n-1}} \widetilde{\mathcal{D}_n} \tau_n^2, \\
&\le \gamma^2 \sum_{n=1}^N \tau_n^2 \widetilde{\mathcal{D}_n}.
\end{aligned}
$$

We can utilize the fact that $h_1(t) = e^{-2Lt}$ is a non-increasing function (due to $L > 0$). Applying the Extreme Value Theorem to the continuous function h_1 in the interval $[t_{n-1}, t_n]$, we can conclude that h_1 attains its maximum value at $t = t_{n-1}$. Hence, we have $e^{-2Lt} \le e^{-2Lt_{n-1}}$ for all t in $[t_{n-1}, t_n]$. Moreover, we know that $\lambda\tau_n = t_n - t$ for all $1 \le n \le N$ (as defined in Definition 3.12). Finally, we can bound the sum as follows:

$$
\sum_{n=1}^N e^{-2Lt_{n-1}} \widetilde{\mathcal{D}_n} \tau_n^2,
$$

to obtain an upper bound: $\gamma^2 \sum_{n=1}^N \tau_n^2 \widetilde{\mathcal{D}_n}$ as in Theorem 4.3 (because $e^{-2Lt_{n-1}} \le 1$, for all $t_n \in [0, T]$). Also, $e^{-2Lt} 2\gamma^2 \lambda \tau_n \widetilde{\mathcal{D}_n}$ is a function with non-negative terms. By the monotony of integral, we obtain:

$$
0 \le \int_0^T e^{-2Lt} 2\gamma^2 \lambda \tau_n \widetilde{\mathcal{D}_n}.
$$

This implies

$$
0 \le |u(0) - U(0)|^2 + \int_0^T e^{-2Lt} 2\gamma^2 \lambda \tau_n \widetilde{\mathcal{D}_n}\, dt \le |u(0) - U(0)|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \widetilde{\mathcal{D}_n}.
$$

Taking square root in both sides of inequality, we obtain:

$$\left(|u(0)-U(0)|^2+\int_0^T e^{-2Lt}2\gamma^2\lambda\tau_n\widetilde{\mathcal{D}_n}\,dt\right)^{\frac{1}{2}}\leq\left(|u(0)-U(0)|^2+\gamma^2\sum_{n=1}^N\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}.$$

In the other side:

$$\begin{aligned}\int_0^T e^{-Lt}L\lambda\tau_n\,|U'(t)|\;dt&=\sum_{n=1}^N L\int_{t_{n-1}}^{t_n}e^{-Lt}\lambda\tau_n\,|U'(t)|\;dt,\\&=L\sum_{n=1}^N\int_{t_{n-1}}^{t_n}e^{-Lt}\lambda\tau_n\,|\delta U_n|\;dt,\\&\leq L\sum_{n=1}^N|\delta U_n|\int_{t_{n-1}}^{t_n}e^{-Lt_{n-1}}(t_n-t)\,dt,\\&\leq L\sum_{n=1}^N|\delta U_n|\,e^{-Lt_{n-1}}\int_{t_{n-1}}^{t_n}(t_n-t)\,dt,\\&\leq L\sum_{n=1}^N|\delta U_n|\,e^{-Lt_{n-1}}\frac{(t_n-t_{n-1})^2}{2},\\&\leq\frac{L}{2}\sum_{n=1}^N e^{-Lt_{n-1}}\,|\delta U_n|\,\tau_n^2,\\&\leq\frac{L}{2}\sum_{n=1}^N\tau_n^2\,|\delta U_n|\,.\end{aligned}$$

We have made use of Proposition 3.15 and the fact that $e^{Lt_{n-1}}\leq 1$ for all $t_n\in[0,T]$. Additionally:

$$\int_0^T e^{-Lt}\left|f(t)-\overline{F}(t)\right|\;dt\leq\int_0^T\left|f(t)-\overline{F}(t)\right|\;dt=\left\|f-\overline{F}\right\|_{L^1((0,T),\mathcal{H})},$$

Finally, note that:

$$e^{-LT}\max_{t\in[0,T]}|u(t)-U(t)|\leq\max_{t\in[0,T]}e^{-Lt}|u(t)-U(t)|.$$

Using the error estimates in Definition 4.1, we obtain $e^{-LT}E_{\mathcal{H}}\leq\max_{t\in[0,T]}e^{-Lt}|u(t)-U(t)|$. Furthermore $u(0)=u^0$ and $U(0)=U^0$. In this way, the inequality in (4.7) can be rewritten as follow:

$$e^{-LT}E_{\mathcal{H}}\leq\left(|u^0-U^0|^2+\gamma^2\sum_{n=1}^N\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}+\frac{L}{2}\sum_{n=1}^N\tau_n^2\,|\delta U_n|+\left\|f(t)-\overline{F}(t)\right\|_{L^1((0,T),\mathcal{H})}.$$

Therefore:

$$E_{\mathcal{H}} \leq e^{LT}\left[\left(|u^0-U^0|^2+\gamma^2\sum_{n=1}^{N}\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}+\frac{L}{2}\sum_{n=1}^{N}\tau_n^2\left|\delta U_n\right|+\left\|f(t)-\overline{F}(t)\right\|_{L^1((0,T),\mathcal{H})}\right].$$

We have obtained the first condition stated by Theorem 4.3. Now for the second condition in Theorem 4.3, we are going to work with the discrete estimators:

$$\widetilde{\mathcal{D}_n}=\tau_n\left\langle \delta G_n-\delta^2U_n,\delta U_n\right\rangle_{\mathcal{H}}=\tau_n\left\langle L\delta U_n+\delta F_n-\delta^2U_n,\delta U_n\right\rangle_{\mathcal{H}}.$$

Indeed, by the inner product properties and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\widetilde{\mathcal{D}_n}&=\tau_n\left\langle L\delta U_n,\delta U_n\right\rangle_{\mathcal{H}}+\tau_n\left\langle \delta F_n,\delta U_n\right\rangle_{\mathcal{H}}-\tau_n\left\langle \delta^2U_n,\delta U_n\right\rangle_{\mathcal{H}},\\
&\leq L\tau_n\left\langle \delta U_n,\delta U_n\right\rangle_{\mathcal{H}}+\tau_n\left|\delta F_n\right|\left|\delta U_n\right|-\tau_n\left\langle \frac{\delta U_n-\delta U_{n-1}}{\tau_n},\delta U_n\right\rangle_{\mathcal{H}},\\
&\leq L\tau_n|\delta U_n|^2+\tau_n\left|\delta F_n\right|\left|\delta U_n\right|-\left\langle \delta U_n-\delta U_{n-1},\delta U_n\right\rangle_{\mathcal{H}},\\
&\leq L\tau_n|\delta U_n|^2+\tau_n\left|\delta F_n\right|\left|\delta U_n\right|-\left\langle \delta U_n,\delta U_n\right\rangle_{\mathcal{H}}+\left\langle \delta U_{n-1},\delta U_n\right\rangle_{\mathcal{H}},\\
&\leq L\tau_n|\delta U_n|^2+\tau_n\left|\delta F_n\right|\left|\delta U_n\right|-\left|\delta U_n\right|^2+\left|\delta U_n\right|\left|\delta U_{n-1}\right|,\\
&= L\tau_n|\delta U_n|^2+\tau_n\left|\delta F_n\right|\left|\delta U_n\right|-\left|\delta U_n\right|(\left|\delta U_n\right|-\left|\delta U_{n-1}\right|).
\end{aligned}$$

This implies:

$$2\left|\delta U_n\right|(\left|\delta U_n\right|-\left|\delta U_{n-1}\right|)+2\widetilde{\mathcal{D}_n}-2L\tau_n|\delta U_n|^2\leq 2\tau_n\left|\delta F_n\right|\left|\delta U_n\right|.$$

Using that $\tau_n\left|\delta F_n\right|=|F_n-F_{n-1}|$, it follows that:

$$2\left|\delta U_n\right|(\left|\delta U_n\right|-\left|\delta U_{n-1}\right|)+2\widetilde{\mathcal{D}_n}-2L\tau_n|\delta U_n|^2\leq 2\left|F_n-F_{n-1}\right|\left|\delta U_n\right|. \tag{4.8}$$

