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PROBLEMAS SOBRE FLUJOS POR CURVATURA EXTRÍNSECA NO LINEALES

By

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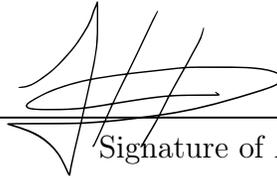
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Resumen y conclusiones

Un flujo geométrico consiste en encontrar soluciones de ecuaciones parabólicas en derivadas parciales que involucran cantidades geométricas de dos o más variedades dadas.

En el caso de los flujos geométricos extrínsecos, las soluciones corresponden a una familia de un parámetro de inmersiones cuyas deformaciones dependen de las curvaturas de la hipersuperficie en una variedad riemanniana ambiente.

El flujo geométrico extrínseco más estudiado en la literatura es el flujo por la curvatura media (FCM), ya que es el flujo gradiente del funcional de área.

En la Introducción de esta tesis presentamos una breve descripción del FCM actuando sobre hipersuperficies en $\mathbb{R}^{n+1}$. En particular, queremos mencionar el trabajo pionero de Huisken en [Hui2] que demostró que cualquier hipersuperficie cerrada y convexa en $\mathbb{R}^{n+1}$, en el sentido de que las curvaturas principales son no negativas, se encoge al evolucionar por FCM hasta un punto en tiempo finito. Este fenómeno se suele denominar como desarrollo de una singularidad bajo el FCM, y es una tarea importante entender por qué aparecen singularidades, y cómo tratar con ellas luego de rescalar la hipersuperficie cerca de la singularidad.

Por otra parte, es natural preguntarse qué ocurre si consideramos otras funciones de curvatura en lugar de la curvatura media en el contexto de flujos geométricos extrínsecos. En este espíritu, los trabajos de B. Andrews y sus colaboradores en [And2], [And1], [AMZ] y [ALM] desarrollaron una potente teoría, similar a la dada para el FCM, para hipersuperficies estrictamente convexas en $\mathbb{R}^{n+1}$, donde la función de curvatura es convexa o cóncava. Por función de curvatura entendemos una función homogénea suave y simétrica cuyo dominio es un cono abierto de $\mathbb{R}^n$.

En esta tesis nos interesamos en soluciones eternas, soluciones que están definidas para todo tiempo, de flujos por curvatura extrínseca completamente no lineales tal que la evolución está dada por traslaciones en una dirección unitaria fija. Además, nos referimos a estas soluciones por solitones de traslación de la función de curvatura que estemos estudiando.

Los solitones de traslación pueden verse como hipersuperficies en $\mathbb{R}^{n+1}$ que satisfacen una ecuación diferencial parcial de la forma $\gamma(\lambda) = \langle \nu, v \rangle$, donde $\gamma(\lambda)$ es la función de curvatura evaluada en las curvaturas príncipales de la hipersuperficie, ν es el vector normal unitario que apunta hacia afuera de la hipersuperficie (mirar Remark 5.0.1), y $v \in \mathbb{S}^n$ (normalmente $v = e_{n+1}$) es la dirección de la traslación del flujo. Es importante destacar que esta ecuación es localmente uniforme elíptica cuando las curvaturas principales pertenecen al cono $\Gamma := \left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \frac{\partial \gamma}{\partial \lambda_i} > 0 \right\}$. Esto nos permite estudiar los trasladores como en el contexto de la geometría diferencial clásica.

Por otro lado, cuando la función de curvatura es la curvatura media, los trasladores son un modelo para las singularidades de tipo II del FCM. Esto significa que el supremo de la norma de la segunda forma fundamental estalla en una tasa mayor que $O\left(\frac{1}{\sqrt{T-t}} \right)$, donde T es el tiempo máximo de existencia del flujo, y luego de escalar la hipersuperficie de manera adecuada, la evolución de la hipersuperficie converge a un H-traslador del espacio ambiente. Además, los trasladores del FCM son hipersuperficies mínimas en $(\mathbb{R}^{n+1}, e^{\langle x, v \rangle} dx^2)$. Este es un hecho notable para el estudio de estas soluciones ya que la teoría local de hipersuperficies mínimas se puede aplicar para construirlas y caracterizarlas.

Desgraciadamente, cuando la función de curvatura no es lineal, sólo se sabe que las singularidades de tipo II pueden modelarse mediante trasladores si además la función de curvatura es convexa como función definida en su dominio. Además, no tenemos esperanzas de que estas hipersuperficies sean mínimas en un espacio euclidiano conforme como en el caso de la curvatura media. Por ello, el estudio de los trasladores para funciones de curvatura no lineales es más complicado, y necesita que se desarrollen otro tipos de técnicas para el desarrollo de esta teoría.

Los resultados de esta tesis están en el espíritu de explotar el hecho de que la ecuación $\gamma(\lambda) = \langle \nu, v \rangle$ es localmente uniformemente elíptica cuando $\lambda \in \Gamma$. En particular, pudimos desarrollar propiedades geométricas para los trasladores contenidas en el capítulo 5.

Una de las propiedades geométricas que obtuvimos fue un principio de tangencia, y como corolarios, también obtuvimos un resultado de no existencia, y un teorema de unicidad cuando el solitón de traslación es un grafo estrictamente convexo definido sobre una bola (mirar Teorema 5.0.5). Este último resultado se obtuvo mediante el método de los planos móviles de Alexandrov aplicado para esta ecuación.

Por otra parte, el resultado principal de esta tesis es una estimación de convexidad en el espíritu de [SS], donde los autores mostraron que un H-traslador 2-convexo con $H > 0$ en $\mathbb{R}^n$ con $n \geq 3$, es convexo. Hasta donde sabemos, sigue siendo un problema abierto si un γ-traslador que es un gráfico en $\mathbb{R}^{n+1}$ tal que $\lambda \in \Gamma$ y $\gamma : \Gamma \to \mathbb{R}$ es una función de curvatura cóncava es convexo o no. Afortunadamente, bajo las hipótesis del Teorema de estimación de convexidad 2.2.12, pudimos demostrar que para una función de curvatura γ estrictamente concava, los γ-solitones de traslación que son uniformente 2-convexos cuyas curvaturas principales satisfacen $\alpha H \leq (\delta + 1)\gamma$ para constantes α y δ positivas, se tiene que el mínimo de las curvaturas principales es asintóticamente cero al infintio del soliton de traslación.

Además, para la familia de las funciones de curvatura $Q_k = \dfrac{S_{k+1}}{S_k}$, donde S_k denota el polinomio simétrico elemental de grado k en n-variables, mostramos en el Capítulo 3 estimaciones de gradiente y de segundo orden en el espíritu del trabajo de Ecker y Huisken en [EH1]. La principal contribución de este capítulo es un resultado de tipo Liouville Teorema 3.0.3 para Q_k-trasladores que son planos en el infinito.

Finalmente, también construimos trasladores rotacionalmente simétricos para la función de curvatura $\sqrt[n]{S_n}$ y Q_{n-1} en $\mathbb{R}^{n+1}$. Estas soluciones son de tipo "bowl" ya que son gráficos estrictamente convexos definidos en una bola o en todo $\mathbb{R}^n$.

Merece la pena mencionar un trabajo reciente de [Ren], en el que el autor construye soluciones de tipo "bowl" para una clase general de funciones de curvatura que son α-homogéneas con $\alpha \geq \frac{1}{2}$. Además, caracterizó cuándo la solución de tipo "bowl" estará definida en una bola o

en todo el hiperplano en términos de la función de curvatura.

En conclusión, la teoría relacionada con los solitones de traslación para funciones de curvatura no lineales está todavía en sus primeras etapas. Queremos terminar este resumen con algunos problemas abiertos relacionados con esta teoría que causarán un gran impacto como en el caso de los solitones de traslación de la FCM:

1. ¿Existen ejemplos de solitones de traslación no convexos tales que $\lambda \in \Gamma$?

2. ¿Son las soluciones de tipo "bowl" los únicos solitones de traslación estrictamente convexos que son enteros o definidos en una bola?

3. ¿Son los solitones de traslación modelos de singularidades de tipo II de los flujos de curvatura cuando la función de curvatura es cóncava?

Abstract and conclusions

A geometric flow consists in finding solutions to parabolic partial differential equations that involve geometric quantities from given manifolds.

In the case of extrinsic geometric flows, solutions correspond to a one-parameter family of immersions whose deformations depend on the curvatures of the hypersurface in a target riemannian manifold.

The most studied extrinsic geometric flow in the literature is the mean curvature flow (MCF), since it is the gradient flow of the area functional.

