Distorted positive isometries

by

Ariel Reyes Pardo

A thesis submitted in partial fulfillment of the requirements for the degree of

Master

in

Mathematics

Examined and approved on

____

by the following examiners:
Dr Daniel Coronel
Dr Godofredo Iommi

____

Dr Mario Ponce (supervisor)

Faculty of Mathematics

Pontifical Catholic University of Chile

Santiago, 2021

1 Acknowledgements

Firstly, I would like to thank my advisor Mario Ponce for his continued guidance, patience and encouragements throughout all this work. Specially for the invaluable time he spent talking about mathematics with me. During this process he helps me learn about mathematics, proofs, typing and life; thank you for all that, and more. I am always going to be grateful for his constant support during the previous years.

I would like to thank the professors Daniel Coronel and Godofredo Iommi for accept to be readers of this thesis, and for their helpful comments.

Futhermore, I would like to thank to all my friends from the PUC. Especially to Sebastian Burgos, Renato Velozo, Nicolas Alvarado, Camilo Gonzalez, Sandra Garrido, Cesar Gonzalez, Eduardo Oregon, Nicolas Vilches, Javiera Vaccaro, Vicente Lavandero and Miguel Aburto. They made me feel at home during all the years I have been here at the PUC. Also, I would like to thank to my friends Melany Reyes, Marcelo Aravena, Randal Santibañez, Brian Nolan, Calvin Arancibia, Alexander Blamey for their company and continuous support. They helped me to gain perspective about life and friendship.

Finally, I would like to thank to the most important people in my life: my family. I would like to thank my brothers Adrian and Moises for cheer me up each time we spend together. I would like to thank my parents for their inconditional love. Lastly, I would like to thank Carolina Orellana for looking after me continuously. I also thank Hirai Momo for his incredible talent and charisma and also Jason Frank who is an example of leadership and kindness. They are a central part of my life.

This work was partially funded by project "ANID FONDECYT 1180922" and by project "Anillo New Trends in Ergodic Theory", ACT 172001-1, ANID-Chile.

Thank you to all the people which have helped me throughout this process.

Contents

2 Introduction

In this work we will study a more general version of isometry in different metric spaces. Isometries by themselves are a very useful object of study in various areas and the number of areas in which they are used are very diverse, from geometry to Special Relativity [1]. For this reason, studying a more general version of isometry that manages to preserve the dynamic behavior of isometries is so attractive.

Let $(M_1, d_1), (M_2, d_2)$ metrics spaces an isometry $T\colon (M_1, d_1) \to (M_2, d_2)$ is a continuous function such that preserves distance, then generalization would be to define a function that almost preserves distances between points, this is how we have the following functions:

Definition 2.1. *Quasi-isometry Suppose that g is a (not necessarily continuous) function from one metric space (M_1, d_1) to a second metric space (M_2, d_2). Then f is called a quasi-isometry from (M_1, d_1) to (M_2, d_2) if there exist constants $A \geq 1, B \geq 0,$ and $C \geq 0$ such that the following two properties both hold:*
1. For every two points x and y in M_1, the distance between their images is up to the additive constant B within a factor of A of their original distance. More formally

$$\forall x, y \in M_1 : \frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B$$

2. Every point of M_2 is within the constant distance C of an image point. More formally: $\forall z \in M_2 : \exists x \in M_1 : d_2(z, f(x)) \leq C$

As we have just defined quasi-isometry, we can detect the first drawback for our objective of preserving dynamic behavior, and that is the continuity of quasi-isometries. Without continuity in the functions we do not have access to classical results in dynamic systems (and analysis in general) such as the fixed point theorems or invariance of sets.

Then we will define a type of function more general than an isometry but not as general as a quasi-isometry. This class of function will be continuous, therefore we can use the classical results of Dynamical Systems.

Definition 2.2. *Distorted isometry. Suppose that f is a continuous function from one metric space (M_1, d_1) to a second metric space (M_2, d_2). Then f is called a distorted isometry from (M_1, d_1) to (M_2, d_2) if there exists an isometry T_f from (M_1, d_1) to (M_2, d_2) and $\varepsilon > 0$ such that*

$$\forall x \in M_1 : d_2(f(x), T_f(x)) < \varepsilon.$$

Therefore the set of distorted isometries from (M_1, d_1) to (M_2, d_2) is a subsets of quasi-isometries from (M_1, d_1) to (M_2, d_2) , see lemma 5.5.
In the way we define this new class of functions we can observe a difference

with the quasi-isometries, distorted isometries are continuous function, in particular we have fixed points theorems. This is the main reason why we will study distorted isometries and not quasi-isometries. Given the continuity of distorted isometries we have more opportunity to preserve dynamic behaviors of isometries.

We have divided our thesis into three sections: Section 2 we set distorted positive isometries in $\mathbb{R}^2$ and some propierties of smoothnes, also we also give an example of distorted isometry which pushes us to study this type of functions in other types of metric spaces, in order to find more desired dynamic behaviors. In Section 3 we examine such basic definitions about isometries on the Half plane model and we will classify them. Then we study the distorted isometries depending on what type of isometry is related. In Section 4 it is initially motivated by the geometry of the distorted isometries but we were able to establish dynamics results on the dynamic behavior as a limit process of composition of these functions.

We would like to make it clear that it is a work on dynamicals systems strongly motivated by the geometry of isometries, but our objective is always to obtain information on classical dynamic properties such as fixed points, global attractors, global repulsors and invariant sets.

3 Dynamics of distorted positive isometries in $\mathbb{R}^2$.

For this it is important to remember that every isometry can be uniquely written as $\Psi v + t$ where $\Psi \in O_n(\mathbb{R})$ and $t \in \mathbb{R}^n$.

Definition 3.1. *Distorted isometry. We call distorted isometry to a continuous function* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that exists an isometry* A, $f(x) = A(x) + \rho(x)$ *with* $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ *a bounded function.*

Lemma 3.1. *A distorted isometry can be uniquely written as* $\Psi + r$ *where* Ψ *is ortonormal and* r *is a bounded function.*

Proof. It is sufficient to prove that the ortonormal part is unique.

Let $f(x) = A(x) + \rho$ with $A(x) = \Psi(x) + b$ then taking $r = b + \rho(x)$ we can write $f(x) = \Psi(x) + r$. Suppose by contradiction there exists Ψ_1, Ψ_2, r_1 and r_2 such that $\Psi_1 \neq \Psi_2$ and $r_1 \neq r_2$. Thus

$$f = \Psi_1 + r_1$$
$$f = \Psi_2 + r_2.$$

Then

$$\Psi_1 + r_1 = \Psi_2 + r_2$$

Now we get

$$\Psi_1 - \Psi_2 = r_2 - r_1$$

The RHS is the difference of 2 bounded functions and hence bounded. The LHS is a linear map. So we get $\Psi_1 - \Psi_2$ is a linear map which is bounded as a function (please do not confuse this with bounded linear map). This forces $\Psi_1 - \Psi_2 = 0$ and hence $r_1 = r_2$

Thus we prove tha uniqueness of the ortonormal part. $\square$

Remark 1. *If* ρ *is not bounded, but it is a function such that* $\lim_{x\to\infty} \frac{|\rho(x)|}{|x|} \to 0$ *then the previous lemma also holds.*

We can work with distorted isometry in the specific case when $n = 2$ then $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Is very common to identifies $\mathbb{R}^2$ with $\mathbb{C}$, for us will be helpful this identification because we are interested with behavior of distorted isometry near "infinity". Then a compactification $\mathbb{R}^2$ is the Riemann Sphere $\mathbb{C}_\infty$. Note that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function, also we will introduce the following notation: Let $\Psi \in O(n)$ and ρ bounded a continuous, then $\Psi(z) \mapsto \Psi z$ with $\Psi = e^{i\alpha}$.
Now we will ask for continuity of $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$.

Lemma 3.2. *Let* $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ *such that*

$$f(z) = \begin{cases} \Psi z + \rho(z) & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty \end{cases}$$

is a continuous function.

Proof. Let $\{z_n\}$ a sequence of complex numbers such that $z_n \xrightarrow{n\to\infty} \infty$. Now we have to study the behavior of f near ∞ point, for that

$$\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} \Psi z_n + \rho\left(z_n\right).$$

Remember that ρ is a bounded function, therefore this function is not important in the dynamic behavior of f at infinity. Thus we can ignore that term and just focus on the linear part $\Psi \in O_2(\mathbb{R})$. Indeed for all x we get $|\Psi x| = |x|$. Now we take a point far away from the origin because of that Ψx is far away from the origin i.e. $\lim_{n\to\infty} \Psi z_n = \infty$, then we can conclude the following:

$$\lim_{n\to\infty} f(z_n) = \infty.$$

Thus f is continuous function in $\mathbb{C}_\infty$. □

Since linear systems are simplest and well studied, we are always interested to find a good change of coordinates to make, in this case, distorted isometry compare it to a linear system. Indeed we are looking for a change of coordinates, so if we look at distorted isometry from a very great distance somehow we only expect to see the isometry.

Example 3.1. The first (non trivial) case if we considerer ρ to be a function of compact support K. Then for all $z_0 \notin K$ we have $f(z_0) = \Psi(z_0)+\rho(z_0) = \Psi(z_0)$, so in this case our desire change of coordinates h when we look from a very great distance is $h(z) = z$ because for all $z \notin K$ we have

$$\Psi(z) = h \circ f \circ h^{-1}(z) = f(z)$$

To find the changes of coordinates desire we will use a technique from Inversive geometry. We will map the ∞ point to the origin and then we study the dynamical behavior in a neighborhood around the origin and finally return to infinity. The technique is called Reciprocation and it is the following:

$$h\colon \mathbb{C}_\infty \to \mathbb{C}_\infty, z \mapsto \frac{1}{z}.$$

Note that $h(w) = \frac{1}{w} = h^{-1}(w)$ then we can compose this function with f in the following form:

$$h \circ f \circ h(w) = g(w) = \frac{1}{\Psi\frac{1}{w} + \rho(\frac{1}{w})}.$$

Definition 3.2. *We call g defined above as the extension to infinity of f.*

Remark 2. *$g(0) = 0$ and $g\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous function.*

Lemma 3.3. *If ρ is a bounded function then $g\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is differentiable at $w = 0$ with derivative equals to $\Psi^{-1} = e^{-i\alpha}$.*

Proof. Using the previous remark is sufficient to prove the case when Ψ is a rotation then $\Psi(w) = e^{i\alpha}w$. Let $w = \frac{1}{z}$, note that when $z = \infty$ then $w = 0$. So

$$\begin{aligned} g(w) &= \frac{1}{\Psi\frac{1}{w} - (-\rho(\frac{1}{w}))} \\ &= w\left(\frac{1}{\Psi - (-w\rho\left(\frac{1}{w}\right))}\right) \\ &= \Psi^{-1}w\left(\frac{1}{1 - (-\Psi^{-1}w\rho\left(\frac{1}{w}\right))}\right) \end{aligned}$$

Taking $r = -\Psi^{-1}\rho\left(\frac{1}{w}\right))$ note wr converges to 0 when w goes to 0 because ρ is a bounded function. Indeed $|r| < 1$ because Ψ is a isometry and $|w| < 1$, then we can write the last equality as follow:

$$\begin{aligned} g(w) &= \Psi^{-1}w(1 + wr + w^2r^2 + w^3r^3 + \ldots) \\ &= \Psi^{-1}w + \Psi^{-1}w^2r + \Psi^{-1}w^3r^2 + \Psi^{-1}w^4r^3 + \ldots \\ &= \Psi^{-1}w + w\hat{r}(w) \end{aligned}$$

With $\hat{r}(w) = \Psi^{-1}wr + \Psi^{-1}w^2r^2 + \Psi^{-1}w^3r^3 + \ldots$ a bounded function, note that $\lim_{w\to 0}\hat{r}(w) = 0$.