We are going to use the Theorem 2.14 in (4.8). In fact, consider the non-negative numbers $a_n:=|\delta U_n|$, $b_n^2:=2\widetilde{\mathcal{D}_n}$, $c_n:=0$, $d_n:=|F_n-F_{n-1}|$ and $\mu_n:=-L\tau_n$. In virtue of Theorem 2.14, we obtain the follow expression:

$$\max\left(\max_{1\leq n\leq N}e^{\frac{-Lt_n}{1-L\tau}}\left|\delta U_n\right|,\left(2\sum_{n=1}^{N}e^{-\frac{2L(t_n-t_{n-1})}{1-L\tau}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right)\leq\left|\delta U_0\right|+\sqrt{2}\sum_{n=1}^{N}e^{-Lt_{n-1}}\left|F_n-F_{n-1}\right|. \tag{4.9}$$

Next, we have the following inequality:

$$\frac{t_n}{1-L\tau}\leq t_{n-1}+\frac{\tau(1+LT)}{1-L\tau}.$$

This implies:

$$\max\left(\max_{1\le n\le N} e^{-\frac{L\tau(1+LT)}{1-L\tau}} e^{-Lt_{n-1}}\left|\delta U_n\right|, e^{-\frac{L\tau(1+LT)}{1-L\tau}}\left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right) \le$$
$$\max\left(\max_{1\le n\le N} e^{\frac{-Lt_n}{1-L\tau}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-\frac{2L(t_n-t_{n-1})}{1-L\tau}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right).$$

Rearranging

$$e^{-\frac{L\tau(1+LT)}{1-L\tau}}\max\left(\max_{1\le n\le N} e^{-Lt_{n-1}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right) \le$$
$$\max\left(\max_{1\le n\le N} e^{\frac{-Lt_n}{1-L\tau}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-\frac{2L(t_n-t_{n-1})}{1-L\tau}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right).$$

In this way, the inequality (4.9) can be expressed as follow:

$$e^{-\frac{L\tau(1+LT)}{1-L\tau}}\max\left(\max_{1\le n\le N} e^{-Lt_{n-1}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right) \le \left|\delta U_0\right| + \sqrt{2}\sum_{n=1}^{N} e^{-Lt_{n-1}}\left|F_n - F_{n-1}\right|,$$
$$\max\left(\max_{1\le n\le N} e^{-Lt_{n-1}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right) \le e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\left|\delta U_0\right| + \sqrt{2}\sum_{n=1}^{N} e^{-Lt_{n-1}}\left|F_n - F_{n-1}\right|\right).$$

Before bounding (4.7), we rewrite the integrals in this inequality using sums. Indeed:

$$\max_{t\in[0,T]} e^{-Lt}|u(t)-U(t)| \le \left(|u(0)-U(0)|^2 + \gamma^2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}$$
$$+\int_0^T e^{-Lt}\left|f-\overline{F}\right|\,dt + \frac{L}{2}\sum_{n=1}^{N}\tau_n^2 e^{-Lt_{n-1}}\left|\delta U_n\right|.$$

But we know that:

$$e^{-LT}E_{\mathcal{H}} \le \max_{t\in[0,T]} e^{-Lt}|u(t)-U(t)|.$$

In the other side, if $U_0 = u_0$ (which is what we have in our case, in virtue of (3.7), (4.1) and (4.2) we obtain:

$$\left(|u(0)-U(0)|^2+\gamma^2\sum_{n=1}^{N}e^{-2Lt_{n-1}}\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} = \gamma\left(\sum_{n=1}^{N}e^{-2Lt_{n-1}}\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}},$$
$$\leq \gamma\left(\sum_{n=1}^{N}e^{-2Lt_{n-1}}\tau^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}},$$
$$= \gamma\tau\left(2\sum_{n=1}^{N}e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}.$$

Additionally:

$$\frac{L}{2}\sum_{n=1}^{N}\tau_n^2 e^{-Lt_{n-1}}\left|\delta U_n\right| \leq \frac{L}{2}\sum_{n=1}^{N}\tau^2 e^{-Lt_{n-1}}\left|\delta U_n\right|,$$
$$\leq \frac{1}{2}L\tau^2\sum_{n=1}^{N}e^{-Lt_{n-1}}\left|\delta U_n\right|,$$
$$\leq \frac{1}{2}L\tau^2 N\max_{1\leq n\leq N}e^{-Lt_{n-1}}\left|\delta U_n\right|,$$
$$\leq \frac{1}{2}L\tau T\max_{1\leq n\leq N}e^{-Lt_{n-1}}\left|\delta U_n\right|,$$

This implies:

$$\gamma\tau\left(2\sum_{n=1}^{N}e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}+\frac{1}{2}L\tau T\max_{1\leq n\leq N}e^{-Lt_{n-1}}\left|\delta U_n\right| \leq$$
$$\left(\gamma\tau+\frac{1}{2}L\tau T\right)\left[\left(2\sum_{n=1}^{N}e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}+\max_{1\leq n\leq N}e^{-Lt_{n-1}}\left|\delta U_n\right|\right] \leq$$
$$2\left(\gamma\tau+\frac{1}{2}L\tau T\right)\max\left(\max_{1\leq n\leq N}e^{-Lt_{n-1}}\left|\delta U_n\right|,\left(2\sum_{n=1}^{N}e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right),$$

Given that $g\in BV((0,T),\mathcal{H})$ (see Proposition 3.8) and $BV((0,T),\mathcal{H})$ is a Banach space (therefore is a closed vectorial space), we obtain that $f\in BV((0,T),\mathcal{H})$. This implies that the expression can be bounded above as follow:

$$\int_0^T e^{-Lt}\left|f-\overline{F}\right|\,dt \leq \|f-F_n\|_{L^1((0,T),\mathcal{H})} \leq \tau Var(f).$$

Likewise (as in the previous step), we obtain:

$$\sqrt{2}\sum_{n=1}^{N}e^{-Lt_{n-1}}\left|F_n-F_{n-1}\right| \leq \sqrt{2}\sum_{n=1}^{N}\left|F_n-F_{n-1}\right| \leq \sqrt{2}\,Var(F) \leq \sqrt{2}\,Var(f).$$

Finally, with the bounds obtained during the development in pages 47, 48, and 49, we obtain the following inequalities:

$$
\begin{aligned}
e^{-LT}E_{\mathcal{H}} \leq & \max_{t\in[0,T]} e^{-Lt}|u(t)-U(t)|,\\
\leq & \gamma \left(\sum_{n=1}^{N} e^{-2Lt_{n-1}}\tau_n^2\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} + \frac{L}{2}\sum_{n=1}^{N}\tau_n^2 e^{-Lt_{n-1}}\left|\delta U_n\right| + \int_0^T e^{-Lt}\left|f-\overline{F}\right| dt,\\
\leq & \gamma\tau \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} + \frac{1}{2}L\tau^2 N \max_{1\leq n\leq N} e^{-Lt_{n-1}}\left|\delta U_n\right| + \tau\, Var(f),\\
= & \gamma\tau \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} + \frac{1}{2}L\tau T \max_{1\leq n\leq N} e^{-Lt_{n-1}}\left|\delta U_n\right| + \tau\, Var(f),\\
\leq & \left(\gamma\tau + \frac{1}{2}L\tau T\right)\left[\left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}} + \max_{1\leq n\leq N} e^{-Lt_{n-1}}\left|\delta U_n\right|\right] + \tau\, Var(f),\\
\leq & 2\left(\gamma\tau + \frac{1}{2}LT\tau\right)\max\left(\max_{1\leq n\leq N} e^{-Lt_{n-1}}\left|\delta U_n\right|, \left(2\sum_{n=1}^{N} e^{-2Lt_{n-1}}\widetilde{\mathcal{D}_n}\right)^{\frac{1}{2}}\right) + \tau\, Var(f),\\
\leq & 2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma\tau + \frac{1}{2}LT\tau\right)\left(\left|\delta U_0\right| + \sqrt{2}\sum_{n=1}^{N} e^{-Lt_{n-1}}\left|F_n - F_{n-1}\right|\right) + \tau\, Var(f),\\
\leq & 2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma\tau + \frac{1}{2}LT\tau\right)\left(\left|\delta U_0\right| + \sqrt{2}\, Var(f)\right) + \tau\, Var(f),\\
= & \tau\left[2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma + \frac{1}{2}LT\right)\left(\left|\delta U_0\right| + \sqrt{2}\, Var(f)\right) + Var(f)\right],\\
= & \tau\left[2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma + \frac{1}{2}LT\right)\left|\delta U_0\right| + \left[1 + 2\sqrt{2}e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma + \frac{1}{2}LT\right)\right] Var(f)\right],\\
= & \tau\left[\widetilde{C}\left|\delta U_0\right| + \left(1+\sqrt{2}\widetilde{C}\right) Var(f)\right].
\end{aligned}
$$

