PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE.
FACULTAD DE MATEMÁTICAS.
DEPARTAMENTO DE MATEMÁTICA.

On certain cuspidal forms whose Fourier coefficients are special values of twisted shifted Rankin-Selberg convolutions

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After reading a lot of overheated puffery
about your new cook, you know what I'm craving?
A little perspective. That's it. I'd like some
fresh, clear, well seasoned perspective.
Can you suggest a good wine to go with that?

Anton Ego

Contents

1. Introduction

Consider the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on the Poincaré upper-half plane $\mathbb{H}$ as a group of fractional linear transformations. Our main interest throughout this thesis is a certain construction of modular forms over Γ which is described explicitly as the image of the adjoint map of some naturally occurring linear operators.

To be more precise, we recall that any modular form f is required to have a Fourier series representation of type

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} .$$

This Fourier series determines the modular form f completely. If $a_0 = 0$ one says that f is cuspidal.

We denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the set of modular (resp. cuspidal) forms of weight k over Γ. These sets are finite dimensional $\mathbb{C}$-vector spaces. The space $S_k(\Gamma)$ is endowed with the scalar product

$$\langle f, g\rangle_k := \int_F f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) ,$$

where $F \subseteq \mathbb{H}$ is any fundamental domain for the action of Γ on $\mathbb{H}$. This is the Petersson scalar product.

Now, for fixed $l \in \mathbb{Z}$ and $g \in M_l(\Gamma)$ one can define the $\mathbb{C}$-linear map

$$\begin{array}{rcl} T_g : S_k(\Gamma) & \to & S_{k+l}(\Gamma) \\ h & \mapsto & hg \end{array}$$

where hg is the usual product of functions. Denote by $T_g^* : S_{k+l}(\Gamma) \to S_k(\Gamma)$ its adjoint map, i.e. the unique $\mathbb{C}$-linear map which satisfies

$$\langle T_g(h), f\rangle_{k+l} = \langle h, T_g^*(f)\rangle_k \text{ for any } h \in S_k(\Gamma) \text{ and } f \in S_{k+l}(\Gamma).$$

In [**6**] W. Kohnen showed the following theorem:

Theorem (Kohnen). Let k, l be even integers with $k \geq 6$ and $l \geq 0$. If

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \text{ is in } S_l(\Gamma),$$

then the image of any

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \text{ in } S_{k+l}(\Gamma)$$

under T_g^* is

$$T_g^*(f) = \frac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{q=1}^{\infty} \frac{a_{m+q}\overline{b_q}}{(m+q)^{k+l-1}} \right) e^{2\pi i m z} .$$

Kohnen obtains in this way a cuspidal form whose Fourier coefficients involve the values at $s = k + l - 1$ of the functions

$$L_{f,g,m}(s) = \sum_{q=1}^{\infty} \frac{a_{m+q}\overline{b_q}}{(m+q)^s} .$$

In general, to f and g as above one associates the Dirichlet series

$$L_f(s) = \sum_{q=1}^{\infty} \frac{a_q}{q^s} \quad \text{and} \quad L_g(s) = \sum_{q=1}^{\infty} \frac{b_q}{q^s}$$

respectively. Then one defines the Rankin-Selberg convolution of L_f and L_g as

$$L_f \otimes L_g(s) = \sum_{q=1}^{\infty} \frac{a_q\overline{b_q}}{q^s} .$$

The functions $L_{f,g,m}$ are sometimes called shifted Rankin-Selberg convolutions.

The theorem given above has been generalized by several authors to other automorphic forms. In particular, Choie, Kim and Knopp [**2**] have obtained an analogous result for Jacobi cusp forms.

In this thesis Kohnen's result is generalized in two directions within the context of modular forms over Γ. Namely:

i) We drop the cuspidal restriction on g, but we impose an extra condition on the weight of g. This variation allows us to get new applications. For example, we get the identity

$$\frac{\tau(m)}{m^{11}} = -240 \sum_{q=1}^{\infty} \frac{\tau(q+m)\sigma_3(q)}{(q+m)^{11}} \quad \text{for all } m \geq 1,$$

where τ is the Ramanujan function and $\sigma_3(q) = \sum_{\substack{d/q \\ d>0}} d^3$.

ii) We consider not only the product of functions for the definition of T_g, but the n-th Rankin-Cohen bracket

$$[\![\ ,\ ]\!]_n^{(k,l)} : S_k(\Gamma) \times M_l(\Gamma) \to S_{k+l+2n}(\Gamma)$$

for every integer $n \geq 0$. This map is $\mathbb{C}$-linear in each entry and for $n = 0$ it is the usual product of functions. In general, the bracket $[\![h, g]\!]_n^{(k,l)}$ involves not only h and g but also their derivatives. For example

$$\begin{aligned} [\![h, g]\!]_1^{(k,l)} &= khDg - lgDh, \\ [\![h, g]\!]_2^{(k,l)} &= \frac{k(k+1)}{2} hD^2g - (k+1)(l+1)DhDg + \frac{l(l+1)}{2} gD^2h, \end{aligned}$$

where $2\pi i Dh = \frac{\partial h}{\partial z}$. In this way one can define, for fixed $l \in \mathbb{Z}$ and $g \in M_l(\Gamma)$, the $\mathbb{C}$-linear map

$$\begin{aligned} T_{g,n} : S_k(\Gamma) &\to S_{k+l+2n}(\Gamma) \\ h &\mapsto [\![h, g]\!]_n^{(k,l)} \end{aligned}$$

Denote by $T_{g,n}^* : S_{k+l+2n}(\Gamma) \to S_k(\Gamma)$ its adjoint map. We prove:

Theorem (Chapter 2). Let k, l, n be integers with k, l even, $k \geq 6$ and $l, n \geq 0$. If

$$g(z) = \sum_{p=0}^{\infty} b_p e^{2\pi i p z} \text{ in } M_l(\Gamma)$$

satisfies:

(a) g is cuspidal, or
(b) g is not cuspidal and $l < k-3$,

then the image of any

$$f(z) = \sum_{p=1}^{\infty} a_p e^{2\pi i p z} \text{ in } S_{k+l+2n}(\Gamma)$$

under $T^*_{g,n}$ is

$$T^*_{g,n}(f) = \frac{\Gamma(k+l+2n-1)}{\Gamma(k-1)(4\pi)^{l+2n}} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{q=0}^{\infty} \frac{a_{m+q}\overline{b_q}\varepsilon^{(k,l,n)}_{m,q}}{(m+q)^{k+l+2n-1}} \right) e^{2\pi i m z} \ ,$$

where

$$\varepsilon^{(k,l,n)}_{m,q} = \left[\!\left[e^{2\pi i m(\cdot)}, e^{2\pi i q(\cdot)} \right]\!\right]^{(k,l)}_n (0) = \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^r \binom{k+n-1}{s} \binom{l+n-1}{r} m^r q^s.$$

In particular we obtain a cuspidal form whose Fourier coefficients involves special values of shifted Rankin-Selberg convolutions "twisted" with certain combinatorial expressions.

In this thesis we also find integral representations for the operators $T^*_{g,n}$. Specifically, we prove:

Theorem (Chapter 3). Under the same hypothesis of the theorem above we have

$$T^*_{g,n}(f)(z) = \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k}, g \right]\!\right]^{(k,l)}_n (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w)$$

and

$$T^*_{g,n}(f)(z) = \frac{i^k 2^{k-3}(k-1)}{\pi} \int_F f(w) \overline{\left[\!\left[h_k(\cdot, -\overline{z}), g \right]\!\right]^{(k,l)}_n (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ ,$$

where

$$h_k(z_1, z_2) = \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=1}} (cz_1+d)^{-k} \left(\frac{az_1+b}{cz_1+d} + z_2 \right)^{-k} .$$

Notice that for $g = 1 \in M_0(\Gamma)$ and $n = 0$ we recover the classical reproduction formulas

$$f(z) = \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} f(w) \frac{1}{(\overline{w} - z)^k} \mathrm{Im}(w)^k d\mu(w)$$

and

$$f(z) = \frac{i^k 2^{k-3}(k-1)}{\pi} \int_F f(w)\overline{h_k(w,-\overline{z})}\mathrm{Im}(w)^k d\mu(w) \ ,$$

valid for any $f \in S_k(\Gamma)$ (see [**4**] and [**10**]). New partial integral reproduction formulas are also given.

This thesis is organized as follows: In Chapter 1 we recall the general definitions, properties and examples of modular forms which are needed in this work. Everything here is part of the classical theory of modular forms and we give no proofs since they can be found in many books (see for example [**1**], [**3**], [**5**], [**7**] and [**8**]). In any case, in section 8 of Chapter 1 we recall the notion of Rankin-Cohen brackets and for convenience we prove their basic properties (see also [**9**]).

Chapters 2 and 3 are devoted to prove our results. Theorem 5 is our generalization of Kohnen's statement and Theorems 6 and 7 are the integral representations mentioned above. In each case we derive some applications of these results; Propositions 2 and 3 from Theorem 5 and Proposition 4 from the other two Theorems.

2. List of symbols

$\mathbb{Z}$	the set of integer numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{H}$	the Poincaré upper-half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
Γ	the full modular group $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$.
$\mathbb{D}^*$	the pointed open disc $\{q \in \mathbb{C} : 0 < \lvert q \rvert < 1\}$.
$j(\gamma, z)$	the factor of automorphy $cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$.
$\frac{\partial f}{\partial z}$	the usual derivative of a function $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is a domain.
$M_k(\Gamma)$	the space of modular forms of weight k for Γ.
$S_k(\Gamma)$	the space of cuspidal forms of weight k for Γ.
$a_n = O(b_n)$	the big O notation: there exist $C > 0$ and $N \in \mathbb{N}$ such that $\lvert a_n \rvert \leq C \lvert b_n \rvert$ for all $n \geq N$.
$\zeta(s)$	Riemann's zeta function, which is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.
σ_k	the arithmetic function $\sigma_k(n) = \sum_{\substack{d/n \\ d>0}} d^k$ for $n \in \mathbb{N}$.
$\Gamma(s)$	Euler's gamma function, which is $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.
Γ_∞	the subgroup $\left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\} \subseteq \Gamma$.
F	a fundamental domain for Γ.
F_∞	a fundamental domain for Γ_∞.
$d\mu(z)$	the hyperbolic measure $\frac{dxdy}{y^2}$ where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

CHAPTER 1

Modular Forms and Cuspidal Forms for $\mathbf{SL}_2(\mathbb{Z})$

1. Basic definitions and properties

Throughout this thesis we denote the **Poincaré upper-half plane** by $\mathbb{H}$ and the **full modular group** by Γ, i.e.

$$\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\} \ \text{ and } \ \Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\}.$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$ we write

$$\gamma z := \frac{az+b}{cz+d} \ .$$

Simple calculations show that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|j(\gamma,z)|^2} \ ,$$

where $j(\gamma, z) := cz + d$. It is also easy to check the identities

$$\gamma_1(\gamma_2 z) = (\gamma_1\gamma_2)z \ \text{ and } \ Iz = z$$

for all $\gamma_1, \gamma_2 \in \Gamma, \ z \in \mathbb{H}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, Γ acts on $\mathbb{H}$ as a group of linear fractional transformations (also called Möbius transformations).

Now, to any function $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$ and $\gamma \in \Gamma$ we associate the function $f|_k[\gamma] : \mathbb{H} \to \mathbb{C}$ defined as

$$f|_k[\gamma](z) := j(\gamma, z)^{-k} f(\gamma z).$$

This gives, for any fixed k, a right action of Γ over the set of functions $f : \mathbb{H} \to \mathbb{C}$. Indeed, one has

$$f|_k[\gamma_1\gamma_2] = (f|_k[\gamma_1])|_k[\gamma_2]$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $f|_k[I] = f$. Moreover

$$(f + \alpha g)|_k[\gamma] = f|_k[\gamma] + \alpha g|_k[\gamma] \ \text{ and } \ f|_k[\gamma]g|_l[\gamma] = (fg)|_{k+l}[\gamma]$$

for any pair of functions $f, g : \mathbb{H} \to \mathbb{C}$, $\gamma \in \Gamma$, $\alpha \in \mathbb{C}$ and $k, l \in \mathbb{Z}$.

Definition: A function $f : \mathbb{H} \to \mathbb{C}$ is called a **modular form of weight k over Γ** if:

(1) f is holomorphic in $\mathbb{H}$,
(2) $f|_k[\gamma] = f$, for all $\gamma \in \Gamma$ and
(3) f admits an expansion of type

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

In addition, if $a_0 = 0$ in (3) then f is called a **cuspidal form of weight k over Γ**.

The set of modular (respectively cuspidal) forms of weight k over Γ is denoted by $M_k(\Gamma)$ (respectively $S_k(\Gamma)$).

Some remarks on this definition:

(a) Condition (2) above can be rephrased as

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } z \in \mathbb{H}.$$

Since the full modular group is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, such condition is equivalent to the transformation formulas

$$f(z+1) = f(z) \text{ and } f\left(-\frac{1}{z}\right) = z^k f(z).$$

Notice that every modular form over Γ is $\mathbb{Z}$-periodic.

(b) For a better interpretation of condition (3) consider the function

$$V : \mathbb{H} \to \mathbb{D}^* = \{q \in \mathbb{C} \ : \ 0 < |q| < 1\} \text{ given by } V(z) = e^{2\pi i z}.$$

Then V is surjective, holomorphic and satisfies

$$V(z_1) = V(z_2) \Longleftrightarrow z_1 = z_2 \bmod \mathbb{Z}.$$

Moreover, V is locally a conformal transformation by the local existence of analytic branches of the logarithm. In particular, we can take a right inverse function of V, say U, and consider the map $\hat{f} := f \circ U$. This definition is independent of the choice of U because f is $\mathbb{Z}$-periodic. Moreover,

$$\hat{f} : \mathbb{D}^* \to \mathbb{C}$$

is holomorphic (for any $q \in \mathbb{D}^*$ take a small open disc $B_q \subseteq \mathbb{D}^*$ around q, small enough to have $U_q : B_q \to \mathbb{H}$ a local conformal inverse of V, in this way $\hat{f}$ restricted to B_q is equal to $f \circ U_q$ which is holomorphic at q). Thus we can write the power series of $\hat{f}$ in terms of the local variable $q = e^{2\pi i z}$ and get

$$f(z) = \hat{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}.$$

Now, it should be clear that condition (3) in the definition above is equivalent to say that $\hat{f}$ is analytic at $q = 0$, which is equivalent to the statement

$$\lim_{Im(z)\to+\infty} f(z) \text{ exists and is finite.}$$

In that case we say that f is analytic at $i\infty$. Following the same ideas we observe that f is cuspidal if and only if $\hat{f}$ vanishes at $q = 0$, which is equivalent to

$$\lim_{Im(z)\to+\infty} f(z) = 0 \ .$$

In that case we say that f vanishes at $i\infty$.