Now to compute the derivative:

$$\begin{aligned} Dg|_{w=0} = \lim_{w\to 0}\frac{g(w) - g(0)}{w - 0} &= \lim_{w\to 0}\frac{\Psi^{-1}w + w\hat{r}(w) - g(0)}{w - 0} \\ &= \lim_{w\to 0}\Psi^{-1} + \hat{r}(w) \\ &= \Psi^{-1}. \end{aligned}$$

As we desire. ☐

Remark 3. *Note that we just get another proof for uniqueness for the ortonormmal part Ψ of f.*

3.1 The Yoccoz example.

We can think that for every distorted isometry f will have a good behavior at $z = \infty$ in terms that f, through the changes of coordinates, can be conjugated with a rotation in some neighbourhood of the origin (infinity). Now we will an example by Yoccoz example, that is the well known non linearizable quadratic polynomial $\lambda z - \lambda z^2$ to exhibit an example that distorted isometry f needs some extra conditions to ensure the desired linearization. Lets build the distorted isometry f such that through the changes coordinates $w = \frac{1}{z}$

will have the form $g(w) = e^{i\alpha}w - e^{i\alpha}w^2$ near the origin.

$$\begin{aligned}
g(w) &= \frac{1}{\hat{f}(z)} \\
e^{i\alpha}w - e^{i\alpha}w^2 &= \frac{1}{\hat{f}(z)} \\
\hat{f}\,(z) &= \frac{1}{e^{i\alpha}\frac{1}{z} - e^{i\alpha}(\frac{1}{z})^2} \\
&= \frac{z}{e^{i\alpha} - e^{i\alpha}(\frac{1}{z})} \\
&= e^{-i\alpha}z\left(\frac{1}{1-\frac{1}{z}}\right)
\end{aligned}$$

Note when z is near to ∞ then $|\frac{1}{z}| < 1$ so from the last equality we get:

$$\hat{f}\,(z) = e^{-i\alpha}z(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots).$$

Then taking $\Phi(z) = e^{-i\alpha}z(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots)$ we get from the last equation

$$\hat{f}(z) = e^{-i\alpha}z + \Phi(z).$$

Note that $\Phi(z)$ is not a bounded function but through the change of coordinates we get the desire quadratic polynomial. To fix that issue we will multiply $\Phi(z)$ with another function to get a bounded function, also we can ask to this function to be C^∞, for that we will need the following lemma.

Lemma 3.4. *C^∞ Urysohn lemma. If K is a compact subset of $\mathbb{R}^d$ and U is an open set containing K, then there exists a function $\phi \in C^\infty(\mathbb{R}^d)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on K, and supp$\phi \subset U$.*

Proof. Let

$$\delta = d\,(K, U^c)$$

which is positive because K is compact and U^c is closed. Let

$$V = \left\{x \in \mathbb{R}^d : d(x, K) < \frac{\delta}{3}\right\} = K + B_{\delta/3}$$

and define f on $\mathbb{R}^d$ by

$$f = \left(\int_{\mathbb{R}^d} \psi(x)dx\right)^{-1} \psi_{\delta/3}$$

whose support is

$$\operatorname{supp} f = \operatorname{supp} \psi_{\delta/3} = \overline{B_{\delta/3}}$$

Finally define ϕ on $\mathbb{R}^d$ by

$$\phi = 1_V * f$$

Because V is bounded and f is C^∞, the function ϕ is C^∞. The support of ϕ is

$$\operatorname{supp}\phi = \operatorname{supp}(1_V * f) \subset \overline{\operatorname{supp}1_V + \operatorname{supp}f} = \overline{V + \overline{B_{\delta/3}}} = K + \overline{B_{2\delta/3}} \subset U$$

Because 1_V and f are nonnegative, so is their convolution ϕ. For any x

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(x-y)f(y)dy \leq \int_{\mathbb{R}^d} f(y)dy = 1$$

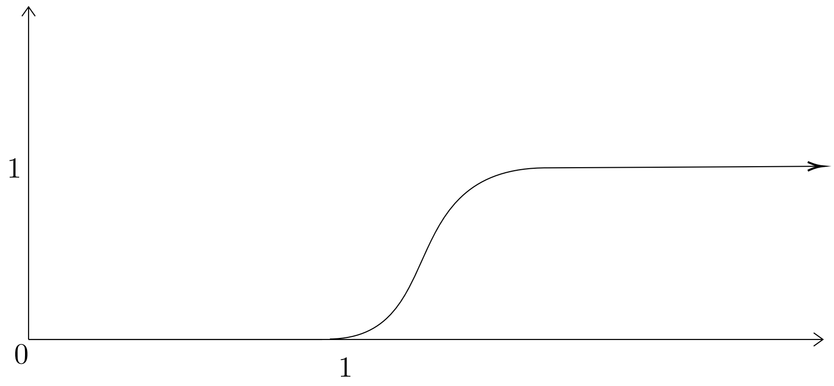
so $0 \leq \phi \leq 1$. For $x \in K$, if $y \in V^c$ then $|x-y| \geq \delta/3$. But $f(u) = 0$ for $|u| \geq \delta/3$, so in this case $f(x-y) = 0$. This implies that for $x \in K$ the functions $y \mapsto 1_V(y)f(x-y)$ and $y \mapsto f(x-y)$ are equal, hence

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(y)f(x-y)dy = \int_{\mathbb{R}^d} f(x-y)dy = \int_{\mathbb{R}^d} f(y)dy = 1$$

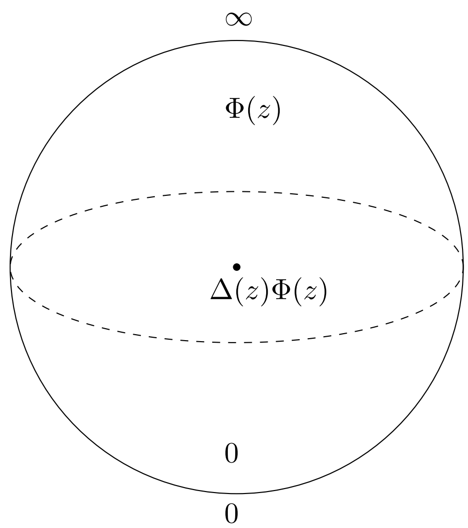
This shows that $\phi = 1$ on K, verifying all the assertions made about ϕ. $\square$

Now given the compact $K = [1,2] \subset \mathbb{R}$ there exists a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi = 1$ on K, then we can define a function $\Delta \in C^\infty(\mathbb{R})$ as follows

$$\Delta(z) = \begin{cases} \phi(|z|) & \text{if } |z| \leq 1 \\ 1 & \text{if } 1 \leq |z| \end{cases}$$



Then we get a bounded function $\rho(z) = \Delta(z)\Phi(z)$ because when z is near to ∞ the function $\Phi(z)$ is bounded function and at the problematic points, the sets of points z which are near to 0, we get $\Delta(z)\Phi(z) = 0$. So we have the following illustration:



Then we can get a C^∞ distorted isometry

$$f(z) = e^{-i\alpha}z + \Delta(z)\Phi(z).$$

Note that f is a distorted isometry such that, thorugh the changes of coordinates, neart to 0 is $g(w) = e^{i\alpha}w - e^{i\alpha}w^2$. Thanks to Yoccoz we know that g needs some arithmetic condition over α to ensure the conjugation with the desire rotation.

Theorem 3.5. *Yoccoz [2]. If α satisfied the Brjuno condition, then g is linearizable. On the other hand if α does not satisfied the Brjuno condition then the quadratic polynomial $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ is not linearizable.*

From the proof made by Yoccoz we know that g has periodic points accumulating at the origin, which means that f has periodic points accumulating at $+\infty$ which makes it different from $z \mapsto e^{i\alpha}z$.

Lemma 3.6. *If α is a Diophantine number and g is an analytic map with a fixed point at 0 with multiplier $e^{2\pi i\alpha}$ then g is conjugate to a complex rotation in angle α near the origin.*

Proof. It is a direct application of Siegel Theorem (see [3]). □

Lemma 3.7. *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a Brjuno number then g is linearizable near the origin.*

Proof. Note that if g is a holomorphic function near the origin and satisfy all arithmetical conditions needed to apply the Brjuno Theorem. □

3.2 Smoothness of Distorted Isometries at infinity.

From a dynamical point of view, there exists a lot of well known results about linearization, so in order to apply these classical theorems we need some kind of regularity over our distorted isometry g.

In a previous lemma we prove that g is differentiable at the origin and moreover we compute the derivative, the purpose of the following lemma is to ensure that g is a C^1 function.

Notation: We write ρ' to refer to the derivative of ρ with respect to z.

Lemma 3.8. *Let $g(w) = \frac{1}{\Psi(\frac{1}{w})+\rho(\frac{1}{w})}$ with ρ bounded and C^1 function. If $\lim_{z\to+\infty}\rho'(z) = 0$ then g is a C^1 function.*

Proof. Let

$$g'(w) = \frac{\Psi + \rho'(\frac{1}{w})}{(\Psi + w\rho(\frac{1}{w}))^2}$$

Using the fact that $Dg|_{w=0} = \Psi^{-1}$ in the the following limit we get

$$\lim_{w\to 0} g'(w) = \lim_{w\to 0} \frac{\Psi + \rho'(\frac{1}{w})}{(\Psi + w\rho(\frac{1}{w}))^2} = \Psi^{-1}.$$

Therefore $g'(w)$ is a continuous function then $g(w)$ is a C^1 as we desire. $\square$

Remark 4. *Note that if ρ' is a convergent function as $z \to +\infty$, necessarily has to converges to 0. Let c any non zero constant. If $\rho' \to c$ when $z \to \infty$ that implies that ρ is not a bounded function, then ρ' has to converges to 0.*

Remark 5. *Note that the condition ρ' converges to 0 when w goes to 0 in fact is a if only if condition to ensure the continuity of the first derivative of g. Also note that condition is a very strong and restrictive condition and this is a evidence of the limitation of compactification at one point, so we need to explore another type of compactification.*

But a question arises: what condition do we need to ensure that is a C^r function? The following lemma will provide us a condition over ρ to ensure the desired behavior, we will not prove this lemma because the main idea of the proof is contained at the previous lemma and the calculations are too long.

First we need to prove a result that will help us to have simpler expressions.

Lemma 3.9. ***Leibniz formula.*** *Let u, v functions such that u and v have the derivatives up to nth order. Then*

$$(uv)^{(n)} = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} u^{(n-i)} v^{(i)}$$

Proof. The first derivative is described by the following formula

$$(uv)' = u'v + uv'.$$

Then suppose by induction the desire formula hold up to n, we have to prove it also holds for $n+1$. Let $y = uv$ and suppose u and v have the derivatives of $(n+1)$th order. Using the recurrence relation we write the $(n+1)$th derivative in the following form:

$$y^{(n+1)} = \left[y^{(n)}\right]' = \left[(uv)^{(n)}\right]' = \left[\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} u^{(n-i)} v^{(i)}\right]'$$

After differentiation we obtain:

$$y^{(n+1)} = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} u^{(n-i+1)} v^{(i)} + \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} u^{(n-i)} v^{(i+1)}$$

Both sums in the right-hand side can be combined into a single sum. Indeed let m such that $1 \le m \le n$. The first term when $i = m$ is the following:

$$\begin{pmatrix} n \\ m \end{pmatrix} u^{(n-m+1)} v^{(m)},$$

and the second term when $i = m-1$:

$$\begin{pmatrix} n \\ m-1 \end{pmatrix} u^{(n-(m-1))} v^{((m-1)+1)} = \begin{pmatrix} n \\ m-1 \end{pmatrix} u^{(n-m+1)} v^{(m)}$$

And the sum of these two terms is:

$$\begin{pmatrix} n \\ m \end{pmatrix} u^{(n-m+1)} v^{(m)} + \begin{pmatrix} n \\ m-1 \end{pmatrix} u^{(n-m+1)} v^{(m)} = \left[\begin{pmatrix} n \\ m \end{pmatrix} + \begin{pmatrix} n \\ m-1 \end{pmatrix}\right] \cdot u^{(n-m+1)} v^{(m)}.$$

Then

$$\left[\begin{pmatrix} n \\ m \end{pmatrix} + \begin{pmatrix} n \\ m-1 \end{pmatrix}\right] \cdot u^{(n-m+1)} v^{(m)} = \begin{pmatrix} n+1 \\ m \end{pmatrix} u^{(n+1-m)} v^{(m)}.$$

Note that when m changes from 1 to n this combination will cover all terms of both sums except the term for $i = 0$ in the first sum equal to

$$\begin{pmatrix} n \\ 0 \end{pmatrix} u^{(n-0+1)} v^{(0)} = u^{(n+1)} v^{(0)},$$

and the term $i = n$ in the second sum equal to

$$\left(\begin{array}{c} n \\ n \end{array}\right) u^{(n-n)} v^{(n+1)} = u^{(0)} v^{(n+1)}.$$

Therefore the $(n+1)$th derivative of y have the following form:

$$y^{(n+1)} = u^{(n+1)} v^{(0)} + \sum_{m=1}^{n} \left(\begin{array}{c} n+1 \\ m \end{array}\right) u^{(n+1-m)} v^{(m)} + u^{(0)} v^{(n+1)},$$

then

$$y^{(n+1)} = \sum_{m=0}^{n+1} \left(\begin{array}{c} n+1 \\ m \end{array}\right) u^{(n+1-m)} v^{(m)}.$$

Thus the desire formula is proved for an arbitrary natural number n.