We have considered $\widetilde{C} := 2e^{\frac{L\tau(1+LT)}{1-L\tau}}\left(\gamma + \frac{1}{2}LT\right)$. This implies:

$$E_{\mathcal{H}} \leq \tau e^{LT}\left[\widetilde{C}\left|\delta U_0\right| + \left(1+\sqrt{2}\widetilde{C}\right) Var(f)\right],$$

which is the second condition of Theorem 4.3 establishes. Therefore, we have successfully demonstrated the fundamental result of our thesis.

□

4.6 Consequences of Theorem 4.3.

As emphasized throughout this thesis, Theorem 4.3 serves as the cornerstone of this document due to the wealth of information it provides, enabling us to draw significant conclusions.

In Chapter 1 (Introduction), we have set the objectives of this thesis. The first one is to demonstrate that the solution $U(t)$ of the discrete problem (4.2) converges to the weak solution $u(t)$ of the continuous problem (4.1) in order to validate the Backward-Euler method in time.

As an immediate consequence of Theorem 4.3, we obtain the following result:

Corollary 4.19. *Let us consider the continuous problem defined by the evolution equation associated with the FitzHugh-Nagumo system:*

$$\begin{cases} u'(t) + \mathcal{F}(u(t)) = g(t), \quad t \in (0,T) \\ \\ u(0) = u^0, \end{cases}$$

where $u(t)$ is the unique weak solution and let:

$$\begin{cases} U'(t) + \mathcal{F}(\overline{U}(t)) = \overline{G}(t), \quad t \in [t_{n-1}, t_n) \\ \\ U(0) = U^0, \end{cases}$$

be the discrete problem where $U(t)$ is the numerical solution (linear interpolation). Considering the assumptions of Theorem 4.3 and assuming that $u_0 = U_0$, $f \in BV((0,T), \mathcal{H})$, and $\tau \to 0$ where $\tau = \max\limits_{1 \leq n \leq N} \tau_n$ and $\tau_n = t_n - t_{n-1}$ for all $1 \leq n \leq N$. Then:

$$\lim_{\tau \to 0} \max_{t \in [0,T]} |u(t) - U(t)| = 0.$$

In other words, we have $U(t) \to u(t)$ as $\tau \to 0$ in $L^\infty(\Omega)$ norm.

Proof. Indeed, considering the assumptions of Theorem 4.3 and the conditions $u_0 = U_0$ and $f \in BV((0,T), \mathcal{H})$, the error estimates satisfies the following inequality:

$$0 \leq E_{\mathcal{H}} \leq \tau e^{LT} \left[\widetilde{C} \, |\delta U_0| + \left(1 + \sqrt{2}\widetilde{C}\right) Var(f) \right],$$

Taking $\tau \to 0$ and using the squeeze theorem, we can conclude that:

$$\lim_{\tau \to 0} \max_{t \in [0,T]} |u(t) - U(t)| = 0.$$

□

From Corollary 4.19, we obtain a very interesting consequence. If we considering the time steps increasingly smaller when obtaining the temporal discretization of the evolution equation associated to the FitzHugh-Nagumo system (thus, $\mathcal{P}$ becomes a finer partition of the interval $[0,T]$), it follows that the solution $U(t)$ of the discrete problem (4.2) converges to the weak solution of the continuous problem (4.1).

And the fact of achieving this convergence from discrete-time solutions to continuous-time solutions is very advantageous. Indeed, we have validated the Backward-Euler method in time: this time-scheme provides a series of solutions that accurately approximate the original solution of the continuous problem with rate of convergence equals to 1.

Instead of attempting to obtain an analytical solution for the FitzHugh-Nagumo equations (3.1), we can obtain a discrete solution using the Backward-Euler scheme in time. In this way, we will have all the advantages provided by numerical methods (flexibility, computational efficiency, practical solutions, validation and applications).

For this reason, in the next chapter we will carry out a numerical simulation with a generalized version of the FitzHugh-Nagumo equations (3.1) in order to analyze and study the behavior of the variables involved in FitzHugh-Nagumo system: the membrane potential and a recovery variable (associated to the plasma membrane).

Chapter 5

Numerical Simulations

Based on the work of article [1], the objective of this chapter is to show a numerical simulation for the generalized FitzHugh-Nagumo equations in cardiac electrophysiology. As a primary goal, we are interested in studying and researching the behavior of membrane potential $\phi(x,t)$ and the plasma membrane recovery current $r(x,t)$ in a three-dimensional domain under fixed initial conditions.

This simulation holds a crucial role in our understanding of cardiac electrophysiology, making significant contributions to the development of treatments to improve cardiac disorders.

For simulation purposes, consider the set $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ has a outward-pointing normal vector $\mathbf{n}$ and $T > 0$ is the final time. We define $\Omega_T := \Omega \times (0,T)$ and $\partial\Omega_T := \partial\Omega \times (0,T)$ as the domain and boundary of the problem (respectively). The boundary $\partial\Omega$ satisfies the conditions:

$$
\begin{aligned}
\partial\Omega_\phi \cap \partial\Omega_q &= \emptyset, \\
\overline{\partial\Omega_\phi} \cup \overline{\partial\Omega_q} &= \partial\Omega,
\end{aligned}
$$

where $\partial\Omega_\phi$, $\partial\Omega_q$ are relatively open and smooth in $\partial\Omega$. We will emphasize that $\partial\Omega_q$ is the Neumann part of boundary $\partial\Omega$ and $\partial\Omega_\phi$ is the Dirichlet part of boundary $\partial\Omega$.

Let $\phi = \phi(x,t) \in H^1(\Omega)$ be the membrane potential and $r = r(x,t) \in H^1(\Omega)$ be the plasma membrane recovery current. We define the generalized FitzHugh-Nagumo equations by the following system:

$$
\begin{cases}
\dfrac{\partial \phi}{\partial t} - \operatorname{div}(\mathcal{D}\nabla\phi) - c_1\phi(\phi-\alpha)(1-\phi) + c_2 r = 0, \quad (x,t) \in \Omega_T \\[2ex]
\dfrac{\partial r}{\partial t} - b(\phi - dr) = 0, \quad (x,t) \in \Omega_T \\[2ex]
\phi = \overline{\phi}, \quad x \in \partial\Omega_\phi \\[2ex]
\mathcal{D}\nabla\phi \cdot \mathbf{n} = 0, \quad x \in \partial\Omega_q \\[2ex]
\phi(x,0) = \phi_0(x), \quad x \in \Omega, \\[2ex]
r(x,0) = r_0(x), \quad x \in \Omega.
\end{cases}
\tag{5.1}
$$

In the expression (5.1), we incorporate the symmetric and positive-definite tensor $\mathcal{D} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ as a fundamental element distinguishing it from equation (1.1), where D is only a positive real number. Introducing this additional factor, our intention is to achieve a higher degree of accuracy and realism in representing the phenomena observed in cardiac electrophysiology.

5.1 Numerical Method

Since the (generalized) FitzHugh-Nagumo equations (5.1) constitute a spatial-temporal problem, we will utilize the following numerical method to approximate a solution and carry out the computational simulations:

- Spatial problem: To address the spatial nature of the (generalized) FitzHugh-Nagumo equations, we will employ the Lagrange Finite Element Method (FEM) in order to discretize the domain Ω.
- Temporal problem: For the temporal discretization, the simulation will be performed using a Forward-Euler time step in the interval $[0, T]$.