Some further remarks:

(a) $M_k(\Gamma)$ is a $\mathbb{C}$-vector space and $S_k(\Gamma)$ is a subspace of $M_k(\Gamma)$.

(b) Let $f \in M_k(\Gamma)$. Since $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, we must have $f(z) = (-1)^k f(z)$. Therefore $M_k(\Gamma) = S_k(\Gamma) = \{0\}$ for k odd.

(c) If $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma)$, then $fg \in M_{k+l}(\Gamma)$. Moreover, if one of them is cuspidal, then fg is also cuspidal.

(d) The expression $j(\gamma, z)$ is called **factor of automorphy** and satisfies the identities

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z) \quad \text{and} \quad j(\gamma, \gamma^{-1}z) = j(\gamma^{-1}, z)^{-1}$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$. It is also related to γ by the equation

$$\frac{\partial}{\partial z}(\gamma z) = \frac{1}{j(\gamma, z)^2}$$

2. On the Fourier coefficients of a modular form

Let $f \in M_k(\Gamma)$ with

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

The complex numbers a_n are called the Fourier coefficients of f and it is not hard to prove that

$$a_n = O(n^{k/2})$$

whenever f is a cuspidal form. Using this fact and the existence of certain modular forms called Eisenstein series (see below) one can also show the estimate

$$a_n = O(n^{k-1})$$

for any f in $M_k(\Gamma)$. In any case, a better result can be found in [**3**] for cuspidal forms.

Theorem 1. *For any $f \in S_k(\Gamma)$ with Fourier coefficients $(a_n)_{n \geq 1}$ and $\delta > 0$ one has*

$$a_n = O(n^{k/2+\delta-1/4}).$$

3. First examples: Eisenstein Series

Let k be an even integer with $k \geq 4$. The **k-th Eisenstein series for** Γ is

$$G_k(z) := \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}, \qquad \text{for } z \in \mathbb{H}.$$

This series converges absolutely and uniformly on any compact subset of $\mathbb{H}$. Therefore, it defines a holomorphic function on $\mathbb{H}$. Moreover, it is easy to check that $G_k|_k[\gamma] = G_k$, for all $\gamma \in \Gamma$. Some standard extra work yields the Fourier expansion of G_k. Namely

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}\right),$$

where ζ is the Riemann's zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

defined in this way for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, B_k is the k-th Bernoulli number defined by the equality

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m,$$

and σ_{k-1} is the arithmetic function

$$\sigma_{k-1}(n) := \sum_{\substack{d|n \\ 0<d}} d^{k-1}.$$

All these properties show that $G_k \in M_k(\Gamma)$.

For convenience we also consider the **k-th normalized Eisenstein series for** Γ;

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

In this way we have a non-cuspidal modular form $E_k \in M_k(\Gamma)$ with very simple rational Fourier coefficients. The first few E_k are:

$$\begin{aligned}
E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z}, \\
E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n z}, \\
E_8(z) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) e^{2\pi i n z}, \\
E_{10}(z) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) e^{2\pi i n z}, \\
E_{12}(z) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) e^{2\pi i n z}, \\
E_{14}(z) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) e^{2\pi i n z}.
\end{aligned}$$

4. Another example: The discriminant cuspidal form Δ

We define the discriminant cuspidal form $\Delta \in S_{12}(\Gamma)$ as follows:

$$\Delta := \frac{E_4^3 - E_6^2}{1728}.$$

Another (equivalent) way of defining Δ is by the infinite product formula

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}.$$

As every cuspidal form, Δ has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z},$$

which defines the **Ramanujan function** τ. The first few values of it are:

n	$\tau(n)$
1	1
2	-24
3	252
4	-1472
5	4830
6	-6048
7	-16744
8	84480
9	-113643
10	-115920
11	534612
12	-370944

5. A brief description of $M_k(\Gamma)$ and $S_k(\Gamma)$

The Eisenstein series and the discriminant cuspidal form are all we need in order to describe $M_k(\Gamma)$ and $S_k(\Gamma)$ as $\mathbb{C}$-vector spaces. More precisely, we can give a basis for each one of these spaces using only products of E_k (with $k \geq 4$ even) and Δ. This should be clear from the following result.

Theorem 2. *For all $k \in \mathbb{Z}$, the $\mathbb{C}$-vector spaces $M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional. Moreover:*

(1) $M_k(\Gamma) = \{0\}$, *for $k < 0$ or $k = 2$ or k odd.*
(2) $M_0(\Gamma) = \mathbb{C}$ *(constant functions).*
(3) $M_k(\Gamma) = \mathbb{C}E_k$, *for $k \in \{4, 6, 8, 10, 14\}$.*
(4) $S_k(\Gamma) = \{0\}$, *for $k < 12$ or $k = 14$ or k odd.*
(5) $S_{12}(\Gamma) = \mathbb{C}\Delta$.
(6) $M_k(\Gamma) = \mathbb{C}E_k \oplus S_k(\Gamma)$, *for $k \geq 4$ even.*
(7) $S_k(\Gamma) = \Delta M_{k-12}(\Gamma)$.

In particular, the exact dimension of each space is the following:

$$dim_{\mathbb{C}} M_k(\Gamma) = \begin{cases} 0 & \text{for } k < 0 \text{ or } k = 2 \text{ or } k \text{ odd}, \\ \left[\frac{k}{12}\right] & \text{for } k \text{ even}, \ k \geq 0, \ k = 2 \ (mod\ 12), \\ \left[\frac{k}{12}\right] + 1 & \text{for } k \text{ even}, \ k \geq 0, \ k \neq 2 \ (mod\ 12), \end{cases}$$

$$dim_{\mathbb{C}} S_k(\Gamma) = \begin{cases} 0 & \text{for } k < 12 \text{ or } k = 14 \text{ or } k \text{ odd}, \\ \left[\frac{k}{12}\right] - 1 & \text{for } k \text{ even}, \ k \geq 12, \ k = 2 \ (mod\ 12), \\ \left[\frac{k}{12}\right] & \text{for } k \text{ even}, \ k \geq 12, \ k \neq 2 \ (mod\ 12). \end{cases}$$

Remark: As a consequence of this theorem we get the following identities:

$$\begin{aligned} E_4^2 &= E_8, \\ E_4 E_6 &= E_{10}, \\ E_4 E_{10} &= E_{14}, \\ E_6 E_8 &= E_{14}. \end{aligned}$$

Each one of these identities give a relation between the corresponding arithmetical functions σ_k, merely by comparing the Fourier coefficients in both sides.

6. Fundamental domains

In many cases it is not necessary to work with the whole complex upper-half plane $\mathbb{H}$ when studying modular forms. We can restrict ourselves to a smaller subset containing at least one point in each orbit of the Γ-action on $\mathbb{H}$ (or of the action of a subgroup of Γ on $\mathbb{H}$).

Definition: Let Γ' be a subgroup of Γ. A measurable subset $\mathfrak{F}$ of $\mathbb{H}$ is called a **fundamental domain** for Γ' if:

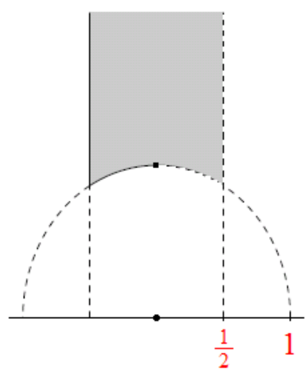
(1) There are no two points $z_1, z_2 \in \mathfrak{F}$ related by Γ' except in the trivial case, i.e.:

$$\text{If } z_1,\ z_2 \in \mathfrak{F} \text{ and } \gamma' \in \Gamma' \text{ with } \gamma' z_1 = z_2, \text{ then } z_1 = z_2 \text{ and } \gamma' \in \{\pm I\} \cap \Gamma'.$$

(2) For each point $z \in \mathbb{H}$ there exists $z_0 \in \mathfrak{F}$ and $\gamma' \in \Gamma'$ such that $\gamma' z_0 = z$.

Explicit examples of connected fundamental domains for Γ are

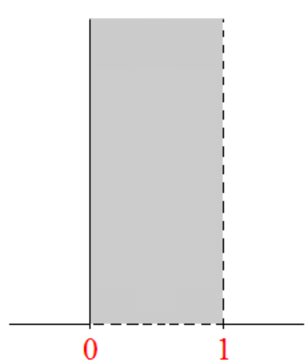
$$F = \left\{ z \in \mathbb{H} \ : \ |z| > 1, -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \right\} \cup \left\{ e^{2\pi i x} \ : \ x \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right] \right\}$$



and any image $\gamma F = \{\gamma z \ : \ z \in F\}$ where $\gamma \in \Gamma$.

Similarly, explicit examples of connected fundamental domains for $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}$ are

$$F_\infty = \{ z \in \mathbb{H} \ : \ 0 \leq \mathrm{Re}(z) < 1 \}$$



and any $F_\infty + r$ where $r \in \mathbb{R}$.

Remark: If F is a fundamental domain for Γ, Γ' is a subgroup of Γ and Λ is a complete set of representatives for $\Gamma'\backslash\Gamma$, then

$$F' = \bigcup_{\gamma\in\Lambda} \gamma F$$

is a disjoint union and it is a fundamental domain for Γ'.

7. The Petersson scalar product and Poincaré series

Let us define

$$\langle f, g\rangle_k := \int_F f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z),$$

where $F \subseteq \mathbb{H}$ is a fundamental domain for Γ, $d\mu(z)$ denotes the hyperbolic measure $d\mu(z) = \frac{dxdy}{y^2}$, with $x = Re(z)$, $y = Im(z)$, and f, g are modular forms in $M_k(\Gamma)$ with at least one of them cuspidal (in this way the integral above is absolutely convergent).

The Γ-invariance of the integrand implies that such a definition is independent of the choice of the fundamental domain and yields a non-degenerate scalar product in $S_k(\Gamma)$ called the **Petersson scalar product**. The set $S_k(\Gamma)$ with $\langle\ ,\ \rangle_k$ is a finite-dimensional complex Hilbert space.

Now, for any $m \geq 0$ and any even integer $k \geq 4$ we introduce the **m-th Poincaré series of weight k over Γ**:

$$P_{k,m}(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,\tau)^{-k} e^{2\pi i m\gamma(z)}.$$

This series is independent of the choice of coset representatives and it converges absolutely and uniformly on compact subsets of $\mathbb{H}$. Therefore, it defines an holomorphic function on $\mathbb{H}$.

It is not difficult to check that $P_{k,m}$ is a modular form in $M_k(\Gamma)$ and that $P_{k,m}$ is cuspidal for $m \geq 1$.

The following proposition relates Petersson scalar product with Poincaré series.

Proposition 1. *For $m \geq 1$, $k \geq 4$ even and $f \in S_k(\Gamma)$ with Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

we have

$$\langle f, P_{k,m}\rangle_k = \frac{\Gamma(k-1)a_m}{(4\pi m)^{k-1}}.$$

Here and from now on $\Gamma(s)$ denotes Euler's gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, \qquad \textit{for } s \in \mathbb{C} \textit{ with } Re(s) > 0.$$

Remarks:

(a) We often use that the hyperbolic measure is invariant under the action of Γ, that is

$$\int_A F(z)d\mu(z) = \int_{\gamma^{-1}A} F(\gamma z)d\mu(\gamma z) = \int_{\gamma^{-1}A} F(\gamma z)d\mu(z)$$

for any measurable set $A \subseteq \mathbb{H}$, any measurable function $F : A \to \mathbb{C}$ and any $\gamma \in \Gamma$.

(b) For convenience we recall here the existence of adjoint operators between finite dimensional $\mathbb{C}$-vector spaces endowed with a non-degenerate scalar product: if V and W are finite dimensional $\mathbb{C}$-vector spaces and $\langle\ ,\ \rangle_V$, $\langle\ ,\ \rangle_W$ are non-degenerate scalar products in V and W respectively, then for any $\mathbb{C}$-linear operator $T : V \to W$ there exists a unique $\mathbb{C}$-linear operator $T^* : W \to V$ such that

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V, \quad \text{for any } v \in V, w \in W.$$

In that case one says that T^* is the adjoint operator of T with respect to the scalar products $\langle\ ,\ \rangle_V$ and $\langle\ ,\ \rangle_W$. In particular, for any $\mathbb{C}$-linear operator $T : S_k(\Gamma) \to S_l(\Gamma)$ there exists the adjoint operator $T^* : S_l(\Gamma) \to S_k(\Gamma)$ with respect to the Petersson scalar product of each space.

8. Rankin-Cohen brackets

In general it is not true that the derivative $\frac{\partial f}{\partial z}$ of a modular form f in $M_k(\Gamma)$ is again a modular form. But in any case, it satisfies the functional equation

$$\frac{\partial f}{\partial z}(\gamma z) = ckj(\gamma, z)^{k+1} f(z) + j(\gamma, z)^{k+2} \frac{\partial f}{\partial z}(z)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We will see that a suitable combination of two modular forms and their derivatives yields a third modular form.

Definition: For two holomorphic function $f, g : U \to \mathbb{C}$ (where $U \subseteq \mathbb{C}$ is an open domain) and $k, l, n \in \mathbb{Z}$ with $n \geq 0$, we define the n**-th Rankin-Cohen bracket of index** (k, l) as

$$[\![f, g]\!]_n^{(k,l)} := \sum_{\substack{r,s\geq 0 \\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r} D^r f D^s g.$$

Here D^r denotes the r-th normalized differential operator

$$D^r f = \frac{1}{(2\pi i)^r} \frac{\partial^r f}{\partial z^r},$$

(in particular $D^0 f = f$) and

$$\binom{m}{s} = \begin{cases} \frac{m(m-1)\cdots(m-s+1)}{s!} & \text{if } s \geq 1, \\ 1 & \text{if } s = 0. \end{cases}$$

Remarks:

(a) We have normalized the differential operator because, in this way, if f has a Fourier expansion of type $f(z) = \sum a_n e^{2\pi i n z}$ then $Df = \sum n a_n e^{2\pi i n z}$.