□

Lemma 3.10. *Let* $g(w) = \frac{1}{\Psi(\frac{1}{w})+\hat{\rho}(\frac{1}{w})}$ *with* $\hat{\rho}$ *a* C^r *function. If* $\lim_{z\to+\infty} \hat{\rho}^{(i)}(z) = 0$ *for all* $i \in \{0, 1, \ldots, r\}$ *then* g *is a* C^r *function.*

Proof. We want to compute the derivative of higher order of

$$g(w) = h \circ f \circ h(z) = h(f(h(z))).$$

The first derivative of $h(f(h(z)))$ have the following form:

$$h'(f(h(z)))f'(h(z))h'(z).$$

Using the previous lemma with $u = h'(f(z))$ and $v = f'(h(z))h'(z)$ we can write the rth derivative of $h \circ f \circ h(z) = h(f(h(z)))$:

$$(uv)^{(n)} = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) \Big(h'(f(h(z))\Big)^{(n-i)} \Big(f'(h(z))h'(z)\Big)^{(i)}.$$

Now taking the limit when $w \to 0$ is equivalent to $z \to \infty$, so we need to compute the following limit to ensure the continuity of the derivative of higher order:

$$\lim_{w\to 0} g^{(n)}(w) = \lim_{z\to\infty} \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) \Big(h'(f(h(z))\Big)^{(n-i)} \Big(f'(h(z))h'(z)\Big)^{(i)}.$$

Note that $\lim_{z\to\infty} \rho^{(i)}(h(z)) = 0$ implies that $\lim_{z\to\infty} f'(h(z)) = 0$ on the other hand we know that $\lim_{z\to\infty} h'(z) = 0$ then

$$\lim_{z\to\infty} \Big(f'(h(z))h'(z)\Big)^{(i)} = 0,$$

for every $i \in \{0, 1, 2, \ldots, n\}$.

Is easy to check that $\lim_{z\to\infty} h'(f(h(z)) = 0$ for every $i \in \{0, 1, 2, \ldots, r\}$.

Therefore all derivatives of $g(w)$ are continuous function then g is a C^r function.

$\square$

3.3 Discussion on the hypotheses of ρ for quadratics maps.

As we have seen in previous sections, reducing itself to making the corresponding coordinate change and applying the KAM theory to our problem is not enough to obtain satisfactory answers from the dynamic behavior of distorted isometries. Applying the desired coordinate change is not enough to achieve the smoothness requirements necessary to apply classical results of the KAM theory.

Previously we construct the Yoccoz example for distorted isometry, in that case the condition $\lim_{z\to\infty} \rho^{(i)}(z) = 0$ is achieved, but a question arises: what happens when we look for this condition in a general quadratic map? Let $g(w) = e^{i\theta}w - k(w)w^2$, so we proceed to find the desired function $\hat{f}(z)$.

Let $k(w) = k(h(z)) = r(z)$ a bounded function

$$\begin{aligned}\hat{f}(z) &= \frac{1}{e^{i\theta}\frac{1}{z} - \frac{r(z)}{z^2}} \\ &= e^{-i\theta} z \left(\frac{1}{1 - \frac{r(z)}{z}} \right)\end{aligned}$$

Since $r(z)$ is a bounded function when $z \to \infty$ then $|\frac{r(z)}{z}| < 1$ so we get the following:

$$\hat{f}(z) = e^{-i\theta} z(1 + \frac{r(z)}{z} + \left(\frac{r(z)}{z}\right)^2 + \left(\frac{r(z)}{z}\right)^3 + \dots)$$

Taking $\Phi(z) = e^{-i\theta}(\frac{r(z)}{z} + \left(\frac{r(z)}{z}\right)^2 + \left(\frac{r(z)}{z}\right)^3 + \dots)$ then $\hat{f}(z) = e^{-i\theta}z + \Phi(z)$. In the same way we did with the Yoccoz example we get the following function:

$$f(z) = e^{-i\theta}z + \Delta(z)\Phi(z),$$

and that f coincides with $\hat{f}$ in a neighbourhood of ∞. And, as we expected, $\lim_{z\to\infty} \rho^{(i)}(z) = \lim_{z\to\infty}(\Delta(z)\Phi(z))^{(i)} = 0$ for all $i \in \{0, 1, \dots, r\}$.

Throughout these sections we have studied and become convinced of the limitations of compactification at one point, since we have seen that the conditions necessary to ensure the desired smoothness on distorted isometries are not arbitrary. That is why the next step is to use another type of compactification where we can find interesting dynamic behaviors and with weaker conditions.

4 Distorted isometries of the hyperbolic plane.

In the sequel we will recall the fundamentals of isometries on hyperbolic geometry. With this, we will start the study of distorted versions of these maps.

4.1 Möbius transformation.

What are the analogous orientation- and distance-preserving functions in hyperbolic geometry? In particular, what are the orientation and distance-preserving functions in the Poincaré model? Since all rigid Euclidean isometries can be realized as certain one-to-one and onto complex functions, a good place to look for hyperbolic transformations might be in the entire class of one-to-one and onto complex functions. But, which functions should we consider? Since we are concentrating on the Poincaré model, we need to find one-to-one and onto orientationpreserving functions that preserve the Euclidean notion of angle, but do not preserve Euclidean length. Thus our first set of candidates are conformal maps. Euclidean rigid motions such as rotations and translations preserve angles, and preserve length globally. Such motions comprise a subset of all conformal maps.

If we consider the entire set of all conformal maps of the plane onto itself, then such maps must have the form $f(z) = az + b$, where $a \neq 0$ and b is a complex constant. since $a = |a|e^{i\theta}$, then f is the composition of a translation, a rotation, and a scaling by $|a|$. Thus, f maps figures to similar figures. The set of all such maps forms a group called the group of similitudes or similarity transformations of the plane. If $b = 0(f(z) = az, a \neq 0)$, we call f a dilation of the plane. Most similarity transformations cannot be isometries of the Poincaré model since most similarities (like translations and scalings) do not fix the boundary circle of the Poincaré disk. Clearly, we must expand our set of possible transformations. One way to do this is to consider the set of all one-to-one and onto conformal maps of the extended complex plane to itself.

Definition 4.1. *The general form of a Möbius transformation is a function on the extended complex plane defined by:*

$$f(z) = \frac{az + b}{cz + d}, ad - bc \neq 0$$

The set of Möbius transformations forms a group called the Möbius group.

Every Möbius transformation is composed of simpler transformations.

Theorem 4.1. *Let T be a Möbius transformation. Then T is the composition of translations, dilations, and inversion $\left(h(z) = \frac{1}{z}\right)$.*

Proof. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is the composition of a translation with a dilation.

If $c \neq 0$, then $f(z) = \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}\cdot$ Thus, f is the composition of a translation (by $\frac{d}{c}$), an inversion, a dilation (by $-\frac{ad-bc}{c^2}$), and a translation (by $\frac{a}{c}$) $\cdot \square$. $\square$

Note that the Möbius group includes the group of Euclidean rigid motions ($|a| = 1, c = 0, d = 1$), and the group of similarities ($a \neq 0, c = 0, d = 1$) as subgroups. Also note that we could define Möbius transformations as those transformations of the definition with $ad - bc = 1$, by dividing the numerator by an appropriate factor.

4.2 Geometric properties of Möbius transformation.

Of particular interest to us will be the effect of a Möbius transformation on a circle or line.

Theorem 4.2. *Given any three distinct complex numbers z_1, z_2, z_3, there is a unique Möbius transformation f that maps these three values to a specified set of three distinct complex numbers w_1, w_2, w_3.*

Proof. Let $g_1(z) = \frac{z-z_2}{z-z_3}\frac{z_1-z_3}{z_1-z_2}$. Then g_1 is a Möbius transformation and g_1 maps z_1 to $1, z_2$ to 0, and z_3 to the point at infinity.

Let $g_2(w) = \frac{w-w_2}{w-w_3}\frac{w_1-w_3}{w_1-w_2}$. We see that g_2 is a Möbius transformation mapping w_1 to $1, w_2$ to 0, and w_3 to ∞. Then $f = g_2^{-1} \circ g_1$ will map z_1 to w_1, z_2 to w_2, and z_3 to w_3.

Is f unique? Suppose f' also mapped z_1 to w_1, z_2 to w_2, and z_3 to w_3. Then $f^{-1} \circ f'$ has three fixed points, and so $f^{-1} \circ f' = id$ and $f' = f$ $\square$

Definition 4.2. *The cross ratio of four complex numbers z_0, z_1, z_2, and z_3 is denoted by $(z0, z1, z2, z3)$ and is the value of*

$$\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}.$$

Definition 4.3. *A subset of the plane is called a cline if it is either a circle or a line.*

The cross ratio can be used to identify clines.

Lemma 4.3. *Let z_0, z_1, z_2, and z_3 be four distinct points. Then the cross ratio $(z0, z1, z2, z3)$ is real if and only if the four points lie on a cline.*

Proof. Let $f(z) = (z, z_1, z_2, z_3)$. Then since f is a Möbius transformation, we can write

$$f(z) = \frac{az + b}{cz + d}$$

Now $f(z)$ is real if and only if

$$\frac{az+b}{cz+d} = \frac{\overline{az}+\bar{b}}{\overline{cz}+\bar{d}}$$

Multiplying this out, we get

$$(a\bar{c}-c\bar{a})|z|^2 + (a\bar{d}-c\bar{b})z - (d\bar{a}-b\bar{c})\bar{z} + (b\bar{d}-d\bar{b}) = 0. \tag{1}$$

If $(a\bar{c}-c\bar{a}) = 0$, let $\alpha = (a\bar{d}-c\bar{b})$ and $\beta = b\bar{d}$. Equation (1) simplifies to

$$\operatorname{Im}(\alpha z + \beta) = 0$$

This is the equation of a line. If $(a\bar{c}-c\bar{a}) \neq 0$, then dividing through by this term we can write equation (1) in the form

$$|z|^2 + \frac{a\bar{d}-c\bar{b}}{a\bar{c}-c\bar{a}}z - \frac{d\bar{a}-b\bar{c}}{a\bar{c}-c\bar{a}}\bar{z} + \frac{b\bar{d}-d\bar{b}}{a\bar{c}-c\bar{a}} = 0$$

Let $\gamma = \frac{a\bar{d}-c\bar{b}}{a\bar{c}-c\bar{a}}$ and $\delta = \frac{b\bar{d}-d\bar{b}}{a\bar{c}-c\bar{a}}$. since $a\bar{c}-c\bar{a}$ is pure imaginary, we have that

$$\bar{\gamma} = (-)\frac{d\bar{a}-b\bar{c}}{a\bar{c}-c\bar{a}} = \frac{d\bar{a}-b\bar{c}}{c\bar{a}-a\bar{c}}$$

Equation (1) becomes

$$|z|^2 + \gamma z + \overline{\gamma z} + \delta = 0$$

Or

$$|z+\bar{\gamma}|^2 = -\delta + |\gamma|^2$$

After multiplying and regrouping on the right, we get

$$|z+\bar{\gamma}|^2 = \left|\frac{ad-bc}{a\bar{c}-c\bar{a}}\right|^2$$

since $ad-bc \neq 0$, this gives the equation of a circle centered at $-\bar{\gamma}$. □

Theorem 4.4. *A Möbius transformation f will map clines to clines. Also, given any two clines c_1 and c_2, there is a Möbius transformation f mapping c_1 to c_2.*

Proof. Let c be a cline and let z_1, z_2, and z_3 be three distinct points on c. Let $w_1 = f(z_1), w_2 = f(z_2)$, and $w_3 = f(z_3)$. These three points will lie on a line or determine a unique circle. Thus, w_1, w_2, and w_3 will lie on a cline c'. Let z be any point on c different than z_1, z_2, or z_3. By the previous theorem we have that (z, z_1, z_2, z_3) is real. Also, $(f(z), w_1, w_2, w_3) = (f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3)$, and thus $f(z)$ is on the cline through w_1, w_2, and w_3.

