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Quantization of Edge Currents along Magnetic Interfaces: A K-Theory Approach

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Contents

Chaper 1: Introduction

This chapter contains both the main ideas of this work explained in a heuristic manner and a formal formulation of the results obtained.

1.1 Main Ideas

In this work a full frame to study the propagation of *topological currents* along *magnetic interfaces* in a 2-dimensional material subject to a orthogonal magnetic field is introduced. By a magnetic interface we mean a thin region of the space that separates the material, at least locally, into two parts subjected to different constant magnetic fields (see Figure 1.1), and the propagated topological current is a quantized charge current generated by the difference between the magnetic fields at each side of the magnetic interface. In this way, one can think on topological currents as a phenomenon that occur in the edge of certain materials and can be described by the properties in the bulk (deep in the interior) of those, making it a *bulk-edge correspondence*. In such frame, several examples are included together with an extensive study to the simplest non trivial magnetic interface. Such case appear naturally when the *Iwatsuka magnetic field* is considered, which was first introduced in [Iwa] and consists of a magnetic field whose strenght is constant in one of the directions of the material, let us say vertically, and admits horizontal limits[1]. The computation of the topological current for the Iwatsuka magnetic field has been studied in a continuous setting in [DGR] and more generally but in a discrette setting in [KS]. Such problem is stated in this work through a K-theoretic framework, which is consistent with the fact that bulk-boundary correspondences in condensed matter can be succesfully explained through the K-theory of the C^*-algebras of observables involved (see [Hat, EG, KRS]).

[1] These are just the main hipothesis considered in [Iwa], however this simplifaction will be adopted and actually simplified even more in this work.

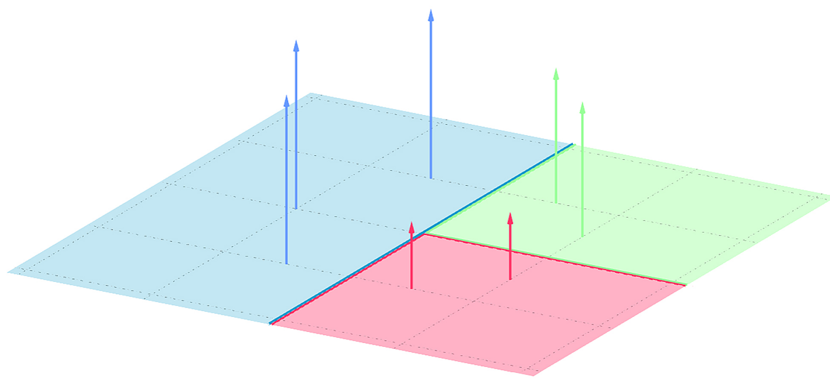

Figure 1.1: Coloured vectors represent different manetic fields applied to the same material. Coloured segments represent the magnetic interfaces induced by the different magnetic fields.

1.2 Formulation and Results

The material will be represented with the discrete lattice $\mathbb{Z}^2$ and the magnetic field acting on it will be described by its strength on every point of the material as a function $B : \mathbb{Z}^2 \to \mathbb{R}$. *Magnetic potentials* associated to the magnetic field B will be defined and will induce the *magnetic translations* as unitary operators $S_{B,1}, S_{B,2}$ on $\ell^2(\mathbb{Z})$. Our definition of the magnetic translations is of course consistent with the one given in most literature (see *e. g.* [PS, DS]) and can also be regarded as a discrete version of its continuous analog (*cf.* [Ara, Theorem 3.2]). In turn, the *Magnetic C^*-algebra* $\mathcal{A}_B$ consisting on the algebra of observables describing the kinematics of charged particles on the material will be defined as the the one generated by the magnetic translations.

Section 2 contains several structural facts about magnetic C^*-algebras, some of which are proving that such algebras: admit a crossed product structure, a differential structure and a well made integration theory; and most importanly, reassemble in a very rich (non-commutative way) the Fourier theory of $C(\mathbb{T})$, proving that the elements of magnetic C^*-algebras admit a unique representation as a Cesàro sum of noncommutative polynomials on the magnetic translations. In the same section, a very important commutative subalgebra of $\mathcal{A}_B$ is introduced, namely the *Flux algebra* $\mathcal{F}_B$. Such algebra depends uniquely on the magnetic field B and is proven to enconde all the information needed to study the entire algebra $\mathcal{A}_B$ for the purposes of this work. Such property lead us to define the *magnetic Hull* as the compact Hausdorff space Ω_B such that $\mathcal{F}_B \simeq C(\Omega_B)$, which exists in virtue of the Gelfand-Neimark Theorem.

Section 3 sits the fundamental blocks of the machinery used to study the topological currents propagated through arbitrary interfaces. The main result consists on the capability to "separate" the magnetic C^*-algebra in two parts: one consisting of an algebra that represents the magnetic interfaces of the magnetic field, which will be called the *interface algebra* and denoted by $\mathcal{I}_B$; and other one which describes the material as its components associated to the different constant magnetic fields of the material **with no interactions whatsoever** (see Figure 1.2). Such algebra will be called the *Bulk Algebra* and denoted $\mathcal{A}_{\text{Bulk}}$, and the interactionless property will be builded upon taking orthogonal direct sums of C^*-algebras describing the constant magnetic fields of each component[2], *i.e.*,

$$\mathcal{A}_{\text{bulk}} = \bigoplus_{j=1}^{N} \mathcal{A}_{b_j} \ ,$$

where each $\mathcal{A}_{b_j}$ stands as the magnetic C^*-algebra associated to the constant magnetic field $B \equiv b_j$ for $j = 1, \ldots, N$.

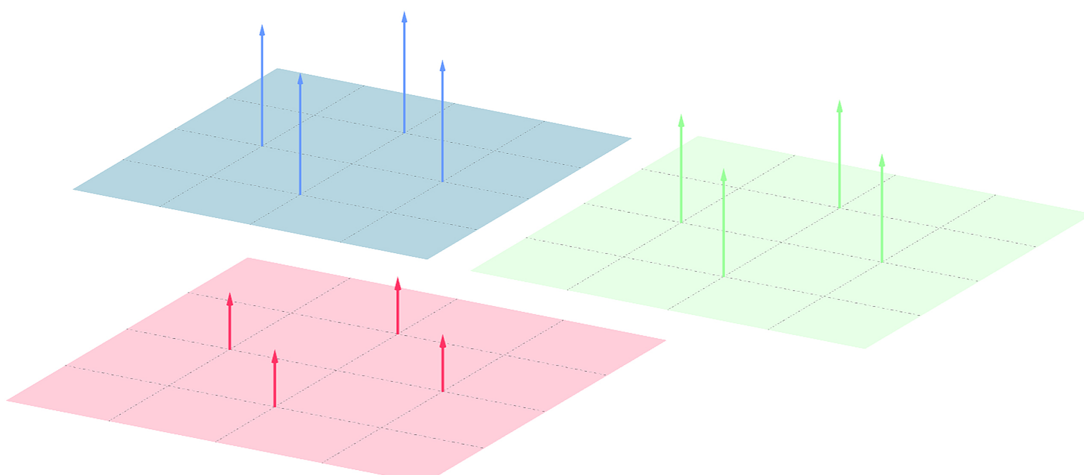

Figure 1.2: Separation of the material of Figure 1.1 into three interactionless materials.

The "separation" argument is made through a short-exact sequence of C^*-algebras in the form

$$0 \longrightarrow \mathcal{I}_B \longrightarrow \mathcal{A}_B \longrightarrow \mathcal{A}_{\text{Bulk}} \longrightarrow 0 \ . \tag{1.1}$$

This type of argument is usual when a bulk-edge correspondance is trying to be achieved (*cf.* [PS, Corollary 5.5.2], [DS, Theorem 2] and throughout the entire work [Thi]), since it implies the existence of a six-term exact sequence (see [Weg, Theorem 9.3.2]) of the form

$$\begin{array}{ccccc} K_0(\mathcal{I}_B) & \longrightarrow & K_0(\mathcal{A}_B) & \longrightarrow & K_0(\mathcal{A}_{\text{bulk}}) \\ \text{ind}\big\uparrow & & & & \big\downarrow\text{exp} \\ K_1(\mathcal{A}_{\text{bulk}}) & \longleftarrow & K_1(\mathcal{A}_B) & \longleftarrow & K_1(\mathcal{I}_B) \ , \end{array} \tag{1.2}$$

[2]Fortunately, not every component is relevant. Actually, just the asymptotic behaviour of the magnetic field B is important. This will be discussed in Section 3.1.

inducing *a fortiori* a connection between the magnetic interface and the bulk algebra via the index and exponential maps.
Section 3 also contains a full description of what is meant by studying topological currents. In a nutshell, consider a Hamiltonian $\hat{H} \in \mathcal{A}_B$ which contains the full information of the magnetic field B and $H = \{H_j\}_{j=1,\ldots,N} \in \mathcal{A}_{\text{Bulk}}$ its image through the exact-sequnce (1.1). Let Δ be an energy domain (open set) that sits inside a non-trivial gap of $\cap_{j=1}^{N}\sigma(H_j)$, $J_{\mathcal{I}}(\Delta)$ be the *interface current* carried by the eigenstates of $\widehat{H}$ in Δ and $\sigma_{\mathcal{I}}(\Delta) := eJ_{\mathcal{I}}(\Delta)$ the associated *interface conductance*, where $e > 0$ is the magnitude of the electron charge. Now consider $\mu \in \Delta$ be a given *Fermi energy* and let $\sigma_{\text{bulk}}(\mu) = \{\sigma_j(\mu)\}_{j=1,\ldots,N}$ denotes the set of *Hall conductances* at energy μ for the "bulk system" described by the Hamiltonian H. The constant $\sigma_{\mathcal{I}}(\Delta)$ and set $\sigma_{\text{Bulk}}(\mu)$ have a meaningful physical interpretation as their name suggests. In this work, they are introduced as the result between "topological maps" on certain elements in the algebras $\mathcal{I}^+$ and $\mathcal{A}_{\text{Bulk}}$ respectively. The "topological property" of such maps, is that they can lift into the groups $K_1(\mathcal{I})$ and $K_0(\mathcal{A}_{\text{Bulk}})$, which is where the sequence (1.2), and particularly the exponential map, result of fundamental utility. In certain cases, it is possible to find a reasonable correspondence between $\sigma_{\mathcal{I}}(\Delta)$ and $\sigma_{\text{bulk}}(\mu)$. This will be the case when considering a Iwatsuka magnetic field, and consequently a *bulk-interface duality* will be said to hold. This name is of course based on the fact that this can be thought as a manifestation of the celebrated bulk-boundary correspondence [Hat, EG, KRS].

Finally, section 4 is dedicated to the particular case of considering the Iwatsuka magnetic field. In such case one can do many explicit computations regarding the theory introduced in the past sections. Among other things one can: determine explicitely the Flux algebra and its magnetic Hull; describe the maps involved in the short-exact sequence (1.1) and understand the evaluation map restricted to $\mathcal{F}_B$ actually as an evaluation at $\pm\infty$; compute every map and group involved in (1.2) explicitely (which results extremely satisfactory); and finally as a consequence of the latter, proving the bulk-interface correspondence for the Iwatsuka C^*-algebra through K-theory. Since the latter is in a way the main result of this work, let us explain a little more about it.
When the Iwatsuka magnetic field is considered, the magnetic interface is (as one could imagine) a line that separates the material in two parts, that is $\mathcal{A}_{\text{Bulk}} = \mathcal{A}_{b_-} \oplus \mathcal{A}_{b_+}$. Moreover, once the Hall conductances $\sigma_\pm(\mu)$ and the Interface conductance $\sigma_{\mathcal{I}}(\Delta)$ are computed, it is rutinary to check that

$$\sigma_{\mathcal{I}}(\Delta) = \sigma_{b_-}(\mu) - \sigma_{b_+}(\mu), \tag{1.3}$$

which is an example of a bulk-interface duality.
Finally, let us remark that equation (1.3) is roughly the same result obtained in both [DGR] and [KS], however such result is proved here using K-theory. When condensed matter is consider, the use of K-Theory to state bulk-edge correspondences is pretty consistent through the current literature, so the proof of (1.3) using such tools is also a extra, but small, contribution.

Chapter 2: Magnetic C^*-Algebras and their properties

In this chapter the C^*-algebra of operators on $\ell^2(\mathbb{Z}^2)$ that describe the kinematics of charged particles subjected to a given orthogonal magnetic field in the tight-binding approximation will be briefly described. This algebra will be called the *magnetic C^*-algebra*, it was first introduced in [DS] and can be regarded as a generalization of the well known *noncommutative torus* preserving most of its structure. For a brief introduction to the noncommutative torus including the computation of its K-theory see appendix D.

2.1 Magnetic fields and potentials

In the tight-binding approximation the two-dimensional position space is $\mathbb{Z}^2$ and an orthogonal *magnetic field* is any function $B : \mathbb{Z}^2 \to \mathbb{R}$. In order to introduce the notion of potentials in the discrete setting let us first fix some notation. When working with magnetic potentials, and magnetic translations later, the notation $n := (n_1, n_2), m := (m_1, m_2) \in \mathbb{Z}^2$ for arbitrary points of $\mathbb{Z}^2$ results particularly useful. It will also be written $e_1 := (1, 0)$, $e_2 := (0, 1)$ for the canonical linear basis of $\mathbb{R}^2$.

Definition 2.1.1 (Tight-binding vector potential)**.** Let $B : \mathbb{Z}^2 \to \mathbb{R}$ be a magnetic field. A *magnetic potential* for B is a function

$$A_B : \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}$$

satisfying:

1. $A_B(n, m) = 0$ for $n, m \in \mathbb{Z}^2$ such that $|n - m| \neq 1$;

2. $A_B(m,n) = -A_B(n,m)$ for all $n,m \in \mathbb{Z}^2$; and

3. $B(n) = \mathfrak{C}[A_B](n)$ for all $n \in \mathbb{Z}^2$ where

$$\begin{aligned}\mathfrak{C}[A_B](n) :=& A_B(n, n-e_1) + A_B(n-e_1, n-e_1-e_2)\\ &+ A_B(n-e_1-e_2, n-e_2) + A_B(n-e_2, n)\end{aligned}$$

is the (counterclockwise) *circulation* of A_B along the boundary of the unit cell of $\mathbb{Z}^2$ with right upper corner equals to n.

From [DS, Proposition 1] we know that every magnetic field B admit an infinite number of vector potentials, and every two potentials A_B and A'_B associated with the same magnetic field B are related by a *gauge function* $G : \mathbb{Z}^2 \to \mathbb{R}$ according to

$$A'_B(n,m) = A_B(n,m) + G(n) - G(m), \quad |n-m| = 1. \tag{2.1}$$

The importance of the gauge function is that a pair of different magnetic potentials for the same magnetic field will generate isomorphic algebras with isomorphism induced by the gauge function. Finally, and as in most of the literature, the word gauge will also be used to refer to different potentials.

The following examples represent two cases already studied together with the main example of this work and will be used several times from now on.

Example 2.1.2 (Potentials for the constant magnetic field)**.** A constant magnetic field of strength b is described by the constant function $B_b(n) = b$ for all $n \in \mathbb{Z}^2$. Among the infinite number of vector potentials associated to the constant magnetic field B_b, there are two of special utility in practical applications. The first one is the so-called *Landau potential* defined by

$$A(n, n-e_j) := \delta_{j,1} n_2 b, \quad \text{for all } n \in \mathbb{Z}^2. \tag{2.2}$$

The second is the *symmetric potential* defined by

$$A_{sym}(n, n-e_j) := (\delta_{j,1} n_2 - \delta_{j,2} n_1)\, \frac{b}{2}, \quad \text{for all } n \in \mathbb{Z}^2. \tag{2.3}$$

A simple computation shows that $\mathfrak{C}[A] = \mathfrak{C}[A_{sym}] = B_b$. Moreover, one can check that the Landau and symmetric potentials are related by the gauge function $G_b(n) := -n_1 n_2 \frac{b}{2}$ as in equation (2.1), that is,

$$A_{sym}(n, n-e_j) = A(n, n-e_j) + G_b(n) - G_b(n-e_j).$$

◁

Example 2.1.3 (Potentials for the Iwatsuka magnetic field)**.** The *Iwatsuka magnetic field* [Iwa] models systems with a magnetic field only depending on its first variable, that is

$B(n_1,n_2) = B(n_1)$, together with some asymptotical behaviour. For the sake of simplicity let us define the Iwatsuka magnetic field as

$$B_{\mathrm{I}}(n) := b_-\delta_-(n) + b_0\delta_0(n) + b_+\delta_+(n), \quad \text{for all } n \in \mathbb{Z}^2 \tag{2.4}$$

where $b_-, b_0, b_+ \in \mathbb{R}$ are real constants and the three functions $\delta_\pm, \delta_0$ are defined by

$$\delta_\pm(n_1,n_2) := \begin{cases} 1 & \text{if} \pm n_1 > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \delta_0(n_1,n_2) := \begin{cases} 1 & \text{if } n_1 = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.5}$$

As in the constant magnetic field case, there are Landau and symmetric like potentials. More precisely let us define the *Landau-Iwatsuka potential* as

$$A_{\mathrm{I}}(n, n-e_j) := \delta_{j,1} n_2 B_{\mathrm{I}}(n), \quad \text{for all } n \in \mathbb{Z}^2 \tag{2.6}$$

and the *symmetric Iwatsuka potential* as

$$A_{\mathrm{I},sym}(n, n-e_j) = A^{(0)}_{\mathrm{I},sym}(n, n-e_j) + \Delta_{\mathrm{I},sym}(n, n-e_j) \tag{2.7}$$

where

$$A^{(0)}_{\mathrm{I},sym}(n, n-e_j) := \left(\frac{\delta_{j,1}}{2} n_2 - \frac{\delta_{j,2}}{2} n_1\right) B_{\mathrm{I}}(n), \quad \text{for all } n \in \mathbb{Z}^2,$$

and

$$\Delta_{\mathrm{I},sym}(n, n-e_j) := \delta_{j,1} \frac{b_0 - b_-}{2} n_2 \delta_0^{(1)}(n), \quad \text{for all } n \in \mathbb{Z}^2.$$

In both cases easy but annoying computations lead us to $\mathfrak{C}[A_{\mathrm{I}}] = \mathfrak{C}[A_{\mathrm{I},sym}] = B_{\mathrm{I}}$. Let us explain a little about the symmetric case. The exact same computations as in the constant magnetic field case provide

$$\mathfrak{C}[A^{(0)}_{\mathrm{I},sym}](n) = B_{\mathrm{I}}(n) \quad \text{for } n_1 \neq 0,$$

however

$$\mathfrak{C}[A^{(0)}_{\mathrm{I},sym}](0, n_2) = (b_0 + b_-)\frac{1}{2},$$

so the term $\Delta_{\mathrm{I},sym}$ is included to correct the mismatch.
This time, one can check that the relation between the symmetric Iwatsuka potential and the Landau Iwatsuka potential is given by the gauge function $G_{\mathrm{I}}(n) := -\frac{n_1 n_2}{2} B_{\mathrm{I}}(n)$ as in equation (2.1). $\triangleleft$

Example 2.1.4 (Potentials for the localized magnetic field)**.** Let $\mathcal{P}_0(\mathbb{Z}^2)$ be the collection of finite subsets of $\mathbb{Z}^2$. For every $\Lambda \in \mathcal{P}_0(\mathbb{Z}^2)$ let δ_Λ be the characteristic function defined by

$$\delta_\Lambda(n) := \begin{cases} 1 & \text{if } n \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$

A *localized* magnetic field of strength $b \in \mathbb{R}$ is defined by the function

$$B_\Lambda(n) := b\delta_\Lambda(n), \quad \text{for all } n \in \mathbb{Z}^2. \tag{2.8}$$

Observe that

$$B_\Lambda = \sum_{\lambda \in \Lambda} B_{\{\lambda\}}$$

where $B_{\{\lambda\}} := b\delta_{\{\lambda\}}$ is the magnetic field localized on the singleton $\{\lambda\} \in \mathcal{P}_0(\mathbb{Z}^2)$. This is convenient since the case localized on one point has already been studied in [DS] and a magnetic potential for such case has been introduced as the *half-line potential* defined by

$$A_{\{\lambda\}}(n, n - e_j) := b\delta_{j,1} \sum_{t=0}^{+\infty} \delta_{\{\lambda + te_2\}}(n).$$

By linearity of the circulation one gets that

$$A_\Lambda := \sum_{\lambda \in \Lambda} A_{\{\lambda\}}$$

provides a vector potential for the localized magnetic field B_Λ. Observe that A_Λ is well defined in view of the finiteness of the sum in λ. ◁

2.2 The magnetic translations

In this section the *magnetic translations*, which are the fundamental blocks of the *magnetic C^*-algebras*, are defined. Let S_j and N_j, $j = 1, 2$, be the canonical *shift operators* and *position operators* respectively defined on the Hilbert space $\ell^2(\mathbb{Z}^2)$ by

$$(S_j\psi)(n) := \psi(n - e_j), \quad j = 1, 2, \quad \psi \in \ell^2(\mathbb{Z}^2),$$

and for suitable $\psi \in \ell^2(\mathbb{Z}^2)$[1]

$$(N_j\psi)(n) := n_j\psi(n), \quad j = 1, 2.$$

Also, we will consider the vector of position operators $X = (X_1, X_2)$.

Definition 2.2.1. Let B a magnetic field with associated vector potential A_B. The *magnetic phases* in the gauge A_B are the unitary operators $Y_{A_B,j} = \mathrm{e}^{iA_B(N,N-e_j)}$, $j = 1, 2$, that is,

$$(Y_{A_B,j}\psi)(n) := \mathrm{e}^{iA_B(n,n-e_j)}\,\psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

[1]This obeys to the fact that the position operators on $\ell^2(\mathbb{Z}^2)$ are not bounded, however since in this work we introduce the position operators just to compute bounded functions of those, there is no need to have any discussion on the domain.

The *magnetic translations* in the gauge A_B are the operators defined by $S_{A_B,j} := Y_{A_B,j} S_j$, or more explicitly as

$$(S_{A_B,j}\psi)(n) := \mathrm{e}^{iA_B(n,n-e_j)}\,\psi(n-e_j), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

Finally, the *magnetic flux* operator is defined by $f_B = \mathrm{e}^{iB(N)}$, that is,

$$(f_B\psi)(n) := \mathrm{e}^{iB(n)}\,\psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

Remarks 2.2.2. 1. The magnetic phases and flux are easily seen to be unitary operators. It follows that the magnetic translations are also unitary operators, since they are a composition of unitary operators. Moreover, one can check that their adjoints (and inverses) are explicitely given by

$$\big(S^*_{A_B,j}\psi\big)(n) := \mathrm{e}^{iA_B(n,n+e_j)}\,\psi(n+e_j), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

2. The magnetic flux does not depend on the choice of the magnetic potential. This simple but fundamental observation will be of high importance several times later on this work.

3. The connection between in magnetic translations and the magnetic flux is given by the equation

$$S_{A_B,1}S_{A_B,2}S^*_{A_B,1}S^*_{A_B,2} = f_B. \tag{2.9}$$

no matter what magnetic potentials were chosen. This will introduce some sort of universality in the magnetic C^*-algebra (see Section 2.5).

4. If two different magnetic potentials, let say A_B and A'_B, induce the same magnetic field B, it has been mentioned that they must be connected by a gauge function G as in equation (2.1). It follows by simple computations that

$$S_{A'_B,j} = \mathrm{e}^{-iG(N)}\,S_{A_B,j}\,\mathrm{e}^{iG(N)}, \quad j = 1, 2,$$

which implies that the gauge function induces a unitary equivalence between the magnetic translations associated to different magnetic potentials but equal magnetic field. This observation allow us to refer to the magnetic translations $S_{A_B,j}$ simply as $S_{B,j}$ unless explicit computations were needed. ◁

Example 2.2.3 (Magnetic translations for a constant magnetic field)**.** One can of course introduce the magnetic translations in the constant magnetic field case with both the Landau and symmetric magnetic potentials defined in 2.1.2. In both cases the magnetic translations are not exceptionally hard to manipulate as operators, however it is remarkable that in the Landau gauge $S_{A_b,2} = S_2$.
No matter the case, equation (2.9) tells us that

$$S_{A_b,1}S_{A_b,2} = \mathrm{e}^{ib}\,S_{A_b,2}S_{A_b,1} \tag{2.10}$$

for every magnetic potential A_B associated to the magnetic field $B = B_b$, since the flux operator coincides with the multiplication by the constant phase e^{ib}. The importance of equation (2.10) is that it proves that the magnetic translations satisfy the canonical commutation relation of the generators of the noncommutative torus with angle $\theta = b/2\pi$ (see appendix D, [GBVF, Chapter 12] or [Weg, Chapter 12.3]). ◁

Example 2.2.4 (Iwatsuka magnetic translations)**.** As in the constant magnetic field case one can introduce the magnetic translations associated to both the Landau and symmetric Iwatsuka potentials, however this time the simplicity of the computations changes radically between gauges and because of that from now just the Landau Iwatsuka potential will be considered, and for the sake of notation let us fix the Iwatsuka magnetic translations as

$$S_{\mathrm{I},j} = S_{A_{\mathrm{I}},j}, \quad \text{for } j = 1, 2.$$

Observe that exactly as in the Landau gauge for the constant magnetic field case, we have that $S_{\mathrm{I},2} = S_2$, which is a relief because it eases the manipulation of the iwatsuka magnetic translations. ◁

Example 2.2.5 (Magnetic translations for a localized field)**.** The magnetic translations associated to a localized magnetic field B_Λ of the type (2.8) can be defined exactly as in the previous examples by using the vector potential A_Λ. Also in this case one obtains that $S_{A_\Lambda,2} = S_2$. ◁

2.3 Construction of the magnetic C^*-algebra

Throughout this section consider $S_{A_B,1}$ and $S_{A_B,2}$ be the magnetic translations associated to the magnetic field B through the magnetic potential A_B.

Definition 2.3.1 (The tight-binding magnetic C^*-algebra)**.** The *magnetic C^*-algebra* $\mathcal{A}_{A_B}$ of the magnetic field B in the gauge A_B is the unital C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}^2))$ generated by $S_{A_B,1}$ and $S_{A_B,2}$, *i. e.*,

$$\mathcal{A}_{A_B} := C^*\left(S_{A_B,1}, S_{A_B,2}\right).$$

Remarks 2.3.2.

1. In more detail the C^*-algebra $\mathcal{A}_{A_B}$ is constructed by closing the collection of the Laurent polynomials in $S_{A_B,1}$ and $S_{A_B,2}$ with respect to the operator norm and then one would be tempted to describe the elements of $\mathcal{A}_{A_B}$ as a series on the magnetic translations and their adjoints, however this is in general not true. The missing elements can be thought as limits of sequences of elements that are already infinite series. The problem of representing the elements as series will be broadly discuss in 2.7.

2. As remarked in 2.2.2, the gauge function induces an unitary equivalence between magnetic translations. This can of course be lifted into the entire magnetic C^*-algebra, proving that if A_B and A'_B are two magnetic potentials for the same magnetic field

$$\mathcal{A}_{A_B} \simeq \mathcal{A}_{A'_B},$$

with isomorphism given by $T \mapsto \mathrm{e}^{-\mathrm{i}\,G(N)}\,T\,\mathrm{e}^{\mathrm{i}\,G(N)}$. This observation allow us to refer to the magnetic C^*-algebra $\mathcal{A}_{A_B}$ simply as $\mathcal{A}_B$ unless we need to make explicit computations regarding the magnetic translations. $\triangleleft$

Example 2.3.3 (Noncommutative torus)**.** In the case of a constant magnetic field of strength b the associated magnetic C^*-algebra will be denoted simply with

$$\mathcal{A}_b := C^*\left(S_{b,1}, S_{b,2}\right)$$

where the the magnetic translations $S_{b,1}$ and $S_{b,2}$ are described in Example 2.2.3. As mentioned in 2.2.3, the magnetic translations $S_{b,1}$ and $S_{b,2}$ satisfy the commutation relation of the generators of the noncommutative torus A_θ with $\theta = b/2\pi$. It follows from the universality of the noncommutative torus (see [Weg, Theorem 12.3.2]) that

$$\mathcal{A}_b \simeq \mathrm{A}_\theta.$$

$\triangleleft$

Example 2.3.4 (Iwatsuka C^*-Algebra)**.** The *Iwatsuka C^*-Algebra* is defined as the magnetic C^*-algebra associated to the Iwatsuka magnetic field B_{I} in the Landau Iwatsuka Gauge, that is,

$$\mathcal{A}_{\mathrm{I}} := C^*\left(S_{\mathrm{I},1}, S_{\mathrm{I},2}\right).$$

$\triangleleft$

Example 2.3.5. The magnetic C^*-algebra associated to a localized magnetic field will be denoted with

$$\mathcal{A}_\Lambda := C^*\left(S_{\Lambda,1}, S_{\Lambda,2}\right)$$

where the magnetic translations $S_{\Lambda,1}$ and $S_{\Lambda,2}$ are defined through the magnetic potential A_Λ described in Example 2.1.4. The special case $\Lambda = \{\lambda\}$ has been exhaustive studied in [DS]. $\triangleleft$

2.4 Relevant $*$-subalgebras

Any magnetic C^*-algebra $\mathcal{A}_B$ contains several relevant $*$-subalgebras. Let us start by defining a $\mathbb{Z}^2$-action over $\ell^\infty(\mathbb{Z}^2)$ by

$$(\tau_m g)(n) = g(n-m), \quad \text{for } n, m \in \mathbb{Z}^2.$$

Observe that, since the elements of $\ell^\infty(\mathbb{Z}^2)$ can be regarded as elements of $\mathcal{B}(\ell^2(\mathbb{Z}^2))$ with the identification $g \mapsto g(N)$, the action τ induces an action on the algebra $\mathcal{M} := \{g(N) \mid g \in \ell^\infty(\mathbb{Z}^2)\}$. Since this identification is so transparent, no difference will be made between neither the elements g and $g(N)$ nor the action τ and its induced action on $\mathcal{M}$.

The importance of the action τ for this work is contained in the simple but fundamental relation

$$\tau_m(g) = (S_{B,1})^{m_1}(S_{B,2})^{m_2} g (S_{B,2})^{-m_2}(S_{B,1})^{-m_1}, \quad \text{for } m = (m_1, m_2) \text{ and } g \in \mathcal{M}, \tag{2.11}$$

where we have used the notation $U^{-n} := (U^{-1})^n$ whenever $n \in \mathbb{N}$ and U is unitary, and the convention $U^0 = 1$ is just the identity operator and it is important to remark that equation (2.11) does not depend on the choice of the magnetic potential for the magnetic field B, which sustains the notation $S_{B,j}$.
Before defining the first and possibly the most relevant C^*-subalgebra of the magnetic C^*-algebra for this work let us observe that because of equation (2.11) $\tau_m f_B \in \mathcal{A}_B$ for any $m \in \mathbb{Z}^2$, and consequently linear combinations and even limits of elements of the form $\tau_m(f_B) \in \mathcal{A}_B$ are also in $\mathcal{A}_B$.

Definition 2.4.1. The *flux C^*-subalgebra* of the magnetic C^*-algebra $\mathcal{A}_B$ is defined as

$$\mathcal{F}_B := C^*(\tau_m(f_B), m \in \mathbb{Z}^2) \subset \mathcal{A}_B.$$

Remarks 2.4.2. Regarding the C^*-algebra $\mathcal{F}_B$ the next remarks result of particular importance:

1. The flux C^*-subalgebra is generated by elements that commute with each other, hence it is a commutative C^*-algebra. It follows from the Gelfand-Neimark theorem [BR, Theorem 2.1.11A] that $\mathcal{F}_B \simeq C(\Omega_B)$, where Ω_B is a compact Hausdorff space. The space Ω_B will be of high importance later (see Section 2.6).