(b) If $m \geq 0$ and $0 \leq s \leq m$, then $\binom{m}{s}$ agrees with the usual combinatorial number $\frac{m!}{(m-s)!s!}$. Here we have chosen a definition that makes sense even if $m < s$ or $m < 0$. This may be the case since in the above definition of the n-th Rankin-Cohen bracket one may have, for example, $k + n - 1 < 0$.

(c) $\binom{m}{s} \in \mathbb{Z}$ for any $m, s \in \mathbb{Z}$ with $s \geq 0$.

Examples:

$$\begin{aligned}
[\![f,g]\!]_0^{(k,l)} &= fg, \\
[\![f,g]\!]_1^{(k,l)} &= kfDg - lgDf, \\
[\![f,g]\!]_2^{(k,l)} &= \frac{k(k+1)}{2} fD^2g - (k+1)(l+1)DfDg + \frac{l(l+1)}{2} gD^2f.
\end{aligned}$$

If f and g are modular forms over Γ, we take k as the weight of f and l as the weight of g in order to get a new modular form. Indeed,

Theorem 3. *If $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma)$, then $[\![f,g]\!]_n^{(k,l)} \in M_{k+l+2n}(\Gamma)$.*

For the proof of this theorem we first establish some technical lemmas.

Lemma 1. *Let k and r be integers with $r \geq 0$. Define*

$$\lambda_{k,t,r} = \begin{cases} \binom{k+r-1}{r} r!(k+r) & \text{if } t = 0, \\ \binom{k+r-1}{r-t} \frac{r!}{t!}(k+r+t) + \binom{k+r-1}{r+1-t} \frac{r!}{(t-1)!} & \text{if } 1 \leq t \leq r, \\ 1 & \text{if } t = r+1. \end{cases}$$

Then $\lambda_{k,t,r} = \binom{k+r}{r+1-t} \frac{(r+1)!}{t!}$.

Proof of Lemma 1. For $t = 0$ we have

$$\lambda_{k,0,r} = (k+r-1)\cdots k(k+r) = \binom{k+r}{r+1-0} \frac{(r+1)!}{0!} \ .$$

For $t = r+1$ we have

$$\lambda_{k,r+1,r} = 1 = \binom{k+r}{r+1-(r+1)} \frac{(r+1)!}{(r+1)!} \ .$$

Now, for $1 \leq t \leq r$ one has to consider the two different cases; $1 \leq t \leq r-1$ and $t = r$ (because the definition of $\binom{m}{s}$ depends whether $s > 0$ or $s = 0$). Thus, for $1 \leq t \leq r-1$ we have

$$\begin{aligned}
\lambda_{k,t,r} &= \frac{(k+r-1)\cdots(k+t)}{(r-t)!} \frac{r!}{t!}(k+r+t) + \frac{(k+r-1)\cdots(k+t-1)}{(r+1-t)!} \frac{r!}{(t-1)!} \\
&= \frac{(k+r-1)\cdots(k+t)r!}{(r-t)!(t-1)!} \left(\frac{k+r+t}{t} + \frac{k+t-1}{r+1-t} \right) \\
&= \frac{(k+r-1)\cdots(k+t)r!}{(r-t)!(t-1)!} \left(\frac{kr+k-kt+r^2+r-rt+tr+t-t^2+tk+t^2-t}{t(r+1-t)} \right) \\
&= \frac{(k+r-1)\cdots(k+t)r!}{(r-t)!(t-1)!} \left(\frac{kr+k+r^2+r}{t(r+1-t)} \right) \\
&= \frac{(k+r-1)\cdots(k+t)r!}{(r-t)!(t-1)!} \left(\frac{(k+r)(r+1)}{t(r+1-t)} \right) \\
&= \frac{(k+r)(k+r-1)\cdots(k+t)(r+1)!}{(r+1-t)!t!} \\
&= \binom{k+r}{r+1-t} \frac{(r+1)!}{t!} \ .
\end{aligned}$$

For $t=r$ we have $\lambda_{k,r,r}=(k+2r)+(k+r-1)r=(k+r)(r+1)=\binom{k+r}{r+1-r}\frac{(r+1)!}{r!}$.
In any case the desired equality holds. $\square$

Lemma 2. *Let $f:\mathbb{H}\to\mathbb{C}$ be a holomorphic function, $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$ and $r\in\mathbb{Z}$ with $r\geq 0$. Then*

$$\frac{\partial^r(f|_k[\gamma])}{\partial z^r}=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r-t}\left.\left(\frac{\partial^t f}{\partial z^t}\right)\right|_{k+r+t}[\gamma].$$

Proof of Lemma 2. We use induction on r.
For $r=0$ we have $f|_k[\gamma]$ in each side of the equation and there is nothing to prove.
For $r=1$ we just differentiate both sides of

$$f|_k[\gamma](z)=j(\gamma,z)^{-k}f(\gamma z)$$

and get

$$\frac{\partial(f|_k[\gamma])}{\partial z}(z)=-kcj(\gamma,z)^{-k-1}f(\gamma z)+j(\gamma,z)^{-k-2}\frac{\partial f}{\partial z}(\gamma z)$$

$\Big($recall that $\frac{\partial}{\partial z}(\gamma z)=\frac{1}{j(\gamma,z)^2}\Big)$. This is exactly

$$\frac{\partial(f|_k[\gamma])}{\partial z}=k(-c)f|_{k+1}[\gamma]+\left.\left(\frac{\partial f}{\partial z}\right)\right|_{k+2}[\gamma],$$

which is the desired equality.
Suppose next that we have

$$\frac{\partial^r(f|_k[\gamma])}{\partial z^r}=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r-t}\left.\left(\frac{\partial^t f}{\partial z^t}\right)\right|_{k+r+t}[\gamma].$$

Differentiating on both sides we get

$$\frac{\partial^{r+1}(f|_k[\gamma])}{\partial z^{r+1}}=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r-t}\frac{\partial}{\partial z}\left(\left.\left(\frac{\partial^t f}{\partial z^t}\right)\right|_{k+r+t}[\gamma]\right).$$

Now we recall the case $r=1$ computed above and obtain

$$\frac{\partial}{\partial z}\left(\left.\left(\frac{\partial^t f}{\partial z^t}\right)\right|_{k+r+t}[\gamma]\right)=(k+r+t)(-c)\left.\left(\frac{\partial^t f}{\partial z^t}\right)\right|_{k+r+1+t}[\gamma]+\left.\left(\frac{\partial^{t+1} f}{\partial z^{t+1}}\right)\right|_{k+r+2+t}[\gamma].$$

Thus

$$\begin{aligned}
&\frac{\partial^{r+1}(f|_k[\gamma])}{\partial z^{r+1}} \\
&=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r-t}\left((k+r+t)(-c)\left(\frac{\partial^t f}{\partial z^t}\right)\bigg|_{k+r+1+t}[\gamma]+\left(\frac{\partial^{t+1} f}{\partial z^{t+1}}\right)\bigg|_{k+r+2+t}[\gamma]\right) \\
&=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r+1-t}(k+r+t)\left(\frac{\partial^t f}{\partial z^t}\right)\bigg|_{k+r+1+t}[\gamma]+ \\
&\qquad\qquad\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r-t}\left(\frac{\partial^{t+1} f}{\partial z^{t+1}}\right)\bigg|_{k+r+2+t}[\gamma] \\
&=\sum_{t=0}^{r}\binom{k+r-1}{r-t}\frac{r!}{t!}(-c)^{r+1-t}(k+r+t)\left(\frac{\partial^t f}{\partial z^t}\right)\bigg|_{k+r+1+t}[\gamma]+ \\
&\qquad\qquad\sum_{t=1}^{r+1}\binom{k+r-1}{r+1-t}\frac{r!}{(t-1)!}(-c)^{r+1-t}\left(\frac{\partial^t f}{\partial z^t}\right)\bigg|_{k+r+1+t}[\gamma] \\
&=\sum_{t=0}^{r+1}\lambda_{k,t,r}(-c)^{r+1-t}\left(\frac{\partial^t f}{\partial z^t}\right)\bigg|_{k+r+1+t}[\gamma]\ ,
\end{aligned}$$

where $\lambda_{k,t,r}$ is the number introduced in the previous Lemma.

By Lemma 1 we have $\lambda_{k,t,r}=\binom{k+r}{r+1-t}\frac{(r+1)!}{t!}$ and this proves the equality in the case $r+1$. $\square$

Lemma 3. *Let $f,g:\mathbb{H}\to\mathbb{C}$ be two holomorphic functions and $\gamma\in\Gamma$. Then*

$$[\![f,g]\!]_n^{(k,l)}\big|_{k+l+2n}[\gamma]=[\![\ f|_k[\gamma],\ g|_l[\gamma]\ ]\!]_n^{(k,l)}.$$

Proof of Lemma 3. By definition of the n-th Rankin-Cohen bracket we have

$$(2\pi i)^n[\![\ f|_k[\gamma]\ ,\ g|_l[\gamma]\ ]\!]_n^{(k,l)}\ =\ \sum_{\substack{r,s\geq 0\\ r+s=n}}(-1)^r\binom{k+n-1}{s}\binom{l+n-1}{r}\frac{\partial^r(f|_k[\gamma])}{\partial z^r}\frac{\partial^s(g|_l[\gamma])}{\partial z^s}.$$

In this expression we use Lemma 2 with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and get

$$
\begin{aligned}
&(2\pi i)^n [\![\, f|_k[\gamma] \,,\, g|_l[\gamma] \,]\!]_n^{(k,l)} \\
&= \sum_{\substack{r,s\geq 0\\ r+s=n}} \sum_{p=0}^{r}\sum_{q=0}^{s} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r}\binom{k+r-1}{r-p}\binom{l+s-1}{s-q}\frac{r!}{p!}\frac{s!}{q!} \\
&\qquad\qquad \times (-c)^{r+s-p-q} \left(\frac{\partial^p f}{\partial z^p}\right)\bigg|_{k+r+p}[\gamma] \left(\frac{\partial^q g}{\partial z^q}\right)\bigg|_{l+s+q}[\gamma] \\
&= \sum_{\substack{r,s\geq 0\\ r+s=n}} \sum_{p=0}^{r}\sum_{q=0}^{s} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r}\binom{k+r-1}{r-p}\binom{l+s-1}{s-q}\frac{r!}{p!}\frac{s!}{q!}(-c)^{n-p-q} \\
&\qquad\qquad \times \left(\frac{\partial^p f}{\partial z^p}\frac{\partial^q g}{\partial z^q}\right)\bigg|_{k+l+n+p+q}[\gamma] \\
&= \sum_{\substack{p,q\geq 0\\ p+q\leq n}} (-c)^{n-p-q} \left(\frac{\partial^p f}{\partial z^p}\frac{\partial^q g}{\partial z^q}\right)\bigg|_{k+l+n+p+q}[\gamma] \sum_{r=p}^{n-q} (-1)^r \binom{k+n-1}{n-r}\binom{l+n-1}{r}\binom{k+r-1}{r-p} \\
&\qquad\qquad \times \binom{l+n-r-1}{n-r-q}\frac{r!}{p!}\frac{(n-r)!}{q!}.
\end{aligned}
$$

But direct calculations show that

$$\binom{\alpha}{\beta}\binom{\alpha-\beta}{\gamma}\beta! = \binom{\alpha}{\beta+\gamma}\frac{(\beta+\gamma)!}{\gamma!}$$

for any $\alpha, \beta, \gamma \in \mathbb{Z}$ with $\beta, \gamma \geq 0$. Thus

$$
\begin{aligned}
\binom{k+n-1}{n-r}\binom{k+r-1}{r-p}(n-r)! &= \binom{k+n-1}{n-p}\frac{(n-p)!}{(r-p)!}, \\
\binom{l+n-1}{r}\binom{l+n-r-1}{n-r-q}r! &= \binom{l+n-1}{n-q}\frac{(n-q)!}{(n-r-q)!}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
(2\pi i)^n [\![\, f|_k[\gamma] \,,\, g|_l[\gamma] \,]\!]_n^{(k,l)} &= \sum_{\substack{p,q\geq 0\\ p+q\leq n}} (-c)^{n-p-q} \left(\frac{\partial^p f}{\partial z^p}\frac{\partial^q g}{\partial z^q}\right)\bigg|_{k+l+n+p+q}[\gamma]\binom{k+n-1}{n-p}\binom{l+n-1}{n-q} \\
&\qquad \times \frac{(n-p)!(n-q)!}{p!q!}\sum_{r=p}^{n-q}\frac{(-1)^r}{(r-p)!(n-r-q)!}.
\end{aligned}
$$

Now

$$\sum_{r=p}^{n-q}\frac{(-1)^r}{(r-p)!(n-r-q)!} = (-1)^p\sum_{r=0}^{n-q-p}\frac{(-1)^r}{r!(n-r-q-p)!} = \frac{(-1)^p}{(n-q-p)!}\sum_{r=0}^{n-q-p}(-1)^r\binom{n-q-p}{r}$$

and

$$\sum_{r=0}^{n-q-p}(-1)^r\binom{n-q-p}{r} = \begin{cases} 1 & \text{if } p+q=n, \\ (1-1)^{n-p-q}=0 & \text{if } p+q<n, \end{cases}$$

thus

$$\sum_{r=p}^{n-q} \frac{(-1)^r}{(r-p)!(n-r-q)!} = \begin{cases} (-1)^p & \text{if } p+q=n, \\ 0 & \text{if } p+q<n. \end{cases}$$

Hence

$$\begin{aligned}
&(2\pi i)^n \left[\!\left[\, f|_k[\gamma] \, , \, g|_l[\gamma] \, \right]\!\right]_n^{(k,l)} \\
&= \sum_{\substack{p,q\geq 0 \\ p+q=n}} (-c)^{n-p-q} \left(\frac{\partial^p f}{\partial z^p} \frac{\partial^q g}{\partial z^q} \right)\Big|_{k+l+n+p+q} [\gamma] \binom{k+n-1}{n-p} \binom{l+n-1}{n-q} \frac{(n-p)!(n-q)!}{p!q!} (-1)^p \\
&= \sum_{\substack{p,q\geq 0 \\ p+q=n}} \left(\frac{\partial^p f}{\partial z^p} \frac{\partial^q g}{\partial z^q} \right)\Big|_{k+l+2n} [\gamma] \binom{k+n-1}{q} \binom{l+n-1}{p} (-1)^p \\
&= (2\pi i)^n \left[\!\left[f, g \right]\!\right]_n^{(k,l)} \Big|_{k+l+2n} [\gamma] \ .
\end{aligned}$$

□

Proof of Theorem 3. Clearly $[\![f,g]\!]_n^{(k,l)}$ is a holomorphic function on $\mathbb{H} \cup \{i\infty\}$ by the definition of Rankin-Cohen bracket. The identity

$$[\![f,g]\!]_n^{(k,l)} \Big|_{k+l+2n} [\gamma] = \left[\!\left[\, f|_k[\gamma] \, , \, g|_l[\gamma] \, \right]\!\right]_n^{(k,l)} = [\![f,g]\!]_n^{(k,l)}$$

for any $\gamma \in \Gamma$ follows from Lemma 3 and the fact that $f|_k[\gamma] = f$ and $g|_l[\gamma] = g$. □

Some further remarks:

(a) The Rankin-Cohen brackets are $\mathbb{C}$-linear in each entry.