5.1.1 Finite Element Method and Geometry of the Problem

It is important to note that in Finite Element Method, the geometry of the problem corresponds to the spatial domain in which the numerical simulation takes place. This geometry is typically represented using a mesh, which provides a discretized approximation of the problem domain.

The mesh accurately captures the shape and boundaries of the problem domain, allowing for a precise representation of the geometry. For this purpose, let K be a unit simplex in $\mathbb{R}^3$ with vertices a_i for $i = \{0, 1, 2, 3\}$ given by:

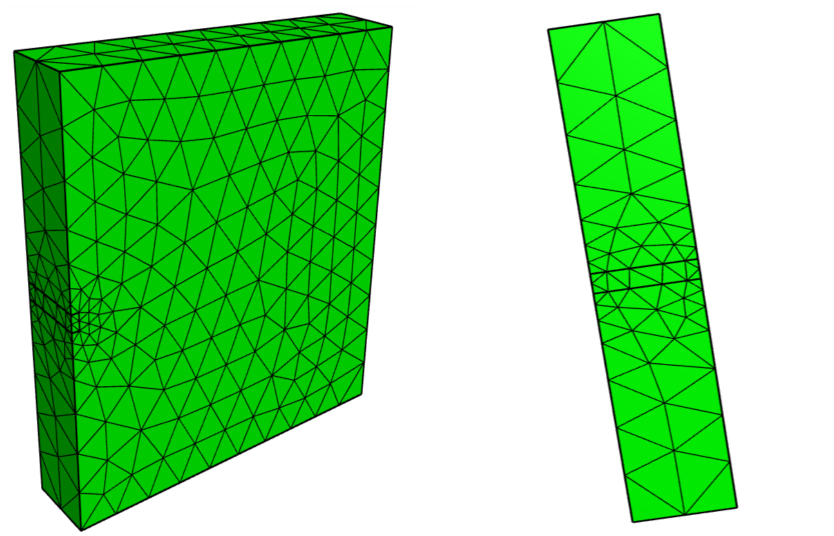
$$
K := \left\{ \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_i \geq 0 \text{ and } \sum_{i=1}^{3} x_i \leq 1, \text{ for } 1 \leq i \leq 3 \right\}.
$$

In order to use the Lagrange Finite Element Method, we will discretize the space:

$$\Omega := [0, 100] \times [0, 100] \times [0, 20] \subset \mathbb{R}^3,$$

with a non-homogeneous mesh $\mathcal{T}_h$ of $\partial\Omega$, where $\partial\Omega$ is the boundary of Ω and the length units are in millimeters. The global mesh size h is set to 10 and the mesh $\mathcal{T}_h$ for the spatial domain $\Omega = [0, 100] \times [0, 100] \times [0, 20]$ is given by the Figure 5.1 (a).

With the aim to introduce a specific condition for the membrane potential $\phi(x, t)$ in a region of the three-dimensional domain $\Omega = [0, 100] \times [0, 100] \times [0, 20]$, we will consider a brick-shaped region in $\{0\} \times [-2, 48, -2] \times [2, 52, 22]$; see Figure 5.1 (b).



(a) Mesh $\mathcal{T}_h$ for Ω. (b) Brick-shaped region in domain Ω.

Figure 5.1: Mesh and brick-shaped region.

Since the FitzHugh-Nagumo equations correspond to a coupled system of a partial parabolic differential equation (PDE) and an ordinary differential equation (ODE) with variables ϕ (membrane potential) and r (plasma membrane recovery current), we will work with the product space of finite element:

$$V := H^1(\Omega) \times H^1(\Omega).$$

The $H^1(\Omega)$ finite element space consists of continuous and piecewise polynomial functions. Based on Galerkin's theory, consider the finite dimension subspace $V_h^1 \subset V$ defined as follows:

$$V_h^1 := \{v \in V \,:\, v|_K \in P^1(K), \text{ for all } K \in \mathcal{T}_h\},$$

where $P^1(K)$ is the space of piecewise polynomial functions of degree one over the elements of $\mathcal{T}_h$. According to [4], we know that $(K, \mathcal{P}^1(K), \Sigma)$ is a Lagrange finite element where $\Sigma = \{\sigma_1, \sigma_2, ...\sigma_n\}$ and $\sigma_i(p) = p(a_i)$.

In order to obtain the spatial discretization for the FitzHugh-Nagumo equations (5.1), let's consider the test functions $p, q \in V$ and the trial functions $\phi, r \in H^1(\Omega)$. If we consider the Neumann boundary condition: $D\nabla\phi \cdot \mathbf{n} = 0$ for $(x,t) \in \partial\Omega_T$, we derive the following weak form for the generalized FitzHugh-Nagumo equations:

$$
\begin{aligned}
&\int_\Omega \frac{\partial \phi}{\partial t} p\, dx + \int_\Omega D(\nabla\phi)\nabla p\, dx - \int_\Omega [c_1\phi(\phi-\alpha)(1-\phi) - c_2 r]\, p\, dx = 0, \\
&\int_\Omega \frac{\partial r}{\partial t} q\, dx - \int_\Omega b(\phi - dr) q\, dx = 0,
\end{aligned} \tag{5.2}
$$

From (5.2), we define:

- Bilinear forms:

$$
\begin{aligned}
\mathfrak{m}((\phi, r), (p, q)) &:= \int_\Omega (\phi p + rq)\, dx, \\
\mathfrak{a}((\phi, r), (p, q)) &:= \int_\Omega D(\nabla\phi)\nabla p\, dx - \int_\Omega b(\phi - dr) q\, dx.
\end{aligned}
$$

- The operator: $\mathcal{F}((\phi, r), (p, q)) := \int_\Omega [c_1\phi(\phi-\alpha)(1-\phi) - c_2 r]\, p\, dx.$

In this way, (5.2) can be rewritten as follow:

$$
\mathfrak{m}\left(\left(\frac{\partial\phi}{\partial t}, \frac{\partial r}{\partial t}\right), (p,q)\right) + \mathfrak{a}((\phi, r), (p, q)) - \mathcal{F}((\phi, r), (p, q)) = \mathcal{L}(p, q). \tag{5.3}
$$

Next, we will utilize the finite subspace V_h^1 (based on the theory of Finite Element Method) in order to solve the following problem:

$$
\mathfrak{m}\left(\left(\frac{d\phi_h}{dt}, \frac{dr_h}{dt}\right), (p,q)\right) + \mathfrak{a}((\phi_h, r_h), (p, q)) - \mathcal{F}((\phi_h, r_h), (p, q)) = 0, \tag{5.4}
$$

where $(\phi_h, r_h) \in V_h^1$, for all $(p, q) \in V_h^1$. In this way, we can find a approximate solution for (5.3). The expression (5.4) corresponds to the spatial discretization of (5.1) and the main associated advantage is that the derivative in (5.3) is total, turning (5.4) into an ordinary differential equation.

5.1.2 Temporal Discretization: Forward-Euler Time Step

Now, let's proceed with the temporal discretization of (5.4). To accomplish this, we will employ a Forward-Euler time-step method over the interval $[0, T]$. More information about this numerical method can be found in [4].

The main advantage of employing the Forward-Euler scheme in time is that it allows us to solve a linear problem with reduced computational cost, unlike Backward-Euler time step (where we solve a nonlinear problem). However, we are aware that Forward-Euler has important limitations. One of them is associated with the accuracy. Forward-Euler scheme in time is a first-order method, which means that it has a lower accuracy compared to higher-order methods (for example, Backward-Euler).

For a better understanding of the temporal discretization using the Forward-Euler method in time and since (5.4) corresponds to an ordinary differential equation, let's consider the following definition:

Definition 5.1. (Forward-Euler scheme in time) *Consider the Cauchy problem:*

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ \\ y(t_0) = y^0, \end{cases}$$

The function $f(t, x)$ and the initial data $y(t_0) = y^0$ are known. The Forward-Euler finite difference scheme in time gives a sequence $\{y^n\}_{n=0}^N$ such that y^n approximates the solution $y(t_0 + nh)$ where h is the step-time, $1 \leq n \leq N$ and:

$$\frac{y^{n+1} - y^n}{h} = f(t_n, y^n).$$

Remark. The information at time $t = t_n$ is assumed to be known.