In the Introduction of this thesis we present a brief description of the MCF acting on hypersurfaces in $\mathbb{R}^{n+1}$. Nevertheless, we want to mention the pioneer work of Huisken in [Hui2] which showed that any closed and convex hypersurface in $\mathbb{R}^{n+1}$, in the sense of the principal curvatures are non-negative, will shrink to a point in finite time. This phenomena is usually called the development of a singularity under the MCF, and it is an important task to understand how the singularities appear and behave after rescaling the hypersurface.

On the other hand, it is natural to ask what happens if we considerate others curvature functions rather than the mean curvature in the extrinsic geometric flow setting. In this spirit, the works of Andrews and his collaborators in [And2], [And1], [AMZ] and [ALM] developed a powerful theory, similar to the one given for the MCF, for strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ where the curvature function is convex or concave. By a curvature function we mean a smooth symmetric homogeneous function whose domain is an open cone of $\mathbb{R}^n$.

In this thesis we study eternal solutions, solutions that exist for all time, for fully nonlinear curvature flows such that the evolution is given by translations by a fixed unitary direction, and we refer to these solutions by translating solitons of the particular curvature function that we are studying.

Additionally, translating solitons can be seen as hypersurfaces in $\mathbb{R}^{n+1}$ which satisfy a partial differential equation of the form $\gamma(\lambda) = \langle \nu, v \rangle$, where $\gamma(\lambda)$ is the curvature function

evaluated in the princial curvatures, ν is the outward unit normal vector (see Remark 5.0.1), and $v \in \mathbb{S}^n$ (normally $v = e_{n+1}$) is the direction of the translation of the flow. It is important to remark that this equation is locally uniformly elliptic when the principal curvatures belong to the cone $\Gamma := \left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \ \dfrac{\partial \gamma}{\partial \lambda_i} > 0 \right\}$, this permits us to study translators as in the classical differential geometry context.

On the other hand, when the curvature function is the mean curvature of the hypersurface, translating solitons are models of type II singularities of the MCF. This means that the second fundamental form of the evolution under the MCF blows up in a larger rate than $O\left(\dfrac{1}{\sqrt{T - t}}\right)$, where T is the maximal time of existence of the flow. Moreover, translating solitons of the MCF are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x, v\rangle} dx^2)$. This is a remarkable fact for the study of these solutions since the local theory of minimal hypersurfaces can be applied to construct and characterize them.

Unfortunately, when the curvature function is not linear, it is only known that type II singularities can be modeled by translating solitons if in addition the curvature function is convex. Furthermore, there is little hope for these hypersurfaces to be minimal in a conformal euclidean space as in the case of the mean curvature. That is why the study of translating solitons for non-linear curvature function is more difficult, and needs other type of techniques to be developed in the theory.

The results of this thesis are in the spirit of exploiting the fact that equation $\gamma(\lambda) = \langle \nu, v \rangle$ is locally uniformly elliptic when $\lambda \in \Gamma$. In particular, we will be able to develop geometric properties for translating solitons contained in Chapter 5.

One of the geometric properties that we will obtain is a tangency principle, and as corollaries, we also obtained a non-existence result, and a uniqueness theorem when the translating soliton is a strictly convex graph defined on a ball. This last result was obtained by the method of moving planes of Alexandrov applied to this equation.

Moreover, the main result of this thesis is a convexity estimate in the spirit of [SS], where

the authors showed that a 2-convex mean convex translating soliton of the MCF in $\mathbb{R}^{n+1}$ with $n \geq 3$, is convex. To the best of our knowledge, it remains an open problem if a translating soliton which is a graph in $\mathbb{R}^{n+1}$ such that $\lambda \in \Gamma$ and $\gamma : \Gamma \to \mathbb{R}$ is a concave curvature function is convex or not. Fortunately, under the hypotheses of the convexity estimate Theorem 2.2.12, we were able to show that these translating solitons are asymptotically convex, in the sense that the minimum of the principal curvatures goes to zero for a sequence of points in the hypersurface that goes to infinity.

In addition, for the family of the curvature functions $Q_k = \dfrac{S_{k+1}}{S_k}$, where S_k denotes the elementary symmetric polynomial of degree k in n-variable, we showed in Chapter 3 gradient and second order estimates in the spirit of the work of Ecker and Huisken in [EH1]. The main contribution of this chapter is a Liouville's type result Theorem 3.0.3 for translating solitons of the Q_k curvature flow which are flat at infinity.

Finally, we also constructed rotationally symmetric translating solitons for the curvature function $\sqrt[n]{S_n}$ and Q_{n-1} in $\mathbb{R}^{n+1}$. These solutions are of "bowl"-type since they are strictly convex graph defined in a ball or in the whole $\mathbb{R}^n$.

It is worth mentioning a recent paper [Ren], where the author constructs "bowl"-type solutions for a general class of curvature functions which are α-homogeneous with $\alpha \geq \frac{1}{2}$. Furthermore, he characterized when the "bowl"-type solution will be defined in a ball or in the whole hyperplane in terms of the curvature function.

In conclusion, the theory related to translation solitons for nonlinear curvature functions is still in its early stages. We want to finish this abstract with some open problems related to this theory which will cause a big impact as in the case of translating solitons of the MCF:

1. Are there examples of non convex translating solitons such that $\lambda \in \Gamma$?

2. Are the "bowl"-type solutions the only strictly convex translating solitons which are entire or defined in a ball?

3. Are translating solitons models of type II singularities of curvature flows when the curvature function is concave?

Chapter 1

Introducción

Un flujo geométrico es una ecuación diferencial parcial (EDP) dependiente del tiempo que involucra cantidades geométricas de una variedad dada. En las últimas décadas del siglo pasado, este tema cobró mucho interés en la comunidad matemática a partir de sus aplicaciones en Geometría Diferencial, Topología, Física y, en un contexto más aplicado, al Procesamiento de Imágenes.

Desde la perspectiva geométrica, se pueden distinguir dos tipos de flujos geométricos: intrínsecos y extrínsecos. Se entiende por un flujo geométrico intrínseco como una EDP que considera la evolución de la cantidades geométricas intrínsecas de una variedad riemanniana.

Un ejemplo importante de este tipo de flujos consiste en modificar una métrica riemanniana g_0 a otra con algún tensor de curvatura constante. Más concretamente, encontrar una familia de métricas de 1-parámetro $\{g(t)\}_{0 \leq t < T}$ para una variedad M tal que

$$
\begin{cases}
\partial_t g = F(g) & , \ t \in (0, T) \\
g(0) = g_0.
\end{cases}
,
$$

donde F es un tensor que depende de la métrica $g(t)$ y sus derivadas. Por último, se estudia

la métrica límite $g(T)$, donde $T \in (0, \infty]$ es el tiempo máximo de existencia del flujo, con la expectativa de que satisfaga la condición de curvatura constante.

En este contexto, el flujo intrínseco más notable y estudiado es $F(g) = -2\text{Ric}(g)$, donde $\text{Ric}(g)$ denota el tensor de curvatura de Ricci de $g(t)$, y se conoce como el flujo de Ricci. El flujo de Ricci fue introducido por R. Hamilton en la década de 1980 como herramienta para demostrar la conjetura de geometrización de Thurston para 3-variedades cerradas. Esta conjetura establece que cualquier 3-variedad cerrada puede descomponerse canónicamente en piezas, donde cada pieza posee una de ocho tipos de estructuras geométricas diferentes. En 2003, gracias al trabajo de G. Perelman, se demostró finalmente esta conjetura y, como consecuencia, la conjetura de Poincaré, que era un problema abierto desde 1904. Remitimos a [CZ] para un excelente libro autocontenido sobre este tema.

Figure 1.1: Evolución de la superficie de una mancuerna bajo el flujo de Ricci. Imagen cortesía de Ignacio Mcmanus.

Por otro lado, un flujo geométrico extrínseco corresponde a una familia de inmersiones de un parámetro cuyas deformaciones dependen de las curvaturas extrínsecas de una variedad en una variedad riemanniana ambiente.

Más concretamente, en el caso de codimensión uno, si $F_0 : M^n \to N^{n+1}$ es una inmersión isométrica desde una variedad a una variedad riemanniana (N, g), uno quiere encontrar una

familia de 1-parámetro de inmersiones $F : M \times (0, T) \to N$ tal que

$$
\begin{cases}
\partial_t F = -\gamma(\lambda)\nu, \ (x,t) \in M \times (0, T), \\
F(x,0) = F_0(x),
\end{cases}
\tag{1.1}
$$

donde $\lambda = (\lambda_1, \ldots, \lambda_n)$ y ν son el vector de curvaturas principales y el vector normal unitario[1] de $M_t = F(M, t)$ en N, respectivamente. En esta tesis nos referiremos a la ecuación 1.1 como un γ-flujo para abreviar.