(b) As we already showed, $[\![f,g]\!]_0^{(k,l)} = fg$. Thus the Rankin-Cohen brackets are a generalization of the usual product of functions.

(c) If $f \in M_k(\Gamma), g \in M_l(\Gamma)$ and $n \geq 1$, then $[\![f,g]\!]_n^{(k,l)} \in S_{k+l+2n}(\Gamma)$.

(d) $[\![f,g]\!]_n^{(k,l)} = (-1)^n [\![g,f]\!]_n^{(l,k)}$. Thus $[\![f,f]\!]_n^{(k,k)} = 0$, whenever n is odd.

(e) Suppose that $\{f_n\}_n, \{g_n\}_n$ are any two sequences of holomorphic functions on a fixed domain $U \subseteq \mathbb{C}$ and $\sum_n f_n, \sum_n g_n$ converge absolutely and uniformly on compact subsets of U. Since the Rankin-Cohen brackets are finite combinations of products and derivatives, one has

$$\left[\!\left[\, \sum_n f_n, \, \sum_n g_n \, \right]\!\right]_n^{(k,l)} = \sum_{n,m} [\![f_n, g_m]\!]_n^{(k,l)} \ .$$

In particular, if $g : U \to \mathbb{C}$ is holomorphic, then

$$\left[\!\left[\, \sum_n f_n, \, g \, \right]\!\right]_n^{(k,l)} = \sum_n [\![f_n, g]\!]_n^{(k,l)} \ .$$

Some examples:

$$\begin{aligned}
[\![E_4, E_6]\!]_1^{(4,6)} &= -3456\Delta \\
[\![E_4, E_4]\!]_2^{(4,4)} &= -1435200\Delta
\end{aligned}$$

9. On a theorem of W. Kohnen

Winfried Kohnen considered in [**6**] certain linear maps between spaces of cuspidal forms and computed their adjoint maps. He obtained in this way some cusp forms whose Fourier coefficients are essentially special values of certain Dirichlet series. Specifically, Kohnen showed the theorem below.

Theorem 4 (Kohnen)**.** *Let k and l be even integers with $k \geq 6$ and $l \geq 0$. Fix*

$$g(z) = \sum_{m=1}^{\infty} b_m e^{2\pi i m z} \quad \textit{in} \quad S_l(\Gamma)$$

and define $T_g : S_k(\Gamma) \to S_{k+l}(\Gamma)$ by $T_g(h) = hg$. Denote by $T_g^* : S_{k+l}(\Gamma) \to S_k(\Gamma)$ its adjoint map. Then the image of any

$$f(z) = \sum_{m=1}^{\infty} a_m e^{2\pi i m z} \in S_{k+l}(\Gamma)$$

under T_g^* is

$$T_g^*(f)(z) = \frac{\Gamma(k+l-1)}{\Gamma(k-1)(4\pi)^l} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{q=1}^{\infty} \frac{a_{q+m}\overline{b_q}}{(q+m)^{k+l-1}} \right) e^{2\pi i m z}.$$

After this result a natural question arises: What can be said if we change the linear map T_g and consider not only the multiplication of cusp forms but other Rankin-Cohen brackets?

A complete answer to this question is given in the next chapter of this thesis. Furthermore, we will see that the cuspidal condition on g can be dropped.

CHAPTER 2

A generalization of a theorem of Kohnen

1. Theorem 5: The statement

The following is our first result.

Theorem 5. *Let k, l, n be integers with k, l even, $k \geq 6$ and $l, n \geq 0$. Fix*

$$g(z) = \sum_{m=0}^{\infty} b_m e^{2\pi imz} \text{ in } M_l(\Gamma)$$

and suppose that either:

(a) *g is cuspidal, or*
(b) *g is not cuspidal and $l < k - 3$.*

Define $T_{g,n} : S_k(\Gamma) \to S_{k+l+2n}(\Gamma)$ by $T_{g,n}(h) = [\![h, g]\!]_n^{(k,l)}$. Denote by $T^*_{g,n} : S_{k+l+2n}(\Gamma) \to S_k(\Gamma)$ its adjoint map. Then the image of any

$$f(z) = \sum_{m=1}^{\infty} a_m e^{2\pi imz} \text{ in } S_{k+l+2n}(\Gamma)$$

under $T^*_{g,n}$ is

$$T^*_{g,n}(f)(z) = \frac{\Gamma(k+l+2n-1)}{\Gamma(k-1)(4\pi)^{l+2n}} \sum_{m=1}^{\infty} m^{k-1} \left(\sum_{q=0}^{\infty} \frac{a_{q+m}\overline{b_q}\varepsilon^{(k,l,n)}_{m,q}}{(q+m)^{k+l+2n-1}} \right) e^{2\pi imz} \ ,$$

where

$$\varepsilon^{(k,l,n)}_{m,q} = [\![e^{2\pi im(\cdot)}, e^{2\pi iq(\cdot)}]\!]_n^{(k,l)}(0) = \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^r \binom{k+n-1}{s} \binom{l+n-1}{r} m^r q^s$$

(in this last expression, and hereafter, we should consider $q^0 = 1$ even if $q = 0$).

Remarks:

(a) If we choose hypothesis (a) and $n = 0$ in this Theorem, we obtain Kohnen's result.

(b) $\varepsilon^{(k,l,n)}_{m,q} \in \mathbb{Z}$ for any $m, q \in \mathbb{Z}$ with $m \geq 1$ and $q \geq 0$ (recall that $\binom{m}{s} \in \mathbb{Z}$ if $m, s \in \mathbb{Z}$ and $s \geq 0$).

For the proof of Theorem 5 we first establish some lemmas.

2. Lemmas for the proof of Theorem 5

In this section we fix an integer m with $m \geq 1$ and recall that for any $z \in \mathbb{H}$ we write $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Moreover, k, l, n, g and f are as in Theorem 5.

Lemma 4. *For any $\gamma \in \Gamma$ we have*

$$f(\gamma^{-1}z)\overline{[\![j(\gamma,\cdot)^{-k}e^{2\pi im\gamma(\cdot)},\ g]\!]_n^{(k,l)}(\gamma^{-1}z)}\ Im(\gamma^{-1}z)^{k+l+2n} = f(z)\overline{[\![e^{2\pi im(\cdot)},\ g]\!]_n^{(k,l)}(z)}\ Im(z)^{k+l+2n}\ .$$

Proof of Lemma 4. By Lemma 3 we have

$$[\![j(\gamma,\cdot)^{-k}e^{2\pi im\gamma(\cdot)},\ g]\!]_n^{(k,l)}|_{k+l+2n}[\gamma^{-1}] = [\![\ j(\gamma,\cdot)^{-k}e^{2\pi im\gamma(\cdot)}|_k[\gamma^{-1}]\ ,\ g|_l[\gamma^{-1}]\ ]\!]_n^{(k,l)}\ .$$

Using that $g|_l[\gamma^{-1}] = g$, this gives

$$j(\gamma^{-1},z)^{-k-l-2n}[\![j(\gamma,\cdot)^{-k}e^{2\pi im\gamma(\cdot)},g]\!]_n^{(k,l)}(\gamma^{-1}z) = [\![j(\gamma^{-1},\cdot)^{-k}j(\gamma,\gamma^{-1}(\cdot))^{-k}e^{2\pi im(\cdot)},g]\!]_n^{(k,l)}(z)\ .$$

But

$$j(\gamma^{-1},\cdot)j(\gamma,\gamma^{-1}(\cdot)) = 1\ ,$$

thus

$$[\![j(\gamma,\cdot)^{-k}e^{2\pi im\gamma(\cdot)},g]\!]_n^{(k,l)}(\gamma^{-1}z) = j(\gamma^{-1},z)^{k+l+2n}[\![e^{2\pi im(\cdot)},g]\!]_n^{(k,l)}(z)\ .$$

Combining this expression with the identities

$$f(\gamma^{-1}z) = j(\gamma^{-1},z)^{k+l+2n}f(z) \ \text{ and } \ \mathrm{Im}(\gamma^{-1}z) = \frac{\mathrm{Im}(z)}{|j(\gamma^{-1},z)|^2}\ ,$$

we get the desired equality. □

For convenience let us define

$$\beta = \begin{cases} \frac{l}{2}-\frac{1}{8} & \text{if } g \text{ satisfy hypothesis (a),} \\ \\ l-1 & \text{if } g \text{ satisfy hypothesis (b).} \end{cases}$$

Then the Fourier coefficients of g satisfy $b_n = O(n^\beta)$ (see Section 2 of Chapter 1. In particular for g cuspidal use Theorem 1 with $\delta = \frac{1}{8}$).

Lemma 5. *There exists a constant $C > 0$, depending on f and g, such that*

$$\int_{F_\infty}\sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_qe^{2\pi iz(p+q+m)}|Im(z)^{k+l+2n}d\mu(z)$$
$$\leq C\frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}}\left(\sum_{p=1}^{\infty}\frac{1}{(p+m)^{(k+l+2n)/2-7/8}}+\sum_{p,q=1}^{\infty}\frac{1}{(p+q+m)^{(k+l+2n)/2-7/8-\beta}}\right)\ .$$

Moreover, for $1\leq s\leq n$ one has

$$\int_{F_\infty}\sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_qe^{2\pi iz(p+q+m)}|q^sIm(z)^{k+l+2n}d\mu(z)\leq C\frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}}\sum_{p,q=1}^{\infty}\frac{1}{(p+q+m)^{(k+l)/2-7/8-\beta}}\ ,$$

and each one of these series converges.

As in Section 6 of Chapter 1, F_∞ denotes any fundamental domain for Γ_∞. Since the integrands are $\mathbb{Z}$-periodic functions, these integrals are well defined. That is, these integrals are independent of the choice of F_∞ . We choose

$$F_\infty = \{z\in\mathbb{H}\ :\ 0\leq \mathrm{Re}(z) < 1\}\ .$$

Proof of Lemma 5. Since the integrands are non-negative we can freely interchange the integral and summation symbols.

$$\begin{aligned}\int_{F_\infty}\sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_qe^{2\pi iz(p+q+m)}|\mathrm{Im}(z)^{k+l+2n}d\mu(z) &= \sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_q|\int_{F_\infty}|e^{2\pi iz(p+q+m)}|\mathrm{Im}(z)^{k+l+2n}d\mu(z)\\ &= \sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_q|\int_0^\infty\int_0^1 e^{-2\pi y(p+q+m)}y^{k+l+2n-2}dxdy\\ &= \sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_q|\int_0^\infty e^{-2\pi y(p+q+m)}y^{k+l+2n-2}dy.\end{aligned}$$

A simple change of variables yields

$$\int_0^\infty e^{-2\pi y(p+q+m)}y^{k+l+2n-2}dy = \frac{\Gamma(k+l+2n-1)}{(2\pi(p+q+m))^{k+l+2n-1}}\ .$$

Thus

$$\sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_q|\int_0^\infty e^{-2\pi y(p+q+m)}y^{k+l+2n-2}dy = \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}}\sum_{\substack{p=1\\q=0}}^{\infty}\frac{|a_pb_q|}{(p+q+m)^{k+l+2n-1}}\ .$$

Now, by Theorem 1 with $\delta=\frac{1}{8}$ we have $a_p = O(p^{(k+l+2n)/2-1/8})$, and since $b_p = O(p^\beta)$, there exists a constant $C_1>0$ such that

$$|a_p|\le C_1p^{(k+l+2n)/2-1/8}\quad\text{and}\quad |b_p|\le C_1p^\beta\ ,$$

for all $p\ge 1$. This implies

$$\begin{aligned}\sum_{\substack{p=1\\q=0}}^{\infty}\frac{|a_pb_q|}{(p+q+m)^{k+l+2n-1}} &= \sum_{p=1}^{\infty}\frac{|a_pb_0|}{(p+m)^{k+l+2n-1}}+\sum_{p,q=1}^{\infty}\frac{|a_pb_q|}{(p+q+m)^{k+l+2n-1}}\\ &\le |b_0|C_1\sum_{p=1}^{\infty}\frac{p^{(k+l+2n)/2-1/8}}{(p+m)^{k+l+2n-1}}+C_1^2\sum_{p,q=1}^{\infty}\frac{p^{(k+l+2n)/2-1/8}q^\beta}{(p+q+m)^{k+l+2n-1}}\\ &\le |b_0|C_1\sum_{p=1}^{\infty}\frac{(p+m)^{(k+l+2n)/2-1/8}}{(p+m)^{k+l+2n-1}}+C_1^2\sum_{p,q=1}^{\infty}\frac{(p+q+m)^{(k+l+2n)/2-1/8}(p+q+m)^\beta}{(p+q+m)^{k+l+2n-1}}\\ &= |b_0|C_1\sum_{p=1}^{\infty}\frac{1}{(p+m)^{(k+l+2n)/2-7/8}}+C_1^2\sum_{p,q=1}^{\infty}\frac{1}{(p+q+m)^{(k+l+2n)/2-7/8-\beta}}\ .\end{aligned}$$