For the second claim let z_1, z_2, and z_3 be three distinct points on c_1 and w_1, w_2, and w_3 be three distinct points on c_2. By Theorem 3.2 there is a Möbius transformation f taking z_1, z_2, z_3 to w_1, w_2, w_3. It follows from the first part of this proof that f maps all points on c_1 to points on c_2.

□

Remark 6. *Möbius transformations of* $\mathbb{H}$ *such that* a, b, c *and* d *are real numbers are orientation-preserving. And from now on we will refer to them as Möbius transformations.*

Now we proceed to define a metric on our space in term of the cross ratio.

Definition 4.4. *Let* $d\colon \mathbb{H}^2 \to \mathbb{R}^+ \cup \{0\}$ *defined as follows*

$$d(z_1, z_2) = \log\left(z_1, z_2; z_1^\times, z_2^\times\right)$$

Here, $z_1^\times$ *and* $z_2^\times$ *are the endpoints, on the real number line, of the geodesic joining* z_1 *and* z_2*. These are numbered so that* z_1 *lies in between* $z_1^\times$ *and* z_2*.*

Lemma 4.5. *If* $z_1, z_2,$ *and* z_3 *are distinct points and* T *is a Möbius transformation, then* $(z, z_1, z_2, z_3) = (T(z), T(z_1), T(z_2), T(z_3))$ *for any* z*.*

Proof. Proof: Let $g(z) = (z, z_1, z_2, z_3)$. Then $g \circ T^{-1}$ will map $T(z_1)$ to 1, $T(z_2)$ to 0, and $T(z_3)$ to ∞. But, $h(z) = (z, T(z_1), T(z_2), T(z_3))$ also maps $T(z_1)$ to $1, T(z_2)$ to 0, and $T(z_3)$ to ∞. since $g \circ T^{-1}$ and h are both Möbius transformations, and both agree on three points, then $g \circ T^{-1} = h$. Since $g \circ T^{-1}(T(z)) = (z, z_1, z_2, z_3)$ and $h(T(z)) = (T(z), T(z_1), T(z_2), T(z_3))$ the result follows. □

Lemma 4.6. *Möbius transformations are isometries in* $\mathbb{H}$*.*

Proof. Since we define our metrics by cross ratio and using the previous lemma the result is direct. □

It can be shown that every orientation preserving isometry of $\mathbb{H}$ is a Möbius transformation.

4.3 Compactification of the Poincaré disk.

In this section we will review some tools necessary for the compactification of hyperbolic spaces.The reference for this sequell is [4].
We will be working with very general definitions but it is important to keep in mind that the motivation of this section is always the compactification of the Poincaré disk (or equivalently the Poincaré Halfplane) and then develop results as we did in the last section.

Definition 4.5. *Two geodesic rays $c, c'\colon [0,\infty) \to X$ in a metric space X are said to be asymptotic if $\sup_t d(c(t), c'(t))$ is finite.*

Remark 7. *This condition is equivalent to the Hausdorff distance between the images of c and c' is finite.*

One of the main themes of this work is the large scale dynamics and geometry of metric spaces. In this context one needs a language that will lend precision to observations such as the following: if one places a dot at each integer point along a line in the Euclidean plane, then the line and the set of dots become indistinguishable when viewed from afar, whereas the line and the plane remain visibly distinct. One makes this observation precise by saying that the set of dots is quasi-isometric to the line whereas the line is not quasi-isometric to the plane.

Definition 4.6. *Quasi-Isometry. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necessarily continuous) map $f : X_1 \to X_2$ is called a (λ, ε)-quasi-isometric embedding if there exist constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for all $x, y \in X_1$*

$$\frac{1}{\lambda} d_1(x,y) - \varepsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x,y) + \varepsilon$$

If, in addition, there exists a constant $C \geq 0$ such that every point of X_2 lies in the $C-$ neighbourhood of the image of f, then f is called a (λ, ε) -quasi-isometry. When such a map exists, X_1 and X_2 are said to be quasi-isometric.

Definition 4.7. *Quasi-Geodesics. A (λ, ε) -quasi-geodesic in a metric space X is* a(λ, ε) *-quasi-isometric embedding $c : I \to X$, where I is an interval of the real line (bounded or unbounded) or the intersection of $\mathbb{Z}$ with such an interval. More explicitly,*

$$\frac{1}{\lambda}\left|t - t'\right| - \varepsilon \leq d\left(c(t), c\left(t'\right)\right) \leq \lambda\left|t - t'\right| + \varepsilon$$

for all $t, t' \in I$. If $I = [a, b]$ then $c(a)$ and $c(b)$ are called the endpoints of c. If $I = [0, \infty)$ then c is called a quasi-geodesic ray.

Quasi-geodesics will play an important role in this section. In particular we will see that quasi-geodesics in hyperbolic spaces such as $\mathbb{H}^n$ follow geodesics closely.

Definition 4.8. *Two quasi-geodesic rays are said to be asymptotic if the Hausdorff distance between their images is finite.*

Being asymptotic is an equivalence relation on quasi-geodesic rays. We write ∂X to denote the set of equivalence classes of geodesic rays in X and we write $\partial_q X$ to denote the set of equivalence classes of quasi-geodesic rays. In each case we write $c(\infty)$ to denote the equivalence class of c.

Definition 4.9. *Slim Triangles. Let $\delta > 0$. A geodesic triangle in a metric space is said to be δ -slim if each of its sides is contained in the δ - neighbourhood of the union of the other two sides. A geodesic space X is said to be δ -hyperbolic if every triangle in X is δ -slim. (If X is δ -hyperbolic for some $\delta > 0$, one often says simply that X is hyperbolic.)*

Theorem 4.7. *(Stability of Quasi-Geodesics). For all $\delta > 0, \lambda \geq 1, \varepsilon \geq 0$ there exists a constant $R = R(\delta, \lambda, \varepsilon)$ with the following property: If X is a δ -hyperbolic geodesic space, c is a (λ, ε) -quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is less than R.*

Lemma 4.8. *If X is a proper geodesic space that is δ -hyperbolic, then the natural map from ∂X to $\partial_q X$ is a bijection. For each $p \in X$ and $\xi \in \partial X$ there exists a geodesic ray $c : [0, \infty) \to X$ with $c(0) = p$ and $c(\infty) = \xi$*

Proof. The natural map $\partial X \to \partial_q X$ is obviously injective. To prove the remaining assertions, given $p \in X$ and a quasi-geodesic ray $c : [0, \infty) \to X$, let c_n be a geodesic with $c_n(0) = p$ that joins p to $c(n)$. since X is proper, a subsequence of the c_n converges to a geodesic ray $c_\infty : [0, \infty) \to X$ (by the Arzelà-Ascoli Theorem (I.3.10)). The previous theorem provides a constant k such that the Hausdorff distance between $c([0, n])$ and the image of c_n is less than k; thus we obtain a bound on the Hausdorff distance between c and c_∞. □

Lemma 4.9. *(Visibility of ∂X). If the metric space X is proper, geodesic and δ-hyperbolic, then for each pair of distinct points $\xi_1, \xi_2 \in \partial X$ there exists a geodesic line $c : \mathbb{R} \to X$ with $c(\infty) = \xi_1$ and $c(-\infty) = \xi_2$*

Proof. Fix $p \in X$ and choose geodesic rays $c_1, c_2 : [0, \infty) \to X$ from p with $c_1(\infty) = \xi_1$ and $c_2(\infty) = \xi_2$. Let T be such that the distance from $c_1(T)$ to the image of c_2 is greater than δ. For each $n > T$ we choose a geodesic segment $[c_1(n), c_2(n)]$ and consider the geodesic triangle with sides $c_1([0, n]), c_2([0, n])$ and $[c_1(n), c_2(n)]$. Since this triangle is δ-slim, $[c_1(n), c_2(n)]$ must intersect the closed (hence compact) ball of radius δ about $c_1(T)$, at a point p_n. By the Arzelà-Ascoli Theorem, as $n \to \infty$ a subsequence of the geodesics $[p_n, c_2(n)] \subset [c_1(n), c_2(n)]$ will converge. By passing to a further subsequence we may assume that the sequence $[c_1(n), c_2(n)]$ converges. The limit is a geodesic line which we call c. Since each $[c_1(n), c_2(n)]$ is contained in the δ-neighbourhood of the union of the images of c_1 and c_2, the image of c is also contained in this neighbourhood. Thus the endpoints of c are ξ_1 and ξ_2 □

Lemma 4.10. *(Asymptotic Rays are Uniformly Close). Let X be a proper δ -hyperbolic space and let $c_1, c_2 : [0,\infty) \to X$ be geodesic rays with $c_1(\infty) = c_2(\infty)$ (1) If $c_1(0) = c_2(0)$ then $d(c_1(t), c_2(t)) \leq 2\delta$ for all $t > 0$ (2) In general, there exist $T_1, T_2 > 0$ such that $d(c_1(T_1 + t), c_2(T_2 + t)) \leq 5\delta$ for all $t \geq 0$*

Now we proceed to work on the main objective of this section, compactifaction of the $\delta-$hyperbolic space. The following description that we will give of the topology on $\bar{X} = X \cup \partial X$. For this task it will be convenient to define a more general idea of rays.

Definition 4.10. *A generalized ray is a geodesic $c : I \to X$, where either $I = [0, R]$ for some $R \geq 0$ or else $I = [0,\infty)$. In the case $I = [0, R]$ it is convenient to define $c(t) = c(R)$ for $t \in [R, \infty]$. Thus $\bar{X} := X \cup \partial X$ is the set $\{c(\infty) \mid c$ a generalized ray $\}$*

Definition 4.11. *(The Topology on $\bar{X} = X \cup \partial X$). Let X be a proper geodesic space that is δ -hyperbolic. Fix a basepoint $p \in X$. We define convergence in $\bar{X}$ by: $x_n \to x$ as $n \to \infty$ if and only if there exist generalized rays c_n with $c_n(0) = p$ and $c_n(\infty) = x_n$ such that every subsequence of (c_n) contains a subsequence that converges (uniformly on compact subsets) to a generalized ray c with $c(\infty) = x$. This defines a topology on $\bar{X}$: the closed subsets $B \subset \bar{X}$ are those which satisfy the condition $[x_n \in B, \forall n > 0$ and $x_n \to x] \implies x \in B$*

Lemma 4.11. *(Neighbourhoods at Infinity). Let X and $p \in X$ be as above. Let $k > 2\delta$. Let $c_0 : [0,\infty) \to X$ be a geodesic ray with $c_0(0) = p$ and for each positive integer n let $V_n(c_0)$ be the set of generalized rays c such that $c(0) = p$ and $d(c(n), c_0(n)) < k$ Then $\{V_n(c_0) \mid n \in \mathbb{N}\}$ is a fundamental system of (not necessarily open) neighbourhoods of $c(\infty)$ in $\bar{X}$.*

Proof. Let c' be a ray in X with $c'(0) = p$. It follows from a previous lemma that $c'(\infty) = c_0(\infty)$ if and only if $c'(n) \in V_n(c_0)$ for all $n > 0$. And if c_i is a sequence of generalized rays in X with $c_i(0) = p$ and $c_i \notin V_n(c_0)$, then by the Arzelà-Ascoli theorem there is a subsequence $c_{i(j)}$ that converges to some $c \notin V_n(c_0)$, hence c_i does not converge to c_0 in $\bar{X}$. Thus $c_i \to c_0$ in $\bar{X}$ if and only if for every $n > 0$ there exists $N_n > 0$ such that $c_i \in V_n(c_0)$ for all $i > N_n$. □

Lemma 4.12. *Let X be a proper geodesic space that is $\delta-$hyperbolic then $\bar{X} = X \cup \partial X$ is compact.*

Proof. The balls $B(x, r)$, with $r > 0$ rational, form a fundamental system of neighbourhoods about $x \in X \subset \bar{X}$. This observation, together with the preceding lemma, shows that the topology on $\bar{X}$ satisfies the first axiom of countability. Thus it suffices to prove that $\bar{X}$ is sequentially compact, and this is obvious by Arzelà-Ascoli. □

Do not forget that all the tools reviewed in this section were motivated by looking for an analog of compactification by a point on the Poincaré disk.