To achieve the temporal discretization of (5.4), let $\mathcal{P}$ denote a partition of the time interval $[0, T]$ defined as follows:

$$\mathcal{P} = \{0 = t_0 < t_1 < ... < t_{N-1} < t_N = T\}.$$

Consider the equation (5.4). Our goal is to approximate the total derivatives. Using a Forward-Euler finite-difference scheme in time (based in Definition 5.1) with step-time $\Delta t = t_{n+1} - t_n$ and initial values $\phi(x, 0) = 0.2$, $r(x, 0) = 0$ for the next iterations, we obtain the following discrete equation:

$$\frac{\mathfrak{m}(\phi_h^{n+1} - \phi_h^n, r_h^{n+1} - r_h^n), (p, q))}{\Delta t} + \mathfrak{a}((\phi_h^n, r_h^n), (p, q)) - \mathcal{F}((\phi_h^n, r_h^n), (p, q)) = 0, \tag{5.5}$$

where $\{\phi_h^n\}_{n=0}^N$ and $\{r_h^n\}_{n=0}^N$ are two sequences whose first terms are given (by the initial conditions) and the other ones are recursively defined for $1 \leq n \leq N$. Furthermore $\phi_h^n \approx \phi_h(t_n)$ and $r_h^n \approx r_h(t_n)$. Multiplying (5.5) by Δt and rearranging terms, we obtain:

$$\mathfrak{m}((\phi_h^{n+1}, r_h^{n+1}), (p, q)) = \Delta t[\mathcal{F}((\phi_h^n, r_h^n), (p, q)) - \mathfrak{a}((\phi_h^n, r_h^n), (p, q))] + \mathfrak{m}((\phi_h^n, r_h^n), (p, q)). \quad (5.6)$$

The expression (5.6) corresponds to the temporal discretization of (5.4), where we need to solve for $(\phi_h^{n+1}, r_h^{n+1})$. It is important to emphasize that all information at time $t = t_n$ is assumed to be known. Therefore, the right-hand side of equation (5.6) is known.

5.1.3 Parameters for Simulations and CFL Condition

Let us recall that the main objective of this chapter is to present a numerical simulation for the FitzHugh-Nagumo equations (5.1) in cardiac electrophysiology. To accomplish this, we have discretized the equations using Finite Element Method (FEM) in space and using Forward-Euler method in time.

With the implementation of this numerical simulation, our goal is to observe and analyze the behavior of the membrane potential $\phi(x,t)$ and the plasma membrane recovery current $r(x,t)$ in the spatial domain $\Omega = [0, 100] \times [0, 100] \times [0, 20] \subset \mathbb{R}^3$. To accomplish this, we will consider specific parameters and their corresponding values in (5.1) based on the article [1]:

- Initial conditions:

$$\begin{aligned} \phi(x, 0) &= 0.2 \text{ mV}, \\ r(x, 0) &= 0. \end{aligned}$$

- Dirichlet boundary condition: $\phi(x,t) = 1$ mV in $x \in \{0\} \times [0, 20] \times [4.8, 5.2]$.
- Final time: $t_{\text{end}} = 400$ ms.
- Tensor:

$$\mathcal{D} = d_{\text{iso}} I + d_{\text{ort}} n \otimes n,$$

  where $d_{\text{iso}} = 0.1$ mm^2/ms is an isotropic conductivity, $d_{\text{ort}} = 0.9$ mm^2/ms is the increase of conductivity and $n = (1, 0, 0)$ is the direction of d_{ort}.

- Parameters in (5.1):

Parameter	Value	Description
α	0.08	Normalized threshold potential
c_1	0.0875	Excitation rate constant
c_2	0.03	Excitation decay constant
b	0.0055	Recovery rate constant
d	0.55	Recovery decay constant

Table II: Parameters value for numerical simulation.

Now it's time to select an appropriate time step Δt for the numerical simulation. To ensure that Δt is compatible with the spatial discretization and guarantees accurate results, we must consider the CFL condition (Courant – Friedrichs – Lewy condition).

The CFL condition for parabolic problems states that the time step Δt must satisfy the inequality:

$$\Delta t \leq Ch^2,$$

where h is the global mesh size and C is the CFL number (where $C \leq 1$). We must consider this CFL condition because Forward-Euler is a numerical method prone to be unstable. Given that h is fixed at 10 and consider the experimental CFL number $C = 0.0005$, the CFL condition will tell us that $\Delta t \leq 0.05$. In this way, we will choose the time step size:

$$\Delta t = 0.05 \text{ ms}.$$

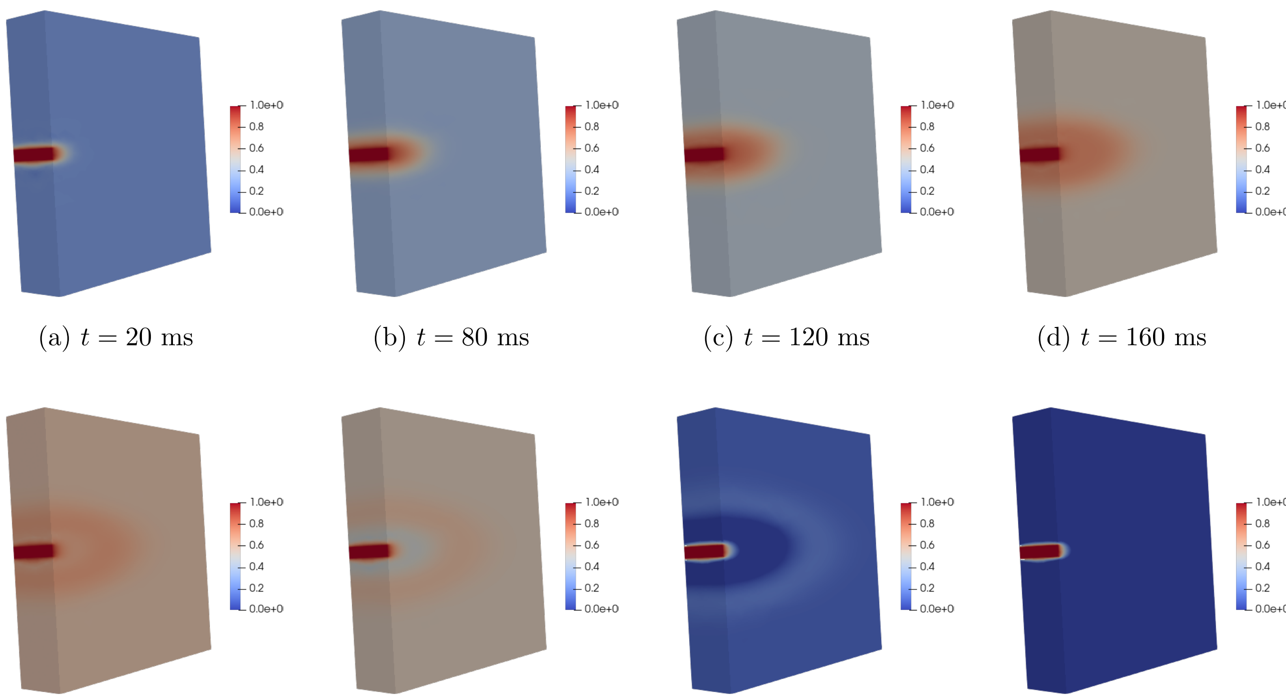
Selecting a time step size $\Delta t = 0.05$ ms, we satisfy the CFL condition. This ensures that the numerical solution remains stable and accurate. We must emphasize that the CFL number was obtained experimentally. Several numerical simulations were conducted for the membrane potential ϕ, which turned out to be inconsistent. Considering $C = 0.0005$, we obtain an appropriate Δt that falls in the stability region of the numerical method.

