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DIRECTED POLYMERS IN COMPLEX-VALUED ENVIRONMENTS

by

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DEDICATION

A mi madre Susana Espinosa Noguera.
Si llegue donde está la nieve, solo es el comienzo, aún falta la cumbre.

Contents

Summary

In usual directed polymer models, each step of the path receives a weight given by $\xi = \exp(\beta\,\omega(t,x))$, where $\{\omega(t,x) : t \in \mathbb{N}, x \in \mathbb{Z}^d\}$ is a family of real-valued random variables and $\beta > 0$ is the inverse temperature of the system. The partition function corresponds to the normalization constant of a random measure on paths, known as the polymer measure. It is known that the model exhibits a phase transition: for small values of β, polymer paths resemble a random walk, whereas, for large β, paths are concentrated in energy-advantageous areas. These two opposite phases are known as the region of weak and strong disorder, respectively, and can be identified through the asymptotics of the partition function or, equivalently, by the value of the free energy.

In this work, we examine a model of directed polymers with complex random weights introduced by Cook and Derrida in [7]. Derrida, Evans, and Speer in [8] prove the existence of a third regime governed by interferences due to random phases that cannot observe in the model with positive weights.

We partially extend the results of [8], in two different directions. First, the authors of [8] assume that the random phases and radii of the environment are independent, i.e., $\xi = \exp(\omega + i\,\theta)$, where the ω and θ random variables are independent. We remove this limitation in most of the phase diagram, except in a part of the region of strong disorder, where the model with weights $|\xi|$ is in the region of weak disorder but where sufficiently disordered random phases produce a phase transition. Secondly, under mild regularity assumptions on the law of the environment, we show that the convergence to the free energy proved to hold in probability in [8] can be reinforced to an almost sure convergence. We also show that the standard martingale techniques used for positive weights can still be applied in the region of weak disorder, resulting in a direct calculation of the free energy and almost sure convergence without additional assumptions beyond those of [8].

Chapter 1

Introduction

One of the main problems in disordered systems theory is the study of the behavior of directed polymers in a random environment. The model was introduced in the physics literature by Huse and Henley, in [13], and reached the mathematics community by Imbrie and Spencer, in [14]. We refer the reader to the Comets' book [6] for an account of the main mathematical results up to 2016.

Let us first introduce the model following [6].

- **The random walk:** $\left(S = \{S_n\}_{n\in\mathbb{N}}, P_x\right)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. Precisely, the random sequence S is defined on the probability space $\Omega_{\text{traj}} = \left(\mathbb{Z}^d\right)^{\mathbb{N}}$ with the cylindric σ-algebra $\mathcal{F}$ and a probability measure P_x such that, under P_x, the jumps $S_1 - S_0, \ldots, S_n - S_{n-1}$ are independent with

$$P_x\left(S_0 = x\right) = 1, \qquad P_x\left(S_n - S_{n-1} = \pm e_j\right) = (2d)^{-1}, \quad j = 1, 2, \ldots, d,$$

  where $e_j = \left(\delta_{kj}\right)_{k=1}^{d}$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, $E_x[X]$ denotes the P_x-expectation of a random variable (r.v.) X, and P_0 will be simply written by P.

- **The random environment:** $\omega = \left\{\omega(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}^d\right\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d. (independent identically distributed), defined on a probability space $\left(\Omega = \mathbb{R}^{\mathbb{N}^* \times \mathbb{Z}^d}, \mathcal{G}, \mathbb{P}\right)$ such that

$$\forall \beta \in \mathbb{R} : \quad \mathbb{E}[\exp(\beta\, \omega(n,x))] < \infty.$$

  From these two basic ingredients we define the object we consider in this work.

- **The polymer measure:** For any $n \in \mathbb{N}^*$, define the probability measure $P_n^{\beta,\omega}$ on the path space $(\Omega_{\mathrm{traj}}, \mathcal{F})$ by

$$P_n^{\beta,\omega}(d\mathbf{x}) = \frac{1}{\mathsf{Z}_n(\omega,\beta)} \exp\left[\beta\, \mathsf{H}_n(\mathbf{x})\right] \; P(d\mathbf{x}), \tag{1.1}$$

 where $\beta > 0$ is a parameter (the inverse temperature), where

$$\mathsf{H}_n(\mathbf{x}) = \mathsf{H}_n^{\omega}(\mathbf{x}) = \sum_{j=1}^{n} \omega\left(j, x_j\right),$$

 is the energy of the path $\mathbf{x}$ in environment ω (Hamiltonian potential) and

$$\mathsf{Z}_n = \mathsf{Z}_n(\omega,\beta) = E\left[\exp\left(\beta \sum_{j=1}^{n} \omega\left(j, S_j\right)\right)\right],$$

 is the normalizing constant to make $P_n^{\beta,\omega}$ a probability measure. From its definition (1.1), $P_n^{\beta,\omega}$ is the Gibbs measure with Boltzmann weight $\exp\left[\beta \mathsf{H}_n\right]$, and Z_n is the so-called partition function. Of course, in the present context, the above expectation is simply a finite sum,

$$\mathsf{Z}_n(\omega,\beta) = \sum_{\mathbf{x}} (2d)^{-n} \exp\left(\beta\, \mathsf{H}_n(\mathbf{x})\right),$$

 where $\mathbf{x}$ ranges over the $(2d)^n$ possible paths of length n for the simple random walk.

This problem arose in statistical mechanics, specifically in the thermal equilibrium of this system. As such, our main object of interest is the effect of the disorder on the asymptotic of the partition function. The model is known to transition from a diffusive regime on $\mathbb{Z}^d$ at high temperatures to a superdiffusive behavior at lower temperatures (see [6]).

1.1 Physical interest of the model

The problem is associated with several physical phenomena, such as the study of an Ising ferromagnet with random impurities below the T_c temperature [13], or the growth patterns of clusters and solidification fronts, described by Kardar, Parisi and Zhang in [16].

In [9], Derrida and Spohn show that the problem also shares many features with the spin-glass phase, especially in the mean-field approximation. In the language of statistical mechanics, the mean field version of directed polymers corresponds to the model on a Cayley tree. So far, this is one of the few disordered systems for which it has been proved that the predictions of replica theory, in the case of the broken symmetry, provide the correct free energy on the tree.

1.2 Model on the Cayley Tree with positive weights

Various techniques have been used to treat the model on the Cayley tree with positive weights. From the physics perspective, Derrida and Spohn in [9] showed that the study of directed polymers on trees can be reduced to the analysis of nonlinear reaction-diffusion equations. Later, Chauvin and Rouault, in [5], used traveling waves, while, on the mathematics side, Buffet, Patrick, and Pulé in [4] focus on the computation of non-integer moments of the partition function, along with martingale arguments, which have the advantage of being transparent and rigorous. All these approaches lead to the same phase diagram and the same expressions for the free energy in the different regions. See also [11] or [15].

The weights assigned to the lattice bonds are positive in the original version of the directed polymer problem. In [7], the problem was generalized by removing this limitation, e.g., weights may have random signs or take complex random values. Goldschmidt and Blum, in [12], work with this generalization as a reasonable model for the conductivity of strongly localized jumping electrons since the transmission of such electrons is dominated by directed paths, and interference effects occur by summing the contributions of individual paths.

In [7], the authors predicted a phase diagram consisting in three regions, which we will denote $\mathcal{R}1$, $\mathcal{R}2$ and $\mathcal{R}3$. The regions $\mathcal{R}1$ and $\mathcal{R}2$ are the only regions found in the case of positive weights; in this case, they are the high and low-temperature regions, respectively. The region $\mathcal{R}3$ is characterized by a new high temperature phase, characterized by substantial interference effects, which occur when the weight-sign fluctuations in the region are significant.

Goldschmidt and Blum, in [12], have examined the problem of continuous directed polymers with complex weights in finite dimensions using a replica approach. At high dimensoins, where results are expected to match those of the model on the tree, they found a phase diagram quite different from that predicted in [7]. Specifically, they obtained two additional regions ($\mathcal{R}4$ and $\mathcal{R}5$), which correspond to different schemes for the broken symmetry of the replicas. The validity of these results were questioned in [7].

The model was solved on mathematical grounds by Derrida, Evans, and Speer, in [8]. It was proved that there are only three regions in the phase diagram. However, the precise phase diagram on the lattice is still an open question.

We note that the phase diagram of the model shares similarities with Complex Multiplicative Chaos [17, 18, 19]. It can also be interpreted as multiplicative cascades [1, 2, 3].

In this thesis, we extend the results of [8] by incorporating arguments from [4]. We were able to relax the hypothesis on the independence of the random radii and phases. Along the way, we simplified many of the arguments from [8].

1.3 Outline of the document

The next chapter presents a mathematical introduction, including some tools used in the proofs. Specifically, we discuss some aspects of martingales, concluding with the Martingale Convergence Theorem for complex-valued martingales.

Our results are presented in Chapter 3, which is divided into six sections. In Section 3.1, we define precisely the model and the free energy that we consider, we expose our hypotheses on the environment then state the main results. A brief comparison is made with the results holding for a random environment with positive values.

In Section 3.2, we obtain general estimates regarding the partition function and its moments. Then, we prove some extensions of the key estimates of [8]. We end that section exposing our main scheme of proof.

The region $\mathcal{R}1$ is presented in Section 3.3, where we use martingale techniques. We continue with region $\mathcal{R}3$ in Section 3.4. The region $\mathcal{R}2$ is treated in Section 3.5, which is divided into two parts. In the second part, a monotonicity argument is used to compute the free energy, which reduces the problem to the results obtained for the region $\mathcal{R}1$ and the model with positive-valued random environments. Finally, Section 3.6 characterizes the three regions for the model with random independent phases and radii.

Chapter 4 presents conclusions and discusses possible extensions of our results.

Finally, the Appendix contains the proof of a lemma which was omitted in the core of the thesis.

Chapter 2

Notations and Preliminary Results

This chapter presents definitions and basic results of probability theory, which will allow us to understand more efficiently the demonstrations made throughout this document.

It is also intended to standardize the notations used in this document.

2.1 Discrete time Martingales

Let $(\Omega, \mathcal{F})$ be a measurable space. A **filtration** is an increasing sequence of sub σ-algebras $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$, i.e., for all $n \in \mathbb{N}$ we have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. We will suppose that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ is the **trivial σ-algebra**, and we will denote

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right).$$

Definition 2.1 (Natural filtration)**.** *We say that $\mathbb{F}$ is a **natural filtration**, or canonical filtration, for the process $\mathbb{X} = (X_n)_{n\in\mathbb{N}}$ if $\mathcal{F}_n = \sigma(X_k:\ 1 \le k \le n)$, for all $n \in \mathbb{N}$.*

We say that a process $\mathbb{X} = (X_n)_{n\in\mathbb{N}}$ is an **adapted process** with respect to $\mathbb{F}$, if

$$\forall n \in \mathbb{N}: \quad X_n \in \mathcal{F}_n.$$

Definition 2.2 (Martingale, Submartingale and Supermartingale)**.** *We say that $\mathbb{X}$ is an $\mathbb{F}$-**martingale** if each X_n is integrable, $\mathbb{X}$ adapted and satisfies the **martingale property**, i.e.,*

$$\forall n \in \mathbb{N}: \quad \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = X_n.$$

*If we replace $=$ by $\geq$ or $\leq$, in the previous identity, we will say that the process $\mathbb{X}$ is an $\mathbb{F}$-**submartingale** or an $\mathbb{F}$-**supermartingale**, respectively.*

REMARK. When the filtration $\mathbb{F}$ is clear, we eliminate "$\mathbb{F}$" from the notation.

THEOREM 2.1. *For all $m, n \in \mathbb{N}$ such that $m \leq n$ we have:*

(i) If $\mathbb{X}$ is an $\mathbb{F}$-martingale, then $\mathbb{E}\left[X_n \mid \mathcal{F}_m\right] = X_m$.

(ii) If $\mathbb{X}$ is an $\mathbb{F}$-submartingale, then $\mathbb{E}\left[X_n \mid \mathcal{F}_m\right] \geq X_m$.

(iii) If $\mathbb{X}$ is an $\mathbb{F}$-supermartingale, then $\mathbb{E}\left[X_n \mid \mathcal{F}_m\right] \leq X_m$.

For the proof, see [10, Theorems 5.2.1-2, pp. 233].

THEOREM 2.2. *Suppose that $\mathbb{X}$ is an $\mathbb{F}$-martingale (more generally a submartingale) and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function (respectively convex and increasing). Let $\phi(\mathbb{X}) = (\phi(X_n))_{n\in\mathbb{N}}$. If each $\phi(X_n)$ is integrable, then $\phi(\mathbb{X})$ is an $\mathbb{F}$-submartingale.*

For the proof, see [10, Theorems 5.2.3-4, pp. 233-234].

2.2 Martingale Convergence Theorem

We say that a collection of random variables $(X_i)_{i\in I}$ that satisfies

$$\lim_{n\to\infty} \sup_{i\in I} \mathbb{E}\left[|X_i| \mathbb{1}_{(|X_i|>n)}\right] = 0.$$

it is **uniformly integrable (u.i.)**.

THEOREM 2.3 (Martingale Convergence Theorem)**.** *Let $\mathbb{X}$ be an $\mathbb{F}$-martingale (supermartingale, submartingale) such that $\sup\limits_{n\in\mathbb{N}} \mathbb{E}\left[|X_n|\right] < \infty$, then there exists $X \in \mathbb{L}^1$ such that $X_n \xrightarrow{\text{a.s.}} X$.*

For the proof, see [21, Theorem 9.8.12, pp. 343-344].

THEOREM 2.4. *Suppose that $\mathbb{X}$ is a u.i. $\mathbb{F}$-martingale (supermartingale, submartingale). Then there exists an integrable random variable X such that $X_n \xrightarrow{\text{a.s.}} X$ and in $\mathbb{L}^1$. Furthermore,*

$$\forall n \in \mathbb{N} : \quad X_n = \mathbb{E}\left[X \mid \mathcal{F}_n\right],$$

and with the corresponding inequalities in the super and submartingale cases.

For the proof, see [21, Theorem 9.8.14, pp. 344].

COROLLARY 2.5. *Let $r \in (1,\infty)$ and let $\mathbb{X}$ be an $\mathbb{F}$-martingale which is bounded in $\mathbb{L}^r$, where $r \in (1,\infty)$. Then, $\mathbb{X}$ is u.i. and there exists a random variable $X \in \mathbb{L}^r$ such that $X_n \xrightarrow{\text{a.s.}} X$ and in $\mathbb{L}^r$.*

For the proof, see [21, Corollary 9.8.22, pp. 348].

We say that $\mathbb{M} = (M_n)_{n\in\mathbb{N}}$ is a complex-valued $\mathbb{F}$-martingale is $M_n \in \mathbb{C}$ for all $n \in \mathbb{N}$, $\mathbb{E}[|M_n|] < \infty$ for all $n \in \mathbb{N}$ and

$$\forall n \in \mathbb{N}: \quad M_n = \mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right].$$

LEMMA 2.6. *Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}}$ be a complex-valued $\mathbb{F}$-martingale. Then the processes*

$$\mathrm{Re}(\mathbb{M}) = (\mathrm{Re}(M_n))_{n\in\mathbb{N}} \quad \textit{and} \quad \mathrm{Im}(\mathbb{M}) = (\mathrm{Im}(M_n))_{n\in\mathbb{N}},$$

are (real-valued) $\mathbb{F}$-martingales.

Proof. Let us prove each condition in the definition of a martigales:

(i) *(Integrability)* Since $\mathbb{M}$ is integrable we have

$$\infty > \mathbb{E}\left[|M_n|\right] = \mathbb{E}\left[\sqrt{|\mathrm{Re}(M_n)|^2 + |\mathrm{Im}(M_n)|^2}\right] \geq \mathbb{E}\left[|\mathrm{Re}(M_n)|\right].$$

Similarly, $\mathbb{E}\left[|\mathrm{Im}(M_n)|\right] < \infty$.

(ii) *(Adaptability)* As $M_n \in \mathcal{F}_n$ and the functions $z \mapsto \mathrm{Re}(z)$ and $z \mapsto \mathrm{Im}(z)$ are continuous, we conclude that $\mathrm{Re}(M_n)$, $\mathrm{Im}(M_n) \in \mathcal{F}_n$.

(iii) *(Martingale property)* Applying the linearity of the conditional expectation, we have

$$\begin{aligned}\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] &= \mathbb{E}\left[\mathrm{Re}(M_{n+1}) + i\,\mathrm{Im}(M_{n+1}) \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[\mathrm{Re}(M_{n+1}) \mid \mathcal{F}_n\right] + i\,\mathbb{E}\left[\mathrm{Im}(M_{n+1}) \mid \mathcal{F}_n\right]. \end{aligned} \tag{2.1}$$

On the other hand, since $\mathbb{M}$ is a martingale, we have

$$\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] = M_n = \mathrm{Re}(M_n) + i\ \mathrm{Im}(M_n). \tag{2.2}$$

Replacing (2.2) in (2.1) we obtain

$$\mathbb{E}\left[\mathrm{Re}(M_{n+1}) \mid \mathcal{F}_n\right] = \mathrm{Re}(M_n) \quad \text{and} \quad \mathbb{E}\left[\mathrm{Im}(M_{n+1}) \mid \mathcal{F}_n\right] = \mathrm{Im}(M_n).$$

We conclude from the above points that the processes $\mathrm{Re}(\mathbb{M})$ and $\mathrm{Im}(\mathbb{M})$ are $\mathbb{F}$-martingales. $\square$

PROPOSITION 2.7. *Let $r > 1$ and let $\mathbb{M} = (M_n)_{n\in\mathbb{N}}$ be a complex-valued $\mathbb{F}$-martingale such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|M_n|^r\right] < \infty.$$

Then there exist random variables $\mathrm{Re}(M), \mathrm{Im}(M) \in \mathbb{L}^r$ such that

$$\mathrm{Re}(M_n) \xrightarrow{\text{a.s.}} \mathrm{Re}(M) \quad \textit{and} \quad \mathrm{Im}(M_n) \xrightarrow{\text{a.s.}} \mathrm{Im}(M),$$

and in $\mathbb{L}^r$. As a consequence, $M_n \xrightarrow{\text{a.s.}} M := \mathrm{Re}(M) + i\,\mathrm{Im}(M)$ and in $\mathbb{L}^r$.

Proof. From the above lemma the processes $\mathrm{Re}(\mathbb{M})$ and $\mathrm{Im}(\mathbb{M})$ are $\mathbb{F}$-martingales.

It is enough to verify the hypothesis of Corollary 2.5, i.e. we will show that $\mathrm{Re}(\mathbb{M}), \mathrm{Im}(\mathbb{M}) \in \mathbb{L}^r$. Let us note

$$|M_n|^r = \left[\sqrt{|\mathrm{Re}(M_n)|^2 + |\mathrm{Im}(M_n)|^2}\right]^r \geq |\mathrm{Re}(M_n)|^r.$$

Similarly, $|\mathrm{Im}(M_n)|^r \leq |M_n|^r$. This finishes the proof. $\square$

Chapter 3

Directed polymers in complex-valued environments on trees

3.1 The model and main results

The model we consider throughout this chapter is that of directed polymers on the Cayley tree $(\mathbb{T}, \mathcal{E})$ with **branching relation** $b \in \mathbb{N}$.

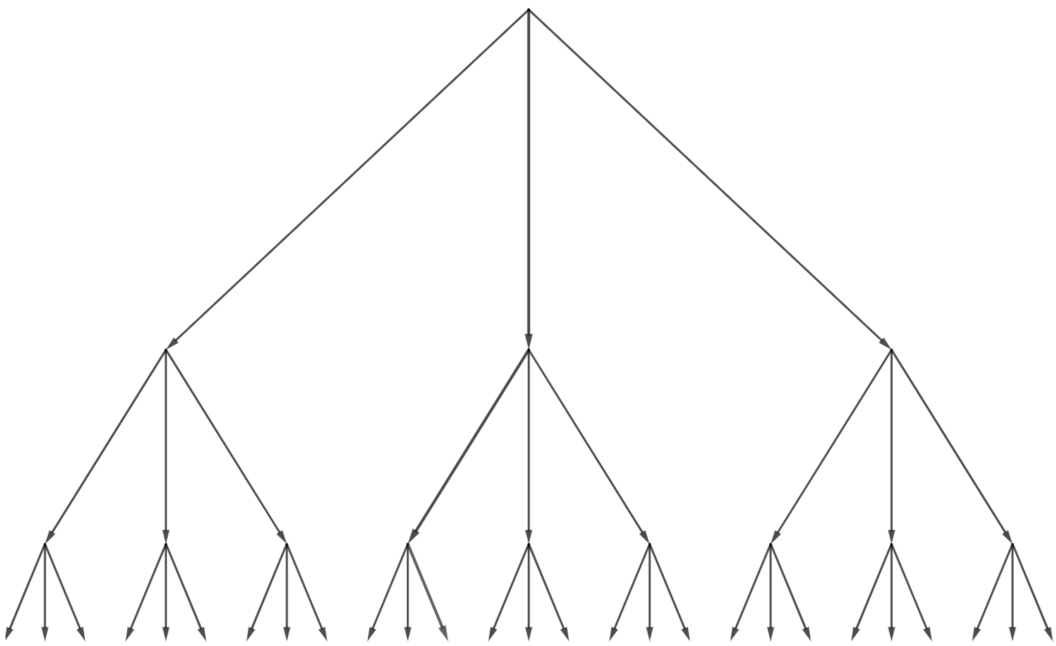
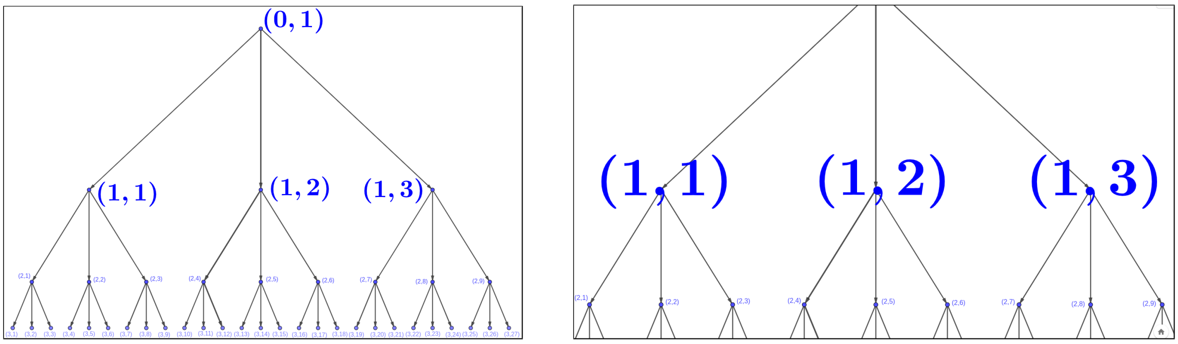


Figure 3.1: Cayley tree with branching ratio three

We will label the vertices with pairs (j, k) where j belongs to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and identifies the **generation** while k is in $\{1, \ldots, b^j\}$ and lists the vertices or sites from left to right in the j-th generation.