2. Since the generators of $\mathcal{F}_B$ are elements of $\mathcal{A}_B$ no matter what magnetic potential has been chosen (because of equations (2.9) and (2.11)), it follows that the flux C^*-subalgebra is also a C^*-subalgebra of $\mathcal{A}_B$ for every choice of the magneti potentials. This fact is not trivial and it will be proved later that $\mathcal{F}_B$ encodes most of the information of the magnetic C^*-algebra needed for this work. $\triangleleft$

Example 2.4.3 (Flux algebra for the constant magnetic field)**.** If the magnetic field is constant, let us say $B(n) = b$ for all $n \in \mathbb{Z}^2$, the magnetic flux is simply the constant function $f_b := \mathrm{e}^{\mathrm{i}\, b}$. It follows that $\tau_m(f_b) = f_b$ for every $m \in \mathbb{Z}^2$ and then the flux algebra associated is just $\mathcal{F}_b := C^*(\mathrm{e}^{\mathrm{i}\, b}) \simeq \mathbb{C}$. $\triangleleft$

Example 2.4.4 (Flux algebra for the Iwatsuka C^*-algebra)**.** The magnetic flux of the Iwatsuka C^*-algebra is the function

$$f_{\mathrm{I}} := \mathrm{e}^{\mathrm{i}\, B_{\mathrm{I}}} = \mathrm{e}^{\mathrm{i}\, b_-}\, \delta_- + \mathrm{e}^{\mathrm{i}\, b_0}\, \delta_0 + \mathrm{e}^{\mathrm{i}\, b_+}\, \delta_+ ,$$

where δ_-, δ_0 and δ_+ are defined in Example 2.1.3. We claim that

$$\mathcal{F}_{\mathrm{I}} := \mathcal{F}_{B_{\mathrm{I}}} \simeq c(\mathbb{Z}) = \{g \in \ell^\infty(\mathbb{Z}) \mid \text{the limits } \lim_{n \to \pm\infty} g_n \text{ exists}\}.$$

First note that the generators of $\mathcal{F}_{\rm I}$ just depend on its first variable, so we can think of them as functions on $\mathbb{Z}$ via the identification $g \mapsto g(\cdot, 0)$. Also note that $\tau_m(f_{\rm I})$ has left and right limits (when $n \to -\infty$ and $n \to \infty$ respectively), then since $c(\mathbb{Z})$ is a C^*-algebra we have the inclusion $\mathcal{F}_{\rm I} \subset c(\mathbb{Z})$.
For the other inclusion let us first make some spoilers related to some properties of the Flux algebra $\mathcal{F}_{\rm I}$. In lemma 4.1.1 it is proved that δ_-, δ_0 and δ_+ are elements of $\mathcal{F}_{\rm I}$, consequently, because of the invariance of $\mathcal{F}_{\rm I}$ through the action τ, $\delta_j := \tau_{(j,0)}\delta_0 \in \mathcal{F}_{\rm I}$ for all $j \in \mathbb{Z}$. Now let any $g \in c(\mathbb{Z})$ with left and right limits $a, b \in \mathbb{C}$. It follows that $g - (a\delta_- + b\delta_+) \in c_0(\mathbb{Z}) = \{g \in \ell^\infty(\mathbb{Z}) \mid \lim_{n \to \pm\infty} g_n = 0\}$, and since $c_0(\mathbb{Z})$ is generated by the projections δ_j (indentifying them as functions of one variable) the inclusion $c(\mathbb{Z}) \subset \mathcal{F}_{\rm I}$ is proved. $\triangleleft$

Example 2.4.5 (Flux algebra for a localized magnetic field)**.** In the case of a localized magnetic field (2.8) one has

$$f_\Lambda \;:=\; \mathrm{e}^{\mathrm{i}\,B_\Lambda} \;=\; (\,\mathrm{e}^{\mathrm{i}\,b} - 1)\,\delta_\Lambda \;+\; 1\;,$$

so in order to make this flux not trivial it is necessary to consider $b \notin 2\pi\mathbb{Z}$.
For the sake of simplicity let us start with the case $\Lambda = \{n\}$ is a singleton in $\mathbb{Z}^2$. In such case we claim that $\mathcal{F}_\Lambda := \mathcal{F}_{B_\Lambda} = c(\mathbb{Z}^2) := \{g \in \ell^\infty(\mathbb{Z}^2) \mid \text{the limit } \lim_{||n|| \to \infty} g_n \text{ exists}\}$. On one hand, and as in the past example, the generators of $\mathcal{F}_\Lambda$ are in $c(\mathbb{Z}^2)$ and $c(\mathbb{Z}^2)$ is a C^*-algebra, proving the inclusion $\mathcal{F}_\Lambda \subset c(\mathbb{Z}^2)$. For the other inclusion let us point out that as in every flux algebra, $1 \in \mathcal{F}_\Lambda$ and

$$(\,\mathrm{e}^{\mathrm{i}\,b} - 1)^{-1}(f_\Lambda - 1) = \delta_\Lambda \in \mathcal{F}_\Lambda.$$

Now, since $\mathcal{F}_\Lambda$ is τ-invariant, $\tau_{m-n}\delta_\Lambda = \delta_m \in \mathcal{F}_\Lambda$ for every $m \in \mathbb{Z}^2$. Let $g \in c(\mathbb{Z}^2)$ with limit $a \in \mathbb{C}$. It follows that $g - a \cdot 1 \in c_0(\mathbb{Z}) = \{g \in \ell^\infty(\mathbb{Z}^2) \mid \lim_{||n|| \to \infty} g_n = 0\}$, and since $c_0(\mathbb{Z}^2)$ is generated by the elements δ_m the inclusion $c_0(\mathbb{Z}^2) \subset \mathcal{F}_\Lambda$ is proved.
If Λ is not a singleton one can also show that it is always possible to build a projector supported in a single point. Let $\lambda_0, \lambda \in \Lambda$ be two distinct points and $\gamma_0 := \lambda_0 - \lambda$. Then $\lambda_0 \in \gamma_0 + \Lambda$ and $\delta_\Lambda(1 - \delta_{\gamma_0+\Lambda})$ is a projection which projects on a subset $\Lambda' \subset \Lambda$ where the strict inclusion is justified by the fact that $\lambda_0 \notin \Lambda'$. By iterating the procedure a sufficient number of times one ends with a projection on a singleton, and using the same proof as in the case $|\Lambda| = 1$ one can also conclude that $\mathcal{F}_\Lambda = c(\mathbb{Z}^2)$. $\triangleleft$

Before defining the next subalgebra let us consider a *finite monomial* on $\mathcal{A}_B$, that is, an element of the form

$$U_{x,y} := (S_{A_B,1})^{x_1}(S_{A_B,2})^{y_1} \ldots (S_{A_B,1})^{x_d}(S_{A_B,2})^{y_d} \tag{2.12}$$

with $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$ and $d \in \mathbb{N}$. In view of equation (2.11) the monomial $U_{x,y}$ can always be rearranged in the form

$$U_{x,y} = g_{x,y}(S_{A_B,1})^{|x|}(S_{A_B,2})^{|y|} \tag{2.13}$$

where $|x| := x_1 + \ldots + x_d$, $|y| := y_1 + \ldots + y_d$ and $g_{x,y}$ is a suitable element of $\mathcal{F}_B$. From its very definition it follows that $\mathcal{A}_B$ is linearly generated by the family of monomials (2.12), or equivalently (2.13). This observation allows to define the next dense $*$-subalgebra of $\mathcal{A}_B$.

Definition 2.4.6. The *$*$-subalgebra of noncommutative polynomials* $\mathcal{A}_B^0$ is defined

$$\mathcal{A}_B^0 = \left\{ \sum_{(x,y)\in\Lambda} \alpha_{x,y} U_{x,y} \;\middle|\; \Lambda \subset \mathbb{Z}^{2d} \text{ is a finite set and } \alpha_{x,y} \in \mathbb{C} \right\}, \tag{2.14}$$

that is, the collection of finite linear combinations of the finite monomials of $\mathcal{A}_B$.

Remark 2.4.7. Let us point out that

$$U_{(1,-1),(1,-1)} = S_{A_B,1} S_{A_B,2} S^*_{A_B,1} S^*_{A_B,2} = f_B \in \mathcal{A}_B^0.$$

Moreover,

$$\begin{aligned} U_{(m_1,1,-1,-m_1),(m_2,1,-1,-m_2)} &= (S_{A_B,1})^{m_1} (S_{A_B,2})^{m_2} f_B (S_{A_B,1})^{-m_1} (S_{A_B,2})^{-m_2} \\ &= \tau_m(f_B) \in \mathcal{A}_B^0, \end{aligned}$$

proving that the generators of $\mathcal{F}_B$ are noncommutative polynomials. ◁

Finally, let us introduce the *operator-valued Schwartz space* $\mathcal{S}(\mathbb{Z}^2, \mathcal{F}_B)$ made of the *rapidly descending* sequences $\mathbf{g} = \{g_{(r,s)}\}_{(r,s)\in\mathbb{Z}^2} \subset \mathcal{F}_B$ such that

$$r_k(\mathbf{g})^2 := \sup_{(r,s)\in\mathbb{Z}^2} \left(1 + r^2 + s^2\right)^k \|g_{(r,s)}\|^2 < \infty, \tag{2.15}$$

for all $k \in \mathbb{N}_0$. It turns out that the system of seminorms (2.15) endows $\mathcal{S}(\mathbb{Z}^2, \mathcal{F}_B)$ with the structure of a Fréchet space (see Appendix A). Now we are in position to define the last $*$-subalgebra of this section.

Definition 2.4.8. The *smooth $*$-subalgebra* $\mathcal{A}_B^\infty$ is defined as

$$\mathcal{A}_B^\infty := \left\{ T_{\mathbf{g}} := \sum_{(r,s)\in\mathbb{Z}^2} g_{(r,s)} (S_{B,1})^r (S_{B,2})^s \;\middle|\; \mathbf{g} = \{g_{(r,s)}\}_{n\in\mathbb{Z}^2} \in \mathcal{S}(\mathbb{Z}^2, \mathcal{F}_B) \right\}. \tag{2.16}$$

Remark 2.4.9. If we consider the set of seminorms on $\mathcal{A}_B^\infty$ defined as

$$|||T_{\mathbf{g}}|||_k = r_k(\mathbf{g}),$$

it is easy to see that $\mathcal{A}_B^\infty$ inherit the Fréchet space structure. This follows from the mere fact that the map $\mathbf{g} \mapsto T_{\mathbf{g}}$ is a bijection. ◁

Before finishing this discussion let us summarize some information about the $*$-subalgebras introduced in this section.

Proposition 2.4.10. *It holds true that*

$$\mathcal{A}_B^0, \mathcal{F}_B \subset \mathcal{A}_B^\infty \subset \mathcal{A}_B,$$

both $\mathcal{A}_B^0$ and $\mathcal{A}_B^\infty$ are dense in $\mathcal{A}_B$, and $\mathcal{A}_B^0 \cap \mathcal{F}_B$ is dense in $\mathcal{F}_B$.

2.5 Crossed Product Structure

In this section it is proved that the magnetic C^*-algebra admits an *Iterated crossed product*. For a brief description of the crossed product of an arbitrary C^*-algebra by $\mathbb{Z}$ see Appendix B which also contains an important application.

First consider the C^*-algebra

$$\mathcal{A}_{B,j} := \mathcal{F}_B \rtimes_{\alpha_j} \mathbb{Z}, \quad \text{for } j = 1, 2,$$

where the actions α_j are defined as $\alpha_j := \tau_{e_j}$. By the definition of the crossed product, $\mathcal{A}_{B,j}$ is generated by the entire C^*-algebra $\mathcal{F}_B$ and a unitary, let us say U_j, such that

$$\alpha_j(g) = U_j g U_j^*, \quad \text{for any } g \in \mathcal{F}_B.$$

However, equation (2.11) tells us that for any $g \in \mathcal{F}_B$

$$\alpha_j(g) = \tau_{e_j}(g) = S_{B,j} g S_{B,j}^*,$$

which implies that

$$\mathcal{A}_{B,j} = \mathcal{F}_B \rtimes_{\alpha_j} \mathbb{Z} \simeq C^*(\mathcal{F}_B, S_{B,j}).$$

In a similar way let us define

$$\mathcal{A}_{B,j,k} := \mathcal{A}_{B,j} \rtimes_{\beta_k} \mathbb{Z}, \quad \text{for } \{j,k\} = \{1,2\},$$

where the actions β_k are defined as $\beta_k(gU_j) = \tau_{e_k}(g) f_B^{(-1)^{k+1}} U_j$[2]. As before, $\mathcal{A}_{B,j,k}$ is generated by the whole C^*-algebra $\mathcal{A}_{B,j}$ and an unitary U_k such that

$$\beta_k(V) = U_k V U_k^*, \quad \text{for any } V \in \mathcal{A}_{B,j}.$$

However, for any $g \in \mathcal{F}_B$

$$\beta_k(gU_j) = \tau_{e_k}(g) f_B^{(-1)^{k+1}} U_j = S_{B,k} g S_{B,k}^* S_{B,k} U_j S_{B,k}^*,$$

provided that $f_B^{(-1)^{k+1}} U_j = S_{B,k} U_j S_{B,k}^*$, but up to representation it has already been stated that $U_k = S_{B,j}$, which proves that

$$\mathcal{A}_{B,j,k} = \mathcal{A}_{B,j} \rtimes_{\beta_k} \mathbb{Z} \simeq C^*(\mathcal{A}_{B,j}, S_{B,k}) \simeq C^*(\mathcal{F}_B, S_{B,1}, S_{B,2}) = \mathcal{A}_B.$$

The next proposition is just a summary of the results already stated.

Proposition 2.5.1. *It holds true that*

$$\mathcal{A}_B \simeq (\mathcal{F}_B \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\beta_2} \mathbb{Z} \simeq (\mathcal{F}_B \rtimes_{\alpha_2} \mathbb{Z}) \rtimes_{\beta_1} \mathbb{Z}.$$

Let us finish this section by noting that the Proposition 2.5.1 can be interpreted by saying that $\mathcal{F}_B$ encode all the information of the algebra $\mathcal{A}_B$. This fact has already been anticipated in remark 2.4.2.

[2]The nature of the exponent of f_B in the action is needed since the commutation relation between the magnetic translations are $S_{B,1}S_{B,2} = f_B S_{B,2} S_{B,1}$ or $S_{B,2}S_{B,1} = f_B^{(-1)} S_{B,1} S_{B,2}$.

2.6 Magnetic hull

The first task of this section is to define the *magnetic hull* by following the construction sketched in [BBD, Section 2.4]. Then we will construct *traces* on the magnetic algebra by following the ideas of [Dav, VIII.3]. Finally, before getting into the magnetic hull, let us point out that in this section the C^*-algebra $\mathcal{F}_B$ will be regarded as an algebra of functions instead of operators. This will clarify the use of certain Theorems canonically written in terms of algebras of functions instead of operators.

Let $B : \mathbb{Z}^2 \to \mathbb{R}$ be a magnetic field and $f_B \in \ell^\infty(\mathbb{Z}^2)$ be its magnetic flux. The natural discrete topology of $\mathbb{Z}^2$ implies that $\ell^\infty(\mathbb{Z}^2) = \mathcal{C}_{\rm b}(\mathbb{Z}^2)$, and the C^*-algebra $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$ carries the $\mathbb{Z}^2$-action defined by (2.11). Since elements of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$ are uniformly continuous one has that

$$\lim_{m\to 0} \|\tau_m(g) - g\|_\infty \;=\; \lim_{m\to 0}\left(\sup_{n\in\mathbb{Z}^2} |g(n-m) - g(n)|\right) \;=\; 0\;,$$

for all $g \in \mathcal{C}_{\rm b}(\mathbb{Z}^2)$. This means that the $\mathbb{Z}^2$-action $m \mapsto \tau_m$ acts continuously on $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$. It is worth recalling that the Gelfand-Naĭmark Theorem [GBVF, Theorem 1.4] provides the isomorphism $\mathcal{C}_{\rm b}(\mathbb{Z}^2) \simeq \mathcal{C}(\beta\mathbb{Z}^2)$ where $\beta\mathbb{Z}^2$ is the *Stone-Čech compactification* of $\mathbb{Z}^2$ [GBVF, Section 1.3]. In particular, one has a canonical inclusion $\mathbb{Z}^2 \hookrightarrow \beta\mathbb{Z}^2$, which identifies the lattice $\mathbb{Z}^2$ with an open and dense subset of $\beta\mathbb{Z}^2$.

As pointed out in Remarks 2.4.2, there exists a compact Hausdorff space Ω_B such that $\mathcal{F}_B \simeq C(\Omega_B)$. Since $\mathcal{F}_B$ is generated by a countable family, it follows that it is a separable and in turn Ω_B is second countable and metrizable as a separable complete metric space (see [GBVF, Proposition 1.11] and [Arv, Section 2.2]). We will refer to the topological space Ω_B as the *hull* of the magnetic field B, or the *magnetic hull* for short.

Actually, if follows from the Gelfand-Neimark Theorem that Ω_B is built as the *Gelfand spectrum* of $\mathcal{F}_B$, namely the set of characters defined as the $*$-homomorphisms $\omega : \mathcal{F}_B \to \mathbb{C}$. As a consequence, $\mathbb{Z}^2$ acts by duality on Ω_B, that is, for every $m \in \mathbb{Z}^2$ let $\tau^*_m : \Omega_B \to \Omega_B$ be the map defined as $\tau^*_m(\omega)(g) := \omega(\tau_{-m}(g))$ for all $g \in \mathcal{F}_B$. It is straightforward to shows that $\tau^*_m \in {\rm Homeo}(\Omega_B)$ are homeomorphisms of Ω_B and that the mapping $m \mapsto \tau^*_m$ provides a continuous $\mathbb{Z}^2$-action by homeomorphisms. As a result $(\Omega_B, \tau^*, \mathbb{Z}^2)$ is a *topological dynamical system* (see e. g. [Wal, Chapter 5]). In Ω_B there is a remarkable point ω_0, called the *evaluation at* 0, defined by $\omega_0(g) := g(0)$ for all $g \in \mathcal{F}_B$. Let $\omega_m := \tau^*_m(\omega_0) = \omega_0 \circ \tau_{-m}$ be the m-translated of ω_0, which is of course the *evaluation at* m and ${\rm Orb}(\omega_0) := \{\omega_m \in \Omega_B \mid m \in \mathbb{Z}^2\}$ the $\mathbb{Z}^2$-*orbit* of ω_0. The next result provides a relevant property of the dynamical system $(\Omega_B, \tau^*, \mathbb{Z}^2)$.

Proposition 2.6.1. *The $\mathbb{Z}^2$-orbit of ω_0 is dense, i. e.*

$$\overline{{\rm Orb}(\omega_0)} \;=\; \Omega_B\;.$$

Proof. In view of the Gelfand-Naĭmark isomorphism $\mathcal{C}_{\rm b}(\mathbb{Z}^2) \simeq \mathcal{C}(\beta\mathbb{Z}^2)$, since the Gelfand spectrum of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$ can be identified with the Stone-Čech compactification $\beta\mathbb{Z}^2$. The inclusion $\jmath : \mathcal{F}_B \hookrightarrow \mathcal{C}_{\rm b}(\mathbb{Z}^2)$ provides, by duality, a continuous map $\jmath' : \beta\mathbb{Z}^2 \to \Omega_B$ defined by $\jmath'(\tilde{\omega}) := \tilde{\omega} \circ \jmath$ where $\tilde{\omega} \in \beta\mathbb{Z}^2$ is a character of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$. More precisely, $\jmath'(\tilde{\omega})$ is by definition the restriction of the character $\tilde{\omega}$ to the subalgebra $\mathcal{F}_B$. On the other hand, every character ω of $\mathcal{F}_B$ admits a (not necessarily unique) extension $\tilde{\omega}$ to a character of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$ [BR, Proposition 2.3.24]. As a result, it turns out that $\jmath'$ is a continuous surjection. Therefore, if $X \subset \beta\mathbb{Z}^2$ is dense in $\beta\mathbb{Z}^2$ then $\jmath'(X) \subset \Omega_B$ is dense in Ω_B. In view of the Riesz-Markov-Kakutani representation Theorem [RS, Theorem IV.14], the Gelfand spectrum of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$ consists of the evaluations (Dirac measures) at the points of $\beta\mathbb{Z}^2$. Since $\mathbb{Z}^2$ can be identified with a dense open subset of $\beta\mathbb{Z}^2$, it follows that the set of characters $\{\tilde{\omega}_m \mid m \in \mathbb{Z}^2\}$, defined by $\tilde{\omega}_m(f) := f(m)$ for $f \in \mathcal{C}_{\rm b}(\mathbb{Z}^2)$, is dense in the Gelfand spectrum of $\mathcal{C}_{\rm b}(\mathbb{Z}^2)$. On the other hand, it holds true that $\omega_m = \jmath'(\tilde{\omega}_m)$, and consequently

$$\jmath'\big(\{\tilde{\omega}_m \mid m \in \mathbb{Z}^2\}\big) \;=\; {\rm Orb}(\omega_0)\;.$$

The last equality proves the density of ${\rm Orb}(\omega_0)$. $\square$

Let $g \in \mathcal{F}_B$. Its Gelfand transform $\hat{g} \in \mathcal{C}(\Omega_B)$ is defined by the equation $\hat{g}(\omega) := \omega(g)$ for all $\omega \in \Omega_B$. The density of the orbit of ω_0 implies that the Gelfand transform is entirely defined by the equation $g(m) = \omega_m(g) = \hat{g}(\tau_m^*(\omega_0))$ for all $m \in \mathbb{Z}^2$.

Remark 2.6.2 (Topological transitivity)**.** Proposition 2.6.1 can be rephrased by saying that the dynamical system $(\Omega_B, \tau^*, \mathbb{Z}^2)$ is *topologically transitive* [Wal, Definition 5.6]. As a consequence, every invariant element of $\mathcal{C}(\Omega_B)$ is automatically constant [Wal, Theorem 5.14]. In our specific setting (Ω_B compact and second countable) the notion of topological transitivity for $(\Omega_B, \tau^*, \mathbb{Z}^2)$ is equivalent to the following property: Whenever U and V are nonempty open subsets of Ω_B, then there exists a $m \in \mathbb{Z}^2$ such that $\tau_m^*(U) \cap V \neq \emptyset$ [Wal, Theorem 5.8]. The latter, is the usual definition of topological transitivity in the context of the general theory of topological dynamical systems (see *e.g.* [KSn, AC] and references therein). It is also worth remembering that Ω_B has no isolated points if and only if it is infinite [KSn, AC, pg. 6]. $\triangleleft$

The subsets ${\rm Orb}(\omega_0)$ and $\partial\Omega_B := \Omega_B \setminus {\rm Orb}(\omega_0)$ are disjoint and τ^*-invariant by construction. Moreover, $\partial\Omega_B$ is nowhere dense [Wal, Theorem 5.8] and it is contained in the subset of *non-wandering points* of the dynamical system [Wal, Theorem 5.6]. Let ${\rm Mes}_{1,\tau^*}(\Omega_B)$ be the set of the normalized and τ^*-invariant regular Borel measures[3] of the dynamical system $(\Omega_B, \tau^*, \mathbb{Z}^2)$. It is well known that ${\rm Mes}_{1,\tau^*}(\Omega_B)$ is a non-empty, convex and compact set (*i. e.* a Choquet simplex) whose extreme points are exactly the *ergodic* measures [Wal, Corollary 6.9.1 & Theorem 6.10]. Let ${\rm Erg}(\Omega_B)$ be the subset of the ergodic probability measures of $(\Omega_B, \tau^*, \mathbb{Z}^2)$. It is worth recalling that ergodic measures $\mathbb{P} \in {\rm Erg}(\Omega_B)$ are characterized by the dichotomy $\mathbb{P}(X) = 1$ or $\mathbb{P}(X) = 0$ for every given τ^*-invariant subset $X \subseteq \Omega_B$. A measure $\mathbb{P} \in {\rm Erg}(\Omega_B)$ such that $\mathbb{P}(\partial\Omega_B) = 1$ will be called a *mesure at infinity.*

[3]By the Riesz-Markov-Kakutani representation Theorem [RS, Theorem IV.14], ${\rm Mes}_{1,\tau^*}(\Omega_B)$ provides the space of τ^*-invariant states of the Abelian C^*-algebra $\mathcal{C}(\Omega_B)$.

Example 2.6.3 (Magnetic hull for a constant magnetic field)**.** In the case of a constant magnetic field of strength b we have that $\mathcal{F}_b = \mathbb{C}$ (see Example 2.4.3). Therefore the associated magnetic hull $\Omega_b \simeq \{\omega_0\}$ is a singleton (or one point set) on which the τ^*-action is trivial. Finally, the unique normalized ergodic measure on Ω_b is entirely specified by $\mathbb{P}(\{\omega_0\}) = 1$. ◁

Example 2.6.4 (Iwatsuka magnetic hull)**.** In the case of the Iwatsuka magnetic field one has $\mathcal{F}_\mathrm{I} \simeq c(\mathbb{Z})$ (see 2.4.4) and consequently the $\mathbb{Z}^2$-action on $\mathcal{F}_\mathrm{I}$ reduces to a $\mathbb{Z}$-action. It follows that the Gelfand isomorphisms $\mathcal{F}_\mathrm{I} \simeq \mathcal{C}(\Omega_\mathrm{I})$ is given by the *Iwatsuka magnetic hull*

$$\Omega_\mathrm{I} \ \simeq \cup \{-\infty\} \ \cup \ \ \mathbb{Z} \, \{+\infty\}, \tag{2.17}$$

which is the two-point compactification of $\mathbb{Z}$. This should be no surprise, since $c(\mathbb{Z}) \simeq C(\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\})$.
Just to make it explicit let us point out that the inclusion $\mathbb{Z} \ni n \mapsto \omega_{(n,0)} \in \Omega_\mathrm{I}$ is given by the *evaluation at* $(n, 0)$, and the two limit points $\{\pm\infty\}$ are identified with the evaluations at infinity $\omega_{\pm\infty} \in \Omega_\mathrm{I}$ defined by

$$\omega_{\pm\infty}\big(\tau_m(f_\mathrm{I})\big) \ := \ \mathrm{e}^{\mathrm{i}\, b_\pm}$$

for every $m \in \mathbb{Z}^2$. From the construction it follows that $\mathbb{Z} \simeq \mathrm{Orb}(\omega_0)$ and in turn $\{\pm\infty\} \simeq \partial\Omega_\mathrm{I}$. Therefore, equation (2.17) provides a decomposition of Ω_I in three invariant subsets. Since $\mathbb{Z}^2$ acts on $\mathrm{Orb}(\omega_0)$ as a one dimensional shift it follows that $\mathrm{Orb}(\omega_0)$ is made by wondering points [Wal, Definition 5.5]. As a consequence every ergodic measure $\mathbb{P} \in \mathrm{Erg}(\Omega_\mathrm{I})$ necessarily must satisfy $\mathbb{P}(\mathrm{Orb}(\omega_0)) = 0$ [Wal, Theorem 6.15]. This implies that the set $\mathrm{Erg}(\Omega_\mathrm{I}) = \{\mathbb{P}_{\pm\infty}\}$ is made by two ergodic measures at infinity specified by the condition $\mathbb{P}_{\pm\infty}(\pm\infty) = 1$. ◁

Example 2.6.5 (Magnetic hull for a localized magnetic field)**.** In the case of a localized magnetic field one has that $\mathcal{F}_\Lambda = c(\mathbb{Z}^2)$ (see Example 2.4.5). It follows that in this case the Gelfand isomorphisms $\mathcal{F}_\Lambda \simeq \mathcal{C}(\Omega_\Lambda)$ is given by the *localized magnetic hull*

$$\Omega_\Lambda \ \simeq \ \mathbb{Z}^2 \ \cup \ \{\infty\}, \tag{2.18}$$

which is the one-point compactification of $\mathbb{Z}^2$. The inclusion $\mathbb{Z}^2 \ni n \mapsto \omega_n \in \Omega_\Lambda$ is given by the evaluations at n and the limit point $\{\infty\}$ is identified with the evaluation at infinity $\omega_\infty \in \Omega_\Lambda$ given by

$$\omega_\infty\big(\tau_m(f_\Lambda)\big) \ := \ 1$$

for every $m \in \mathbb{Z}^2$. From the construction it follows that $\mathbb{Z}^2 \simeq \mathrm{Orb}(\omega_0)$ and $\{\infty\} \simeq \partial\Omega_\Lambda$ are the two invariant subsets of Ω_Λ. Since $\mathrm{Orb}(\omega_0)$ is made of wondering points under the action of $\mathbb{Z}^2$ it follows that $\mathrm{Erg}(\Omega_\Lambda) = \{\mathbb{P}_\infty\}$ where the measure at infinity $\mathbb{P}_\infty$ is specified by $\mathbb{P}_\infty(\infty) = 1$. ◁

The ergodic measures of $(\Omega_B, \tau^*, \mathbb{Z}^2)$ play a crucial role in the construction of the integration theory of the magnetic algebra $\mathcal{A}_B$ developed in section 2.9.

2.7 Fourier theory

Considering that the magnetic C^*-algebra has (hopefully) already proved to be a generalization of the noncommutative torus, it is nothing but natural to expect it to have a rich Fourier Theory (for the Fourier Theory of the noncommutative torus see [PS, Chapter 3.3]). Actually, the fact that the noncommutative torus has a very rich Fourier Theory can be considered equally natural, since it is a generalization of the continuous functions on the two-dimmensional Torus $C(\mathbb{T}^2)$ (for the Fourier theory of $C(\mathbb{T})$ see [Kat, Gra]). In this section it will be showed that some of the results of the classical Fourier theory extend to the magnetic C^*-algebra $\mathcal{A}_B$. Similar results can also be found in [Dav, Section VIII.2].

First let us fix notations. From now on the torus $\mathbb{T}$ will appear several times in this work, and unless the opposite is mentioned, by the torus we will refer to $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\theta := (\theta_1, \theta_2)$ will be a point of $\mathbb{T}^2$.
Consider the unitary operator $Z_\theta := \mathrm{e}^{-i\theta\cdot X} = \mathrm{e}^{-i(\theta_1 X_1 + \theta_2 X_2)}$ and define the $\mathbb{T}^2$-action on $\mathcal{B}(\ell^2(\mathbb{Z}^2))$ by

$$\rho_\theta(T) := Z_\theta T Z_\theta^*. \tag{2.19}$$

Let us observe that this action is not continuous. For this consider the *rotation operator* R_π defined by $(R_\pi\psi)(n) = \psi(-n)$. Since $\rho_\theta(R_\pi) = W_{2\theta}R_\pi$, it follows that $\|\rho_\theta(R_\pi) - R_\pi\| = \|W_{2\theta} - 1\| = 2$ whenever one of θ_1 or θ_2 are irrational. Things go differently if the action of $\mathbb{T}^2$ is restricted to $\mathcal{A}_B$.

Proposition 2.7.1. *The formula* (2.19) *defines a continuous group action of* $\mathbb{T}^2$ *into the magnetic* C^**-algebra* $\mathcal{A}_B$*.*

Proof. A direct computation shows that

$$\rho_\theta(g) = g \quad \text{for all } g \in \mathcal{F}_B, \tag{2.20}$$

independently of $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, and

$$\rho_\theta\big((S_{A_B,1})^r (S_{A_B,2})^s\big) = \mathrm{e}^{-i(r\theta_1 + s\theta_2)}\,(S_{A_B,1})^r (S_{A_B,2})^s \tag{2.21}$$

for all $(r,s) \in \mathbb{Z}^2$. The relations (2.20) and (2.21) along with the definition (2.16) of $\mathcal{A}_B^\infty$ imply that $\rho_\theta(\mathcal{A}_B^\infty) = \mathcal{A}_B^\infty$ for all $\theta \in \mathbb{T}^2$. Finally, the density of $\mathcal{A}_B^\infty$ and the fact that ρ_θ is norm-preserving imply that $\rho_\theta(\mathcal{A}_B) = \mathcal{A}_B$, namely $\rho_\theta \in \mathrm{Aut}(\mathcal{A}_B)$ for all $\theta \in \mathbb{T}^2$. Let us prove now the continuity of the group action. Let $T_{\mathbf{g}} = \sum_{(r,s)\in\mathbb{Z}^2} g_{r,s}(S_{B,1})^r(S_{B,2})^s$ according to (2.16). Then

$$\|\rho_\theta(T_{\mathbf{g}}) - T_{\mathbf{g}}\| \leqslant \sum_{(r,s)\in\mathbb{Z}^2} \big|\,\mathrm{e}^{-i(r\theta_1+s\theta_2)} - 1\big|\|g_{r,s}\| \leqslant 2\sum_{(r,s)\in\mathbb{Z}^2}\|g_{r,s}\|$$

and from the dominated convergence Theorem (for series) it follows that

$$\lim_{\theta\to 0}\|\rho_\theta(T_{\mathbf{g}}) - T_{\mathbf{g}}\| \leqslant \sum_{(r,s)\in\mathbb{Z}^2}\lim_{\theta\to 0}\big|\,\mathrm{e}^{-i(r\theta_1+s\theta_2)} - 1\big|\|g_{r,s}\| = 0 \tag{2.22}$$

for all $T_{\mathbf{g}} \in \mathcal{A}_B^\infty$. Now, let $T \in \mathcal{A}_B$ be a generic element and $\varepsilon > 0$. By density it exists a $T_{\mathbf{g}} \in \mathcal{A}_B^\infty$ such that $\|T - T_{\mathbf{g}}\| < \frac{\varepsilon}{2}$. Moreover,

$$\begin{aligned}\|\rho_\theta(T) - T\| &\leqslant \|\rho_\theta(T_{\mathbf{g}}) - T_{\mathbf{g}}\| + \|\rho_\theta(T - T_{\mathbf{g}}) - (T - T_{\mathbf{g}})\| \\ &< \|\rho_\theta(T_{\mathbf{g}}) - T_{\mathbf{g}}\| + \varepsilon.\end{aligned}$$

Therefore, from (2.22) it follows that $\lim_{\theta\to 0} \|\rho_\theta(T) - T\| < \varepsilon$, independently of $\varepsilon > 0$ and for all $T \in \mathcal{A}_B$. This proves that the group action $\theta \mapsto \rho_\theta$ is continuous on $\mathcal{A}_B$. □

Let

$$\mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B) := \left\{T \in \mathcal{A}_B \middle| \rho_\theta(T) = T, \text{for all } \theta \in \mathbb{T}^2\right\}$$

be the set of invariant elements of $\mathcal{A}_B$. From (2.20) one gets that $\mathcal{F}_B \subseteq \mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$. The next goal is to characterize $\mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$. For that let us denote with $\mathrm{d}\mu(\theta) := (2\pi)^{-2}\,\mathrm{d}\theta$ the normalized Haar measure on $\mathbb{T}^2$ and consider the averaging

$$\langle T\rangle := \int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\rho_\theta(T), \quad T \in \mathcal{A}_B$$

where the integral is meant in the Bochner sense. From the invariance of the Haar measure it follows that $\langle T\rangle \in \mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$ by construction. Moreover, $\langle T\rangle = T$ if and only if $T \in \mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$. This means that every element of $\mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$ can be always represented as the averaging of some element in $\mathcal{A}_B$. The next result characterizes the set of invariant elements.

Lemma 2.7.2. *It holds true that*

$$\mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B) = \mathcal{F}_B.$$

Proof. Since we already know that $\mathcal{F}_B \subseteq \mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$ we only need to prove the opposite inclusion. Since every element in $\mathrm{Inv}_{\mathbb{T}^2}(\mathcal{A}_B)$ can be represented as the averaging of some element in $\mathcal{A}_B$ it is enough to prove that $\langle T\rangle \in \mathcal{F}_B$ for all $T \in \mathcal{A}_B$. Since the map $T \mapsto \langle T\rangle$ is a continuous, *i. e.* $\|\langle T\rangle\| \leqslant \|T\|$ and $\mathcal{F}_B$ is closed, it is sufficient to prove that the averaging of the monomials (2.13) takes value in $\mathcal{F}_B$. Based on (2.20) and (2.21), a direct computation shows that

$$\begin{aligned}\langle g(S_{B,1})^r(S_{B,2})^s\rangle &= g(S_{B,1})^r(S_{B,1})^s \int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\,\mathrm{e}^{-\mathrm{i}\,(r\theta_1+s\theta_2)} \\ &= g\delta_{r,0}\delta_{s,0}\end{aligned} \tag{2.23}$$

for all $g \in \mathcal{F}_B$ and for all $(r,s) \in \mathbb{Z}^2$. This completes the proof. □

We are now in position to prove that every element of $\mathcal{A}_B$ can be represented as a Fourier-type series in the generating monomials (2.13). To make precise this statement, we need to introduce some notation. Given a $T \in \mathcal{A}_B$ let us define the $\mathcal{F}_B$-valued coefficients

$$\begin{aligned}\hat{T}_{r,s} :&= \langle T(S_{B,2})^{-s}(S_{B,1})^{-r}\rangle \\ &= \left(\int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\,\mathrm{e}^{\mathrm{i}\,(r\theta_1+s\theta_2)}\,\rho_\theta(T)\right)(S_{B,2})^{-s}(S_{B,1})^{-r}.\end{aligned} \tag{2.24}$$

consider the family of boxes $\Lambda_N := [-N,N]^2 \cap \mathbb{Z}^2$ with $N \in \mathbb{N}$, and the associated *Cesàro means*

$$\sigma_N(T) := \sum_{(r,s)\in\Lambda_N} \left(1 - \frac{|r|}{N+1}\right)\left(1 - \frac{|s|}{N+1}\right) \hat{T}_{r,s}(S_{B,1})^r (S_{B,2})^s. \tag{2.25}$$

Theorem 2.7.3 (Fourier expansion - Cesàro mean)**.** [4] *For every element $T \in \mathcal{A}_B$ it holds true that*

$$\lim_{N\to\infty} \|\sigma_N(T) - T\| = 0.$$

Proof. By combining (2.24) and (2.25) one gets

$$\sigma_N(T) = \int_{\mathbb{T}^2} \mathrm{d}\mu(\theta) K_N(\theta)\rho_\theta(T)$$

where

$$\begin{aligned} K_N(\theta) : &= \sum_{(r,s)\in\Lambda_N} \left(1 - \frac{|r|}{N+1}\right)\left(1 - \frac{|s|}{N+1}\right) \mathrm{e}^{i(r\theta_1 + s\theta_2)} \\ &= F_N(\theta_1)F_N(\theta_2) \end{aligned}$$

and

$$F_N(\theta_j) := \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \mathrm{e}^{ik\theta_j} = \frac{1}{N+1}\left(\frac{\sin\left(N\theta_j + \frac{\theta_j}{2}\right)}{\sin\left(\frac{\theta_j}{2}\right)}\right)^2$$

is the Fejér kernel, with $j = 1, 2$ [Kat, Chapter I, Section 2.5] or [Gra, Chapter I, Section 3.1.3]. Since $(2\pi)^{-1}\int_0^{2\pi} \mathrm{d}\theta_j F_N(\theta_j) = 1$, and consequently $\int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)K_N(\theta) = 1$, one gets that

$$\sigma_N(T) - T = \int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)K_N(\theta)\big[\rho_\theta(T) - T\big].$$

Using the identity $\rho_\theta(T) - T = \rho_{(\theta_1,0)}(\rho_{(0,\theta_2)}(T) - T + T - \rho_{(-\theta_1,0)}(T))$ and the fact that the $\mathbb{T}^2$-action is isometric one gets

$$\begin{aligned} \|\sigma_N(T) - T\| \leqslant &\int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} F_N(\theta_1)\|\rho_{(\theta_1,0)}(T) - T\| \\ &+ \int_0^{2\pi} \frac{\mathrm{d}\theta_2}{2\pi} F_N(\theta_2)\|\rho_{(0,\theta_2)}(T) - T\|. \end{aligned}$$

Since the functions $f_1(\theta_1) := \|\rho_{(\theta_1,0)}(T) - T\|$ and $f_2(\theta_2) := \|\rho_{(0,\theta_2)}(T) - T\|$ are continuous with $f_1(0) = 0 = f_2(0)$ and the the Fejér kernel is a summability kernel [Kat, Chapter I, Section 2.2] one obtains that the two integrals on the right go to zero when $N \to \infty$. This concludes the proof. ∎

Theorem 2.7.3 states that every element of $T \in \mathcal{A}_B$ can be approximated by the sequence $\sigma_N(T) \in \mathcal{A}_B^0$ obtained from the "Fourier" coefficients $\hat{T}_{r,s}$. It follows that two elements with the same $\mathcal{F}_B$-valued coefficients are identical. Equivalently, one has that

[4] The proof of this theorem is adapted from [Wea, Theorem 5.5.7].