5.2 Simulation and Conclusions

Based on the analysis presented in Section 5.1, we have the necessary tools to proceed with our numerical simulation. Linking with (5.6) and utilizing the mesh $\mathcal{T}_h$, we can develop a code (see Appendix: Code for Simuation, page 67). This code yields an approximate solution for the behavior of the membrane potential ϕ, considering the fixed parameters outlined in Subsection 5.1.3.

The intention behind including this code is to make it freely available for further enhancements in order to achieve a more precise numerical simulation. The simulation was implemented using the open-source finite element library Netgen/NGSolve. The graphical visualization of the simulation results was obtained using the Paraview software.

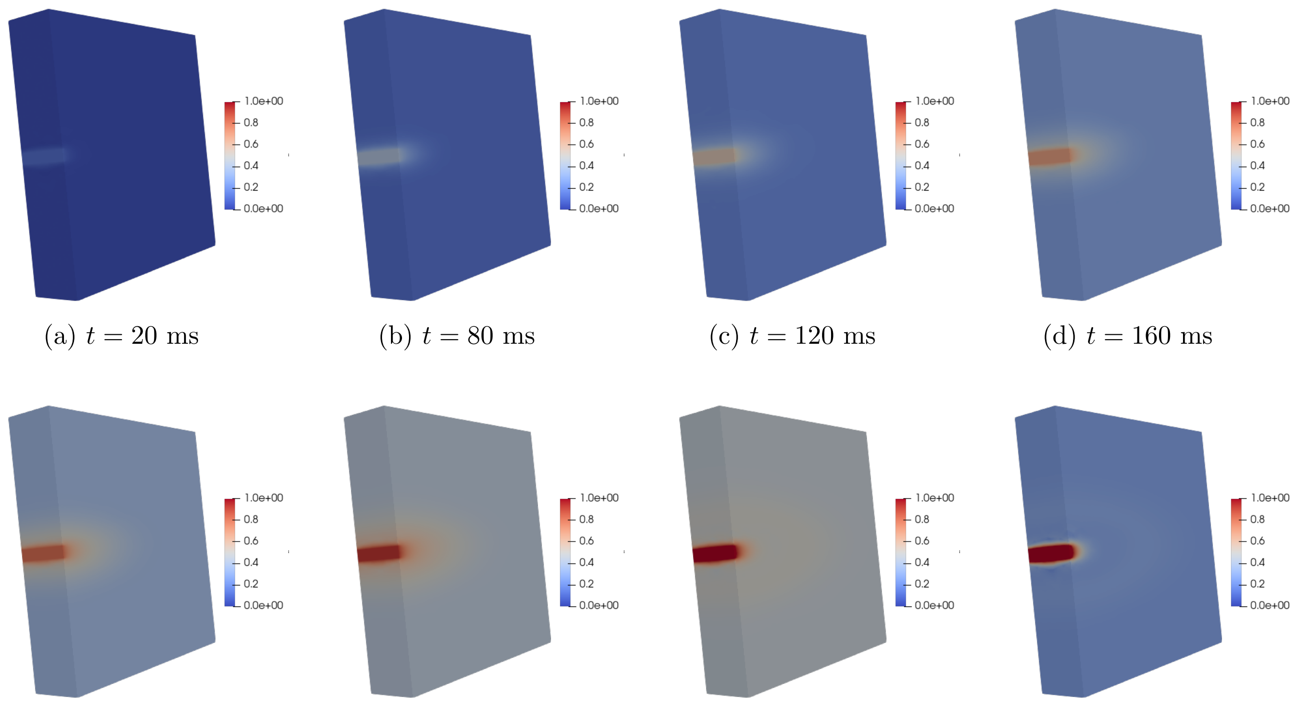
Figures 5.2 and 5.3 illustrates the evolution of the membrane potential ϕ and the plasma membrane recovery current r using a Finite Element Method (FEM) in space and Forward-Euler scheme in time. Recall that in brick-shaped region $\{0\} \times [-2, 48, -2] \times [2, 52, 22]$, the membrane potential ϕ is setted to 1 mV.



(a) $t = 20$ ms (b) $t = 80$ ms (c) $t = 120$ ms (d) $t = 160$ ms

(e) $t = 200$ ms (f) $t = 240$ ms (g) $t = 310$ ms (h) $t = 400$ ms

Figure 5.2: Evolution of membrane potential ϕ.



(a) $t = 20$ ms (b) $t = 80$ ms (c) $t = 120$ ms (d) $t = 160$ ms

(e) $t = 200$ ms (f) $t = 240$ ms (g) $t = 310$ ms (h) $t = 400$ ms

Figure 5.3: Evolution of plasma membrane recovery current r.

Based on the numerical simulation (whose results are displayed in Figures 5.2 and 5.3), we can make the following comments:

- In the time interval $[20, 80]$ ms, we can observe a gradual increase of the membrane potential ϕ across the spatial domain Ω. The potential starts to 0.1 mV at $t = 20$ ms and reaches 0.3 mV at $t = 80$ ms. This change in potential is clearly influenced by the fixed condition $\phi(x, t) = 1$. However, for the plasma membrane recovery current the situation is different. There is no significant increase for r.

- During the time $[80, 160]$ ms, the membrane potential ϕ continues to dissipate across the spatial domain Ω, reaching 0.6 mV at $t = 160$ ms. Notably in the neighborhood of the brick-shaped region, the membrane potential exhibits a higher value approaching 0.8 mV compared to the rest of the domain Ω. In the other side, there is a significant increase of the plasma membrane recovery current over Ω; especially in the brick-shaped region (reaching 0.6 mV).

- In the time interval $[200, 240]$ ms, the membrane potential along Ω is close to 0.8 mV. For the recovery variable, there still exists a slight increase for r, with values close to 0.5 mV. However, at $t = 310$ ms; the potential begins to decay (from 0.8 mV) noticeably around the neighborhood of brick-shaped region. The same situation occurs for the recovery variable r, except in $\{0\} \times [-2, 48, -2] \times [2, 52, 22]$.

- At $t = 400$ ms, the membrane potential drops to 0 mV throughout the spacial domain Ω domain, except for the specified region $0 \times [-2, 48, -2] \times [-2, 52, 22]$. The recovery variable r drops to 0.3 mV across the spacial domain Ω domain, except for the brick-shaped region.

Based on the numerical simulation (with results shown in Figures 5.2 and 5.3), we can conclude that the voltage (in mV) decays due to the absence of external stimulation. This is equivalent to what would occur in a plasma membrane only if there were an initial voltage condition with no external current.

Chapter 6

Conclusions

We have observed that the FitzHugh-Nagumo equations (which derived from the Hodgkin-Huxley model) correspond to a coupled system of a parabolic partial differential equation (for the variable ϕ representing the membrane potential) and an ordinary differential equation (for the plasma membrane recovery current r).

Due the fact that $\phi(x,t) = \phi = \overline{\phi} + u_1 \in L^2(\Omega)$ (where $u_1 \in \mathcal{S}(\Omega)$, $\overline{\phi}$ is defined in $\overline{\Omega}$) and $r(x,t) = r \in L^2(\Omega)$, we were able to rewrite the FitzHugh-Nagumo equations in (1.1) as the system given in (3.1). If we consider the temporal evolution of (3.1), we obtain the following evolution equation:

$$\begin{cases} u'(t) + \mathcal{F}(u(t)) = g(t), \quad t \in (0,T) \\ \\ u(0) = u^0. \end{cases}$$

The concrete achievement of this thesis consists in determining a specific variational structure for the FitzHugh-Nagumo equations. To do this, we must work with our evolution operator. Indeed, the operator $\mathcal{F}$ describes the temporal behavior of the FitzHugh-Nagumo equations (3.1). Recall that $\mathcal{F} : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ is given by:

$$\mathcal{F}(u(t)) := \begin{pmatrix} -D\Delta u_1 + F'_{\text{ion}}(\overline{\phi} + u_1) + c_2 r + L u_1 \\ (c_2 d + L) r - c_2 u_1 \end{pmatrix}.$$

where $L > 0$ and $D(\mathcal{F}) := \left\{ \begin{pmatrix} u_1(t) \\ r(t) \end{pmatrix} \in \mathcal{S}(\Omega) \times L^2(\Omega) \, : \, -\Delta u_1(t) \in L^2(\Omega) \right\}.$