(a) Cayley tree labeled

(b) $\mathbb{T}_1$ or First Generation

We will define the set of all nodes of the n-th generation, where $n \in \mathbb{N}$, as

$$\mathbb{T}_n = \{(j,k) \in \mathbb{T} : \ j = n\} .$$

A **path** s starting at vertex $x \in \mathbb{T}$ and of **length** $|s| = n$ is a finite sequence of vertices $[(j_0, k_0), (j_1, k_1), \ldots, (j_n, k_n)]$, where $(j_0, k_0) = x$ and

$$\forall i \in \{1, \ldots, n\} : \qquad k_i = b\, k_{i-1} + a, \quad j_i = j_{i-1} + 1,$$

where a takes values in the set $\{-(b-1), \ldots, 0\}$.

We denote by $S_{n,x}$ the **space of paths** of length n starting at vertex $x \in \mathbb{T}$, and simply by S_n when x is the **root node**, i.e., $x = (0, 1)$.

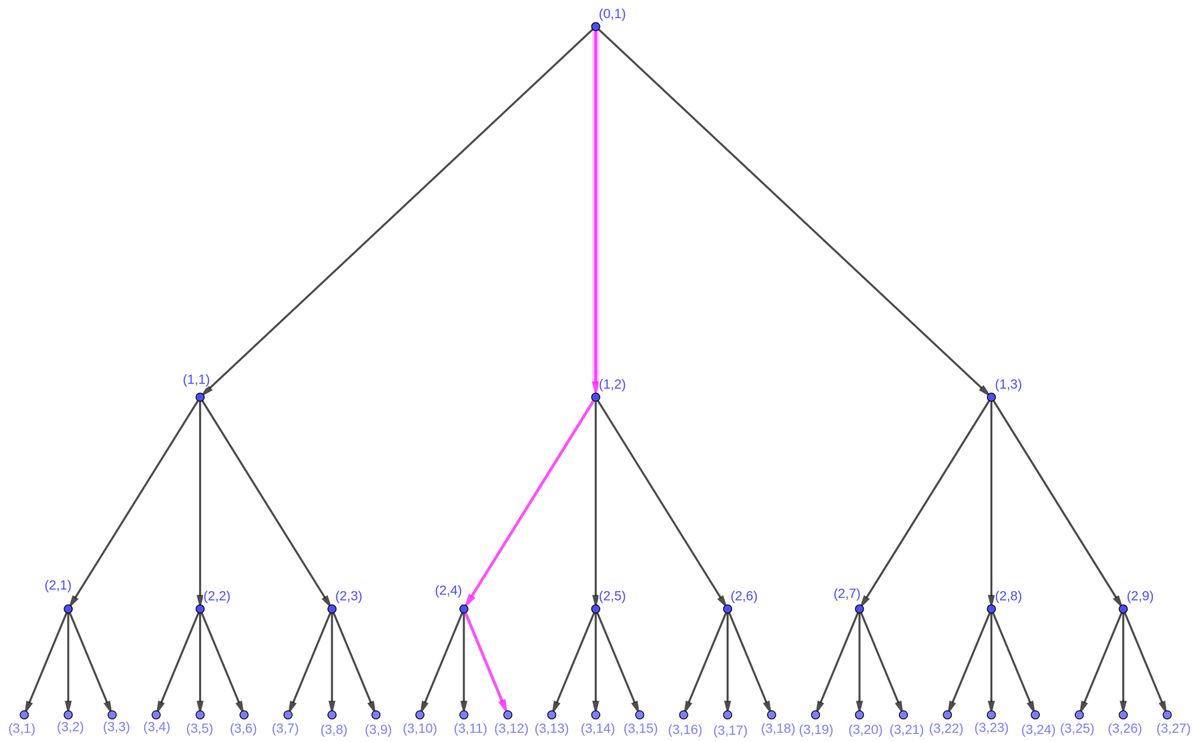


Figure 3.3: A path over the Cayley Tree.

Our **random environment** on the tree (minus its root) is a set of i.i.d. complex valued non-trivial random variables indexed by $\mathbb{T}$:

$$\xi = \{\xi(x) : x \in \mathbb{T}\}, \qquad \xi(0,1) = 1.$$

With a slight abuse of notation, we will define their common law by ξ.

Remark. To be more precise, these random variables are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{C}^{\mathbb{T}}$, $\mathcal{F}$ is the Borel σ-algebra over Ω, and $\mathbb{P} = \mathrm{P}_0^{\otimes \mathbb{T}}$ for some probability measure P_0 defined on the Borel σ-algebra of $\mathbb{C}$. We then take ξ as the canonical process.

We will further assume that ξ satisfies the following assumptions:

- **(HA1)** The distribution of the amplitude $|\xi|$ is continuous, i.e.,

$$\forall a \in \mathbb{R} : \quad \mathbb{P}\left(|\xi| = a\right) = 0.$$

- **(HA2)** All moments of the amplitude of ξ are well defined, i.e.,

$$\forall \alpha \geq 0 : \quad \mathbb{E}\left[|\xi|^{\alpha}\right] < \infty.$$

We will denote by $\mathrm{Z}_{n,x}(\xi)$ the partition function given by

$$\mathrm{Z}_{n,x}(\xi) = \sum_{s \in S_{n,x}} \prod_{t=1}^{n} \xi(s_t),$$

We also denote $\mathrm{Z}_n(\xi) = \mathrm{Z}_{n,(0,1)}(\xi)$.

Remark. Fixing $m \in \mathbb{N}$, for all $x \in \mathbb{T}_m$ it is clear that the random variables $\mathrm{Z}_{n,x}(\xi)$ are i.i.d. and distributed as $\mathrm{Z}_n(\xi)$.

Remark. Throughout this paper, as we will always work with fixed ξ, we will often denote $\mathrm{Z}_{n,x} = \mathrm{Z}_{n,x}(\xi)$. We will use the full notation $\mathrm{Z}_{n,x}(\xi)$ whenever it is necessary to highlight the dependence on environment.

3.1.1 The Free Energy

Turning now to a discussion of the phase diagram of the model, let us define the function

$$\begin{aligned} G \colon \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \tfrac{1}{\alpha} \ln\left(b\, \mathbb{E}\left[|\xi|^{\alpha}\right]\right). \end{aligned}$$

Note that hypothesis HA2 guarantees that G is well-defined.

We start with some simple properties of the function G.

PROPOSITION 3.1.

(i) The function $\alpha \mapsto \alpha\, G(\alpha) = \ln\left(b\, \mathbb{E}\left[|\xi|^{\alpha}\right]\right)$ *is convex, with* $G(0) = \ln b$.

(ii) The function G satisfies exactly one of the following properties:

- *There exists a unique minimizer of G denoted by* $\alpha_{\min} > 0$ *, i.e., G is strictly decreasing in* $(0, \alpha_{\min}]$ *and strictly increasing in* $[\alpha_{\min}, \infty)$.
- *G is strictly decreasing in* $\mathbb{R}_+$.

The proof is deferred to the appendix.

REMARK. If G is strictly decreasing, we adopt the convention $\alpha_{\min} = \infty$.

Our goal is to study the behavior of the **free energy**

$$\frac{1}{n}\, \ln |\mathsf{Z}_n|,$$

of infinitely long polymers, i.e., when $n \to \infty$. The system may exist in three regions or lie on their boundaries. Distinct regions are characterized by distinct analytic expressions for the free energy value per step in the limit of infinitely long polymers.

The following theorem is our main result and extends [8, Theorems 6.5 and 7.4].

THEOREM 3.2 (Regions)**.** *Assume that* ξ *satisfies hypotesis HA1 and HA2. Therefore, the* ***free energy limit****, i.e.,*

$$\mathsf{f}(\xi) = \lim_{n\to\infty} \frac{1}{n}\, \ln |\mathsf{Z}_n(\xi)|, \tag{3.1}$$

exists in probability. Furthermore:

- ***(REGION*** $\mathcal{R}1$***)*** *If there exists* $\alpha \in (1, 2]$ *such that* $G(\alpha) < \ln\left(b\, \left|\mathbb{E}\left[\xi\right]\right|\right)$, *we have*

$$\mathsf{f}(\xi) = f_{\mathrm{I}} = \ln\left(b\, \left|\mathbb{E}\left[\xi\right]\right|\right). \tag{3.2}$$

- ***(REGION*** $\mathcal{R}2$***)*** *If* $\alpha_{\min} < 1$ *or if* $1 \le \alpha_{\min} < 2$, $G\left(\alpha_{\min}\right) > \ln\left(b\, \left|\mathbb{E}\left[\xi\right]\right|\right)$ *and the families of random variables* $|\xi(x)|$ *and* $\frac{\xi(x)}{|\xi(x)|}$ *are independent, then*

$$\mathsf{f}(\xi) = f_{\mathrm{II}} = G\left(\alpha_{\min}\right). \tag{3.3}$$

- ***(REGION $\mathcal{R}3$)*** *If $\alpha_{\min} > 2$ and $G(2) > \ln(b\ |\mathbb{E}[\xi]|)$, we obtain*

$$\mathsf{f}(\xi) = f_{\mathrm{III}} = G(2). \tag{3.4}$$

Furthermore, the limit (3.1) *holds $\mathbb{P}$-a.s. in the region $\mathcal{R}1$ and in the part of the region $\mathcal{R}2$ where $\alpha_{\min} < 1$.*

The above result was obtained by Derrida, Evans, and Speer, in [8], under the hypothesis that the random variables $|\xi(x)|$ and $\frac{\xi(x)}{|\xi(x)|}$ are independent. As can be seen above, we remove this constraint from most of the phase diagram, except for the part of the region $\mathcal{R}2$ where $1 \leq \alpha_{\min} < 2$. As noted below, under this hypothesis, the model with positive weights, $|\xi(x)|$, is in the weak disorder regime. Here, the addition of sufficiently disordered random phases can induce strong disorder. It is then quite natural to expect that some hypothesis is required to observe such behavior.

REMARK. Let us note that if the law of the environment is nontrivial, applying Jensen's inequality, we have

$$\ln(b\ |\mathbb{E}[\xi]|) < \ln(b\,\mathbb{E}[|\xi|]) = G(1).$$

Then, if $\alpha_{\min} < 1$, we have $G(\alpha) > G(1)$ for all $\alpha > 1$ and the system cannot be in the region $\mathcal{R}1$. Therefore, in the region $\mathcal{R}1$, one has $\alpha_{\min} > 1$.

However, in the region $\mathcal{R}1$, if there exists $\alpha \in (1, 2]$ such that $G(\alpha) < \ln(b\ |\mathbb{E}[\xi]|)$, we have again $G(\alpha_{\min}) < \ln(b\ |\mathbb{E}[\xi]|)$, so that the system cannot be in the region $\mathcal{R}2$. In the same vein, if $\alpha_{\min} > 2$, we obtain $G(2) < G(\alpha) < \ln(b\ |\mathbb{E}[\xi]|)$ so that the system cannot be in the region $\mathcal{R}3$.

The observation above shows that the three regions are mutually exclusive.

REMARK. From the exact calculations of Section 3.2, we will see that, in the $\mathcal{R}1$ region of weak disorder, it holds that

$$\mathsf{f}(\xi) = \lim_{n\to\infty} \frac{1}{n} \ln |\mathbb{E}[\mathsf{Z}_n(\xi)]|.$$

The right-hand side is generally referred to as the annealed free energy. In the Section below, we will recall the possible values of the free energy for the model with positive weights. We can then remark that, in the regions $\mathcal{R}2$ and $\mathcal{R}3$, it holds that

$$\mathsf{f}(\xi) = \lim_{n\to\infty} \frac{1}{2n} \ln \mathsf{Z}_n(|\xi|^2),$$

where $\mathsf{Z}_n(|\xi|^2)$ denotes the partition function of the model with random environment

$$|\xi|^2 = \left\{ |\xi(x)|^2 : \ x \in \mathbb{T} \right\}, \tag{3.5}$$

which is in the regime of strong disorder (respect. weak disorder) in the region $\mathcal{R}2$ (respect. $\mathcal{R}3$). In the case of the $\mathcal{R}3$ region, it also holds that

$$\mathsf{f}(\xi) = \lim_{n\to\infty} \frac{1}{2n} \ln \mathbb{E}\left[\mathsf{Z}_n(|\xi|^2)\right],$$

which corresponds to (half of) the annealed free energy for the model with random environment (3.5).

The limit (3.1) is, in fact, an almost surely limit in the region $\mathcal{R}1$ and in the case $\alpha_{\min} \leq 1$. This can be extended to the entire phase diagram under additional assumptions on the law of the environment.

DEFINITION 3.1 (τ-property). *Let $\tau \in (0,2]$ be fixed, and let us denote the ball of center $z \in \mathbb{C}$ and radius r as $B(z,r)$. We say that the environment satisfies the **τ-property** if there exists a finite constant $C > 0$ such that*

$$(\forall (z,r) \in \mathbb{C} \times \mathbb{R}_+^* : \quad \mathbb{P}[\xi \in B(z,r)] \leq Cr^\tau) \qquad \wedge \qquad \mathbb{E}\left[|\xi|^{-\tau}\right] < \infty.$$

THEOREM 3.3. *If the law of the environment satisfies the τ-property, for some $\tau \in (0,2]$, then the limit* (3.1) *almost surely exists.*

For positive weights, martingale techniques are a powerful tool for obtaining information about the model. It remains valid for complex weights, at least in the $\mathcal{R}1$ region.

We will denote $\mathbb{T}_{\leq n} = \bigcup_{i=0}^{n} \mathbb{T}_i$ the **set of all nodes** up to the n-th generation, and the **natural filtration** $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{F}_n = \sigma\left(\xi(x) : \ x \in \mathbb{T}_{\leq n}\right).$$

It is standard, when $\mathbb{E}[\xi] \neq 0$, to define the process $\mathbb{M} = (\mathsf{M}_n)_{n\in\mathbb{N}}$ such that

$$\mathsf{M}_n = \frac{\mathsf{Z}_n}{\mathbb{E}[\mathsf{Z}_n]},$$

which turns out to be a $\mathbb{F}$-martingale. Moreover, we can say that in the region $\mathcal{R}1$:

PROPOSITION 3.4. *The martingale $\mathbb{M}$ is uniformly integrable in the region $\mathcal{R}1$. More-*

over, if

$$b\ |\mathbb{E}[\xi]|^2 > \mathbb{E}\left[|\xi|^2\right], \tag{3.6}$$

then

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|\mathsf{M}_n|^2\right] < \infty. \tag{3.7}$$

The **region** $\mathbb{L}^2$ is given by the condition (3.6).

3.1.2 An Example of Goldschmidt and Blum (1992)

Goldschmidt and Blum, in [12] (see also [8]), examined the problem of directed walks with a random potential with real and imaginary parts ω and θ respectively, i.e., ξ given by the expression

$$\xi = \exp\left(-\beta\omega + i\gamma\theta\right),$$

where $(\beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}$ is fixed, and the energy ω and phase θ are standard independent Gaussian random variables (i.e., with distribution function $N(0,1)$). Note that:

- β allows to adjust the width of the amplitude of ξ, i.e., $|\xi|$.
- γ allows us to adjust the **phase** width of ξ, i.e., $s = \xi/|\xi|$.

We now verify that hypotheses HA1 and HA2 are satisfied.

- *(HA1)* Since $\omega \sim N(0,1)$, we have $-\beta\omega \sim N(0,\beta^2)$. Then,

$$\begin{aligned} |\xi| &= |\exp\left(-\beta\omega + i\gamma\theta\right)| = |\exp\left(-\beta\omega\right)|\ |\exp\left(i\gamma\theta\right)| \\ &= \exp\left(-\beta\omega\right) \sim Log - N(0,\beta^2), \end{aligned}$$

  i.e., the amplitude of ξ has a continuous (lognormal) distribution.

- *(HA2)* For all $\alpha > 0$,

$$\mathbb{E}\left[|\xi|^\alpha\right] = \mathbb{E}\left[\exp\left(-\beta\alpha\omega\right)\right] = \exp\left(\frac{(\beta\alpha)^2}{2}\right) < \infty.$$

Recall that the random variables ω and θ are independent. Therefore, the amplitude and phase of ξ are independent, i.e.,

$$|\xi| = \exp\left(-\beta\omega\right), \tag{3.8}$$

$$s = \frac{\xi}{|\xi|} = \exp\left(i\gamma\theta\right), \tag{3.9}$$

are independent.

We will continue studying G. By using (3.8) and the moment-generating function of the normal distribution, we have that G can be expressed by

$$G(\alpha) = \frac{1}{\alpha}\ \ln\left(b\ \mathbb{E}\left[|\xi|^{\alpha}\right]\right) = \frac{1}{\alpha}\ \ln\left(b\ \exp\left(\frac{(\beta\alpha)^2}{2}\right)\right) = \frac{\ln b}{\alpha} + \frac{\alpha\,\beta^2}{2}, \tag{3.10}$$

which generates the following table showing the characteristics of G:

		α			
		0		β_0/β	
$G(\alpha)$	$\frac{\ln b}{\alpha} + \frac{\alpha\,\beta^2}{2}$	$\nexists$	$+$	$\beta\,\beta_0$	$+$
$G'(\alpha)$	$\frac{\beta^2}{2} - \frac{\ln b}{\alpha^2}$	$\nexists$	$-$	0	$+$
$G''(\alpha)$	$\frac{2\ln b}{\alpha^3}$	$\nexists$	$+$	β^3/β_0	$+$

Table 3.1: Characteristic of G

where $\beta_0 = (2\ln b)^{1/2}$. From the above table we can see that G is convex and reaches its minimum at $\alpha_{\min} = \beta_0/\beta$.

We will continue with the characterization of $\ln\left(b\ |\mathbb{E}\left[\xi\right]|\right)$. By (3.8) and (3.9) together with the moment generating and characteristic functions of the Gaussian variables ω and θ, and their independence, we obtain

$$\begin{aligned}
\ln\left(b\,|\mathbb{E}\left[\xi\right]|\right) &= \ln\left(b\,|\mathbb{E}\left[|\xi|\ s\right]|\right) = \ln\left(b\,|\mathbb{E}\left[|\xi|\right]|\ \ |\mathbb{E}\left[s\right]|\right) \\
&= \ln\left(b\mathbb{E}\left[|\xi|\right]\right) + \ln\left(|\mathbb{E}\left[s\right]|\right) \\
&= \ln\left(b\mathbb{E}\left[\exp\left(-\beta\omega\right)\right]\right) + \ln\left(|\mathbb{E}\left[\exp\left(i\gamma\theta\right)\right]|\right) \\
&= \ln\left(b\exp\left(\frac{\beta^2}{2}\right)\right) + \ln\left(\exp\left(-\frac{\gamma^2}{2}\right)\right) \\
&= \ln b + \frac{\beta^2 - \gamma^2}{2}.
\end{aligned} \tag{3.11}$$

Thanks to (3.10) and (3.11) our conditions for the regions will be:

- *(REGION $\mathcal{R}1$)* Since $G(1) \geq \ln\left(b\ |\mathbb{E}\left[\xi\right]|\right)$, then applying the intermediate value theorem, the condition (3.2) is equivalent to

  $$\left(2 < \alpha_{\min}\ \wedge\ G(2) < \ln\left(b\ |\mathbb{E}\left[\xi\right]|\right)\right) \vee \left(1 < \alpha_{\min} \leq 2\ \wedge\ G(\alpha_{\min}) < \ln\left(b\ |\mathbb{E}\left[\xi\right]|\right)\right),$$

and using (3.11) together with (3.10) we obtain

$$\left(\beta < \frac{\beta_0}{2} \wedge \beta^2 + \gamma^2 < \left(\frac{\beta_0}{\sqrt{2}}\right)^2\right) \vee \left(\frac{\beta_0}{2} \leq \beta < \beta_0 \wedge \beta + \gamma < \beta_0\right).$$

- *(REGION $\mathcal{R}2$)* The condition (3.3) is equivalent to

$$\beta_0 < \beta \qquad \vee \qquad \left(\frac{\beta_0}{2} \leq \beta < \beta_0 \quad \wedge \quad \beta + \gamma > \beta_0\right).$$

- *(REGION $\mathcal{R}3$)* The condition (3.4) is equivalent to

$$\frac{\beta_0}{2} > \beta \qquad \wedge \qquad \beta^2 + \gamma^2 > \left(\frac{\beta_0}{\sqrt{2}}\right)^2.$$

Then the values of the free energy $\mathrm{f}(\xi)$, for the three regions are

$$f_{\mathrm{I}} = \ln b + \frac{\beta^2 - \gamma^2}{2}, \qquad f_{\mathrm{II}} = \beta\,\beta_0, \qquad f_{\mathrm{III}} = \frac{\beta_0^2}{4} + \beta^2.$$

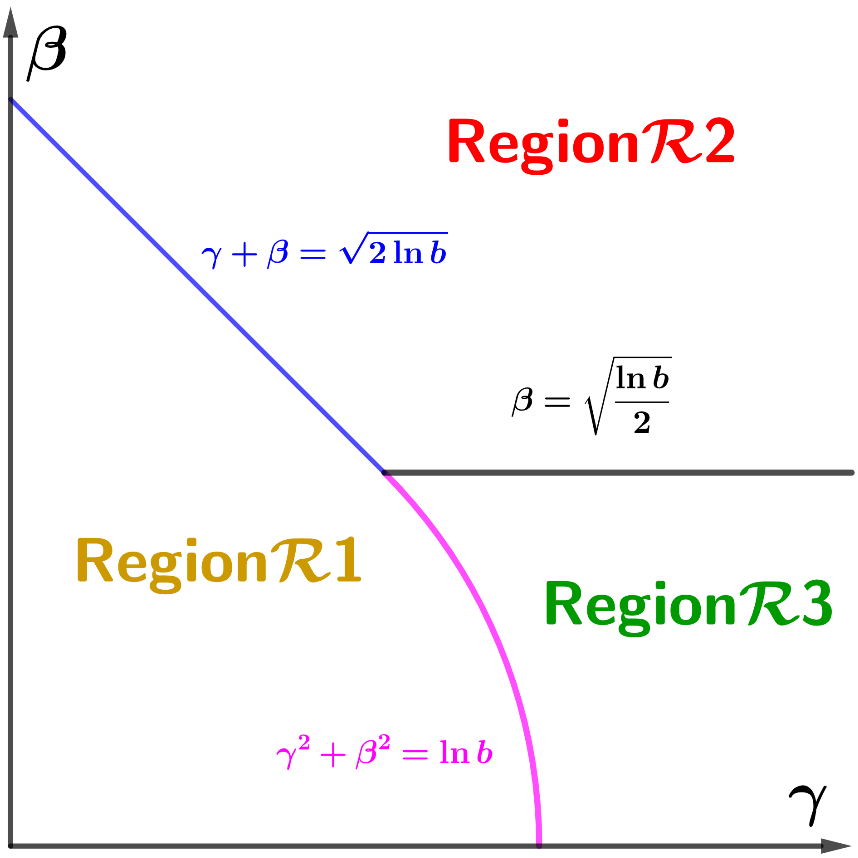


Figure 3.4: Regions using Gaussian random variables.