Corollary 2.7.4. *Let $T \in \mathcal{A}_B$. Then $T = 0$ if and only if $\hat{T}_{r,s} = 0$ for all $(r,s) \in \mathbb{Z}^2$.*

Remark 2.7.5 (Cesàro vs. uniform convergence)**.** By observing that

$$K_N(\theta) = \frac{1}{(N+1)^2} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} D_{(n_1,n_2)}(\theta),$$

where

$$D_{(n_1,n_2)}(\theta) := \sum_{(r,s)\in\Lambda_{(n_1,n_2)}} \mathrm{e}^{i(r\theta_1+s\theta_2)} = \frac{\sin\left(n_1\theta_1 + \frac{\theta_1}{2}\right)\sin\left(n_1\theta_2 + \frac{\theta_2}{2}\right)}{\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)}$$

is the *Dirichlet kernel* of the rectangular domain $\Lambda_{(n_1,n_2)} := ([-n_1, n_1] \times [-n_2, n_2]) \cap \mathbb{Z}^2$, one can rewrite (2.25) in the form

$$\sigma_N(T) = \frac{1}{(N+1)^2} \sum_{(n_1,n_2)\in\Lambda_N} S_{(n_1,n_2)}(T) \tag{2.26}$$

where

$$S_{(n_1,n_2)}(T) : = \sum_{(r,s)\in\Lambda_{(n_1,n_2)}} \widehat{T}_{r,s}(S_{B,1})^r(S_{B,2})^s$$

is the partial Fourier-type expansion of T. Therefore, Theorem 2.7.3 provides a justification of the series representation

$$T \overset{\sigma}{=} \lim_{(n_1,n_2)\to\infty} S_{(n_1,n_2)}(T) := \sum_{(r,s)\in\mathbb{Z}^2} \widehat{T}_{r,s}(S_{B,1})^r(S_{B,2})^s$$

where the symbol $\overset{\sigma}{=}$ means that the limit must be understood in the sense of Cesàro, as given by equation (2.26). This is the best that one can generally hope for a generic element $T \in \mathcal{A}_B$. Indeed, let $f \in \mathcal{C}(\mathbb{T})$ be the Fejér-type function constructed as in [Kat, Chapter II, Section 2.1]. Then, the sequence of the partial Fourier-type expansions of the element $f(S_1^{\mathcal{A}_B}) \in \mathcal{A}_B$ cannot be convergent in norm. $\triangleleft$

In view of the lack of a Fourier series representation for the elements of $\mathcal{A}_B$ it may be useful to characterize the collection of elements of $\mathcal{A}_B$ with such property, that is, the elements having an absolutely convergent Fourier series of $\mathcal{F}_B$-valued coefficients. More precisely, let us introduce the space

$$\mathcal{A}_B^{\text{a.c.}} := \left\{ T \in \mathcal{A}_B \;\middle|\; \|T\|_{\ell^1} := \sum_{(r,s)\in\mathbb{Z}^2} \|\widehat{T}_{r,s}\| < \infty \right\}$$

where the coefficients $\widehat{T}_{r,s}$ are defined by (2.24). Since $\mathcal{A}_B^\infty \subset \mathcal{A}_B^{\text{a.c.}} \subset \mathcal{A}_B$ it follows that $\mathcal{A}_B^{\text{a.c.}}$ is dense in $\mathcal{A}_B$. The main properties of $\mathcal{A}_B^{\text{a.c.}}$ are described in the next result.

Proposition 2.7.6. *The space $\mathcal{A}_B^{\text{a.c.}}$, endowed with the norm $\|\|_{\ell^1}$, is a Banach $*$-algebra isomorphic to $\ell^1(\mathbb{Z}^2, \mathcal{F}_B)$. In particular every $T \in \mathcal{A}_B^{\text{a.c.}}$ coincides with its Fourier-type expansion, i. e.*

$$T = \sum_{(r,s)\in\mathbb{Z}^2} T_{r,s}(S_{B,1})^r (S_{B,2})^s$$

Proof. Every $T \in \mathcal{A}_B^{\text{a.c.}}$ defines an element $\{\widehat{T}_{r,s}\} \in \ell^1(\mathbb{Z}^2, \mathcal{F}_B)$ by definition. Moreover, the map $T \mapsto \{\widehat{T}_{r,s}\}$ is injective in view of Corollary 2.7.4. The surjectivity follows by observing that every $\{\widehat{T}_{r,s}\} \in \ell^1(\mathbb{Z}^2, \mathcal{F}_B)$ defines an element

$$T := \lim_{N\to\infty} \sum_{(r,s)\in\Lambda_N} \widehat{T}_{r,s}(S_{B,1})^r (S_{B,2})^s \in \mathcal{A}_B^{\text{a.c.}}.$$

with $\mathcal{F}_B$-valued coefficients $\{\widehat{T}_{r,s}\}$. Finally, a straightforward computation as in [Kat, Chapter I, Section 6.1] shows that $\mathcal{A}_B^{\text{a.c.}}$ is closed under the operations inherited by the $*$-algebraic structure of $\mathcal{A}_B$. □

2.8 Differential structure

As stated in proposition 2.7.1 , the map $\theta \mapsto \rho_\theta$ defines a strongly continuous $\mathbb{T}^2$-action on the C^*-agebra $\mathcal{A}_B$, so one can think on the action ρ_θ as a continuous group of operators on the space $\mathcal{A}_B$ (see [BR, Definition 3.1.2]).This allows us to introduce the *infinitesimal generators* ∂_1 and ∂_2 defined by

$$\begin{aligned}\partial_1(T) &:= \lim_{\theta_1\to 0} \frac{\rho_{(\theta_1,0)}(T) - T}{\theta_1} \\ \partial_2(T) &:= \lim_{\theta_2\to 0} \frac{\rho_{(0,\theta_2)}(T) - T}{\theta_2}\end{aligned}$$

for suitable elements $T \in \mathcal{A}_B$ [BR, Definition 3.1.5]. Indeed, ∂_1 and ∂_2 are unbounded linear maps on $\mathcal{A}_B$, defined on dense domains $\mathcal{D}(\partial_1)$ and $\mathcal{D}(\partial_2)$, respectively [BR, Proposition 3.1.6]. Moreover, they are *(symmetric) derivations* [BR, Definition 3.2.21], in the sense that

$$\begin{aligned}\partial_j(T^*) &= \partial_j(T)^*; \\ \partial_j(TR) &= T\,\partial_j(R) + \partial_j(T)\,R\,,\end{aligned} \qquad T, R \in \mathcal{D}(\partial_j)\ , \quad j = 1, 2\ . \tag{2.27}$$

Since the subalgebra $\mathcal{F}_B$ is invariant under the action α_θ it follows that

$$\partial_1(g) = \partial_2(g) = 0\ , \qquad \forall\, g \in \mathcal{F}_B\ . \tag{2.28}$$

Moreover, a direct computation shows

$$\begin{aligned}\partial_1\big((S_{B,1})^r\,(S_{B,2})^s\big) &= -\,\mathrm{i}\,r\,(S_{B,1})^r\,(S_{B,2})^s \\ \partial_2\big((S_{B,1})^r\,(S_{B,2})^s\big) &= -\,\mathrm{i}\,s\,(S_{B,1})^r\,(S_{B,2})^s\,,\end{aligned} \qquad \forall\,(r,s) \in \mathbb{Z}^2\ . \tag{2.29}$$

In particular, one can check that

$$\partial_j\big(g\;(S_{B,1})^r\;(S_{B,2})^s\big)\;=\;\mathrm{i}\,\big[g\;(S_{B,1})^r\;(S_{B,2})^s, N_j\big]\;, \tag{2.30}$$

where [,] denotes the commutator. Indeed equation (2.30) is a special case of a more general result [BR, Definition 3.2.55], which justifies the name of *spatial derivation* for ∂_1 and ∂_2.

From (2.28) and (2.29) it follows that

$$\mathcal{A}_B^0\;\subset\;\mathcal{A}_B^\infty\;\subset\;\mathcal{D}(\partial_1)\;\cap\;\mathcal{D}(\partial_2)\;.$$

Moreover, the elements of $\mathcal{A}_B^\infty$ support several iterated derivations. Let $\partial_j^a := \partial_j \circ \ldots \circ \partial_j$ be the a-times iteration of the derivation ∂_j. Since the group $\mathbb{T}^2$ is abelian, it follows that $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$ whenever the product of the derivatives is well defined. It follows that the expression $\partial_1^a\partial_2^b$, for $a, b \in \mathbb{N}_0$, it is not ambiguous in suitable domains like $\mathcal{A}_B^\infty$. Let us introduce the spaces

$$\mathcal{C}^k(\mathcal{A}_B)\;:=\;\overline{\mathcal{A}_B^0}^{\;\|\;\|_k}\;,$$

obtained by closing the noncommutative polynomials with respect to the norm

$$\|T\|_k\;:=\;\sum_{j=0}^{k}\sum_{a+b=j}\|\partial_1^a\partial_2^b(T)\|\;.$$

A standard argument shows that $T \in \mathcal{C}^k(\mathcal{A}_B)$ if and only if $\partial_1^a\partial_2^b(T) \in \mathcal{A}_B$ is well defined for all $a, b \in \mathbb{N}_0$ such that $a + b \leqslant k$, namely

$$\mathcal{C}^k(\mathcal{A}_B)\;=\;\big\{T \in \mathcal{A}_B\;\big|\;\theta \mapsto \rho_\theta(T)\quad\text{is } k\text{-differentiable}\big\}\;.$$

The regularity of an element is reflected on the decay property of its $\mathcal{F}_B$-valued coefficients. This is the content of the next result.

Lemma 2.8.1. *Let* $T \in \mathcal{C}^k(\mathcal{A}_B)$ *then*

$$\sup_{(r,s)\in\mathbb{Z}^2}\big(1 + r^2 + s^2\big)^k\,\|\widehat{T}_{r,s}\|^2\;<\;\infty \tag{2.31}$$

where the $\widehat{T}_{r,s}$ *are defined by* (2.24)*. In particular*

$$\mathcal{C}^k(\mathcal{A}_B)\;\subset\;\mathcal{A}_B^{\text{a.c.}}$$

for all $k > 2$.

Proof. Let $a, b \in \mathbb{N}_0$ such that $a + b \leqslant k$. Then $\partial_1^a \partial_2^b(T) \in \mathcal{A}_B$ and we can calculate the $\mathcal{F}_B$-valued coefficients according to (2.24). An iterated integration by parts provides

$$\begin{aligned}
\widehat{\partial_1^a \partial_2^b(T)}_{r,s} &= \left(\int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\ \mathrm{e}^{\mathrm{i}\,(r\theta_1 + s\theta_2)}\ \rho_\theta(\partial_1^a \partial_2^b(T)) \right) (S_{B,2})^{-s} (S_{B,1})^{-r} \\
&= (-\mathrm{i}\,)^{a+b} r^a s^b \left(\int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\ \mathrm{e}^{\mathrm{i}\,(r\theta_1 + s\theta_2)}\ \rho_\theta(T) \right) (S_{B,2})^{-s} (S_{B,1})^{-r} \\
&= (-\mathrm{i}\,)^{a+b} r^a s^b\ \widehat{T}_{r,s}\ .
\end{aligned}$$

Since $\|\widehat{\partial_1^a \partial_2^b(T)}_{r,s}\| \leqslant \|\partial_1^a \partial_2^b(T)\| =: C_{a,b}$ for all $(r, s) \in \mathbb{Z}^2$, we can define $C := \max_{a+b=k}\{C_{a,b}\}$. It follows that $r^{2a} s^{2b} \|\widehat{T}_{r,s}\|^2 \leqslant C^2$ for all a, b such that $a + b = k$. Then, by using the formula for the binomial expansion one gets

$$(r^2 + s^2)^k \|\widehat{T}_{r,s}\|^2 \ \leqslant\ 2^k C^2\ . \tag{2.32}$$

From (2.32), a second application of the formula for the binomial expansion provides (2.31) with bound given by $4^k C^2$. From (2.32) one gets

$$\begin{aligned}
\sum_{(r,s)\in\mathbb{Z}^2} \|\widehat{T}_{r,s}\| &\leqslant \|\widehat{T}_{0,0}\| + 2^k C^2 \sum_{(r,s)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(r^2+s^2)^{\frac{k}{2}}} \\
&= \|\widehat{T}_{0,0}\| + 2^{k+1} C^2 \left(2 \sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \frac{1}{(r^2+s^2)^{\frac{k}{2}}} + \sum_{r=1}^{\infty} \frac{1}{r^k} + \sum_{s=1}^{\infty} \frac{1}{s^k} \right) \\
&= \|\widehat{T}_{0,0}\| + 2^{k+2} C^2 \left(\sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \frac{1}{(r^2+s^2)^{\frac{k}{2}}} + \sum_{r=1}^{+\infty} \frac{1}{r^k} \right) \\
&\leq \|\widehat{T}_{0,0}\| + 2^{\frac{k}{2}+2} C^2 \left(\left(\sum_{r=1}^{+\infty} \frac{1}{r^{\frac{k}{2}}} \right) \left(\sum_{s=1}^{+\infty} \frac{1}{s^{\frac{k}{2}}} \right) + \sum_{r=1}^{+\infty} \frac{1}{r^k} \right)
\end{aligned}$$

where in the last inequality we used $2rs \leqslant r^2 + s^2$. This concludes the proof. □

Remark 2.8.2. The results provided in Lemma 2.8.1 are not optimal, in general. For instance, in the case of a zero magnetic field described in Example 2.3.3 one can replace (2.31) with $(1 + r^2 + s^2)^k \|\widehat{T}_{r,s}\|^2 \to 0$ when $(r, s) \to \infty$ [Gra, Theorem 3.3.9]. Moreover, the absolute convergence of the series of coefficients is generally guaranteed by a degree of regularity weaker than $k > 2$ [Gra, Theorem 3.3.16]. However, for the purposes of this work we will not need such a kind of generalization.

The space of the smooth elements is defined by

$$\mathcal{C}^\infty(\mathcal{A}_B) \ :=\ \bigcap_{k\in\mathbb{N}_0} \mathcal{C}^k(\mathcal{A}_B)\ .$$

For $T \in \mathcal{C}^\infty(\mathcal{A}_B)$ the map $\theta \mapsto \rho_\theta(a)$ turns out to be smooth. ◁

Proposition 2.8.3. *The dense subalgebra $\mathcal{A}_B^\infty$ defined by* (2.16) *coincides with the algebra of the smooth elements with respect to the $\mathbb{T}^2$-action, i. e.*

$$\mathcal{A}_B^\infty \ = \ \mathcal{C}^\infty(\mathcal{A}_B) \ .$$

Proof. Let $T \in \mathcal{A}_B^\infty$. Then the computation of the $\mathcal{F}_B$-valued coefficients of $\partial_1^a \partial_2^b(T)$ provided in the proof of Lemma 2.8.1 shows that $\partial_1^a \partial_2^b(T) \in \mathcal{A}_B^{\text{a.c.}}$ for all $a, b \in \mathbb{N}_0$. This implies that $\mathcal{A}_B^\infty \subset \mathcal{C}^k(\mathcal{A}_B)$ for all $k \in \mathbb{N}_0$, and so $\mathcal{A}_B^\infty \subseteq \mathcal{C}^\infty(\mathcal{A}_B)$. On the other hand it is also true that $\mathcal{C}^\infty(\mathcal{A}_B) \subseteq \mathcal{A}_B^\infty$. In fact, if $T \in \mathcal{C}^\infty(\mathcal{A}_B)$ then (2.31) applies for all $k \in \mathbb{N}_0$, showing that $T \in \mathcal{A}_B^\infty$. This concludes the proof. □

The last result justifies the name of *smooth algebra* for $\mathcal{A}_B^\infty$. Let us recall that a pre-C^*-algebra is a dense subalgebra of a C^*-algebra which is stable under holomorphic functional calculus (see [GBVF, Definition 3.26]).

Proposition 2.8.4. *The smooth algebra $\mathcal{A}_B^\infty$ defined by* (2.16) *is a unital Fréchet pre-C^*-algebra of $\mathcal{A}_B$.*

Proof. Since $\mathbb{T}^2$ is a Lie group, the criterion established in [GBVF, Proposition 3.45] applies proving the claim. □

The Fréchet topology of the pre-C^*-algebra $\mathcal{A}_B^\infty$ is provided by the system of norms described in Proposition 2.4.9.

2.9 Integration Theory

Before constructing the integration theory of the magnetic C^*-algebra $\mathcal{A}_B$ the next lemma will be of high importance.

Lemma 2.9.1. *Let $\mathscr{G} \ : \ \mathcal{F}_B \longrightarrow \mathcal{C}(\Omega_B)$ be the Gelfand isomorphism described before. It follows that every invariant measure $\mathbb{P} \in \mathrm{Mes}_{1,\tau^*}(\Omega_B)$ defines a trace $t_\mathbb{P}$ on $\mathcal{F}_B$ through the formula*

$$t_\mathbb{P}(g) \ := \ \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \mathscr{G}(g)(\omega) \ , \qquad\qquad g \in \mathcal{F}_B \ .$$

The trace $t_\mathbb{P}$ is $\mathbb{Z}^2$-invariant in the sense that

$$t_\mathbb{P}\left(\tau_m(g)\right) \ = \ t_\mathbb{P}(g) \ , \qquad \forall\, m = (m_1, m_2) \in \mathbb{Z}^2$$

where $\tau_m(g) := (S_{A_B,1})^{m_1} (S_{A_B,2})^{m_2} g (S_{A_B,2})^{-m_2} (S_{A_B,1})^{-m_1}$

Proof. The fact that $t_{\mathbb{P}}$ define a trace is straightforward and follows from the properties of the integral. Let $g \in \mathcal{F}_B$ and observe that

$$\begin{aligned} t_{\mathbb{P}}(\tau_m(g)) &= \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \mathscr{G}(\tau_m(g))(\omega) \\ &= \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \omega(\tau_m(g)) \\ &= \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \tau^*_{-m}(\omega)(g) \\ &= \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \mathscr{G}(g)(\tau^*_{-m}(\omega)) \\ &= \int_{\Omega_B} \mathrm{d}\mathbb{P}(\omega)\, \mathscr{G}(g)(\omega) \end{aligned}$$

where in the last equality we have used the τ^*-invariance of $\mathbb{P} \in \mathrm{Mes}_{1,\tau^*}(\Omega_B)$. □

We are now in position to construct the integration theory of the magnetic algebra $\mathcal{A}_B$. Let $\mathbb{P} \in \mathrm{Mes}_{1,\tau^*}(\Omega_B)$ be an invariant measure and define the map $\mathscr{T}_{\mathbb{P}} : \mathcal{A}_B \to \mathbb{C}$ by

$$\mathscr{T}_{\mathbb{P}}(T) := t_{\mathbb{P}}(\hat{T}_{0,0}), \qquad T \in \mathcal{A}_B \tag{2.33}$$

where the $\mathcal{F}_B$-valued coefficient $\hat{T}_{0,0}$ is defined by (2.24).

Proposition 2.9.2. *The map $\mathscr{T}_{\mathbb{P}} : \mathcal{A}_B \to \mathbb{C}$ defined by (2.33) is a tracial state of the C^*-algebra $\mathcal{A}_B$. Moreover, it holds true that:*

(i) $\mathscr{T}_{\mathbb{P}}(\partial_j(T)) = 0$ for all $T \in \mathcal{C}^1(\mathcal{A}_B)$ and $j = 1, 2$;

(ii) $\mathscr{T}_{\mathbb{P}}(R\partial_j(T)) = -\mathscr{T}_{\mathbb{P}}(T\partial_j(R))$ for all $T, R \in \mathcal{C}^1(\mathcal{A}_B)$ and $j = 1, 2$.

Proof. The map $\mathscr{T}_{\mathbb{P}}$ is evidently linear (composition of linear maps) and normalized, *i.e.* $\mathscr{T}_{\mathbb{P}}(1) = 1$. The positivity follows by observing that

$$(\widehat{T^*T})_{0,0} = \int_{\mathbb{T}^2} \mathrm{d}\mu(\theta)\, \rho_\theta(T^*)\, \rho_\theta(T) \geqslant 0$$

and consequently $\mathscr{T}_{\mathbb{P}}(T^*T) = t_{\mathbb{P}}((\widehat{T^*T})_{0,0}) \geqslant 0$ since $t_{\mathbb{P}}$ is a trace state (hence positive) on $\mathcal{F}_B$. Since $\mathscr{T}_{\mathbb{P}}$ is linear and positive then it is automatically continuous [BR, Proposition 2.3.11]. To prove the cyclic property of the trace let us consider two monomials $U_j := g_j\, (S_{A_B,1})^{r_j} (S_{A_B,2})^{s_j}$ with $g_j \in \mathcal{F}_B$ and $j = 1, 2$. Observe that

$$U_1 U_2 = g_1 \tau_{(r_1,s_1)}(g_2)\, (S_{B,1})^{r_1} (S_{B,2})^{s_1} (S_{B,1})^{r_2} (S_{B,2})^{s_2}$$

where $\tau_{(r_1,s_1)}(g_2) := (S_{B,1})^{r_1} (S_{B,2})^{s_1} g_2 (S_{B,1})^{-r_1} (S_{B,2})^{-s_1}$ and by mimicking the computation of (2.23) one gets

$$(\widehat{U_1 U_2})_{0,0} = g_1\, \tau_{(r_1,s_1)}(g_2)\, (S_{B,1})^{r_1} (S_{B,2})^{s_1} (S_{B,1})^{-r_1} (S_{B,2})^{-s_1}\, \delta_{r_1,-r_2} \delta_{s_1,-s_2}\,.$$

A similar argument provides

$$(\widehat{U_2 U_1})_{0,0} \;=\; \tau_{(-r_1,-s_1)}(g_1)\; g_2\; (S_{B,1})^{-r_1}(S_{B,2})^{-s_1}(S_{B,1})^{r_1}(S_{B,2})^{s_1}\;\delta_{r_1,-r_2}\delta_{s_1,-s_2}\;.$$

An iterated application of the commutation relation (2.9) provides

$$(S_{B,1})^{r_1}(S_{B,2})^{s_1}(S_{B,1})^{-r_1}(S_{B,2})^{-s_1} \;=:\; g_{(r_1,s_1)} \;\in\; \mathcal{F}_B\;.$$

This implies

$$(\widehat{U_1 U_2})_{0,0} \;=\; g_{(r_1,s_1)}\; g_1\; \tau_{(r_1,s_1)}(g_2)\; \delta_{r_1,-r_2}\delta_{s_1,-s_2}$$

and

$$\begin{aligned}(\widehat{U_2 U_1})_{0,0} \;&=\; \tau_{(-r_1,-s_1)}(g_{(r_1,s_1)})\; \tau_{(-r_1,-s_1)}(g_1)\; g_2\; \delta_{r_1,-r_2}\delta_{s_1,-s_2}\\ &=\; \tau_{(-r_1,-s_1)}((\widehat{U_1 U_2})_{0,0})\;.\end{aligned}$$

From the invariance property of Lemma 2.9.1 it follows that

$$t_{\mathbb{P}}((\widehat{U_1 U_2})_{0,0}) \;=\; t_{\mathbb{P}}((\widehat{U_2 U_1})_{0,0})$$

and in turn

$$\mathscr{T}_{\mathbb{P}}(U_1 U_2) \;=\; \mathscr{T}_{\mathbb{P}}(U_2 U_1)$$

for all pair of monomials U_1, U_2. It turns out that $\mathscr{T}_{\mathbb{P}}$ satisfies the cyclic property of the trace on the dense subalgebra $\mathcal{A}_B^0$ of the noncommutative polynomials, and by continuity on the whole algebra $\mathcal{A}_B$. Property (i) follows from the computation at the beginning of of the proof of Lemma 2.8.1 which provides $\widehat{\partial_j(T)}_{0,0} = 0$ for $j = 1, 2$. Property (ii) follows by the application of property (i) along with the Leibniz's rule (2.27). □

The trace property of the map $\mathscr{T}_{\mathbb{P}}$ is guaranteed by the invariance property of the measure $\mathbb{P}$. The ergodicity of $\mathbb{P}$ plays a role for the physical interpretation of $\mathscr{T}_{\mathbb{P}}$. For the next result we need to introduce some notation. Let $\{\Lambda_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_0(\mathbb{Z}^2)$ be a sequence of bounded subsets of cardinality $|\Lambda_i|$. The family $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a *Følner sequence* [Gre] if: (i) it is increasing, *i.e.* $\Lambda_i \subseteq \Lambda_{i+1}$ for all $i \in \mathbb{N}$; (ii) it is exhaustive, *i.e.* $\Lambda_i \nearrow \mathbb{Z}^2$; (iii) it meets the Følner condition. *i.e.*

$$\lim_{i\to\infty} \frac{|(m+\Lambda_i)\triangle\Lambda_i|}{|\Lambda_i|} \;=\; 0\;, \qquad\quad \forall\, m \in \mathbb{Z}^2\;,$$

where $m + \Lambda_i$ is the m-translated of Λ_i and $\triangle$ is the symmetric difference.

Let $\mathbb{P} \in \mathrm{Erg}(\Omega_B)$ be an ergodic measure and $\{\Lambda_i\}_{i\in\mathbb{N}}$ a Følner sequence. The Birkhoff's Ergodic Theorem [Wal, Lemma 6.13] assures that there exists a Borelian subset $Y \subseteq \Omega_B$ such that $\mathbb{P}(Y) = 1$ and

$$t_{\mathbb{P}}(g) \;=\; \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \sum_{m\in\Lambda_i} \mathscr{G}(g)\big(\tau_m^*(\omega)\big)\;, \qquad \forall\, \omega \in Y\;,\;\; \forall\, g \in \mathcal{F}_B\;,$$

where the isomorphism $\mathscr{G}$ and the mapping $t_{\mathbb{P}}$ are defined in Lemma 2.9.1. By observing that $\mathscr{G}(g) \circ \tau_m^* = \mathscr{G}(\tau_{-m}(g))$ and recalling the definition of the trace $\mathscr{T}_{\mathbb{P}}$ given by (2.33) one gets

$$\mathscr{T}_{\mathbb{P}}(T) \;=\; \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \sum_{m\in\Lambda_i} \mathscr{G}\big(\tau_{-m}\big(\hat{T}_{0,0}\big)\big)(\omega) \;, \qquad \forall\, \omega \in Y \;,\;\; \forall\, T \in \mathcal{A}_B \;,$$

where the $\mathcal{F}_B$-valued coefficient $\hat{T}_{0,0}$ is defined by (2.24). Finally, by observing that the extraction of the $\mathcal{F}_B$-valued coefficient commutes with the translations one gets

$$\mathscr{T}_{\mathbb{P}}(T) \;=\; \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \sum_{m\in\Lambda_i} \mathscr{G}\left(\widehat{\tau_{-m}(T)}_{0,0}\right)(\omega) \;, \qquad \forall\, \omega \in Y \;,\;\; \forall\, T \in \mathcal{A}_B \;.$$

The latter formula becomes physically meaningful in the special case $\Omega_B = \{\omega_0\}$ as for the constant magnetic field (*cf.* Example 2.6.3).

Proposition 2.9.3 (Trace per unit volume)**.** *Assume that $\Omega_B = \{\omega_0\}$ and let $\mathbb{P}$ be the (ergodic) measure supported on $\{\omega_0\}$. Let $\{\Lambda_i\}_{i\in\mathbb{N}}$ be a Følner sequence and for every Λ_i let P_{Λ_i} be the associated projection defined by $(P_{\Lambda_i}\psi)(n) = \delta_{\Lambda_i}(n)\psi(n)$ for all $\psi \in \ell^2(\mathbb{Z}^2)$. Then, it holds true that*

$$\mathscr{T}_{\mathbb{P}}(T) \;:=\; \lim_{i\to\infty} \frac{1}{|\Lambda_i|} \mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}\big(P_{\Lambda_i}\; T\; P_{\Lambda_i}\big) \;, \qquad \forall\, T \in \mathcal{A}_B \;.$$

Proof. Let $\psi_m \in \ell^2(\mathbb{Z}^2)$ be the normalized vector defined by $\psi_m(n) := \delta_{n,m}$. Then, it holds true that $\imath(\hat{T}_{0,0})(\omega_0) = \langle \psi_0, T\psi_0\rangle$ for all $T \in \mathcal{A}_B$ where ω_0 can be identified with the evaluation at $0 \in \mathbb{Z}^2$. Indeed, from (2.25) one gets that $\imath(\hat{T}_{0,0})(\omega_0) = \langle \psi_0, \sigma_N(T)\psi_0\rangle$ for all $N \in \mathbb{N}$ and the continuity of the scalar product concludes the argument. Therefore after some straightforward computation one obtains

$$\begin{aligned}
\sum_{m\in\Lambda_i} \imath\left(\widehat{\tau_{-m}(T)}_{0,0}\right)(\omega_0) \;&=\; \sum_{m\in\Lambda_i} \langle \psi_0, \tau_{-m}(T)\psi_0\rangle \\
&=\; \sum_{m\in\Lambda_i} \langle \psi_m, T\psi_m\rangle \;=\; \mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}\big(P_{\Lambda_i}\; T\; P_{\Lambda_i}\big) \;,
\end{aligned}$$

and this concludes the proof ∎

Chapter 3: Magnetic interfaces, Toeplitz extensions and K-theory

In this chapter the fundamental blocks needed to study *topological currents* are introduced. This includes giving a precise definition for the Interface and Bulk algebras, together with their connection through K-theory.

3.1 Evaluation and Interface Algebras

In this section we will study a family of C^*-homomorphisms between the magnetic algebras $\mathcal{A}_{B_1}$ and $\mathcal{A}_{B_2}$ associated the different magnetic fields. This family of homomorphisms will be of central importance in the rest of the work.

Definition 3.1.1 (Evaluation homomorphisms)**.** A C^*-homomorphism $\mathrm{ev} : \mathcal{A}_{B_1} \to \mathcal{A}_{B_2}$ such that

$$\begin{aligned} \mathrm{ev}\big(S_{B_1,1}\big) &:= S_{B_2,1} \\ \mathrm{ev}\big(S_{B_1,2}\big) &:= S_{B_2,2}. \end{aligned}$$

will be called an *evaluation map* from $\mathcal{A}_{B_1}$ to $\mathcal{A}_{B_2}$.

Remarks 3.1.2. The last definition hides tons of information, some of which is remarked here:

1. Note that nothing is said about the existence. The existence of such homomorphisms is not trivial and will be the main focus of section 3.3. To clarify the existence problem of the evaluation map, consider $b_1 \neq b_2$ and observe that because of equation (3.1)

$$\mathrm{ev}(\,\mathrm{e}^{\mathrm{i}\,b_1}\,) = \mathrm{e}^{\mathrm{i}\,b_2}\,,$$

however it is easily seen that any evaluation map must map the identity into the identity, which lead to a contradiction.

2. It follows from equation (2.9) that any evaluation map has the property that

$$\mathrm{ev}(f_{B_1}) = f_{B_2}. \tag{3.1}$$

Moreover, using equations (2.11) it will also follow that

$$\mathrm{ev}(\tau_m(f_{B_1})) = \tau_m(f_{B_2}), \quad \text{for } m \in \mathbb{Z}^2. \tag{3.2}$$

3. Evaluation maps are gauge-invariant in the sense that, if $\mathrm{ev} : \mathcal{A}_{B_1} \to \mathcal{A}_{B_2}$ is a evaluation map and $\Gamma : \mathcal{A}'_{B_1} \to \mathcal{A}_{B_1}$ is a gauge transformation, where $\mathcal{A}'_{B_1}$ is a magnetic C^*-algebra associated to the magnetic field B in a different gauge than $\mathcal{A}_{B_1}$, it follows that

$$\begin{aligned} \mathrm{ev}\ \big(\Gamma\big(S'_{B_1,1}\big)\big) = \mathrm{ev}\big(S_{B_1,1}\big) \ &= S_{B_2,1} \\ \mathrm{ev}\ \big(\Gamma\big(S'_{B_1,2}\big)\big) = \mathrm{ev}\big(S_{B_1,2}\big) \ &= S_{B_2,2}\ , \end{aligned}$$

proving that $\widetilde{\mathrm{ev}} := \mathrm{ev} \circ \Gamma$ is an evaluation map connecting $\mathcal{A}'_{B_1}$ and $\mathcal{A}_{B_2}$. $\triangleleft$

Equation (3.2) tel us that there must be certain correspondence between $\mathcal{F}_{B_1}$ and $\mathcal{F}_{B_2}$ as long as an evaluation map exists. This notion is precisely written in the next lemma.

Lemma 3.1.3. *Let* $\mathrm{ev} : \mathcal{A}_{B_1} \to \mathcal{A}_{B_2}$ *be an evaluation homomorphism. Then*

$$\mathrm{ev}|_{\mathcal{F}_{B_1}} : \mathcal{F}_{B_1} \longrightarrow \mathcal{F}_{B_2}$$

restricts to a surjective C^**-homomorphism.*

Proof. Let $\mathcal{F}^0_{B_j} \subseteq \mathcal{F}_{B_j}$, $j = 1, 2$, be the dense subalgebra generated by the finite polynomials in the generators $\tau_\gamma(f_{B_j})$. From (3.2) it follows that $\mathcal{F}^0_{B_2} \subseteq \mathrm{ev}(\mathcal{F}^0_{B_1}) \subset \mathcal{F}_{B_2}$. Since by assumption ev is a C^*-homomorphism and $\mathcal{F}^0_{B_1}$ is dense, one gets that

$$\mathcal{F}^0_{B_2} \subseteq \mathrm{ev}(\mathcal{F}^0_{B_1}) \subseteq \mathrm{ev}(\mathcal{F}_{B_1}) \subseteq \overline{\mathrm{ev}\big(\mathcal{F}^0_{B_1}\big)} \subseteq \mathcal{F}_{B_2}.$$

From the chain of inclusions above it follows that $\mathrm{ev}(\mathcal{F}_{B_1})$ is a C^*-subalgebra of $\mathcal{F}_{B_2}$ [BR, Proposition 2.3.1] which contains the dense set $\mathcal{F}^0_{B_2}$. This implies that: (i) the restriction $\mathrm{ev}|_{\mathcal{F}_{B_1}}$ is well defined, and (ii) $\mathrm{ev}(\mathcal{F}_{B_1}) = \mathcal{F}_{B_2}$, *i.e.* the surjectivity of the map. $\square$

Since $\mathrm{ev}|_{\mathcal{F}_{B_1}}$ is a well defined C^*-homomorphism between $\mathcal{F}_{B_1}$ and $\mathcal{F}_{B_2}$ it follows that $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_{B_1}}) \subset \mathcal{F}_{B_1}$ is a closed (two-sided) ideal.