Hence

$$\begin{aligned}&\int_{F_\infty}\sum_{\substack{p=1\\q=0}}^{\infty}|a_pb_qe^{2\pi iz(p+q+m)}|\mathrm{Im}(z)^{k+l+2n}d\mu(z)\\ \le&\frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}}\left(|b_0|C_1\sum_{p=1}^{\infty}\frac{1}{(p+m)^{(k+l+2n)/2-7/8}}+C_1^2\sum_{p,q=1}^{\infty}\frac{1}{(p+q+m)^{(k+l+2n)/2-7/8-\beta}}\right)\ .\end{aligned}$$

Similarly, for every $1 \le s \le n$ we have

$$\begin{aligned}
\int_{F_\infty} \sum_{\substack{p=1\\q=0}}^{\infty} |a_p b_q e^{2\pi i z(p+q+m)}| q^s \mathrm{Im}(z)^{k+l+2n} d\mu(z) &= \sum_{p,q=1}^{\infty} |a_p b_q| q^s \int_{F_\infty} |e^{2\pi i z(p+q+m)}| \mathrm{Im}(z)^{k+l+2n} d\mu(z) \\
&= \sum_{p,q=1}^{\infty} |a_p b_q| q^s \int_0^{\infty} e^{-2\pi y(p+q+m)} y^{k+l+2n-2} dy \\
&= \sum_{p,q=1}^{\infty} |a_p b_q| q^s \frac{\Gamma(k+l+2n-1)}{(2\pi(p+q+m))^{k+l+2n-1}} \\
&= \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{|a_p b_q| q^s}{(p+q+m)^{k+l+2n-1}} \\
&\le C_1^2 \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{p^{(k+l+2n)/2-1/8} q^{\beta+n}}{(p+q+m)^{k+l+2n-1}} \\
&\le C_1^2 \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{1}{(p+q+m)^{(k+l)/2-7/8-\beta}} \ .
\end{aligned}$$

(using that $p, q \le p+q+m$). If we put $C = \max\{|b_0|C_1, C_1^2\}$, we get the inequalities in the Lemma. Finally we observe that each one of these series converges because

$$(k+l+2n)/2 - 7/8 \ge 17/8 \quad \text{and} \quad (k+l+2n)/2 - 7/8 - \beta \ge (k+l)/2 - 7/8 - \beta > 2 \ .$$

This completes the proof of Lemma 5. □

Lemma 6. *The integral*

$$\int_F \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |f(z) [\![j(\gamma,\cdot)^{-k} e^{2\pi i m \gamma(\cdot)}, g]\!]_n^{(k,l)}(z) \ Im(z)^{k+l+2n} | d\mu(z)$$

converges.

As in Section 6 of Chapter 1, F denotes any fundamental domain for Γ. One can check that the integrand is invariant under the action of Γ over the variable z thus, this integral is well defined. Also, the inner sum is well-defined since it is independent of the choice of representatives for the cosets in $\Gamma_\infty \backslash \Gamma$.

Proof of Lemma 6. Using that the integrands are non-negative, we write

$$\begin{aligned}
&\int_F \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |f(z) [\![j(\gamma,\cdot)^{-k} e^{2\pi i m \gamma(\cdot)}, g]\!]_n^{(k,l)}(z) \ \mathrm{Im}(z)^{k+l+2n} | d\mu(z) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_F |f(z) [\![j(\gamma,\cdot)^{-k} e^{2\pi i m \gamma(\cdot)}, g]\!]_n^{(k,l)}(z) \ \mathrm{Im}(z)^{k+l+2n} | d\mu(z) \ .
\end{aligned}$$

Then, in each integral, we make the change of variables $z = \gamma^{-1}z'$ and use lemma 4 in order to write the last expression as

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma F} |f(z) [\![e^{2\pi i m(\cdot)}, g]\!]_n^{(k,l)}(z) \ \mathrm{Im}(z)^{k+l+2n}| d\mu(z)$$
$$= \int_{F_\infty} |f(z) [\![e^{2\pi i m(\cdot)}, g]\!]_n^{(k,l)}(z) \ \mathrm{Im}(z)^{k+l+2n}| d\mu(z)$$

(here we have used that the disjoint union $\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma} \gamma F$ gives a fundamental domain for Γ_∞).

Now, replacing f and g by their Fourier series, using the definition and properties of the n-th Rankin-Cohen bracket, and applying the triangular inequality, we have

$$\int_{F_\infty} |f(z) [\![e^{2\pi i m(\cdot)}, g]\!]_n^{(k,l)}(z) \ \mathrm{Im}(z)^{k+l+2n}| d\mu(z)$$
$$\leq \int_{F_\infty} \sum_{\substack{r,s\geq 0\\ r+s=n}} \sum_{\substack{p=1\\ q=0}}^{\infty} \left| \binom{k+n-1}{s}\binom{l+n-1}{r} \right| m^r |a_p b_q e^{2\pi i z(p+m+q)}| q^s \mathrm{Im}(z)^{k+l+2n} d\mu(z)$$
$$= \sum_{\substack{r,s\geq 0\\ r+s=n}} \left| \binom{k+n-1}{s}\binom{l+n-1}{r} \right| m^r \int_{F_\infty} \sum_{\substack{p=1\\ q=0}}^{\infty} |a_p b_q e^{2\pi i z(p+m+q)}| q^s \mathrm{Im}(z)^{k+l+2n} d\mu(z) \ .$$

The last expression is a sum of finitely many terms, indexed by r and s, and each one of these terms is finite by lemma 5. □

Lemma 7. *For any integer s with $0 \leq s \leq n$, one has*

$$\int_{F_\infty} f(z)\overline{e^{2\pi i m z} D^s g(z)} \ Im(z)^{k+l+2n} d\mu(z) = \frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}} \sum_{q=0}^{\infty} \frac{a_{q+m}\overline{b_q} q^s}{(q+m)^{k+l+2n-1}}.$$

Proof of Lemma 7. Replacing f and g by their Fourier series we have

$$\int_{F_\infty} f(z)\overline{e^{2\pi i m z} D^s g(z)} \ \mathrm{Im}(z)^{k+l+2n} d\mu(z) = \int_{F_\infty} \sum_{\substack{p=1\\ q=0}}^{\infty} a_p \overline{b_q} e^{2\pi i z p} \overline{e^{2\pi i z(m+q)}} q^s \mathrm{Im}(z)^{k+l+2n} d\mu(z).$$

By Lemma 5 the last expression converges absolutely. Thus Fubini's Theorem yields

$$\int_{F_\infty} f(z)\overline{e^{2\pi i m z} D^s g(z)} \ \mathrm{Im}(z)^{k+l+2n} d\mu(z) = \sum_{\substack{p=1\\ q=0}}^{\infty} a_p \overline{b_q} q^s \int_{F_\infty} e^{2\pi i z p} \overline{e^{2\pi i z(m+q)}} \mathrm{Im}(z)^{k+l+2n} d\mu(z).$$

But

$$\int_{F_\infty} e^{2\pi i z p} \overline{e^{2\pi i z(m+q)}} \ \mathrm{Im}(z)^{k+l+2n} d\mu(z) \ = \ \int_0^\infty \int_0^1 e^{2\pi i x(p-m-q)} e^{-2\pi y(p+m+q)} y^{k+l+2n-2} dx dy$$

and

$$\int_0^1 e^{2\pi i x(p-m-q)} dx = \begin{cases} 0 & \text{if } p \neq q+m, \\ 1 & \text{if } p = q+m, \end{cases}$$

thus

$$\int_{F_\infty} e^{2\pi i z p} \overline{e^{2\pi i z(m+q)}} \ \mathrm{Im}(z)^{k+l+2n} d\mu(z) = \begin{cases} 0 & \text{if } p \neq q+m, \\ \int_0^\infty e^{-4\pi y(m+q)} y^{k+l+2n-2} dy = \frac{\Gamma(k+l+2n-1)}{(4\pi(m+q))^{k+l+2n-1}} & \text{if } p = q+m. \end{cases}$$

Hence

$$\begin{aligned}\int_{F_\infty} f(z)\overline{e^{2\pi imz} D^s g(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z) &= \sum_{q=0}^{\infty} a_{q+m}\overline{b_q} q^s \frac{\Gamma(k+l+2n-1)}{(4\pi(m+q))^{k+l+2n-1}} \\ &= \frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}} \sum_{q=0}^{\infty} \frac{a_{q+m}\overline{b_q}q^s}{(q+m)^{k+l+2n-1}}.\end{aligned}$$

This completes the proof of Lemma 7. □

3. Proof of Theorem 5

Proof of Theorem 5. Let us write

$$T^*_{g,n}(f)(z) = \sum_{m=1}^{\infty} c_m e^{2\pi imz}$$

and let $P_{k,m}$ be the m-th Poincaré series of weight k for Γ $(m \geq 1)$. Then, by Proposition 1, we have

$$\frac{\Gamma(k-1)c_m}{(4\pi m)^{k-1}} = \langle T^*_{g,n}(f), P_{k,m}\rangle_k \ .$$

But, by definition of the adjoint map,

$$\langle T^*_{g,n}(f), P_{k,m}\rangle_k = \langle f, T_{g,n}(P_{k,m})\rangle_{k+l+2n} \ .$$

Now,

$$\begin{aligned}\langle f, T_{g,n}(P_{k,m})\rangle_{k+l+2n} &= \langle f, [\![P_{k,m}, g]\!]_n^{(k,l)}\rangle_{k+l+2n} \\ &= \int_F f(z)\overline{[\![P_{k,m}, g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z) \\ &= \int_F f(z)\overline{\left[\!\!\left[\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,\cdot)^{-k} e^{2\pi im\gamma(\cdot)},\ g\right]\!\!\right]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z) \\ &= \int_F \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(z)\overline{[\![j(\gamma,\cdot)^{-k} e^{2\pi im\gamma(\cdot)}, g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z).\end{aligned}$$

By Lemma 6 the last expression converges absolutely. Thus, applying Fubini's Theorem, it is equal to

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_F f(z)\overline{[\![j(\gamma,\cdot)^{-k} e^{2\pi im\gamma(\cdot)}, g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z).$$

Now, in each integral, we make the change of variables $z = \gamma^{-1}z'$ and use lemma 4 in order to get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma F} f(z)\overline{[\![e^{2\pi im(\cdot)}, g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z) = \int_{F_\infty} f(z)\overline{[\![e^{2\pi im(\cdot)}, g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n} d\mu(z)$$

(recall that $\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma F$ is a disjoint union and it gives a fundamental domain for Γ_∞). Using the definition of the n-th Rankin-Cohen bracket we get

$$\begin{aligned}&\int_{F_\infty} f(z)\overline{[\![e^{2\pi im(\cdot)},g]\!]_n^{(k,l)}(z)}\ \mathrm{Im}(z)^{k+l+2n}d\mu(z)\\ &=\sum_{\substack{r,s\geq 0\\ r+s=n}}(-1)^r\binom{k+n-1}{s}\binom{l+n-1}{r}m^r\int_{F_\infty} f(z)\overline{e^{2\pi imz}D^s g(z)}\mathrm{Im}(z)^{k+l+2n}d\mu(z).\end{aligned}$$

Next we apply Lemma 7 to the integral in this expression and obtain

$$\begin{aligned}&\sum_{\substack{r,s\geq 0\\ r+s=n}}(-1)^r\binom{k+n-1}{s}\binom{l+n-1}{r}m^r\frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}}\sum_{q=0}^{\infty}\frac{a_{q+m}\overline{b_q}q^s}{(q+m)^{k+l+2n-1}}\\ =&\frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}}\sum_{q=0}^{\infty}\frac{a_{q+m}\overline{b_q}}{(q+m)^{k+l+2n-1}}\sum_{\substack{r,s\geq 0\\ r+s=n}}(-1)^r\binom{k+n-1}{s}\binom{l+n-1}{r}m^r q^s\\ =&\frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}}\sum_{q=0}^{\infty}\frac{a_{q+m}\overline{b_q}\varepsilon_{m,q}^{(k,l,n)}}{(q+m)^{k+l+2n-1}}.\end{aligned}$$

Hence

$$\begin{aligned}c_m &= \frac{(4\pi m)^{k-1}}{\Gamma(k-1)}\frac{\Gamma(k+l+2n-1)}{(4\pi)^{k+l+2n-1}}\sum_{q=0}^{\infty}\frac{a_{q+m}\overline{b_q}\varepsilon_{m,q}^{(k,l,n)}}{(q+m)^{k+l+2n-1}}\\ &= \frac{\Gamma(k+l+2n-1)}{\Gamma(k-1)(4\pi)^{l+2n}}m^{k-1}\sum_{q=0}^{\infty}\frac{a_{q+m}\overline{b_q}\varepsilon_{m,q}^{(k,l,n)}}{(q+m)^{k+l+2n-1}},\end{aligned}$$

as desired. □

4. Applications of Theorem 5

In this section we give some applications of Theorem 5. Observe that even in the case $n=0$ these results do not follow from Kohnen's theorem.