3.1.3 Independent Radii and Phases

With the above example in mind, we can formulate the problem of independent radii and phases in general. Let $\omega(0,1) = \theta(0,1) = 0$,

$$\{\omega(x) : \ x \in \mathbb{T}\} \quad \text{and} \quad \{\theta(x) : \ x \in \mathbb{T}\},$$

be independent families of i.i.d. real-valued random variables. We assume that

- The distributions of ω and θ are continuous, i.e.,

$$\forall a \in \mathbb{R} : \qquad \mathbb{P}\left(\omega = a\right) = 0, \quad \mathbb{P}\left(\theta = a\right) = 0.$$

- The moment-generating function of ω is well defined, i.e.,

$$\forall \beta \geq 0 : \quad \mathbb{E}\left[e^{\beta\omega}\right] < \infty.$$

We define our random environment as

$$\xi_{\beta,\gamma}(x) = \exp(\beta\omega(x) + i\gamma\theta(x)),$$

where β and γ are parameters. We further define the logarithmic moment-generating and characteristic functions as

$$\begin{aligned} \lambda_{\mathbb{R}} \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ \beta &\longmapsto \ln\left(\mathbb{E}\left[\exp(\beta\omega)\right]\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \lambda_{\mathbb{C}} \colon \mathbb{R} &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto -\ln\left|\mathbb{E}\left[\exp(i\gamma\theta)\right]\right|. \end{aligned}$$

Let us note that using (4.1) below, we can prove that $\lambda''_{\mathbb{R}}$ is equal to a variance, i.e., $\lambda_{\mathbb{R}}$ is a convex function.

As in [4], we define $\beta_c \in (0, \infty]$ as the solution of the equation

$$\beta_c\lambda'_{\mathbb{R}}(\beta_c) - \lambda_{\mathbb{R}}(\beta_c) = \ln b,$$

if it exists, with the convention $\beta_c = \infty$ if it does not. In order to keep our discussion concise, we assume that $\beta_c < \infty$. We also define $\beta_0 = \beta_c/2$.

Next, we define $0 < \gamma_0 < \gamma_c$ such that

$$\lambda_{\mathbb{R}}(2\beta_0) - 2\lambda_{\mathbb{R}}(\beta_0) + 2\lambda_{\mathbb{C}}(\gamma_0) = \ln b, \qquad 2\lambda_{\mathbb{C}}(\gamma_c) = \ln b.$$

In Section 3.6, we will discuss the proof of the following corollary that helps us to better characterize the regions of the phase diagram in this case.

COROLLARY 3.5. *Under the hypotheses above, the three regions can be characterized as follows:*

- *(REGION $\mathcal{R}1$)* $\beta_0 \leq \beta < \beta_c$ *and* $\gamma \geq 0$ *satisfy*

$$\beta\lambda'_{\mathbb{R}}(\beta_c) - \lambda_{\mathbb{R}}(\beta) + \lambda_{\mathbb{C}}(\gamma) < \ln b,$$

  or $0 \leq \beta \leq \beta_0$ *and* $\gamma \geq 0$ *satisfy*

$$\lambda_{\mathbb{R}}(2\beta) - 2\lambda_{\mathbb{R}}(\beta) + 2\lambda_{\mathbb{C}}(\gamma) < \ln b.$$

  Furthermore, the value of the free energy $\mathsf{f}(\xi)$ *is*

$$f_{\mathrm{I}}(\beta,\gamma) = \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma).$$

- *(REGION $\mathcal{R}2$)* $\beta > \beta_c$ *or* $\beta_0 \leq \beta < \beta_c$ *and* $\gamma \geq 0$ *satisfy*

$$\beta\lambda'_{\mathbb{R}}(\beta_c) - \lambda_{\mathbb{R}}(\beta) + \lambda_{\mathbb{C}}(\gamma) > \ln b.$$

  Furthermore, the value of the free energy $\mathsf{f}(\xi)$ *is*

$$f_{\mathrm{II}}(\beta,\gamma) = \beta\lambda'_{\mathbb{R}}(\beta_c).$$

- *(REGION $\mathcal{R}3$)* $0 \leq \beta < \beta_0$ *and* $\gamma \geq 0$ *satisfy*

$$\lambda_{\mathbb{R}}(2\beta) - 2\lambda_{\mathbb{R}}(\beta) + 2\lambda_{\mathbb{C}}(\gamma) > \ln b.$$

  Furthermore, the value of the free energy $\mathsf{f}(\xi)$ *is*

$$f_{\mathrm{III}}(\beta,\gamma) = \frac{1}{2}\left(\ln b + \lambda_{\mathbb{R}}(2\beta)\right).$$

3.1.4 Comparison with positive-valued environments

Buffet, Patrick and Pulé, in [4], worked with a family $\{\omega(x) : x \in \mathbb{T}\}$ of i.i.d. real-valued random variables and ξ given by the expression

$$\xi = \xi_\beta = \exp\left(\beta\omega\right), \qquad \beta \geq 0.$$

Define the moment-generating functions $\lambda\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\lambda(\beta) = \ln\mathbb{E}\left[\exp(\beta\,\omega)\right],$$

and write $\xi_\beta(x) = \exp(\beta\omega(x))$. The partition function is given by

$$\mathsf{Z}_n(\xi_\beta) = \sum_{s\in S_n} \prod_{t=1}^{n} \xi_\beta(s_t) = \sum_{s\in S_n} \exp\left[\beta \sum_{t=1}^{n} \omega(s_t)\right].$$

Let β_c be the positive solution of the equation

$$\beta\,\lambda'(\beta) - \lambda(\beta) = \ln b,$$

if it exists; otherwise, let $\beta_c = \infty$.

The following is a result proved in [4].

THEOREM 3.6. *For all $\beta \geq 0$, the limit of*

$$\mathsf{f}(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln \mathsf{Z}_n(\xi_\beta),$$

exists almost surely. In addition, f *is characterized by*

$$\mathsf{f}(\beta) = \begin{cases} \ln b + \lambda(\beta), & 0 \leq \beta \leq \beta_c, \\ \beta\lambda'(\beta_c), & \beta > \beta_c. \end{cases}$$

Let us rewrite all this in our language to understand the behavior of the model with positive weights $\{|\xi(x)| : x \in \mathbb{T}\}$. We define

$$Z_n(|\xi|) = \sum_{s\in S_n} \prod_{t=1}^{n} |\xi(s_t)| \qquad \text{and} \qquad Z_n(|\xi|^2) = \sum_{s\in S_n} \prod_{t=1}^{n} |\xi(s_t)|^2.$$

These correspond to directed polymer partition functions in positive-valued environments. We write $|\xi(x)| = \exp(\omega(x))$, $\lambda(\beta) = \ln \mathbb{E}[|\xi|^\beta]$, and already defined β_c as the inverse critical temperature for the model with weights $|\xi(x)|^\beta = \exp(\beta\omega(x))$. Therefore,

$$\begin{aligned} G(\alpha) &= \frac{1}{\alpha} \ln\left(b\,\mathbb{E}\left[|\xi|^\alpha\right]\right) = \frac{\ln b + \lambda(\alpha)}{\alpha}, \\ G'(\alpha) &= \frac{\alpha\lambda'(\alpha) - \lambda(\alpha) - \ln b}{\alpha^2}. \end{aligned}$$

In particular, $\alpha_{\min} = \beta_c$ and $G(\alpha_{\min}) = \lambda'(\beta_c)$. Let us note that the model with positive weights $|\xi(x)|^\beta$ is in the weak disorder regime if and only if $G'(\beta) < 0$.

We can then identify the behaviour of the model with positive weights $|\xi|$ in each of the regions of the phase diagram of the model with complex weights ξ.

- *(REGION $\mathcal{R}1$)* Here, there exists $\alpha > 1$ such that $G(\alpha) < \ln(b|\mathbb{E}[\xi]|)$, in particular, $G(\alpha) < \ln(b\mathbb{E}[|\xi|]) = G(1)$ and $G'(1) < 0$. Therefore, the model with positive weights $|\xi|$ is in the weak disorder regime and

$$\frac{1}{n} \ln Z_n(|\xi|) \xrightarrow{\text{a.s.}} \mathsf{f}(|\xi|) = \ln b + \ln \mathbb{E}[|\xi|].$$

- *(REGIÓN $\mathcal{R}2$)* If $\alpha_{\min} < 1$, then $G'(1) > 0$, which means that the model with positive weights $|\xi|$ is in the strong disorder regime, and

  $$\frac{1}{n} \ln Z_n(|\xi|) \xrightarrow{\text{a.s.}} \mathsf{f}(|\xi|) = \beta\lambda'(\beta_c) = G(\alpha_{\min}).$$

  The remaining part of the region corresponds to $1 \leq \alpha_{\min} < 2$ and $G(\alpha_{\min}) > \ln(b|\mathbb{E}[\xi]|)$. In particular, $G'(1) < 0$ and $G'(2) > 0$, which means that the model with positive environments $|\xi|$ (resp. $|\xi|^2 = \exp(2\omega)$) is in the weak disorder regime (resp. strong disorder) and

  $$\begin{aligned} \frac{1}{n} \ln Z_n(|\xi|) &\xrightarrow{\text{a.s.}} \mathsf{f}(|\xi|) = \ln b + \ln \mathbb{E}[|\xi|], \\ \frac{1}{n} \ln Z_n(|\xi|^2) &\xrightarrow{\text{a.s.}} \mathsf{f}(|\xi|^2) = 2G(\alpha_{\min}). \end{aligned}$$

- *(REGION $\mathcal{R}3$)* We have $\alpha_{\min} > 2$, then $G'(1) < 0$ and $G'(2) < 0$, so the models with positive environments $|\xi|$ and $|\xi|^2$ are in the weak disorder regime, and as a consequence

  $$\begin{aligned} \frac{1}{n} \ln Z_n(|\xi|) &\xrightarrow{\text{a.s.}} \mathsf{f}(|\xi|) = \ln b + \ln \mathbb{E}[|\xi|], \\ \frac{1}{n} \ln Z_n\left(|\xi|^2\right) &\xrightarrow{\text{a.s.}} \mathsf{f}\left(|\xi|^2\right) = \ln b + \ln \mathbb{E}\left[|\xi|^2\right]. \end{aligned}$$

3.2 General Estimates

Although the random variables $\xi(x)$ are i.i.d., if we take different paths $s,\ s' \in S_n$, the random variables

$$\prod_{i=1}^{n} \xi(s_i) \qquad \text{and} \qquad \prod_{i=1}^{n} \xi(s'_i),$$

are not independent. However, the dependence between the summands is manageable.

Since the random variables $\xi(x)$ are independent, we can write Z_n, for all $n \geq 2$, as

$$Z_n = \sum_{x \in \mathbb{T}_1} \xi(x)\ Z_{n-1,x}, \tag{3.12}$$

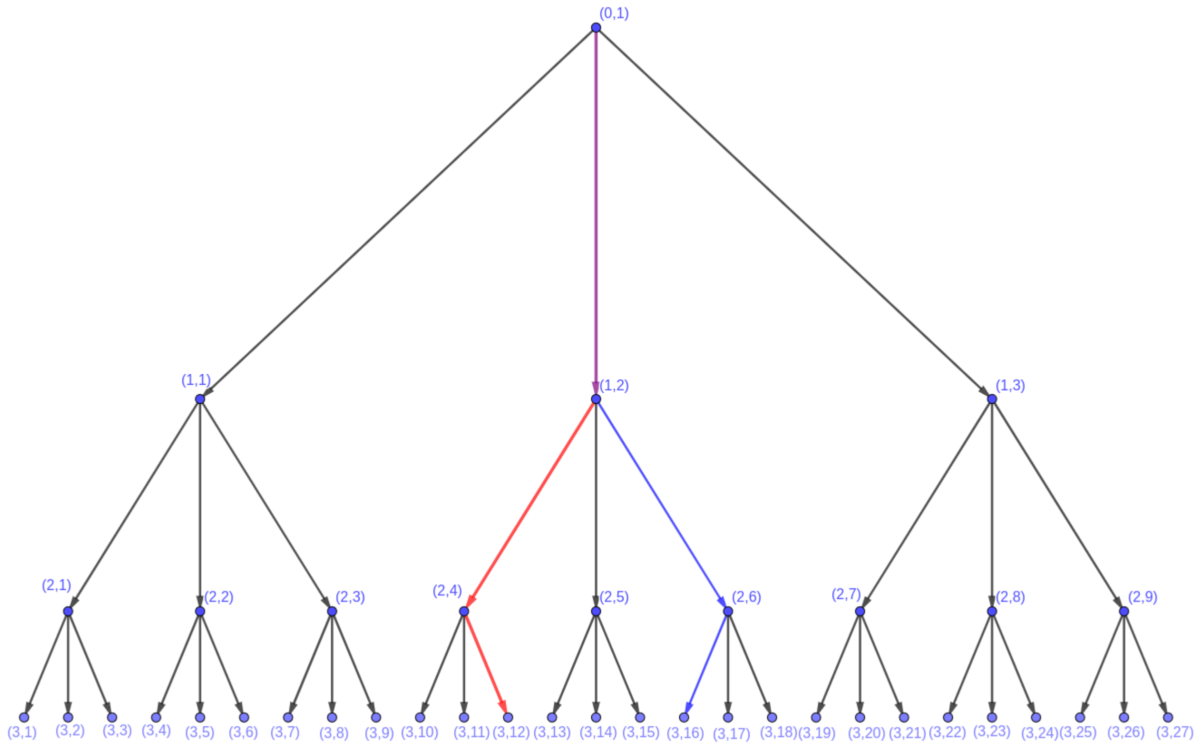


Figure 3.5: Example of two paths that start identically (violet color) but separate in the first generation (red and blue colors).

where the $\mathsf{Z}_{n-1,x}$ are independent, with the same law as Z_{n-1}.

Lemma 3.7. *We have* $\mathbb{E}\left[\mathsf{Z}_1\right] = b\,\mathbb{E}\left[\xi\right]$, *and*

$$\mathbb{E}\left[\mathsf{Z}_n\right] = b\,\mathbb{E}\left[\xi\right]\ \mathbb{E}\left[\mathsf{Z}_{n-1}\right], \qquad n \geq 2 \tag{3.13}$$

$$\mathbb{E}\left[\mathsf{Z}_n\right] = \left(b\,\mathbb{E}\left[\xi\right]\right)^n, \qquad n \geq 0. \tag{3.14}$$

Proof. Using the fact that the random variables $\xi(x)$ are i.i.d., with law ξ, we obtain

$$\mathbb{E}\left[\mathsf{Z}_1\right] = \mathbb{E}\left[\sum_{x\in\mathbb{T}_1} \xi(x)\right] = \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[\xi(x)\right] = \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[\xi\right] = b\,\mathbb{E}\left[\xi\right].$$

On the other hand, for $n \geq 2$, we apply reasoning analogous to the previous one: from (3.12), recalling that the random variables $\mathsf{Z}_{n-1,x}$ are independent and distributed as Z_{n-1},

$$\begin{aligned}\mathbb{E}\left[\mathsf{Z}_n\right] &= \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[\xi(x)\ \mathsf{Z}_{n-1,x}\right] = \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[\xi(x)\right]\ \mathbb{E}\left[\mathsf{Z}_{n-1,x}\right] \\ &= \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[\xi\right]\ \mathbb{E}\left[\mathsf{Z}_{n-1}\right] = b\,\mathbb{E}\left[\xi\right]\ \mathbb{E}\left[\mathsf{Z}_{n-1}\right],\end{aligned}$$

thus we conclude (3.13).

To prove the (3.14), we will proceed by induction:

(i) *(Case $n = 1$)* By the above lemma we prove that indeed $\mathbb{E}[Z_1] = b\,\mathbb{E}[\xi]$.

(ii) *(Case $n+1$)* We will check that the structure of (3.14) holds for $n+1$, assuming that the formula (3.14) holds for n. Indeed,

$$\mathbb{E}[\mathsf{Z}_{n+1}] = b\,\mathbb{E}[\xi]\ \mathbb{E}[\mathsf{Z}_n] = (b\,\mathbb{E}[\xi])\ (b\,\mathbb{E}[\xi])^n = (b\,\mathbb{E}[\xi])^{n+1}. \qquad \square$$

The following lemma gives us a recursive formula for the second moment of the modulus of Z_n.

LEMMA 3.8. *The second moment of the modulus of Z_{n+1} satisfies*

$$\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right] = b\,\mathbb{E}\left[|\xi|^2\right]\ \mathbb{E}\left[|\mathsf{Z}_n|^2\right] + b(b-1)\ |\mathbb{E}[\xi]|^2\ |\mathbb{E}[\mathsf{Z}_n]|^2. \tag{3.15}$$

Proof. First, we note that

$$|\mathsf{Z}_n|^2 = \mathsf{Z}_n\,\overline{\mathsf{Z}_n} = \sum_{x\in\mathbb{T}_1} |\xi(x)\ \mathsf{Z}_{n-1,x}|^2 + \sum_{y,y'\in\mathbb{T}_1, y\neq y'} \xi(y)\ \mathsf{Z}_{n-1,y}\ \overline{\xi(y')\ \mathsf{Z}_{n-1,y'}}. \tag{3.16}$$

Together with the independence of the random variables $\xi(x)$ and $\mathsf{Z}_{n-1,x}$, $x \in \mathbb{T}_1$, we obtain

$$\begin{aligned}\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right] &= \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[|\xi(x)|^2\right]\ \mathbb{E}\left[|\mathsf{Z}_{n,x}|^2\right] \\ &\quad + \sum_{y',y\in\mathbb{T}_1, y'\neq y} \mathbb{E}[\xi(y')]\ \mathbb{E}\left[\mathsf{Z}_{n,y'}\right]\ \overline{\mathbb{E}[\xi(y)]\ \mathbb{E}[\mathsf{Z}_{n,y}]}.\end{aligned}$$

Remember that$\xi(x) \overset{\mathcal{L}}{=} \xi(y) \overset{\mathcal{L}}{=} \xi(y') \overset{\mathcal{L}}{=} \xi$ and $\mathsf{Z}_{n,x} \overset{\mathcal{L}}{=} \mathsf{Z}_{n,y} \overset{\mathcal{L}}{=} \mathsf{Z}_{n,y'} \overset{\mathcal{L}}{=} \mathsf{Z}_n$, then

$$\begin{aligned}\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right] &= \sum_{x\in\mathbb{T}_1} \mathbb{E}\left[|\xi|^2\right]\ \mathbb{E}\left[|\mathsf{Z}_n|^2\right] + \sum_{y',y\in\mathbb{T}_1, y'\neq y} \mathbb{E}[\xi]\ \mathbb{E}[\mathsf{Z}_n]\ \overline{\mathbb{E}[\xi]\ \mathbb{E}[\mathsf{Z}_n]} \\ &= b\,\mathbb{E}\left[|\xi|^2\right]\ \mathbb{E}\left[|\mathsf{Z}_n|^2\right] + \sum_{y',y\in\mathbb{T}_1, y'\neq y} |\mathbb{E}[\xi]|^2\ |\mathbb{E}[\mathsf{Z}_n]|^2 \\ &= b\,\mathbb{E}\left[|\xi|^2\right]\ \mathbb{E}\left[|\mathsf{Z}_n|^2\right] + b(b-1)\ |\mathbb{E}[\xi]|^2\ |\mathbb{E}[\mathsf{Z}_n]|^2,\end{aligned}$$

whereupon we conclude (3.15). $\square$

We will adopt the notation m_α for the **moments** of ξ, i.e.,

$$m_\alpha = \mathbb{E}[\xi^\alpha], \quad \alpha \geq 0.$$

Recall that we defined the following objects

$$\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}, \qquad \mathcal{F}_n = \sigma\left(\xi(x) :\ x \in \mathbb{T}_{\leq n}\right), \qquad \mathbb{T}_{\leq n} = \bigcup_{i=0}^{n} \mathbb{T}_i.$$

LEMMA 3.9. *The discrete-time process $(\mathsf{Z}_n)_{n\in\mathbb{N}}$ is integrable, adapted to the filtration $\mathbb{F}$ and satisfies*

$$\mathbb{E}\left[\mathsf{Z}_{n+1} \middle| \mathcal{F}_n\right] = bm_1\, \mathsf{Z}_n. \tag{3.17}$$

Proof.