Remark 8. *The Poincaré disk is a $log3$-hyperbolic space.*

4.4 Distorted Möbius transformations.

We want to study some distorted version of Möbius transformations as we did before with the orientation-preserving isometries of $\mathbb{R}^2$.

Definition 4.12. *A distorted Möbius transformations is a continuous function $f\colon \mathbb{H} \to \mathbb{H}$ such that there exists a Möbius transformation T and a small number $\epsilon > 0$ such that*

$$d(f(z), T(z)) < \epsilon, \text{ for all } z \in \mathbb{H}$$

Lemma 4.13. *Given f a distorted Möbius transformation and $\varepsilon > 0$, $T, H \in Mob(\mathbb{D})$ such that $dist(f(z), T(z)) < \varepsilon$ and $dist(f(z), H(z)) < \varepsilon$ $\forall z \in \mathbb{D}$, then $T = H$.*

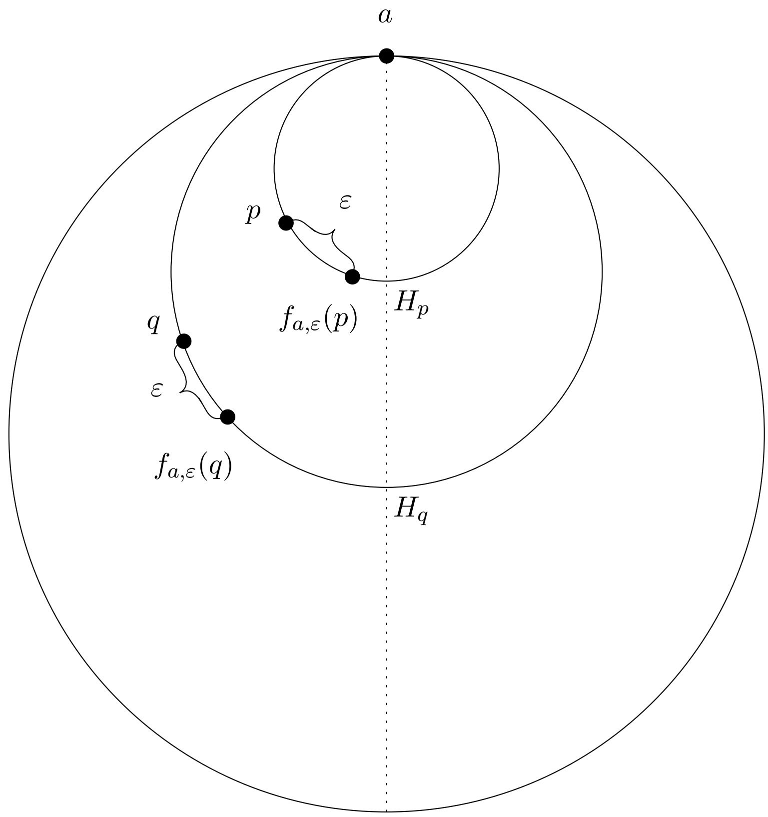
Definition 4.13. *Given f a distorted Möbius transformation we write T_f the unique Möbius transformation of the lemma* 4.13.

We will not give the proof of this result yet, we will do it in the next section when we have a general result that will give an elegant and simple proof.

Remark 9. *Note the previous lemma is different than lemma 3.1 because in lemma 3.1 the isometry was not unique, only the lineal part was unique.*

Example 4.1. Let a a point in $\partial\mathbb{H}$ and a positive constant ε. Let considerer the following function:

$f_{a,\varepsilon}\colon \mathbb{H} \to \mathbb{H}$ a function such that for a point p the image $f_{a,\varepsilon}(p)$ is in the horocycle H_p wich contains a and p such that the arc length between p and $f_{a,\varepsilon}(p)$ is ε, note we can consider two points to be $f_{a,\varepsilon}(p)$ (because we can move along to H_p to the right and to the left) but we will only consider the point such that the first coordinates in the Half plane model of the Poincaré disk is bigger than the pre-image, that means we will consider the point to the right of the pre-image.

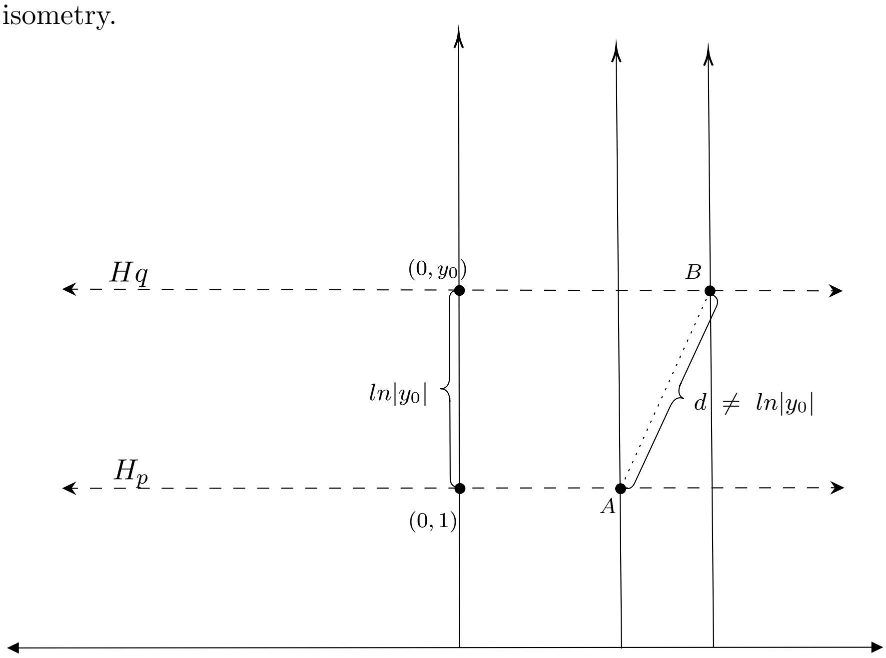


The previous function is a distorted Möbius transformation because the distance between the two points q and $f_{a,\varepsilon}(q)$ is bounded because these two points belongs to the same horocycle H_q. The previous function is a distorted Möbius transformation but it is not an isometry.

Lemma 4.14. *The function $f_{a,\varepsilon}$ is not an isometry.*

Proof. To prove that $f_{a,\epsilon}$ is not an isometry, it would be easiest to work in the upper half plane model using $a = \infty$, in which the horocycles are the horizontal lines $y = y_0$ along which the horocyclic distance is $\Delta x / y_0$. We may take $p = (0, 1)$ and so H_p is the $y = 1$ horocycle. Take $q = (0, y_0)$, and so H_q is the $y = y_0$ horocycle, where $y_0 \neq 1$. We have $d(p, q) = \ln |y_0|$, and $A = f_{a,\varepsilon}(p) = (\varepsilon, 1)$, and $B = f_{a,\varepsilon}(q) = (\varepsilon y_0, y_0)$.

Every distance mimizing line segment from the $y = 1$ horocycle H_p to the $y = y_0$ horocycle H_q is a "vertical" line segment in the upper half plane, from a point $(x, 1)$ to the point (x, y_0), all having the same length $\ln |y_0|$. Every "nonvertical" path from a point on H_p to a point on H_q has length strictly larger than $\ln |y_0|$. Since the points A and B do not lie on the same vertical segment (because $\varepsilon \neq \varepsilon y_0$), every path between those two points A, B has length strictly larger than $\ln |y_0|$; its not hard to prove this by simply looking at the path length integral for any parameterized path from A to B, and using the fact that the path must have a nonvertical tangent line along a subinterval of the parameter interval. It follows that $f_{a,\varepsilon}$ is not an isometry.



$\square$

4.5 Continuous extension of distorted Möbius.

In a previous section we prove that distorted isometries are continuous in the compact metric space $\bar{\mathbb{C}}$, so we are able to give a similar answer to the boundary at infinity of the compactification of the Poincaré disk.

Recall that a distorted Möbius is a function f such that $d(f(z), T(z))$ is bounded with, T a Möbius transformation. Now we will give the proof to the lemma(4.13) that we left pending.

Lemma 4.15. *If $\gamma : [0, \infty) \to X$ is geodesic ray then $B \circ \gamma : [0, \infty) \to X$ is a quasi-geodesic ray.*

Lemma 4.16. *Let the geodesic ray γ then the geodesic ray $T \circ \gamma$ and the quasi-geodesic ray $f \circ \gamma$ belongs to the same equivalence class of geodesic ray.*

Proof. By definition $f \circ \gamma$ is close to $T \circ \gamma$ (because f is a distorted Möbius transformation). And using theorem of Stability of quasi-geodesics (Theorem(4.7)) there exist γ' close to $f \circ \gamma$. Then $T \circ \gamma$ is close to γ' then they have the same class of equivalence. Therefore $f \circ \gamma$ and $T \circ \gamma$ have the same class of equivalence. □

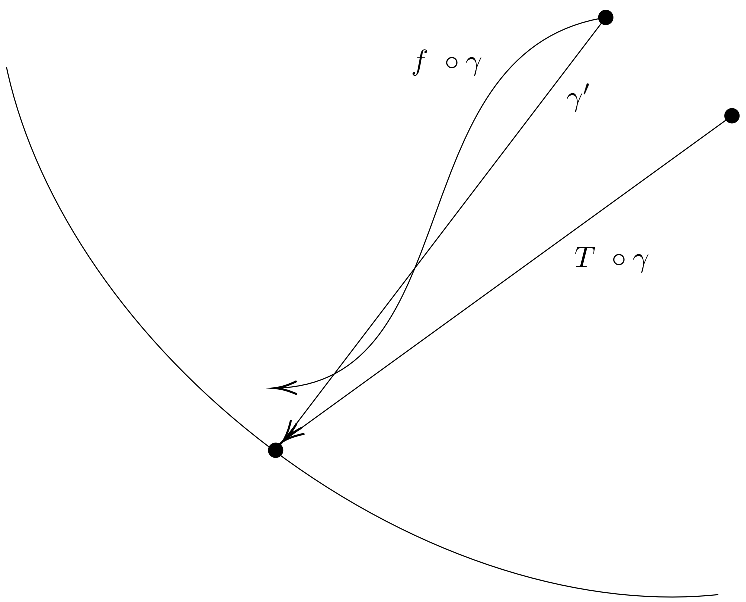


Figure 1: Construction

Remark 10. *f at the boundary, $\partial\mathbb{H}$, looks like the Möbius transformation T. Note that we have just given the proof that we had left pending on the uniqueness of T.*

Definition 4.14. *Given a function f let define f^* as the action in $\partial\mathbb{D}$.*

Lemma 4.17. *Let the continuous function* $(-)^*$ *then it has some properties:*

1. $T_1^* \circ T_2^* = (T_1 \circ T_2)^*$.
2. $(T^{-1})^* = (T^*)^{-1}$.