Proposition 2. *Let n be an integer with $n\geq 0$, $f\in S_{12+2n}(\Gamma)$ with Fourier coefficients $(a_m)_{m\geq 1}$ and $\sigma_3(m)=\sum_{\substack{d/m\\ d>0}} d^3$. Then*

$$\frac{a_m}{m^{11+n}}=(-1)^{n+1}\frac{1440}{(n+1)(n+2)(n+3)}\sum_{q=1}^{\infty}\frac{a_{q+m}\sigma_3(q)\varepsilon_{m,q}^{(8,4,n)}}{(q+m)^{11+2n}}\ ,\ \textit{for all } m\geq 1.$$

Proof of Proposition 2. Choosing $k=8$, $l=4$ and $g=E_4\in M_4(\Gamma)$ in Theorem 5, we have $T^*_{E_4,n}(f)\in S_8(\Gamma)=\{0\}$. Thus

$$\frac{\Gamma(11+2n)}{\Gamma(7)(4\pi)^{4+2n}}\sum_{m=1}^{\infty}m^7\left(\sum_{q=0}^{\infty}\frac{a_{q+m}b_q\varepsilon_{m,q}^{(8,4,n)}}{(q+m)^{11+2n}}\right)e^{2\pi imz}=0\ ,$$

where $(b_q)_{q\geq 0}$ are the Fourier coefficients of E_4. Since

$$E_4(z)=1+240\sum_{n=1}^{\infty}\sigma_3(n)e^{2\pi inz}\ ,$$

we have

$$\frac{a_m \varepsilon_{m,0}^{(8,4,n)}}{m^{11+2n}} + 240 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_3(q)\varepsilon_{m,q}^{(8,4,n)}}{(q+m)^{11+2n}} = 0 \ , \ \text{for all } m \geq 1.$$

Simple calculations yield $\varepsilon_{m,0}^{(8,4,n)} = (-1)^n \frac{(n+3)(n+2)(n+1)}{6} m^n$, thus

$$(-1)^n \frac{(n+3)(n+2)(n+1)}{6} \frac{a_m}{m^{11+n}} = -240 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_3(q)\varepsilon_{m,q}^{(8,4,n)}}{(q+m)^{11+2n}} \ , \ \text{for all } m \geq 1.$$

This gives the desired equality. □

Examples:

(1) For $n = 0$ and $f = \Delta$ one has

$$\frac{\tau(m)}{m^{11}} = -240 \sum_{q=1}^{\infty} \frac{\tau(q+m)\sigma_3(q)}{(q+m)^{11}} \ , \ \text{for all } m \geq 1.$$

(2) For $n = 2$ and $f = \Delta E_4$ one has

$$\frac{a_m}{m^{13}} = -24 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_3(q)(36q^2 - 45mq + 10m^2)}{(q+m)^{15}} \ , \ \text{for all } m \geq 1.$$

Proposition 3. *Let n be an integer with $n \geq 0$, $f \in S_{16+2n}(\Gamma)$ with Fourier coefficients $(a_m)_{m\geq 1}$ and $\sigma_5(m) = \sum_{\substack{d/m \\ d>0}} d^5$. Then*

$$\frac{a_m}{m^{15+n}} = (-1)^n \frac{60480}{(n+1)(n+2)(n+3)(n+4)(n+5)} \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_5(q)\varepsilon_{m,q}^{(10,6,n)}}{(q+m)^{15+2n}} \ , \ \textit{for all } m \geq 1.$$

Proof of Proposition 3. Choosing $k = 10$, $l = 6$ and $g = E_6 \in M_6(\Gamma)$ in Theorem 5, we have $T^*_{E_6,n}(f) \in S_{10}(\Gamma) = \{0\}$. Thus

$$\frac{\Gamma(15+2n)}{\Gamma(9)(4\pi)^{6+2n}} \sum_{m=1}^{\infty} m^9 \left(\sum_{q=0}^{\infty} \frac{a_{q+m} b_q \varepsilon_{m,q}^{(10,6,n)}}{(q+m)^{15+2n}} \right) e^{2\pi i m z} = 0 \ ,$$

where $(b_q)_{q\geq 0}$ are the Fourier coefficients of E_6. Since

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n z} \ ,$$

we have

$$\frac{a_m \varepsilon_{m,0}^{(10,6,n)}}{m^{15+2n}} - 504 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_5(q)\varepsilon_{m,q}^{(10,6,n)}}{(q+m)^{15+2n}} = 0 \ , \ \text{for all } m \geq 1.$$

Simple calculations yield $\varepsilon_{m,0}^{(10,6,n)} = (-1)^n \frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{120} m^n$, thus

$$(-1)^n \frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{120} \frac{a_m}{m^{15+n}} = 504 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_5(q)\varepsilon_{m,q}^{(10,6,n)}}{(q+m)^{15+2n}} \ , \ \text{for all } m \geq 1.$$

This gives the desired equality. $\square$

EXAMPLES:

(1) For $n = 0$ and $f = \Delta E_4$ one has

$$\frac{a_m}{m^{15}} = 504 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_5(q)}{(q+m)^{15}} \ , \text{ for all } m \geq 1.$$

(2) For $n = 1$ and $f = \Delta E_6$ one has

$$\frac{a_m}{m^{16}} = -84 \sum_{q=1}^{\infty} \frac{a_{q+m}\sigma_5(q)(10q-6m)}{(q+m)^{17}} \ , \text{ for all } m \geq 1.$$

CHAPTER 3

Integral representations for the operators $T^*_{g,n}$

In this chapter we present two integral representations for the linear operators $T^*_{g,n}$ introduced in Theorem 5.

1. Theorem 6: First integral representation

Theorem 6. *Under the hypothesis of Theorem 5 we have*

$$T^*_{g,n}(f)(z) = \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k},\ g \right]\!\right]_n^{(k,l)} (w)}\ Im(w)^{k+l+2n} d\mu(w)\ .$$

Here and hereafter, if necessary, we write $F(\cdot)$ to empasize that F must be considered as a function of the variable placed at "$\cdot$". In particular, in the above Theorem, the derivatives involved in the Rankin-Cohen bracket must be taken with respect to that variable.

2. Lemmas for the proof of Theorem 6

In this section we fix $z \in \mathbb{H}$ and we suppose k, l, n, g and f as in Theorem 5.

Lemma 8. *The series*

$$\sum_{m=1}^{\infty} \int_{F_\infty} \left| m^{k-1} f(w) \left[\!\left[e^{2\pi i m(\cdot)}, g \right]\!\right]_n^{(k,l)} (w)\ Im(w)^{k+l+2n} e^{2\pi i m z} \right| d\mu(w)$$

converges.

Proof of Lemma 8. Clearly

$$\begin{aligned} & \sum_{m=1}^{\infty} \int_{F_\infty} \left| m^{k-1} f(w) \left[\!\left[e^{2\pi i m(\cdot)}, g \right]\!\right]_n^{(k,l)} (w)\ \mathrm{Im}(w)^{k+l+2n} e^{2\pi i m z} \right| d\mu(w) \\ = \ & \sum_{m=1}^{\infty} m^{k-1} |e^{2\pi i z}|^m \int_{F_\infty} \left| f(w) \left[\!\left[e^{2\pi i m(\cdot)}, g \right]\!\right]_n^{(k,l)} (w)\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w)\ . \end{aligned}$$

By definition of the n-th Rankin-Cohen bracket and applying the triangular inequality we get

$$\begin{aligned} & \sum_{m=1}^{\infty} m^{k-1} |e^{2\pi i z}|^m \int_{F_\infty} \left| f(w) \left[\!\left[e^{2\pi i m(\cdot)}, g \right]\!\right]_n^{(k,l)} (w)\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\ \leq \ & \sum_{m=1}^{\infty} m^{k-1} |e^{2\pi i z}|^m \sum_{\substack{r,s \geq 0 \\ r+s=n}} \left| \binom{k+n-1}{s} \binom{l+n-1}{r} \right| m^r \int_{F_\infty} \left| f(w) e^{2\pi i m w} D^s g(w)\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w)\ . \end{aligned}$$

Now,

$$\int_{F_\infty} \left| f(w) e^{2\pi i m w} D^s g(w)\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \leq \int_{F_\infty} \sum_{\substack{p=1 \\ q=0}}^{\infty} \left| a_p b_q e^{2\pi i w(p+q+m)} \right| q^s\ \mathrm{Im}(w)^{k+l+2n} d\mu(w)\ ,$$

and by Lemma 5 there exists $C > 0$ such that

$$\int_{F_\infty} \sum_{\substack{p=1\\q=0}}^{\infty} \left|a_p b_q e^{2\pi i w(p+q+m)}\right| q^s \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \leq C$$

for all $m \geq 1$ and $0 \leq s \leq n$ (the bounds in Lemma 5 decrease with respect to m). Thus

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{k-1} |e^{2\pi i z}|^m \sum_{\substack{r,s\geq 0\\ r+s=n}} \left| \binom{k+n-1}{s} \binom{l+n-1}{r} \right| m^r \int_{F_\infty} \left| f(w) e^{2\pi i m w} D^s g(w) \ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\
\leq \ & C \sum_{\substack{r,s\geq 0\\ r+s=n}} \left| \binom{k+n-1}{s} \binom{l+n-1}{r} \right| \sum_{m=1}^{\infty} m^{k-1+r} |e^{2\pi i z}|^m ,
\end{aligned}$$

where the last expression is a sum of finitely many terms, indexed by r and s, and each one of these terms is finite (using the ratio test for series and recalling that $|e^{2\pi i z}| < 1$). This completes the proof of Lemma 8. $\square$

Lemma 9. *Let r be an integer with $r \geq 0$. There exists a constant $C > 0$, depending on f,g,r and z, such that*

$$\begin{aligned}
&\int_{\mathbb{H}} \sum_{\substack{p=1\\q=0}}^{\infty} \left| \frac{a_p b_q e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| Im(w)^{k+l+2n} d\mu(w) \\
\leq \ & C \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \left(\sum_{p=1}^{\infty} \frac{1}{p^{(k+l+2n)/2-7/8}} + \sum_{p,q=1}^{\infty} \frac{1}{(p+q)^{(k+l+2n)/2-7/8-\beta}} \right).
\end{aligned}$$

Moreover, for $1 \leq s \leq n$ one has

$$\int_{\mathbb{H}} \sum_{\substack{p=1\\q=0}}^{\infty} \left| \frac{a_p b_q q^s e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| Im(w)^{k+l+2n} d\mu(w) \leq C \frac{\Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{1}{(p+q)^{(k+l)/2-7/8-\beta}} ,$$

and each one of these series converges.

Proof of Lemma 9. First we take $C_1 > 0$ such that

$$|a_p| \leq C_1 p^{(k+l+2n)/2-1/8} \quad \text{and} \quad |b_p| \leq C_1 p^{\beta}$$

for all $p \geq 1$ (as before use Theorem 1 with $\delta = \frac{1}{8}$ for the Fourier coefficients of f and recall that $b_p = O(p^\beta)$). We also take $C_2 = \int_{-\infty}^{\infty} |u - \overline{z}|^{-k-r} du$, which is finite because $k + r \geq 6$ and $|u - \overline{z}|$

never vanishes. Writing $w = u + iv$ with $u, v \in \mathbb{R}$ and $v \geq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{H}} \sum_{\substack{p=1\\q=0}}^{\infty} \left| \frac{a_p b_q e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
=& \int_{\mathbb{H}} \sum_{p=1}^{\infty} \left| \frac{a_p b_0 e^{2\pi i w p}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) + \int_{\mathbb{H}} \sum_{\substack{p=1\\q=1}}^{\infty} \left| \frac{a_p b_q e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
=& |b_0| \sum_{p=1}^{\infty} |a_p| \int_{\mathbb{H}} \frac{e^{-2\pi v p} v^{k+l+2n-2}}{|u+iv-\overline{z}|^{k+r}} du dv + \sum_{p,q=1}^{\infty} |a_p b_q| \int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u+iv-\overline{z}|^{k+r}} du dv \ .
\end{aligned}$$

Since $|u + iv - \overline{z}| \geq |u - \overline{z}|$ (because $v \geq 0$), the last expression is clearly less than or equal to

$$|b_0| \sum_{p=1}^{\infty} |a_p| \int_{\mathbb{H}} \frac{e^{-2\pi v p} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv + \sum_{p,q=1}^{\infty} |a_p b_q| \int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv \ .$$

But

$$\begin{aligned}
\int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv &= \int_0^{\infty} \int_{-\infty}^{\infty} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv \\
&= \left(\int_0^{\infty} e^{-2\pi v(p+q)} v^{k+l+2n-2} dv \right) \left(\int_{-\infty}^{\infty} \frac{1}{|u-\overline{z}|^{k+r}} du \right) \\
&= \frac{\Gamma(k+l+2n-1)}{(2\pi(p+q))^{k+l+2n-1}} C_2 \ ,
\end{aligned}$$

for any pair of integers p, q with $p \geq 1$ and $q \geq 0$. Thus

$$\begin{aligned}
&|b_0| \sum_{p=1}^{\infty} |a_p| \int_{\mathbb{H}} \frac{e^{-2\pi v p} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv + \sum_{p,q=1}^{\infty} |a_p b_q| \int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv \\
=& |b_0| \sum_{p=1}^{\infty} |a_p| \frac{\Gamma(k+l+2n-1)}{(2\pi p)^{k+l+2n-1}} C_2 + \sum_{p,q=1}^{\infty} |a_p b_q| \frac{\Gamma(k+l+2n-1)}{(2\pi(p+q))^{k+l+2n-1}} C_2 \\
=& \frac{C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \left(|b_0| \sum_{p=1}^{\infty} \frac{|a_p|}{p^{k+l+2n-1}} + \sum_{p,q=1}^{\infty} \frac{|a_p b_q|}{(p+q)^{k+l+2n-1}} \right) \\
\leq& \frac{C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \left(|b_0| C_1 \sum_{p=1}^{\infty} \frac{p^{(k+l+2n)/2-1/8}}{p^{k+l+2n-1}} + C_1^2 \sum_{p,q=1}^{\infty} \frac{p^{(k+l+2n)/2-1/8} q^{\beta}}{(p+q)^{k+l+2n-1}} \right) \\
\leq& \frac{C_1 C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \left(|b_0| \sum_{p=1}^{\infty} \frac{p^{(k+l+2n)/2-1/8}}{p^{k+l+2n-1}} + C_1 \sum_{p,q=1}^{\infty} \frac{(p+q)^{(k+l+2n)/2-1/8} (p+q)^{\beta}}{(p+q)^{k+l+2n-1}} \right) \\
=& \frac{C_1 C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \left(|b_0| \sum_{p=1}^{\infty} \frac{1}{p^{(k+l+2n)/2-7/8}} + C_1 \sum_{p,q=1}^{\infty} \frac{1}{(p+q)^{(k+l+2n)/2-7/8-\beta}} \right) .
\end{aligned}$$