El γ-flujo más estudiado en la literatura es el flujo de curvatura media, donde

$$
\gamma = H = \lambda_1 + \ldots + \lambda_n.
$$

Además, el H-flujo también aparece como el flujo de gradiente negativo del funcional de área de M en N, y en consecuencia, se puede utilizar para deformar hipersuperficies en hipersuperficies mínimas en diferentes espacios ambientes.

A continuación, queremos dar una breve descripción del análisis de las singularidades relacionadas con el H-flujo en $\mathbb{R}^{n+1}$ que están relacionadas con los resultados obtenidos en esta tesis.

En primer lugar, para contextualizar el análisis de las singularidades que aparecen bajo la evolución a través del H-flujo, queremos mencionar un resultado pionero de G. Huisken en [Hui2], que prueba que cualquier hipersuperficie cerrada y convexa[2] colapsará a un punto en tiempo finito bajo la evolucion del H-flujo.

---

[1]La elección de la orientación de la hipersuperficie es tal que las curvaturas principales están contenidas en un cono simétrico abierto que contiene el cono positivo $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$.

[2]Las curvaturas principales de la hipersuperficie son no negativas.

Además, este fenómeno se caracterizó por la explosión de la cantidad

$$\sup_{M_t} |A|,$$

donde $|A| = \sqrt{\lambda_1^2 + \ldots + \lambda_n^2}$, a medida que $t \to T$, donde T es el tiempo máximo de existencia del flujo. Para ejemplificar esto, la evolución de una esfera de radio R_0 bajo el H-flujo en $\mathbb{R}^{n+1}$ viene dada por

$$F(p,t) = R(t)p,$$

donde $R(t) = \sqrt{R_0^2 - 2nt}$ y $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Observamos que el tiempo máximo de existencia viene dado por $T = \dfrac{R_0^2}{2n}$, y por tanto, se cumple que

$$|A|(t) = \frac{\sqrt{n}}{R(t)} = \frac{1}{\sqrt{2}\sqrt{T-t}}.$$

De hecho, las singularidades bajo el H-flujo en $\mathbb{R}^{n+1}$ se clasifican en dos tipos relacionados con la tasa de explosión de $\sup_{M_t} |A|$:

a) Tipo I: Existe una constante $C > 0$ tal que $\sup_{M_t} |A| \leq \dfrac{C}{\sqrt{T-t}}$.

b) Tipo II: No es del tipo I, es decir: $\lim_{t \to T} \sup_{M_t} |A|\sqrt{T-t} = \infty$.

La clasificación de estas singularidades es un problema abierto en toda su generalidad. Sin embargo, cuando los datos iniciales son cerrados y satisfacen $H > 0$, los autores en [HS1] demostraron que bajo un reescalamiento parabólico[3] adecuado, la singularidad puede ser

---

[3]Si $(x_n, t_n) \in M \times (0, T)$ es tal que $a_n = |A|(x_n, t_n) \to \infty$, entonces un rescalamiento parabolico es de la forma

$$M_n = a_n \left(M_{\frac{t_n}{a_n^2}} - x_n \right).$$
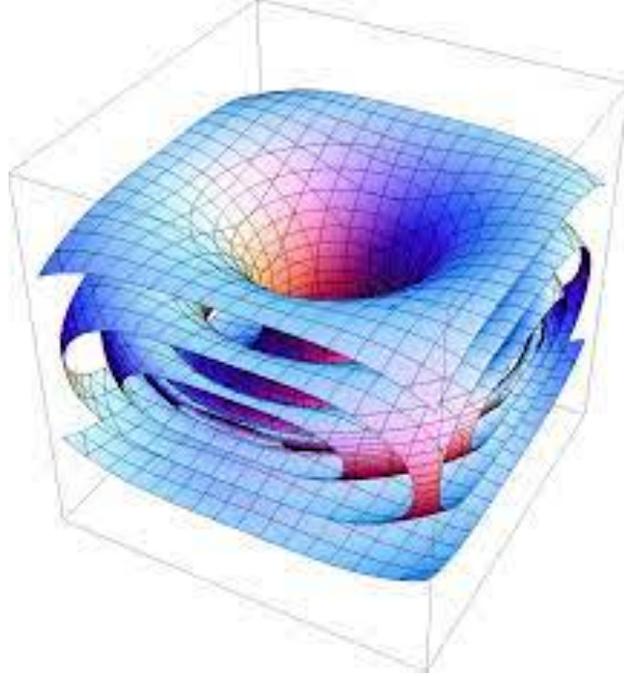
Figure 1.2: La evolución de una superficie bajo el flujo H en $\mathbb{R}^3$. Imagen por cortesía de Francisco Martín.

modelada por una solución autosimilar[4] por el H-flujo. En efecto, estas soluciones son:

a) Auto-contráctiles: Aparecen bajo un escalamiento parabólico alrededor de una singularidad de tipo I. Después de pasar al límite estas convergen a una solución antigua[5] autosimilar que evoluciona sólo por dilatación, es decir: $F(x,t) = \sqrt{-t}F_0(x)$ con $t \in (-\infty, 0]$.

Además, cada corte de tiempo, la solución M_t es convexa y satisface la ecuación $H = -\langle p, \nu \rangle$ para todo $p \in M_t$.

b) Solitones de traslación: Aparecen bajo un escalamiento parabólico alrededor de una singularidad de tipo II. Después de pasar al límite estas convergen a una solución

---

[4]Una solución al flujo H de la forma $F(x,t) = r(t)O(t)F_0(x) + V(t)$, donde $r(t) \geq 0$, $V(t) \in \mathbb{R}^{n+1}$ y $O(t) \in \mathcal{O}(n+1)$ es una matriz ortogonal. Visualmente esto significa que la hipersuperficie inicial F_0 evoluciona bajo una dilatación $r(t)$, una isometría lineal $O(t)$, y una traslación $V(t)$ en $\mathbb{R}^{n+1}$.

[5]Solución de la ecuación (1.1) que está definida para $(-\infty, a]$ para algún $a \in \mathbb{R}$.

eterna[6] autosimilar que evoluciona por traslación en una dirección unitaria, es decir: $F(x,t) = F_0(x) + vt$ para $t \in \mathbb{R}$ y algún vector unitario $v \in \mathbb{S}^n$.

Además, cada corte de tiempo M_t puede escribirse como $\mathbb{R}^{n-k} \times S^k$ donde S^k es una hipersuperficie estrictamente convexa, y la ecuación $H = \langle \nu, v \rangle$ se cumple en cada M_t.

La herramienta principal utilizada en la demostración de esta clasificación fue la fórmula de monotonicidad de Huisken (véase [Hui1]), relacionada con la evolución de un funcional con un núcleo de calor,

$$\frac{d}{dt} \int\limits_M \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}} dx = -\int\limits_M \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}} \left| H + \frac{\langle x-p, \nu \rangle}{2|t_0-t|} \right|^2 dx \le 0.$$

Por otro lado, queremos mencionar que existe un tipo adicional de solución autosimilar del H-flujo llamado auto-expansiva. En [EH1], los autores mostraron que la evolución de un gráfico entero Lipschitz existe para todo $t > 0$ y, después de un rescalamiento adecuado, la evolución converge a una solución auto-expansiva como $t \to \infty$, es decir: el flujo límite converge a $F(x,t) = \sqrt{t} F_0(x)$ y cada trozo de tiempo M_t satisface $H = \langle p, \nu \rangle$. Para un estudio reciente de solcuiones auto-expansivas nos referimos a [Smo].

Finalmente, la clasificación de las soluciones autosimilares del H-flujo es una tarea importante para entender el comportamiento de las hipersuperficies bajo la evolución del H-flujo. Actualmente, este problema se ha completado para algunos casos particulares. Por ejemplo, en el caso de soluciones auto-contractantes convexas las posibles soluciones, módulo isometrías y reparametrizaciones, son esferas, cilindros o hipersuperficies de Abresch-Langer[7].

Por otra parte, recientemente se ha completado la clasificación de los solitones de traslación que son grafos en $\mathbb{R}^3$, véase [HIMW2] para más detalles. Las posibles soluciones son (véase la figura 1.3 a continuación). :

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[6] Solución de la ecuación (2.1) que está definida para todo $t \in (-\infty, \infty)$.
[7] Cilindros de la forma $C_{n,m} \times \mathbb{R}^{n-1}$ donde $C_{n,m}$ es una curva auto-contráctil del flujo de acortamiento de curvas en $\mathbb{R}^2$ (referimos al lector revisar [AL] para más detalles).