Proof. We will not give a detailed demonstration but we will give an idea of how to prove the first part since point two is a direct consequence of the first. For the first point the main idea is take an arbitrary geodesic γ and then see the image of γ under the composition of $T_1 \circ T_2$ and observe that on $\partial\mathbb{D}$ the image the same under $T_1^* \circ T_2^*$. □

Lemma 4.18. *Distorted Möbius transformation extends continuously to the boundary.*

Proof. Because f is a distorted Möbius transformation, in particular is a continuous function on $\mathbb{H}$. Recall that $f = T_f$ on $\partial\mathbb{H}$ because of continuity of T_f the distorted Möbius transformation f is continuous on $\partial\mathbb{H}$. Now let $\gamma\colon \mathbb{R} \to \mathbb{H} \cup \partial\mathbb{H}$ a geodesic with end points $a, b \in \partial\mathbb{H}$, then $T_f(\gamma)$ is a geodesic with end points $T_f(a), T_f(b) \in \partial\mathbb{H}$. Let $\{\gamma(n)\}$ a sequence such that $\lim_{n\to+\infty} \gamma(n) = a$, because of continuity of f we have the following:

$$\lim_{n\to+\infty} f(\gamma(n)) = f\left(\lim_{n\to+\infty} \gamma(n)\right) = f(a).$$

Since $f = T_f$ we have $f(a) = T(a)$ therefor f is a continuous function on $\mathbb{H} \cup \partial\mathbb{H}$. □

Lemma 4.19. *Let* T^* *the action in* $\partial\mathbb{H}$ *related to a isometry* T. *Then the function* $T \mapsto T^*$ *is injective.*

Proof. We will prove that a given an isometry it is determined by its action at the boundary. Let γ_1 and γ_2 two geodesic such that $\gamma_1 \cap \gamma_2 \neq \varnothing$ with $A, B \in \partial\mathbb{H}\cap\gamma_1$ and $C, D \in \partial\mathbb{H}\cap\gamma_2$. We know that isometry maps geodesic to geodesic then image of γ_1 is determined by the geodesic through $T^*(A)$ and $T^*(B)$, analogous for γ_2 and the points $T^*(C)$ and $T^*(D)$. But $\gamma_1 \cap \gamma_2 \neq \varnothing$ then there exists a point x such that $x \in \gamma_1\cap\gamma_2$ then $T(x) \in T^*(\gamma_1)\cap T^*(\gamma_2)$ then T^* is unique for T. □

Lemma 4.20. *Let* f *be a distorted Möbius transformation and* T *a isometry related to* f *then* $T^* = f^*$. *In particular* T *is unique.*

Proof. Note for a given geodesic γ with end points A, B $f(\gamma)$ is of the same class of equivalence as $T(\gamma)$ then by the previous lemma the action in the boundary determines that f and T are the same in the boundary. Moreover T is unique ($f^* = T^*$). □

Is important to notice that in the case of Euclidean isometries when we consider the one point compactification we give an algebraic proof about uniqueness of the linear part but in this compactification we give a geometric proof about uniqueness of T.

Definition 4.15. *We say that a distorted Möbius transformation f is "hyperbolic", "parabolic" or "elliptic" when T_f is "hyperbolic", "parabolic" or "elliptic" respectively.*

4.6 Classification of Isometries.

In this section we give a detailed account of this classical topic see [5] for further references. When given a description of an isometry, it is possible to classify it into one of several categories which closely resemble the well-known types of Euclidean isometries. Note that it is conventional to classify Möbius transformations in a similar fashion, but using different terms. To highlight similarities with and distinctions to Euclidean geometry, I'll primarily use names like those used for Euclidean geometry, but mention the names from the Möbius group classifications as well. The key to either classification is an analysis of the fixed points of the transformation. Let $z = x + yi$ be the position of a point. In order for this point to be a fixed point, its homogeneous coordinates must be an eigenvector of the transformation matrix.

$$\begin{pmatrix} a-bi & c+di \\ c-di & a+bi \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} z \\ 1 \end{pmatrix} \qquad a,b,c,d \in \mathbb{R}$$

$$\lambda = (c-di)z + (a+bi)$$

$$(a-bi)z + (c+di) = \lambda z = ((c-di)z + (a+bi))z$$

$$(c-di)z^2 + (2bi)z - (c+di) = 0$$

If $c = d = 0$, then the matrix will represent a multiplication by a fixed complex root of unity $\frac{a-bi}{a+bi}$, which corresponds to a rotation around the origin, as described above. In this case, the second fixed point would be the point at infinity, which isn't described by the equation as its coordinates $(1,0)^T$ don't match the prescribed form. In the classification of Möbius transformations, such a transformation is a special case of the elliptic transformations. If we even have $b = c = d = 0$, then the equation will hold for any z, thus representing the identity transformation. The identity transformation can be seen as a special case of most of the other classes, so in a complete classification it makes sense to consider it as a distinct class by itself. If $c + di \neq 0$, then there will in general be two fixed points.

$$z_{1,2} = \frac{-2bi \pm \sqrt{(2bi)^2 - 4(c-di)(c+di)}}{2(c-di)} = \frac{-bi \pm \sqrt{c^2+d^2-b^2}}{c-di}$$

In case the discriminant $c^2 + d^2 - b^2$ is positive, both results will be located on the unit circle, as the following computation verifies.

$$|z_1|^2 = z_1 \cdot \overline{z_1} = \frac{\sqrt{c^2+d^2-b^2} - bi}{c-di} \cdot \frac{\sqrt{c^2+d^2-b^2} + bi}{c+di} = \frac{c^2+d^2-b^2+b^2}{c^2+d^2} = 1$$

A similar equation holds for z_2. Those two points on the unit disk can be considered the ideal "endpoints" of a hyperbolic line, which is uniquely defined by those two points. The corresponding transformation is a hyperbolic translation, moving all points away from one of the fixed points and towards the other, keeping their connecting line as a whole invariant. In terms of the usual classification of Möbius transformations, such a group would be called hyperbolic, although the use of this term here has only a very remote connection to its use in hyperbolic geometry. These two uses should not be confused. If the discriminant is negative, the square root will result in a purely imaginary number. Conjugating that number will change its sign. For this reason, the computation above now expresses a slightly different product, namely

$$z_1 \cdot \overline{z_2} = \frac{\sqrt{c^2 + d^2 - b^2} - bi}{c - di} \cdot \frac{\sqrt{c^2 + d^2 - b^2} + bi}{c + di} = \frac{c^2 + d^2 - b^2 + b^2}{c^2 + d^2} = 1$$

This means that z_1 and z_2 are related to one another via an inversion in the unit circle, those two points in the model are in fact different representatives of the same point in the hyperbolic plane. That single fixed point is the center of a hyperbolic rotation. In the common classification of Möbius transformations, this would be called an elliptic transformation. The two fixed points coincide in a single point on the unit circle. This denotes a so-called limit rotation. Like a rotation, it has no finite fixed lines, but like a translation, it has no finite fixed points either. Instead, all points will be moved along horocycles which pass through the single ideal fixed point. There is no obvious counterpart to this in Euclidean geometry, although depending on the way one translates concepts, one can think of this as a special case of either a translation or a rotation. In the nomenclature of Möbius transformations, this would be called a parabolic transformation.

	No. of fixed points in $\mathbb{D}$	No. of fixed points in $\partial\mathbb{D}$	Conjugate to
Hyperbolic.	0	2	a dilation $z \mapsto kz, k \neq 1$
Parabolic.	0	1	the translation $z \mapsto z + 1$
Elliptic.	1	0	a rotation

4.6.1 Fixed points of distorted Möbius transformation.

The objective of this section is to answer the question about the fixed points of a distorted Möbius transformation since for each distorted Möbius transformation there is a unique Möbius transformation related to it. That is why the work of the previous section of classifying Möbius transformation by the number of fixed points and even more about the location of these fixed points was important.

Theorem 4.21. *Let f a distorted Möbius transformation and the Möbius transformation T_f. If T_f is a Hyperbolic/Parabolic transformation and z is fixed point for T_f then $f(z) = z$. If T_f is a Elliptic Möbius transformation and z is fixed point for T_f then f has at least a fixed point.*

Proof. We have two cases. First case is when T_f a Hyperbolic/Parabolic Möbius transformation, recall that we prove in a previous section that f and T_f are the same at $\partial\mathbb{D}$ then if $z \in \partial\mathbb{D}$ is fixed point for T_f then it will be a fixed point for f. The last case if the distorted Möbius transformation f is related to a Elliptic Möbius transformation T_f. Recall that f is a continuous function from the compact Poincaré Disk to itself then by the Schauder fixed point theorem f has a fixed point. $\square$

4.7 Dynamics of distorted Möbius transformation.

As we studied previously, each type of isometry has a different dynamic behavior and different invariant set, Hyperbolic transformation, for example one fixed point is a repulsor and the other points is an atractor, then we can interpret this as that the dynamics of the transformation is that the points in space move from the repulsor to the attractor and left invariant the geodesic passing through this two fixed points. On the other hand, elliptic transformation has one fixed point and left invariant $\partial\mathbb{D}$, then we can say that the dynamics is similar to a rotation in the euclidian case. And finally we have parabolic transformations that left invariant the horocycles passing through the fixed point, parabolic case can be considered as a limit case of the previous situation where the fixed point goes to infinity, that means the fixed point belongs to $\partial\mathbb{D}$.

Now a natural question appears: given a distorted Möbius transformation f can we say something about the dynamics of f? In general we are not able to ensure that invariant sets between T and the distorted Möbius transformation f are the same.

Definition 4.16. *Given a Möbius transformation T we define the displacement of T as the following number*

$$disp(T) := \inf\{d_{\mathbb{H}}(z, T(z))\}.$$

The previous definition is somehow (from the metric point of view) to measure how much a function moves the points in space.

Remark 11. *It is important to note that the definition 4.16 is a way to classify isometries.*

If $disp(T) = 0$ and the infimum is attained then T is a elliptic transformation, if $disp(T) = 0$ and the infimum is not attained then T is a parabolic transformation and if $disp(T) = c$ with positive constant c then T hyperbolic.

Definition 4.17. *Given and a distorted Möbius transformation f and T_f the related Möbius transformation. We define the isometric distortion to the following number:*

$$distor(f) := \sup\{d_{\mathbb{H}}(f, T_f)\}.$$

Note that the above definition is well defined because f is a distorted Möbius trasnformation.

4.7.1 Hyperbolic case.

The first case we will study will be the hyperbolic case. In this case, the dynamics of T can be summarized in that there are two fixed points on the boundary $\partial\mathbb{D}$ and T moves all the points away from one and takes them to the other fixed point, leaving the geodesic that passes through these two fixed points invariant. Therefore we would like to say that the fixed points of f act in the same way as they did for T.

Now we will introduce some definitions and results to give the desired answer.

Definition 4.18. *Busseman functions. Let $\mathbb{H}$ be a unique geodesic hyperbolic space. Given $\alpha \in \partial\mathbb{H}$ and a point $p \in \mathbb{H}$, we define the Busseman function $B_{p,\alpha} : \mathbb{H} \to \mathbb{R}$ in the direction α and with base point p as*

$$B_{p,\alpha}(h) = \lim_{n\to\infty} d_{\mathbb{H}}\left(x_n, h\right) - d_{\mathbb{H}}\left(x_n, p\right)$$

where $(x_n) \subset \mathbb{H}$ is any sequence such that $x_n \to \alpha$. The convergence and independence on (x_n) relies on the triangle inequality and the hyperbolicity of $\mathbb{H}$.

Lemma 4.22. *Let $\alpha \in \partial\mathbb{H}$, $x_n \to \alpha$ if and only if $B_{x_0,\alpha}(x_n) \to -\infty$.*

Proof. See [6]. $\square$

Lemma 4.23. *Let $\gamma : \mathbb{R} \to \mathbb{H}$ be a geodesic with end points $\alpha, \beta \in \partial\mathbb{H}$ then*

$$B_{x_0,\alpha}(f(x)) \leq B_{x_0,\alpha}(x) - (disp(T_f) - \varepsilon),$$

where $\varepsilon \geq distor(f)$.

Proof. By definition for a point $x \in \gamma$ we have

$$B_{x_0,\alpha}(T(x)) + d_{\mathbb{H}}\left(T(x), x\right) = B_{x_0,\alpha}(x).$$

Because of hyperbolicity we have the following property:

$$B_{x_0,\alpha}(T(x)) - B_{x_0,\alpha}(f(x)) \leq d_{\mathbb{H}}\left(T(x), f(x)\right) \leq \varepsilon.$$

Using the first part of this proof with the last inequality we get:

$$\begin{aligned} B_{x_0,\alpha}(f(x)) - B_{x_0,\alpha}(x) + d_{\mathbb{H}}\left(T(x), x\right) &\leq \varepsilon \\ B_{x_0,\alpha}(f(x)) - B_{x_0,\alpha}(x) + disp(T) &\leq \varepsilon \end{aligned}$$

Then we get

$$B_{x_0,\alpha}(f(x)) \leq B_{x_0,\alpha}(x) - (disp(T) - \varepsilon)$$

$\square$

Proposition 1. Let T a hyperbolic Möbius transformation and f be a distorted Möbius transformation with $T_f = T$, and the point $\alpha \in \partial\mathbb{H}$ be the global attractor for T. If $distor(f) < disp(T)$ then $\alpha \in \partial\mathbb{H}$ is a global attractor point for f.