Similarly, we now observe that for $1 \le s \le n$ we have

$$\begin{aligned}
\int_{\mathbb{H}} \sum_{\substack{p=1\\q=0}}^{\infty} \left| \frac{a_p b_q q^s e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) &= \int_{\mathbb{H}} \sum_{p,q=1}^{\infty} \left| \frac{a_p b_q q^s e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
&= \sum_{p,q=1}^{\infty} |a_p b_q| q^s \int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u+iv-\overline{z}|^{k+r}} du dv \\
&\le \sum_{p,q=1}^{\infty} |a_p b_q| q^s \int_{\mathbb{H}} \frac{e^{-2\pi v(p+q)} v^{k+l+2n-2}}{|u-\overline{z}|^{k+r}} du dv \\
&= \sum_{p,q=1}^{\infty} |a_p b_q| q^s \frac{\Gamma(k+l+2n-1)}{(2\pi(p+q))^{k+l+2n-1}} C_2 \\
&= \frac{C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{|a_p b_q| q^s}{(p+q)^{k+l+2n-1}} \\
&\le \frac{C_1^2 C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{p^{(k+l+2n)/2-1/8} q^{\beta+s}}{(p+q)^{k+l+2n-1}} \\
&\le \frac{C_1^2 C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{(p+q)^{(k+l+2n)/2-1/8} (p+q)^{\beta+s}}{(p+q)^{k+l+2n-1}} \\
&= \frac{C_1^2 C_2 \Gamma(k+l+2n-1)}{(2\pi)^{k+l+2n-1}} \sum_{p,q=1}^{\infty} \frac{1}{(p+q)^{(k+l)/2-7/8-\beta}}.
\end{aligned}$$

If we now put $C = \max\{C_1^2 C_2, C_1 C_2 |b_0|\}$, we get the inequalities in the Lemma. The statement about convergence is obtained as in the proof of Lemma 5. $\square$

Lemma 10. *If $F : \mathbb{H} \to \mathbb{C}$ is any holomorphic function and d is an integer, then*

$$f(z-d) [\![F, g]\!]_n^{(k,l)}(z-d)\ Im(z-d)^{k+l+2n} = f(z) [\![\ F(\ \cdot\ -d)\ ,\ g\ ]\!]_n^{(k,l)}(z)\ Im(z)^{k+l+2n}\ .$$

Proof of Lemma 10. Since f is a modular form, it is $\mathbb{Z}$-periodic. Thus $f(z-d) = f(z)$. Clearly $\mathrm{Im}(z-d) = \mathrm{Im}(z)$. Hence, we only have to prove

$$[\![F, g]\!]_n^{(k,l)}(z-d) = [\![\ F(\ \cdot\ -d)\ ,\ g\ ]\!]_n^{(k,l)}(z)\ .$$

Let $T_d = \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix}$. Then $T_d \in \Gamma$ and by Lemma 3

$$[\![F, g]\!]_n^{(k,l)}|_{k+l+2n}[T_d] = [\![\ F|_k[T_d]\ ,\ g|_l[T_d]\ ]\!]_n^{(k,l)}\ .$$

Since $g|_l[T_d] = g$ and $T_d z = z - d$, we have

$$j(T_d, z)^{-k-l-2n} [\![F, g]\!]_n^{(k,l)}(z-d) = [\![\ j(T_d, \cdot)^{-k} F(\ \cdot\ -d)\ ,\ g\ ]\!]_n^{(k,l)}(z)\ .$$

But

$$j(T_d,\ \cdot\ ) = 1\ ,$$

thus

$$[\![F, g]\!]_n^{(k,l)}(z-d) = [\![\ F(\ \cdot\ -d)\ ,\ g\ ]\!]_n^{(k,l)}(z)\ .$$

□

Lemma 11. *The integral*

$$\int_{F_\infty} \sum_{d\in\mathbb{Z}} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z}+d)^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ Im(w)^{k+l+2n} \right| d\mu(w)$$

converges.

As usual, one can check that the integrand is $\mathbb{Z}$-periodic, so the integral is well defined (it is independent of the choice of the fundamental domain F_∞).

Proof of Lemma 11. Using that the integrand is non-negative, we write

$$\begin{aligned}
&\int_{F_\infty} \sum_{d\in\mathbb{Z}} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z}+d)^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\
= &\sum_{d\in\mathbb{Z}} \int_{F_\infty} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z}+d)^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w)\ .
\end{aligned}$$

Next, in each integral, we make the change of variables $w' = w + d$ and use Lemma 10 in order to obtain

$$\begin{aligned}
&\sum_{d\in\mathbb{Z}} \int_{F_\infty} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z}+d)^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\
= &\sum_{d\in\mathbb{Z}} \int_{F_\infty+d} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\
= &\int_{\mathbb{H}} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w)\ .
\end{aligned}$$

Here we have used that $\bigcup_{d\in\mathbb{Z}} F_\infty + d = \mathbb{H}$. Now, by the triangular inequality and the definition of the n-th Rankin-Cohen bracket, we get

$$\begin{aligned}
&\int_{\mathbb{H}} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot\ -\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\
\le &\sum_{\substack{r,s\ge 0\\ r+s=n}} \left| \binom{k+n-1}{s} \binom{l+n-1}{r} \right| \int_{\mathbb{H}} \left| f(w) D^r \frac{1}{(w-\overline{z})^k} D^s g(w) \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w)\ .
\end{aligned}$$

By simple calculations we have

$$\left| D^r \frac{1}{(w-\overline{z})^k} \right| = \binom{k+r-1}{r} \frac{r!}{(2\pi)^r} \left| \frac{1}{(w-\overline{z})^{k+r}} \right| ,$$

thus

$$\begin{aligned}
&\sum_{\substack{r,s\geq 0\\ r+s=n}} \left| \binom{k+n-1}{s}\binom{l+n-1}{r} \right| \int_{\mathbb{H}} \left| f(w) D^r \frac{1}{(w-\overline{z})^k} D^s g(w) \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
= &\sum_{\substack{r,s\geq 0\\ r+s=n}} \left| \binom{k+n-1}{s}\binom{l+n-1}{r} \right| \binom{k+r-1}{r} \frac{r!}{(2\pi)^r} \int_{\mathbb{H}} \left| \frac{f(w) D^s g(w)}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w)
\end{aligned}$$

But

$$\int_{\mathbb{H}} \left| \frac{f(w) D^s g(w)}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \leq \int_{\mathbb{H}} \sum_{\substack{p=1\\ q=0}}^{\infty} \left| \frac{a_p b_q q^s e^{2\pi i w(p+q)}}{(w-\overline{z})^{k+r}} \right| \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ ,$$

(recall that $(a_p)_{p\geq 1}$ and $(b_q)_{q\geq 0}$ are the Fourier coefficients of f and g respectively) and by Lemma 9 this expression is finite. Thus, above we have a sum of finitely many terms, indexed by r and s, and each one of them is finite. This proves Lemma 11. □

3. Proof of Theorem 6

As in the proof of Theorem 5 we let

$$T^*_{g,n}(f)(z) = \sum_{m=1}^{\infty} c_m e^{2\pi i m z} \ .$$

In the first part of Section 3, Chapter 2, we proved that

$$\frac{\Gamma(k-1)c_m}{(4\pi m)^{k-1}} = \int_{F_\infty} f(w) \overline{[\![e^{2\pi i m(\cdot)},\ g]\!]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ .$$

Thus

$$\begin{aligned}
T^*_{g,n}(f)(z) &= \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \sum_{m=1}^{\infty} m^{k-1} \left(\int_{F_\infty} f(w) \overline{[\![e^{2\pi i m(\cdot)},\ g]\!]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \right) e^{2\pi i m z} \\
&= \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \sum_{m=1}^{\infty} \int_{F_\infty} m^{k-1} f(w) \overline{[\![e^{2\pi i m(\cdot)},\ g]\!]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} e^{2\pi i m z} d\mu(w) \ .
\end{aligned}$$

By Lemma 8 this last expression converges absolutely. Thus, by Fubini's theorem

$$\begin{aligned}
T^*_{g,n}(f)(z) &= \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \int_{F_\infty} \sum_{m=1}^{\infty} m^{k-1} f(w) \overline{[\![e^{2\pi i m(\cdot)}, g]\!]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} e^{2\pi i m z} d\mu(w) \\
&= \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \int_{F_\infty} f(w) \overline{\left[\!\!\left[\sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m(\ \cdot\ -\overline{z})}\ ,\ g \right]\!\!\right]_n^{(k,l)} (w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ .
\end{aligned}$$

Now, we use the identity

$$\sum_{m=1}^{\infty} m^{\kappa-1} e^{2\pi i m \xi} = \frac{(\kappa-1)!}{(-2\pi i)^{\kappa}} \sum_{d\in\mathbb{Z}} \frac{1}{(\xi+d)^{\kappa}} \qquad \text{for } \xi\in\mathbb{H}, \kappa\in\mathbb{Z}, \kappa\geq 2 \ ,$$

(which is a consequence of the Poisson summation formula) with $\xi = w - \overline{z}$ and $\kappa = k$ in the previous equation and obtain

$$\begin{aligned} T^*_{g,n}(f)(z) &= \frac{(4\pi)^{k-1}(k-1)!}{\Gamma(k-1)(-2\pi i)^k} \int_{F_\infty} f(w) \overline{\left[\!\left[\sum_{d\in\mathbb{Z}} \frac{1}{(\cdot - \overline{z} + d)^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ &= \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{F_\infty} \sum_{d\in\mathbb{Z}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z} + d)^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ . \end{aligned}$$

By Lemma 11 this integral converges absolutely, so we can apply Fubini's theorem again.

$$\begin{aligned} &\int_{F_\infty} \sum_{d\in\mathbb{Z}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z} + d)^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ = &\sum_{d\in\mathbb{Z}} \int_{F_\infty} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z} + d)^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ . \end{aligned}$$

Next, en each integral we make the change of variables $w' = w + d$ and use Lemma 10 in order to write the last series as

$$\begin{aligned} &\sum_{d\in\mathbb{Z}} \int_{F_\infty + d} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ = &\int_{\mathbb{H}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)}(w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \end{aligned}$$

(recall that $\bigcup_{d\in\mathbb{Z}} F_\infty + d = \mathbb{H}$). This completes the proof of Theorem 6. $\square$

4. Theorem 7: Second integral representation

For an even integer $k \geq 4$ let us define

$$h_k(z_1, z_2) := \sum_{\gamma\in\Gamma} j(\gamma, z_1)^{-k}(\gamma z_1 + z_2)^{-k} = \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ ad-bc=1}} (cz_1z_2 + az_1 + dz_2 + b)^{-k} \ , \ \text{where } z_1, z_2 \in \mathbb{H}.$$

Then h_k is a holomorphic function in each variable and it is symmetric, i.e. $h(z_1, z_2) = h(z_2, z_1)$. Moreover, $h_k(z_1, z_2)$ is a cuspidal form over Γ in each variable (see Lemmas 13 and 14 below). Don Zagier considered this function in [**10**], where he obtained integral representations for certain operators called Hecke operators.

Our second integral representation for the operators $T^*_{g,n}$ also involves the function h_k.

Theorem 7. *Under the hypothesis of Theorem 5 we also have*

$$T^*_{g,n}(f)(z) = \frac{i^k 2^{k-3}(k-1)}{\pi} \int_F f(w) \overline{[\![h_k(\cdot, -\overline{z}), \ g]\!]_n^{(k,l)}(w)} \ Im(w)^{k+l+2n} d\mu(w) \ .$$

5. Basic properties of h_k

Lemma 12. *For any $x, y \in \mathbb{R}$ with $y > 0$*

$$\sum_{d\in\mathbb{Z}} \frac{1}{|x+d+iy|^k} \leq \sum_{d\in\mathbb{Z}} \frac{1}{|d+iy|^k} + \frac{1}{y^k} \ .$$

Proof of Lemma 12. Let $\varrho = x - \max\{l \in \mathbb{Z} \ : \ l \leq x\}$ (i.e. the fractional part of x). Then $\varrho \in [0,1[$ and $x = l_0 + \varrho$ where $l_0 \in \mathbb{Z}$. Hence

$$\begin{aligned} \sum_{d\in\mathbb{Z}} \frac{1}{|x+d+iy|^k} &= \sum_{d\in\mathbb{Z}} \frac{1}{|l_0+\varrho+d+iy|^k} \\ &= \sum_{d\in\mathbb{Z}} \frac{1}{|\varrho+d+iy|^k} \\ &= \sum_{d=0}^{\infty} \frac{1}{|\varrho+d+iy|^k} + \sum_{d=-1}^{-\infty} \frac{1}{|\varrho+d+iy|^k} \ . \end{aligned}$$

Since $|\varrho+d+iy| \geq |d+iy|$ for all $d \geq 0$ we obtain

$$\sum_{d=0}^{\infty} \frac{1}{|\varrho+d+iy|^k} \leq \sum_{d=0}^{\infty} \frac{1}{|d+iy|^k} \ .$$

For $d < 0$ we have $|\varrho+d+iy| \geq |d+1+iy|$, thus

$$\sum_{d=-1}^{-\infty} \frac{1}{|\varrho+d+iy|^k} \leq \sum_{d=0}^{-\infty} \frac{1}{|d+iy|^k} \ .$$

Hence

$$\sum_{d\in\mathbb{Z}} \frac{1}{|x+d+iy|^k} \leq \sum_{d=0}^{\infty} \frac{1}{|d+iy|^k} + \sum_{d=0}^{-\infty} \frac{1}{|d+iy|^k} = \sum_{d\in\mathbb{Z}} \frac{1}{|d+iy|^k} + \frac{1}{y^k} \ .$$

□

Lemma 13. *For any fixed $z_2 \in \mathbb{H}$, the series*

$$\sum_{\gamma\in\Gamma} j(\gamma, z_1)^{-k} (\gamma z_1 + z_2)^{-k}$$

converges absolutely and uniformly on any compact subset of $\mathbb{H}$.