1. La familia de los "grim reapers" inclinados dados por el gráfico de la función $g_\theta(x, y) = \csc^2(\theta) \ln(\cos(y\cos(\theta))) + x\tan(\theta)$ definidos en la banda $\left\{(x, y) \in \mathbb{R}^2 : |y| < \dfrac{\pi}{2\cos(\theta)}\right\}$ donde $\theta \in [0, \pi/2)$. Notar que para $\theta = 0$, $g_0(x, y)$ es conocido como el cilindro grim reaper.

2. El "bowl" soliton que es el único (modulo traslaciones) gráfico entero que es rotacionalmente simétrico que se comportan como un paraboloide en infinito. Más precisamente, si $u : \mathbb{R}^2 \to \mathbb{R}$ denota el gráfico que define el "bowl" soliton, entonces $u(r) = \dfrac{r^2}{2} - \ln(r) + O(r^{-2})$ cuando $r := \sqrt{x^2 + y^2} \to \infty$.

3. La familia de delta-wings, dadas por el gráfico de una función $u_b : \{(x, y) \in \mathbb{R}^2 : |y| < b\} \to \mathbb{R}$ con $b > \dfrac{\pi}{2}$. Estos solitones por traslación satisfacen $u_b(x, y) = u_b(-x, y) = u_b(x, -y)$ y alcanza su máximo en el origen.

Es importante mencionar que en la clasificación de estas soluciones autosimilares del H-flujo se utilizó la teoría de las superficies mínimas. Más concretamente, en [Ilm], T. Ilmanen demostró que estas soluciones también pueden verse como hipersuperficies mínimas en $(\mathbb{R}^{n+1}, g)$, donde g corresponde a $e^{-|p|^2}dx^2$ para los auto-contractantes, y $e^{-\langle p,v\rangle}dx^2$ para los solitones de traslación. También observamos que estas métricas no son completas, lo que significa que la teoría global de las superficies mínimas no se aplica a estas soluciones auto-similares, pero sí la teoría local.
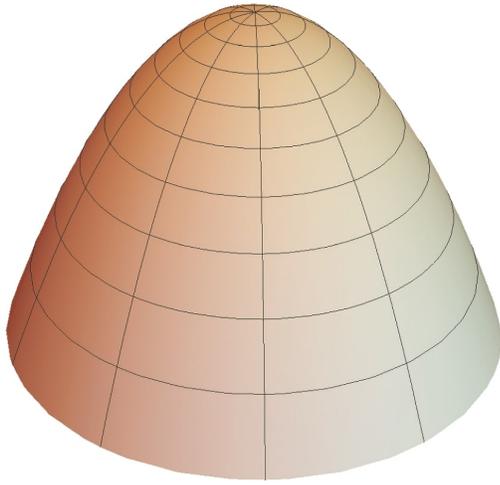
(a) Bowl soliton en $\mathbb{R}^3$.
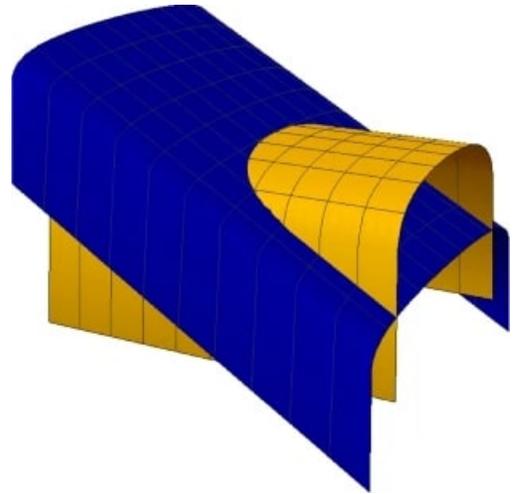
(b) Grim reaper y un grim reaper incli-
nado con ángulo $\theta = \dfrac{\pi}{6}$ in $\mathbb{R}^3$.
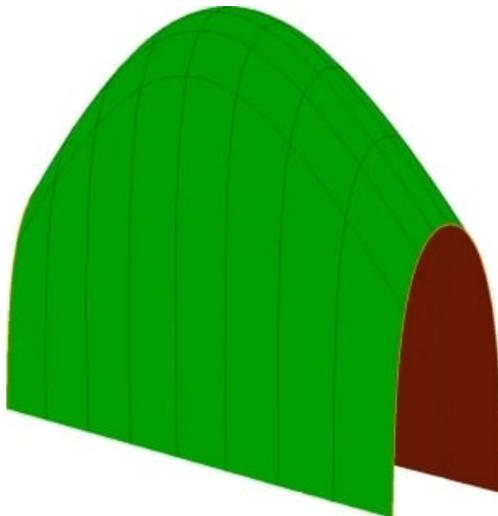
(c) Delta wing en $\mathbb{R}^3$.

Figure 1.3: Estas superficies están tomadas como soluciones de $H = \langle \nu, -e_3 \rangle$. Imágenes por cortesía de Francisco Martín.

1.1 Objetivos

Los objetivos generales propuestos en esta tesis son los siguientes:

- Estudiar teoremas de clasificación para solitones por traslación de γ-flujos en espacios Euclideos donde γ es una función no lineal.

- Estudiar comportamiento asintótico y principios de comparación en infinito para dichos solitones.

- Caracterización de soluciones tipo "bowl" para flujos por curvatura no lineales.

Además, los objetivos específicos propuestos en esta tesis son los siguientes:

1. Encontrar estimaciones locales interiores hasta segundo orden para solitones por traslación del flujo Q_k. Ver siguiente sección para una definición de Q_k y S_k.

2. Construir ejemplos de solitones por traslación para los flujos dados por las funciones de curvatura $\sqrt[n]{S_n}$ y Q_{n-1} en espacios euclídeos.

3. Probar un principio de tangencia para solitones de traslación de γ-flujos en $\mathbb{R}^{n+1}$.

4. Probar que los únicos γ-solitones por traslación que son grafos estrictamente convexos definidos en una bola en $\mathbb{R}^n$ son rotacionalmente simétricos asintóticos a cilindros, y por tanto únicos gracias al teorema de unicidad expuesto en [Ren].

5. Encontrar y probar una estimación de convexidad para γ-trasladores siendo γ es una función de curvatura cóncava, i.e: encontrar una cota inferior puntual para el mínimo de las curvaturas principales en términos de γ y H.

1.2 Metodología y Resultados

Esta tesis trata de problemas relacionados con flujos geométricos extrínsecos (1.1) donde la función de curvatura $\gamma : \Gamma \to \mathbb{R}$ es una función suave, simétrica y 1-homogénea definida en un cono abierto $\Gamma \subset \mathbb{R}^n$ que contiene al cono positivo $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0,\ i = 1, \ldots, n\}$.

Además, es importante mencionar que la metodología utilizada en esta tesis es la usual utilizada en investigación de matemáticas puras, o sea, revisión en la literatura relacionada con los objetivos propuestos, luego dar una formulación de los problemas observados, para finalmente dar un listado de teoremas y demostraciones que resuelven los problemas propuestos en la tesis.

Por otra parte, elegimos estudiar este tipo de funciones de curvatura porque son un análogo geométrico natural de la teoría de las EDP. De hecho, queremos remarcar la teoría desarrollada por Nirenberg, Caffarelli y Spruck en [CNS1] y [CNS2] para prescribir ecuaciones de curvatura completamente no lineales[8] en dominios acotados en $\mathbb{R}^n$ con condiciones de contorno de Dirichlet. Esta teoría sirvió de inspiración para el trabajo desarrollado en esta tesis.

Una clase importante de funciones de curvatura 1-homogéneas consideradas en esta tesis son de tipo cociente hessiano[9], es decir

$$\sqrt[k]{S_k} \text{ and } Q_k = \frac{S_{k+1}}{S_k}, \text{ o de manera general, } Q_{k,l} = \sqrt[k-l]{\frac{S_k}{S_l}}. \tag{1.2}$$

Recordemos que estas funciones se evalúan en las curvaturas principales de una hipersuper-

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[8]Esto significa que en coordenadas $(x, u(x))$, la ecuación de curvatura prescrita es una EDP elíptica completamente no lineal.