Proof. Using the lemma 4.23 is easy to see that

$$B_{x_0,\alpha}(f^n(x)) \leq B_{x_0,\alpha}(x) - n(disp(T) - \varepsilon). \tag{2}$$

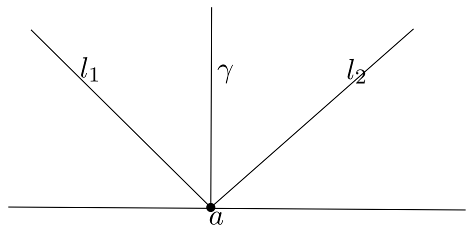
Now using $x = x_0$ in (3) and recall that $B_{x_0,\alpha}(x_0) = 0$ we get:

$$B_{x_0,\alpha}(f^n(x)) \leq -n(disp(T) - \varepsilon)$$

Now taking $n \to \infty$ and using lemma 4.22 we are done. $\square$

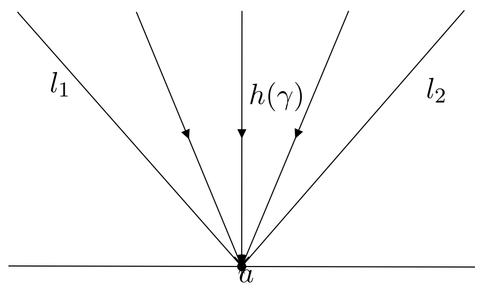
Remark 12. *An analogous result for the global repulsor can be stated.*

Example 4.2. We will construct an example where $distor(f) \geq disp(T)$, and such that the dynamics of f is different from the dynamics of T, showing that Propositon 1 is sharp. Considerer the upper half plane model and T hyperbolic Möbius transformation T with the imaginary axis γ the invariant geodesic with end points ∞ and the origin. Suppose that the origin acts as



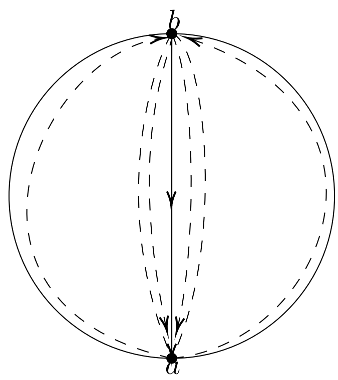
a repulsor and the infinity point acts as an attractor for T. Here we can considerer T as the dilation $z \mapsto kz$ with $k > 1$. Let K the compact set determined by two equidistants lines l_1 and l_2 such that $\partial K = l_1 \cup l_2$ and $\gamma \subset K$.

Let the continuous function h such that $h|_\gamma = T^{-1}$ this means that the origin is now an attractor for $z \in \gamma$ and the ∞ is a repulsor for γ. Now we extends continuously to ∂K such that $h|_{\partial K} = Id$ and then we extends continuously to the boundary such that $h = T$.



Then our distorted Möbius transformation f is the composition between h and the continuous bounded function B such that $\lim_{n\to\infty} Re(B^n(z)) = 0$, this means that $|Re(z)| > |Re(B(z))|$.

Note that $distor(f) \geq disp(T)$ and f on K the origin acts as an attractor and the infinity points acts as a repulsor point.



Note that for all z there exists a number $N \in \mathbb{N}$ such that for all $n > N$, $f^n(z) \in K$, then the origin acts as an attractor point for all the point on the interior of the Poincaré Disk, analogous for the infinity points acts as a repulsor for the points inside of the Poincaré Disk. Thus there are no open sets for the previous construction such that the behavior of T is preserved under f.

4.7.2 Parabolic case.

Now we will discuss the case of a parabolic distorted Möbius transformation.

Recall for a parabolic Möbius transformation T the two invariant sets are: the pencil of all geodesics passing through the fixed point in the boundary (each element of this family maps to another) and horocycles which are orthogonal to the geodesics from the first family and they are tangents to the boundary at the fixed point. If T moves points of the horocycle clockwise, that means $Re(z) < Re(T(z))$ in the upper half plane model, then f also moves points clockwise.

But a question arises and it is if we can ensure that the dynamic behavior of f is similar to that of T in some subset of space? For example, a subset we could refer to a family of Horcycles near the imaginary axis in the model of the upper half plane.

We will give an example sketch where the above is not true, that is, there is no subset where the fixed point acts as a repulsor / attractor. Let T a parabolic Möbius transformation and f the distorted Möbius transformation related and by definition $distor(f) > disp(T) = 0$.

To give the desire example, it would be easiest to illustrate the situation first in the upper half model.

Recall that the hyperbolic distance of z and $T(z)$ is:

$$d_{\mathbb{H}}(z, w) = Arc\cosh\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(T(z))}\right)$$

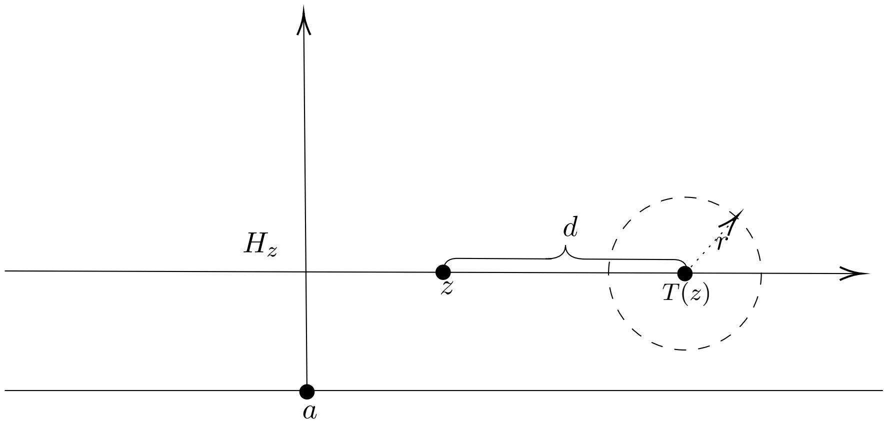
When $|Im(z)| \to 0$ then $d_{\mathbb{H}}(z, T(z)) := d \to \infty$. WLOG we can take

$$T : z \mapsto z + 1,$$

note that for all z, we have $Im(z) = Im(T(z))$, so if $|Im(z)|$ is small enough in the worst case the distance between $f(z)$ and z is equal to $d - r > 0$ with $r = d_{\mathbb{H}}(T(z), f(z))$. Thus $Re(f(z)) > Re(z)$ in particular $Re(f^{k+1}(z)) > Re(f^k(z))$ with $k \geq 0$. And let f a distorted Möbius transformation such that f it can have an erratic behavior compared to T, for example it can transform the repulsor into an attractor for a family of horcycles. An important observation is given the distance in this model there will be a coordinate in the imaginary axis such that for every coordinate smaller than this the function T will move the points much more than what distorts f.

So now we concentrate on looking for an f such that it is a distorted and that it somehow manages to move the points such that the images of the iterates have increasingly larger imaginary coordinates.

And this is easy to imagine consider the translation T named above and consider f a distortion of this such that the points under f are at a bounded distance r from $T(z)$.



Let y_0 the imaginary coordinate such that for all $y \geq y_0$ f has an erratic behavior, that is, we cannot assure that it has a behavior similar to that of the function T.

Let z an arbitrary point such that $Im(z) < y_0$ is easy to see that there exists an iterative of f such that $Im(f^k(z)) \geq y_0$. Take k large enough such that $ky_0 \geq y_0$ and this just means that iterated $k-$th has the desired imaginary coordinate then we are done.

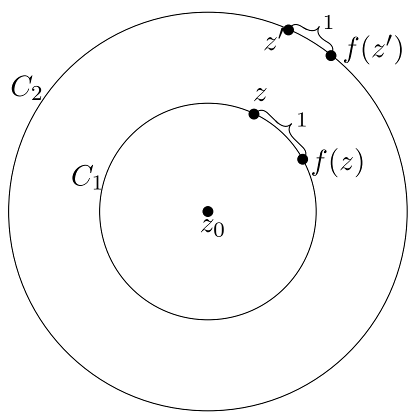
From what we have discussed in this section we cannot be sure if the behaviors between T and f are similar, either because of their invariant sets or because of the nature of their fixed point.

4.7.3 Elliptic case.

In the previous cases we were able to establish some relationship between the dynamics of T and its related f, but we are going to consider a couple of examples for the elliptic case in which we cannot establish some relationship with the dynamics of T.

One way of wanting to establish some relationship between the dynamics of T and f is to try to see if there is any relationship between the rotation numbers of both functions. Let us consider the following example.

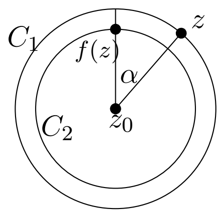
Example 4.3. Considerer $T = Id$ with rotation number equals to 0 and considerer the function f such that leaves the circles centered at a point z_0 invariant, $f : z' \mapsto f(z')$ such that the arclenght between z' and $f(z')$ is equal to 1.



Note that $f|_{\partial\mathbb{D}} = T|_{\partial\mathbb{D}}$ and the invariants sets for f are the same for T but on every circle C centered in z_0 f has a different rotation number. So the only thing we can conclude is that f and T are equal on the boundary.

Now let's build an example where the rotation number is preserved but the invariant sets are not preserved.

Example 4.4. Let T the rotation around the fixed point z_0 in angle α and let f the distorted rotation related to T such that for z the image under f is a rotation in angle α and then we move the point to another circle centered in z_0 which is at a bounded distance.



Note that in this example the rotation number is preserved but the invariant sets are not, again as in the previous example the only thing we can conclude is that both functions are equal on the boundary.

4.8 Resume

Now we will make a brief summary of what we could conclude throughout this section of the dynamic behaviors between f and T_f.

Hyperbolic: We prove Proposition 1 so if $distor(f) < disp(T_f)$ then $\alpha \in \partial\mathbb{H}$ the global attractor for T_f it is also the global attractor for f.

Also we construct an example where $distor(f) \geq disp(T_f)$ and we conclude that f and T_f they did not share dynamic behavior.

Parabolic: We construct an example such that we cannot be sure if the behaviors between T_f and f are similar, either because of their invariant sets or because of the nature of their fixed point.

Elliptic: We gave two examples, the first one we conserve the invariant sets of T_f but f did not preserve the rotation number of T_f. The second one, we conserve the rotation number of T_f but f did not preserve the invariants sets of T_f. Therefore we cannot ensure that f and T_f have the same dynamic behavior.

The only thing we can be sure of in these three cases is something that we already knew and that is that T_f and f are equal on the boundary $\partial\mathbb{H}$.

5 Cocycles

Let's consider a continuous transformation $R\colon \Omega \to \Omega$ where Ω is a compact metric space, Let $A\colon \Omega \to dIsom(\mathbb{H})$ a function thats takes value on the space of distorted Möbius transformation. We write$d_{\mathbb{H}}(\cdot,\cdot), d_{\Omega}(\cdot,\cdot)$ for the distances in $\mathbb{H}$ and Ω respectively and we $A(\omega) = f_\omega$.

Definition 5.1. *We say that $A : \Omega \to dI\,\mathrm{som}(\mathbb{H})$ is continuous for the topology of the uniform convergence on bounded sets when, given any bounded set $K \subset \mathbb{H}$, any $\omega_0 \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in K$ we have*

$$d_{\Omega}\left(\omega_0, \omega\right) < \delta \Rightarrow d_{\mathbb{H}}\left(A(\omega) \cdot h, A\left(\omega_0\right) \cdot h\right) < \varepsilon$$

Whenever A is continuous for the topology of the uniform convergence on bounded sets we will simply say that the pair (R, A) is a continuous cocycle by distorted isometries of the fiber $\mathbb{H}$ over the base space Ω.