Definition 3.1.4 (Interface algebra)**.** Let $\mathrm{ev} : \mathcal{A}_{B_1} \to \mathcal{A}_{B_2}$ be an evaluation homomorphism. The *interface algebra* $\mathcal{I} \subset \mathcal{A}_{B_1}$ is the closed two-sided ideal generated in $\mathcal{A}_{B_1}$ by $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_{B_1}})$.

In other words $\mathcal{I}$ coincides with the C^*-subalgebra of $\mathcal{A}_{B_1}$ generated by elements of the type TgR with $g \in \mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_{B_1}})$ and $T, R \in \mathcal{A}_{A_{B_1}}$. This justifies the following notation

$$\mathcal{I} := \mathcal{A}_{B_1}\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_{B_1}})\mathcal{A}_{B_1}.$$

Let $\mathcal{K}(\mathcal{H})$ be the closed ideal of compact operators on the separable Hilbert space $\mathcal{H}$. It is worth recalling that $\mathcal{K}(\mathcal{H})$ is an *essential ideal* in $\mathcal{B}(\mathcal{H})$ [Mur, Example 3.1.2] and $\mathcal{C}(\mathbb{T})$ be the C^*-algebra of the continuous function on the Torus.

Definition 3.1.5 (Localized and straight-line interfaces)**.** Let $\mathcal{I}$ be the interface algebra associated to a given evaluation homomorphism. We will say that the interface is *localized* if $\mathcal{I} = \mathcal{K}(\ell^2(\mathbb{Z}^2))$. The interface will be called *straight-line* if $\mathcal{I} \simeq \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ up to a unitary transformation.

The motivation for the terminology introduced in Definition 3.1.5 will be clarified partially in the next example and Section 4.1.

Example 3.1.6 (Interface algebra for a localized magnetic field)**.** According to the notations and results introduced in the examples 2.1.4, Example 2.2.5 and Example 2.4.5, let $\mathcal{A}_\Lambda$ be the magnetic algebra associated to a localized magnetic field B_Λ of strenght $b \notin 2\pi\,\mathrm{i}\,\mathbb{Z}$ (in order to make the algebra $\mathcal{A}_\Lambda$ not trivial) and $\mathcal{A}_0$ be the algebra associated to a constant zero magnetic field (see Example 2.3.3). Once more and for the sake of simplicity, let $\Lambda = \{n\}$ be a singleton in $\mathbb{Z}^2$. The map defined by $\mathrm{ev}(S_{\Lambda,j}) = S_j$, where S_j are the canonical shift operators, extends to a C^*-homomorphism $\mathrm{ev} : \mathcal{A}_\Lambda \to \mathcal{A}_0$ (see the proof of Proposition 3.3.2). Therefore, from (3.2) one has that $\mathrm{ev}(f_\Lambda) = 1$, and in turn

$$\mathrm{ev}(\delta_n) = \mathrm{ev}\left((\,\mathrm{e}^{ib}\, - 1)^{-1}(f_\Lambda - 1)\right) = 0\,. \tag{3.3}$$

This shows that $\delta_n \in \mathcal{I}$ is an element of the interface algebra. Moreover, by acting with the magnetic translations $S_{\Lambda,j}$ one obtains that also $\delta_m \in \mathcal{I}$ for every $m \in \mathbb{Z}^2$. Now by multiplying by the appopiate constants it follows that the elements of the form $\delta_m(S_1)^r(S_2)^s$, where $r, s \in \mathbb{Z}$, are elements of the interface algebra for the localized magnetic field $\mathcal{I}$, and since those elements generate $\mathcal{K}(\ell^2(\mathbb{Z}^2))$ we have the inclusion $\mathcal{K}(\ell^2(\mathbb{Z}^2)) \subset \mathcal{I}$. The other inclusion is also true and its proof follows pretty much the same steps made to prove Proposition 4.2.4, so just the main ideas are written here: First, one can prove that $\mathrm{ev}|_{\mathcal{F}^0_\Lambda}$ acts by taking limits of functions; then a density argument together with the continuity of the evaluation map imply that $\mathrm{ev}|_{\mathcal{F}_\Lambda}$ acts also by taking limits (which is reasonable since $\mathcal{F}_\Lambda = c(\mathbb{Z}^2)$ as proved in 2.4.5); then if an element is in $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\Lambda})$ it must be a function in $c_0(\mathbb{Z}^2) = \{g \in \ell^\infty \mid \lim_{||n||\to\infty} g(n) = 0\}$; finally one can check that since $c_0(\mathbb{Z}^2) \subset \mathcal{K}(\ell^2(\mathbb{Z}^2))$ (regarding the functions in $c_0(\mathbb{Z}^2)$ as multiplication operators), it follows that the ideal generated by $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\Lambda})$ in $\mathcal{A}_\Lambda$ must also be contained in $\mathcal{K}(\ell^2(\mathbb{Z}^2))$, proving that $\mathcal{I} \subset \mathcal{K}(\ell^2(\mathbb{Z}^2))$.
If $|\Lambda| > 1$ it is easy to see that for every $m \in \Lambda$ we have that $\delta_m = \delta_m\delta_\Lambda$. It follows from equation (3.3) that

$$\mathrm{ev}(\delta_m) = \mathrm{ev}(\delta_m\delta_\Lambda) = \mathrm{ev}(\delta_m)\mathrm{ev}(\delta_\Lambda) = 0,$$

so one can repeat the exact same proof as in the case $|\Lambda| = 1$ to prove that $\mathcal{I} = \mathcal{K}(\ell^2(\mathbb{Z}^2))$. In summary, a localized magnetic field always provides a localized interface in the sense of Definition 3.1.5. $\triangleleft$

3.2 Toeplitz extensions by an interface

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ three C^*-algebras fitting into the short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \longrightarrow 0. \tag{3.4}$$

In such a case we will say that $\mathcal{B}$ is the *Toeplitz extension* of $\mathcal{A}$ by $\mathcal{C}$. For a simple and complete review of the theory of extension of C^*-algebras we refer to [Weg, Chapter 3]. It is worth pointing out that we are proposing the use of the expression Toeplitz extension in a extremely generalized sense. Indeed the original notion of Toeplitz extension refers to a very specific example of extension of C^*-algebras (see Appendix B for an applied example or [Mur, Section 3.5], [Weg, Exercise 3.F] for more general structure). However, such a generalized use of the name Toeplitz extension is becoming standard in condensed matter problems (see *e. g.* [AM]) and we decided to adhere to this use.

The main aim of this section is to show that an evaluation homomorphism automatically provides a Toeplitz extension.

Theorem 3.2.1. *Every evaluation homomorphism* $\mathrm{ev} : \mathcal{A}_{B_1} \to \mathcal{A}_{B_2}$ *fits into the short exact sequence*

$$0 \longrightarrow \mathcal{I} \xrightarrow{\imath} \mathcal{A}_{B_1} \xrightarrow{\mathrm{ev}} \mathcal{A}_{B_2} \longrightarrow 0 \tag{3.5}$$

where $\mathcal{I}$ *is the related interface algebra and* $\imath$ *is the (natural) inclusion map.*

Proof. The map $\imath$ is injective by definition. Therefore, to complete the proof we need to prove that the evaluation homomorphism is surjective and that $\mathrm{Ker}(\mathrm{ev}) = \mathcal{I}$. The surjectivity is a consequence of Lemma 3.1.3 which ensures $\mathrm{ev}(\mathcal{A}^0_{B_1}) = \mathcal{A}^0_{B_2}$ (or equivalently $\mathrm{ev}(\mathcal{A}^\infty_{B_1}) = \mathcal{A}^\infty_{B_2}$). Then, as in the proof of Lemma 3.1.3, the chain of inclusions

$$\mathcal{A}^0_{B_2} = \mathrm{ev}(\mathcal{A}^0_{B_1}) \subseteq \mathrm{ev}(\mathcal{A}_{B_1}) \subseteq \overline{\mathrm{ev}(\mathcal{A}^0_{B_1})} \subseteq \mathcal{A}_{B_2}$$

implies $\mathrm{ev}(\mathcal{A}_{B_1}) = \mathcal{A}_{B_2}$. The description of the kernel of ev is a consequence of Corollary 2.7.4 which guarantees that $T \in \mathrm{Ker}(\mathrm{ev})$ if and only if all the $\mathcal{F}_{B_2}$-coefficients of $\mathrm{ev}(T)$ are zero. From the definition (2.24), the linearity of the integral and the fact that the evaluation homomorphism ev commutes (by construction) with the family of automorphisms ρ_θ, it follows that $\widehat{\mathrm{ev}(T)}_{r,s} = \mathrm{ev}(\widehat{T}_{r,s})$. Then $T \in \mathrm{Ker}(\mathrm{ev})$ if and only if $\mathrm{ev}(\widehat{T}_{r,s}) = 0$ for all $(r, s) \in \mathbb{Z}^2$. This implies that $T \in \mathrm{Ker}(\mathrm{ev})$ if and only if $\sigma_N(T) \in \mathcal{I}$ for every $N \in \mathbb{N}$, where $\sigma_N(T)$ is the Cesàro mean (2.25) which converges to T. Since $\mathcal{I}$ is a closed ideal it follows that $T \in \mathrm{Ker}(\mathrm{ev})$ if and only if $T \in \mathcal{I}$. ■

By using the terminology introduced at the beginning of this section we will say that $\mathcal{A}_{B_1}$ is the Toeplitz extension of the *interface* $\mathcal{I}$ by $\mathcal{A}_{B_2}$.

Corollary 3.2.2. $1 \in \mathcal{I}$ *if and only if* $\mathrm{ev} = 0$.

Proof. Since $\mathcal{I}$ is an ideal one has that $1 \in \mathcal{I}$ if and only if $\mathcal{I} = \mathcal{A}_{B_1}$. □

Example 3.2.3 (Localized interface and discrete spectrum)**.** In the case of a localized interface $\mathcal{I} = \mathcal{K}(\ell^2(\mathbb{Z}^2))$ (like in Example 3.1.6) the short exact sequence (3.5) provides the isomorphism

$$\mathcal{A}_{A_{B_2}} \simeq \mathcal{A}_{A_{B_1}}/\mathcal{K}(\ell^2(\mathbb{Z}^2)) \, .$$

This means that the elements of $\mathcal{A}_{A_{B_1}}$ are compact perturbations of elements of the (bulk) algebra $\mathcal{A}_{A_{B_2}}$. Since $\mathcal{K}(\mathcal{H})$ is an *essential ideal* in $\mathcal{B}(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$ (see eg. [Mur, Example 3.1.2]), $\mathcal{K}(\ell^2(\mathbb{Z}^2))$ is an essential ideal of $\mathcal{A}_{A_{B_1}}$ and it follows that the short exact sequence (3.5) is *essential* [Weg, Definition 3.2.1]. The isomorphism above is useful to analyze the spectrum of elements $\mathfrak{a} \in \mathcal{A}_{A_{B_1}}$. In fact it holds true that the evaluation $\mathrm{ev}(\mathfrak{a}) \in \mathcal{A}_{A_{B_2}}$ contains the information about the essential spectrum $\sigma_{\mathrm{ess}}(\mathfrak{a})$ while the discrete spectrum $\sigma_{\mathrm{d}}(\mathfrak{a})$ is generated by the part of $\mathfrak{a}$ which belongs to the interface. Usually, the discrete spectrum of $\mathfrak{a}$ is located in the gaps of the spectrum of $\mathrm{ev}(\mathfrak{a})$. $\triangleleft$

Remark 3.2.4 (Split exact squences of C^*-algebras)**.** Before continuing, let us briefly discuss the three ways a short exact sequence can split[1]. Consider the exact sequence (3.4) and suppose there is a function $\gamma : \mathcal{C} \to \mathcal{B}$ such that $\beta \circ \gamma = \mathrm{Id}_{\mathcal{C}}$. The three cases are the following:

1. Since C^*-algebras are also $\mathbb{C}$-modules, and actually vector spaces over $\mathbb{C}$, the first way in that the C^*-algebras could split would be γ being a linear transformation. In such case one would have that both $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$ are vector spaces and

   $$\mathcal{B} = \alpha(\mathcal{A}) \oplus \gamma(\mathcal{C})$$

   is the direct sum of the vector spaces $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$.

2. If now γ is actually a $*$-homomorphism, then both $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$ would be $*$-subalgebras of $\mathcal{B}$ and

   $$\mathcal{B} = \alpha(\mathcal{A}) + \gamma(\mathcal{C})$$

   is the Banach sum space between the $*$-algebras $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$.

3. Finally, if γ is a $*$-homomorphism and it occurs that $\gamma(\mathcal{C})$ is an ideal in $\mathcal{B}$ it holds that

   $$\mathcal{B} = \alpha(\mathcal{A}) \oplus \gamma(\mathcal{C})$$

   is the direct sum space of the $*$-algebras $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$. Let us remark that the main difference between this case and the last one is the orthogonality between elements of $\alpha(\mathcal{A})$ and $\gamma(\mathcal{C})$.

[1]This discussion is based on [Weg, Chapter 3.1]

In any case, the sequence (3.4) is said to be *split exact* but since they have different properties it is important to also say if it splits as a sequence of $\mathbb{C}$-modules or $*$-algebras. Indeed, let us recall that if the second or third cases occur, then

$$0 \longrightarrow K_j(\mathcal{A}) \xrightarrow{\alpha_*} K_j(\mathcal{B}) \xrightarrow{\beta_*} K_j(\mathcal{C}) \longrightarrow 0, \quad \text{for } j = 1, 2,$$

(see [Weg, Corollary 8.2.2]), but this does not hold in general if the splitting function γ is just linear (a counter example for this is provided by the Toeplitz extension associated to the Iwatsuka algebra discussed in Chapter 4). $\triangleleft$

Example 3.2.5 (Toeplitz extension for a localized magnetic field)**.** From Example 3.1.6 one infers that a localized magnetic field provides the Toeplitz extension

$$0 \longrightarrow \mathcal{K}\big(\ell^2(\mathbb{Z}^2)\big) \xrightarrow{\imath} \mathcal{A}_\Lambda \xrightarrow{\text{ev}} \mathcal{A}_0 \longrightarrow 0. \tag{3.6}$$

Let us claim that the short exact sequence (3.6) splits as a sequence of C^*-algebras. In [DS, Proposition 2] it is proved that when the algebra $\mathcal{A}_\Lambda$ is generated by the magnetic potentials in the *Aharonov–Bohm gauge*, it holds true that the difference

$$S'_{\Lambda,j} - S_j = (Y'_{A_\Lambda,j} - 1)S_j \ \in \ \mathcal{K}(\ell^2(\mathbb{Z}^2)), \quad \text{for } j = 1, 2, \tag{3.7}$$

where the apostrophe has been placed to state the difference with the gauge used until now defined in 2.1.4. Also note that since the interface algebra $\mathcal{I}_\Lambda = \mathcal{K}(\ell^2(\mathbb{Z}^2))$ in at least one gauge, it follows immediately that $\mathcal{I}_\Lambda = \mathcal{K}(\ell^2(\mathbb{Z}^2))$ for every gauge, since the algebra of compact operators is invariant through the transformation $T \mapsto \mathrm{e}^{-iG(N)}\, T\, \mathrm{e}^{-iG(N)}$ for every gauge function G and the evaluation map transforms into other evaluation map when the gauge is changed (see 3.1.2). The reader could find useful to see the next exact sequence to clarify the gauge equivalence of the problem:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}\big(\ell^2(\mathbb{Z}^2)\big) & \xrightarrow{\imath} & \mathcal{A}_\Lambda & \xrightarrow{\text{ev}} & \mathcal{A}_0 & \longrightarrow & 0 \\ & & \Gamma\big\downarrow & & \Gamma\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathcal{K}\big(\ell^2(\mathbb{Z}^2)\big) & \xrightarrow{\imath} & \mathcal{A}'_\Lambda & \xrightarrow{\tilde{\text{ev}}} & \mathcal{A}_0 & \longrightarrow & 0, \end{array} \tag{3.8}$$

where $\mathcal{A}'_\Lambda$ is the magnetic C^*-algebra associated to the localized magnetic field in the Aharonov-Bohm gauge, $\Gamma : \mathcal{A}'_\Lambda \to \mathcal{A}_\Lambda$ is the gauge transformation and $\tilde{\text{ev}} := \Gamma^{-1} \circ \text{ev}$ is the evaluation map from $\mathcal{A}'_\Lambda$ into $\mathcal{A}_0$.

Now observe from equation (3.7) that $S_j \in \mathcal{A}_\Lambda$ (in the Aharonov–Bohm gauge) for $j = 1, 2$, and also that $\text{ev}(S_j) = \text{ev}(S'_{\Lambda,j}) = S_j$, leading us to consider the trivial lift $\gamma = \text{Id}_{\mathcal{A}_0}$. It is of course trivial that γ is a $*$-homomorphism, so we conclude that the sequence (3.6) splits as a sequence of C^*-algebras.

Finally let us note that $\gamma(\mathcal{A}_0) = \mathcal{A}_0$ can not be an ideal in $\mathcal{A}'_\Lambda$ because it is unital, and in such case, one would have that $\mathcal{A}_0 \simeq \mathcal{A}_\Lambda$ (with equality in the Aharonov–Bohm gauge) which is not true since $\mathcal{A}_0$ is commutative but $\mathcal{A}_\Lambda$ is not. In summary

$$\mathcal{A}_\Lambda \simeq \mathcal{I}_\Lambda + \mathcal{A}_0 \simeq \mathcal{K}(\mathcal{H}) + C(\mathbb{T}^2),$$

where $\mathcal{H}$ is an arbitrary separable Hilbert space, the sums are Banach sums of $*$-algebras, the first isomorphism is an equality in the Aharonov-Bohn gauge and the sum is not direct (in the sense of C^*-algebras). $\triangleleft$

Example 3.2.5 is somehow special since the Toeplitz extensions (3.5) considered in this work, which connect the Iwatsuka C^*-algebra with the orthogonal sum of two noncommutative torus, will be not split exact in general as a sequence of C^*-algebras. Nevertheless, it will be possible to find a linear lift. This will be developed in 4.3.

3.3 Existence of Toeplitz Extensions and Dynamics

In the previous section we described the consequences of having an evaluation homomorphism between two magnetic algebras. In this section we will analyze the relation between the existence of evaluation homomorphisms and the dynamical properties of the dynamical systems generated by the magnetic hulls. As a result we will provide a generalized definition of *magnetic multi-interface* based on purely dynamical properties of the magnetic hulls.

Let $(\Omega_{B_1}, \tau^*, \mathbb{Z}^2)$ and $(\Omega_{B_2}, \tau^*, \mathbb{Z}^2)$ be the two topological dynamical systems associated to the magnetic fields B_1 and B_2. An *equivariant* map from Ω_{B_2} to Ω_{B_1} is a continuous function $\phi^* : \Omega_{B_2} \to \Omega_{B_1}$ such that

$$\phi^* \circ \tau^*_\gamma = {\tau^*}_\gamma \circ \phi^*, \qquad \text{for all } \gamma \in \mathbb{Z}^2.$$

Proposition 3.3.1. *Every evaluation homomorphism* $\mathrm{ev} : \mathcal{A}_{A_{B_1}} \to \mathcal{A}_{A_{B_2}}$ *defines an injective closed equivariant map* $\phi^* : \Omega_{B_2} \hookrightarrow \Omega_{B_1}$.

Proof. Let us consider the Gelfand trasforms $\mathscr{G}_j : \mathcal{F}_{B_j} \to \mathcal{C}(\Omega_{B_j})$, with $j = 1, 2$. The map $\phi : \mathcal{C}(\Omega_{B_1}) \to \mathcal{C}(\Omega_{B_2})$ defined by

$$\phi := \mathscr{G}_2 \circ \mathrm{ev}|_{\mathcal{F}_{B_1}} \circ \mathscr{G}_1^{-1}$$

is the composition of surjective C^*-homomorphisms, hence it is a surjective C^*-homomorphism. By duality, ϕ induces a continuous map $\phi^* : \Omega_{B_2} \to \Omega_{B_1}$ defined by

$$\phi^*(\omega) := \omega \circ \phi.$$

Indeed, if $\omega \in \Omega_2$ is meant as a character of $\mathcal{C}(\Omega_{B_2})$, then $\phi^*(\omega)$ is a character of $\mathcal{C}(\Omega_{B_1})$, hence a point of Ω_1. The surjectivity of ϕ implies the injectivity of ϕ^*. Indeed, $\phi^*(\omega_1) = \phi^*(\omega_2)$ implies that $\omega_1(\hat{g}) = \omega_2(\hat{g})$ for all $\hat{g} \in \mathcal{C}(\Omega_{B_2})$ which is exactly $\omega_1 = \omega_2$. Finally ϕ^* is closed in view of the *Closed Map Lemma* [Lee, Lemma 4.25] since Ω_{B_1} and Ω_{B_2} are both compact Hausdorff spaces. $\square$

Let us recall that a continuous closed injection between topological spaces is usually called a *(topological) embedding*. Let $\phi^* : \Omega_{B_2} \hookrightarrow \Omega_{B_1}$ be the equivariant embedding of Proposition (3.3.1). The subset $\Omega_* := \phi^*(\Omega_{B_2})$ is evidently a closed invariant subset of Ω_{B_1} and $(\Omega_*, \tau^*, \mathbb{Z}^2)$ becomes a dynamical subsystem of $(\Omega_{B_1}, \tau^*, \mathbb{Z}^2)$. Moreover

$$\Omega_* = \phi^*\left(\overline{\mathrm{Orb}(\omega_0)}\right) = \overline{\mathrm{Orb}(\omega_*)}$$

where $\omega_* := \phi^*(\omega_0)$ and $\omega_0 \in \Omega_{B_2}$ is the evaluation at 0. In conclusion, Proposition (3.3.1) states that every evaluation homomorphism identifies (up to isomorphisms) a dynamical subsystem of the initial magnetic hull. However, in view of Proposition 2.6.1, the only possibilities for a closed and invariant subset Ω_* are $\Omega_* \subseteq \partial\Omega_{B_1}$ or $\Omega_* = \Omega_{B_1}$. The latter circumstance corresponds to the case of ϕ^* being an isomorphism and, as a consequence of Proposition 3.3.1 and the short exact sequence of Theorem 3.2.1, this is equivalent to the isomorphism $\mathcal{A}_{B_1} \simeq \mathcal{A}_{B_2}$. This case will be called trivial as opposite to the non trivial case in which ϕ^* defines a proper dynamical subsystem of the initial dynamical system. The next result provides a sort of converse of Proposition (3.3.1).

Proposition 3.3.2. *Let $\mathcal{A}_B$ be a magnetic algebra and $(\Omega_B, \tau^*, \mathbb{Z}^2)$ the topological dynamical system associated to its magnetic hull. Let $\Omega_* \subseteq \partial\Omega_B$ be a proper invariant closed subset. Assume that $\Omega_* = \overline{\mathrm{Orb}(\omega_*)}$ for some $\omega_* \in \partial\Omega_B$. Then, there is a magnetic algebra $\mathcal{A}_{B_*}$ with magnetic hull Ω_* and an evaluation homomorphism* $\mathrm{ev} : \mathcal{A}_B \to \mathcal{A}_{B_*}$.

Proof. Let $\phi : \mathcal{C}(\Omega_B) \to \mathcal{C}(\Omega_*)$ be the surjective *restriction* C^*-homomorphism defined by $\phi(\hat{g}) := \hat{g}|_{\Omega_*}$ for all $g \in \mathcal{C}(\Omega_B)$. Let $\widehat{f}_B$ be the Gelfand transform of the generator f_B of $\mathcal{F}_B$ and define the function $f_{B_*} : \mathbb{Z}^2 \to \mathbb{C}$ by

$$f_{B_*}(m) := \widehat{f}_B(\tau^*_m(\omega_*)), \qquad m \in \mathbb{Z}^2.$$

The function f_{B_*} provides a magnetic flux with an associated (non unique) magnetic field $B_* : \mathbb{Z}^2 \to \mathbb{R}$. Let A_{B_*} be a suitable vector potential for B_* and $\mathcal{A}_{A_{B_*}}$ the associated magnetic algebra. The surjective C^*-homomorphism ϕ and the Gelfand isomorphism provide a surjective C^*-homomorphism $\widetilde{\mathrm{ev}} : \mathcal{F}_B \to \mathcal{F}_{B_*}$ characterized by $\widetilde{\mathrm{ev}}(f_B) = f_{B_*}$. It turns out that the map $\mathrm{ev} : \mathcal{A}_B^\infty \to \mathcal{A}_{B_*}^\infty$ defined by

$$\mathrm{ev}\left(\sum_{(r,s)\in\mathbb{Z}^2} g_{r,s}(S_{B,1})^r(S_{B,2})^s\right) = \sum_{(r,s)\in\mathbb{Z}^2} \widetilde{\mathrm{ev}}(g_{r,s})(S_{B_*,1})^r(S_{B_*,2})^s$$

is a $*$-homomorphism of pre-C^*-algebras (Proposition 2.8.4). Therefore, the claim follows from [GBVF, Lemma 3.41]. $\square$

Remark 3.3.3 (Non-uniqueness of the magnetic field). The magnetic algebra $\mathcal{A}_{A_{B_*}}$ which enters in Proposition 3.3.2 is non unique for two reasons. First of all $\mathcal{A}_{B_*}$ depends of the election of a vector potential A_{B_*} for the magnetic field B_* and this involves the election of gauge. However, magnetic algebras related to different gauges are unitarily equivalent as discussed in Section 2.3. The second source of ambiguity is more subtle and is related with the determination of the magnetic field B_* from the magnetic flux f_{B_*}. Indeed, the natural candidate would be $B_* = -i\log(f_{B_*})$ but the the logarithm is not univocally defined in the complex plane. In particular, given a magnetic field B_* compatible with the magnetic flux f_{B_*} and a (not necessarily bounded) function $\zeta : \mathbb{Z}^2 \to \mathbb{Z}$ one gets that $B'_* := B_* + 2\pi\zeta$ provides the same magnetic flux. A way to solve this ambiguity is to fix the convention that $B_* := \mathrm{Arg}(f_{B_*}) \in [0, 2\pi)$ is given by the *principal argument* of the flux f_{B_*}. This correspond to a sort of *minimal growth* assumption for the magnetic field at infinity and we will use this convention in the rest of this work. $\triangleleft$

We are now in position to introduce a key definition for this work.

Definition 3.3.4 (Magnetic multi-interface). A system subjected to a magnetic field $B : \mathbb{Z}^2 \to [0, 2\pi)$ and with the boundary of the magnetic hull given by a finite collection of invariant points

$$\partial\Omega_B = \{\omega_{*,1}, \ldots, \omega_{*,N+1}\}$$

will be called a N*-interface* magnetic system. In this case the associated Toeplitz extension is given by

$$0 \longrightarrow \mathcal{I} \xrightarrow{\imath} \mathcal{A}_B \xrightarrow{\mathrm{ev}} \mathcal{A}_{\mathrm{bulk}} \longrightarrow 0 \tag{3.9}$$

where $\mathcal{A}_B$ is any magnetic algebra associated to the magnetic field B and the *bulk algebra*

$$\mathcal{A}_{\mathrm{bulk}} := \mathcal{A}_{b_1} \oplus \ldots \oplus \mathcal{A}_{b_{N+1}} \tag{3.10}$$

is given by the orthogonal direct sum of $N+1$ magnetic algebras of constant magnetic fields of strengths given by

$$b_j := \mathrm{Arg}(\widehat{f}_B(\omega_{*,j})), \qquad \text{for every } j = 1, \ldots, N+1,$$

where $\widehat{f}_B$ is the Gelfand transform of the flux function f_B as described in the proof of Proposition 3.3.2. Finally the evaluation map and the interface algebra $\mathcal{I}$ are completely specified by

$$\mathrm{ev}(f_B) := \left(\mathrm{e}^{ib_1}, \ldots, \mathrm{e}^{ib_{N+1}}\right)$$

as discussed in Section 3.1.

As showed in Example 2.6.5 and Example 3.1.6, a localized magnetic field provides an example of a magnetic interface with order $N = 0$. On the other hand, Example 2.6.4 shows the Iwatsuka magnetic field provides an example of magnetic interface of order $N = 1$. The case of the Iwatsuka magnetic field will be discussed extensively in Section 4.

3.4 K-theory of Magnetic C^*-Algebras

In this section we will discuss some aspects of the K-theory of magnetic interfaces. There is a large literature concerning the K-theory for C^*-algebras. We will refer to the classic monographs [Mur, Weg, Bla, GBVF] as well as [PS] for a stronger connection with the condensed matter problems.

3.4.1 Six-Term Exact Sequence Regarding the Crossed Product Structure

The study of the K-theory associated to C^*-algebras that are crossed product algebras by $\mathbb{Z}$ was studied in [PV2] and the main results are summarized in Appendix B. Since magnetic C^*-algebras are indeed iterated crossed product algebras by $\mathbb{Z}$ one is allowed to use these techniques in order to compute their K-groups.

More precisely, let $\mathcal{A}_B$ be a magnetic C^*-algebra and $\mathcal{A}_{B,j}, \alpha_j$ and β_j as defined in 2.5. Since $\mathcal{A}_{B_j} = \mathcal{F}_B \rtimes_{\alpha_j} \mathbb{Z}$, it follows that we can build the Pimsner-Voiculescu exact sequence (see Appendix B for notations)

$$\begin{array}{ccccc} K_0(\mathcal{F}_B) & \xrightarrow{1-\alpha_{j,*}} & K_0(\mathcal{F}_B) & \xrightarrow{\imath_*} & K_0(\mathcal{A}_{B,j}) \\ \delta\Big\uparrow & & & & \Big\downarrow\epsilon \\ K_1(\mathcal{A}_{B,j}) & \xleftarrow[\imath_*]{} & K_1(\mathcal{F}_B) & \xleftarrow[1-\alpha_{j,*}]{} & K_1(\mathcal{F}_B). \end{array} \tag{3.11}$$

In some cases, as for the Iwatsuka C^*-algebra, the last six-term exact sequence is builded with maps that can be explicitely computed. This allows to compute the groups $K_0(\mathcal{A}_{B,j})$ and $K_1(\mathcal{A}_{B,j})$. If this is the case for at least one choice of $j \in \{1,2\}$, one can recall that $\mathcal{A}_B = \mathcal{A}_{B,j} \rtimes_{\beta_k} \mathbb{Z}$ for k such that $\{j,k\} = \{1,2\}$, and analogously the Pimsner-Voiculescu six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{A}_{B,j}) & \xrightarrow{1-\beta_{k,*}} & K_0(\mathcal{A}_{B,j}) & \xrightarrow{\imath_*} & K_0(\mathcal{A}_B) \\ \delta\Big\uparrow & & & & \Big\downarrow\epsilon \\ K_1(\mathcal{A}_B) & \xleftarrow[\imath_*]{} & K_1(\mathcal{A}_{B,j}) & \xleftarrow[1-\beta_{k,*}]{} & K_1(\mathcal{A}_{B,j}). \end{array} \tag{3.12}$$

can be used to compute the actual K-theory of the magnetic C^*-algebra $\mathcal{A}_B$.

The main goal of this work is finding a correspondence between the bulk and interface algebras. This correspondance must be understood in a K-theoretic level, and because of that it is important to have as much information as possible regarding the K-groups of the algebras in the exact sequence (3.9). When it is about the Iwatsuka algebra, the Toeplitz extension can be used to determine the K-groups associated to such extension (see sections 3.4.2 and 4.4), however the methods developed in this section are way easier to apply and capture the K-groups of the magnetic C^*-algebra $\mathcal{A}_{\rm I}$ with not much effort (see 4.4.1).

3.4.2 Six-Term Exact Sequence Associated to the Toeplitz Extension

Let us recall that to each Toeplitz extension of type (3.4) there is an associated *six term sequence* in K-theory [Weg, Theorem 9.3.2]. Therefore, there is a six term sequence for every magnetic Toeplitz extension of type (3.5) or (3.9). We will focus here on the latter case concerning a magnetic multi-interface.

From the exact sequence (3.9) one obtains the six term sequence

$$
\begin{tikzcd}
K_0(\mathcal{I}) \arrow[r, "\imath_*"] & K_0(\mathcal{A}_B) \arrow[r, "\mathrm{ev}_*"] & K_0(\mathcal{A}_{\rm bulk}) \arrow[d, "\exp"] \\
K_1(\mathcal{A}_{\rm bulk}) \arrow[u, "\mathrm{ind}"] & K_1(\mathcal{A}_B) \arrow[l, "\mathrm{ev}_*"] & K_1(\mathcal{I}) \arrow[l, "\imath_*"]
\end{tikzcd}
\tag{3.13}
$$

where the canonical maps ind and exp are called *index map* and *exponential map* respectively. The role of the six term sequence (3.13) is twofold: first of all it allows to reconstruct the K-theory of $\mathcal{A}_B$ from the knowledge of the K-theory of $\mathcal{I}$ and $\mathcal{A}_{\rm bulk}$; secondly it defines how the K-theory of $\mathcal{A}_B$ intertwines the K-theories of $\mathcal{I}$ and $\mathcal{A}_{\rm bulk}$ through the maps ind and exp. The latter aspect is known in condensed matter as *bulk-boundary correspondence* [PS].

The K-theory of the bulk algebra $\mathcal{A}_{\rm bulk}$ can be easily computed. Indeed the $\mathcal{A}_{\rm bulk}$ is an orthogonal direct sum of noncommutative tori (see Example 2.3.3) and the K-theory of the noncommutative torus is well-known (*cf.* Appendix D).

Proposition 3.4.1. *Let $\mathcal{A}_{\rm bulk}$ be the bulk algebra* (3.10) *of an N-interface magnetic system. Then*

$$
K_0(\mathcal{A}_{\rm bulk}) = \bigoplus_{j=1}^{N+1} K_0(\mathcal{A}_{b_j}) \simeq \bigoplus_{j=1}^{N+1} \mathbb{Z}^2,
$$
$$
K_1(\mathcal{A}_{\rm bulk}) = \bigoplus_{j=1}^{N+1} K_1(\mathcal{A}_{b_j}) \simeq \bigoplus_{j=1}^{N+1} \mathbb{Z}^2.
$$

The isomorphisms above are given by the K-theory of the noncommutative tori

$$K_0(\mathcal{A}_{b_j}) = \mathbb{Z}[1] \oplus \mathbb{Z}[P_{\theta_j}] \simeq \mathbb{Z}^2,$$
$$K_1(\mathcal{A}_{b_j}) = \mathbb{Z}[S_{b_j,1}] \oplus \mathbb{Z}[S_{b_j,2}] \simeq \mathbb{Z}^2,$$

where $S_{b_j,1}$ and $S_{b_j,2}$ are the magnetic translations which generates $\mathcal{A}_{b_j}$ and the projection P_{θ_j} is described in Appendix D.