[9]La razón del nombre proviene de la teoría de prescripción de los polinomios simétricos elementales evaluados en los valores propios de $\text{Hess}(u)$. También se conocen como de tipo Monge-Ampère ya que los polinomios simétricos elementales aparecen en expresiones de la forma $\det(\text{Hess}(u) - \lambda \text{Id})$.

ficie dada, y por definición, estos son los valores propios del operador de forma, es decir

$$\mathcal{W}_p : T_pM \to T_pM$$

$$\vec{v} \to -\nabla_{\vec{v}}\nu,$$

donde ∇ y ν son la conexión Levi-Civita de $\mathbb{R}^{n+1}$ y el vector normal unitario interior de M en $\mathbb{R}^{n+1}$. En coordenadas locales, el operador de forma puede escribirse por $h_j^i = g^{ik}h_{kj}$, donde g^{ij} denota los coeficientes de la inversa del tensor métrico g_{ij} de la hipersuperficie, y h_{ij} denota los coeficientes de la segunda forma fundamental de la hipersuperficie, respectivamente. Además, los coeficientes de la segunda forma fundamental corresponden a las componentes tangentes de la derivada covariante del campo normal unitario (módulo de orientación), es decir: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ donde $\{e_i\}$ es una base ortonormal del espacio tangente en algún punto de la hipersuperficie.

Por otra parte, el símbolo

$$S_k(\lambda) = \sum_{1 \le i_1 < \ldots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k} \tag{1.3}$$

denota el polinomio simétrico elemental de grado k en n-variables, y utilizamos la convención $S_0 = 1$ y $S_j = 0$ para $j > n$.
Nótese que la curvatura media $H = S_1(\lambda)$, la curvatura escalar $R_g = \dfrac{2}{n(n-1)} S_2(\lambda)$ y la curvatura de Gauss-Kronecker $K = S_n(\lambda)$.

Por último, es importante destacar que estos flujos son parabólicos cuando las curvaturas principales de la evolución M_t pertenecen a los conos de Gärding dados por

$$\Gamma_k := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : S_j(\lambda) > 0. j = 1, \ldots, k\}, \tag{1.4}$$

También mencionamos que para nosotros Γ_k siempre denotará la componente conexa que

contiene el cono positivo Γ_+.

Ahora queremos discutir algunos trabajos y resultados anteriores relacionados con los γ-flujos en la literatura.

En primer lugar, cuando los datos iniciales son compactos y estrictamente convexos, la evolución bajo un γ-flujo presenta un comportamiento similar al H-flujo. De hecho, en [And2] el autor demostró que bajo algunos supuestos naturales sobre γ (ver por ejemplo las Propiedades 1-7 en el capítulo 5) la evolución colapsará a un punto en tiempo finito, y después de un escalamiento parabólico adecuado, la solución converge a una esfera unitaria en $\mathbb{R}^{n+1}$.

Sin embargo, hasta el presente de esta tesis, el análisis de las singularidades relacionadas con la evolución bajo γ-flujos no se entiende tan bien como en el caso del H-flujo. En efecto, uno de los principales problemas es la no linealidad relacionada con la geometría implicada. Por ejemplo, se dispone de la fórmula de monotonicidad de Huisken para describir las singularidades del H-flujo, mientras que no se conoce una fórmula similar para otras funciones de curvatura.

Alentadoramente, hay pocos resultados relacionados con el estudio de las singularidades bajo γ-flujo que describimos brevemente. En [Lyn3] el autor clasificó soluciones antiguas convexas uniformemente parabólicas de γ-flujos que presentan una singularidad de tipo I. En concreto, estas soluciones son esferas o cilindros que se contraen. Un aspecto importante de su resultado es que incluye funciones de curvatura convexas y cóncavas que satisfacen propiedades de concavidad adicionales y admite un teorema de separación del tipo $M = \Sigma^k \times \mathbb{R}^{n-k}$.

Además, si la función de curvatura es convexa (y satisface las Propiedades (1)-(3) y 5 descritas en el capítulo 5), se conocen más resultados relacionados con las singularidades de Tipo II. En [ALM], los autores mostraron que las soluciones eternas que presentan una singularidad de Tipo II se descomponen como $M = \Sigma^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ y evolucionan bajo

traslación por una dirección unitaria fija. Todavía es una problema abierto si este fenómeno se produce para las funciones cóncavas.

Por último, es importante mencionar que para γ-flujos donde γ es una función de curvatura no lineal suceden otro tipo de singularidades relacionadas con la pérdida de convexidad de los datos iniciales.

Para describir este fenómeno mencionamos en primer lugar que si la hipersuperficie inicial satisface que las curvaturas principales pertenecen al cono de simetría

$$\left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \frac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0, \text{ for } i = 1, \ldots, n \right\}^{[10]},$$

entonces bajo el γ-flujo esta propiedad también es satisfecha por la evolución M_t. De hecho, esto puede verse como una aplicación sencilla del principio de máximo fuerte aplicado a la ecuación

$$(\partial_t - \Delta_\gamma)\gamma = |A|_\gamma^2 \gamma,$$

donde $\Delta_\gamma = \frac{\partial \gamma}{\partial h_{ij}} \nabla_i \nabla_j$ y $|A|_\gamma^2 = \frac{\partial \gamma}{\partial h_{ij}} h_{ia} h_{aj}$. Nótese que para este argumento utilizamos que la matriz $\left(\frac{\partial \gamma}{\partial h_{ij}} \right)_{i,j}$ es una matriz localmente uniformemente definida positiva.

Sin embargo, hay ejemplos de funciones de curvatura que no preservan los conos positivos Γ_+ bajo la evolución del flujo.

En [AMZ], los autores construyeron una hipersuperficie convexa en $\mathbb{R}^{n+1}$ tal que la evolución bajo el flujo γ dado por

$$\gamma(\lambda) = \sum_{1 \leq i < j \leq n} \frac{\lambda_i \lambda_j}{\sqrt{\lambda_i^2 + \lambda_j^2}},$$

---

[10]Se trata de un cono abierto cuando γ es una función continua simétrica 1-homogénea. Además, si γ es cóncava, entonces este cono es también convexo. Esto se puede ver por el hecho de que bajo la hipótesis de 1-homogénea, ser cóncava es equivalente a ser subaditivo (es decir: $\gamma(a\lambda + b\mu) \geq a\gamma(\lambda) + b\gamma(\mu)$).

pierde convexidad para $t > 0$ pequeños. Nótese que $\gamma(\lambda)$ es positiva, estrictamente creciente en cada variable, y cóncava en el cono $\{\lambda \in \mathbb{R}^n : \lambda_i + \lambda_j > 0 \,\forall i \neq j\}$.

Además, los autores caracterizaron la pérdida de convexidad bajo la evolución de las funciones de curvatura cóncava si no se satisfacía la hipótesis de ser inversamente cóncavo en la restricción de la frontera de Γ_+. Esto significa que si $\gamma(1,...,1,0) > 0$, entonces la función

$$\tilde{\gamma}(\lambda_1,\ldots,\lambda_{n-1}) = \gamma(\lambda_1^{-1},\ldots,\lambda_{n-1}^{-1},0)^{-1}$$

es cóncava en el cono positivo $(n-1)$-dimensional $\lambda_+^{n-1} = \{\lambda \in \mathbb{R}^{n-1} : \lambda_i > 0\}$.

Notamos que bajo el H-flujo este tipo de singularidades no ocurren. De hecho, en [HS2] los autores demostraron que todos los conos Γ_k se conservan bajo H-flujo, dando en particular que los datos iniciales no perderán su convexidad bajo el flujo. En consecuencia, el estudio de los γ-flujos para funciones de curvatura no lineales puede utilizarse para evolucionar hipersuperficies de forma que sólo algunos conos de convexidad se conserven bajo la evolución.

En esta tesis también nos interesamos en los solitones de traslación del γ-flujo en $\mathbb{R}^{n+1}$. Al igual que mencionamos para el H-flujo, se trata de soluciones autosimilares eternas del γ-flujo que evolucionan bajo traslaciones por el campo vectorial $V : \mathbb{R} \to \mathbb{S}^n$, es decir

$$F(x,t) = F_0(x) + V(t).$$

Además, podemos comprobar que $V(t) = \vec{v}t$ para algún vector constante $\vec{v} \in \mathbb{S}^n$. En efecto, observamos que

$$\partial_t V = \partial_t F(x,t) = -\gamma(\lambda)\nu.$$

Luego, como el operador de forma $\mathcal{W}_p$ es invariante bajo traslaciones, las curvaturas prin-

cipales también son invariantes por traslaciones. En particular, $\partial_t \gamma(\lambda) = 0$, y esto implica que $\partial_t^2 V(t) = 0$, o, equivalentemente, $V(t) = \vec{v}t$ para algún vector constante $v \in \mathbb{S}^n$. Sin embargo, en esta tesis sólo consideraremos solitones de traslación del γ-flujo con vector de traslación $\vec{v} = e_{n+1}$ por simplicidad.