(i) *(Integrability)* We need to prove $\mathbb{E}\left[|\mathsf{Z}_n|\right] < \infty$. Applying the triangular inequality, i.i.d. property of the random variables $\xi(s_t)$, HA2 and Jensen's inequality we have

$$\begin{aligned}\mathbb{E}\left[|\mathsf{Z}_n|\right] &\leq \sum_{s\in S_n}\left|\mathbb{E}\left[\prod_{t=1}^{n}\xi(s_t)\right]\right| = \sum_{s\in S_n}\prod_{t=1}^{n}\left|\mathbb{E}\left[\xi(s_t)\right]\right| \leq \sum_{s\in S_n}\prod_{t=1}^{n}\left|\mathbb{E}\left[\xi\right]\right| \\ &= \sum_{s\in S_n}\left|\mathbb{E}\left[\xi\right]\right|^n = \left|\mathbb{E}\left[\xi\right]\right|^n \sum_{s\in S_n} 1 = \left(b\ \left|\mathbb{E}\left[\xi\right]\right|\right)^n \leq \left(b\,\mathbb{E}\left[|\xi|\right]\right)^n < \infty.\end{aligned}$$

(ii) *(Adaptivity)* We need $\mathsf{Z}_n \in \mathcal{F}_n$, for all $n \geq 1$. But this trivial because $\mathbb{F}$ is the natural filtration of the random variables $\xi(s_t)$.

(iii) To prove (3.17), let us note

$$\mathsf{Z}_{n+1} = \sum_{s\in S_{n+1}}\prod_{t=1}^{n+1}\xi(s_t) = \sum_{s\in S_n,}\left(\prod_{t=1}^{n}\xi(s_t)\right)\ \mathsf{Z}_{1,s_n}. \tag{3.18}$$

Then, calculating the conditional expectation, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathsf{Z}_{n+1} \middle| \mathcal{F}_n\right] &= \sum_{s\in S_n}\left(\left(\prod_{t=1}^{n}\xi(s_t)\right)\ \mathbb{E}\left[\mathsf{Z}_{1,s_n} \middle| \mathcal{F}_n\right]\right) = \sum_{s\in S_n}\left(\prod_{t=1}^{n}\xi(s_t)\,\mathbb{E}\left[\mathsf{Z}_1\right]\right) \\ &= \sum_{s\in S_n}\left(\left(\prod_{t=1}^{n}\xi(s_t)\right)\ \left(b\,\mathbb{E}\left[\xi\right]\right)\right) = bm_1\ \sum_{s\in S_n}\prod_{k=1}^{n}\xi(s_t) = bm_1\mathsf{Z}_n.\square\end{aligned}$$

We will denote by σ^2 the **variance** of ξ, i.e., $\sigma^2 = \mathrm{Var}(\xi) = \mathbb{E}\left[|\xi - |\mathbb{E}\left[\xi\right]||^2\right]$.

LEMMA 3.10. *The second conditional moment of the modulus of Z_{n+1} satisfies*

$$\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2 \middle| \mathcal{F}_n\right] = b^2\,|m_1|^2\ |\mathsf{Z}_n|^2 + b\sigma^2\,\mathsf{Z}_n(|\xi|^2). \tag{3.19}$$

Proof. Applying properties of the conditional variance and (3.17), we have

$$\mathbb{E}\left[|Z_{n+1}|^2\middle|\mathcal{F}_n\right] = |\mathbb{E}\left[Z_{n+1}|\mathcal{F}_n\right]|^2 + \mathrm{Var}\left[Z_{n+1}|\mathcal{F}_n\right] \tag{3.20}$$

$$= (b\,|m_1|)^2\ |Z_n|^2 + \mathrm{Var}\left[Z_{n+1}|\mathcal{F}_n\right]. \tag{3.21}$$

We will continue calculating the conditional variance of Z_{n+1}. We will use (3.18) and the fact that the random variables $Z_{1,x}$, where $x \in \mathbb{T}_n$, are i.i.d.

$$\begin{aligned}\mathrm{Var}\left[Z_{n+1}|\mathcal{F}_n\right] &= \sum_{s\in S_n}\left|\prod_{t=1}^{n}\xi(s_t)\right|^2\ \mathrm{Var}\left[Z_{1,s_n}|\mathcal{F}_n\right] = \sum_{s\in S_n}\left(\left(\prod_{t=1}^{n}|\xi(s_t)|^2\right)\ \mathrm{Var}\left[Z_1\right]\right)\\ &= \sum_{s\in S_n}\left(\left(\prod_{t=1}^{n}|\xi(s_t)|^2\right)\ (b\ \mathrm{Var}\left[\xi\right])\right) = b\sigma^2\sum_{s\in S_n}\prod_{t=1}^{n}|\xi(s_t)|^2\\ &= b\sigma^2\ Z_n(|\xi|^2).\end{aligned} \tag{3.22}$$

Therefore, we conclude (3.19) by replacing the recent equality in (3.21). □

By computing the expected value in (3.19) and by properties of conditional expectation, we obtain the following problem in finite differences

$$\begin{cases}\mathbb{E}\left[|Z_n|^2\right] = (b\,|m_1|)^2\ \mathbb{E}\left[|Z_{n-1}|^2\right] + b\sigma^2\,(b\tilde{m}_2)^{n-1}, \quad n\geq 2,\\ \mathbb{E}\left[|Z_1|^2\right] = (b\,|m_1|)^2 + b\sigma^2,\end{cases}$$

where $\tilde{m}_\alpha$ denotes the **moment of the modulus** of ξ, i.e.,

$$\tilde{m}_\alpha = \mathbb{E}\left[|\xi|^\alpha\right], \quad \alpha\geq 0.$$

From this problem, we can obtain the following recursive equations

$$\begin{cases}\mathbb{E}\left[|Z_n|^2\right] - (b\,|m_1|)^2\ \mathbb{E}\left[|Z_{n-1}|^2\right] &= b\sigma^2\,(b\tilde{m}_2)^{n-1},\\ (b\,|m_1|)^2\ \mathbb{E}\left[|Z_{n-1}|^2\right] - (b\,|m_1|)^{2\cdot 2}\ \mathbb{E}\left[|Z_{n-2}|^2\right] &= b\sigma^2\,(b\tilde{m}_2)^{n-2}\ (b\,|m_1|)^2,\\ &\vdots\\ (b\,|m_1|)^{2(n-2)}\ \mathbb{E}\left[|Z_2|^2\right] - (b\,|m_1|)^{2(n-1)}\ \mathbb{E}\left[|Z_1|^2\right] &= b\sigma^2\,(b\tilde{m}_2)^1\ (b\,|m_1|)^{2(n-2)}.\end{cases}$$

By adding all these equations on the left (respect. right) side of the equalities, we will have a telescopic (respect. geometric) summation, i.e.,

$$\mathbb{E}\left[|Z_n|^2\right] = (b\,|m_1|)^{2(n-1)}\ \mathbb{E}\left[|Z_1|^2\right] + b\sigma^2\sum_{i=0}^{n-2}(b\tilde{m}_2)^{n-1-i}\ (b\,|m_1|)^{2i}$$

$$= (b\,|m_1|)^{2n} + b\sigma^2\ (b\tilde{m}_2)^{n-1} \sum_{i=0}^{n-1} \left(\frac{b\,|m_1|^2}{\tilde{m}_2}\right)^i. \tag{3.23}$$

With this equation, we can infer that the behavior $\mathbb{E}\left[|\mathsf{Z}_n|^2\right]$ of an infinitely large polymer will depend on the value of $\dfrac{b\,|m_1|^2}{\tilde{m}_2}$.

LEMMA 3.11. *It hold that*

$$\lim_{n\to\infty} \frac{1}{n}\ \ln \mathbb{E}\left[|\mathsf{Z}_n|^2\right] = \begin{cases} 2f_{\mathrm{I}}, & b\,|\mathbb{E}[\xi]|^2 > \mathbb{E}\left[|\xi|^2\right], \\ 2f_{\mathrm{III}}, & b\,|\mathbb{E}[\xi]|^2 < \mathbb{E}\left[|\xi|^2\right], \\ 2f_{\mathrm{I}} = 2f_{\mathrm{III}}, & b\,|\mathbb{E}[\xi]|^2 = \mathbb{E}\left[|\xi|^2\right]. \end{cases} \tag{3.24}$$

Proof. First,

$$(b\tilde{m}_2)^{n-1} \sum_{i=0}^{n-1} \left(\frac{b\,|m_1|^2}{\tilde{m}_2}\right)^i \approx \begin{cases} (b\,|m_1|)^{2(n-1)}, & b\,|m_1|^2 > \tilde{m}_2, \\ (b\tilde{m}_2)^{n-1}, & b\,|m_1|^2 < \tilde{m}_2, \\ n\,(b\tilde{m}_2)^{n-1} = n\,(b\,|m_1|)^{2(n-1)}, & b\,|m_1|^2 = \tilde{m}_2. \end{cases}$$

Therefore, using these approximations and (3.23) we get

$$\mathbb{E}\left[|\mathsf{Z}_n|^2\right] \approx \begin{cases} (b\,|m_1|)^{2n}\left(1+\frac{\sigma^2}{b|m_1|^2}\right), & b\,|m_1|^2 > \tilde{m}_2, \\ (b\tilde{m}_2)^n\,\frac{\sigma^2}{\tilde{m}_2}, & b\,|m_1|^2 < \tilde{m}_2, \\ (b\tilde{m}_2)^n\left(1+\frac{\sigma^2 n}{b|m_1|^2}\right) = (b\,|m_1|)^{2n}\left(1+\frac{\sigma^2 n}{b|m_1|^2}\right), & b\,|m_1|^2 = \tilde{m}_2. \end{cases}$$

This proves (3.24). □

REMARK. Assuming $\mathbb{E}[\xi] = 0$, by the previous lemma, we will only have

$$\lim_{n\to\infty} \frac{1}{n}\ \ln \mathbb{E}\left[|\mathsf{Z}_n|^2\right] = 2f_{\mathrm{III}}.$$

With that in mind, from this point on **we will assume** $\mathbb{E}[\xi] \neq 0$.

3.2.1 Moments Estimates

Recall that we define the process

$$\mathbb{M} = (\mathsf{M}_n)_{n\in\mathbb{N}}, \qquad \mathsf{M}_n = \frac{\mathsf{Z}_n}{\mathbb{E}[\mathsf{Z}_n]}.$$

Then by setting $m \in \mathbb{N}$, for $x \in \mathbb{T}_m$ it is clear that the random variables $\mathsf{M}_{n,x}$ are i.i.d. and distributed as M_n.

LEMMA 3.12. *The process* $\mathbb{M}$ *is an* $\mathbb{F}$*-martingale. In other words,* $\mathbb{M}$ *is integrable, adapted, and satisfies the martingale property, i.e.,*

$$\mathbb{E}\left[\mathsf{M}_{n+1} \middle| \mathcal{F}_n\right] = \mathsf{M}_n. \tag{3.25}$$

Proof. This proposition can be proved using Lemma 3.9.

(i) *(Integrability)* We need to prove $\mathbb{E}\left[|\mathsf{M}_n|\right] < \infty$. This literal is obtained by the integrability of Z_n, because

$$\mathbb{E}\left[|\mathsf{M}_n|\right] = \frac{1}{|bm_1|^n}\, \mathbb{E}\left[|\mathsf{Z}_n|\right] < \infty.$$

(ii) *(Adaptivity)* We will continue by showing that $\mathsf{M}_n \in \mathcal{F}_n$. Now let us note that $\mathsf{M}_n = \dfrac{1}{(bm_1)^n}\, \mathsf{Z}_n$, so that, by the adaptivity of Z_n we have

$$\mathsf{M}_n = \frac{1}{(bm_1)^n}\, \mathsf{Z}_n \in \mathcal{F}_n.$$

(iii) *(Martingale property)* To prove (3.25) we use (3.17), whereby

$$\mathbb{E}\left[\mathsf{M}_{n+1} \middle| \mathcal{F}_n\right] = \frac{1}{(bm_1)^{n+1}}\mathbb{E}\left[\mathsf{Z}_{n+1} \middle| \mathcal{F}_n\right] = \frac{bm_1}{(bm_1)^{n+1}}\mathsf{Z}_n = \mathsf{M}_n. \qquad \square$$

By using (3.20), we can obtain a formula analogous to 3.19 for $\mathbb{M}$.

LEMMA 3.13. *The second conditional moment of modulus* M_{n+1} *satisfies*

$$\mathbb{E}\left[|\mathsf{M}_{n+1}|^2 \middle| \mathcal{F}_n\right] = |\mathsf{M}_n|^2 + \frac{\sigma^2}{b\,|m_1|^2}\left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^n \mathsf{M}_n(|\xi|^2),$$

where $\mathsf{M}_n = \dfrac{\mathsf{Z}_n(|\xi|^2)}{\mathbb{E}\left[\mathsf{Z}_n(|\xi|^2)\right]}$ *and* $Z_n(|\xi|^2) = \sum_{s \in S_n} \prod_{t=1}^{n} |\xi(s_t)|^2$.

Proof. Applying properties of the conditional variance, the fact that $\mathbb{M}$ is a martingale, and (3.14) together with (3.22) we have

$$\begin{aligned}
\mathbb{E}\left[|\mathsf{M}_{n+1}|^2 \middle| \mathcal{F}_n\right] &= \left|\mathbb{E}\left[\mathsf{M}_{n+1} \middle| \mathcal{F}_n\right]\right|^2 + \operatorname{Var}\left[\mathsf{M}_{n+1} \middle| \mathcal{F}_n\right] \\
&= |\mathsf{M}_n|^2 + \operatorname{Var}\left[\frac{\mathsf{Z}_{n+1}}{(b\,\mathbb{E}\left[\xi\right])^{n+1}} \middle| \mathcal{F}_n\right]
\end{aligned}$$

$$
\begin{aligned}
&= |\mathsf{M}_n|^2 + \frac{1}{|b\,\mathbb{E}\,[\xi]|^{2(n+1)}}\ \mathrm{Var}\,[\mathsf{Z}_{n+1}|\,\mathcal{F}_n] \\
&= |\mathsf{M}_n|^2 + \frac{b\sigma^2}{|b\,\mathbb{E}\,[\xi]|^{2(n+1)}}\ \mathsf{Z}_n(|\xi|^2) \\
&= |\mathsf{M}_n|^2 + \frac{\sigma^2}{b\,|m_1|^2}\ \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^n\ \mathsf{M}_n(|\xi|^2). \qquad \square
\end{aligned}
$$

Remember that Proposition 3.1 tells us that the function

$$
G(\alpha) = \frac{1}{\alpha}\ \ln\left(b\,\mathbb{E}\,[|\xi|^\alpha]\right),
$$

satisfies exactly one of the following properties:

- There exists a unique minimizer of G denoted by $\alpha_{\min} > 0$, i.e., G is strictly decreasing in $(0, \alpha_{\min}]$ and strictly increasing in $[\alpha_{\min}, \infty)$.
- G is strictly decreasing in $\mathbb{R}_+$, in this case we note $\alpha_{\min} = \infty$.

PROPOSITION 3.14. *Suppose $b\ \ |\mathbb{E}\,[\xi]|^2 > \mathbb{E}\left[|\xi|^2\right]$ which is equivalent to $G(2) < \ln(b\ \ |\mathbb{E}\,[\xi]|)$. For all $\alpha \in [0,2]$ we have*

$$
\sup_{n\in\mathbb{N}} \mathbb{E}\left[|\mathsf{M}_n|^\alpha\right] < \infty. \tag{3.26}
$$

Moreover for all $\alpha \in (0,2)$ such that $G(\alpha) < \ln(b\ \ |\mathbb{E}\,[\xi]|)$, (3.26) *is also satisfied.*

Proof. Let us take $0 < \alpha \leq 2$. Using Jensen's inequality and Lemma 3.13, we have

$$
\begin{aligned}
\mathbb{E}\left[|\mathsf{M}_n|^\alpha\middle|\,\mathcal{F}_{n-1}\right] &\leq \left(\mathbb{E}\left[|\mathsf{M}_n|^2\middle|\,\mathcal{F}_{n-1}\right]\right)^{\alpha/2} \\
&= \left(|\mathsf{M}_{n-1}|^2 + \frac{\sigma^2}{b\,|m_1|^2}\ \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^n\ \mathsf{M}_{n-1}(|\xi|^2)\right)^{\alpha/2} \\
&\leq |\mathsf{M}_{n-1}|^\alpha + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha\ \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{n\alpha/2}\ \mathsf{M}_{n-1}^{\alpha/2}(|\xi|^2).
\end{aligned} \tag{3.27}
$$

From the last inequality, we can obtain a recursive inequality, using the properties of conditional expectation and Jensen's inequality, with $\alpha/2 \in [0,1]$: we have

$$
\begin{aligned}
\mathbb{E}\left[|\mathsf{M}_n|^\alpha\right] &= \mathbb{E}\left[\mathbb{E}\left[|\mathsf{M}_n|^\alpha\middle|\,\mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[|\mathsf{M}_{n-1}|^\alpha\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha\ \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{n\alpha/2}\ \mathbb{E}\left[\mathsf{M}_{n-1}^{\alpha/2}(|\xi|^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[|\mathsf{M}_{n-1}|^{\alpha}\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{n\alpha/2} \left(\mathbb{E}\left[\mathsf{M}_{n-1}(|\xi|^2)\right]\right)^{\alpha/2} \\
&\leq \mathbb{E}\left[|\mathsf{M}_{n-1}|^{\alpha}\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{n\alpha/2},
\end{aligned}
$$

if this inequality is applied recursively, we have

$$
\begin{aligned}
\mathbb{E}\left[|\mathsf{M}_n|^{\alpha}\right] &\leq \mathbb{E}\left[|\mathsf{M}_{n-1}|^{\alpha}\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{n\alpha/2} \\
&\leq \mathbb{E}\left[|\mathsf{M}_{n-2}|^{\alpha}\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \sum_{i=0}^{1} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{(n-i)\alpha/2} \leq \cdots \\
&= \mathbb{E}\left[|\mathsf{M}_1|^{\alpha}\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \sum_{i=2}^{n} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^{i\alpha/2},
\end{aligned}
\tag{3.28}
$$

with this control we conclude (3.26) if $\frac{\tilde{m}_2}{b|m_1|^2} < 1$, i.e., $b\ |\mathbb{E}[\xi]|^2 > \mathbb{E}\left[|\xi|^2\right]$.

On the other hand, by Lemma 3 and since $0 < \alpha \leq 2$, for all $n \geq 1$ we have $\mathsf{Z}_n^{1/2}\left(|\xi|^2\right) \leq \mathsf{Z}_n^{1/\alpha}\left(|\xi|^{\alpha}\right)$ which is equivalent to

$$
\mathsf{Z}_n^{\alpha/2}\left(|\xi|^2\right) \leq \mathsf{Z}_n\left(|\xi|^{\alpha}\right),
$$

so that

$$
\begin{aligned}
\mathsf{M}_n^{\alpha/2}\left(|\xi|^2\right) = \left(\frac{\mathsf{Z}_n\left(|\xi|^2\right)}{\mathbb{E}\left[\mathsf{Z}_n\left(|\xi|^2\right)\right]}\right)^{\alpha/2} &= \left(\frac{\mathsf{Z}_n\left(|\xi|^2\right)}{(b\tilde{m}_2)^n}\right)^{\alpha/2} \\
&\leq \frac{\mathsf{Z}_n\left(|\xi|^{\alpha}\right)}{\left(b\,\mathbb{E}\left[|\xi|^{\alpha}\right]\right)^n} \left(\frac{b\,\mathbb{E}\left[|\xi|^{\alpha}\right]}{\left(b\,\mathbb{E}\left[|\xi|^2\right]\right)^{\alpha/2}}\right)^n \\
&= \frac{\mathsf{Z}_n(|\xi|^{\alpha})}{\mathbb{E}\left[\mathsf{Z}_n\left(|\xi|^{\alpha}\right)\right]} \left(\frac{b\,\mathbb{E}\left[|\xi|^{\alpha}\right]}{\left(b\,\mathbb{E}\left[|\xi|^2\right]\right)^{\alpha/2}}\right)^n \\
&= \mathsf{M}_n\left(|\xi|^{\alpha}\right) \left(\frac{b\,\mathbb{E}\left[|\xi|^{\alpha}\right]}{\left(b\,\mathbb{E}\left[|\xi|^2\right]\right)^{\alpha/2}}\right)^n.
\end{aligned}
\tag{3.29}
$$

Using the identity (3.29) in (3.27) we get

$$
\mathbb{E}\left[|\mathsf{M}_n|^{\alpha} \middle| \mathcal{F}_{n-1}\right] \leq |\mathsf{M}_{n-1}|^{\alpha} + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^{\alpha} \left(\frac{b\,\mathbb{E}\left[|\xi|^{\alpha}\right]}{(b\,|\mathbb{E}[\xi]|)^{\alpha}}\right)^n \mathsf{M}_{n-1}(|\xi|^{\alpha}),
$$

by taking the expected value, we have

$$
\begin{aligned}
\mathbb{E}\left[|\mathsf{M}_n|^\alpha\right] &= \mathbb{E}\left[\mathbb{E}\left[|\mathsf{M}_n|^\alpha \middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[|\mathsf{M}_{n-1}|^\alpha\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha \left(\frac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b\,|\mathbb{E}\left[\xi\right]|)^\alpha}\right)^n \mathbb{E}\left[\mathsf{M}_{n-1}(|\xi|^\alpha)\right] \\
&= \mathbb{E}\left[|\mathsf{M}_{n-1}|^\alpha\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha \left(\frac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b\,|\mathbb{E}\left[\xi\right]|)^\alpha}\right)^n,
\end{aligned}
$$

if we work analogously to the inequality (3.26) we get

$$
\mathbb{E}\left[|\mathsf{M}_n|^\alpha\right] \leq \mathbb{E}\left[|\mathsf{M}_1|^\alpha\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha \sum_{i=2}^{n} \left(\frac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b\,|\mathbb{E}\left[\xi\right]|)^\alpha}\right)^i. \tag{3.30}
$$

Then based on the above inequality we conclude (3.26) if $\frac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b|\mathbb{E}[\xi]|)^\alpha} < 1$, which in turn is equivalent to $G(\alpha) < \ln(b\ \ |\mathbb{E}\left[\xi\right]|)$. □

Proof of Proposition 3.4. We use (3.28) with $\alpha = 2$ to obtain

$$
\mathbb{E}\left[|\mathsf{M}_n(\xi)|^2\right] \leq \mathbb{E}\left[|\mathsf{M}_1|^2\right] + \frac{\sigma^2}{b\,|m_1|^2} \sum_{i=2}^{n} \left(\frac{\tilde{m}_2}{b\,|m_1|^2}\right)^i,
$$

by (3.6), the right-hand side of the above inequality converges, with which we conclude (3.7). □

COROLLARY 3.15. *If there exists* $\alpha \in (1,2)$ *such that* $G(\alpha) < G(2) < \ln(b\ \ |\mathbb{E}\left[\xi\right]|)$*, then we still obtain* (3.26).