Proof. The first part of the claim follows from the additive property of the K-theory with respect to the orthogonal direct sum of C^*-algebras [Weg, Exercise 6.E & Example 7.1.11(4)]. The second part is a consequence of the structure of the K-theory of the noncommutative torus described in Appendix D. □

The K-theory of the interface algebra requires a preliminary observation. In fact, if one assumes that ev is *not* trivial one has that $\mathcal{I}$ is *not* unital (Corollary 3.2.2) and as a consequence $K_j(\mathcal{I})$, $j = 0, 1$, must be understood as the K-groups of the unitalization[2] $\mathcal{I}^+$ of $\mathcal{I}$ [Weg, RLL]. The main case of interest for this work is when there exists a unitary equivalence

$$\mathcal{I} \overset{\varpi}{\simeq} \mathcal{I}_0 \otimes \mathcal{K}(\mathcal{H}_{\text{red}}) \tag{3.14}$$

where $\mathcal{I}_0$ is a unital and abelian C^*-algebra and $\mathcal{K}(\mathcal{H}_{\text{red}})$ is the C^*-algebra of compact operators on the (reduced) separable Hilbert space $\mathcal{H}_{\text{red}}$. In such case one has

$$K_j(\mathcal{I}) \simeq K_j(\mathcal{I}_0), \quad j = 0, 1,$$

because of the stability property of the K-theroy [Weg, Corollary 6.2.11 & Corollary 7.1.9].

Remark 3.4.2. The ansatz (3.14) imposes a quite strong condition on the geometry of the interface. In the Iwatsuka magnetic field case this ansatz is satisfied and can be thought as saying that the interface is a straight line separating the magnetic fields. To handle more general geometries like corners, the ansatz (3.14) must be modified. A quite general discussion for other geometry imperfections is discussed in [Thi]. In such paper the half-plane and quarter-plane cases are modified in a way that the edges are allowed to have a non trivial slope together with other imperfections.

Example 3.4.3 (Six term sequence for a localized magnetic field)**.** The six-term exact sequence associated to the Toeplitz extension (3.6) for a localized magnetic field can be easily computed by observing that it is split exact as a sequence of C^*-algebras (*cf.* Remark 3.2.4 and Example 3.2.5). In this case the interface algebra has the form $\mathcal{I} \simeq \mathbb{C} \otimes \mathcal{K}$ (see Example 3.1.6) and in turn its K-theory is given by

$$K_0(\mathcal{I}) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}, \quad K_1(\mathcal{I}) \simeq K_1(\mathbb{C}) = 0.$$

[2] In our case $\mathcal{I} \subset \mathcal{B}(\ell^2(\mathbb{Z}^2))$ is a concrete C^*-algebra, therefore its unitalization is given by $\mathcal{I}^+ := \{T + \alpha 1 | T \in \mathcal{I}, \alpha \in \mathbb{C}\}$.

(*cf.* Appendix C). It follows from the split exactness of the Toeplitz extension together with the exactnes of its induced six-term exact sequence in K-theory that

$$
\begin{aligned}
K_0(\mathcal{A}_\Lambda) &= K_0(\mathcal{A}_0) \oplus \mathbb{Z}[P_{\{0\}}] \simeq \mathbb{Z}^3, \\
K_1(\mathcal{A}_\Lambda) &= K_1(\mathcal{A}_0) \simeq \mathbb{Z}^2
\end{aligned}
$$

where $\mathcal{A}_0$ is the magnetic algebra for a zero magnetic field (*cf.* Example 2.3.3) and $P_{\{0\}}$ is the projection on the fundamental site $(0,0) \in \mathbb{Z}^2$. This fact has already been proved in [DS, Theorem 12]. ◁

The case of straight-line interface (Definition 3.1.5) will be relevant in the next section. Its K-theory is described below.

Proposition 3.4.4 (K-theory for the straight-line interface)**.** *In the case of a straight-line interface $\mathcal{I} \simeq \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ the K-theory is given by*

$$
K_0(\mathcal{I}) \simeq \mathbb{Z}, \quad K_1(\mathcal{I}) \simeq \mathbb{Z}.
$$

Proof. The result follows from the stability property of K-theory along with $K_0(\mathcal{C}(\mathbb{S}^1)) \simeq \mathbb{Z}[1]$ and $K_1(\mathcal{C}(\mathbb{S}^1)) \simeq \mathbb{Z}[u]$ where $u(t) = \mathrm{e}^{\mathrm{i}\,t}$ (see Appendix C). □

3.5 Bulk and interface currents

Let $\mathcal{A}_B$ be a magnetic algebra endowed with the trace $\mathscr{T}_{\mathbb{P}}$ associated to an ergodic measure $\mathbb{P} \in \mathrm{Erg}(\Omega_B)$ as discussed in Section 2.6. Given a differentiable projection $P \in \mathcal{C}^1(\mathcal{A}_B)$, the (generalized) *transverse Hall conductance* associated to P is defined by

$$
\sigma_{B,\mathbb{P}}(P) \;:=\; \frac{e^2}{h}\,\mathrm{Ch}_{B,\mathbb{P}}(P) \tag{3.15}
$$

where e is the electron charge, $h = 2\pi\hbar$ is the Planck's constant and the dimensionless part, known as *Chern number*, is given by

$$
\mathrm{Ch}_{B,\mathbb{P}}(P) \;:=\; \mathrm{i}\,2\pi\;\mathscr{T}_{\mathbb{P}}\big(P\big[\partial_1(P),\partial_2(P)\big]\big)\,. \tag{3.16}
$$

The projection P is usually obtained as the spectral projection into a gap of a self-adjoint element (Hamiltonian) of $\mathcal{A}_B$ and represents the ground state of the system as described by the Fermi-Dirac distribution in the limit of the temperature $T = 0$ and chemical potential (Fermi energy) sited into the gap. The quantity (3.15) enters in the (microscopic) Ohm's law

$$
J_\perp \;=\; \sigma_{B,\mathbb{P}}(P)\,E \tag{3.17}
$$

which describes the transverse current density $J_\perp$ generated in the material as a response to the external electric perturbation E. The expression (3.17) is usually known as *Kubo's formula* and is obtained in the linear response approximation. There are countless derivations of the Kubo's formula (3.17) in the literature. For our aims we will refer to [BES, SB] for the case of a constant magnetic field and to [DL] for more general casees.

In the case of a constant magnetic field B of strength b there is a unique ergodic measure (*cf.* Example 2.6.3) and the associated trace, simply denoted with $\mathscr{T}$, is given by the trace per unit volume as proved in Proposition 2.9.3. Therefore, it is appropriate to rewrite equations (3.15) and (3.16) with the lighter notation

$$\sigma_b(P) \;=\; \frac{e^2}{h}\,\mathrm{Ch}_b(P) \tag{3.18}$$

In particular, the map Ch_b can be obtained from the trilinear map $\xi_b : \mathcal{C}^1(\mathcal{A}_b)^{\times 3} \to \mathbb{C}$, defined by

$$\xi_b(T_0, T_1, T_2) \;:=\; \mathrm{i}\,2\pi\; \mathscr{T}\big[T_0\big(\partial_1(T_1)\partial_2(T_2) - \partial_2(T_1)\partial_1(T_2)\big)\big]\;, \tag{3.19}$$

according to $\mathrm{Ch}_b(P) = \xi_b(P,P,P)$. Formula (3.19) is crucial in the study of the topology of the algebra $\mathcal{A}_b$ (which coincides with the noncommutative torus). In fact, as discussed in [Con, Chapter 3], [GBVF, Chapter 12] or [PS, Chapter 5] among others, it turns out that the map ξ_b is a cyclic 2-cocycle of the algebra $C^1(\mathcal{A}_b)$ and therefore defines a class $[\xi_b] \in HC^2(C^1(\mathcal{A}_b))$ in the cyclic cohomology of $C^1(\mathcal{A}_b)$. The class $[\xi_b]$ plays a special role in the canonical bilinear pairing

$$\prec\,,\,\succ\,:\; K_0(\mathcal{A}_b)\;\times\; HC^2(\mathcal{A}_b)\;\longrightarrow\;\mathbb{C}$$

between (even) K-theory and (even) cyclic cohomology, defined by

$$\big([P],[\varphi]\big)\;\longmapsto\;\prec [P],[\varphi]\succ\;:=\;(\mathrm{tr}\sharp\varphi)(P,P,P)$$

where the projection $P \in \mathcal{C}^1(\mathcal{A}_b)\otimes \mathrm{Mat}_N(\mathbb{C})$ is a suitable[3] representative of the class $[P]$, $N \in \mathbb{N}$ is a suitable integer[4] and tr denotes the trace on $\mathrm{Mat}_N(\mathbb{C})$ [PS, Theorem 5.1.4]. In the case $N = 1$, a comparison with equations (3.16) and (3.19) shows that

$$\mathrm{Ch}_b(P)\;=\;\prec [P],[\xi_b]\succ\;\in\mathbb{Z} \tag{3.20}$$

where the integrality of the pairing $[P]\mapsto\prec [P],[\xi_b]\succ$ is the celebrated *Index Theorem* for the even K-theory [Con, Section 3.3, Corollary 16]. Equation (3.20) along with (3.15) provides the quantization (in units of e^2h^{-1}) of the transverse Hall conductance for a constant magnetic field [TKNN, BES].

The conductance for the bulk algebra 3.10 can be defined (by linearity) from the case of a constant magnetic field.

[3] The suitability of the projection follows from the need of it being at least once differentiable. In fact, the pairing is initially defined on the K-Theory of the algebra $C^1(\mathcal{A}_b)$ and then extended to the K-Theory of the whole algebra $\mathcal{A}_b$, which are canonically isomorphic.

[4] For a *non-trivial* magnetic field $b(2\pi)^{-1}\in\mathbb{R}\setminus\mathbb{Z}$ it is always possible to fix $N = 1$ since the K-theory is entirely realized inside the algebra $\mathcal{A}_b$ (*cf.* Appendix D).

Definition 3.5.1 (Bulk transverse conductance)**.** Let $\mathcal{A}_{\rm bulk}$ be the bulk algebra defined in equation (3.10) and $P := (P_1, \dots, P_{N+1})$ a projection in $\mathcal{C}^1(\mathcal{A}_{\rm bulk})$. The *bulk transverse conductance* for the projection P is given by the collection

$$\sigma_{\rm bulk}(P) \;:=\; \big\{\sigma_{b_1}(P_1), \dots, \sigma_{b_{N+1}}(P_{N+1})\big\}$$

where every $\sigma_{b_j}(P_j)$ is defined by (3.18).

Let us now consider the current associated with the interface algebra $\mathcal{I}$. We will focus on the case described by the ansatz (3.14) and we will assume that the unital and abelian C^*-algebra $\mathcal{I}_0$ is endowed with a faithful (normalized) trace τ_0 and a suitable (unbounded) derivation δ_0 which meet the compatibility condition $\tau_0 \circ \delta_0 = 0$. In this way one can define a faithful lower-semicontinuous trace $\mathscr{T}_{\mathcal{I}}$ on $\mathcal{I}$ through the prescription

$$\mathscr{T}_{\mathcal{I}}(T) \;:=\; \tau_0 \otimes {\rm Tr}_{\mathcal{H}_{\rm red}}\big(\varpi(T)\big)\;, \qquad T \in \mathscr{D}_{\mathcal{I}}$$

where the ideal $\mathscr{D}_{\mathcal{I}} \subset \mathcal{I}$ is defined by $\mathscr{D}_{\mathcal{I}} := \varpi^{-1}(\mathcal{I}_0 \otimes \mathcal{L}^1(\mathcal{H}_{\rm red}))$ and $\mathcal{L}^1(\mathcal{H}_{\rm red})$ is the ideal of trace class operators on $\mathcal{H}_{\rm red}$). Similarly, one can endow $\mathcal{I}$ with the derivation $\partial_{\mathcal{I}}$ given by

$$\partial_{\mathcal{I}}(T) \;:=\; \delta_0 \otimes {\rm Id}_{\mathcal{K}}\big(\varpi(T)\big)\;, \qquad T \in \mathcal{C}^1_{\mathcal{I}}$$

where $\mathcal{C}^k_{\mathcal{I}} := \varpi^{-1}(\mathcal{C}^k(\mathcal{I}_0) \otimes \mathcal{K}(\mathcal{H}_{\rm red}))$ for every $k \in \mathbb{N}$. Therefore, such a derivation can be extended to the unitalization $\mathcal{I}^+$ by the prescription $\partial_{\mathcal{I}}(1) = 0$. With these structures one can define the map

$$W_{\mathcal{I}}(U) \;:=\; {\rm i}\,\mathscr{T}_{\mathcal{I}}\big((U^*-1)\,\partial_{\mathcal{I}}(U-1)\big) \;=\; {\rm i}\,\mathscr{T}_{\mathcal{I}}\left(U^*\partial_{\mathcal{I}}(U)\right) \qquad (3.21)$$

for every unitary operator $U \in \mathcal{I}^+$ such that $U - 1 \in \mathcal{C}^1_{\mathcal{I}} \cap \mathscr{D}_{\mathcal{I}}$. The map $W_{\mathcal{I}}$ is known as the *(non-commutative) winding number of U*.

Example 3.5.2 (Triviality of the winding number in the localized case)**.** According to Example (3.1.6) the structure of the interface algebra in the case of a localized magnetic field is given by $\mathcal{I} \simeq \mathbb{C} \otimes \mathcal{K}(\ell^2(\mathbb{Z}^2))$ and therefore it satisfies the ansatz (3.14). However, in view of the simple structure of $\mathcal{I}_0 = \mathbb{C}$ one has that the only faithful (normalized) trace τ_0 is the identity $\tau_0(a) = a$ and the only derivation δ_0 is the null-map $\delta_0(a) = 0$ for all $a \in \mathbb{C}$. As a consequence the associated trace on $\mathcal{I}$ coincides with the canonical trace of the Hilbert space $\ell^2(\mathbb{Z}^2)$, while there is no non-trivial derivation compatible with the ansatz (3.14). In view of that one has that $W_{\mathcal{I}} = 0$ identically in the case of a localized magnetic field. $\triangleleft$

Definition 3.5.3 (Interface conductance)**.** Let $\mathcal{I}$ be an interface algebra of type 3.14 endowed with the derivation $\partial_{\mathcal{I}}$ and the trace $\mathscr{T}_{\mathcal{I}}$. Let $U \in \mathcal{I}^+$ be a unitary operator such that $U - 1 \in \mathcal{C}^1_{\mathcal{I}} \cap \mathscr{D}_{\mathcal{I}}$. The *interface conductance* associated to the *configuration* U is defined by

$$\sigma_{\mathcal{I}}(U) \;:=\; \frac{e^2}{h}\,W_{\mathcal{I}}(U)\;. \qquad (3.22)$$

The definition above is justified by the fact that $\sigma_{\mathcal{I}}$ provides the proportionality coefficient for the current that flows along the interface (*cf.* [SKR] or [PS, Section 7.1]). To clarify Definition 4.30 we need some intermediate concept.

Let us call *magnetic Hamiltonians* the self-adjoint elements of $\mathcal{A}_B$. Let $\hat{H} \in \mathcal{A}_B$ be a magnetic Hamiltonian and $H := \mathrm{ev}(\hat{H}) \in \mathcal{A}_{\mathrm{bulk}}$ its image in the bulk algebra. By construction the bulk Hamiltonian $H = (H_1, \ldots, H_{N+1})$ is made by a $N+1$-upla of suitable self-adjoint elements of the constant magnetic field algebras $\mathcal{A}_{b_j}$ and its spectrum is given by $\sigma(H) = \bigcup_{j=1}^{N+1} \sigma(H_j)$.

Definition 3.5.4 (Non-trivial bulk gap)**.** The magnetic Hamiltonian $\hat{H} \in \mathcal{A}_B$ has a *non-trivial bulk gap* if there is a compact set $\Delta \in \mathbb{R}$ such that

$$\min \sigma(H) \ < \ \min \Delta \ < \ \max \Delta \ < \ \max \sigma(H)$$

and $\Delta \cap \sigma(H) = \emptyset$.

According to the above definition Δ lies inside a non-trivial spectral gap of the bulk Hamiltonian H and for every chemical potential $\mu \in \Delta$ the *Fermi projection*

$$P_\mu \ = \ (P_{\mu,1}, \ldots, P_{\mu,N+1}) \ \in \ \mathcal{A}_{\mathrm{bulk}} \,, \qquad P_{\mu,j} \ := \ \chi_{(-\infty,\mu)}(H_j) \ \in \ \mathcal{A}_{b_j}$$

is an element of the bulk algebra. If the magnetic Hamiltonian is smooth, *i. e.* $\hat{H} \in \mathcal{A}_{\mathrm{I}}^{\infty}$, then also $H \in \mathcal{A}_{\mathrm{bulk}}^{\infty}$ (the evaluation map preserves the regularity), and in turn $P_\mu \in \mathcal{A}_{\mathrm{bulk}}^{\infty}$ since $\mathcal{A}_{\mathrm{bulk}}^{\infty}$ is closed under holomorphic functional calculus. Let $[P_\mu] = [(P_{\mu,1}, \ldots, P_{\mu,N+1})] \in K_0(\mathcal{A}_{\mathrm{bulk}})$ be the class of the Fermi projection in the K_0-group of $\mathcal{A}_{\mathrm{bulk}}$. As a first step let us compute the image of $[P_\mu]$ inside $K_1(\mathcal{I})$ under the exponential map.

Proposition 3.5.5. *Assume that the magnetic Hamiltonian $\hat{H} \in \mathcal{A}_B$ has a* non-trivial bulk gap *detected by Δ. Let $g : \mathbb{R} \to [0, 1]$ be a non-decreasing (smooth) function such that $g = 0$ below Δ and $g = 1$ above Δ and consider the unitary operator*

$$U_\Delta \ := \ \mathrm{e}^{\mathrm{i}\, 2\pi g(\hat{H})} \ . \tag{3.23}$$

Then $U_\Delta \in \mathcal{I}^+$ and

$$\exp([P_\mu]) \ = \ -\,[U_\Delta] \ \in \ K_1(\mathcal{I}) \,.$$

Proof. The proof is similar to that of [PS, Proposition 4.3.1]. Since the evaluation map is a homomorphism of C^*-algebras it commutes with the functional calculus and consequently

$$\mathrm{ev}(g(\hat{H})) \ = \ g(H) \ = \ (g(H_1), \ldots, g(H_{N+1})) \ = \ 1 \ - \ P_\mu$$

due to the fact that g is equal to 0 below the bulk gap and to 1 above the bulk gap and therefore $g(H_j) = 1 - P_{\mu,j}$. As a consequence

$$\mathrm{ev}\left(1 - \mathrm{e}^{\mathrm{i}\, 2\pi g(\hat{H})}\right) \ = \ 1 - \mathrm{e}^{\mathrm{i}\, 2\pi g(H)} \ = \ 0$$

showing that U_Δ is a unitary element in $\mathcal{I}^+$. Since $1-g(\hat{H})$ is a self-adjoint lift of P_μ one can compute the exponential map as in [Weg, Definition 9.3.1 & Exercise 9.E] obtaining in this way

$$\exp([P_\mu]) \;=\; \left[\mathrm{e}^{-\mathrm{i}\,2\pi(1-g(\hat{H}))}\right] \;=\; \left[\mathrm{e}^{-\mathrm{i}\,2\pi g(\hat{H})}\right] \;=\; -\,[U_\Delta]$$

where the additive notation[5] for $K_1(\mathcal{I})$ has been used. □

In the case $\hat{H} \in \mathcal{A}_B^\infty$ it follows from the construction that $U_\Delta \in \mathcal{I}^+ \cap \mathcal{A}_B^\infty$ acquires the same regularity. It is worth noting that the element $1-U_\Delta$ can be constructed entirely from the spectral subspace of $\hat{H}$ corresponding to the bulk insulating gap Δ. Indeed, the support of the function $\mathrm{e}^{\mathrm{i}\,2\pi g}-1$ is contained inside the region Δ which lies in the insulating gap.

Remark 3.5.6 (Gap closing as a topological obstraction)**.** The condition $[U_\Delta] \neq 0$ (*cf.* Note 5) measures the obstruction to lift the Fermi projection $P_\mu \in \mathcal{A}_{\text{bulk}}$ to a true projection in $\mathcal{A}_{A_B} \otimes \mathrm{Mat}_{\mathrm{N}}(\mathbb{C})$ (for some N large enough). From the construction emerges that this obstruction detects the presence of spectrum of $\hat{H}$ inside Δ which is generated by the existence of the magnetic interface. Since the election of Δ inside the bulk gap is totally arbitrary, and the Fermi projection does not depend on the specific μ inside the bulk gap, one gets that for any given Δ the related element $1-g(\hat{H})$ is a self-adjoint lift of the Fermi projection. This implies immediately that the condition $[U_\Delta] \neq 0$ guarantees the complete closure of the bulk gap due to the presence of the magnetic interface. ◁

Let g as in the claim of Proposition 3.5.5. The derivative g' is non-negative, supported in Δ and normalization in the sense that $\|g'\|_{L^1} = 1$. By construction the element $g'(\hat{H})$ satisfies the condition $\mathrm{ev}(g'(\hat{H})) = g'(\mathrm{ev}(\hat{\mathrm{H}})) = 0$ and so $g'(\hat{H}) \in \mathcal{I}$ is an element of the interface algebra. Moreover $g'(\hat{H})$ can be regarded as a density matrix which describes a state of the system with energy distributed in the region Δ. If one interprets the operator $\hbar^{-1}\partial_{\mathcal{I}}(\hat{H})$ as the *velocity* along the interface one deduces that

$$J_{\mathcal{I}}(\Delta) \;:=\; -\frac{e}{\hbar}\mathscr{T}_{\mathcal{I}}\left(g'(\hat{H})\,\partial_{\mathcal{I}}(\hat{H})\right) \tag{3.24}$$

is the *current density* along the interface carried by the "extended states" in Δ and, as a consequence, $\sigma_{\mathcal{I}} = eJ_{\mathcal{I}}$ provides the associated conductance (we are assuming that $e > 0$ is the magnitude of the electron charge). The connection between the latter formula and Definition 3.5.3 is provided by the following result originally proved in [SKR].

Proposition 3.5.7. *It holds true that*

$$\mathscr{T}_{\mathcal{I}}\left(g'(\hat{H})\,\partial_{\mathcal{I}}(\hat{H})\right) \;=\; -\frac{1}{2\pi}\,W_{\mathcal{I}}\big(U_\Delta\big)\,.$$

[5] In terms of the additive notation of $K_1(\mathcal{I})$, the trivial element is $[1]=0$ and $-[U]=[U^*]$ denotes the inverse of $[U]$.

Proof. The result can be obtained by adapting step by step the proof of [PS, Proposition 7.1.2]. Indeed the proof is purely algebraic and only uses the properties of the trace $\mathscr{T}_{\mathcal{I}}$ and the derivation $\partial_{\mathcal{I}}$ assumed by hypothesis at the beginning of this section. $\square$

By combining definition 3.24 (which is motivated by physics) with Proposition 3.5.7 one gets that the interface conductance generated the "extended states" in Δ is given by

$$\sigma_{\mathcal{I}}(\Delta) \;:=\; \frac{e^2}{h}\, W_{\mathcal{I}}(U_\Delta)\;. \tag{3.25}$$

This equation justifies the "abstract" Definition 3.5.3.

The relevance of Definition 3.5.3 lies in its topological interpretation. Consider the map $\eta_{\mathcal{I}} : (\mathcal{C}^1(\mathcal{I}) \cap \mathscr{D}_{\mathcal{I}})^{\times 2} \to \mathbb{C}$, defined by

$$\eta_{\mathcal{I}}(T_0, T_1) \;:=\; \mathrm{i}\;\mathscr{T}_{\mathcal{I}}\big(T_0\;\partial_{\mathcal{I}}(T_1)\big)\;. \tag{3.26}$$

In view of the properties of $\mathscr{T}_{\mathcal{I}}$ and $\partial_{\mathcal{I}}$ assumed by hypothesis $\eta_{\mathcal{I}}$ turns out to be a cyclic 1-cocycle and therefore defines a class $[\eta_{\mathcal{I}}] \in HC^1(\mathcal{I})$ in the cyclic cohomology of the C^*-algebra $\mathcal{I}$ [Con, Chapter 3]. Let

$$\prec\,,\,\succ\,:\; K_1(\mathcal{I}) \;\times\; HC^1(\mathcal{I}) \;\longrightarrow\; \mathbb{C}$$

be the canonical bilinear pairing between (odd) K-theory and (odd) cyclic cohomology, defined by

$$\big([U],[\phi]\big) \;\longmapsto \prec [U],[\phi] \succ\, :=\; (\mathrm{tr}\sharp\phi)\big((U-1)^*, U-1\big)$$

where the unitary $U \in \mathcal{C}^1(\mathcal{I}^+) \otimes \mathrm{Mat}_N(\mathbb{C})$ is a suitable[6] representative of the class $[U]$ [Con, Section 3.3, Proposition 3]. Since every unitary $U \in \mathcal{I}^+$ (like U_Δ) defines an element $[U] \in K_1(\mathcal{I})$ in the K_1-group of the interface algebra, one gets that

$$W_{\mathcal{I}}(U) \;= \prec [U],[\eta_{\mathcal{I}}] \succ \tag{3.27}$$

only depends on the class $[U]$. In particular, by combining Proposition 3.5.5 and equation (3.27) one gets

$$\sigma_{\mathcal{I}}(\Delta) \;:=\; \frac{e^2}{h}\; \prec [U_\Delta],[\eta_{\mathcal{I}}] \succ\, =\; -\frac{e^2}{h}\; \prec \exp([P_\mu]),[\eta_{\mathcal{I}}] \succ\;\;. \tag{3.28}$$

Equation (3.28) is the topological essence of the *bulk-interface duality* and will be used in Sections 4.5 to prove equation (1.3), that is, a bulk-interface correspondance for the Iwatsuka magnetic field (*cf.* Theorem 4.5.3).

[6]The suitability of the unitary follows from the need of it being at least once differentiable. In fact, the pairing is initially defined on the K-Theory of the algebra $C^1(\mathcal{I}^+)$ and then extended to the K-Theory of the whole algebra $\mathcal{I}^+$, which are canonically isomorphic.

Chapter 4: The Iwatsuka C^*-algebra

In this chapter most of what has been achieved for general magnetic C^*-algebras is written down as explicitely as possible for the Iwatsuka magnetic field. This contains a full description of the evaluation map and interface algebra, which builds a full understanding of its Toeplitz extension, together with the explicit computation of every map and group involved in its six-term exact sequence associated defined in (3.9).
Let us recall that the magnetic translations associated to the Iwatsuka magnetic field has been described in Example 2.2.4 and the Iwatsuka C^*-algbera has been defined in Example 2.3.4.

4.1 Toeplitz extension and structure of the Iwatsuka C^*-algebra

The simplest examples of a magnetic multi-interface system as described in Definition 3.3.4 is provided by the Iwatsuka magnetic B_{I} defined by (2.4) with the condition

$$b_- \neq b_+. \tag{4.1}$$

In fact, according to the content of Example 2.6.4 one has that the boundary of the Iwatsuka magnetic hull Ω_{I} can be represented as $\partial\Omega_{\mathrm{I}} = \{\omega_{-\infty}, \omega_{+\infty}\}$ with $\omega_{\pm\infty}$ two distinct invariant points. As a consequence the associated Toeplitz extension is given by

$$0 \longrightarrow \mathcal{I} \xrightarrow{\imath} \mathcal{A}_{\mathrm{I}} \xrightarrow{\mathrm{ev}} \mathcal{A}_{\mathrm{bulk}} \longrightarrow 0 \tag{4.2}$$

with *bulk algebra* given by

$$\mathcal{A}_{\mathrm{bulk}} := \mathcal{A}_{b_-} \oplus \mathcal{A}_{b_+} \tag{4.3}$$

and evaluation map defined by

$$\begin{aligned} \mathrm{ev}(S_{\mathrm{I},1}) &:= (S_{b_-,1}, S_{b_+,1}) \\ \mathrm{ev}(S_{\mathrm{I},2}) &:= (S_{b_-,2}, S_{b_+,2}) \\ \mathrm{ev}(f_{\mathrm{I}}) &:= \left(\mathrm{e}^{ib_-}, \mathrm{e}^{ib_+}\right) \end{aligned} \tag{4.4}$$

where $S_{\mathrm{I},1}$ and $S_{\mathrm{I},2}$ are the Iwatsuka magnetic translations and f_{I} is the associated flux operator as defined in Example 2.2.4.

In section 2.7 was proved that every element of a magnetic C^*-algebra can be expressed as a limit of Cesàro like sums (see Theorem 2.7.3). The Iwatsuka C^*-algebra is no exception to this fact, however we can also prove that $\mathcal{A}_{\mathrm{I}}$ contains several relevant projections. Let us start by introducing the more operator-like notation

$$\begin{aligned} (P_\pm\psi)(n) &:= \delta_\pm(n)\psi(n) \\ (P_0\psi)(n) &:= \delta_0(n)\psi(n) \end{aligned}, \quad \psi \in \ell^2(\mathbb{Z}^2)$$

where the functions $\delta_\pm$ and δ_0 are defined in Example 2.1.3. It is of course true that $P_l = \delta_l(N)$, where $l \in \{-, 0, +\}$ and $N = (N_1, N_2)$ is the vector of position operators on $\ell^2(\mathbb{Z}^2)$. This notation is introduced mainly because the operators P_l are projections, and such kind of operators are really important when working with K-theory, so writing them with the letter P clarifies their importance and make this work consistent with most of the literature.

Lemma 4.1.1. *Under the assumption* (4.1) *the projections $P_\pm$ and P_0 are elements of $\mathcal{A}_{\mathrm{I}}$.*

Proof. The identity 1 and the flux operator f_{I} defined in (2.4.4) are elements of $\mathcal{A}_{\mathrm{I}}$. Let us start with the case $b_0 \neq b_+$ and $b_0 \neq b_-$. A straightforward computation shows that

$$P_0 = \left(\mathrm{e}^{ib_-} - \mathrm{e}^{ib_0}\right)^{-1}\left(\mathrm{e}^{ib_+} - \mathrm{e}^{ib_0}\right)^{-1}\left(\mathrm{e}^{ib_-} - f_{\mathrm{I}}\right)\left(\mathrm{e}^{ib_+} - f_{\mathrm{I}}\right),$$

hence $P_0 \in \mathcal{A}_{\mathrm{I}}$. Similarly, one can check that

$$P_\pm = \left(\mathrm{e}^{ib_\mp} - \mathrm{e}^{ib_\pm}\right)^{-1}\left(\mathrm{e}^{ib_\mp} - f_{\mathrm{I}}\right)(1 - P_0).$$

Let us assume now $b_0 = b_+$ and consider the projection $P_\geqslant := P_0 + P_+$. As above one can check that

$$\begin{aligned} P_\geqslant &= \left(\mathrm{e}^{ib_-} - \mathrm{e}^{ib_+}\right)^{-1}\left(\mathrm{e}^{ib_-}1 - f_{\mathrm{I}}\right) \\ P_- &= \left(\mathrm{e}^{ib_+} - \mathrm{e}^{ib_-}\right)^{-1}\left(\mathrm{e}^{ib_+}1 - f_{\mathrm{I}}\right) \end{aligned} \tag{4.5}$$

are both elements of $\mathcal{A}_{\mathrm{I}}$. Moreover, the equality

$$P_0 = P_\geqslant - S_{\mathrm{I},1}P_\geqslant S_{\mathrm{I},1}^* \tag{4.6}$$

shows that also $P_0 \in \mathcal{A}_{\mathrm{I}}$. Finally $P_+ = P_\geqslant - P_0$. The case $b_0 = b_-$ is analogous. □

For every $j \in \mathbb{Z}$ let us introduce the projection

$$(P_j\psi)(n) := \delta_0(n - je_1)\psi(n) \quad \psi \in \ell^2(\mathbb{Z}^2).$$

From the definition it follows that P_j is the translation of P_0 along the vertical line located at $n_1 = j$. The projections P_j are mutually orthogonal.

Corollary 4.1.2. *Under the assumption* (4.1) *it holds true that $P_j \in \mathcal{A}_{\mathrm{I}}$ for all $j \in \mathbb{Z}$.*

Proof. From Lemma 4.1.1 we know that $P_0 \in \mathcal{A}_{\mathrm{I}}$. Moreover, a direct computation shows that

$$P_j = \begin{cases} (S_{\mathrm{I},1})^j P_0 (S_{\mathrm{I},1}^*)^j & \text{if } j > 0 \\ (S_{\mathrm{I},1}^*)^{|j|} P_0 (S_{\mathrm{I},1})^{|j|} & \text{if } j < 0\,. \end{cases} \tag{4.7}$$

This completes the proof. □

Finally let us observe that from (4.7) one gets the useful formula

$$P_j S_{\mathrm{I},1} = S_{\mathrm{I},1} P_{j-1}, \qquad j \in \mathbb{Z}. \tag{4.8}$$

4.2 Evaluation and Interface Algebra for the Iwatsuka Magnetic Field

In this section the main features of the evaluation and interface algebras for the Iwatsuka magnetic field are proved. The next result provides a first step for the understanding of the evaluation map.

Lemma 4.2.1. *Under the assumption* (4.1) *it holds true that*

$$\begin{aligned} \mathrm{ev}(P_+) \ &= (0,1) \\ \mathrm{ev}(P_-) \ &= (1,0) \end{aligned} \tag{4.9}$$

and

$$\mathrm{ev}(P_j) = (0,0), \quad \textit{for all } j \in \mathbb{Z}. \tag{4.10}$$

Proof. Let us start with the case $b_0 \neq b_+$ and $b_0 \neq b_-$. Then the result follows from the last equation in (4.4), the formulas for $P_\pm$ and P_j in Lemma 4.1.1 and Corollary 4.1.2 along with the fact that ev is a C^*-homomorphism. In the case $b_0 = b_+$ one obtains from (4.5) that $\mathrm{ev}(P_\geqslant) = (0,1)$ and $\mathrm{ev}(P_-) = (1,0)$. Moreover, from (4.6) one gets that

$$\mathrm{ev}(P_+) = (0,1) - (0, S_{b_+,1} 1 S_{b_+,1}^*) = 0.$$

The case $b_0 = b_-$ is similar. □

Let $\Lambda \subset \mathbb{Z}$ be a finite subset and

$$P_\Lambda := \bigoplus_{j \in \Lambda} P_j. \tag{4.11}$$

the next result is a direct consequence of Lemma 4.2.1.

Corollary 4.2.2. *Under the assumption* (4.1) *it holds true that*

$$\mathrm{ev}(TP_\Lambda R) = 0$$

for all $T, R \in \mathcal{A}_\mathrm{I}$ *and for all finite subset* $\Lambda \subset \mathbb{Z}$.

The next result characterizes the evaluation map when it is restricted to the flux algebra $\mathcal{F}_B$ and will be of extreme importance to give a proper description of the Interface algebra. Recall from Example 2.4.4 that $\mathcal{F}_B \simeq c(\mathbb{Z})$, where the identification is given by simply fixing the second variable, so we are allowed to compute left and right limits for the elements of the flux algebra.