También observamos que los solitones de traslación del flujo γ, o γ-trasladores para abreviar, satisfacen en cada corte de tiempo M_t la ecuación

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle. \tag{1.5}$$

Esto significa que los γ-trasladores pueden ser estudiados tanto por la teoría de las ecuaciones parabólicas o con la teoría de las ecuaciones elípticas.

Los problemas que estudiamos en esta tesis están más relacionados con la perspectiva elíptica de los γ-trasladores. De hecho, los supuestos que asumiremos sobre la función de curvatura $\gamma : \Gamma \to \mathbb{R}$ (ver Sección 5.1) están en el espíritu de los problemas clásicos de geometría diferencial:

- γ se evalúa en el cono $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$.
- Para cada $i = 1, \ldots, n$ y $\forall \lambda \in \Gamma$, $\dfrac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0$.

En coordenadas locales, esta propiedad da que la Ecuación (1.5) es localmente uniformemente elíptica, y además de que todo γ-soliton por traslación conexo y simplemente conexo es un grafo vertical sobre algún conjunto abierto conexo $\Omega \subset \mathbb{R}^n$.

Por otro lado, cuando consideramos funciones de curvatura que tienen una extensión continua en la frontera de $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$, como la familia $\sqrt[k]{S_k}$ soportada en Γ_k, tenemos como soluciones triviales a la ecuación (1.5) hiperplanos verticales y cilindros redondos (el vector normal unitario es ortogonal al hiperplano que contiene e_{n+1}). Nótese

que hay funciones de curvatura que no tienen esta propiedad, como Q_k soportada en Γ_k, dado que las soluciones triviales son cilindros redondos verticales.

Por último, queremos mencionar que la teoría de las EDP elípticas degeneradas es útil para el estudio de las funciones de curvatura que se anulan en la frontera de Γ. Remitimos al lector a ver [CNS1], [CNS2], [Ivo], [SUW] para ver buenos artículos introductorios sobre este tema.

Cabe mencionar que si consideramos el caso parabólico, existen estudios que permiten iniciar el γ-flujo para funciones de curvatura γ que están soportadas en $\{\lambda \in \mathbb{R}^n : \gamma(\lambda) \geq 0\} \cap \overline{\Gamma}_+$. De hecho, esto lo hizo la autora en [Die] para el Q_k-flujo. Ella demostró que el flujo Q_k puede iniciarse para datos iniciales convexos cerrados tales que $Q_k \geq 0$ y las curvaturas principales pertenecen a Γ_{k-1}. Su principal contribución fue el uso de barreras cilíndricas que permiten iniciar el flujo H, dando lugar a que la evolución se vuelva estrictamente convexa para tiempos positivos, permitiendo iniciar el flujo Q_k. En particular, la evolución posee el mismo comportamiento que lo expuesto en [And1].

Sigue siendo un problema abierto si este procedimiento puede aplicarse para una función de curvatura diferente, o si puede extenderse a diferentes conos abiertos en lugar de Γ_{k-1}. Remitimos al lector a [ACGL] para un libro introductorio sobre los flujos γ en diferentes entornos.

1.2.1   Aproximación no variacional para Q_k-trasladores $\mathbb{R}^{n+1}$

En esta subsección vamos a demostrar que cierta aproximación variacional no funciona para los solitones por traslación del Q_k-flujo en $\mathbb{R}^{n+1}$, dando pocas esperanzas para que estos solitones sean punto críticos de algún funcional.

Recordemos que los Q_k-trasladores satisfacen

$$Q_k = \langle \nu, e_{n+1} \rangle \Leftrightarrow S_{k+1} - S_k \langle \nu, e_{n+1} \rangle = 0, \tag{1.6}$$

nótese que el si, y sólo si, ocurre cuando las curvaturas principales pertenecen a Γ_k.

Por tanto, para comprobar si podemos encontrar una funcional tal que la ecuación (1.6) aparezca como una ecuación de Euler-Lagrange, utilizaremos las fórmulas obtenidas en el capítulo 4 de [Ros].

En primer lugar, para una función suave dada $f : \mathbb{R}^{n+1} \to \mathbb{R}$, consideramos el funcional sobre hipersuprefícies compactas $M \subset \mathbb{R}^{n+1}$ dado por

$$\mathcal{F}(M) = \int_M f(S_1, \dots, S_n, \langle X, e_{n+1} \rangle) dA,$$

donde S_i son los polinomios elementales simétricos evaluados en las curvaturas principales de $M \subset \mathbb{R}^{n+1}$ y X es el vector de posición en $\mathbb{R}^{n+1}$.

Entonces, eligiendo una variación normal $\varphi : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ con $\varphi_0 = \mathrm{Id}$ y $\left. \dfrac{d\varphi_t}{dt} \right|_{t=0} = g\nu$, donde $\varphi_t(x) = \varphi(x, t)$ y $g \in \mathcal{C}_c^\infty(M)$, podemos ver que

$$\left. \frac{d}{dt} \mathcal{F}(M_t) \right|_{t=0} = \int_M \left. \frac{d}{dt} f \right|_{t=0} dA + \int_M \left. f \frac{d}{dt} dA \right|_{t=0}, \tag{1.7}$$

donde $M_t = \varphi_t(M)$.

A continuación, calculamos cada término de la ecuación 1.7 por separado. Utilizaremos el teorema 4.1 de [Ros] para calcular el primer término de la ecuación anterior

$$
\begin{aligned}
\left. \frac{d}{dt} f \right|_{t=0} &= \sum_{i=1}^n D_i f \left. \frac{d}{dt} S_i \right|_{t=0} + D_{n+1} f \left. \frac{d}{dt} \langle X, e_{n+1} \rangle \right|_{t=0} \\
&= \sum_{i=1}^n D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1) S_{i+1}) \right) + g D_{n+1} f \langle \nu, e_{n+1} \rangle,
\end{aligned}
$$

donde $L_{i-1}(g) = \mathrm{div}_M (T_{i-1} \nabla g)$ y T_i es la i-ésima transformación de Newton de la segunda

forma fundamental de M en $\mathbb{R}^{n+1}$ definida inductivamente por la relación

$$\begin{cases} T_i(\mathcal{W}_p) = S_i Id - \mathcal{W}_p T_{i-1}(\mathcal{W}_p), \\[2ex] T_0 = Id. \end{cases}$$

Por otro lado, es un resultado bien conocido que la fórmula de variación de la forma de volumen bajo variaciones normales viene dada por

$$\frac{d}{dt} dA \bigg|_{t=0} = -gH dA.$$

En consecuencia, obtenemos la siguiente fórmula para la primera variación de $\mathcal{F}$ dada por

$$\frac{d}{dt} \mathcal{F}(M_t) \bigg|_{t=0} \tag{1.8}$$
$$= \int_M \sum_{i=1}^n D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1) S_{i+1}) \right) + g D_{n+1} f \langle \nu, e_{n+1} \rangle - gH f dA.$$

Como hemos mencionado antes, los H-trasladores son puntos críticos del funcional

$$F_0(M) = \int_M e^{\langle X, e_{n+1} \rangle} dA.$$

Por lo tanto, tiene sentido considerar el siguiente funcional para Q_k-solitones por traslación dada por

$$\mathcal{F}_k(M) = \int_M S_k e^{(k+1)\langle X, e_{n+1} \rangle} dA, \ \text{con } k = 0, \ldots, n.$$

Entonces, tomando como $f(x_1, \ldots, x_n) = x_k e^{(k+1)x_{n+1}}$ en $\mathcal{F}_k(M)$, la primera variación es

fácil de calcular. En efecto, por la fórmula (1.8), resulta

$$\frac{d}{dt}\mathcal{F}_k(M_t)\bigg|_{t=0} \tag{1.9}$$
$$= \int_M -(k+1)g\left(S_{k+1} - S_k\langle\nu, e_{n+1}\rangle\right)e^{(k+1)\langle X, e_{n+1}\rangle} + L_{k-1}(g)e^{(k+1)\langle X, e_{n+1}\rangle}dA.$$

Obsérvese que el término $L_{k-1}(g)$ en (1.9) sólo desaparece cuando $k = 0$, ya que $T_{-1} = 0$ por definición, y esto concuerda con el hecho de que los solitones por traslación del H-flujo son hipersuperficies mínimas en $(\mathbb{R}^{n+1}, e^{\langle X, e_{n+1}\rangle}dx^2)$.