Proof. By using (3.30) in the recent inequality we get

$$
\mathbb{E}\left[|\mathsf{M}_n|^\alpha\right] \leq \mathbb{E}\left[|\mathsf{M}_1|^\alpha\right] + \left(\frac{\sigma}{\sqrt{b}\,|m_1|}\right)^\alpha \sum_{i=2}^{n} \left(\frac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b\,|\mathbb{E}\left[\xi\right]|)^\alpha}\right)^i,
$$

to conclude, it is sufficient to remember that $G(\alpha) < G(2) < \ln(b\ \ |\mathbb{E}\left[\xi\right]|)$, then we have $\dfrac{b\,\mathbb{E}\left[|\xi|^\alpha\right]}{(b\,|\mathbb{E}\left[\xi\right]|)^\alpha} < 1$, so that the summation on the right converges. We obtain (3.26). □

In Proposition (3.14) we considered the hypothesis $b\ \ |\mathbb{E}\left[\xi\right]|^2 > \mathbb{E}\left[|\xi|^2\right]$. Then it is natural to ask what happens when the inequality is in the opposite direction.

We will continue studying the case $b\ \ |\mathbb{E}\left[\xi\right]|^2 < \mathbb{E}\left[|\xi|^2\right]$ by defining a new process

$$\mathbb{X} = (\mathsf{X}_n)_{n\in\mathbb{N}}, \qquad \mathsf{X}_n = \frac{|\mathsf{Z}_n|^2}{\mathbb{E}\left[|\mathsf{Z}_n|^2\right]}.$$

Because of the linearity of the expected value, it is trivial to note that $\mathbb{E}[\mathsf{M}_n] = \mathbb{E}[\mathsf{X}_n] = 1$.

The following lemma gives us a result similar to Proposition 3.14.

LEMMA 3.16. *The conditional expectation of* X_{n+1} *is given by*

$$\mathbb{E}[\mathsf{X}_{n+1}|\,\mathcal{F}_n] = b^2|m_1|^2 \frac{\mathbb{E}\left[|\mathsf{Z}_n|^2\right]}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathsf{X}_n + b\sigma^2 \frac{(b\tilde{m}_2)^n}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathsf{M}_n(|\xi|^2).$$

Proof. It is sufficient to use (3.19) together with the respective denominators of the processes $\mathbb{X}$ and $\mathbb{Y}$

$$\begin{aligned}
\mathbb{E}[\mathsf{X}_{n+1}|\,\mathcal{F}_n] &= \frac{1}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\middle|\,\mathcal{F}_n\right] \\
&= \frac{b^2|m_1|^2}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}|\mathsf{Z}_n|^2 + \frac{b\sigma^2}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathsf{Z}_n(|\xi|^2) \\
&= b^2|m_1|^2 \frac{\mathbb{E}\left[|\mathsf{Z}_n|^2\right]}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathsf{X}_n + b\sigma^2 \frac{(b\tilde{m}_2)^n}{\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right]}\mathsf{M}_n(|\xi|^2). \qquad \square
\end{aligned}$$

Following the idea of presenting similar results, we have one for Corollary 3.15.

THEOREM 3.17. *Suppose* $\alpha_{\min} > 2$*, then there exists* $\alpha > 2$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|\mathsf{X}_n|^{\alpha/2}\right] < \infty. \tag{3.31}$$

Proof. Applying the triangular inequality together with the Multinomial Theorem, we have

$$\begin{aligned}
|\mathsf{Z}_{n+1}|^4 &\le \left(\sum_{x\in\mathbb{T}_1} |\xi(x)\mathsf{Z}_{n,x}|\right)^4 \\
&= \sum_{k_1+k_2+\cdots+k_b=4} \binom{4}{k_1,k_2,\ldots,k_b} \prod_{i=1}^{b} \left|\xi(1,k_i)\mathsf{Z}_{n,(1,k_i)}\right|^{k_i}.
\end{aligned}$$

Now taking $\alpha \le 8/3$, we have $\alpha/4 < 1$; applying Lemma 3 together with the above and Jensen's inequality, by defining $P_k^b = \dfrac{b!}{(b-k)!}$ and $\lambda_{\mathbb{R}}(\beta) = \ln\left(\mathbb{E}\left[|\xi|^\beta\right]\right)$ we obtain

$$\begin{aligned}
\mathbb{E}\left[|\mathsf{Z}_{n+1}|^{\alpha}\right] &\leq \mathbb{E}\left[\left(|\mathsf{Z}_{n+1}|^{4}\right)^{\alpha/4}\right] \\
&\leq \mathbb{E}\left[\sum_{k_1+\cdots+k_b=4}\binom{4}{k_1,\ldots,k_b}\prod_{i=1}^{b}\left|\xi(1,k_i)\mathsf{Z}_{n,(1,k_i)}\right|^{k_i\alpha/4}\right] \\
&= \sum_{k_1+\cdots+k_b=4}\binom{4}{k_1,\ldots,k_b}\prod_{i=1}^{b}\mathbb{E}\left[\left|\xi(1,k_i)\mathsf{Z}_{n,(1,k_i)}\right|^{k_i\alpha/4}\right] \\
&= P_1^b\, e^{\lambda_{\mathbb{R}}(\alpha)}\ \mathbb{E}\left[|\mathsf{Z}_n|^{\alpha}\right] + 3\ P_2^b\, e^{2\lambda_{\mathbb{R}}(\alpha/2)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{\alpha/2}\right]\right)^2 \\
&\quad + 4\ P_2^b\, e^{\lambda_{\mathbb{R}}(\alpha/4)+\lambda_{\mathbb{R}}(3\alpha/4)}\ \mathbb{E}\left[|\mathsf{Z}_n|^{\alpha/4}\right]\mathbb{E}\left[|\mathsf{Z}_n|^{3\alpha/4}\right] \\
&\quad + 6\ P_3^b\, e^{2\lambda_{\mathbb{R}}(\alpha/4)+\lambda_{\mathbb{R}}(\alpha/2)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{\alpha/4}\right]\right)^2\mathbb{E}\left[|\mathsf{Z}_n|^{\alpha/2}\right] \\
&\quad + P_4^b\, e^{4\lambda_{\mathbb{R}}(\alpha/4)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{\alpha/4}\right]\right)^4 \\
&\leq P_1^b\, e^{\lambda_{\mathbb{R}}(\alpha)}\ \mathbb{E}\left[|\mathsf{Z}_n|^{\alpha}\right] + 3\ P_2^b\, e^{\alpha/2\lambda_{\mathbb{R}}(2)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/2} \\
&\quad + 4\ P_2^b\, e^{\lambda_{\mathbb{R}}(2)(\alpha/8+3\alpha/8)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/8}\left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{3\alpha/8} \\
&\quad + 6\ P_3^b\, e^{\lambda_{\mathbb{R}}(2)(\alpha/4+\alpha/4)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/4}\left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/4} \\
&\quad + P_4^b\, e^{\alpha/2\lambda_{\mathbb{R}}(2)}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/2} \\
&= b\tilde{m}_\alpha\ \mathbb{E}\left[|\mathsf{Z}_n|^{\alpha}\right] + b(b-1)(b^2+3)\tilde{m}_2^{\alpha/2}\ \left(\mathbb{E}\left[|\mathsf{Z}_n|^{2}\right]\right)^{\alpha/2},
\end{aligned}$$

applying (3.15), we have $\mathbb{E}\left[|\mathsf{Z}_{n+1}|^2\right] \geq b\tilde{m}_2\ \mathbb{E}\left[|\mathsf{Z}_n|^2\right]$. In the above inequality, we obtain the following estimate

$$\forall n \in \mathbb{N}: \quad \mathbb{E}\left[\mathsf{X}_{n+1}^{\alpha/2}\right] \leq \frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\ \mathbb{E}\left[\mathsf{X}_{n}^{\alpha/2}\right] + c,$$

where $c = \frac{b(b-1)(b^2+3)}{b^{\alpha/2}}$, then with the above recursion, we can pose the following system of inequations

$$\begin{cases}
\mathbb{E}\left[\mathsf{X}_{n}^{\alpha/2}\right] - \frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\ \mathbb{E}\left[\mathsf{X}_{n-1}^{\alpha/2}\right] & \leq c, \\
\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\ \mathbb{E}\left[\mathsf{X}_{n-1}^{\alpha/2}\right] - \left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^2\ \mathbb{E}\left[\mathsf{X}_{n-2}^{\alpha/2}\right] & \leq c\ \frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}, \\
 & \vdots \\
\left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{n-2}\ \mathbb{E}\left[\mathsf{X}_{2}^{\alpha/2}\right] - \left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{n-1}\ \mathbb{E}\left[\mathsf{X}_{1}^{\alpha/2}\right] & \leq c\ \left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{n-2}.
\end{cases}$$

If we add all these inequations on the left-hand side, we will have a telescopic summation and on the right-hand side, a geometric series, i.e., we have

$$\mathbb{E}\left[\mathsf{X}_n^{\alpha/2}\right] \leq \left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{n-1} \mathbb{E}\left[\mathsf{X}_1^{\alpha/2}\right] + c\sum_{i=0}^{n-2}\left(\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{i}.$$

Recall that $\alpha_{\min} > 2$ and G is convex so that taking $\alpha \in (2, 8/3)$, we have $G(2) > G(\alpha)$, i.e., $\frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}} < 1$, and consequently the summation of the right-hand side of the above inequality converges to

$$\tilde{B}_\alpha = \frac{b(b-1)(b^2+3)}{b^{\alpha/2}}\left(1 - \frac{b\tilde{m}_\alpha}{(b\tilde{m}_2)^{\alpha/2}}\right)^{-1}. \qquad \square$$

REMARK. Based on the assumptions of the above theorem, we still obtain (3.31) in the region $\mathcal{R}3$ and on the boundary of the regions $\mathcal{R}1$ and $\mathcal{R}3$.

3.2.2 Lower tail bounds

The following results correspond to [8, Lemma 6.1] and [8, Theorem 6.4]. We include the proofs for the convenience of the reader.

Let us begin by introducing some notation and establishing two preliminary results. Given $(z_1, z_2, \ldots, z_b) = z \in \mathbb{C}^b$ we define $\|z\| = \|z\|_\infty = \max_{1\leq k\leq b} |z_k|$ and

$$q(w,z) = \frac{w \cdot z}{\|z\|} = \frac{1}{\|z\|}\sum_{k=1}^{b} w_k\, z_k, \qquad w, z \in \mathbb{C}^b, \|z\| \neq 0.$$

LEMMA 3.18. *Let $x_1, x_2, \ldots, x_b$ be i.i.d. complex-valued random variables such that their moduli $|x_k|$ have continuous distribution. Then for all $\nu > 0$ there exists $c = c(\nu) > 0$ such that for all $z \in \mathbb{C}^b - \{0\}$ we have*

$$\mathbb{P}\left(|q(x,z)| < c\right) < \nu, \tag{3.32}$$

where $x = (x_1, x_2, \ldots, x_b)$.

Proof. First, let us study the distribution function of $|x_1|$. We will denote

$$\begin{aligned} F\colon \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ t &\longmapsto F(t) = \mathbb{P}\left(|x_1| \leq t\right). \end{aligned}$$

By hypothesis, F is continuous. Furthermore,

$$\lim_{t\to\infty} F(t) = 1 \quad \text{and} \quad \lim_{t\to-\infty} F(t) = 0.$$

Hence, F is uniformly continuous so that, for all $\nu > 0$, there exists $\delta = \delta(\nu) > 0$ such that

$$\forall t, t' \geq 0: \ |t - t'| < \delta \Rightarrow \mathbb{P}\left(\min\{t, t'\} < |x_1| \leq \max\{t, t'\}\right) = |F(t) - F(t')| < \nu,$$

if we take, in particular, $t = d + c$ and $t' = d - c$, where $c = c(\nu) = \delta/3$ and $d \in \mathbb{R}_+$, in the above proposition we obtain

$$\sup_{d \in \mathbb{R}_+} \mathbb{P}\left(|x_1| \in B(d, c)\right) < \nu. \tag{3.33}$$

Now let $z = (z_1, \ldots, z_b) \in \mathbb{C}^b - \{0\}$. Without loss of generality, we will assume that $|z_1| = \|z\|$. Since $z \neq 0$, then $|z_1| > 0$.

Using the definition of q, the inverse triangular inequality, properties of the conditional expectation along with independence, we have

$$\begin{aligned}
\mathbb{P}\left(|q(x,z)| < c\right) &= \mathbb{P}\left(\left|x_1 + \sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right| < c\right) \leq \mathbb{P}\left(\left||x_1| - \left|\sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right|\right| < c\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\left||x_1| - \left|\sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right|\right| < c\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\left||x_1| - \left|\sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right|\right| < c \,\middle|\, \sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right)\right].
\end{aligned}$$

We conclude (3.32) by applying (3.33) to the above equality, as

$$\mathbb{P}\left(|q(x,z)| < c\right) = \mathbb{E}\left[\mathbb{P}\left[|x_1| \in B\left(\left|\sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right|, c\right) \,\middle|\, \sum_{k=2}^{b} \frac{x_k\, z_k}{|z_1|}\right]\right] \leq \nu. \qquad \square$$

Now, let us take $\nu \in (0,1)$ and define the function

$$\begin{aligned}
\phi_\nu \colon \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\
t &\longmapsto t^b + \nu.
\end{aligned}$$

Let us check that ϕ_ν has two fixed points in the interval $[0,1]$ for ν small enough. Indeed, let

$$\begin{aligned}
\Phi_\nu \colon \mathbb{R}_+ &\longrightarrow \mathbb{R} \\
t &\longmapsto t^b - t + \nu.
\end{aligned}$$

Now, $\Phi'_\nu(t_{\text{crit}}) = 0$ is equivalente to

$$b\left(t_{\text{crit}}\right)^{b-1} - 1 = 0 \quad \text{i.e.} \quad t_{\text{crit}} = \frac{1}{b^{1/(b-1)}}.$$

As $\Phi_\nu'' \geq 0$, then Φ_ν is convex, so $\Phi_\nu(t_{\text{crit}}) = \inf_{t \in \mathbb{R}_+} \Phi_\nu(t)$. To find the two fixed points, it is enough to require that $\Phi_\nu(t_{\text{crit}}) < 0$, which is equivalent to

$$0 < \nu < \frac{1}{b^{1/(b-1)}} - \frac{1}{b^{b/(b-1)}} = \frac{b-1}{b^{b/(b-1)}}.$$

From now on, we will assume that ν fulfills the previous inequality with which we have the existence of two fixed points for ϕ_ν which we will denote t_ν^- and t_ν^+ such that $0 < t_\nu^- < t_\nu^+ < 1$. Now,

$$\phi_\nu(2\nu) = 2^b \nu^b + \nu < 2\nu \quad \Leftrightarrow \quad \nu^{b-1} < \frac{1}{2^b}.$$

Hence, if this estimate is satisfied, $\phi_\nu(2\nu) < 2\nu$. As $\phi_\nu(0) > 0$, we conclude that $t_\nu^- < 2\nu$ for ν small enough. In particular, $\lim_{\nu \downarrow 0} t_\nu^- = 0$. Similarly, $\lim_{\nu \downarrow 0} t_\nu^+ = 1$.

On the other hand, if $t \in (t_\nu^-, t_\nu^+)$, it holds that $\phi_\nu(t) < t$, so that

$$\forall 0 \leq t < t_\nu^+ : \quad \lim_{n \to \infty} \phi_\nu^n(t) = t_\nu^-.$$

The following results correspond to [8, Lemma 6.3].

LEMMA 3.19. *Suppose that for some $n \in \mathbb{N}$ there exists $(F, a) \in \mathbb{R} \times (0, 1)$ such that*

$$\mathbb{P}\left(\ln |Z_n| \leq F\right) < a. \tag{3.34}$$

Then for all $\eta > 0$ there exists $K = K(\eta, a) \in \mathbb{N}_0$ and $C = C(\eta) \in \mathbb{R}$ such that for all $k > K$ we have

$$\mathbb{P}\left(\ln |Z_{n+k}| \leq F - kC\right) < \eta.$$

Proof. Take a generic $\eta > 0$. By the previous discussion, there exist $\nu_\eta, \nu_a > 0$ such that

$$\forall 0 < \nu < \nu_\eta : \quad t_\nu^- < \frac{\eta}{2},$$
$$\forall 0 < \nu < \nu_a : \quad t_\nu^+ > a.$$

Then, let us set $\nu = \nu(\eta, a) \in \left(0, \min\left\{\nu_\eta, \nu_a\right\}\right)$ whereby

$$\lim_{n \to \infty} \phi_\nu^n(a) < \frac{\eta}{2}. \tag{3.35}$$

On the other hand, by Lemma 3.18, there exists a constant $c = c(\nu) > 0$, such that

$$\sup_{z \in \mathbb{C}^b - \{0\}} \mathbb{P}\left(\ln \left| q\left(\underline{\xi}, z\right)\right| < -C\right) < \nu, \qquad \underline{\xi} = (\xi(x))_{x \in \mathbb{T}_1},$$

where $C = \ln(1/c)$.

Now we will prove by induction that for all $j \in \mathbb{N}_0$ we have

$$\mathbb{P}\left(\ln\left|\mathsf{Z}_{n+j}\right| \le F - jC\right) < \phi_\nu^j(a). \tag{3.36}$$

The case $j = 0$ follows trivially, from the hypothesis (3.34). Now we will assume that (3.36) is satisfied, with j, and we will prove that the estimate holds for $j+1$. For this, let us note the following points:

1. Applying (3.12), we have

$$\begin{aligned}\ln\left|\mathsf{Z}_{n+j+1}\right| &= \ln\left|\sum_{x\in\mathbb{T}_1}\xi(x)\,\mathsf{Z}_{n+j,x}\right| = \ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| + \ln\max_{x\in\mathbb{T}_1}\left|\mathsf{Z}_{n+j,x}\right| \\ &= \ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| + \max_{x\in\mathbb{T}_1}\ln\left|\mathsf{Z}_{n+j,x}\right|,\end{aligned} \tag{3.37}$$

   where $\underline{\mathsf{Z}_{n+j}} = \left(\mathsf{Z}_{n+j,x}\right)_{x\in\mathbb{T}_1}$.

2. By the induction assumption (3.36) and since the random variables $\mathsf{Z}_{n+j,x}$ are i.i.d., we obtain

$$\begin{aligned}\mathbb{P}\left(\max_{x\in\mathbb{T}_1}\ln\left|\mathsf{Z}_{n+j,x}\right| \le F - jC\right) &= \mathbb{P}\left(\bigcap_{x\in\mathbb{T}_1}\left\{\ln\left|\mathsf{Z}_{n+j,x}\right| \le F - jC\right\}\right) \\ &= \prod_{x\in\mathbb{T}_1}\mathbb{P}\left(\ln\left|\mathsf{Z}_{n+j,x}\right| \le F - jC\right) \\ &= \prod_{x\in\mathbb{T}_1}\mathbb{P}\left(\ln\left|\mathsf{Z}_{n+j}\right| \le F - jC\right) \\ &< \prod_{x\in\mathbb{T}_1}\phi_\nu^j(a) = \left(\phi_\nu^j(a)\right)^b.\end{aligned} \tag{3.38}$$

3. We will continue with a control for the event $\left\{\ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| < -C\right\}$. By Lemma 3.18, we have

$$\begin{aligned}\mathbb{P}\left(\ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| < -C\right) &= \mathbb{E}\left[\mathbb{P}\left[\ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| < -C\,\middle|\,\underline{\mathsf{Z}_{n+j}}\right]\right] \\ &< \mathbb{E}\left[\nu\right] = \nu.\end{aligned}$$

4. Finally, to conclude the proof by induction, we apply (3.37), (3.38), the above inequality, together with the definition of ϕ, and we obtain

$$\mathbb{P}\left(\ln\left|\mathsf{Z}_{n+j+1}\right| \le F - (j+1)C\right) \le \mathbb{P}\left(\left\{\ln\left|q\left(\underline{\xi},\underline{\mathsf{Z}_{n+j}}\right)\right| < -C\right\} \cup\right.$$

$$
\left.\left\{\max_{x\in\mathbb{T}_1}\ln\left|Z_{n+j,x}\right| \le F - jC\right\}\right)
$$

$$
\begin{aligned}
&\le \mathbb{P}\left(\ln\left|q\left(\underline{\xi},\underline{Z_{n+j}}\right)\right| < -C\right) \\
&\quad + \mathbb{P}\left(\max_{x\in\mathbb{T}_1}\ln\left|Z_{n+j,x}\right| \le F - jC\right) \\
&< \left(\phi_\nu^j(a)\right)^b + \nu = \phi_\nu^{j+1}(a).
\end{aligned}
$$

To conclude, we use (3.36) and the limit (3.35), so that there exists $K = K(\eta, a) \in \mathbb{N}_0$ such that for all $k > K$ we have

$$
\mathbb{P}\left(\ln\left|Z_{n+k}\right| \le F - kC\right) < \phi_\nu^k(a) < \eta. \qquad \square
$$

COROLLARY 3.20. *Let $\varepsilon > 0$. Suppose that there exists $(N_1, \mathsf{f}, a) \in \mathbb{N} \times \mathbb{R} \times (0,1)$ such that*

$$
\mathbb{P}\left(\frac{1}{n}\ln\left|Z_n\right| \le \mathsf{f} - \frac{\varepsilon}{2}\right) < a, \tag{3.39}
$$

for all $n \ge N_1$. Then, for all $\eta > 0$ there exists $N = N(\eta, a, \varepsilon, N_1) \in \mathbb{N}$ such that

$$
\mathbb{P}\left(\frac{1}{n}\ln\left|Z_n\right| \le \mathsf{f} - \varepsilon\right) < \eta, \tag{3.40}
$$

for all $n > N$.