We know that $\mathcal{F}$ can be decomposed as the sum of two operators: $\mathcal{F}_1$ and $\mathcal{F}_2$. Indeed $\mathcal{F}_1, \mathcal{F}_2 : D(\mathcal{F}) \to L^2(\Omega) \times L^2(\Omega)$ are given by:

$$\mathcal{F}_1(u(t)) := \begin{pmatrix} -D\Delta u_1 + c_2 r \\ (c_2 d + L) r - c_2 u_1 \end{pmatrix},$$

$$\mathcal{F}_2(u(t)) := \begin{pmatrix} F'_{\text{ion}}(\overline{\phi} + u_1) + L u_1 \\ 0 \end{pmatrix}.$$

such that $\mathcal{F}(u(t)) = (\mathcal{F}_1 + \mathcal{F}_2)(u(t)) = \mathcal{F}_1(u(t)) + \mathcal{F}_2(u(t))$, for all $u(t) \in D(\mathcal{F})$. Remember that $\mathcal{F}_1$ can be associated to a bilinear form $\mathfrak{a}$, where the antisymmetric part $\mathfrak{a}_{AS}$ of the bilinear form is controlled by the symmetric part $\mathfrak{a}_{AS}$. Indeed, by Theorem 4.9:

$$|\mathfrak{a}_{AS}| \leq 2\mu|\mathfrak{a}_S|, \quad \mu \geq 0.$$

This allow us to conclude that $\mathcal{F}_1$ is γ_1^2-angle-bounded operator.

On the other side, in $\mathcal{F}_2$ we have added a multiple of u_1 in order to make it the gradient of a convex potential. As demonstrated in Theorem 4.10, the gradients of a convex potential are always γ_2^2-angle-bounded, where $\gamma_2^2 = 1$. In virtue of Proposition 4.4, we can conclude that $\mathcal{F}$ is γ^2-angle-bounded with $\gamma = \max\left\{\frac{\mu^2+1}{4}, 1\right\}$.

Based on the properties satisfied by our evolution operator $\mathcal{F}$, we can conclude that the FitzHugh-Nagumo equations do not possess a gradient flow structure. However, they do exhibit a specific variational structure. Indeed:

- From the linear operator $\mathcal{F}_1$, we conclude that FitzHugh-Nagumo is a system whose dissipative part dominates the conservative part.
- Module -add a multiple of the identity-, the non-linear operator $\mathcal{F}_2$ is the first variation of a convex functional.

If we employ the Backward-Euler scheme for the temporal discretization for the evolution equation (3.2), we obtain the discrete problem:

$$\begin{cases} U'(t) + \mathcal{F}(\overline{U}(t)) = \overline{G}(t), \quad t \in [t_{n-1}, t_n) \\ \\ U(0) = U^0, \end{cases}$$

We know that the evolution equation has a unique weak solution $u(t)$. The discrete problem provides us with a sequence of solutions $U(t)$. This motivates us to introduce an estimation error given by:

$$E_{\mathcal{H}} := \max_{t\in[0,T]} |u(t) - U(t)|.$$

If we consider the assumptions of Theorem 4.3 and assuming that $u_0 = U_0$, $f \in BV((0,T), \mathcal{H})$ and $\tau \to 0$ (where $\tau = \max_{1\leq n\leq N} \tau_n$, $\tau_n = t_n - t_{n-1}$ for all $1 \leq n \leq N$), we obtain:

$$\lim_{\tau\to 0} \max_{t\in[0,T]} |u(t) - U(t)| = 0.$$

In other words, we have $U(t) \to u(t)$ as $\tau \to 0$ in $L^\infty(\Omega)$ norm. From Corollary 4.19, we derive a crucial consequence. Considering an increasingly smaller time steps in the temporal discretization for the evolution equation associated to the FitzHugh-Nagumo system, we can obtain that the solution $U(t)$ of the discrete problem (4.2) converges to the weak solution of the continuous problem (4.1).

And the fact of achieve this convergence from discrete-time solutions to continuous-time solutions is highly advantageous. In fact, we have successfully validated the Backward-Euler method in time as scheme generates a sequence of solutions that accurately approximate the original solution of the continuous problem.

Based on the simulations obtained in Chapter 5 for the membrane potential $\phi(x,t)$ and the plasma membrane recovery current $r(x,t)$, we can conclude that using the Forward-Euler method in time has its advantages and disadvantages. It is associated with lower computational cost, but it does not accurately describe the evolution of $\phi(x,t)$ and $r(x,t)$ as the Backward-Euler method would. This is because Forward-Euler scheme in time is a first-order method, which means that it has a lower accuracy compared to higher-order methods.

The simulations (whose results can be observed in Figures 5.2 and 5.3) allow us to observe the evolution of the membrane potential $\phi(x,t)$ and the plasma membrane recovery current $r(x,t)$ in the domain Ω. We can see that the voltage (in mV) decays due to the absence of external stimulation. This is equivalent to what would happen in a plasma membrane if only an initial voltage condition were given without any external current. Also, this numerical simulations are consistent with the variational structure of the FitzHugh-Nagumo equations.

As possible generalizations and improvements of this thesis, the following projects/ideas are proposed to be developed:

- Consider D as a tensor diffusion (instead of a real parameter) in the mathematical analysis of FitzHugh-Nagumo equations.
- Use Backward-Euler or Runge-Kutta method in the temporal discretization for FitzHugh-Nagumo equations in order to obtain the numerical simulations.

Appendix: Electrophysiological Glossary

Excitable cells (such as neurons and muscle cells) are encompassed by a plasma membrane, whose main function is to control the passage of ions and molecules into and out of the cell. The **plasma membrane** is a structure that bounds the cell. The membrane is mainly made of lipid, which often represents as much as 70% of the membrane volume depending on cell type. The membrane lipid itself prevents the passage of ions through the membrane.

The plasma membrane is a heterogeneous structure that contains numerous large complex proteins (some of which consist of approximately 2.500 amino acids). These proteins are embedded within the membrane and serve as constituents of pumps and channels responsible for the exchange of ions between the intracellular and extracellular spaces. Figure A.1 shows the structure and composition of the plasma membrane.

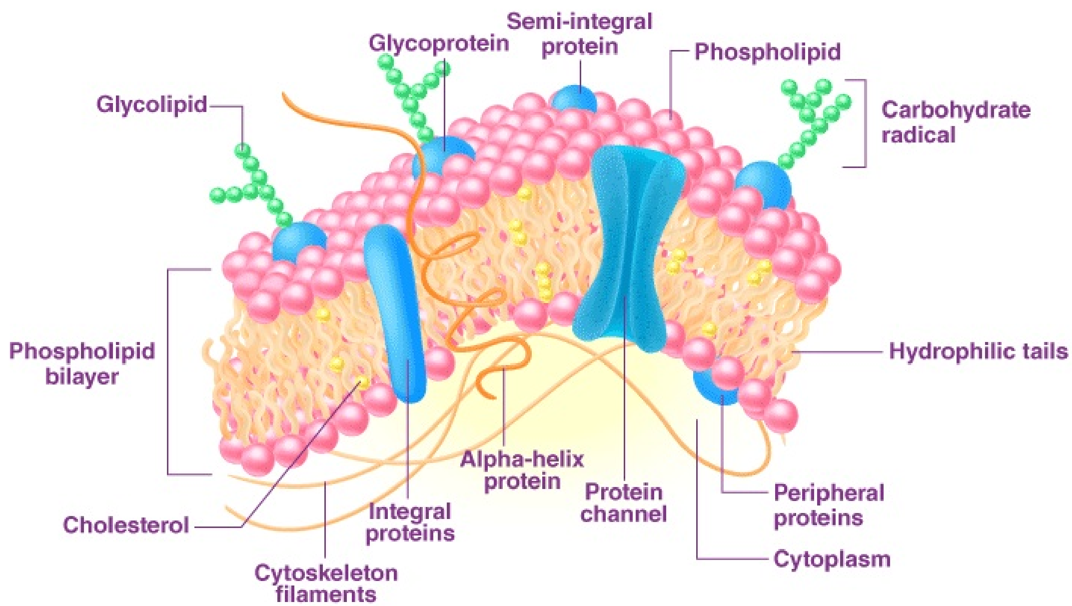


Figure A.1: Structure and composition of plasma membrane.