Remark 13. *We can generate the previous topology by all $B_K(f,\varepsilon)$, where $B_K(f,\varepsilon)$ is the set $\left\{g \in \mathbb{H}^{\Omega} : \sup_{x\in K}\ \mathrm{d}_{\mathbb{H}}(f(x), g(x)) < \varepsilon\right\}$ for a given bounded set K and $\varepsilon > 0$ and $f \in \mathbb{H}^{\Omega}$.*

We will ask that the function A be continuous under the topology of the uniform convergence on bounded sets.

Lemma 5.1. *A is a measurable function.*

Proof. **Sketch:** If A is continuous it is also measurable because the inverse image of sets $B_K(f,\varepsilon)$ are open sets in the topology of Ω then the inverse images of sets $B_K(f,\varepsilon)$ are measurable sets, therefore the function A is measurable. □

For $\omega \in \Omega$ we write the following function that measures how far f_ω is from T_ω

$$\begin{aligned} G : \Omega &\longrightarrow \mathbb{R}_0^+ \\ \omega &\longmapsto \varepsilon_\omega := \sup_{h\in\mathbb{H}} d_{\mathbb{H}}(f_\omega(h), T(h)). \end{aligned}$$

Remark 14. *G is a measurable function because of the continuity of A un der the topology of uniform convergence on bounded sets, since the inverse image under G of open sets in $\mathbb{R}_0^+$ are measurable sets in Ω, therefore G is a measurable function.*

5.1 Skew product dynamical system.

Given a cocycle (R, A) we can construct the following dynamics

$$\begin{aligned} F : \Omega \times \mathbb{H} &\longrightarrow \Omega \times \mathbb{H} \\ (\omega, h) &\longmapsto (R\omega, A(\omega) \cdot h), \end{aligned}$$

where $A(\omega) \cdot h = f_\omega(h)$ with f_ω a distorted Möbius transformation related to T such that $d_{\mathbb{H}}(T(h), f_\omega(h)) < \varepsilon_\omega$ for all $\omega \in \Omega$.

In the sequell we will assume that there exists $T \in Mob(\mathbb{H})$ such that $T_\omega = T$ for every $\omega \in \Omega$.

Lemma 5.2. *Let $F^n(\omega, h) = (R^n(\omega), f_{R^{n-1}(\omega)} \circ \cdots \circ f_\omega(h))$. If $disp(T) > disp_{isom}(f_\omega)$ for all $\omega \in \Omega$ then $\lim_{n\to\infty} f_{R^{n-1}(\omega)} \circ \cdots \circ f_\omega(h) = b$ where $b \in \partial\mathbb{H}$ is the global attractor.*

Proof. Because for all $\omega \in \Omega$ f_ω is a distorted Möbius transformation related to the same Möbius transformation T, then we can choose $\sup_\omega disp_{isom}(f_\omega)$ and apply lemma 1. $\square$

Theorem 5.3. *Given ergodic $f_\omega : \Omega \to \Omega$ be a measurable transformation and let μ be a finite R-invariant ergodic measure in Ω, and $b \in \partial\mathbb{H}$ the global attractor for T. If*

$$disp(T) > \int_\Omega G(\omega) d\mu$$

then $\Omega \times \{b\}$ is a.e-global attractor for the cocycle (R, A).

Proof. Let $x_0 \in \mathbb{H}$ an arbitrary point. By lemma 4.23

$$\begin{aligned}
B_{x_0,\alpha}(f_\omega(x)) &\leq B_{x_0,\alpha}(x) - (disp(T) - \varepsilon_\omega) \\
B_{x_0,\alpha}(f_{R_\omega}(f_\omega(x)) &\leq B_{x_0,\alpha}(x) - (2disp(T) - \varepsilon_\omega - \varepsilon_{R_\omega}) \\
&\vdots \\
B_{x_0,\alpha}(f_{R^{n-1}_\omega} \circ \cdots \circ f_\omega(x)) &\leq B_{x_0,\alpha}(x) - (ndisp(T) - \sum_{i=1}^{n} \varepsilon_{R^{n-1}\omega}) \\
&= B_{x_0,\alpha}(x) - n(disp(T) - \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{R^{n-1}\omega})
\end{aligned}$$

In the last inequality we take the limit when $n \to \infty$.

$$\lim_{n\to\infty} B_{x_0,\alpha}(f_{R^{n-1}_\omega} \circ \cdots \circ f_\omega(x)) \leq B_{x_0,\alpha}(x) - \lim_{n\to\infty} n(disp(T) - \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{R^{n-1}\omega})$$

The Birkhoff Ergodic Theorem yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{R^{n-1}\omega} = \int_\Omega G(\omega) d\mu.$$

Then $n(disp(T) - \int_\Omega G(\omega) d\mu) \to \infty$ when $n \to \infty$. Therefore

$$\lim_{n\to\infty} B_{x_0,\alpha}(f_{R^{n-1}_\omega} \circ \cdots \circ f_\omega(x)) = -\infty,$$

because lemma 4.22 and taking $x = x_0$ we have α is a μ-a.e global atracttor. Then $\Omega \times \{\alpha\}$ is a global atracttor for (R, A). $\square$

Now we will give a more general version to the previous theorem, for that we will suppose that for each ω we will have a hyperbolic Möbius transformation T_ω such that for $\omega_1 \neq \omega_2$ we do not necessarily have that $disp(T_{\omega_1}) = disp(T_{\omega_2})$, but for all $\omega \in \Omega$ we have that $\alpha \in \partial\mathbb{H}$ is the global attractor for T_ω.

Now lets define the following function which, given $\omega \in \Omega$, measures the displacement of T_ω.

$$\begin{aligned} D : \Omega &\longrightarrow \mathbb{R}_0^+ \\ \omega &\longmapsto disp(T_\omega), \end{aligned}$$

Remark 15. *D is a continous function and measureable function under the topology of uniform convergence on bounded sets.*

Theorem 5.4. *Given $f_\omega : \Omega \to \Omega$ and $T_\omega : \Omega \to \Omega$ be measurables funtions and let μ be a finite R-invariant and ergodic in Ω. For all $\omega \in \Omega$ let $\alpha \in \partial\mathbb{H}$ the global attractor for T_ω. If*

$$\int_\Omega D(\omega) d\mu > \int_\Omega G(\omega) d\mu$$

then $\Omega \times \{\alpha\}$ is a.e-global attractor for (R, A).

Proof. Let $x_0 \in \mathbb{H}$ an arbitrary point. By lemma 4.23

$$\begin{aligned} B_{x_0,\alpha}(f_\omega(x)) &\leq B_{x_0,\alpha}(x) - (disp(T_\omega) - \varepsilon_\omega) \\ B_{x_0,\alpha}(f_{R_\omega}(f_\omega(x)) &\leq B_{x_0,\alpha}(x) - (disp(T_\omega) + disp(T_{R\omega}) - \varepsilon_\omega - \varepsilon_{R_\omega}) \\ &\vdots \end{aligned}$$

$$\begin{aligned} B_{x_0,\alpha}(f_{R_\omega^{n-1}} \circ \cdots \circ f_\omega(x)) &\leq B_{x_0,\alpha}(x) - \left(\sum_{i=1}^n disp(T_{R^{n-1}\omega}) - \sum_{i=1}^n \varepsilon_{R^{n-1}\omega} \right) \\ &= B_{x_0,\alpha}(x) - n \left(\frac{1}{n} \sum_{i=1}^n disp(T_{R^{n-1}\omega}) - \frac{1}{n} \sum_{i=1}^n \varepsilon_{R^{n-1}\omega} \right) \end{aligned}$$

In the last inequality we take the limit when $n \to \infty$.

$$\lim_{n\to\infty} B_{x_0,\alpha}(f_{R_\omega^{n-1}} \circ \cdots \circ f_\omega(x)) \leq B_{x_0,\alpha}(x) - \lim_{n\to\infty} n \left(\frac{1}{n} \sum_{i=1}^n disp(T_{R^{n-1}\omega}) - \frac{1}{n} \sum_{i=1}^n \varepsilon_{R^{n-1}\omega} \right)$$

Because Birkhoff Ergodic Theorem we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \varepsilon_{R^{n-1}\omega} = \int_\Omega G(\omega) d\mu$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n disp(T_{R^{n-1}\omega}) = \int_\Omega D(\omega) d\mu$$

Then $n\left(\frac{1}{n}\sum_{i=1}^{n} disp(T_{R^{n-1}\omega}) - \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{R^{n-1}\omega}\right) \to \infty$ when $n \to \infty$. Therefore

$$\lim_{n\to\infty} B_{x_0,\alpha}(f_{R_\omega^{n-1}} \circ \cdots \circ f_\omega(x)) = -\infty,$$

because lemma 4.22 and taking $x = x_0$ we have α is a μ-a.e global atracttor. Then $\Omega \times \{\alpha\}$ is a global atracttor for (R, A). $\square$

5.2 Properties of Distorted Möbius transformation.

Lemma 5.5. *Let f a distorted Möbius transformation and $x, y \in \mathbb{H}$ then*

$$d_{\mathbb{H}}(f(x), f(y)) - d_{\mathbb{H}}(x, y) \leq \varepsilon.$$

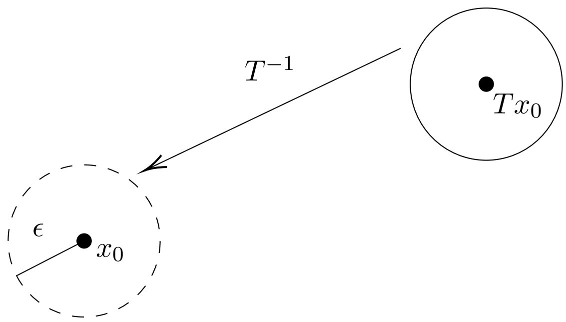
Proof. Recall that $d_{\mathbb{H}}(T(x), T(y)) = d_{\mathbb{H}}(x, y)$ then

$$\begin{aligned} d_{\mathbb{H}}(f(x), f(y)) - d_{\mathbb{H}}(x, y) &\leq d_{\mathbb{H}}(f(x), T(x)) + d_{\mathbb{H}}(T(x), f(y)) - d_{\mathbb{H}}(x, y) \\ &\leq d_{\mathbb{H}}(f(x), T(x)) + d_{\mathbb{H}}(T(x), T(y)) + d_{\mathbb{H}}(T(y), f(y)) - d_{\mathbb{H}}(x, y) \\ &= d_{\mathbb{H}}(f(x), T(x)) + d_{\mathbb{H}}(T(y), f(y)) \\ &\leq 2\epsilon. \end{aligned}$$

Taking $\varepsilon = 2\epsilon$ we are done. □

Lemma 5.6. *Let f a distorted and T the Möbius transformation related then $T^{-1} \circ f = h$ with h a bounded displacement.*

Proof. Given a point x_0 is sufficient to show that $h(x_0)$ is at a bounded distance from x_0. So by definition $f(x_0) \in B_\epsilon(T(x_0))$ and using that T is an isometry then T^{-1} preserves distances, in particular preserves balls so $T^{-1} \circ f(x_0) \in B_\epsilon(x_0)$. Because x_0 is an arbitrary point we can say that $T^{-1} \circ f$ is a bounded function and taking $T^{-1} \circ f = h$ we are done. □



References

[1] N.-H. Lee, "Geometry: from isometries to special relativity," 2020.

[2] J. YOCCCOZ, "Linéarisation des germes de difféomorphismes holomorphes de (c, 0)," *CR Acad. Sci. Paris*, vol. 306, pp. 55–58, 1988.

[3] C. L. Siegel and J. K. Moser, *Lectures on celestial mechanics.* Springer Science & Business Media, 2012.

[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, vol. 319. Springer Science & Business Media, 2013.

[5] J. Cederberg, *A course in modern geometries.* Springer Science & Business Media, 2004.

[6] A. Moraga and M. Ponce, "Relating boundary and interior solutions of the cohomological equation for cocycles by isometries of negatively curved spaces. the livsic case," *arXiv preprint arXiv:2103.10945*, 2021.