Proof. By (3.39) and Lemma 3.19, there exists $(K, C) \in \mathbb{N} \times \mathbb{R}$ (depending only on η and independent of n) such that, for all $k > K$, we have

$$
\begin{aligned}
\eta &> \mathbb{P}\left[\ln\left|Z_{n+k}\right| \le n\left(\mathsf{f} - \frac{\varepsilon}{2}\right) - kC\right] \\
&= \mathbb{P}\left[\frac{1}{n+k}\ln\left|Z_{n+k}\right| \le \frac{1}{n+k}\left(n\left(\mathsf{f} - \frac{\varepsilon}{2}\right) - kC\right)\right] \\
&= \mathbb{P}\left[\frac{1}{n+k}\ln\left|Z_{n+k}\right| \le \mathsf{f} - \frac{\varepsilon}{2} - \frac{k}{n+k}\left(\mathsf{f} - \frac{\varepsilon}{2} + C\right)\right].
\end{aligned} \tag{3.41}
$$

Recall that K is independent of n. Then as $\lim_{n\to\infty}\frac{k}{n+k}\left(\mathsf{f} - \frac{\varepsilon}{2} + C\right) = 0$, there exists $N_2 = N_2(\varepsilon) \in \mathbb{N}$ such that, for all $n > N_2$, we obtain

$$
\left|\frac{k}{n+k}\left(\mathsf{f} - \frac{\varepsilon}{2} + C\right)\right| < \frac{\varepsilon}{2}, \tag{3.42}
$$

whereby, defining $N_3 = \max\{N_1, N_2\}$, for all $n > N_3$, we have

$$
\mathbb{P}\left[\frac{1}{n+k}\ln\left|Z_{n+k}\right| \le \mathsf{f} - \varepsilon\right] = \mathbb{P}\left[\frac{1}{n+k}\ln\left|Z_{n+k}\right| \le \mathsf{f} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right]
$$

$$
\begin{aligned}
&\leq \mathbb{P}\left[\frac{1}{n+k}\ln|\mathsf{Z}_{n+k}| \leq \mathsf{f} - \frac{\varepsilon}{2} - \frac{k}{n+k}\left(\mathsf{f} - \frac{\varepsilon}{2} + \mathsf{C}\right)\right] \\
&< \eta,
\end{aligned}
$$

where we applied (3.42) and (3.41). We conclude (3.40) using the above inequality and defining $N = N(\eta, a, \varepsilon, N_1) = K + N_3$. □

3.2.3 Almost-Sure Lower Bounds

The following result improves the lemmas of the previous section with the τ-property; recall that we defined this property as follows:

Definition 3.2 (τ-property). *Let $\tau \in (0,2]$ be fixed, and let us denote the ball of center $z \in \mathbb{C}$ and radius r as $B(z,r)$. We say that the environment satisfies the* ***τ-property*** *if there exists a finite constant $C > 0$ such that*

$$
(\forall (z,r) \in \mathbb{C} \times \mathbb{R}_+^* : \quad \mathbb{P}[\xi \in B(z,r)] \leq Cr^\tau) \qquad \wedge \qquad \mathbb{E}\left[|\xi|^{-\tau}\right] < \infty.
$$

The next result will be the key to show the almost sure convergence of the free energy.

Lemma 3.21. *Let $\xi_1, \xi_2, \ldots, \xi_k$ be i.i.d. random variables, with law ξ satisfying the τ-property. Then,*

$$
\mathbb{P}\left[\xi_{[1,k]} \in B(z,r)\right] \leq Cr^\tau \left(\mathbb{E}\left[|\xi|^{-\tau}\right]\right)^{k-1}, \tag{3.43}
$$

where $\xi_{[1,k]} = \prod_{i=1}^{k} \xi_i$.

Proof. We will proceed to develop this proof by induction.

(i) *(Case $k = 1$)* This case is just the definition of the τ-property.

(ii) *(Case $k+1$)* Now we will assume our induction hypothesis, i.e., suppose that (3.43) holds for k and show that the structure of the inequality holds for $k+1$:

$$
\begin{aligned}
\mathbb{P}\left[\xi_{[1,k+1]} \in B(z,r)\right] &= \mathbb{P}\left[\xi_{[1,k]}\xi_{k+1} \in B(z,r)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\xi_{[1,k]} \in B\left(\frac{z}{\xi_{k+1}}, \frac{r}{|\xi_{k+1}|}\right) \middle| \xi_{k+1}\right]\right] \\
&\leq \mathbb{E}\left[C\left(\frac{r}{|\xi_{k+1}|}\right)^\tau \left(\mathbb{E}\left[|\xi|^{-\tau}\right]\right)^{k-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= Cr^{\tau}\left(\mathbb{E}\left[|\xi|^{-\tau}\right]\right)^{k-1}\ \mathbb{E}\left[|\xi_{k+1}|^{-\tau}\right] \\
&= Cr^{\tau}\left(\mathbb{E}\left[|\xi|^{-\tau}\right]\right)^{k}. \qquad \square
\end{aligned}
$$

LEMMA 3.22. *Under the assumptions of the previous lemma, assume that, for all $c > 0$, there exists $N \in \mathbb{N}$ such that*

$$
\mathbb{P}\left(\frac{1}{n}\ \ln|\mathsf{Z}_n| < \mathsf{f} - c\right) \le p < 1, \tag{3.44}
$$

for all $n \ge N$. Then, almost surely,

$$
\liminf_{n\to\infty} \frac{1}{n}\ \ln|\mathsf{Z}_n| \ge \mathsf{f}. \tag{3.45}
$$

Proof. Let $c > 0$. Let $m \in \mathbb{N}$ and set $l \in \{0, 1, \ldots, m\}$. Then, by decomposing $\mathsf{Z}_{l+(m+1)k}$ into the generation $l+k$, we obtain

$$
\begin{aligned}
\ln\left|\mathsf{Z}_{l+(m+1)k}\right| &= \ln\left|\sum_{s\in S_{l+k}} \xi_{l+k}(s)\ \mathsf{Z}_{mk,s_{l+k}}\right| \\
&= \ln\left|q\left(\underline{\xi}_{l+k}, \underline{\mathsf{Z}_{mk}}\right)\right| + \ln \max_{x\in\mathbb{T}_{l+k}} |\mathsf{Z}_{mk,x}| \\
&= \ln\left|q\left(\underline{\xi}_{l+k}, \underline{\mathsf{Z}_{mk}}\right)\right| + \max_{x\in\mathbb{T}_{l+k}} \ln|\mathsf{Z}_{mk,x}| \\
&= \ln|A_{k,l}| + \max_{x\in\mathbb{T}_{l+k}} \ln|\mathsf{Z}_{mk,x}|,
\end{aligned} \tag{3.46}
$$

where

$$
\begin{aligned}
A_{k,l} &= q\left(\underline{\xi}_{l+k}, \underline{\mathsf{Z}_{mk}}\right), & \underline{\xi}_{l+k} &= (\xi_{l+k}(s))_{s\in S_{l+k}}, \\
\xi_{l+k}(s) &= \prod_{t=1}^{l+k} \xi(s_t), & \underline{\mathsf{Z}_{mk}} &= (\mathsf{Z}_{mk,x})_{x\in\mathbb{T}_{l+k}}.
\end{aligned}
$$

Applying the fact that the random variables $\mathsf{Z}_{n+j,x}$ are i.i.d. and the hypothesis (3.44) we obtain

$$
\begin{aligned}
\mathbb{P}\left(\max_{x\in\mathbb{T}_{l+k}} \ln|\mathsf{Z}_{mk,x}| \le mk(\mathsf{f}-c)\right) &= \mathbb{P}\left(\bigcap_{x\in\mathbb{T}_{l+k}} \{\ln|\mathsf{Z}_{mk,x}| \le mk(\mathsf{f}-c)\}\right) \\
&= \prod_{x\in\mathbb{T}_{l+k}} \mathbb{P}\left(\ln|\mathsf{Z}_{mk,x}| \le mk(\mathsf{f}-c)\right) \\
&= \prod_{x\in\mathbb{T}_{l+k}} \mathbb{P}\left(\ln|\mathsf{Z}_{mk}| \le mk(\mathsf{f}-c)\right) \\
&\le \prod_{x\in\mathbb{T}_{l+k}} p = p^{b^{l+k}}.
\end{aligned}
$$

Then, by Borel-Cantelli's First Lemma, there exists an almost surely finite value $K \in \mathbb{N}$ such that

$$\max_{x \in \mathbb{T}_{l+k}} \ln |\mathsf{Z}_{mk,x}| \geq mk(\mathsf{f} - c),$$

for all $k \geq K$. Applying the above inequality in (3.46), it is true that

$$\begin{aligned}
\frac{1}{l+(m+1)k} \ln |\mathsf{Z}_{l+(m+1)k}| &\geq \frac{1}{l+(m+1)k} \ln |A_{k,l}| + \frac{mk}{l+(m+1)k}(\mathsf{f} - c) \\
&= \mathsf{f} - c + \frac{1}{l+(m+1)k} \ln |A_{k,l}| \\
&\quad - \frac{l+k}{l+(m+1)k}(\mathsf{f} - c) \\
&\geq \mathsf{f} - 2c + \frac{1}{l+(m+1)k} \ln |A_{k,l}|,
\end{aligned}$$

for k or m, sufficiently large. Then, we have

$$\liminf_{k\to\infty} \frac{1}{l+(m+1)k} \ln |\mathsf{Z}_{l+(m+1)k}| \geq \mathsf{f} - 2c + \liminf_{k\to\infty} \frac{1}{l+(m+1)k} \ln |A_{k,l}|. \quad (3.47)$$

Take $\kappa > 0$ and let $x_* \in \mathbb{T}_{l+k}$ be such that $|\mathsf{Z}_{mk,x_*}| = \max_{x \in \mathbb{T}_{l+k}} |\mathsf{Z}_{mk,x}|$. Then,

$$\begin{aligned}
\mathbb{P}\left[|A_{k,l}| < \kappa^{k+l}\right] &= \sum_{x \in \mathbb{T}_{l+k}} \mathbb{P}\left[|A_{k,l}| < \kappa^{k+l} \,\middle|\, x_* = x\right] \mathbb{P}\left[x_* = x\right] \\
&= \sum_{x \in \mathbb{T}_{l+k}} \mathbb{P}\left[\xi_{l+k}(x) \in B\left(\underline{\mathsf{Z}_{mk}}, \kappa^{l+k}\right)\right] \mathbb{P}\left[x_* = x\right] \\
&= \sum_{x \in \mathbb{T}_{l+k}} \mathbb{P}\left[\xi(x)\, \tilde{\xi}(x) \in B\left(\underline{\mathsf{Z}_{mk}}, \kappa^{l+k}\right)\right] \mathbb{P}\left[x_* = x\right], \qquad (3.48)
\end{aligned}$$

where $\tilde{\xi}(x) = \prod_{t=1}^{l+k-1} \xi(x_t)$. Furthermore

$$\begin{aligned}
\mathbb{P}\left[\xi(x)\, \tilde{\xi}(x) \in B\left(\underline{\mathsf{Z}_{mk}}, \kappa^{l+k}\right)\right] &= \mathbb{E}\left[\mathbb{P}\left[\xi(x) \in B\left(\tilde{\xi}(x)^{-1}\underline{\mathsf{Z}_{mk}}, |\tilde{\xi}(x)|^{-1}\kappa^{l+k}\right) \middle|\, \underline{\xi}_{l+k-1}\right]\right] \\
&\leq C\, \mathbb{E}\left[|\tilde{\xi}(x)|^{-\tau} \kappa^{\tau(l+k)}\right] \leq C\, \mathbb{E}[|\xi|^{-\tau}]^{l+k-1} \kappa^{\tau(l+k)},
\end{aligned}$$

therefore, taking κ small enough, we apply the recent inequality in (3.48) and together with the Borel-Cantelli First Lemma, there exists an almost surely finite $k_1 = k_1(\xi) \in \mathbb{N}$ such that

$$\forall k \geq k_1 : \quad |A_{k,l}| \geq \kappa^{l+k}.$$

As a consequence, we have

$$\liminf_{k\to\infty} \frac{1}{l+(m+1)k} \ln |A_{k,l}| \geq \liminf_{k\to\infty} \frac{l+k}{l+(m+1)k} \ln \kappa = \frac{1}{m+1} \ln \kappa.$$

Therefore, if m is sufficiently large, we apply the above inequality in (3.47) so that

$$\liminf_{k\to\infty} \frac{1}{l+(m+1)k} \ln |\mathsf{Z}_{l+(m+1)k}| \geq \mathsf{f} - 3c.$$

We conclude (3.45) by taking $c \to 0$ in the above inequality. □

3.2.4 A Generic Computation

Let us see how the arguments from the previous sections will be applied. The following scheme will be used in the proofs of Lemma 3.28 for the region $\mathcal{R}3$ and Lemma 3.32 for the region $\mathcal{R}2$, where details will be omitted.

We consider the usual partition function $\mathsf{Z}_n(\xi)$ and a quantity of the form

$$\mathsf{W}_n = \mathbb{E}\left[|\mathsf{Z}_n|^2 \middle| \mathcal{G} \right],$$

where $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra. Assume that

$$\mathbb{E}\left[\frac{|\mathsf{Z}_n|^{2\eta}}{|\mathsf{W}_n|^{\eta}} \middle| \mathcal{G} \right] \leq C, \tag{3.49}$$

almost surely, for some $\eta > 1$ and all $n \geq 1$.

PROPOSITION 3.23. *If the limit*

$$\lim_{n\to\infty} \frac{1}{2n} \ln \mathsf{W}_n(\xi) = \mathsf{f},$$

holds almost surely, then the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln |\mathsf{Z}_n(\xi)| = \mathsf{f},$$

holds in probability. Furthermore, under the τ-condition, this limit holds almost surely.

Proof. We remove ξ from the notation. Integrating (3.49) and applying Chebyshev's inequality, we obtain

$$\mathbb{P}\left[|\mathsf{Z}_n|^2 \geq n\mathsf{W}_n \right] \leq \frac{C}{n^{\eta}},$$

for some finite $C > 0$, so that, by Borel-Cantelli, there exists an almost surely finite quantity $C(\omega) > 0$ such that

$$|\mathsf{Z}_n|^2 \leq C(\omega) n \mathsf{W}_n,$$

for all $n \geq 1$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\mathsf{Z}_n|^2 \leq \lim_{n\to\infty} \frac{1}{n} \ln \mathsf{W}_n = \mathsf{f}, \qquad \mathbb{P} - \text{almost surely.} \tag{3.50}$$

To obtain a lower bound, let $c > 0$. By Paley-Zygmund's inequality (Lemma 3.24), (3.49) also implies that there exists $p \in (0, 1)$ such that

$$\mathbb{P}\left[|\mathsf{Z}_n|^2 \geq e^{-cn} \mathsf{W}_n\right] \geq 1 - p > 0,$$

for all $n \geq 1$. This entails that

$$\mathbb{P}\left[\ln |\mathsf{Z}_n|^2 < \ln \mathsf{W}_n - cn\right] \leq p < 1,$$

for all $n \geq 1$. Hence, if $p < p' < 1$,

$$\begin{aligned}
\mathbb{P}\left[\ln |\mathsf{Z}_n|^2 < n\mathsf{f} - 2cn\right] &= \mathbb{P}\left[\ln |\mathsf{Z}_n|^2 < n\mathsf{f} - 2cn, \ln \mathsf{W}_n \geq n(\mathsf{f} - c)\right] \\
&\quad + \mathbb{P}\left[\ln |\mathsf{Z}_n|^2 < n\mathsf{f} - 2cn, \ln \mathsf{W}_n < (\mathsf{f} - c)\right] \\
&\leq \mathbb{P}\left[\ln |\mathsf{Z}_n|^2 < \ln \mathsf{W}_n - cn\right] + \mathbb{P}\left[\ln \mathsf{W}_n < n(\mathsf{f} - c)\right] \leq p',
\end{aligned}$$

for n large enough. By Corollary 3.20, we can make the right hand side as small as we want by taking n large enough. This yields convergence in probability. Under the τ-condition, we can apply Lemma 3.22 to obtain the almost sure lower bound. □

To continue our study, we will present the following lemma, whose proof is based on the Paley-Zygmund inequality.

Lemma 3.24. *Suppose that the non-negative random variable Y satisfies*

$$\frac{\mathbb{E}\left[Y^\gamma\right]}{\left(\mathbb{E}\left[Y\right]\right)^\gamma} \leq B, \qquad (B, \gamma) \in \mathbb{R}_+ \times [1, \infty).$$

Then, for all $0 \leq \theta \leq 1$,

$$\mathbb{P}\left[\frac{Y}{\mathbb{E}\left[Y\right]} > \theta\right] \geq \left(\frac{1-\theta}{1+\theta}\right)^{\gamma/(\gamma-1)} \frac{1}{B^{1/(\gamma-1)}}.$$

Proof. First, let us take a $0 \leq \theta \leq 1$, then applying the inequalities of Hölder, Minkowski, and Jensen, we have

$$\begin{aligned}(1-\theta)\,\mathbb{E}\,[Y] &= \mathbb{E}\,[Y-\theta\mathbb{E}\,[Y]] \le \mathbb{E}\left[(Y-\theta\mathbb{E}\,[Y])\ \mathbb{1}_{\{Y>\theta\mathbb{E}[Y]\}}\right] \\ &\le \left(\mathbb{E}\left[|Y-\theta\mathbb{E}\,[Y]|^{\gamma}\right]\right)^{1/\gamma}\ \left(\mathbb{P}\,[Y>\theta\mathbb{E}\,[Y]]\right)^{(\gamma-1)/\gamma} \\ &\le \left(\left(\mathbb{E}\,[Y^{\gamma}]\right)^{1/\gamma} + \left(\mathbb{E}\left[(\theta\mathbb{E}\,[Y])^{\gamma}\right]\right)^{1/\gamma}\right)\ \left(\mathbb{P}\,[Y>\theta\mathbb{E}\,[Y]]\right)^{(\gamma-1)/\gamma} \\ &\le \left(\left(\mathbb{E}\,[Y^{\gamma}]\right)^{1/\gamma} + \left(\mathbb{E}\,[(\theta^{\gamma}\mathbb{E}\,[Y^{\gamma}])]\right)^{1/\gamma}\right)\ \left(\mathbb{P}\,[Y>\theta\mathbb{E}\,[Y]]\right)^{(\gamma-1)/\gamma} \\ &= (1+\theta)\ \left(\mathbb{E}\,[Y^{\gamma}]\right)^{1/\gamma}\ \left(\mathbb{P}\,[Y>\theta\mathbb{E}\,[Y]]\right)^{(\gamma-1)/\gamma},\end{aligned}$$

which is equivalent to writing

$$\begin{aligned}\mathbb{P}\left[\frac{Y}{\mathbb{E}\,[Y]}>\theta\right] = \mathbb{P}\,[Y>\theta\mathbb{E}\,[Y]] &\ge \left(\left(\frac{1-\theta}{1+\theta}\right)^{\gamma}\ \frac{(\mathbb{E}\,[Y])^{\gamma}}{\mathbb{E}\,[Y^{\gamma}]}\right)^{1/(\gamma-1)} \\ &\ge \left(\frac{1-\theta}{1+\theta}\right)^{\gamma/(\gamma-1)}\ \frac{1}{B^{1/(\gamma-1)}}.\end{aligned}$$

□

3.3 Region $\mathcal{R}1$

For this region we will assume that there exists $\alpha \in (1,2]$ such that $G(\alpha) < \ln(b\ |\mathbb{E}\,[\xi]|)$, and recall that $\mathbb{E}\,[\xi] \neq 0$. Since the process $(\mathsf{M}_n(\xi))_{n\in\mathbb{N}}$ is a uniformly integrable martingale, by the Martingale Convergence Theorem, there exists $\mathsf{M}_\infty(\xi) \in \mathbb{L}^1$ such that

$$\mathsf{M}_n(\xi) \xrightarrow{\text{a.s.}} \mathsf{M}_\infty(\xi), \tag{3.51}$$

and in $\mathbb{L}^1$.

The following lemma allows us to calculate the free energy in this regime.

LEMMA 3.25. *Assume that there exists $\alpha \in (1,2]$ such that $G(\alpha) < \ln(b\ |\mathbb{E}\,[\xi]|)$ and the law of ξ is continuous. Then the probability of the event $\{\mathsf{M}_\infty(\xi) = 0\}$ is zero or one.*

We will postpone the demonstration until the end of the section and calculate the free energy in the region $\mathcal{R}1$.

LEMMA 3.26. *Under the assumptions of Theorem 3.2, it holds that*

$$\frac{1}{n}\ \ln|\mathsf{Z}_n(\xi)| \xrightarrow{\text{a.s.}} \ln\left(b\ |\mathbb{E}[\xi]|\right).$$

Proof. Applying (3.51) and u.i. we obtain

$$\mathbb{E}[\mathsf{M}_\infty] = \lim_{n\to\infty} \mathbb{E}[\mathsf{M}_n] = 1,$$

then $\mathbb{P}\left[\mathsf{M}_n = 0\right] = 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathsf{M}_n = \lim_{n\to\infty} \frac{1}{n} \ln \mathsf{M}_\infty = 0, \quad \mathbb{P} - \text{a. s.}.$$

We thus conclude

$$\frac{1}{n} \ln |\mathsf{Z}_n| = \frac{1}{n} \ln |\mathbb{E}\left[\mathsf{Z}_n\right] \mathsf{M}_n| = \frac{1}{n} \ln \left|(b\mathbb{E}\left[\xi\right])^n \mathsf{M}_n\right| \xrightarrow{\text{a.s.}} \ln\left(b \mid \mathbb{E}\left[\xi\right]|\right),$$

by applying the definition of M_n and (3.14). □

The following technical lemma is key to proving Lemma 3.25.