Proof of Lemma 13. We have

$$\begin{aligned} \sum_{\gamma\in\Gamma} |j(\gamma, z_1)|^{-k} |\gamma z_1 + z_2|^{-k} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma, z_1)|^{-k} \sum_{\alpha\in\Gamma_\infty} \frac{1}{|\alpha\gamma(z_1) + z_2|^k} \\ &= 2 \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma, z_1)|^{-k} \sum_{d\in\mathbb{Z}} \frac{1}{|\gamma(z_1) + d + z_2|^k} \ , \end{aligned}$$

since every translation appears twice in Γ_∞. Now,

$$|\gamma(z_1) + d + z_2| = |\mathrm{Re}(\gamma(z_1) + z_2) + d + i\mathrm{Im}(\gamma(z_1) + z_2)| \geq |\mathrm{Re}(\gamma(z_1) + z_2) + d + i\mathrm{Im}(z_2)|$$

and by Lemma 12

$$\sum_{d\in\mathbb{Z}} \frac{1}{|\mathrm{Re}(\gamma(z_1) + z_2) + d + i\mathrm{Im}(z_2)|^k} \leq \sum_{d\in\mathbb{Z}} \frac{1}{|d + i\mathrm{Im}(z_2)|^k} + \frac{1}{\mathrm{Im}(z_2)^k} \ .$$

Thus

$$\sum_{d\in\mathbb{Z}} \frac{1}{|\gamma(z_1)+d+z_2|^k} \leq \sum_{d\in\mathbb{Z}} \frac{1}{|d+i\mathrm{Im}(z_2)|^k} + \frac{1}{\mathrm{Im}(z_2)^k} \ .$$

Since $k \geq 4$ this last series converges. Hence, there exists $D > 0$ (depending only on z_2) such that

$$\sum_{\gamma\in\Gamma} |j(\gamma, z_1)|^{-k} |\gamma z_1 + z_2|^{-k} \leq 2D \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma, z_1)|^{-k} \ .$$

One observes that for $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ in Γ,

$$\Gamma_\infty \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \Gamma_\infty \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \Longleftrightarrow (c_1, d_1) = \pm(c_2, d_2) \ .$$

Thus

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma, z_1)|^{-k} \leq \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{|cz+d|^k}$$

and this last series converges uniformly on any compact subset of $\mathbb{H}$ (as in the case of the k-th Eisenstein series). $\square$

Lemma 14. *h_k is symmetric and it is a cuspidal form of weight k over Γ in each variable.*

Proof of Lemma 14. By Lemma 13 h_k is a well defined complex-valued function on $\mathbb{H} \times \mathbb{H}$. The symmetry follows immediately from the definition

$$h_k(z_1, z_2) = \sum_{ad-bc=1} (cz_1z_2 + az_1 + dz_2 + b)^{-k} \ .$$

Now, fix $z_2 \in \mathbb{H}$. By Lemma 13, $h_k(z_1, z_2)$ is a holomorphic function of z_1 on $\mathbb{H}$. For $\alpha \in \Gamma$ we have

$$h_k(\alpha z_1, z_2) = \sum_{\gamma\in\Gamma} j(\gamma, \alpha z_1)^{-k} (\gamma\alpha z_1 + z_2)^{-k} \ .$$

Since

$$j(\gamma\alpha, z_1) = j(\gamma, \alpha z_1) j(\alpha, z_1) \ ,$$

we obtain

$$h_k(\alpha z_1, z_2) = j(\alpha, z_1)^k \sum_{\gamma\in\Gamma} j(\gamma\alpha, z_1)^{-k} (\gamma\alpha z_1 + z_2)^{-k} = j(\alpha, z_1)^k h_k(z_1, z_2) \ .$$

Moreover, using that h_k is symmetric we have

$$\lim_{Im(z_1)\to+\infty} h_k(z_1, z_2) = \lim_{Im(z_1)\to+\infty} h_k(z_2, z_1) = \lim_{Im(z_1)\to+\infty} \sum_{\gamma\in\Gamma} j(\gamma, z_2)^{-k} (\gamma z_2 + z_1)^{-k} = 0 \ ,$$

thus $h_k(z_1, z_2)$ vanishes at $i\infty$ with respect to z_1. All these properties yield that h_k is a cuspidal form over Γ in the first variable. Since h_k is symmetric, the same result holds for the second variable. $\square$

6. Proof of Theorem 7

By Theorem 6 we have

$$T^*_{g,n}(f)(z) = \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ .$$

Thus

$$\begin{aligned} T^*_{g,n}(f)(z) &= \frac{i^k 2^{k-2}(k-1)}{\pi} \frac{1}{2} \sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ &= \frac{i^k 2^{k-3}(k-1)}{\pi} \sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \ , \end{aligned}$$

where the factor $\frac{1}{2}$ appears because $\gamma F = (-\gamma)F$ for every $\gamma \in \Gamma$. Now, in each integral we make the change of variables $w = \gamma w'$ and we use the invariance of the hyperbolic measure, so that

$$\begin{aligned} &\sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ =&\sum_{\gamma \in \Gamma} \int_{F} f(\gamma w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (\gamma w)} \ \mathrm{Im}(\gamma w)^{k+l+2n} d\mu(w) \ . \end{aligned}$$

Since

$$\begin{aligned} &\sum_{\gamma \in \Gamma} \int_{F} \left| f(\gamma w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (\gamma w)} \ \mathrm{Im}(\gamma w)^{k+l+2n} \right| d\mu(w) \\ =&\sum_{\gamma \in \Gamma} \int_{\gamma F} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \\ =&2 \int_{\mathbb{H}} \left| f(w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k} \ , \ g \right]\!\right]_n^{(k,l)} (w)} \ \mathrm{Im}(w)^{k+l+2n} \right| d\mu(w) \end{aligned}$$

the expression above converges absolutely (this integral appears in the proof of Lemma 11). Thus, by Fubini's theorem

$$\begin{aligned} &\sum_{\gamma \in \Gamma} \int_{F} f(\gamma w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k}, \ g \right]\!\right]_n^{(k,l)} (\gamma w)} \ \mathrm{Im}(\gamma w)^{k+l+2n} d\mu(w) \\ = &\int_{F} \sum_{\gamma \in \Gamma} f(\gamma w) \overline{\left[\!\left[\frac{1}{(\cdot - \overline{z})^k}, \ g \right]\!\right]_n^{(k,l)} (\gamma w)} \ \mathrm{Im}(\gamma w)^{k+l+2n} d\mu(w) \ . \end{aligned}$$

On the other hand,

$$
\begin{aligned}
& f(\gamma w)\overline{\left[\!\left[\frac{1}{(\cdot-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(\gamma w)}\ \mathrm{Im}(\gamma w)^{k+l+2n} \\
=& j(\gamma,w)^{k+l+2n} f(w)\overline{\left[\!\left[\frac{1}{(\cdot-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(\gamma w)}\ \frac{\mathrm{Im}(w)^{k+l+2n}}{|j(\gamma,w)|^{2(k+l+2n)}} \\
=& f(w)\overline{j(\gamma,w)^{-(k+l+2n)}\left[\!\left[\frac{1}{(\cdot-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(\gamma w)}\ \mathrm{Im}(w)^{k+l+2n} \\
=& f(w)\overline{\left[\!\left[\frac{1}{(\cdot-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}\Bigg|_{k+l+2n}[\gamma](w)}\ \mathrm{Im}(w)^{k+l+2n} \\
=& f(w)\overline{\left[\!\left[\frac{1}{(\cdot-\overline{z})^k}\Bigg|_k[\gamma]\ ,\ g|_l[\gamma] \right]\!\right]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} \\
=& f(w)\overline{\left[\!\left[j(\gamma,\cdot)^{-k}\frac{1}{(\gamma(\cdot)-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n}\ .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int_F \sum_{\gamma\in\Gamma} f(\gamma w)\overline{\left[\!\left[\frac{1}{(\cdot-\overline{z})^k},\ g \right]\!\right]_n^{(k,l)}(\gamma w)}\ \mathrm{Im}(\gamma w)^{k+l+2n} d\mu(w) \\
=& \int_F \sum_{\gamma\in\Gamma} f(w)\overline{\left[\!\left[j(\gamma,\cdot)^{-k}\frac{1}{(\gamma(\cdot)-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
=& \int_F f(w)\overline{\left[\!\left[\sum_{\gamma\in\Gamma} j(\gamma,\cdot)^{-k}\frac{1}{(\gamma(\cdot)-\overline{z})^k}\ ,\ g \right]\!\right]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\
=& \int_F f(w)\overline{[\![h_k(\cdot,-\overline{z}),\ g]\!]_n^{(k,l)}(w)}\ \mathrm{Im}(w)^{k+l+2n} d\mu(w)\ .
\end{aligned}
$$

This completes the proof of Theorem 7. $\square$

7. Applications of Theorems 6 and 7

Proposition 4 (Integral reproduction formulas for cuspidal forms)**.** *Let k be an integer and $f \in S_k(\Gamma)$. Then*

$$f(z) = \frac{i^k 2^{k-2}(k-1)}{\pi}\int_{\mathbb{H}} f(w)\frac{1}{(\overline{w}-z)^k} Im(w)^k d\mu(w)$$

and

$$f(z) = \frac{i^k 2^{k-3}(k-1)}{\pi}\int_F f(w)\overline{h_k(w,-\overline{z})} Im(w)^k d\mu(w).$$

Proof of Proposition 4. We can assume $k \geq 12$ even, otherwise $S_k(\Gamma) = \{0\}$ and the identities hold trivially.

For $k \geq 12$ even, we choose $l = n = 0$ and $g = 1 \in M_0(\Gamma)$ in Theorem 6 (resp. Theorem 7). Since $T_1 : S_k(\Gamma) \to S_k(\Gamma)$ is the identity function, we have $T_1^*(f) = f$ for any $f \in S_k(\Gamma)$ and we get the desired equations. $\square$

Proposition 5 (Partial integral reproduction formulas for cuspidal forms)**.** *Let k, l, n be integers with $k \geq 12$, $l, n \geq 0$ and $g \in M_l$. Suppose that either:*

(a) *g is cuspidal, or*

(b) *g is not cuspidal and $l < k - 3$.*

Then there exists an orthonormal basis $\{f_1, ..., f_d\}$ of $S_k(\Gamma)$ such that

$$\left\| \llbracket f_i, g \rrbracket_n^{(k,l)} \right\|^2_{k+l+2n} f_i(z) = \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} \llbracket f_i, g \rrbracket_n^{(k,l)}(w) \overline{\left\llbracket \frac{1}{(\cdot - \overline{z})^k}, g \right\rrbracket_n^{(k,l)}(w)} \; Im(w)^{k+l+2n} d\mu(w)$$

and

$$\left\| \llbracket f_i, g \rrbracket_n^{(k,l)} \right\|^2_{k+l+2n} f_i(z) = \frac{i^k 2^{k-3}(k-1)}{\pi} \int_{F} \llbracket f_i, g \rrbracket_n^{(k,l)}(w) \overline{\llbracket h_k(\cdot, -\overline{z}), g \rrbracket_n^{(k,l)}(w)} \; Im(w)^{k+l+2n} d\mu(w)$$

for any $i \in \{1, ..., d\}$.

Proof of Proposition 5. Since $T^*_{g,n} T_{g,n} : S_k(\Gamma) \to S_k(\Gamma)$ is self-adjoint, there exists an orthonormal basis $\{f_1, ..., f_d\}$ of $S_k(\Gamma)$ where each f_i is an eigenvector of $T^*_{g,n} T_{g,n}$. Write

$$T^*_{g,n} T_{g,n}(f_i) = \lambda_i f_i, \text{ for } i \in \{1, ..., d\}.$$

Then

$$\langle \lambda_i f_i, f_i \rangle_k = \langle T^*_{g,n} T_{g,n}(f_i), f_i \rangle = \langle T_{g,n}(f_i), T_{g,n}(f_i) \rangle_{k+l+2n} \; ,$$

thus

$$\lambda_i = \frac{\langle T_{g,n}(f_i), T_{g,n}(f_i) \rangle_{k+l+2n}}{\langle f_i, f_i \rangle_k} = \left\| \llbracket f_i, g \rrbracket_n^{(k,l)} \right\|^2_{k+l+2n} \; .$$

Hence

$$T^*_{g,n}(\llbracket f_i, g \rrbracket_n^{(k,l)}) = T^*_{g,n} T_{g,n}(f_i) = \left\| \llbracket f_i, g \rrbracket_n^{(k,l)} \right\|^2_{k+l+2n} f_i \; .$$

But by Theorems 6 and 7 we have

$$\begin{aligned} T^*_{g,n}(\llbracket f_i, g \rrbracket_n^{(k,l)}) &= \frac{i^k 2^{k-2}(k-1)}{\pi} \int_{\mathbb{H}} \llbracket f_i, g \rrbracket_n^{(k,l)}(w) \overline{\left\llbracket \frac{1}{(\cdot - \overline{z})^k}, g \right\rrbracket_n^{(k,l)}(w)} \; \mathrm{Im}(w)^{k+l+2n} d\mu(w) \\ &= \frac{i^k 2^{k-3}(k-1)}{\pi} \int_{F} \llbracket f_i, g \rrbracket_n^{(k,l)}(w) \overline{\llbracket h_k(\cdot, -\overline{z}), g \rrbracket_n^{(k,l)}(w)} \; \mathrm{Im}(w)^{k+l+2n} d\mu(w) \end{aligned}$$

This completes the proof of Proposition 5. $\square$

Example: Choose $k = 12$, $l = 4$ and $g = E_4$. Since $\left\{ \frac{\Delta}{\|\Delta\|_{12}} \right\}$ is the only orthonormal basis of $S_{12}(\Gamma)$, we have for all $n \geq 0$ that

$$\left(\frac{\left\| \llbracket \Delta, E_4 \rrbracket_n^{(12,4)} \right\|_{16+2n}}{\|\Delta\|_{12}} \right)^2 \Delta(z) = \frac{2^{10} 9}{\pi} \int_{\mathbb{H}} \llbracket \Delta, E_4 \rrbracket_n^{(12,4)}(w) \overline{\left\llbracket \frac{1}{(\cdot - \overline{z})^{12}}, E_4 \right\rrbracket_n^{(12,4)}(w)} \; \mathrm{Im}(w)^{16+2n} d\mu(w) \; .$$

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