Además, para $k = 1$, $L_0(g) = \operatorname{div}_M(T_0\nabla g) = \Delta g$, donde Δ es el operador de Laplace-Beltrami de M. Por lo tanto, por el teorema de la divergencia, este término en la fórmula (1.9) se puede escribir por

$$\int_M L_0(g)e^{2\langle X, e_{n+1}\rangle}dA = \int_M g\Delta e^{2\langle X, e_{n+1}\rangle}dA$$
$$= \int_M 2g\left(2|e_{n+1}^\top|^2 - H\langle\nu, e_{n+1}\rangle\right)e^{2\langle X, e_{n+1}\rangle}dA.$$

donde $e_{n+1}^\top$ es la proyección ortogonal sobre T_pM en $T_p\mathbb{R}^{n+1}$.

Por último, como podría sospechar el lector, la ecuación (1.6) no es la de Euler-Lagrange para $\mathcal{F}_k$ para $k \geq 1$, ya que en la ecuación de Euler-Lagrange aparecen más términos de curvatura. En consecuencia, no esperamos la existencia de ninguna funcional tal que un Q_k-translador aparezca como una hipersuperficie mínima en un espacio euclidiano ponderado.

1.2.2 Estimaciones para el operador Q_k

Desde la perspectiva de las EDP, las estimaciones de gradiente y de segundo orden son herramientas importantes para construir soluciones y analizar sus propiedades y comportamiento.

Como se mencionó anteriormente, en [EH1] y [EH2] los autores obtienen estimaciones

locales de gradiente y curvatura para construir soluciones no compactas del H-flujo en $\mathbb{R}^{n+1}$.
Estos resultados y técnicas motivaron el desarrollo de una teoría similar para ser aplicada a los γ-flujos.

En este sentido, queremos mencionar el trabajo de [CD], donde los autores construyeron grafos convexos completos no compactos que evolucionan bajo el flujo Q_k en $\mathbb{R}^{n+1}$, véase la ecuación (1.2) para la definición. En su trabajo el uso de la convexidad fue un hecho importante.

En esta tesis nos ocupamos de relajar el supuesto de convexidad utilizado en [CD] para obtener estimaciones locales de gradiente y de segundo orden para el flujo Q_k, y también para los Q_k-trasladores, tales que las curvaturas principales pertenecen al cono Γ_{k+1}.

Más precisamente, si $F(x,t) = (x, u(x,t))$ donde $u : \Omega \times [0,T) \to \mathbb{R}$ entonces la ecuación (1.1) con $\gamma = Q_k$, puede escribirse como

$$u_t = Q_k(Du, D^2u)\sqrt{1 + |Du|^2}, \tag{1.10}$$

y para Q_k-trasladores, si $F(x) = (x, u(x))$ con $u : \Omega \to \mathbb{R}$, la ecuación (1.5) puede escribirse como

$$Q_k(Du, D^2u) = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{1.11}$$

Por lo tanto, para obtener estimaciones de gradiente y de segundo orden sin la hipótesis de convexidad impuesta en los gráficos, tuvimos que entender mejor el término $Q_k(Du, D^2u)$[11] cuando las curvaturas principales de las gráficas pertenecen al cono Γ_{k+1}.

De hecho, gracias al trabajo realizado en [Hol] para el flujo $\sqrt[k]{S_k}$ cuando las curvaturas

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[11]Observamos que estamos abusando con la notación al escribir $Q_k(Du, D^2u)$ en lugar de $Q_k(\lambda)$. La razón por la que elegimos escribir de esta manera es para enfatizar la dependencia en las variables del operador Q_k cuando se evalúa en las curvaturas principales de M_t o M.
Además, utilizamos una forma conveniente de trabajar con este término, véase la sección 3.2 en el capítulo 3 para más detalles

principales de M_t pertenecen al cono Γ_k, pudimos obtener la siguiente estimación para el flujo Q_k.

Teorema 1.2.1. *Sea $u \in \mathcal{C}^{3,2}(\Omega)$[12] una solución de la ecuación* (1.10)*, donde $\Omega = B_r(0) \times (0, T]$ con $r > 0$. Supongamos que las curvaturas principales de la grafo de u se encuentran en el cono Γ_{k+1} definido en* (1.4) *para cada t. Entonces, existe una constante $K = K(k, n) > 0$ independiente de u tal que para cada $t > 0$ se cumple que*

$$|Du(0,t)| \leq \exp\left(K + \frac{KMT}{rt}\frac{mrT}{t} + \frac{m^2}{t} + \frac{(T+1)r^2}{t} + \frac{Tm^2}{tr^2} + \frac{Tm}{tr}\right),$$

donde $m = \sup\limits_{\Omega} u$.

Este teorema se enunciarán nuevamente en Thm. 2.2.1 y Thm. 3.0.1, y se probará en la sección 3.2 a partir de la pág. 78.

Además, como hemos podido adaptar la técnica utilizada en el teorema anterior para los solitones por traslación del Q_k-flujo, hemos obtenido la siguiente estimación del gradiente para esa ecuación.

Teorema 1.2.2. *Sea $r > 0$ y $u \in \mathcal{C}^3(B_r(0))$ una solución de* (1.11) *tal que las curvaturas principales de $\mathrm{graph}(u)$ está en Γ_{k+1}. Entonces, existe una constante $C = C(k, n)$ independiente de u tal que*

$$\ln|Du(0)| \leq C\left(\frac{m^2}{r^2} + \frac{m}{r} + 1\right),$$

donde $m = \sup\limits_{B_r(0)} u$.

Este teorema se enunciarán nuevamente en Thm. 2.2.2 y Thm. 3.0.2, y se probará en la sección 3.2 a partir de la pág. 90.

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[12]Esto significa que $u(x, t)$ es de clase $\mathcal{C}^3$ en $x \in B_r(0)$ y de clase $\mathcal{C}^2$ en $t \in (0, T)$.

Por otro lado, adaptando la demostración del teorema anterior, obtuvimos un teorema de tipo Liouville para Q_k-trasladores que dice lo siguiente.

Teorema 1.2.3. *No existen soluciones $u \in \mathcal{C}^3(\mathbb{R}^n)$ tal que*

1. *Las curvaturas principales del grafo de u están en Γ_{k+1}.*

2. *$u(x) = o(|x|)$ para $|x| \to \infty$.*

3. *El grafo de u es Q_k-traslador en $\mathbb{R}^{n+1}$.*

Este teorema se enunciarán nuevamente en Thm. 2.2.3 y Thm. 3.0.3, y se probará en la sección 3.2 a partir de la pág. 94.

Las estimaciones de segundo orden que enunciaremos a continuación dependen de la suposición de que las curvaturas principales pertenecen al cono Γ_{k+2} con $k \geq 1$. La razón es que cualquier hipersuperficie con sus curvaturas principales situadas en Γ_{k+2} con $k \geq 1$, satisface $|A|^2 \leq H^2$ y $|A|_k^2 \leq \tilde{C}(k,n)H^2$, donde $|A|_k^2 = \sum_{i=1}^{n} \dfrac{\partial Q_k}{\partial \lambda_i} \lambda_i^2$ y $\tilde{C}(k,n)$ es una constante positiva definida en la Ecuación (3.58).

Por lo tanto, dado que es más fácil estimar H que $|A|$ o $|A|_k^2$, aplicamos técnicas similares a las de [EH2] para obtener estimaciones de segundo orden para las soluciones de las ecuaciones (2.10) y (2.11), respectivamente.

Además, como en [EH2], supondremos que la evolución M_t puede ser escrita como el gráfico de una función definida sobre sobre un hiperplano, y que existe una función no negativa $h(x,t)$ tal que $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ y satisface

$$|(\partial_t - \Box_k)h| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h$$

para una constante $C(k,n) > 0$, donde $\Box_k f = \dfrac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$, $|v|_k = \dfrac{\partial Q_k}{\partial h_{ij}} v_i v_j$ con $v \in \mathbb{R}^n$.

En las estimaciones de Ecker-Huisken usualmente se considera la función

$$h(x,t) = |x|^2 - \langle x, \vec{w} \rangle + 2\frac{n-k}{k+1}t,$$

que mide la distancia al hyperplano cuyo normal unitario viene dado por $\vec{w} \in \mathbb{S}^n$. Notamos que esta función $h(x,t)$ funciona al menos para el caso cuando M_t es convexo. Referimos al lector a revisar la observación 3.3.2 de la sección 3.3 para más detalles.