LEMMA 3.27. *Let $z_1, \cdots, z_b \in \mathbb{C}$ and $\omega_1, \cdots, \omega_b$ be complex-valued independent random such that their moduli $|\omega_k|$ have continuous distribution.. Then,*

$$\mathbb{P}\left[\sum_{k=1}^{b} z_k \omega_k = 0\right] = 0,$$

if and only if $z_1 = \cdots = z_b = 0$.

Proof. Let us note that Lemma 3.18 proves this statement. □

Proof of Lemma 3.25. By (3.12), we have the simple identity

$$\mathsf{Z}_{n+1} = \sum_{x\in\mathbb{T}_1} \xi(x)\, \mathsf{Z}_{n,x},$$

so that using (3.12) and (3.14) we have

$$\mathsf{M}_{n+1} = \frac{\mathsf{Z}_{n+1}}{\mathbb{E}\left[\mathsf{Z}_{n+1}\right]} = \frac{1}{\left(b\, \mathbb{E}\left[\xi\right]\right)^{n+1}} \left(\sum_{x\in\mathbb{T}_1} \xi(x)\, \mathsf{Z}_{n,x}\right) = \frac{1}{b} \sum_{x\in\mathbb{T}_1} \frac{\xi(x)}{\mathbb{E}\left[\xi\right]} \mathsf{M}_{n,x},$$

with a hopefully self-explanatory notation. Now, all these martingales converge a.s. and we obtain that

$$\mathsf{M}_\infty = b^{-1} \sum_{x\in\mathbb{T}_1} \frac{\xi(x)}{\mathbb{E}[\xi]} \mathsf{M}_{\infty,x}. \tag{3.52}$$

It is clear that the random variables $\mathsf{M}_{\infty,x}$ are i.i.d. and distributed as M_∞.

Studying the event $\{\mathsf{M}_\infty = 0\}$, by the above lemma applied to the random variables $\dfrac{\xi(x)}{\mathbb{E}[\xi]}$, we have

$$\{\mathsf{M}_\infty = 0\} = \left\{\mathsf{M}_{\infty,(1,1)} = 0, \mathsf{M}_{\infty,(1,2)} = 0, \ldots, \mathsf{M}_{\infty,(1,b)} = 0\right\}, \tag{3.53}$$

almost surely. Then, from (3.52) and (3.53), we obtain

$$\begin{aligned}\mathbb{P}\left(\mathsf{M}_\infty = 0\right) &= \mathbb{P}\left(\mathsf{M}_{\infty,(1,1)} = 0, \mathsf{M}_{\infty,(1,2)} = 0, \ldots, \mathsf{M}_{\infty,(1,b)} = 0\right) \\ &= \prod_{x\in\mathbb{T}_1} \mathbb{P}\left(\mathsf{M}_{\infty,x} = 0\right) = \prod_{x\in\mathbb{T}_1} \mathbb{P}\left(\mathsf{M}_\infty = 0\right) = \left(\mathbb{P}\left(\mathsf{M}_\infty = 0\right)\right)^b. \qquad \square\end{aligned}$$

3.4 Region $\mathcal{R}3$

Lemma 3.28. *Under the hypothesis of Theorem 3.25, about the region $\mathcal{R}3$, it is satisfied that*

$$\frac{1}{n}\, \ln|\mathsf{Z}_n(\xi)| \xrightarrow{\;\mathbb{P}\;} \frac{1}{2}\ln\left(b\,\mathbb{E}[|\xi|^2]\right) = G(2). \tag{3.54}$$

Under the τ-property, we ensure convergence almost surely.

Proof. Recall that there exists $\nu > 1$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\mathsf{X}_n^\nu\right] < \infty, \qquad \mathsf{X}_n = \frac{|\mathsf{Z}_n|^2}{\mathbb{E}\left[|\mathsf{Z}_n|^2\right]}.$$

Following a reasoning analogous to the calculation of (3.50), we have

$$\limsup_{n\to\infty} \frac{1}{n}\, \ln \mathsf{X}_n \le 0,$$

almost surely, therefore

$$\limsup_{n\to\infty} \frac{1}{n}\, \ln|\mathsf{Z}_n|^2 \le \lim_{n\to\infty} \frac{1}{n}\, \ln\mathbb{E}\left[|\mathsf{Z}_n|^2\right] = G(2). \tag{3.55}$$

Let $c > 0$. Let us take $\theta = e^{-\frac{cn}{2}}$ and $Y = \mathsf{X}_n$ in Lemma 3.24. Then when $n \to \infty$, there exists $N \in \mathbb{N}^*$ and $p = p(N) \in (0,1)$ such that for all $n \ge N$ we have

$$\mathbb{P}\left[\mathsf{X}_n > e^{-\frac{cn}{2}}\right] \ge p \quad \Rightarrow \quad \mathbb{P}\left[\frac{1}{n}\, \ln|\mathsf{Z}_n|^2 < \frac{1}{n}\, \ln\mathbb{E}\left[|\mathsf{Z}_n|^2\right] - \frac{c}{2}\right] < 1 - p < 1.$$

Let us define $\mathsf{f} = G(2) = \lim_{n\to\infty} \frac{1}{n}\, \ln\mathbb{E}\left[|\mathsf{Z}_n|^2\right]$. Then there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$ we have

$$\mathsf{f} - c < \frac{1}{n}\, \ln\mathbb{E}\left[|\mathsf{Z}_n|^2\right] - \frac{c}{2},$$

whereby, for all $n \geq n_0 = \max\{N_0, N\}$, we have

$$\mathbb{P}\left[\frac{1}{n}\ln|\mathsf{Z}_n|^2 < \mathsf{f} - c\right] < 1 - p < 1.$$

Then, by setting $\varepsilon > 0$, by the Corollary 3.20 applied in the inequality above, there exists $n_1 \geq 1$ such that for all $n \geq n_1$ we have

$$\mathbb{P}\left[\frac{1}{n}\ln|\mathsf{Z}_n|^2 < \mathsf{f} - 2c\right] < \varepsilon. \tag{3.56}$$

On the other hand, let us note that (3.55), for n sufficiently large, implies

$$\mathbb{P}\left[\frac{1}{n}\ln|\mathsf{Z}_n|^2 > \mathsf{f} + 2c\right] < \varepsilon,$$

whereupon we conclude (3.54) by the inequality above and (3.56).

Finally, we ensure almost sure convergence under the τ-property by applying Lemma 3.22. □

3.5 Region $\mathcal{R}2$

Let us see that, in the region $\mathcal{R}2$, for n sufficiently large, we have

$$|\mathsf{Z}_n(\xi)|^2 \approx \mathsf{Z}_n\left(|\xi|^2\right),$$

while the last partition function, which, according to the discussion in Subsection 3.1.4, corresponds to a polymer with a positive environment in the strong disorder regime, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\ln \mathsf{Z}_n\left(|\xi|^2\right) = 2G(\alpha_{\min}).$$

3.5.1 Region $\alpha_{\min} \leq 1$

LEMMA 3.29. *Under the assumptions of Theorem 3.2, if $\alpha_{\min} < 1$, then*

$$\lim_{n\to\infty}\frac{1}{n}\ln|\mathsf{Z}_n(\xi)| \xrightarrow{\text{a.s.}} G(\alpha_{\min}). \tag{3.57}$$

Proof. By the triangle inequality $|\mathsf{Z}_n(\xi)| \leq \mathsf{Z}_n(|\xi|)$, then

$$\limsup_{n\to\infty}\frac{1}{n}\ln|\mathsf{Z}_n(\xi)| \leq \limsup_{n\to\infty}\frac{1}{n}\ln \mathsf{Z}_n(|\xi|) = G(\alpha_{\min}). \tag{3.58}$$

We will define $\mathsf{E}_n(\xi) = \sum_{s \neq s' \in S_n} \prod_{t=1}^{n} \xi(s_t)\overline{\xi(s'_t)}$, then by (3.16) we obtain

$$\begin{aligned} |\mathsf{Z}_n(\xi)|^2 &= \mathsf{Z}_n\left(|\xi|^2\right) + \mathsf{E}_n(\xi) \\ &= \mathsf{Z}_n\left(|\xi|^2\right)\left(1 + \frac{\mathsf{E}_n(\xi)}{\mathsf{Z}_n\left(|\xi|^2\right)}\right) \geq \mathsf{Z}_n\left(|\xi|^2\right)\left(1 + \frac{\mathsf{E}_n(\xi)}{\mathsf{Z}_n\left(|\xi|\right)^2}\right). \end{aligned} \tag{3.59}$$

Now suppose that

$$\liminf_{n\to\infty} \frac{1}{n} \ln\left(1 + \frac{\mathsf{E}_n(\xi)}{\mathsf{Z}_n\left(|\xi|^2\right)}\right) < 0,$$

then there exist $c_1, c_2 > 0$ and an indexed subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$1 + \frac{\mathsf{E}_{n_k}(\xi)}{\mathsf{Z}_{n_k}\left(|\xi|^2\right)} \leq c_1 e^{-c_2 n_k} \qquad \Leftrightarrow \qquad \mathsf{E}_{n_k}(\xi) \leq \left(c_1 e^{-c_2 n_k} - 1\right) \mathsf{Z}_{n_k}(|\xi|)^2,$$

which is negative for k large enough, so that

$$\left(1 - c_1 e^{-c_2 n_k}\right) \mathsf{Z}_{n_k}(|\xi|)^2 \leq |\mathsf{E}_{n_k}(\xi)| \leq \mathsf{E}_{n_k}(|\xi|),$$

by the triangle inequality. Hence,

$$\mathsf{Z}_{n_k}\left(|\xi|^2\right) + \left(1 - c_1 e^{-c_2 n_k}\right) \mathsf{Z}_{n_k}(|\xi|)^2 \leq \mathsf{Z}_n\left(|\xi|^2\right) + \mathsf{E}_{n_k}(|\xi|) = \mathsf{Z}_{n_k}(|\xi|)^2,$$

so that

$$\mathsf{Z}_{n_k}\left(|\xi|^2\right) \leq c_1 e^{-c_2 n_k} \mathsf{Z}_{n_k}(|\xi|)^2,$$

and, after computing the free energies,

$$\mathsf{f}(|\xi|^2) \leq -c_2 + 2\mathsf{f}(|\xi|) < 2\mathsf{f}(|\xi|),$$

which contradicts the fact that $\mathsf{f}(|\xi|^2) = 2\mathsf{f}(|\xi|) = 2G(\alpha_{\min})$. Hence,

$$\liminf_{n\to\infty} \frac{1}{n}\, \ln\left(1 + \frac{\mathsf{E}_n(\xi)}{\mathsf{Z}_n(|\xi|)^2}\right) \geq 0,$$

and, going back to (3.59), we conclude that

$$\liminf_{n\to\infty} \frac{1}{n}\, \ln|\mathsf{Z}_n(\xi)|^2 \geq \liminf_{n\to\infty} \frac{1}{n}\, \ln \mathsf{Z}_n\left(|\xi|^2\right) = 2G(\alpha_{\min}).$$

Together with (3.58), we conclude (3.57). $\square$

3.5.2 Region $1 < \alpha_{\min} \le 2$: Averaging The Phases

Let us define $\omega(x) \in \mathbb{R}$ and $\theta(x) \in [0, 2\pi)$ for $x \in \mathbb{T}$ trough the relations $\xi(x) = \exp(\omega(x) + i\theta(x))$. Therefore, the partition function can be written as

$$|\mathsf{Z}_n(\xi)|^2 = \sum_{s,s' \in S_n} \prod_{t=1}^{n} \xi(s_t)\overline{\xi(s'_t)} = \sum_{s,s' \in S_n} \prod_{t=1}^{n} e^{i[\theta(s_t) - \theta(s'_t)]} \prod_{t=1}^{n} |\xi(s_t)||\xi(s'_t)|.$$

We will adopt the notation $\mathbb{E}\left[\cdot \mid \omega\right] = \mathbb{E}\left[\cdot \mid \sigma\left(\omega(x) :\ x \in \mathbb{T}\right)\right]$. Then

$$\begin{aligned}
\mathbb{E}\left[|\mathsf{Z}_n(\xi)|^2 \middle| \omega \right] &= \mathbb{E}\left[\sum_{s,s' \in S_n} \prod_{t=1}^{n} \xi(s_t)\overline{\xi(s'_t)} \middle| \omega \right] \\
&= \mathbb{E}\left[\sum_{s,s' \in S_n} \prod_{t=1}^{n} e^{i[\theta(s_t) - \theta(s'_t)]} \prod_{t=1}^{n} |\xi(s_t)||\xi(s'_t)| \middle| \omega \right] \\
&= \sum_{s,s' \in S_n} \prod_{t=1}^{n} \mathbb{E}\left[e^{i[\theta(s_t) - \theta(s'_t)]} \right] \prod_{t=1}^{n} |\xi(s_t)||\xi(s'_t)| \\
&= \sum_{s \in S_n} \prod_{t=1}^{n} |\xi(s_t)|^2 + \sum_{s \neq s' \in S_n} \prod_{t > s \wedge s'} \left| \mathbb{E}\left[e^{i\theta} \right] \right|^2 \prod_{t=1}^{n} |\xi(s_t)||\xi(s'_t)| \\
&= \mathsf{Z}_n\left(|\xi|^2 \right) + \mathsf{E}_n(\omega),
\end{aligned}$$

where $\mathsf{E}_n(\omega) = \sum_{s \neq s' \in S_n} \prod_{t > s \wedge s'} \left| \mathbb{E}\left[e^{i\theta} \right] \right|^2 \prod_{t=1}^{n} |\xi(s_t)||\xi(s'_t)| \ge 0$, so that

$$\mathbb{E}\left[|\mathsf{Z}_n(\xi)|^2 \middle| \omega \right] \ge \mathsf{Z}_n\left(|\xi|^2 \right)$$

accordingly

$$\liminf_{n \to \infty} \frac{1}{n} \ln \mathbb{E}\left[|\mathsf{Z}_n(\xi)|^2 \middle| \omega \right] \ge \lim_{n \to \infty} \frac{1}{n} \ln \mathsf{Z}_n\left(|\xi|^2 \right) = 2G(\alpha_{\min}), \quad \mathbb{P} - \text{a.s.}.$$

The following theorem is crucial to our estimates and holds in all regions as long as we assume independence between radii and phase.

LEMMA 3.30. *Let us assume that ω and θ are independent. Then*

$$\forall n \in \mathbb{N} : \quad \frac{\mathbb{E}\left[|\mathsf{Z}_n|^4 \middle| \omega \right]}{\left(\mathbb{E}\left[|\mathsf{Z}_n|^2 \middle| \omega \right] \right)^2} \le 3.$$

Furthermore, in the region $\mathcal{R}1$, there exists $\eta > 1$ such that

$$\sup_{n\in\mathbb{N}} \frac{\mathbb{E}\left[\mathbb{E}\left[|\mathsf{Z}_n|^2\middle|\,\omega\right]^{\eta/2}\right]}{|\mathbb{E}[\mathsf{Z}_n]|^{\eta}} < \infty.$$

Proof. Before we begin, for convenience, we will fix some notations for both the weight of the path $s \in S_n$ and its associated phase, respectively:

$$\mathsf{B}_s = \mathsf{B}_{s,n} = \prod_{t=1}^{n} \xi(s_t) \qquad \text{and} \qquad \Gamma_s = \Gamma_{s,n} = \frac{\mathsf{B}_{s,n}}{|\mathsf{B}_{s,n}|}.$$

With these notations,

$$\begin{aligned} |\mathsf{Z}_n|^{2m} = \left(|\mathsf{Z}_n|^2\right)^m = \prod_{i=1}^{m} \mathsf{Z}_n\,\overline{\mathsf{Z}_n} &= \sum_{s_1,s_2,\dots,s_{2m}\in S_n} \prod_{i=1}^{m} \mathsf{B}_{s_{2i-1}}\,\overline{\mathsf{B}_{s_{2i}}} \\ &= \sum_{s_1,\dots,s_{2m}\in S_n} \left|\prod_{i=1}^{2m} \mathsf{B}_{s_i}\right| \prod_{i=1}^{m} \Gamma_{s_{2i-1}}\,\overline{\Gamma_{s_{2i}}}. \end{aligned}$$

Now, conditioning with respect to ω,

$$\begin{aligned} \mathbb{E}\left[|\mathsf{Z}_n|^4\middle|\,\omega\right] &= \sum_{s_1,s_2,s_3,s_4\in S_n} \mathbb{E}\left[\left(|\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\;\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\,\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right)\middle|\,\omega\right] \\ &= \sum_{s_1,s_2,s_3,s_4\in S_n} |\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\;\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\,\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\middle|\,\omega\right] \\ &= \sum_{s_1,s_2,s_3,s_4\in S_n} |\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\;\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\,\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right]. \qquad (3.60) \\ \mathbb{E}\left[|\mathsf{Z}_n|^2\middle|\,\omega\right] &= \sum_{s_1,s_2\in S_n} |\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}|\;\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\right]. \end{aligned}$$

If we square this last equation, we have

$$\left(\mathbb{E}\left[|\mathsf{Z}_n|^2\middle|\,\omega\right]\right)^2 = \sum_{s_1,s_2,s_3,s_4\in S_n} |\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\;\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\right]\;\mathbb{E}\left[\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right]. \qquad (3.61)$$

Taking a permutation of the four paths s_1, s_2, s_3, and s_4, i.e.,

$$(s_1, s_2, s_3, s_4) \to (s_{P(1)}, s_{P(2)}, s_{P(3)}, s_{P(4)}), \qquad P \in \mathrm{Sym}(4),$$

the amplitude $|\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|$ remains unchanged, so that (3.60) and (3.61) can be written as

$$\mathbb{E}\left[|\mathsf{Z}_n|^4\middle|\,\omega\right] = \sum_{s_1,\dots,s_4\in S_n} \left|\prod_{i=1}^{4} \mathsf{B}_{s_i}\right| \left(\frac{1}{24} \sum_{P\in\mathrm{Sym}(4)} \mathbb{E}\left[\prod_{j=1}^{2} \Gamma_{s_{P(2j-1)}}\,\overline{\Gamma_{s_{P(2j)}}}\right]\right),$$

$$\left(\mathbb{E}\left[|\mathsf{Z}_n|^2\middle|\,\omega\right]\right)^2=\sum_{s_1,\ldots,s_4\in S_n}\left|\prod_{i=1}^{4}\mathsf{B}_{s_i}\right|\left(\frac{1}{24}\sum_{P\in\mathrm{Sym}(4)}\prod_{j=1}^{2}\mathbb{E}\left[\Gamma_{s_{P(2j-1)}}\,\overline{\Gamma_{s_{P(2j)}}}\right]\right).$$

Now for any choice of the four paths s_1, s_2, s_3 and s_4 there exists a permutation $Q\in\mathrm{Sym}(4)$ such that

$$\mathbb{E}\left[\Gamma_{s_{Q(1)}}\,\overline{\Gamma_{s_{Q(2)}}}\right]\ \mathbb{E}\left[\Gamma_{s_{Q(3)}}\,\overline{\Gamma_{s_{Q(4)}}}\right]=\mathbb{E}\left[\Gamma_{s_{Q(1)}}\,\overline{\Gamma_{s_{Q(2)}}}\,\Gamma_{s_{Q(3)}}\,\overline{\Gamma_{s_{Q(4)}}}\right]=|\mathbb{E}\,[\mathrm{Y}]|^{2(2n-\tau_1-\tau_3)},$$

where $\mathrm{Y}=\xi/|\xi|$ and τ_i represents i-th time a path separates from the others (with $\tau_{i+1}=\tau_i$ if more than one path separate at that time).

Let us notice that, for each choice of four paths $s_1,\cdots,s_4$, there exists at least a permutation Q such that $s_{Q(i)}$ is the i-th path that separates from the others. Without lost of generality, suppose that $Q=(1\,2\,3\,4)$. Then,

$$\begin{aligned}|\mathbb{E}\,[\mathrm{Y}]|^{2(2n-\tau_1-\tau_3)}&=\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\,\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right]\\&=\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\right]\ \mathbb{E}\left[\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right]=|\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_1)}\ |\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_3)},\end{aligned}$$

Furthermore, note that any permutation within the set

$$\left\{\begin{matrix}(1\,2\,3\,4), & (1\,2\,4\,3), & (2\,1\,3\,4), & (2\,1\,4\,3)\\(3\,4\,1\,2), & (4\,3\,1\,2), & (3\,4\,2\,1), & (4\,3\,2\,1)\end{matrix}\right\},$$

will yield the same result. In general, for any choice of four paths $s_1,\cdots,s_4$, there exist at least eight permutations such that

$$\mathbb{E}\left[\Gamma_{s_{P(1)}}\,\overline{\Gamma_{s_{P(2)}}}\,\Gamma_{s_{P(3)}}\,\overline{\Gamma_{s_{P(4)}}}\right]=|\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_1)}\ |\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_3)}.$$

As $\mathbb{E}\left[\Gamma_{s_{Q(1)}}\,\overline{\Gamma_{s_{Q(2)}}}\right]\ \mathbb{E}\left[\Gamma_{s_{Q(3)}}\,\overline{\Gamma_{s_{Q(4)}}}\right]\geq 0$, we obtain

$$\left(\mathbb{E}\left[|\mathsf{Z}_n|^2\middle|\,\omega\right]\right)^2\geq\frac{8}{24}\sum_{s_1,\ldots,s_4\in S_n}|\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\ \ |\mathbb{E}\,[\mathrm{Y}]|^{2(2n-\tau_1-\tau_3)}.\tag{3.62}$$

On the other hand, note that for any choice of four paths $s_1,\cdots,s_4$, it always hold that

$$\mathbb{E}\left[\Gamma_{s_1}\,\overline{\Gamma_{s_2}}\,\Gamma_{s_3}\,\overline{\Gamma_{s_4}}\right]\leq|\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_1)}\ |\mathbb{E}\,[\mathrm{Y}]|^{2(n-\tau_3)}.$$

Hence,

$$\mathbb{E}\left[|\mathsf{Z}_n|^4\middle|\,\omega\right]\leq\sum_{s_1,\ldots,s_4\in S_n}|\mathsf{B}_{s_1}\,\mathsf{B}_{s_2}\,\mathsf{B}_{s_3}\,\mathsf{B}_{s_4}|\ \ |\mathbb{E}\,[\mathrm{Y}]|^{2(2n-\tau_1-\tau_3)}.\tag{3.63}$$

Then, we use (3.62) and (3.63) to prove the first estimate. The second one follows by straightforward adaptations of the argument leading to the first estimate in Lemma 3.14. □

LEMMA 3.31. *Suppose that the non-negative random variable Y satisfies*

$$\frac{\mathbb{E}\left[Y^\gamma\right]}{\left(\mathbb{E}\left[Y\right]\right)^\gamma} \leq B,$$

for $B, \gamma \geq 0$. Then, for all $\eta > 0$, there exists $m > 0$ such that

$$\mathbb{P}\left(\frac{Y}{\mathbb{E}\left[Y\right]} < m\right) \geq 1 - \eta.$$

Proof. First, let us take an arbitrary $m \in \mathbb{N}$. By Chebyshev's inequality, we have

$$\begin{aligned} 1 - \mathbb{P}\left(\frac{Y}{\mathbb{E}\left[Y\right]} < m\right) &= \mathbb{P}\left(\frac{Y}{\mathbb{E}\left[Y\right]} \geq m\right) \\ &\leq \frac{1}{m^\gamma}\,\mathbb{E}\left[\left|\frac{Y}{\mathbb{E}\left[Y\right]}\right|^\gamma\right] = \frac{1}{m^\gamma}\,\frac{\mathbb{E}\left[Y^\gamma\right]}{\left(\mathbb{E}\left[Y\right]\right)^\gamma} \leq \frac{B}{m^\gamma} < \eta, \end{aligned}$$

for m enough. □

LEMMA 3.32. *Suppose that $1 \leq \alpha_{\min} < 2$, $G(\alpha_{\min}) > \ln(b\ \mathbb{E}[\xi]|)$, and in addition to the hypotheses of Theorem 3.2, assume that the random variables ω and θ are independent. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[\left|Z_n(\xi)\right|^2 \middle|\, \omega\right] = 2G(\alpha_{\min}),$$

almost surely.