In the context of the plasma membrane, the electrical potential refers to the electric voltage that exists across the plasma membrane of a cell. If the electrical potential at the inside surface of the plasma membrane ϕ_i (in a excitable cell) is compared to the potential at the outside surface ϕ_e, we can define mathematically the membrane potential as follow:

$$\phi = \phi_i - \phi_e.$$

In fact, the **membrane potential** ϕ refers to the electrical potential difference that exists between the interior and exterior of a plasma membrane. The membrane potential is measured in millivolts (mV).

Because the plasma membrane has a resistence, there will be a ionic current $F'_{\text{ion}}(\phi)$. This **ionic current** refers to the flow of ions through a cell membrane or any other conducting medium. Ions -which are electrically charged particles- can move through ion channels present in the cell membrane and other mechanisms of ion transport. Furthermore, this current is considered to have a positive sign when it flows across the membrane potential in the direction from inside to outside.

The membrane potential is crucial for various cellular functions including the transmission of nerve impulses, muscle contractions, and cell signaling. It allows for the generation and propagation of action potentials, which are electrical signals that enable cells to communicate and perform their specialized functions. Changes in the membrane potential, such as depolarization (a decrease in potential) or hyperpolarization (an increase in potential), are important for cellular processes like synaptic transmission, sensory perception, and information processing.

The membrane potential can be altered by various factors, including ion channel activity, ion concentration gradients and the presence of signaling molecules. These changes in membrane potential play a significant role in cell behavior, signal integration and the regulation of physiological processes. It is important to note that the membrane potential is a dynamic property that can be modified in response to stimuli and can vary across different cell types and physiological conditions.

During cell recovery, the membrane potential gradually returns to resting state. This process involves the movement of ions across the cell membrane through ion channels and pumps, which helps restore the proper distribution of charges and ion concentrations. The **recovery current** $r(x,t)$ of the plasma membrane refers to the electrical current that flows across the cell membrane during this restoration phase. It represents the movement of ions, such as potassium (K^+), sodium (Na^+), calcium (Ca^{2+}), and chloride (Cl^-), in and out of the cell to restore the normal balance of charges and ion concentrations. The recovery current is measured in millivolts (mV).

The direction and magnitude of the recovery current of membrane potential depend on the specific ion channels and pumps involved and the specific characteristics of the perturbation or stress that the cell experienced. The restoration of the membrane potential is crucial for the cell to regain its normal functioning, including the ability to generate and transmit electrical signals, maintain proper ion gradients, and carry out various cellular processes.

Appendix: Code for Simulation

```
#Import NGSolve
from ngsolve import *
from ngsolve.webgui import Draw

#Parameters
alpha = 0.08
c1 = 0.0875
c2 = 0.03
b = 0.0055
d = 0.55
Iapp = 0

#Functions
def D(v):
diso = 0.1
dort = 0.9
return diso*v + (dort*v[0],0,0)

#Mesh
from netgen.csg import *
brick = OrthoBrick(Pnt(0,0,0), Pnt(100,100,20))
brick2 = OrthoBrick(Pnt(-2,48,-2), Pnt(2,52,22))
geo = CSGeometry()
geo.Add(brick-brick2)
geo.Add(brick*brick2)
hraw = 10
mesh = Mesh(geo.GenerateMesh(maxh=hraw))
mesh.ngmesh.SetBCName(3,'in')

# Finite Element Space
order = 1
V = H1(mesh, order=order, dirichlet='in')
W = H1(mesh, order=order)
fes = V*W
(phi,r), (p,q) = fes.TnT()

# Bilinear Forms
alinear = BilinearForm(fes)
alinear += (D(grad(phi))*grad(p))*dx

anonlinear = BilinearForm(fes)
anonlinear += -c1*phi*(phi-alpha)*(1-phi)*p*dx
anonlinear += c2*r*p*dx
anonlinear += -(b*phi-b*d*r)*q*dx
```

```
m = BilinearForm(fes, symmetric=True)
m += phi*p*dx
m += r*q*dx

# Assembly Matrices
alinear.Assemble()
anonlinear.Assemble()
m.Assemble()
invm = m.mat.Inverse(freedofs=fes.FreeDofs())

# Time Loop
import time
tend = 400
dt = 0.05
tn = 0
t = Parameter(0)
phi0 = 0.2
r0 = 0
phi in = 1.0
gfu = GridFunction(fes)
gfphi, gfr = gfu.components
gfphi.Set(phi0)
gfr.Set(r0)

# Solution
scene = Draw(gfu.components[0], mesh, min=0, max=1, autoscale=False)
reslinear = gfu.vec.CreateVector()
resnonlinear = gfu.vec.CreateVector()
res = gfu.vec.CreateVector()
while tn < tend - 0.5 * dt:
m.Apply(gfu.vec, res)
alinear.Apply(gfu.vec, reslinear)
anonlinear.Apply(gfu.vec, resnonlinear)
res.data -= dt*reslinear
res.data -= dt*resnonlinear
tn += dt
t.Set(tn)
gfu.components[0].Set(phi in, BND)
res.data -= m.mat*gfu.vec
gfu.vec.data += invm * res
```

Bibliography

[1] D. Hurtado and D. Henao. *Gradient flows and variational principles for cardiac electrophysiology: Toward efficient and robust numerical simulations of the electrical activity of the heart*. Computer Methods in Applied Mechanics and Engineering (2014).

[2] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Mathematics Studies (1973).

[3] H. Brézis and F. Browder. *Nonlinear Integral Equations and Systems of Hammerstein Type*. Advances in Mathematics (1975).

[4] M. Sánchez. *Metodos para Ecuaciones Diferenciales*. Apuntes para el curso de Métodos para Ecuaciones Diferenciales (2022).

[5] R. Plonsey and R. Barr. *Bioelectricity: A Quantitative Approach*. Springer (2007).

[6] R. Nochetto, G. Savaré and C. Verdi. *A Posteriori Error Estimates for Variable Time-Step Discretizations of Nonlinear Evolution Equations*. Pure and Applied Mathemathics (2000).

[7] S. Sanfelici. *Convergence of the Galerkin Approximation of a Degenerate Evolution Problem in Electrocardiology*. Pure and Applied Mathemathics (2002).

[8] L. Evans. *Partial Differential Equations*. Americal Mathematical Society (2010).

[9] H. Brézis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer Universitext (2010).*

[10] P. Colli Franzone and G. Savaré. *Degenerate Evolution Systems Modeling the Cardiac Electric Field at Micro- and Macroscopic Level*. Progress in Nonlinear Differential Equations and Their Applications (2002).

[11] A. Lorenzi and B. Ruf. *Evolution Equations, Semigroups and Functional Analysis*. Springer Basel AG (2000).

[12] V. Anaya, M. Bendahmane, D. Mora and M. Sepúlveda. *A Virtual Element Method for a Nonlocal FitzHugh-Nagumo Model of Cardiac Electrophysiology*. IMA Journal of Numerical Analysis (2019).

[13] I. Ekeland and R. Téman. *Convex Analysis and Variational Problems*. SIAM (1999).

[14] N. Cornejo. *Análisis variacional de las ecuaciones de FitzHugh-Nagumo en electrofisiología cardíaca*. Tesis de Magíster en Matemáticas, Pontificia Universidad Católica de Chile (2014).

[15] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Birkhäuser (2008).

[16] G. Friesecke and G. Dolzmann. *Implicit time discretization and global existence for a quasi-linear evolution equation with nonconvex energy*. SIAM (1997).

[17] Y. Hyona, B. Eisenberg and C. Liu. *An energetic variational approach to ion channel dynamics*. Mathematical Methods in the Applied Sciences (2013).