Lemma 4.2.3. *Let* $g \in \mathcal{F}_B$. *Then,*

$$\mathrm{ev}(g) = \left(\lim_{n_1 \to -\infty} g(n), \lim_{n_1 \to \infty} g(n) \right), \qquad \textit{where } n = (n_1, n_2) \in \mathbb{Z}^2. \tag{4.12}$$

Proof. First let us define

$$L(g) = \left(\lim_{n_1 \to -\infty} g(n), \lim_{n_1 \to \infty} g(n) \right), \qquad \text{where } n = (n_1, n_2) \in \mathbb{Z}^2,$$

and observe that L defines a $*$-homomorphism.
Recall that

$$\tau_{(r,s)}(f_\mathrm{I}) = (S_{1,I})^r \ (S_{2,I})^s \ f_B \ (S_{2,I})^{-s} \ (S_{1,I})^{-r},$$

and observe that

$$\begin{aligned} \mathrm{ev}(\tau_{(\mathrm{r,s})}(\mathrm{f_I})) &= \mathrm{ev}((S_{1,I})^r \ (S_{2,I})^s \ f_\mathrm{I} \ (S_{2,I})^{-s} \ (S_{1,I})^{-r})) \\ &= (\tau_{(r,s)} f_{b_-}, \tau_{(r,s)} f_{b_+}) \\ &= (\mathrm{e}^{\mathrm{i}\, b_-} , \mathrm{e}^{\mathrm{i}\, b_+}) \\ &= L(\tau_{(r,s)}(f_\mathrm{I})), \quad \text{for } (r,s) \in \mathbb{Z}^2. \end{aligned} \tag{4.13}$$

Since last equality holds after taking algebraic combinations it follows that $\mathrm{ev}|_{\mathcal{F}_\mathrm{I}^0} = L|_{\mathcal{F}_\mathrm{I}^0}$, where $\mathcal{F}_\mathrm{I}^0$ is the dense $*$-subalgebra of $\mathcal{F}_\mathrm{I}$ spanned by $\{\tau_m(f_\mathrm{I}) \mid m \in \mathbb{Z}^2\}$, and hence since this maps are continuous they must be equal in the entire flux algebra.

□

Proposition 4.2.4. *The* interface algebra $\mathcal{I}$ *is the closed two-sided ideal of* $\mathcal{A}_\mathrm{I}$ *generated by* P_0, *i. e.*

$$\mathcal{I} = \mathcal{A}_\mathrm{I} P_0 \mathcal{A}_\mathrm{I} := \mathrm{span}\left\{TP_0R | T, R \in \mathcal{A}_\mathrm{I}\right\}.$$

Proof. A comparison with Definition 3.1.4 shows the claim is equivalent to state that P_0 generates $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\mathrm{I}})$. From Corollary 4.2.2 one gets that $P_j \in \mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\mathrm{I}}) \subset \mathcal{F}_\mathrm{I}$ for every $j \in \mathbb{Z}$. From lemma 4.2.3 it follows that $g \in \mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\mathrm{I}})$ if and only if g vanishes when $n_1 \to \pm\infty$. The proof is completed by observing that $c_0(\mathbb{Z}) = \{g \in \ell^\infty \mid \lim_{n \to \pm\infty} g(n) = 0\}$ is generated by the projections $\{\delta_j\}_{j \in \mathbb{Z}}$ and then, up to representation, $\mathrm{Ker}(\mathrm{ev}|_{\mathcal{F}_\mathrm{I}}) = C^*(P_j \mid j \in \mathbb{Z})$. □

The Iwatsuka magnetic field is constant in one direction and therefore one can use the magnetic Bloch-Floquet transform described in Appendix F to study the interface algebra. Indeed, the Iwatsuka magnetic translations commute with the operator $V_f := \mathrm{e}^{if(N_1)}\, S_2$ defined through the function

$$f(m) := \begin{cases} mb_+ & \text{if } m \geqslant 0 \\ (m+1)b_- - b_0 & \text{if } m < 0\,. \end{cases}$$

Let $\mathcal{U}_B : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{S}^1) \otimes \ell^2(\mathbb{Z})$ be the associated magnetic Bloch-Floquet transform as defined in Appendix F. The next result contains the main feature of the Iwatsuka interface algebra.

Proposition 4.2.5. *It holds true that* $\mathcal{U}_B \mathcal{I} \mathcal{U}_B^{-1} = \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. *In particular the Iwatsuka interface algebra is a* straight-line *according to Definition 3.1.5.*

Proof. A direct computation shows that $\mathcal{U}_B P_j \mathcal{U}_B^{-1} = 1 \otimes \pi_j$ where π_j is the rank-one projection on $\ell^2(\mathbb{Z})$ defined by $(\pi_j \phi)(m) := \delta_{m,j}\phi(m)$. Since $\mathcal{U}_B (S_{\mathrm{I},2} P_j)^n \mathcal{U}_B^{-1} = \mathcal{U}_B (S_{\mathrm{I},2})^n P_j \mathcal{U}_B^{-1}$ is proportional to $\mathrm{e}^{ink} \otimes \pi_j$ up to a phase factor one gets that $g \otimes \pi_j \in \mathcal{U}_B \mathcal{I} \mathcal{U}_B^{-1}$ for every $g \in \mathcal{C}(\mathbb{S}^1)$ and $j \in \mathbb{Z}$. Acting with powers of $\mathcal{U}_B S_{\mathrm{I},1} \mathcal{U}_B^{-1}$ on the latter elements one gets that also $g \otimes \pi_{i,j} \in \mathcal{U}_B \mathcal{I} \mathcal{U}_B^{-1}$ where $\pi_{i,j}$ is the rank-one operator defined by $(\pi_{i,j}\phi)(m) := \delta_{m,j}\phi(i)$. The result follows by observing that the rank-one operators generates the compact operators. $\square$

Following the procedure described in Section 3.5 we can use Proposition 4.2.5 to equip $\mathcal{I}$ with a derivation and a trace. The natural derivation on $\mathcal{C}(\mathbb{S}^1)$ is $\delta_0 := -\frac{\mathrm{d}}{\mathrm{d}k}$. With this sign convention a comparison with (2.29) provides

$$\delta_0 \otimes \mathrm{Id}_{\mathcal{K}} \left(\mathcal{U}_B T \mathcal{U}_B^{-1}\right) \;=\; \partial_2(T) \;=\; \mathrm{i}\,[T, N_2]$$

for differentiable elements $T \in \mathcal{I}$. Therefore we obtain that the interface derivation is given by $\partial_{\mathcal{I}} := \mathrm{i}\,[\,\cdot\,, N_2]$. Similarly the natural trace on $\mathcal{C}(\mathbb{S}^1)$ is given by $\tau_0 := \int_{\mathbb{S}^1} \mathrm{d}k$ where $\mathrm{d}k$ is the normalized Haar measure. Since $\tau_0(\,\mathrm{e}^{\mathrm{i}\,nk}\,) = \delta_{n,0}$ one gets that

$$\tau_0 \otimes \mathrm{Tr}_{\ell^2(\mathbb{Z})} \left(\mathcal{U}_B T \mathcal{U}_B^{-1}\right) \;=\; \mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}(Q_0 T Q_0)$$

where the projection Q_0 is given by $(Q_0\psi)(n,m) = \delta_{m,0}\psi(n,m)$ and $T \in \mathcal{I}$ is any suitable integrable elements. In this way one can define the interface trace as

$$\mathscr{T}_{\mathcal{I}}(T) \;:=\; \mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}(Q_0 T Q_0) \;=\; \sum_{n\in\mathbb{Z}} \langle n,0|\; T\; |n,0\rangle \tag{4.14}$$

where the Dirac notation in the right-hand side turns out to be particularly useful.

4.3 Splittings of the Toeplitz Extension

The Toeplitz extension for the Iwatsuka magnetic field admits a natural splitting of the linear space structure which turns out to be useful in applications.

Let us start by recalling that $\mathcal{A}_{\rm bulk}$ is generated, as $*$-linear space, by the linear combinations of monomials of the type $(S^r_{b_-,1}S^s_{b_-,2}, S^p_{b_+,1}S^q_{b_+,2})$ with $r,s,p,q \in \mathbb{Z}$. Consider the *linear* map $\jmath : \mathcal{A}_{\rm bulk} \to \mathcal{A}_{\rm I}$ initially defined on the monomials by

$$\jmath\big(S^r_{b_-,1}S^s_{b_-,2}, S^p_{b_+,1}S^q_{b_+,2}\big) := P_- S^r_{{\rm I},1}S^s_{{\rm I},2}P_- + P_+ S^p_{{\rm I},1}S^q_{{\rm I},2}P_+ \tag{4.15}$$

and then extended linearly to $\mathcal{A}_{\rm bulk}$. Such a map is well defined because both $\mathcal{A}_{\rm bulk}$ and $\mathcal{A}_{\rm I}$ are spanned as Banach spaces by the families of respective monomials. From its very definition it follows that ${\rm ev}\circ\jmath = {\rm Id}_{\mathcal{A}_{\rm bulk}}$, namely $\jmath$ provides a splitting of the linear structures. It follows that

$$\mathcal{A}_{\rm I} = \mathcal{I} + \jmath\big(\mathcal{A}_{\rm bulk}\big)$$

as direct sum of linear spaces [Weg, Proposition 3.1.3].

It is worth noting that the *linear* map $\jmath$ defined by (4.15) cannot be extended to a C^*-homomorphism. For instance, a direct computation shows that

$$\jmath\big(1, S_{b_+,1}\big)\jmath\big(1, S^*_{b_+,1}\big) - \jmath\big(1,1\big) = P_+\big(S_{b_+,1}P_+S^*_{b_+,1} - 1\big)P_+ = -P_1$$

since $S_{b_+,1}P_+S^*_{b_+,1} = P_+ - P_1$. On the other hand,

$$\jmath\big(1, S^*_{b_+,1}\big)\jmath\big(1, S_{b_+,1}\big) - \jmath\big(1,1\big) = P_+\big(S^*_{b_+,1}P_+S_{b_+,1} - 1\big)P_+ = 0$$

due to $S^*_{b_+,1}P_+S_{b_+,1} = P_+ + P_0$.

Remark 4.3.1 (Failure of the C^*-splitting)**.** A linear splitting is the best that we can do since the existence of a lifting being a $*$-homomorphism implies that in the K_0-level we have the short exact sequence (see [Weg, Proposition 8.2.2] or [GBVF, Proposition 3.29])

$$0 \longrightarrow K_0(\mathcal{I}) \xrightarrow{\imath_*} K_0(\mathcal{A}_{\rm I}) \xrightarrow{{\rm ev}_*} K_0(\mathcal{A}_{\rm bulk}) \longrightarrow 0, \tag{4.16}$$

but as will be proved in Proposition 4.4.3 and Remmark 4.4.5, $K_0(\mathcal{I}) \simeq \mathbb{Z}$ and $\imath_* = 0$ respectively, which contradicts the exactness of the sequence (4.16). $\triangleleft$

4.4 K-theory of the Iwatsuka C^*-Algebra

In this section several aspects of the K-theory of the Iwatsuka C^*-algebra are studied. In subsection 4.4.2 the six-term exact sequence associated to the Toeplitz extension (4.2) is entirely described and in turn the K-theory of every concerned algebra is determined, including the Iwatsuka C^*-algebra. On the other side, and as anticipated in Section 3.4, one

can also use the iterated crossed structure of the Iwatsuka C^*-algebra to study its K-theory in a much simpler way. This approach is followed in subsection 4.4.1 just as an example of the use of the iterated crossed product structure of magnetic C^*- algebras (see Section 2.5) and the methods developed in subsection 3.4.1.
The results obtained in both subsections 4.4.1 and 4.4.2 concerning the Iwatsuka C^*-algebra are of course the same, however in subsection 4.4.2 the K-theory of the Interface algebra and the important index and exponential maps are also determined.

4.4.1 Crossed-Product Approach

By adapting the notation of section 2.5 and choosing $j = 1$ and $k = 2$ we have that

$$\mathcal{A}_{\mathrm{I}} \simeq \mathcal{A}_{\mathrm{I},1} \rtimes_{\beta_2} \mathbb{Z} , \qquad \mathcal{A}_{\mathrm{I},1} := \mathcal{F}_{\mathrm{I}} \rtimes_{\alpha_1} \mathbb{Z},$$

where α_1 is defined by $\alpha_1(g) := \tau_{(1,0)}(g)$ for every $g \in \mathcal{F}_{\mathrm{I}}$ and β_2 is defined by $\beta_2(gS_{\mathrm{I},1}) := \tau_{(0,1)}(g)f_{\mathrm{I}}^{-1}S_{\mathrm{I},1}$ for every $g \in \mathcal{F}_{\mathrm{I}}$ and $r \in \mathbb{N}_0$. From now on and for the sake of notation let us just write $\alpha := \alpha_1$ and $\beta := \beta_2$.

In order to use the six-term exact sequence (3.11) it will be necessary to compute first the K-theory of the flux algebra $\mathcal{F}_{\mathrm{I}} \simeq c(\mathbb{Z})$. Since the K-theory of $c(\mathbb{Z})$ can be computed using methods that are not relevant to this work it has been left as an Appendix (see Appendix E). The important information about the K-theory of the flux algebra $\mathcal{F}_B$ is listed here:

$$\begin{aligned} K_0(\mathcal{F}_{\mathrm{I}}) &= \left(\bigoplus_{j\in\mathbb{Z}} \mathbb{Z}[P_j] \right) \oplus \mathbb{Z}[P_-] \oplus \mathbb{Z}[P_+] , \\ K_1(\mathcal{F}_{\mathrm{I}}) &= 0 . \end{aligned} \tag{4.17}$$

The K-theory of the first crossed product $\mathcal{A}_{\mathrm{I},1}$ can be then computed from the Pimsner-Voiculescu exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{F}_{\mathrm{I}}) & \xrightarrow{1-\alpha_*} & K_0(\mathcal{F}_{\mathrm{I}}) & \xrightarrow{\imath_*} & K_0(\mathcal{A}_{\mathrm{I},1}) \\ \delta \Big\uparrow & & & & \Big\downarrow \epsilon \\ K_1(\mathcal{A}_{\mathrm{I},1}) & \xleftarrow[\imath_*]{} & K_1(\mathcal{F}_{\mathrm{I}}) & \xleftarrow[1-\alpha_*]{} & K_1(\mathcal{F}_{\mathrm{I}}) \end{array} \tag{4.18}$$

where the connecting maps ϵ and δ are described in Appendix B.

Proposition 4.4.1. *Consider the six-term exact sequence* (4.18)*. Then, it holds true that:*

(i) The image and kernel of the map $(1 - \alpha_*) : K_0(\mathcal{F}_{\mathrm{I}}) \to K_0(\mathcal{F}_{\mathrm{I}})$ *are given by*

$$\mathrm{Im}(1 - \alpha_*) = \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}[P_j] , \qquad \mathrm{Ker}(1 - \alpha_*) = \mathbb{Z}[1] .$$

(ii) $\delta[S_{\mathrm{I},1}] = -[1]$.

Consequently,

$$K_0(\mathcal{A}_{\mathrm{I},1}) \;=\; \mathbb{Z}[P_-] \oplus \mathbb{Z}[P_+]\;, \qquad K_1(\mathcal{A}_{\mathrm{I},1}) \;=\; \mathbb{Z}[S_{\mathrm{I},1}]\;.$$

Proof. By the definition of α one gets

$$\begin{aligned}
(1-\alpha_*)\big([P_j]\big) \;&=\; [P_j - S^*_{\mathrm{I},1}P_jS_{\mathrm{I},1}] \;=\; [P_j] - [P_{j-1}]\;,\\
(1-\alpha_*)\big([P_-]\big) \;&=\; [P_- - S^*_{\mathrm{I},1}P_-S_{\mathrm{I},1}] \;=\; [P_{-1}]\;,\\
(1-\alpha_*)\big([P_+]\big) \;&=\; [P_+ - S^*_{\mathrm{I},1}P_+S_{\mathrm{I},1}] \;=\; -[P_0]\;.
\end{aligned}$$

It follows that the image of $(1-\alpha_*)$ is $\bigoplus_{j\in\mathbb{Z}}\mathbb{Z}[P_j]$ and

$$(1-\alpha_*)\left(n_-[P_-] + n_+[P_+] + \sum_{j=-M}^{+M} n_j[P_j]\right) \;=\; 0$$

has a non-trivial solution if and only if $n_- = n_0 = n_+$, and $n_j = 0$ in all other cases. As a consequence one has that the kernel of $(1-\alpha_*)$ is generated by $[P_-]+[P_0]+[P_+] = [1]$. For (ii) let us recall that the boundary map $\delta := \kappa_0^{-1}\circ\mathrm{ind}$ is the composition of the index map $\mathrm{ind} : K_1(\mathcal{A}_{\mathrm{I},1}) \to K_0(\mathcal{F}_{\mathrm{I}}\otimes\mathcal{K})$ associated to the Toeplitz extension (B.1) and the inverse of the stabilization isomorphism $\kappa_0 : K_0(\mathcal{F}_{\mathrm{I}}) \to K_0(\mathcal{F}_{\mathrm{I}}\otimes\mathcal{K})$ induced by the identification $g \mapsto g\otimes\pi_0$ (here $\pi_0 \in \mathcal{K}$ is any fixed rank-one projection). The isometry $V := S_{\mathrm{I},1}\otimes v$ which generates the Toeplitz algebra $\mathcal{T}_\alpha$ together with $\mathcal{F}_{\mathrm{I}}\otimes 1$ verifies the condition $\psi(V) = S_{\mathrm{I},1}$. Therefore, V provides a lift of $S_{\mathrm{I},1}$ by an isometry. Consider the unitary matrix

$$w(S_{\mathrm{I},1}) \;:=\; \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} \;\in\; \mathrm{Mat}_2(\mathcal{T}_{\alpha_1})$$

where $P := 1 - VV^*$. By construction $w(S_{\mathrm{I},1})$ is a lift of $\mathrm{diag}(S_{\mathrm{I},1}, S^*_{\mathrm{I},1})$ and $[\mathrm{diag}(S_{\mathrm{I},1}, S^*_{\mathrm{I},1})] \simeq [1]$ as a class in $K_1(\mathcal{A}_{\mathrm{I},1})$. As a consequence we can construct the index map according to [Weg, Definition 8.1.1] and after an explicit computation one gets

$$\begin{aligned}
\mathrm{ind}\big([S_{\mathrm{I},1}]\big) \;&=\; \varphi_*^{-1}([1-V^*V] - [1-VV^*])\\
&=\; \varphi_*^{-1}([0] - [P]) \;=\; -[1\otimes\pi_0]
\end{aligned}$$

where in the last equality we used the property $\varphi(1\otimes\pi_0) = P$. By using the isomorphism κ_0, one finally gets $\delta[S_{\mathrm{I},1}] = -[1]$. Since $K_1(\mathcal{F}_{\mathrm{I}}) = 0$, it follows that δ provides an isomorphism between $K_1(\mathcal{A}_{\mathrm{I},1})$ and $\mathrm{Ker}((1-\alpha_*))$. In view of (i) and (ii) one infers that $K_1(\mathcal{A}_{\mathrm{I},1}) \simeq \mathbb{Z}[S_{\mathrm{I},1}]$. Again, $K_1(\mathcal{F}_{\mathrm{I}}) = 0$ implies the surjectivity of $\imath_* : K_0(\mathcal{F}_{\mathrm{I}}) \to K_0(\mathcal{A}_{\mathrm{I},1})$ and so $K_0(\mathcal{A}_{\mathrm{I},1}) \simeq K_0(\mathcal{F}_{\mathrm{I}})/\mathrm{Im}((1-\alpha_*)) = \mathbb{Z}[P_-]\oplus\mathbb{Z}[P_+]$. □

For the K-theory of the second crossed product $\mathcal{A}_{\mathrm{I}} \simeq \mathcal{A}_{\mathrm{I},1} \rtimes_\beta \mathbb{Z}$ we need the Pimsner-Voiculescu exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{A}_{\mathrm{I},1}) & \stackrel{1-\beta_*}{\longrightarrow} & K_0(\mathcal{A}_{\mathrm{I},1}) & \stackrel{\imath_*}{\longrightarrow} & K_0(\mathcal{A}_{\mathrm{I}}) \\ \delta \Big\uparrow & & & & \Big\downarrow \epsilon \\ K_1(\mathcal{A}_{\mathrm{I}}) & \underset{\imath_*}{\longleftarrow} & K_1(\mathcal{A}_{\mathrm{I},1}) & \underset{1-\beta_*}{\longleftarrow} & K_1(\mathcal{A}_{\mathrm{I},1}) \end{array} \tag{4.19}$$

Theorem 4.4.2 (K-theory of the Iwatsuka C^*-algebra)**.** *Consider the six-term exact sequence* (4.19)*. Then, it holds true that*

(i) Both maps $(1-\beta_*) : K_j(\mathcal{A}_{\mathrm{I},1}) \to K_j(\mathcal{A}_{\mathrm{I},1})$, with $j = 1, 2$, vanish;

(ii) The map δ verifies

$$\begin{aligned} \delta\big([P_- S_{\mathrm{I},2} + P_0 + P_+]\big) &= -[P_-]\ , \\ \delta\big([P_- + P_0 + P_+ S_{\mathrm{I},2}]\big) &= -[P_+]\ ; \end{aligned}$$

(iii) There exists $N \in \mathbb{N}$ and a projection $P_{\mathrm{I}} \in \mathcal{A}_{\mathrm{I}} \otimes \mathrm{Mat}_N(\mathbb{C})$ such that

$$\epsilon[P_{\mathrm{I}}] = [S_{\mathrm{I},1}]\ .$$

Consequently,

$$\begin{aligned} K_0(\mathcal{A}_{\mathrm{I}}) &= \mathbb{Z}[P_-] \oplus \mathbb{Z}[P_-] \oplus \mathbb{Z}[P_{\mathrm{I}}] \\ K_1(\mathcal{A}_{\mathrm{I}}) &= \mathbb{Z}[V_{\mathrm{I},-}] \oplus \mathbb{Z}[V_{\mathrm{I},+}] \oplus \mathbb{Z}[S_{\mathrm{I},1}] \end{aligned}$$

where $V_{\mathrm{I},\pm} := 1 + P_\pm(S_{\mathrm{I},2} - 1)$.

Proof. For (i) it is enough to note that $\beta(P_l) = P_l$ for $l \in \mathbb{Z} \cup \{\pm\}$ and

$$\beta_*^{-1}[S_{\mathrm{I},1}] = [S_{\mathrm{I},2}^* S_{\mathrm{I},1} S_{\mathrm{I},2}] = [(S_{\mathrm{I},2}^* f_{\mathrm{I}} S_{\mathrm{I},2}) S_{\mathrm{I},1}] = [f_{\mathrm{I}}][S_{\mathrm{I},1}] = [S_{\mathrm{I},1}]\ ,$$

since $[f_{\mathrm{I}}] = [1]$ in $K_1(\mathcal{A}_{\mathrm{I},1})$. As a consequence $1 - \beta_*^{-1} = 0$. For (ii) let us observe that the isometry $V = S_{\mathrm{I},2} \otimes v \in \mathcal{T}_\beta$ satisfies $\psi((P_+ \otimes 1)V) = P_+ S_{\mathrm{I},2}$. It follows that $W := (P_- \otimes 1)V + (P_0 + P_+) \otimes 1$ is an isometry which provides a lift of $P_- S_{\mathrm{I},2} + P_0 + P_+$ in $\mathcal{T}_\beta$. The index map of the latter element can be computed as in the proof of Proposition 4.4.1 and after some computation one gets

$$\begin{aligned} \mathrm{ind}\big([P_- S_{\mathrm{I},2} + P_0 + P_+]\big) &= \varphi_*^{-1}([1 - W^*W] - [1 - WW^*]) \\ &= \varphi_*^{-1}([0] - [P_- \otimes P]) = -[P_- \otimes \pi_0]\ . \end{aligned}$$

where we used $P := 1 - VV^*$ and $\varphi(P_- \otimes \pi_0) = P_- \otimes P$. After recalling that $\delta := \kappa_0^{-1} \circ \mathrm{ind}$, with κ_0 stabilization isomorphism, one gets the first equation in (ii). The derivation of the second equation is identical. Item (iii) follows from the fact that $(1-\beta_*)=0$ implies the surjectivity of the map ϵ and so there must be a projection $P_{\rm I}\in\mathcal{A}_{\rm I}\otimes{\rm Mat}_N(\mathbb{C})$ and $M\leq N$ such that

$$\epsilon([P_{\rm I}]-M[1])=[S_{\rm I,1}]$$

(see [Weg, Proposition 6.2.7]). Now, since $[1]=[P_-]+[P_+]\in{\rm Im}(\imath_*)=\ker\epsilon$ it follows that $\epsilon[P_{\rm I}]=[S_{\rm I,1}]$. The exactness of the sequence (4.19) along with $(1-\beta_*)=0$ implies $K_j(\mathcal{A}_{\rm I})=\imath_*(K_j(\mathcal{A}_{\rm I,1}))\oplus\partial_j^{-1}(K_{j+1}(\mathcal{A}_{\rm I,1}))$ with $j=0,1$ (mod. 2). This concludes the proof. ■

Observe that the proof for the existence of the element $P_{\rm I}$ works as well for the case $\epsilon(P'_{\rm I})=[S^*_{\rm I,1}]=-[S_{\rm I,1}]$. Any projection $P_{\rm I}$ with the property

$$\epsilon(P_{\rm I})\in\{[S_{\rm I,1}],[S^*_{\rm I,1}]\}$$

will be called a *Power-Rieffel-Iwatsuka projection* or simply a *PRI-projection*. We can say a little more about $P_{\rm I}$. From its very definition one has that $\exp[P_{\rm I}]=[(S_{\rm I,1}-1)\otimes\pi_0+1]$ where exp is the actual exponential map associated with the Toeplitz exact sequence (B.1).

4.4.2 K-Theory Associated to the Toeplitz Extension

The six-term exact sequence associated with the Toeplitz extension for the Iwatsuka magnetic field (4.2) is given by

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \xrightarrow{\imath_*} & K_0(\mathcal{A}_{\rm I}) & \xrightarrow{{\rm ev}_*} & K_0(\mathcal{A}_{\rm bulk}) \\ {\scriptstyle\rm ind}\Big\uparrow & & & & \Big\downarrow{\scriptstyle\exp} \\ K_1(\mathcal{A}_{\rm bulk}) & \xleftarrow[{\rm ev}_*]{} & K_1(\mathcal{A}_{\rm I}) & \xleftarrow[\imath_*]{} & K_1(\mathcal{I}) \end{array} \qquad (4.20)$$

The K-theory of the bulk algebra is described in Proposition 3.4.1 and is explicitly given by

$$\begin{aligned} K_0(\mathcal{A}_{\rm bulk}) &= \mathbb{Z}[(1,0)]\oplus\mathbb{Z}[(P_{\theta_-},0)]\oplus\mathbb{Z}[(0,1)]\oplus\mathbb{Z}[(0,P_{\theta_+})]\;, \\ K_1(\mathcal{A}_{\rm bulk}) &= \mathbb{Z}[(S_{b_-,1},1)]\oplus\mathbb{Z}[(S_{b_-,2},1)]\oplus\mathbb{Z}[(1,S_{b_+,1})]\oplus\mathbb{Z}[(1,S_{b_+,2})]\;, \end{aligned}$$

where $P_{\theta_\pm}$ are the the Power-Rieffel projections of the C^*-algebras $\mathcal{A}_{b_\pm}$ respectively (*cf.* Appendix D).

The description of the K-theory of the interface algebra follows from Proposition 3.4.4 and Proposition 4.2.5.

Proposition 4.4.3. *It holds true that*

$$K_0(\mathcal{I})\;=\;\mathbb{Z}[P_0]\;,\qquad K_1(\mathcal{I})\;=\;\mathbb{Z}[U_{\mathcal{I}}]\;,$$

where $U_{\mathcal{I}}:=P_-+P_0S_{\rm I,2}+P_+=1+P_0(S_{\rm I,2}-1)\in\mathcal{I}^+$.

Proof. Let us start with the K_0-group. As proved in Appendix C the generator of $K_0(\mathcal{C}(\mathbb{S}^1))$ is the constant function 1. The group isomorphism $K_0(\mathcal{C}(\mathbb{S}^1)) \simeq K_0(\mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))$ is induced by the C^*-homomorphism $\mu : \mathcal{C}(\mathbb{S}^1) \to \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ defined by $\mu : g \mapsto g \otimes \pi_0$ where π_0 is any fixed rank-one the projection on $\ell^2(\mathbb{Z})$ [Weg, Corollary 6.2.11]. Consider $(\pi_0\phi)(m) := \delta_{m,0}\phi(m)$. The result follows by observing that $\mathcal{U}_B^{-1}(1 \otimes \pi_0)\mathcal{U}_B = P_0$ where $\mathcal{U}_B$ is the magnetic Bloch-Floquet transform used in Proposition 4.2.5. The argument for the K_1-group follows a similar structure. We already know that the generator of $K_1(\mathcal{C}(\mathbb{S}^1))$ is the exponential function $u(t) = \mathrm{e}^{\mathrm{i}\,t}$ (see Appendix C) and the isomorphism $K_1(\mathcal{C}(\mathbb{S}^1)) \simeq K_1(\mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))$ is induced by the same homomorphism μ defined above [RLL, Proposition 8.2.8]. However, since the K_1 is computed from the unitalization of the related C^*-algebra one needs to promote $\mathrm{e}^{\mathrm{i}\,k} \otimes \pi_0$ to a unitary in $(\mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))^+$. This can be done through the map

$$\mathrm{e}^{\mathrm{i}\,k} \otimes \pi_0 \;\longmapsto\; \mathrm{e}^{\mathrm{i}\,k} \otimes \pi_0 \;-\; 1 \otimes \pi_0 \;+\; 1 \otimes 1$$

as described in [RLL, Proposition 8.1.6]. As a result one has that the generator of the K_1-group can be identified with the class of $(\,\mathrm{e}^{\mathrm{i}\,k} - 1) \otimes \pi_0 + 1 \otimes 1$. Finally, using the magnetic Bloch-Floquet transform

$$\mathcal{U}_B^{-1}\big((\,\mathrm{e}^{\mathrm{i}\,k} - 1) \otimes \pi_0 + 1 \otimes 1\big)\mathcal{U}_B \;=\; V_f P_0 \;-\; P_0 \;+\; 1$$

along with the identities $V_f P_0 = S_2 P_0 = S_{\mathrm{I},2} P_0$ and $1 - P_0 = P_- + P_+$ provides the desired result □

We now have all the ingredients to study the vertical homomorphisms of the diagram (4.20). Let us start with the index map.

Proposition 4.4.4. *The image of the generators of* $K_0(\mathcal{A}_{bulk})$ *under the map* ind *in diagram* (4.20) *are given by*

$$\begin{aligned} \mathrm{ind}([(\mathrm{S}_{\mathrm{b}_-,2}, 1)]) \;&=\; \mathrm{ind}([(1, \mathrm{S}_{\mathrm{b}_+,2})]) \;=\; 0\;, \\ \mathrm{ind}([(\mathrm{S}_{\mathrm{b}_-,1}, 1)]) \;&=\; -\mathrm{ind}([(1, \mathrm{S}_{\mathrm{b}_+,1})]) \;=\; [\mathrm{P}_0]\;. \end{aligned} \tag{4.21}$$

Consequently the index map is surjective.

Proof. Let us construct the index map according to [Weg, Definition 8.1.1] for the set of generators $A \in \{(S_{b_-,1}, 1), (S_{b_-,2}, 1), (1, S_{b_+,1}), (1, S_{b_+,2})\} \subset \mathcal{A}_{\mathrm{bulk}}$ of the K_1-group of $\mathcal{A}_{\mathrm{bulk}}$. Let $\jmath$ as in (4.15) and define the map

$$w(A) \;:=\; \begin{pmatrix} \jmath(A) & 1 - \jmath(A)\jmath(A)^* \\ 1 - \jmath(A)^*\jmath(A) & \jmath(A)^* \end{pmatrix} \;\in\; \mathrm{Mat}_2(\mathcal{A}_{\mathrm{I}})\;.$$

A direct check shows that $\jmath(A) \in \mathcal{A}_{\mathrm{I}}$ is a partial isometry for every A in the generator set, indeed

$$\begin{aligned} \jmath(S_{b_-,1}, 1)\jmath(S_{b_-,1}, 1)^* \;&=\; 1 - P_0\;, \\ \jmath(S_{b_-,2}, 1)\jmath(S_{b_-,2}, 1)^* \;&=\; 1 - P_0\;, \\ \jmath(1, S_{b_+,1})\jmath(1, S_{b_+,1})^* \;&=\; 1 - (P_0 + P_1)\;, \\ \jmath(1, S_{b_+,2})\jmath(1, S_{b_+,2})^* \;&=\; 1 - P_0\;, \end{aligned} \tag{4.22}$$

and, on the other hand,

$$\begin{aligned}
\jmath(S_{b_-,1},1)^*\jmath(S_{b_-,1},1) &= 1-(P_0+P_{-1})\;, \\
\jmath(S_{b_-,2},1)^*\jmath(S_{b_-,2},1) &= 1-P_0\;, \\
\jmath(1,S_{b_+,1})^*\jmath(1,S_{b_+,1}) &= 1-P_0\;, \\
\jmath(1,S_{b_+,2})^*\jmath(1,S_{b_+,2}) &= 1-P_0\;.
\end{aligned} \tag{4.23}$$

As a consequence one can check that $w(A)$ is a unitary operator for every generator A. Moreover $\mathrm{ev}(w(A)) = \mathrm{diag}(A, A^*)$ showing that $w(A)$ is a unitary lift of $\mathrm{diag}(A, A^*)$. Finally $[\mathrm{diag}(A, A^*)] \simeq [1]$ as a class in the K_1-group. With all these data we can compute the index map of each generators according to $\mathrm{ind}([A]) := [w(A)P_1w(A)^*]-[P_1]$ where $P_1 := \mathrm{diag}(1, 0)$. An explicit computation provides

$$\begin{aligned}
\mathrm{ind}([A]) &= \left[\begin{pmatrix}\jmath(A)\jmath(A)^* & 0\\ 0 & 1-\jmath(A)^*\jmath(A)\end{pmatrix}\right] - \left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right] \\
&= \left[\begin{pmatrix}0 & 0\\ 0 & 1-\jmath(A)^*\jmath(A)\end{pmatrix}\right] - \left[\begin{pmatrix}1-\jmath(A)\jmath(A)^* & 0\\ 0 & 0\end{pmatrix}\right] \\
&= [1-\jmath(A)^*\jmath(A)] - [1-\jmath(A)\jmath(A)^*]
\end{aligned}$$

where the second and third equality are understood in the sense of the K_0-group. The equations (4.21) follow from the latter formula along with the computations (4.22) and (4.23) and the observation that, in view of (4.7), P_j is unitarily equivalent to P_0 for every $j \in \mathbb{Z}$. The latter fact implies $[P_j] = [P_k]$ for every pair $j, k \in \mathbb{Z}$ as elements of $K_0(\mathcal{A}_\mathrm{I})$. Consequently, $\mathrm{ind}([S_{b_-,1}, 1]) = [P_0]$, and $[P_0]$ is the generators of $K_0(\mathcal{I})$. This shows that the map ind is surjective. $\square$

Remark 4.4.5. As a consequence of Proposition 4.4.4 and the exactness of diagram 4.20 one infers that the map $\imath_* : K_0(\mathcal{I}) \to K_0(\mathcal{A}_\mathrm{I})$ is just the zero map. This implies that $[\imath_*([P_j])] = [P_j] = 0$ as element of $K_0(\mathcal{A}_\mathrm{I})$. This fact is in agreement with the description of $K_0(\mathcal{A}_\mathrm{I})$ in Theorem 4.4.2 and can be justified by the following direct argument: From $P_0 = S^*_{\mathrm{I},1}P_+S_{\mathrm{I},1} - P_+$ one gets $[P_0] = [S^*_{\mathrm{I},1}P_+S_{\mathrm{I},1}] - [P_+] = [P_+] - [P_+] = 0$ and $[P_0] = [P_j]$ for every $j \in \mathbb{Z}$ as justified at the end of the proof of Proposition 4.4.4. $\triangleleft$

Now we are in position to study the exponential map of diagram (4.20).