Proof. As noticed above, $\mathbb{E}\left[\left|Z_n(\xi)\right|^2 \middle|\, \omega\right] \geq Z_n\left(|\xi|^2\right)$, so that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[\left|Z_n(\xi)\right|^2 \middle|\, \omega\right] \geq \lim_{n\to\infty} \frac{1}{n} \ln Z_n\left(|\xi|^2\right) = 2G(\alpha_{\min}).$$

Hence, we only have to obtain an upper bound.

Now, assume that $1 \leq \alpha_{\min} \leq 2$ and write $\xi = e^{i\theta}|\xi| = e^{i\theta+\omega}$. We will define a new environment $e^{i\tilde{\theta}}|\xi|$ such that $\gamma \mapsto \left|\mathbb{E}\left[e^{i\gamma\tilde{\theta}}\right]\right|$ is decreasing in $[0,1]$. For this, we let $t = \left|\mathbb{E}\left[e^{i\theta}\right]\right|$, $z = t + i\sqrt{1-t^2}$ and

$$e^{i\tilde{\theta}} = \begin{cases} z, & \text{with probability } 1/2, \\ \bar{z}, & \text{with probability } 1/2. \end{cases}$$

Note that

$$\mathbb{E}\left[|\mathsf{Z}_n(\xi)|^2\middle|\,\omega\right] = \mathbb{E}\left[\left|\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right|^2\middle|\,\omega\right].$$

Now, there exists $0 < \gamma_0 < 1$ such that $\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)$ will be in the region $\mathcal{R}1$ for all $0 \le \gamma < \gamma_0$ and on the $\mathcal{R}1 - \mathcal{R}2$ boundary for $\gamma = \gamma_0$. Hence, for $0 \le \gamma < \gamma_0$, the second estimate in Lemma 3.30 and a Borel-Cantelli argument imply

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[\left|\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right|^2\middle|\,\omega\right] &\le \lim_{n\to\infty} \frac{1}{n} \ln \left|\mathbb{E}\left[\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right]\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \ln \left|\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right| =: f(\gamma),\end{aligned}$$

almost surely. Let $\delta > 0$. As the model falls into the region $\mathcal{R}2$ when $\gamma > \gamma_0$, there exists $0 < \gamma_1 < \gamma_0$ such that $f(\gamma_1) < 2G(\alpha_{\min}) + \delta$. Now, by construction, the function

$$[0,1] \ni \gamma \mapsto \mathbb{E}\left[\left|\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right|^2\middle|\,\omega\right],$$

is decreasing. Hence,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[|\mathsf{Z}_n(\xi)|^2\middle|\,\omega\right] &= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[\left|\mathsf{Z}_n\left(e^{i\tilde{\theta}}|\xi|\right)\right|^2\middle|\,\omega\right] \\ &\le \limsup_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[\left|\mathsf{Z}_n\left(e^{i\gamma_1\tilde{\theta}}|\xi|\right)\right|^2\middle|\,\omega\right] \\ &= f(\gamma_1) < 2G(\alpha_{\min}) + \delta,\end{aligned}$$

almost surely. This finishes the proof. □

The next lemma finishes the proof of Theorem 3.2.

Lemma 3.33. *Assume that* $1 \le \alpha_{\min} < 2$, $G(\alpha_{\min}) > \ln(b\mathbb{E}[\xi]|)$ *and, in addition to the hypotheses of Theorem 3.2, assume that the random variables* ω *and* θ *are independent. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln |\mathsf{Z}_n(\xi)| = G(\alpha_{\min}),$$

in probability. Under the τ*-condition, the convergence holds* $\mathbb{P}$*-almost surely.*

Proof. Thanks to the previous lemma and the first estimate in Lemma 3.30, the result follows by an application of Proposition 3.23 with $\mathcal{G} = \sigma\left(\omega(x) : x \in \mathbb{T}\right)$. □

3.6 Independent Radii and Phases

In this section, we will analyse how the three regions described in Theorem 3.2 are characterized in the case of independent phases and radii. The discussion below contains the proof of Corollary 3.5.

Recall that

$$\xi_{\beta,\gamma} = \exp(\beta\omega + i\gamma\theta), \quad \lambda_{\mathbb{R}}(\beta) = \ln\left(\mathbb{E}\left[\exp(\beta\omega)\right]\right), \quad \lambda_{\mathbb{C}}(\gamma) = -\ln\left|\mathbb{E}\left[\exp(i\gamma\theta)\right]\right|.$$

Let us keep in mind that we define our function G as

$$G(\alpha) = G_\beta(\alpha) = \frac{1}{\alpha}\ln\left(b\,\mathbb{E}\left[|\xi_{\beta,\gamma}|^\alpha\right]\right) = \frac{\ln b + \lambda_{\mathbb{R}}(\alpha\beta)}{\alpha}.$$

Hence,

$$G'_\beta(\alpha) = \frac{(\alpha\,\beta)\,\lambda'_{\mathbb{R}}(\alpha\,\beta) - \lambda_{\mathbb{R}}(\alpha\,\beta) - \ln b}{\alpha^2}.$$

As a preliminary, we note that $\alpha_{\min} > 2$ is equivalent to $G'_\beta(2) < 0$, which is equivalent to

$$2\beta\lambda'_{\mathbb{R}}(2\beta) - \lambda_{\mathbb{R}}(2\beta) < \ln b,$$

i.e. $\beta < \beta_0$. Note that this corresponds to the $\mathbb{L}^2$-region for the model with $\gamma = 0$. In the same vein, the condition $\alpha_{\min} > 1$ is equivalent to $G'_\beta(1) < 0$ which can be rewritten as

$$\beta\lambda'_{\mathbb{R}}(\beta) - \lambda_{\mathbb{R}}(\beta) < \ln b,$$

which corresponds to the weak disorder region for the model with $\gamma = 0$, i.e., $\beta < \beta_c$.

Finally, note that $\alpha_{\min} = \alpha_{\min}(\beta)$ satisfies

$$\alpha_{\min}\beta\lambda'_{\mathbb{R}}(\alpha_{\min}\beta) - \lambda_{\mathbb{R}}(\alpha_{\min}\beta) = \ln b.$$

Hence, $\alpha_{\min} = \beta_c/\beta$. Furthermore,

$$G_\beta(\alpha_{\min}) = \frac{\ln b + \lambda_{\mathbb{R}}(\beta_c)}{\beta_c/\beta} = \frac{\beta_c\lambda'_{\mathbb{R}}(\beta_c)}{\beta_c/\beta} = \beta\lambda'_{\mathbb{R}}(\beta_c).$$

The region $\mathcal{R}1$ is characterized by the condition that there exists $\alpha \in (1,2]$ such that $G_\beta(\alpha) < \ln(b|\mathbb{E}[\xi_{\beta,\gamma}]|)$, i.e.,

$$\frac{\ln b + \lambda_{\mathbb{R}}(\alpha\beta)}{\alpha} < \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma) = G_\beta(1) - \lambda_{\mathbb{C}}(\gamma). \tag{3.64}$$

We split our discussion into three cases. If $1 < \alpha_{\min} \leq 2$, then the above condition is equivalent to

$$G_\beta(\alpha_{\min}) < \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma) \quad \Leftrightarrow \quad \beta\lambda'_{\mathbb{R}}(\beta_c) - \lambda_{\mathbb{R}}(\beta) + \lambda_{\mathbb{C}}(\gamma) < \ln b.$$

We have seen above that $1 < \alpha_{\min} \leq 2$ is equivalent to $\beta_0 \leq \beta < \beta_c$.

Next, assume that $\alpha_{\min} > 2$ (in particular, implies that $\beta < \beta_0$). In this case, the function $\alpha \mapsto G_\beta(\alpha)$ is decreasing in the interval $(1, 2]$ and Condition (3.64) is therefore equivalent to $G_\beta(2) < G_\beta(1) - \lambda_{\mathbb{C}}(\gamma)$, i.e.,

$$\frac{\ln b + \lambda_{\mathbb{R}}(2\beta)}{2} < \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma) \quad \Leftrightarrow \quad \lambda_{\mathbb{R}}(2\beta) - 2\lambda_{\mathbb{R}}(\beta) + 2\lambda_{\mathbb{C}}(\gamma) < \ln b.$$

We are left with the possibility that $\alpha_{\min} \leq 1$, which, in particular, implies that $\beta \geq \beta_c$. In this case, the function $\alpha \mapsto G_\beta(\alpha)$ is increasing in the interval $(1, 2]$ and $G_\beta(\alpha) \geq G(1)$ for all $(1, 2]$. Hence, Condition (3.64) cannot be satisfied.

Finally, note that, in the whole $\mathcal{R}1$ region, we have

$$f(\beta, \gamma) = f_{\mathrm{I}}(\beta, \gamma) = \ln\left(b \ \left|\mathbb{E}[\xi_{\beta,\gamma}]\right|\right) = \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma).$$

The region $\mathcal{R}3$ is characterized by the condition that $\alpha_{\min} > 2$ (so that $\beta > \beta_0$) and $G_\beta(2) > \ln(b|\mathbb{E}[\xi_{\beta,\gamma}]|)$, i.e.,

$$G_\beta(2) = \frac{1}{2}\ln\left(b\,\mathbb{E}\left[|\xi_{\beta,\gamma}|^2\right]\right) = \frac{1}{2}\left(\ln b + \lambda_{\mathbb{R}}(2\beta)\right) > \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma).$$

We can rewrite this as

$$\lambda_{\mathbb{R}}(2\beta) - 2\lambda_{\mathbb{R}}(\beta) + 2\lambda_{\mathbb{C}}(\gamma) > \ln b.$$

Furthermore,

$$f(\beta, \gamma) = f_{\mathrm{III}}(\beta, \gamma) = G_\beta(2) = \frac{1}{2}\left(\ln b + \lambda_{\mathbb{R}}(2\beta)\right).$$

Finally, the region $\mathcal{R}2$ has two parts. The first one is characterized by the condition $\alpha_{\min} < 1$. We have seen above that this is equivalent to

$$\beta\lambda'_{\mathbb{R}}(\beta) - \lambda_{\mathbb{R}}(\beta) > \ln b,$$

i.e., $\beta > \beta_c$.

The second possibility is given by the conditions $1 \leq \alpha_{\min} < 2$ (i.e. $\beta_0 \leq \beta < \beta_c$) and $G_\beta(\alpha_{\min}) > \ln(b|\mathbb{E}[\xi_{\beta,\gamma}]|)$, i.e.,

$$\beta\lambda'_{\mathbb{R}}(\beta_c) > \ln b + \lambda_{\mathbb{R}}(\beta) - \lambda_{\mathbb{C}}(\gamma),$$

which can rewrite as

$$\beta\lambda'_{\mathbb{R}}(\beta_c) - \lambda_{\mathbb{R}}(\beta) + \lambda_{\mathbb{C}}(\gamma) > \ln b.$$

In both cases,

$$f(\beta,\gamma) = f_{\mathrm{II}}(\beta,\gamma) = G_\beta(\alpha_{\min}) = \beta\lambda'_{\mathbb{R}}(\beta_c).$$

The above discussion finishes the proof of the Corollary 3.5.

Chapter 4

Conclusions

In this study, we have established the asymptotic behavior of the partition function Z_n for the directed polymer problem with a complex-valued random environment over the Cayley tree. In particular, we have shown that only the three regions determined by Derrida, Evans, and Speer in [8] are possible and that, in these regions, the free energy for polymers of infinite size converges in probability

$$\frac{1}{n}\ln|\mathsf{Z}_n(\xi)| \xrightarrow{\mathrm{p}} \mathsf{f},$$

and, if we assume the τ-property, almost surely. The main methods used to prove these results are an extension of the martingale method and the estimation of non-integer moments of the partition function in the spirit of Buffet, Patrick, and Pulé in [4], as well as an extension of [8], which allowed us to understand where we can weaken hypotheses and provide the appropriate property to ensure almost sure convergence.

By demonstrating that we only need to consider hypotheses H1 and H2 about the distribution of the environment ξ, we relaxed the hypotheses of [8] in the regions $\mathcal{R}1$ and $\mathcal{R}3$. For the region $\mathcal{R}2$, we showed that it is only necessary to include the hypothesis of independence between radii and phase in the part where $1 < \alpha_{\min} \leq 2$. Moreover, we simplified many of the arguments used in [8].

Recall that the partition function Z_n can be expressed as

$$\mathsf{Z}_n = \sum_{s\in S_n} |\mathsf{B}_s|\ \Gamma_s, \qquad \mathsf{B}_s = \prod_{t=1}^{n} \xi(s_t), \qquad \Gamma_s = \frac{\mathsf{B}_s}{|\mathsf{B}_s|},$$

where $|\mathsf{B}_s|$ represents the amplitude and Γ_s the phase. A better bound was calculated for [8, Theorem 4.2] thanks to a thorough study of its proof. Since such a limit is significant for our study, three questions naturally arise to extend this work:

- Can we make the independence of radii and phases more flexible? For example, would it be enough to assume an estimate of the type
$$e^{-\lambda_-(\gamma)} \left|\xi_{\beta,\gamma}\right| \leq \left|\mathbb{E}\left[\xi_{\beta,\gamma}\middle|\,\omega\right]\right| \leq e^{-\lambda_+(\gamma)} \left|\xi_{\beta,\gamma}\right|.$$

- As these expressions arose several times during the preparation of this work, is it possible to obtain results
$$\lim_{n\to\infty} \frac{1}{n} \ln\left[|Z_n(\xi)|^2\middle|\,\mathcal{F}_{n-1}\right] = 2G(\alpha_{\min}),$$
and
$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\frac{|Z_n(\xi)|^4}{\mathbb{E}\left[|Z_n(\xi)|^2\middle|\,\mathcal{F}_{n-1}\right]^2}\middle|\,\mathcal{F}_{n-1}\right] < \infty?$$

- Is it possible to find an example of an environment such that $1 < \alpha_{\min} \leq 2$ and such that the model is not in the $\mathcal{R}2$ region?

Answering these questions could help us to weaken the hypothesis of independence between radius and phase. We defer them to future work.

Appendix

We prove Proposition 3.1, which we state again for the convenience of the reader.

PROPOSITION 4.1.

(i) The function $\alpha \mapsto \alpha\, G(\alpha) = \ln\left(b\, \mathbb{E}\left[|\xi|^\alpha\right]\right)$ *is convex, with* $G(0) = \ln b$*.*

(ii) The function G *satisfies exactly one of the following properties:*

- *There exists a unique minimizer of* G *denoted by* $\alpha_{\min} > 0$ *, i.e.,* G *is strictly decreasing in* $(0, \alpha_{\min}]$ *and strictly increasing in* $[\alpha_{\min}, \infty)$*.*
- G *is strictly decreasing in* $\mathbb{R}_+$*.*

Proof. It is trivial to note the identity $\ln\left(b\, \mathbb{E}\left[|\xi|^0\right]\right) = \ln b$.

(i) First, we will prove that such a function is differentiable. Indeed,

$$\frac{d}{d\alpha}\mathbb{E}\left[|\xi|^\alpha\right] = \lim_{h\to 0}\mathbb{E}\left[h^{-1}\left(|\xi|^{\alpha+h} - |\xi|^\alpha\right)\right] = \lim_{h\to 0}\mathbb{E}\left[|\xi|^\alpha h^{-1}\left(|\xi|^h - 1\right)\right].$$

Therefore, we must prove that we can enter the limit within the expectation. Let $h > 0$ (the case $h < 0$ is similar). If $|\xi| \geq 1$, write $|\xi|^h = e^{hX}$ (with $X \geq 0$). Then,

$$h^{-1}(|\xi|^h - 1) = \frac{e^{hX} - 1}{h} = X\frac{e^{hX} - 1}{hX} \leq Xe^{hX} \leq Xe^X = |\xi| \ln|\xi| \leq |\xi|^2.$$

If $|\xi| \leq 1$, write $|\xi|^h = e^{-hY}$, with $Y \geq 0$. In this case,

$$\frac{e^{-hY} - 1}{h} \geq -1.$$

In any case,

$$\left|\frac{|\xi|^h - 1}{h}\right| \leq \max\left\{1, |\xi|^2\right\}.$$

Hence, by the dominated convergence theorem,

$$\frac{d}{d\alpha}\mathbb{E}\left[|\xi|^{\alpha}\right] = \mathbb{E}\left[|\xi|^{\alpha}\ln|\xi|\right].$$

Higher-order derivatives can be handled similarly. Now,

$$\begin{aligned}
\frac{d}{d\alpha}\,\ln\left(b\,\mathbb{E}\left[|\xi|^{\alpha}\right]\right) &= \frac{\mathbb{E}\left[|\xi|^{\alpha}\,\ln|\xi|\right]}{\mathbb{E}\left[|\xi|^{\alpha}\right]} = \mathbb{E}\left[\ln|\xi|\,\frac{|\xi|^{\alpha}}{\mathbb{E}\left[|\xi|^{\alpha}\right]}\right] \\
&= \mathbb{E}_{\mathbb{P}_\alpha}\left[\ln|\xi|\right], \\
\frac{d^2}{d\alpha^2}\,\ln\left(b\,\mathbb{E}\left[|\xi|^{\alpha}\right]\right) &= \frac{\mathbb{E}\left[|\xi|^{\alpha}\,\ln^2|\xi|\right]\,\mathbb{E}\left[|\xi|^{\alpha}\right] - \left(\mathbb{E}\left[|\xi|^{\alpha}\,\ln|\xi|\right]\right)^2}{\left(\mathbb{E}\left[|\xi|^{\alpha}\right]\right)^2} \\
&= \mathbb{E}\left[\ln^2|\xi|\,\frac{|\xi|^{\alpha}}{\mathbb{E}\left[|\xi|^{\alpha}\right]}\right] - \left(\mathbb{E}\left[\ln|\xi|\,\frac{|\xi|^{\alpha}}{\mathbb{E}\left[|\xi|^{\alpha}\right]}\right]\right)^2 \\
&= \mathrm{Var}_{\mathbb{P}_\alpha}\left[\ln|\xi|\right] \geq 0, \qquad (4.1)
\end{aligned}$$

where $\dfrac{d\mathbb{P}_\alpha}{d\mathbb{P}} = \dfrac{|\xi|^{\alpha}}{\mathbb{E}\left[|\xi|^{\alpha}\right]}$. By (4.1) we conclude that the function $\alpha\,G(\alpha)$ is convex.

(ii) If ξ is concentrated at a point, then G takes the form

$$G(\alpha) = \frac{1}{\alpha}\,\ln\left(b\,\mathbb{E}\left[|\xi|^{\alpha}\right]\right) = \frac{1}{\alpha}\,\ln\left(b\,|\xi|^{\alpha}\right) = \frac{\ln b}{\alpha} + \ln|\xi|.$$

Therefore, G is strictly decreasing.

In all other cases, $\alpha\ G(\alpha)$ is strictly convex, so for every $\beta, \alpha > 0$ such that $\beta \neq \alpha$ we have

$$\begin{aligned}
\alpha\ G(\alpha) &> \beta G(\beta) + \left[x\ G(x)\right]'\Big|_{x=\beta}\ (\alpha-\beta) \\
&= \beta G(\beta) + \left[G(\beta) + \beta\,G'(\beta)\right]\ (\alpha-\beta).
\end{aligned}$$

– In particular, if G has a local extremum at $\alpha_{\min}$ and $\alpha_{\min} \neq \alpha$, we get

$$\begin{aligned}
\alpha\ G(\alpha) &> \alpha_{\min}G(\alpha_{\min}) + \left[G(\alpha_{\min}) + \alpha_{\min}\ G'(\alpha_{\min})\right]\ (\alpha-\alpha_{\min}) \\
&= \alpha_{\min}G(\alpha_{\min}) + G(\alpha_{\min})\ (\alpha-\alpha_{\min}) \\
&= \alpha\ G(\alpha_{\min}),
\end{aligned}$$

Therefore, $\alpha_{\min}$ is the only value where G reaches its global minimum.

– Finally, if G has no local extrema, G is strictly monotone. Since

$$\ln\left[b\,\mathbb{E}\left[|\xi|^0\right]\right] = \ln b \quad\Rightarrow\quad \lim_{\alpha\to 0^+} G(\alpha) = +\infty.$$

Hence, G must be strictly decreasing. □

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