Proposition 4.4.6. *The map* exp *in diagram* (4.20) *is surjective. Moreover, it holds true that*

$$\begin{aligned}
\exp([(1,0)]) &= \exp([(0,1)]) = [1] = 0\;, \\
\exp([(P_{\theta_-},0)]) &= -\exp([(0,P_{\theta_+})]) = [U_\mathcal{I}]\;,
\end{aligned} \tag{4.24}$$

where the additive notation for the group $K_1(\mathcal{I})$ is used.

Proof. The surjectivity will be a consequence of from formulas (4.24) recalling that $[U_\mathcal{I}]$ generates $K_1(\mathcal{I})$. The construction of the exponential map is described in [Weg, Definition 9.3.1 & Exercise 9.E]. The first step is to construct appropriate lifts of the representatives of the elements of the group $K_0(\mathcal{A}_{\rm bulk})$. Let us start with the two generators $(1,0)$ and $(0,1)$. From Lemma 4.2.1 we get that suitable self-adjoint lifts are given by $\mathrm{lift}(1,0) := P_-$ and $\mathrm{lift}(0,1) := P_+$. Moreover, since $P_\pm$ are genuine projections one gets $\mathrm{e}^{-\,\mathrm{i}\,2\pi P_\pm} = 1 \in \mathcal{I}^+$. As a consequence, one gets the first equation in (4.24). For the second set of equations we need to construct explicitly the element $[q]$ introduced abstractly above. We will follows quite closely the strategy in [PV2, pp. 114-116]. Let us start with the Power-Rieffel projection (*cf.* Appendix D)

$$P_{\theta_+} \;=\; S^*_{b_+,1}\,\mathfrak{d}_1 \;+\; \mathfrak{d}_0 \;+\; \mathfrak{d}_1\,S_{b_+,1} \;\in\; \mathcal{A}_{b_+}$$

where $\mathfrak{d}_1 := g(S_2)$ and $\mathfrak{d}_0 := f(S_2)$ are self-adjoint elements of $\mathcal{A}_{b_+} \cap \mathcal{A}_{\rm I}$ in view of $S_2 = S_{b_+,2} = S_{{\rm I},2}$. Consider the self-adjoint lift of $(0, P_{\theta_+})$ given by

$$Q_+ \;=\; \mathfrak{v}_+^*\,\mathfrak{d}_1 \;+\; \mathfrak{d}_0 P_\geqslant \;+\; \mathfrak{d}_1\,\mathfrak{v}_+\;,$$

where $\mathfrak{v}_+ := S_{{\rm I},1}P_\geqslant = S_{b_+,1}P_\geqslant$ and $P_\geqslant := P_0 + P_+$. It is worth remembering that $[\mathfrak{d}_i, P_0] = [\mathfrak{d}_i, P_+] = 0$ for $i = 0, 1$. A direct computation shows that

$$Q_+^2 \;=\; Q_+ \;-\; \mathfrak{d}_1^2 P_0 \;=\; Q_+ \;+\; (\mathfrak{d}_0^2 - \mathfrak{d}_0)\mathfrak{L}P_0\;. \tag{4.25}$$

The first equality in (4.25) is justified by the relations

$$\begin{aligned}
\mathfrak{d}_1\mathfrak{v}_+\mathfrak{d}_1\mathfrak{v}_+ \;&=\; \mathfrak{v}_+(S^*_{b_+,1}\mathfrak{d}_1 S_{b_+,1}\mathfrak{d}_1)\mathfrak{v}_+ \;=\; 0\;,\\
\mathfrak{d}_0 P_\geqslant \mathfrak{d}_1\mathfrak{v}_+ + \mathfrak{d}_1\mathfrak{v}_+\mathfrak{d}_0 P_\geqslant \;&=\; (\mathfrak{d}_0\mathfrak{d}_1 + \mathfrak{d}_1 S_{b_+,1}\mathfrak{d}_0 S^*_{b_+,1})\mathfrak{v}_+ \;=\; \mathfrak{d}_1\mathfrak{v}_+\;,\\
\mathfrak{d}_0^2 P_\geqslant + \mathfrak{v}_+^*\mathfrak{d}_1\mathfrak{d}_1\mathfrak{v}_+ + \mathfrak{d}_1\mathfrak{v}_+\mathfrak{v}_+^*\mathfrak{d}_1 \;&=\; P_\geqslant(\mathfrak{d}_0^2 + S^*_{b_+,1}\mathfrak{d}_1^2 S_{b_+,1} + \mathfrak{d}_1^2)P_\geqslant - \mathfrak{d}_1^2 P_0\\
&=\; \mathfrak{d}_0 P_\geqslant - \mathfrak{d}_1^2 P_0
\end{aligned}$$

The second equality in (4.25) follows from (D.8) where $\mathfrak{L} := \mathfrak{L}(\mathfrak{d}_1)$ is the support projection of $\mathfrak{d}_1$ (in the von Neumann algebra generated by $\mathcal{A}_{b_+}$). An inductive argument, based on the identities

$$Q_+P_0 \;=\; \mathfrak{d}_0 P_1 + \mathfrak{d}_1 S_{b_+,1}P_0\;, \qquad \mathfrak{d}_1 S_{b_+,1}\mathfrak{L} \;=\; 0$$

and the commutation relations $[\mathfrak{L}, \mathfrak{d}_i] = 0 = [\mathfrak{L}, P_0]$, provides

$$Q_+^N \;=\; Q_+ \;+\; (\mathfrak{d}_0^N - \mathfrak{d}_0)\mathfrak{L}P_0 \;=\; (Q_+ - \mathfrak{d}_0\mathfrak{L}P_0) \;+\; (\mathfrak{d}_0\mathfrak{L})^N P_0\;. \tag{4.26}$$

Equation (4.26) facilitates the computation of the exponential of Q_+. Indeed, one immediately gets

$$\begin{aligned}
\mathrm{e}^{-\,\mathrm{i}\,2\pi Q_+} \;&=\; \sum_{N=0}^{+\infty}\frac{(-\,\mathrm{i}\,2\pi)^N}{N!}Q_+^N\\
&= (\,\mathrm{e}^{-\,\mathrm{i}\,2\pi} - 1)(Q_+ - \mathfrak{d}_0\mathfrak{L}P_0) \;+\; \mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}\,P_0 \;+\; (1 - P_0)\\
&= P_- \;+\; \mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}\,P_0 \;+\; P_+\;.
\end{aligned} \tag{4.27}$$

Before going forward let us make a similar computation for $(P_{\theta_-}, 0)$. Consider the Power-Rieffel projection[1]

$$P_{\theta_-} \;=\; S^*_{b_-,1}\,\mathfrak{d}'_1 \;+\; \mathfrak{d}'_0 \;+\; \mathfrak{d}'_1\,S_{b_-,1} \;\in\; \mathcal{A}_{b_-}$$

and the lift

$$Q_- \;=\; \mathfrak{v}^*_-\,\mathfrak{d}'_1 \;+\; \mathfrak{d}'_0 P_- \;+\; \mathfrak{d}'_1\,\mathfrak{v}_- \;,$$

where $\mathfrak{v}_- := P_- S_{\mathrm{I},1} = P_- S_{b_-,1}$. This time, a direct computation provides

$$Q_-^2 \;=\; Q_- \;-\; (S^*_{b_-,1}\mathfrak{d}'_1 S_{b_-,1})^2 P_{-1} \;=\; Q_- \;+\; ({\mathfrak{d}'_0}^2 - \mathfrak{d}'_0)\mathfrak{L}' P_{-1}\;.$$

where now $\mathfrak{L}' := \mathfrak{L}'(S^*_{b_-,1}\mathfrak{d}'_1 S_{b_-,1})$ is the support projection of $S^*_{b_-,1}\mathfrak{d}'_1 S_{b_-,1}$. After an induction one gets

$$Q_-^N \;=\; Q_- \;+\; ({\mathfrak{d}'_0}^N - \mathfrak{d}'_0)\mathfrak{L}' P_{-1} \;=\; (Q_- - \mathfrak{d}'_0\mathfrak{L}' P_{-1}) \;+\; (\mathfrak{d}'_0\mathfrak{L}')^N P_{-1}\;.$$

and the exponential of Q_- is given by

$$\mathrm{e}^{-\,\mathrm{i}\,2\pi Q_-} \;=\; \mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}'_0\mathfrak{L}'}\,P_{-1} \;+\; (1 - P_{-1})\;. \tag{4.28}$$

Now that we have satisfactory representations for the exponentials of both $\mathrm{e}^{-2\pi i Q_\pm}$, namely equations (4.27) and (4.28), we can state the result by proving that $[\,\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}\,] = [U^*_{\mathcal{I}}]$ and $[\,\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}'_0\mathfrak{L}'}\,P_{-1} \;+\; (1-P_{-1})] = [U_{\mathcal{I}}]$ as elements of $K_1(\mathcal{I})$. For the sake of clarity, let us add such facts in the next lemma.

□

Lemma 4.4.7. *Consider the notation introduced in the last proof. Then it holds true that*

$$[\,\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}\,] = [U^*_{\mathcal{I}}] \qquad \textit{and} \qquad [\,\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}'_0\mathfrak{L}'}\,P_{-1} \;+\; (1-P_{-1})] = [U_{\mathcal{I}}],$$

where the equalities must be understood in $K_1(\mathcal{I})$.

Proof. In lemma D.4 an homotopy of unitaries in $C^*(S_{\mathrm{I},2})$ between $\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}} \sim S^*_{b_+,2} = S^*_{\mathrm{I},2}$ is explicitely found (up to representation). Let u_t denote such homotopy and observe that $U_t := u_t P_0 + (1-P_0)$ is also an homotopy of unitaries in $\mathcal{I}^+$ connecting $\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}$ and $U^*_{\mathcal{I}}$, proving the first equality.
In a similar way, in lemma D.4 an homotopy of unitaries in $C^*(S_{\mathrm{I},2})$ between $\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}'_0\mathfrak{L}'}$ and $S_{\mathrm{I},2}$ is found (up to representation). Let u'_t be such homotopy and observe that $U'_t := u'_t(P_{-1}) + (1-P_{-1})$ is also an homotopy of unitaries in $\mathcal{I}^+$. To finish the proof, let us consider the operator $V := S_{\mathrm{I},1}P_{-1} + S^*_{\mathrm{I},1}P_0 + (1 - P_{-1} - P_0)$. This is an unitary involution in $\mathcal{I}^+$, *i.e.* $\mathfrak{r} = V^{-1} = V^*$. Since $K_1(\mathcal{I}) \simeq \mathbb{Z}$ is torsion-free, this implies that $[\mathfrak{r}] = [1]$ is the trivial element of $K_1(\mathcal{I})$. As a consequence

$$\begin{aligned}
[S_{\mathrm{I},2}P_{-1} + (1-P_{-1})] \;&=\; [V] + [S_{\mathrm{I},2}P_{-1} + (1-P_{-1})] + [V] \\
&=\; [V(S_{\mathrm{I},2}P_{-1} + (1-P_{-1}))V] \\
&=\; [\,\mathrm{e}^{\,\mathrm{i}\,b_0}\,S_{\mathrm{I},2}P_0 + (1-P_0)] \\
&=\; [S_{\mathrm{I},2}P_0 + (1-P_0))]
\end{aligned}$$

[1] Observe that the set of self-adjoint operators $\{\mathfrak{d}_0, \mathfrak{d}_1\}$ which defines P_{θ_+} is in principle different from the set of self-adjoint operators $\{\mathfrak{d}'_0, \mathfrak{d}'_1\}$ which defines P_{θ_-}.

where we used $VP_{-1}V = P_0$, $\mathfrak{r}S_{\mathrm{I},2}P_{-1}V = f_B S_{\mathrm{I},2}P_0 = \mathrm{e}^{\mathrm{i}\,b_0}\,S_{\mathrm{I},2}P_0$ and the fact that $\mathrm{e}^{\mathrm{i}\,b_0}\,S_{\mathrm{I},2}$ is connected to $S_{\mathrm{I},2}$ by the homotopy $[0,1]\ni t\mapsto \mathrm{e}^{\mathrm{i}\,(1-t)b_0}\,S_{\mathrm{I},2}$. □

Remark 4.4.8. The surjectivity of the exponential map can also be deduced directly by the exactness of the diagram (4.20). Since $\mathrm{Ker}(\exp)\simeq K_0(\mathcal{A}_{\mathrm{I}})\simeq\mathbb{Z}^3$ and $K_0(\mathcal{A}_{\mathrm{bulk}})\simeq\mathbb{Z}^4$ it follows that there is an element $[q]\in K_0(\mathcal{A}_{\mathrm{bulk}})$ such that $\exp([q]) = m[P_- + S_{\mathrm{I},2}P_0 + P_+]\in K_1(\mathcal{I})$ for some $m\in\mathbb{Z}\setminus 0$. In such case $\mathbb{Z}_m\simeq\imath_*(K_1(\mathcal{I}))\subset K_1(\mathcal{A}_{\mathrm{I}})\simeq\mathbb{Z}^3$ which is not possible unless $m=\pm1$, and in both cases the exponential map turns out to be surjective. ◁

The surjectivity of the index map (Proposition 4.4.4) and of the exponential map (Proposition 4.4.6) implies that the two maps $\imath_*$ in the diagram (3.13) are just the zero maps. After replacing $\imath_*=0$ in (3.13) one obtains the short exact sequences

$$\begin{aligned} 0\;\longrightarrow\;K_0(\mathcal{A}_{\mathrm{I}})\;&\xrightarrow{\mathrm{ev}_*}\;K_0(\mathcal{A}_{\mathrm{bulk}})\;\xrightarrow{\exp}\;K_1(\mathcal{I})\;\longrightarrow\;0\;,\\ 0\;\longrightarrow\;K_1(\mathcal{A}_{\mathrm{I}})\;&\xrightarrow{\mathrm{ev}_*}\;K_1(\mathcal{A}_{\mathrm{bulk}})\;\xrightarrow{\mathrm{ind}}\;K_0(\mathcal{I})\;\longrightarrow\;0\;. \end{aligned}$$

As a result, one gets further information about the structure of the K-theory of the Iwatsuka C^*-algebra.

Theorem 4.4.9 (K-theory of the Iwatsuka C^*-algebra II)**.** *It holds true that*

$$\begin{aligned} K_0(\mathcal{A}_{\mathrm{bulk}}) \;&=\; \mathrm{ev}_*\big(K_0(\mathcal{A}_{\mathrm{I}})\big)\;\oplus\;\psi_{\exp}\big(K_1(\mathcal{I})\big)\;,\\ K_1(\mathcal{A}_{\mathrm{bulk}}) \;&=\; \mathrm{ev}_*\big(K_1(\mathcal{A}_{\mathrm{I}})\big)\;\oplus\;\psi_{\mathrm{ind}}\big(K_0(\mathcal{I})\big) \end{aligned}$$

where $\psi_{\exp}$ and ψ_{ind} are suitable lifts of the exponential map and of the index map, respectively.

Proof. The two short exact sequences are of the form

$$0\;\longrightarrow\;\mathbb{Z}^3\;\xrightarrow{\alpha}\;\mathbb{Z}^4\;\xrightarrow{\beta}\;\mathbb{Z}\;\longrightarrow\;0$$

meaning that $\mathbb{Z}^4$ is an abelian extension of $\mathbb{Z}$ by $\mathbb{Z}^3$. The possible extensions are classified by $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}^3)=0$ [HS, Chapter III], meaning that only the trivial extension is possible. This in particular ensures the existence of the lifts $\psi_{\exp}$ and ψ_{ind}. ■

Remark 4.4.10. We can provide a more precise presentation of $K_1(\mathcal{A}_{\mathrm{bulk}})$ by combining Theorem 4.4.9 with the computation of the map ev_* and Proposition 4.4.4. One gets that

$$\begin{aligned} \mathrm{ev}_*\big(K_1(\mathcal{A}_{\mathrm{I}})\big)\;&=\;\mathbb{Z}[(S_{b_-,2},1)]\;+\;\mathbb{Z}[(1,S_{b_+,2})]\;+\;\mathbb{Z}[(S_{b_-,1},S_{b_+,1})]\;,\\ \psi_{\mathrm{ind}}\big(K_0(\mathcal{I})\big)\;&=\;\mathbb{Z}[(S_{b_-,1},1)]\;, \end{aligned}$$

where $[(S_{b_-,1},S_{b_+,1})]=[(S_{b_-,1},1)]+[(1,S_{b_+,1})]$ in the sense of the K_1-group and the (non-unique) lift ψ_{ind} has been chosen as $\psi_{\mathrm{ind}}([P_0]):=[(S_{b_-,1},1)]$. A similar analysis for $K_0(\mathcal{A}_{\mathrm{bulk}})$ provides

$$\begin{aligned} \mathrm{ev}_*\big(K_0(\mathcal{A}_{\mathrm{I}})\big)\;&=\;\mathbb{Z}[(1,0)]\;+\;\mathbb{Z}[(0,1)]\;+\;\mathbb{Z}[(P_{\theta_-},P_{\theta_+})]\;,\\ \psi_{\exp}\big(K_1(\mathcal{I})\big)\;&=\;\mathbb{Z}[(0,P_{\theta_+})]\;, \end{aligned}$$

where the (non-unique) lift $\psi_{\rm expo}$ is defined by $\psi_{\rm expo}([P_- + S_{\rm I,2}P_0 + P_+]) := [(0, P_{\theta_+})]$. Finally, we are in position to say something more about the Power-Rieffel-Iwatsuka projection $P_{\rm I} \in \mathcal{A}_{\rm I} \otimes {\rm Mat}_N(\mathbb{C})$ introduced short after Theorem 4.4.2. First consider $P_{\rm I} \in \mathcal{A}_{\rm I} \otimes {\rm Mat}_N(\mathbb{C})$ and I_M the identity matrix in ${\rm Mat}_M(\mathcal{A}_{\rm bulk})$ with $M \leq N$, such that

$$
{\rm ev}_*([P_{\rm I}] - [I_M]) \;=\; [(P_{\theta_-}, P_{\theta_+})] \;.
$$

This relation is satisfied in view of the surjectivity of ${\rm ev}_*$ and the standard picture of K_0-group [Weg, Proposition 6.2.7]. It follows that

$$
{\rm ev}_*([P_{\rm I}]) \;=\; [(P_{\theta_-}, P_{\theta_+})] + M[1] \;.
$$

Evidently $[P_-], [P_+], [P_{\rm I}]$ are a set of generators for $K_0(\mathcal{A}_{\rm I})$. The fact that $[P_{\rm I}]$ is the third generator of $K_0(\mathcal{A}_I)$ can be used in the six-term exact sequence 4.19 which provides $\epsilon[P_{\rm I}] \in \{\pm[S_{\rm I,1}]\}$, showing that $P_{\rm I}$ is actually a *PRI-projection*. It is interesting to note that even though neither $[(P_{\theta_-}, 0)]$ nor $[(0, P_{\theta_+})]$ can be lifted into a projection, the existence of the *PRI-projection* implies that the matrix ${\rm diag}((P_{\theta_-}, P_{\theta_+}), I_M) \in {\rm Mat}_{M+1}(\mathcal{A}_{\rm bulk})$, can actually be lifted into a *PRI-projection*. $\triangleleft$

4.5 Bulk-interface correspondence for the Iwatsuka C^*-algebra

Let us start with a preliminary result which is a direct consequence of Proposition 4.4.6.

Lemma 4.5.1. *Let $P = (P_-, P_+) \in \mathcal{A}_{\rm bulk}$ be a projection and $[P] \in K_0(\mathcal{A}_{\rm bulk})$ the related class in the K_0-group. Let $N_\pm := {\rm Ch}_{b_\pm}(P_\pm) \in \mathbb{Z}$ be the Chern numbers of $P_\pm$ defined by* (3.20)*. Then,*

$$
\exp([P]) \;=\; (N_- - N_+)[U_{\mathcal{I}}]
$$

where $[U_{\mathcal{I}}]$ is the generator of $K_1(\mathcal{I})$ defined in Proposition 4.4.3.

Proof. In terms of the generators of $K_0(\mathcal{A}_{\rm bulk})$ one has that if $[P] \in K_0(\mathcal{A}_{\rm Bulk})$, then

$$
[P] \;=\; M_-\,[(1,0)] \;+\; M_+\,[(0,1)] \;+\; N_-\,[(P_{\theta_-},0)] \;+\; N_+\,[(0,P_{\theta_+})]
$$

for some $M_\pm, N_\pm \in \mathbb{Z}$ suitable integers. Now consider the maps $\hat{\xi}_{b_-} := \xi_{b_-} \oplus 0$, $\hat{\xi}_{b_+} := 0 \oplus \xi_{b_+}$ defined in suitable supspaces of $\mathcal{A}_{\rm Bulk}$ (see sequation 3.19 and the domains there) and observe that both $\hat{\xi}_{b_-}$ and $\hat{\xi}_{b_+}$ are cyclic 2-cocycle. It follows from the canonical pairing between $HC^2(\mathcal{A}_{\rm Bulk})$ and $K_0(\mathcal{A}_{\rm Bulk})$ that

$$
\begin{aligned}
\prec [P], [\hat{\xi}_{b_-}] \succ \; &= \; M_- \prec [1], [\xi_{b_-}] \succ + N_- \prec [P_{\theta_-}], [\xi_{b_-}] \succ \; = \; N_-{\rm Ch}(P_{\theta_-}) \; = \; N_- \;, \\
\prec [P], [\hat{\xi}_{b_+}] \succ \; &= \; M_+ \prec [1], [\xi_{b_+}] \succ + N_+ \prec [P_{\theta_+}], [\xi_{b_+}] \succ \; = \; N_+{\rm Ch}(P_{\theta_+}) \; = \; N_+ \;.
\end{aligned}
\tag{4.29}
$$

Finally, by using that the map $\exp : K_0(\mathcal{A}_{\rm Bulk}) \to K_1(\mathcal{I})$ is a group homomorphism along with formulas (4.24), one gets the result. $\square$

For the next result we need the winding number $W_{\mathcal{I}}$ defined by (3.21). The derivation and the trace on $\mathcal{I}$, needed to build $W_{\mathcal{I}}$, are described at the end of Section 4.2

Lemma 4.5.2. *Let $U_{\mathcal{I}} \in \mathcal{I}^+$ be the unitary operator defined in Proposition 4.4.3. Then, it holds true that*

$$W_{\mathcal{I}}(U_{\mathcal{I}}) \;=\; 1$$

Proof. An explicit computation provides

$$\begin{aligned}(U_{\mathcal{I}}^* - 1)\,\partial_{\mathcal{I}}(U_{\mathcal{I}} - 1) \;&=\; \mathrm{i}\,P_0(S_{\mathrm{I},2}^* - 1)\big[P_0(S_{\mathrm{I},2} - 1), N_2\big] \\ &=\; \mathrm{i}\,P_0(S_{\mathrm{I},2}^* - 1)\big[S_{\mathrm{I},2}, N_2\big] \\ &=\; -\,\mathrm{i}\,P_0(S_{\mathrm{I},2}^* - 1)S_{\mathrm{I},2} \\ &=\; \mathrm{i}\,P_0(S_{\mathrm{I},2} - 1)\;.\end{aligned}$$

By applying formula (4.14) one gets

$$\mathscr{T}_{\mathcal{I}}\big((U_{\mathcal{I}}^* - 1)\partial_{\mathcal{I}}(U_{\mathcal{I}} - 1)\big) \;=\; -\,\mathrm{i}\,\mathscr{T}_{\mathcal{I}}(P_0 Q_0) \;=\; -\,\mathrm{i}$$

since $Q_0 S_{\mathrm{I},2} Q_0 = 0$. This completes the proof. □

We are now in position to provide our main result, namely the proof of equation (1.3).

Theorem 4.5.3 (Bulk-Interface duality for the Iwatsuka magnetic field)**.** *Let $\hat{H} \in \mathcal{A}_{\mathrm{I}}$ be a magnetic Hamiltonian with* non-trivial bulk gap *detected by Δ (cf. Definition 3.5.4). Let $g : \mathbb{R} \to [0,1]$ be a non-decreasing (smooth) function such that $g = 0$ below Δ and $g = 1$ above Δ and consider the unitary operator $U_\Delta := \mathrm{e}^{\mathrm{i}\,2\pi g(\hat{H})}$ and the associated interface conductance (cf. Definition 3.5.3)*

$$\sigma_{\mathcal{I}}(\Delta) \;:=\; \frac{e^2}{h}\,W_{\mathcal{I}}(U_\Delta)\;. \tag{4.30}$$

Let $H := \mathrm{ev}(\hat{H}) = (H_-, H_+) \in \mathcal{A}_{\mathrm{bulk}}$ be the bulk Hamiltonian and for a given Fermi energy inside the bulk gap $\mu \in \Delta$ let $P_\mu := (P_{\mu,-}, P_{\mu,+})$ with $P_{\mu,\pm} := \chi_{(-\infty,\mu]}(H_\pm)$ be the associated Fermi projections. Denote with $N_\pm := \mathrm{Ch}(P_{\mu,\pm}) \in \mathbb{Z}$ the Chern numbers of such projectors. Then it holds true that

$$\sigma_{\mathcal{I}}(\Delta) \;=\; \frac{e^2}{h}(N_+ - N_-)\;.$$

Proof. We can compute $\sigma_{\mathcal{I}}(\Delta)$ with the topological formula (3.28). From Lemma 4.5.1 and the bilinearity of the canonical pairing between $K_1(\mathcal{I})$ an d $HC^1(\mathcal{I})$ one obtains

$$\prec \exp([P_\mu]), [\eta_{\mathcal{I}}] \succ \;=\; (N_- - N_+) \prec [U_{\mathcal{I}}], [\eta_{\mathcal{I}}] \succ\;\;.$$

Then, equation (3.27) and Lemma 4.5.2 provide

$$\prec \exp([P_\mu]), [\eta_{\mathcal{I}}] \succ \;=\; (N_- - N_+) W_{\mathcal{I}}(U_{\mathcal{I}}) \;=\; N_- - N_+\;.$$

This concludes the proof. ■

Appendix

In this chapter a bunch of results that are used in this work, some of them more important than others, are included.

A Discrete Schwartz space

Just because of the lack of references, the discrete Schwartz space over a C^*-algebra will be defined and proved to be a Fréchet algebra.

Definition A.1 (Discrete Schwartz Space)**.** Let $\mathcal{A}$ be a C^*-algebra. The n-dimensional discrete Schwartz space over $\mathcal{A}$ is

$$S(\mathbb{Z}^n, \mathcal{A}) = \{a = \{a_m\}_{m\in\mathbb{Z}^n} \subset \mathcal{A} \mid ||a||_k < \infty \text{ for all } k \in \mathbb{N}_0\},$$

where

$$||a||_k = \sup_{m\in\mathbb{Z}^n} (1 + ||m||^2)^{\frac{k}{2}} ||a_m||.$$

In order to prove that the discrete Schwartz space is a Fréchet algebra it is important to note that each $||\cdot||_k$ is indeed a norm, and that $||\cdot||_0$ induces a banach space.

Lemma A.2. *The space* $\ell_\infty(\mathbb{Z}^n, \mathcal{A}) = \{\{a_m\}_{m\in\mathbb{Z}^n} \mid ||a||_0 < \infty\}$ *is complete with the norm* $||\cdot||_0$.

Proof. It is easy to see that $\ell_\infty(\mathbb{Z}^n, \mathcal{A})$ is a vector space and that $||\cdot||_0$ is actually a norm. We show completeness. Assume $a^{(k)} = \{a_m^{(k)}\}$ is a Cauchy sequence in $\ell_\infty(\mathbb{Z}^n, \mathcal{A})$, that is, $\sup_{m\in\mathbb{Z}^n} ||a_m^{(k)} - a_m^{(l)}|| < \varepsilon$ for $k, l \geq N(\varepsilon)$. Thus, each $\{a_m^{(k)}\}_k$ is a Cauchy sequence on $\mathcal{A}$ and consequently it converges, say, to a_m. Set $a = \{a_m\}_{m\in\mathbb{Z}^n}$ and note that if k is big enough then

$$||a_m|| \leq ||a_m - a_m^{(k)}|| + ||a_m^{(k)}|| \leq 1 + ||a^{(k)}||_0.$$

Since the bound is uniform on $m \in \mathbb{Z}^n$, it follows that $a \in \ell_\infty(\mathbb{Z}^n, \mathcal{A})$.
It remains to prove that $a^{(k)} \to a$. For all $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $k, l > N(\epsilon)$ we have $||a_m^{(k)} - a_m^{(l)}|| < \varepsilon$ for all $m \in \mathbb{Z}^n$. We now let go l to infinity and we get the result. □

Proposition A.3. *$S(\mathbb{Z}^n, \mathcal{A})$ is a Fréchet algebra when it is considered with the topology induced by the family of seminorms $\{|| \cdot ||_k\}_{k \in \mathbb{N}_0}$.*

Proof. That $S(\mathbb{Z}^n, \mathcal{A})$ is a vectorial space follows from the triangle inequality on the norms $|| \cdot ||_k$. In order to see that it is closed under multiplication let $a, b \in S(\mathbb{Z}^n, \mathcal{S}) \subset \ell_\infty(\mathbb{Z}^n, \mathcal{S})$ and note that

$$\begin{aligned} ||ab||_k &= \sup_{m \in \mathbb{Z}^n} (1 + ||m||^2)^{\frac{k}{2}} ||a_m b_m|| \\ &\leq \sup_{m \in \mathbb{Z}^n} (1 + ||m||^2)^{\frac{k}{2}} ||a_m|| \cdot ||b_m|| \\ &\leq ||a||_0 \sup_{m \in \mathbb{Z}^n} (1 + ||m||^2)^{\frac{k}{2}} ||b_m|| \\ &= ||a||_0 ||b||_k. \end{aligned}$$

Now we prove completeness. Assume $a^{(k)} = \{a_m^{(k)}\}_{m \in \mathbb{Z}^n}$ is a Cauchy sequence in $S(\mathbb{Z}^n, \mathcal{S})$, that is, $a^{(k)}$ is a Cauchy sequence for every $|| \cdot ||_l$. Observe that $a^{(k)}$ being a Cauchy sequence in the sense of the norm $||\cdot||_l$ is equivalent to that $a^{(k)}(l) = \{(1 + ||m||)^{\frac{l}{2}} a_m^{(k)}\}_{m \in \mathbb{Z}^n}$ is a Cauchy sequence in the sense of the norm $||\cdot||_0$. It follows from the completeness of $\ell_\infty(\mathbb{Z}^n, \mathcal{A})$ proved in the last lemma that there exist $a(l) \in \ell_\infty(\mathbb{Z}^n, \mathcal{A})$ for every $l \in \mathbb{N}_0$ such that $a^{(k)}(l) \to a(l)$ in the sense of the norm $|| \cdot ||_0$. It follows by mere punctual convergence that

$$a(l) = \{(1 + ||m||^{\frac{l}{2}}) a(0)\} \tag{A.1}$$

and so $a^{(k)} \to a(0)$ in the Fréchet topology. It remains to prove that $a(0) \in S(\mathbb{Z}^n, \mathcal{A})$, however it follows from the equality (A.1) that

$$||a(0)||_k = ||a(k)||_0 < \infty$$

since $a(k) \in \ell_\infty(\mathbb{Z}^n, \mathcal{A})$ for every $k \in \mathbb{N}_0$, which proves the completeness.
Finally, it is necessary to prove that joint multiplication is a continuous map. Since we have already proved that $S(\mathbb{Z}^n, \mathcal{A})$ is a Fréchet space it is enough to prove ([Wael] Chapter VII, proposition I) that multiplication is separately continuous. Let $a, b^{(k)} \in S(\mathbb{Z}^n, \mathcal{A})$ such that $b^{(k)} \to b$ and note that for every $l \in \mathbb{N}_0$

$$||ab^{(k)} - ab||_l \leq ||a||_0 ||b^{(k)} - b||_l,$$

which proves right continuity. Since left continuity is analogous this ends the proof. □

B The Pimsner-Voiculescu exact sequence

In this section we will provide a brief overview on the *Pimsner-Voiculescu six-term exact sequence* which is the main tool to compute the K-theory for crossed product C^*-algebras by $\mathbb{Z}$. For the interested reader we refer to the original work [PV2] and the monograph [Bla, Chapter V].

Let $\mathcal{A}$ be a C^*-algebra, $\alpha \in \mathrm{Aut}(\mathcal{A})$ an automorphism and $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathcal{A}$ by $\mathbb{Z}$, that is, the C^*-algebra generated by $\mathcal{A}$ together with a (abstract) unitary u and the relation

$$\alpha(a) \;=\; uau^* \;, \qquad \forall\, a \in \mathcal{A}\,.$$

The first step of the construction is to define an appropriate short exact sequence of C^*-algebras. This is done by considering the tensor product $\mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space, and the C^*-algebra $\mathcal{T}_\alpha$ generated in $\mathcal{A} \otimes C^*(v)$ by $\mathcal{A} \otimes 1$ and $V = u \otimes v$, where v is a non-unitary (abstract) isometry. It is useful to think of elements of $\mathcal{K}$ as infinite matrices acting on $\ell^2(\mathbb{N}_0^2)$ with respect to its canonical basis.
Let us consider the maps $\varphi : \mathcal{A} \otimes \mathcal{K} \to \mathcal{T}_\alpha$ and $\psi : \mathcal{T}_\alpha \to \mathcal{A} \rtimes_\alpha \mathbb{Z}$ such that

$$\varphi(a \otimes e_{j,k}) \;:=\; V^j(a \otimes (1 - vv^*))(V^*)^k \;=\; (\alpha^j(a) \otimes 1)V^j P (V^*)^k$$

and

$$\psi(a \otimes 1) \;:=\; a\,, \qquad \psi(V) \;:=\; u\,,$$

where P is the (non-trivial) self-adjoint projection given by $P := 1 - VV^* = 1 \otimes (1 - vv^*)$ and $e_{j,k}$ are the rank one operators which generates $\mathcal{K}$.
Both maps ϕ and ψ can be proved to extend as $*$-homomorphism and fit in the short exact sequence of C^*-algebras

$$0 \;\longrightarrow\; \mathcal{A} \otimes \mathcal{K} \;\xrightarrow{\varphi}\; \mathcal{T}_\alpha \;\xrightarrow{\psi}\; \mathcal{A} \rtimes_\alpha \mathbb{Z} \;\longrightarrow\; 0. \tag{B.1}$$

The Pimsner-Voiculescu (six-term) exact sequence is a cyclic sequence which connects the K-theory of $\mathcal{A}$ and $\mathcal{A} \rtimes_\alpha \mathbb{Z}$. Its is given by

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \xrightarrow{1-\alpha_*^{-1}} & K_0(\mathcal{A}) & \xrightarrow{\imath_*} & K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}) \\ \delta\Big\uparrow & & & & \Big\downarrow\epsilon \\ K_1(\mathcal{A} \rtimes_\alpha \mathbb{Z}) & \underset{\imath_*}{\longleftarrow} & K_1(\mathcal{A}) & \underset{1-\alpha_*^{-1}}{\longleftarrow} & K_1(\mathcal{A}) \end{array} \tag{B.2}$$

and it is worth pointing out that this is not exactly the standard six-term exact sequence associated with the short exact sequence (B.1), although it is closely related. The maps $\imath_*$ are induced by the canonical inclusion $\imath : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes_\alpha \mathbb{Z}$ and the maps $(1-\alpha_*^{-1})$ are induced by the map $(1-\alpha^{-1}) : \mathcal{A} \to \mathcal{A}$. For the vertical maps consider ind : $K_1(\mathcal{A} \rtimes_\alpha \mathbb{Z}) \to K_0(\mathcal{A} \otimes \mathcal{K})$ and exp : $K_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}) \to K_1(\mathcal{A} \otimes \mathcal{K})$ to be the index and the exponential maps for the standard six-term exact sequence in K-theory emerging from the short exact sequence (B.1) (*cf.* [Weg, Theorem 9.3.2]), then $\epsilon := \kappa_0^{-1} \circ \mathrm{ind}$ and $\delta := \kappa_0^{-1} \circ \exp$, where $\kappa_0 : K_j(\mathcal{A}) \to K_j(\mathcal{A} \otimes \mathcal{K})$, with $j = 0, 1$, is the stabilization isomorphism induced by $a \mapsto a \otimes e_{0,0}$ for every $a \in \mathcal{A}$.

C K-Theory of $C(\mathbb{S})$

It is a well known fact that $K_0(C(\mathbb{S})) = K_1(C(\mathbb{S})) = \mathbb{Z}$. However, in chapter 4.4 we also need to know what are the generators of these groups, and because of that we'll need to follow the construction of these groups in order to get such information.
First recall that for every C^*-algebra A we have isomorphisms $\theta_A : K_1(A) \to K_0(SA)$ and $\beta_A : K_0(A) \to K_1(SA)$ ([Weg, Chapters 7 and 9]), where $SA = A \otimes C_0(\mathbb{R}) = \{f \in C(\mathbb{S} \to A) \mid f(1) = 0\}$ is the suspension algebra asociated to A.
Now let $A = \mathbb{C}$ and observe that in such case

$$\begin{aligned} \{0\} = K_1(\mathbb{C}) \cong K_0(S\mathbb{C}) \\ \mathbb{Z}[1] = K_0(\mathbb{C}) \cong K_1(S\mathbb{C}), \end{aligned} \tag{C.1}$$

so $K_0(S\mathbb{C}) = \{0\}$ and $K_1(S\mathbb{C}) = \mathbb{Z}[\beta_\mathbb{C}(1)]$. Following chapter 9 of [Weg] it holds true that $\beta_\mathbb{C}(1) = [f_1]$, where $f_1 : \mathbb{S} \to \mathbb{C}$ is given by

$$f_1(z) = 1 - (1 - z) = z,$$

so $\beta_\mathbb{C}(1) = \mathrm{Id}_\mathbb{S}$.
Now observe that $S\mathbb{C} \oplus \mathbb{C} \cong C(\mathbb{S})$, where the isomorphism is given by $f \oplus \lambda \mapsto f + \lambda$ for $f \in C(\mathbb{S} \to A)$, $f(1) = 0$ and $\lambda \in \mathbb{C}$. It follows from the last affirmation and (C.1) that

$$\begin{aligned} K_0(C(\mathbb{S})) \cong K_0(S\mathbb{C} \oplus \mathbb{C}) \cong K_0(S\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \ & \mathbb{Z}[1] \\ K_1(C(\mathbb{S})) \cong K_1(S\mathbb{C} \oplus \mathbb{C}) \cong K_1(S\mathbb{C}) \oplus K_1(\mathbb{C}) \cong \ & \mathbb{Z}[\mathrm{Id}_\mathbb{S}]. \end{aligned} \tag{C.2}$$

D Noncommutative Torus in a Nutshell

In this appendix the structure and K-theory of the noncommutative torus will be briefly summarized. The importance of this appendix relies on the fact that the noncommutative torus is isomorphic to the magnetic C^*-algebra associated to a constant magnetic field $\mathcal{A}_b$ (see 2.3.3), and such algebra is of extreme importance in the sequence 3.9 because of the definition of the bulk algebra $\mathcal{A}_{\mathrm{bulk}}$ (3.10). The K-theory for the noncommutative torus was broadly studied in the 80's and the results stated are mostly based on the original papers [PV1, PV2, Rie] and the posterior monographs [Weg, Chapter 12.3] and [GBVF, Chapter 12].

The noncommutative torus A_θ for $\theta \in [0,1]$ is the universal C^*-algebra generated by the (abstract) unitaries u, v with the commutation relation

$$uv = e^{2\pi \mathrm{i}\,\theta}\; vu. \tag{D.1}$$

The noncommutative torus has a crossed product structure, namely $\mathrm{A}_\theta \simeq C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}$ where

$$\rho_\theta(f)(z) := f(z-\theta), \quad \text{for } f \in C(\mathbb{T}),$$

and by the torus we mean $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$[2]. In this frame we can consider $u(t) := \mathrm{e}^{2\pi \mathrm{i}\, t}$ and v the unitary such that $\rho_\theta(g) = vgv^*$ to reassemble the definition of the noncommutative torus. Finally, let us remark that since the action ρ_θ acts by rotating functions defined on $\mathbb{T}$, A_θ is usually called the *Rotation Algebra*.
The representation of A_θ as a crossed product has two great implications. First, its K-Theory can be fitted in the Pimnser-Voiculescu exact sequence (see Appendix B), and second, most of the terms involved in such exact sequence were already studied in the appendix C, recalling $\mathbb{S} \simeq \mathbb{T}$. The six-term exact sequence in question is

$$\begin{tikzcd}
K_0(C(\mathbb{T})) \arrow[r, "1-\rho_{\theta,*}^{-1}"] & K_0(C(\mathbb{T})) \arrow[r, "\imath_*"] & K_0(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) \arrow[d, "\epsilon"] \\
K_1(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) \arrow[u, "\delta"] & K_1(C(\mathbb{T})) \arrow[l, "\imath_*"'] & K_1(C(\mathbb{T})) \arrow[l, "1-\rho_{\theta,*}^{-1}"']
\end{tikzcd} \tag{D.2}$$

where the maps induced by $1-\rho_\theta^{-1}$ in the K-Theory level vanish because $\rho_\theta(f)$ is homotopically connected to f in $C(\mathbb{T})$.
As a consequence of the exactness of the sequence D.2 we can extract the short exact sequences

$$0 \longrightarrow K_0(C(\mathbb{T})) \xrightarrow{\imath_*} K_0(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) \xrightarrow{\epsilon} K_1(C(\mathbb{T})) \longrightarrow 0, \tag{D.3}$$

$$0 \longrightarrow K_1(C(\mathbb{T})) \xrightarrow{\imath_*} K_1(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) \xrightarrow{\delta} K_0(C(\mathbb{T})) \longrightarrow 0, \tag{D.4}$$

and then conclude that $K_j(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) \simeq K_0(C(\mathbb{T})) \oplus K_1(C(T)) \simeq \mathbb{Z}^2$.

D.1 Generators for the K_1-group

The generators of $K_1(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z})$ can be easily found noting that the exact sequence D.4 actually implies that

$$K_1(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) = K_1(C(\mathbb{T})) \oplus \delta^{-1}(K_0(C(\mathbb{T}))) = \mathbb{Z}[u] \oplus \mathbb{Z}\delta^{-1}[1],$$

[2]In this appendix we change the torus representation mainly to use the same notation used in the original papers [PV1, PV2, Rie].

for what we have used $K_0(C(\mathbb{T})) = \mathbb{Z}[1]$ and $K_1(C(\mathbb{T})) = \mathbb{Z}[u]$ (*cf.* (C.2)).
Finally, it can be proved by a direct computation that $\delta[v] = [1]$ proving the next well known result.

Proposition D.1. *It holds true that*

$$K_1(\mathrm{A}_\theta) = \mathbb{Z}[u] \oplus \mathbb{Z}[v].$$

D.2 Generators for the K_0-group

The construction of the generators of the K_0-group depend strongly on the condition $\theta \neq 0$, because if $\theta = 0$ the C^*-algebra A_θ is easily seen to be isomorphic to $C(\mathbb{T}^2)$, which is projectionless. Due to this fact we will start studying this case, that is, finding generators for $K_0(C(\mathbb{T}^2))$.

The technique that we are about to use was left as an exercise in ([Weg, Exercise 8.b]). First consider the exact sequence

$$0 \longrightarrow SC(\mathbb{T}) \xrightarrow{\imath} C(\mathbb{T} \to C(\mathbb{T})) \xrightarrow{\mathrm{ev}} C(\mathbb{T}) \longrightarrow 0,$$

where $S\mathcal{A} = \{f \in C(\mathbb{T} \to \mathcal{A}) \mid f(1) = 0\}$ is the suspension algebra of the C^*-algebra $\mathcal{A}$, $\imath$ is the identity map and $\mathrm{ev}(f) = f(1)$ is the evaluation at 1. It follows that in the K-Theory level we have

$$\begin{array}{ccccc} K_0(SC(\mathbb{T})) & \xrightarrow{\imath_*} & K_0(C(\mathbb{T} \to C(\mathbb{T}))) & \xrightarrow{\mathrm{ev}_*} & K_0(C(\mathbb{T})) \\ \mathrm{ind}\Big\uparrow & & & & \Big\downarrow\mathrm{exp} \\ K_1(C(\mathbb{T})) & \xleftarrow{\mathrm{ev}_*} & K_1(C(\mathbb{T} \to C(\mathbb{T}))) & \xleftarrow{\imath_*} & K_1(C(\mathbb{T})). \end{array} \tag{D.5}$$

Simple computations imply $\exp[1] = [1]$ and $\mathrm{ind}[u] = [0]$ making the maps ind and exp vanish. Consequently we can extract the short exact sequence

$$0 \longrightarrow K_0(SC(\mathbb{T})) \xrightarrow{\imath_*} K_0(C(\mathbb{T} \to C(\mathbb{T}))) \xrightarrow{\mathrm{ev}_*} K_0(C(\mathbb{T})) \longrightarrow 0,$$

and hence

$$K_0(C(\mathbb{T} \to C(\mathbb{T}))) = K_0(SC(\mathbb{T})) \oplus \mathrm{ev}_*^{-1}(K_0(C(\mathbb{T}))).$$

In order to conclude let's observe that first, the map $T : C(\mathbb{T} \to C(\mathbb{T})) \to C(\mathbb{T}^2)$ defined by

$$(Tf)(x, y) = f(x)(y), \quad x, y \in \mathbb{T}$$

is an isomorphism, and second, the Bott periodicity give us an isomorphism $\theta_{C(\mathbb{T})} : K_1(C(\mathbb{T})) \to K_0(SC(\mathbb{T}))$ (see [Weg, Theorem 7.2.5]), so it would be enough to carefully follow the paths to find generators for $K_0(C(\mathbb{T}^2))$.

Proposition D.2. *It holds true that*

$$K_0(C(\mathbb{T}^2)) \simeq \mathbb{Z}[1] \oplus \mathbb{Z}[\beta],$$

where $\beta \in \mathrm{Mat}_2(C(\mathbb{T}))$ is a projection (explicitely written in the proof).

Proof. Summarizing the relevant maps defined so far we have that

$$T_* \circ (\theta_{C(\mathbb{T})} \oplus \mathrm{ev}_*^{-1}) : \mathrm{K}_1(\mathrm{C}(\mathbb{T})) \oplus \mathrm{K}_0(\mathrm{C}(\mathbb{T})) \to \mathrm{K}_0(\mathrm{C}(\mathbb{T}^2))$$

is an isomorphism, so

$$K_0(C(\mathbb{T}^2)) = \mathbb{Z}((T_* \circ \mathrm{ev}_*^{-1})\,[1]\,) \oplus \mathbb{Z}((T_* \circ \theta_{C(\mathbb{T})})\,[u]\,).$$

Observe that clearly $T_*\,(\mathrm{ev}_*^{-1}\,[1]\,) = [1]$. On the other side, following the proof of [Weg, Theorem 7.2.5] we have

$$\theta_{C(\mathbb{T})}[u] = [q] - [1],$$

where q is a loop of projections defined by $q_t = w_t \mathrm{diag}(1,0) w_t^*$, and w_t is any homotopy of unitaries between $w_0 = \mathrm{diag}(1,1)$ and $w_1 = \mathrm{diag}(u, u^*)$. In particular let us consider

$$w_t = \mathrm{diag}(u,1)\, r_t\, \mathrm{diag}(u^*,1)\, r_t^*$$

where

$$r_t = \begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix},$$

and observe that w_t is unitary for every $t \in [0,1]$, $w_0 = \mathrm{diag}(1,1)$ and $w_1 = \mathrm{diag}(u,u^*)$. Finally define the *Bott projection* as

$$\beta(s,t) = \mathrm{diag}(1,1) - q_t(s),$$

where the evaluation on s of q_t is made in its entries as a matrix (which is possible since each entry is actually an element of $C(\mathbb{T})$) and observe

$$T_*\,(\theta_{C(\mathbb{T})}[u]) = -[\beta].$$

Last equality together with $T_* \circ \mathrm{ev}_*^{-1}[1] = [1]$ imply that a couple of generators for $K_0(C(\mathbb{T}^2))$ are $[1]$ and $-[\beta]$, proving that

$$K_0(C(\mathbb{T}^2)) = \mathbb{Z}[1] \oplus \mathbb{Z}[\beta].$$

□

The case when $\theta \neq 0$ can be developed in a rather different way. The original method for the irrational case was studied in [PV1], [PV2] and [Rie], and consisted on: proving that A_θ admits a particular trace state τ whose image is contained in $\mathbb{Z} \oplus \theta\mathbb{Z}$; finding a projection,

namely the Power-Rieffel projection p_θ; and finally proving that $\tau p_\theta = \theta$. Since every trace state can be extended as a morphism in K_0 and we already know that $K_0(\mathrm{A}_\theta) \simeq \mathbb{Z}^2$ it follows that since the domain and range of the surjective map $\tau_* : K_0(\mathrm{A}_\theta) \to \mathbb{Z} \oplus \theta\mathbb{Z}$ are free groups, the elements $[1], [p_\theta]$ must generate the entire K_0-group. The method stated here corresponds to a latter computation shown in [PV2, Appendix] which luckily doesn't depend on the irrationality of θ.
First note that because of the exactness of the sequence D.3 it follows that

$$K_0(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) = \imath_*(K_0(C(\mathbb{T}))) \oplus \delta^{-1}(K_1(C(\mathbb{T}))) = \imath_*[1] \oplus \delta^{-1}[u],$$

and then since $\imath_*[1] = [\imath(1)] = [1]$, the only missing part is finding a lift for $[u] \in K_1(C(\mathbb{T}))$. In order to find a lift for $[u] \in K_1(C(\mathbb{T}))$ let's define the Power-Rieffel projection as an element of the form

$$p_\theta = v^* g^* + f + gv \in C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z},$$

where $f, g \in C(\mathbb{T})$ are chosen in a way such that p_θ is actually a projection.
The existence of a Power-Rieffel projection is not trivial but it follows from the next elementary observations:

1. p_θ is automatically self-adjoint when f is real valued, and
2. $p_\theta^2 = p_\theta$ is equivalent to

$$\begin{aligned} &\rho_\theta(g) \cdot g = 0 \\ &g \cdot (f + \rho_\theta(f)) = g \\ &f = f^2 + |g|^2 + |\rho_\theta^{-1}(g)|^2. \end{aligned} \tag{D.6}$$

Explicit choices for $f, g \in C(\mathbb{T})$ can be found (see [GBVF, Proposition 2.4] and [Weg, Exercise 12.M]) and it is also noted in [GBVF] that no matter the choice of these functions, the Power-Rieffel projections define the same element in the $K_0(\mathrm{A}_\theta)$.
The final ingredient is the formula proved in [PV2, Appendix] that states

$$\delta([p_\theta]) = [\exp(2\pi\,\mathrm{i}\, f \triangle)],$$

where $\triangle$ is the left support projection of g. The proof of this fact is extremely similar to the one of 4.4.6 and hence it will be omitted in this appendix.

Proposition D.3. *It holds true that*

$$K_0(C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}) = \mathbb{Z}[1] \oplus \mathbb{Z}[p_\theta].$$

Proof. As said before, the only fact remining is finding a lift for $[u] \in K_1(C(\mathbb{T}))$ and we claim that $\delta[p_\theta] = [u]$. Let $\delta \in (0,1)$ such that $\theta + \delta < 1$,

$$f(t) = \frac{t}{\delta} \mathbb{1}_{[0,\delta]} + \mathbb{1}_{(\delta,\theta)} + \left(-\frac{t}{\delta} + 1 + \frac{\theta}{\delta}\right) \mathbb{1}_{[\theta,\theta+\delta]} \quad \text{and} \quad g(t) = \sqrt{f(t)(1-f(t))}\mathbb{1}_{[0,\delta]},$$

and observe that f, g satisfy the conditions given in (D.6), so we have a Power-Rieffel projection p_θ.
Now note that in such case $\triangle = \mathbb{1}_{[0,\delta]}$, so we have that

$$\delta[p_\theta] = [\exp(2\pi\,\mathrm{i}\, f\triangle)] = \left[\exp\left(2\pi i \frac{t}{\delta} \mathbb{1}_{[0,\delta]}\right)\right] .$$

Finally consider

$$u_s(t) = \exp\left(2\pi\,\mathrm{i}\,\frac{t}{\delta + s(1-\delta)} \mathbb{1}_{[0,\delta+s(1-\delta)]}\right)$$

and observe that

1. $u_0(t) = \exp\left(2\pi\,\mathrm{i}\,\frac{t}{\delta}\mathbb{1}_{[0,\delta]}\right)$,
2. $u_1(t) = \exp(2\pi\,\mathrm{i}\,t)$ and
3. $u_s \in C(\mathbb{T})$ is unitary for each $s \in [0,1]$,

so it follows that

$$\left[\exp\left(2\pi\,\mathrm{i}\,\frac{t}{\delta}\mathbb{1}_{[0,\delta]}\right)\right] = [\exp(2\pi\,\mathrm{i}\,t)] = [u]$$

and the proof is complete. □

Before finishing this appendix, let us provide a presentation P_{θ_b} optimized for the aims of this work. We will set

$$P_{\theta_b} \ := S_{b,1}^* \,\mathfrak{d}_1 \ + \ \mathfrak{d}_0 \ + \ \mathfrak{d}_1 \, S_{b,1}$$

where $\mathfrak{d}_1 := g(S_2)$ and $\mathfrak{d}_0 := f(S_2)$[3] are suitable self-adjoint elements of $C^*(S_2) \subset \mathcal{A}_b$. Here we are using the coincidence $S_2 = S_{b,2}$ between the ordinary shift and magnetic translation in view of the election of the Landau gauge for the constant magnetic field. The requirement for P_{θ_b} of being a projection is automatically satisfied if the following relations hold true:

$$\begin{aligned} ((S_{b,1})^*\mathfrak{d}_1 S_{b,1})\mathfrak{d}_1 \ &= \ 0 \ , \\ \mathfrak{d}_1(\mathfrak{d}_0 + S_{b,1}\mathfrak{d}_0 S_{b,1}^*) \ &= \ \mathfrak{d}_1 \ , \\ \mathfrak{d}_0^2 + \mathfrak{d}_1^2 + (S_{b,1}^*\mathfrak{d}_1 S_{b,1})^2 \ &= \ \mathfrak{d}_0 \ . \end{aligned} \tag{D.7}$$

The relations (D.7) provide a useful identity. Let $\mathfrak{L} := \mathfrak{L}(\mathfrak{d}_1)$ be the left support projection of $\mathfrak{d}_1$ (in the von Neumann algebra generated by $\mathcal{A}_b$). This is by definition the smallest projection such that $\mathfrak{L}\mathfrak{d}_1 = \mathfrak{d}_1 = \mathfrak{d}_1\mathfrak{L}$. It is immediate to conclude that $\mathfrak{L}$ is mapped into the characteristic function on the support of $g \circ e$ under the isomorphism used above,

[3]The functions f, g can be chosen to be as in the proof of Lemma D.3 with a little abuse of notation taking advantage of the homeomorphism $\mathbb{T} \simeq \mathbb{S}^1$.

i. e. $\mathfrak{L} \mapsto \chi_{[0,\delta]}$. Combining $\mathfrak{L}$ with the first relation in (D.7) one gets $((S_{b,1})^*\mathfrak{d}_1 S_{b,1})\mathfrak{L} = 0$. This relation combined with the third equation in (D.7) provides

$$\mathfrak{d}_1^2 \;=\; \mathfrak{L}(\mathfrak{d}_0 - \mathfrak{d}_0^2) \;=\; (\mathfrak{d}_0 - \mathfrak{d}_0^2)\mathfrak{L}\,. \tag{D.8}$$

Finally, and mainly to avoid writing a larger proof than needed for Proposition 4.4.6 by getting advantage of the context of this appendix, let us state the next results.

Lemma D.4. *The following statements are true:*

1. *The unitary operators* $\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}}$ *and* $S_{b,2}^*$ *are homotopic equivalent in* $C^*(S_2)$*; and*

2. *The unitary operators* $\mathrm{e}^{-\,\mathrm{i}\,2\pi\mathfrak{d}_0\mathfrak{L}'}$ *and* $S_{b,2}$ *are homotopic equivalent in* $C^*(S_2)$*.*

Proof. The homotopy connecting the elements in the first statement was made, up to representation, in the proof of the last lemma D.3. For the second statement consider the representation on $C(\mathbb{T}) \rtimes_{\rho_\theta} \mathbb{Z}$ by taking the function f and g as in the proof of D.3 and observe that in such case it is enough to find an homotopy between the functions $x(t) := \exp(-\,\mathrm{i}\,2\pi\left(1 + \frac{\theta - t}{\delta}\right)\mathbb{1}_{[\theta,\theta+\delta]}(t))$ and $u(t) = \mathrm{e}^{\mathrm{i}\,2\pi t}$, where we used that in this representation $\mathfrak{L}' \mapsto \mathbb{1}_{[\theta,\theta+\delta]}$. Such an homotopy is explicitly given by

$$[0,1] \ni t \;\longmapsto\; \exp\left(-\,\mathrm{i}\,2\pi\left(\frac{(1-t)(\theta_b - k + \delta - 1) - t\delta k}{\delta}\right)\mathbb{1}_{[(1-t)\theta_b,(1-t)(\theta_b+\delta-1)+1]}(k)\right)\,,$$

completing the proof. $\square$

E The K-theory of $c(\mathbb{Z})$

In this section the K-theory of $c(\mathbb{Z}) = \{g \in \ell^\infty(\mathbb{Z}) \mid \text{the limits } \lim_{n\to\pm\infty} g(n) \text{ exists }\}$ si computed. Let us recall that the importance of this appendix rely in the isomorphism between the latter an the flux algebra associated to the Iwatsuka magnetic field, that is, $\mathcal{F}_{\mathrm{I}} \simeq c(\mathbb{Z})$.

First consider the short exact sequence

$$0 \;\longrightarrow\; c_0(\mathbb{Z}) \;\stackrel{\imath}{\longrightarrow}\; c(\mathbb{Z}) \;\stackrel{\mathrm{ev}}{\longrightarrow} \mathbb{C}\oplus\mathbb{C} \;\longrightarrow 0 \tag{E.1}$$

where $c_0(\mathbb{Z})$ is the C^*-algebra of sequences vanishing at infinity, $\imath$ is the inclusion homomorphism and the evaluation homomorphism ev compute the left and right limits of elements in $c(\mathbb{Z})$. The sequence (E.1) is split exact in view of the $*$-homomorphism

$$\mathbb{C}\oplus\mathbb{C} \;\ni\; (\ell_-,\ell_+) \;\stackrel{\jmath}{\longrightarrow}\; c_{(\ell_-,\ell_+)} \;\in\; c(\mathbb{Z})$$

where the element $c_{(\ell_-,\ell_+)}$ is specified by

$$c_{(\ell_-,\ell_+)}(n) \;:=\; \begin{cases} \ell_- & \text{if } n < 0 \\ \ell_+ & \text{if } n \geqslant 0\,. \end{cases}$$

Then, it follows that [Weg, Corollary 8.2.2]

$$K_j\big(c(\mathbb{Z})\big) \;\simeq\; K_j\big(c_0(\mathbb{Z})\big) \;\oplus\; K_j\big(\mathbb{C}\oplus\mathbb{C}\big)\;, \qquad j=1,2\;.$$

The K-theory of $\mathbb{C}\oplus\mathbb{C}$ is easily calculated as $K_0(\mathbb{C}\oplus\mathbb{C}) = \mathbb{Z}\oplus\mathbb{Z}$ and $K_1(\mathbb{C}\oplus\mathbb{C}) = 0$. The K-theory of $c_0(\mathbb{Z})$ is given by $K_0(c_0(\mathbb{Z})) = \mathbb{Z}^{\oplus\mathbb{Z}}$ and $K_0(c_0(\mathbb{Z})) = 0$. The latter fact follows from the isomorphism $K_j(c_0(\mathbb{Z})) \simeq K^j_{\rm top}(\mathbb{Z}) \simeq K^j_{\rm top}(*)^{\oplus\mathbb{Z}}$ between the algebraic and the topological K-theory [BSG, Theorem 5]. Another way of achieving the same result is to consider the Pontryagin duality $\mathbb{S}^1 = \widehat{\mathbb{Z}}$ and the isomorphism $c_0(\mathbb{Z}) \simeq C^*_r(\mathbb{S}^1)$ where $C^*_r(\mathbb{S}^1)$ is the (reduced) group algebra of the circle [Dav, Proposition VII.1.1]. Therefore, one has that $K_0(C^*_r(\mathbb{S}^1)) \simeq {\rm Rep}(\mathbb{S}^1) \simeq \mathbb{Z}^{\oplus\mathbb{Z}}$ and $K_0(C^*_r(\mathbb{S}^1)) \simeq 0$ where ${\rm Rep}(\mathbb{S}^1)$ denotes the complex representation ring of $\mathbb{S}^1$ [BSG, Section 7]. The generators of $K_j(c_0(\mathbb{Z}))$ are the classes $[\delta_j]$, $j\in\mathbb{Z}$. After putting all the information together, we can describe the K-theory of $\mathcal{F}_{\rm I}$ as

$$K_0\big(\mathcal{F}_{\rm I}\big) \;\simeq\; \bigoplus_{j\in\mathbb{Z}}\mathbb{Z}[\delta_j] \;\oplus\; \mathbb{Z}[\delta_-] \;\oplus\; \mathbb{Z}[\delta_+]\;, \qquad K_1\big(\mathcal{F}_{\rm I}\big) \;=\; 0\;.$$

where δ_- and δ_+ are defined in equation (2.5).

F Magnetic Bloch-Floquet transform

Let $\mathcal{A}_B$ be the magnetic C^*-algebra of the magnetic field B as in Definition 2.3.1. We are interested in the case where the magnetic field is constant along every vertical line on $\mathbb{Z}^2$, *i. e.* $B(n_1,n_2) = B(n_1)$ for every $n := (n_1,n_2)\in\mathbb{Z}^2$. The Iwatsuka magnetic field is of course contained in such a class of examples. Observe that these magnetic fields admit Landau-type potentials given by

$$A_B(n, n-e_j) \;=\; \delta_{j,1}\, n_2\, B(n_1), \quad n\in Z^2,$$

and consequently the pair of magnetic translations which define $\mathcal{A}_{A_B}$ is

$$S_{B,1} := S_{A_B,1} \;=\; {\rm e}^{{\rm i}\, N_2 B(N_1}\, S_1\;, \qquad S_{B,2} := S_{A_B,2} \;=\; S_2\;,$$

where X_1, X_2 are the position operators on $\ell^2(\mathbb{Z}^2)$. Let $V_h := {\rm e}^{{\rm i}\, h(N_1)}\, S_2 \in \mathcal{B}(\ell^2(\mathbb{Z}^2))$ for a given function $h:\mathbb{Z}\to\mathbb{R}$. The operator V_h is unitary and commutes with $S_{B,2}$ by construction. On the other hand one can compute that

$$V_h\; S_{B,1}\; V_h^* \quad=\quad {\rm e}^{{\rm i}\, h(N_1)}\;\; {\rm e}^{-{\rm i}\, B(N_1)}\;\; {\rm e}^{-{\rm i}\, h(N_1-1)}\;\; S_{B,1}\;.$$

The commutation condition $V_h S_{B,j} V_h^* = S_{B,j}$ is then guaranteed by

$$h(m) \;-\; h(m-1) \;=\; B(m)\;, \qquad \forall\, m\in\mathbb{Z}\;. \tag{F.1}$$

Note that equation (F.1) determines the function h up to a constant, that is, by fixing $h(0) = a$ one gets

$$h(m) \;:=\; a \;+\; \delta_{m>0}\sum_{j=1}^{m} B(j) \;-\; \delta_{m<0}\sum_{j=0}^{|m|-1} B(-j)\;.$$

The map $j \mapsto V_h^j$ provides a unitary representation of $\mathbb{Z}$ on $\ell^2(\mathbb{Z}^2)$ which commutes with the magnetic translations $S_{B,j}$, and consequently with the magnetic algebra $\mathcal{A}_B$. This fact can be used to define the *magnetic Bloch-Floquet transform*[4] $\mathcal{U}_B$ as follows:

$$(\mathcal{U}_B\psi)_t \ := \ \sum_{j\in\mathbb{Z}} \mathrm{e}^{-\,\mathrm{i}\,j\,t} \ V_h^j \ \psi \ . \quad t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

The map $\mathcal{U}_B$ is initially defined on the dense domain $\psi \in \mathcal{C}_{\mathrm{c}}(\mathbb{Z}^2) \subset \ell^2(\mathbb{Z}^2)$ of the compactly supported sequences. From its very definition one gets

$$(\mathcal{U}_B(V_h)^s\psi)_t \ = \ \mathrm{e}^{\mathrm{i}\,t\,s}\,(\mathcal{U}_B\psi)_t \ . \tag{F.2}$$

This equation expresses the fact that the transformed vectors $(\mathcal{U}_B\psi)_t$ are generalized eigenvectors of V_h. The latter condition can be rewritten in the form

$$(\mathcal{U}_B\psi)_t(n_1, n_2 - s) \ = \ \mathrm{e}^{\mathrm{i}\,s(t-f(n_1))}\,(\mathcal{U}_B\psi)_t(n_1, n_2).$$

and shows that $(\mathcal{U}_B\psi)_t$ is entirely determined by a single value of n_2, *e.g.* by its value on the horizontal line $n_2 = 0$. The latter observation allows to identify $(\mathcal{U}_B\psi)_t$, for every fixed $t \in \mathbb{T}$, as an element of the *fiber space* $\ell^2(\mathbb{Z})$ by fixing $n_2 = 0$, *i.e.* by setting $(\mathcal{U}_B\psi)_t(r) := (\mathcal{U}_B\psi)_t(r, 0)$ for every $r \in \mathbb{Z}$. More precisely, one can show that $\mathcal{U}_B$ defined in this way provides a unitary equivalence[5]

$$\mathcal{U}_B \ : \ \ell^2(\mathbb{Z}^2) \ \longrightarrow \mathcal{H} := \int_{\mathbb{S}^1}^{\oplus} \mathrm{d}t \ \ell^2(\mathbb{Z}) \ \simeq \ \ell^2(\mathbb{Z}) \otimes L^2(\mathbb{S}^1) \ , \tag{F.3}$$

where $\mathrm{d}t$ is the normalized Haar measure of $\mathbb{S}^1$. In fact a standard computation shows that $\mathcal{U}_B$ is isometric on the dense domain $\mathcal{C}_{\mathrm{c}}(\mathbb{Z}^2)$, hence extends to an isometry on $\ell^2(\mathbb{Z}^2)$. Moreover, the inverse map $\mathcal{U}_B^{-1}$, defined by

$$(\mathcal{U}_B^{-1}\phi)(r, s) \ := \ \int_{\mathbb{S}^1} \mathrm{d}t \ \ \mathrm{e}^{-\,\mathrm{i}\,s(k-f(m))} \ \ \phi_t(r) \ , \qquad \phi = \{\phi_t\}_{t\in\mathbb{T}} \in \mathcal{H}$$

satisfies $\mathcal{U}_B^{-1}\mathcal{U}_B = 1$ on $\mathcal{C}_{\mathrm{c}}(\mathbb{Z}^2)$ and is isometric as well. Hence $\mathcal{U}_B^{-1}$ must be injective and as a consequence $\mathcal{U}_B$ must be surjective and thus unitary.

Since the magnetic translations commute with the unitary V_h they can be decomposed along the direct integral. Let us summarize the representation of the magnetic translations

[4]The theory of the Bloch-Floquet transform is described in full generality in the classic monograph [Kuc]. The results presented in this section are just an adaptation of the general theory to our specific case which includes a magnetic field which is constant along one direction.

[5]For the general theory of direct integrals of Hilbert spaces we refer to the standard monograph [Dix2, Part II, Chapter 1]. In particular, the isomorphism used in the right-hand side of equation (F.3) is proved in the Corollary on [Dix2, p. 175].

together with a general element of the Flux algebra through the magnetic Bloch-Flowuet transform $\mathcal{U}_B$:

$$
\begin{aligned}
S_{B,1} \;\longmapsto\; \hat{S}_1 \;&:=\; \mathcal{U}_B\, S_{B,1}\, \mathcal{U}_B^{-1} \;=\; \int_{\mathbb{S}^1}^{\oplus} \mathrm{d}t\; S \;\simeq\; S \otimes 1 \\
S_{B,2} \;\longmapsto\; \hat{S}_2 \;&:=\; \mathcal{U}_B\, S_{B,2}\, \mathcal{U}_B^{-1} \;=\; \int_{\mathbb{S}^1}^{\oplus} \mathrm{d}t\; \mathrm{e}^{\mathrm{i}\,t}\, \mathrm{e}^{-\mathrm{i}\,h(N)} \;\simeq\; \mathrm{e}^{-\mathrm{i}\,h(N)} \otimes \mathrm{e}^{\mathrm{i}\,t} \\
g \;\longmapsto\; \hat{g} \;&:=\; \mathcal{U}_B\, g\, \mathcal{U}_B^{-1} \;=\; \int_{\mathbb{S}^1}^{\oplus} \mathrm{d}t\; g(N,0) \;\simeq\; g(N,0) \otimes 1, \quad \text{for } g \in \mathcal{F}_B,
\end{aligned}
$$

where S and X are the usual shift and position operator on $\ell^2(\mathbb{Z})$. As a consequence of the formulas above one gets that the magnetic Bloch-Floquet transform maps $\mathcal{A}_B$ into a subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z})) \otimes \mathcal{C}(\mathbb{S}^1)$.

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