Teorema 1.2.4. *Sea $R > 0$ tal que $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ es un gráfico sobre una bola de radio R en el hiperplano ortogonal a $\vec{w} \in \mathbb{S}^n$ para todo $[0,T)$.*
Supongamos que M_t evoluciona bajo el Q_k-flujo y que las curvaturas principales pertenecen al cono Γ_{k+2} con $k \geq 1$. Entonces, para cualquier $t_0 \in (0,T)$ y $\theta \in [0,1)$, se cumple la siguiente estimación

$$\sup_{M_t} H^2(x,t) \leq \frac{c(k,n)}{(1-\theta)^2}\left(\frac{1}{t} + \frac{1}{R^2}\right)\sup_{[0,t_0]}\sup_{M_t} v^4,$$

donde $v = \langle \nu, \vec{w} \rangle^{-1}$.

Este teorema se enunciará de nuevo en Thm. 2.2.4 y Thm. 3.0.4, y se probará en la sección 3.3 a partir de la pág. 104.

De igual manera, para los Q_k-trasladores, consideramos que se es gráfico sobre $B_r(0) \subset \mathbb{R}^n$ y que existe de una función no negativa $h(x)$ tal que $M = \{x \in M; h(x) \leq R^2\}$ y satisface las estimaciones anteriores pero cambiando la suposición $|(\partial_t - \Box_k)h| \leq C(k,n)$ por

$$|\Box_k h + \langle h, e_{n+1} \rangle| \leq C(k,n), \tag{1.12}$$

que es la suposición natural dado que la evolución ocurre por traslación en la dirección e_{n+1}. Es importante mencionar que para estas soluciones, la función $h(x) = |x|^2 - \langle x, e_{n+1} \rangle^2$ no

satisface de igual manera la estimación (1.12). Referimos al lector revisar la observación 3.3.5 en la sección 3.3 para los detalles.

Teorema 1.2.5. *Sea $R > 0$ tal que $M = \{x \in M : h(x) \leq R^2\}$ es un gráfico sobre una bola de radio $R > 0$ en el hiperplano ortogonal a e_{n+1}.*
Supongamos que M es un Q_k-traslador tal que las curvaturas principales pertenecen al cono Γ_{k+2} con $k \geq 1$. Entonces, para cualquier $\theta \in [0, 1)$, se cumple la siguiente estimación

$$\sup_{B_R} H^2(x) \leq \frac{c(k, n)}{(1-\theta)^2} \left(1 + \frac{1}{R^2}\right) \sup_{\{h \leq \theta R^2\}} v^4,$$

donde $v = \langle \nu, e_{n+1} \rangle^{-1}$.

Este teorema se enunciará de nuevo en Thm. 2.2.5 y Thm. 3.0.5, y se probará en la sección 3.3 a partir de la pág. 110.

1.2.3 γ-trasladores rotacionalmente simétricos en $\mathbb{R}^{n+1}$

En esta subsección queremos discutir sobre la construcción de γ-trasladores que son rotacionalmente simétricos en $\mathbb{R}^{n+1}$ para funciones de curvatura particulares.

En primer lugar, un γ-traslador gráfico rotacionalmente simétrico en $\mathbb{R}^{n+1}$ con $\gamma > 0$ corresponde al conjunto $\{(x, u(x)) \in \mathbb{R}^{n+1} : |x| < R\}$ donde $u(x) = \mathbf{u}(r)$, $\mathbf{u} : [0, R) \to \mathbb{R}$ y $r = |x|$.

Entonces, la ecuación (1.5) se convierte en una ecuación diferencial ordinaria (EDO) de la forma

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2) f\left(\frac{\dot{\mathbf{u}}}{r}\right),$$

donde la notación de puntos significa derivadas con respecto a r, y f es una función que aparece aplicando el teorema de la función implícita en las componentes conexas del conjunto

de nivel $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$. Observamos que en este caso, las curvaturas principales son dos, y como $\gamma(\lambda)$ es creciente en cada variable, el teorema de la función implícita garantiza una expresión local de $f(x)$ en cada componente conexa de $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$.

Como mencionamos anteriormente en 2, en el caso de los H-trasladores, los autores en [AW] construyeron una solución estrictamente convexa definida para $R = \infty$ llamada el solitón "bowl" (ver Fig. 2.3a) que se comporta como un paraboloide en el infinito.

En esta tesis, construimos soluciones de tipo "bowl" para $\gamma = Q_{n-1}$ y $\gamma = \sqrt[n]{S_n}$ en $\mathbb{R}^{n+1}$. Estas soluciones son gráficas estrictamente convexas rotacionalmente simétricas, que se definen en bolas o en el hiperplano completo.

En particular, para $n = 2$, estas soluciones son explícitas y vienen dadas por

$$-\ln(1 - r^2) \quad \text{y} \quad \int_0^r \sqrt{e^{s^2} - 1}\, ds$$

para las funciones de curvatura Q_1 y $\sqrt{S_2}$, respectivamente, y se probarán en los Thm. 4.2.1 y Thm. 4.2.2, respectivamente. Observamos que el $\sqrt{S_2}$-traslador de tipo "bowl" no es $\mathcal{C}^2$ en el origen, y Q_1-traslador del tipo "bowl" es $\mathcal{C}^2$-asintótico[13] al cilindro $\mathbb{S}^1 \times \mathbb{R}$.

Teorema 1.2.6. *Para $n \geq 3$[14], existe un único[15] traslador rotacionalmente simétrico suave estrictamente convexo en $\mathbb{R}^{n+1}$ para las funciones de curvatura Q_{n-1} y $\sqrt[n]{S_n}$. Además, se tiene*

1. *Para $n \geq 3$, el Q_{n-1}-traslador está definido en una bola de radio $Q_{n-1}(1, \ldots, 1) = \dfrac{1}{n}$ y es $\mathcal{C}^2$-asintótica al cilindro $\mathbb{S}^{n-1}\left(\frac{1}{n}\right) \times \mathbb{R}$.*

2. *Para $n \geq 3$, el $\sqrt[n]{S_n}$-traslador es entero.*

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[13]Esto significa que las curvaturas principales $\lambda_1 \leq \ldots \leq \lambda_n$ satisfacen $\lambda_1 \to 0$ y $\lambda_i \to 1$ para $i \neq 1$ cuando $|p| \to \infty$.

[14]Notamos que para $n \geq 3$, la singularidad en el origen del $\sqrt{S_2}$-traslador de tipo "bowl" no se extiende para los $\sqrt[n]{S_n}$-trasladores.

[15]La unicidad con respecto a ser de rotacionalmente simétrico.

Este teorema se enunciará nuevamente en Thm. 2.2.6 y Thm. 4.0.1, y se probará la existencia para $\sqrt[n]{S_n}$ en Thm. 4.4.2 (pág. 132) y para Q_{n-1} Thm. 4.5.2 (pág. 138), y la unicidad en Cor. 4.6.3 (pág. 146).

Además, también encontramos una familia de $\sqrt{S_2}$-trasladores rotacionalmente simétricos en $\mathbb{R}^3$ que se asemejan a una catenoide (mirar Fig. 1.4) y satisfacen $H < 0$ y $K > 0$.

Teorema 1.2.7. *Sea* $a \in [0,1)$. *Entonces, la familia de superficies*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi), \ z \in \mathbb{R} \right\},$$

donde la ecuacion que define a $r_a(z)$ *es*

$$z = \int\limits_{1}^{r_a(z)} \sqrt{e^{s^2 - a} - 1}\, ds$$

con $r_a(0) = 1$, *son* $\sqrt{S_2}$-*trasladores completos en* $\mathbb{R}^3$ *que satisfacen* $H < 0$ *y* $K > 0$.

Este teorema se enunciará nuevamente en Thm. 2.2.7 y Thm. 4.0.2, y se probará en Thm. 4.7.1.

Más recientemente, en [Ren], el autor demostró la existencia de soluciones de tipo "bowl" para una clase general de funciones de curvatura simétrica α-homogéneas[16], sin ninguna suposición sobre convexidad o concavidad en la función de curvautra. Además caracterizó cuando la solución estará definida en una bola o será entera en términos de la función de curvatura, véase la introducción del capítulo 4 más adelante.

1.2.4 Propiedades geométricas de los γ-trasladores en $\mathbb{R}^{n+1}$

Otro aspecto importante que estudiamos en esta tesis son las propiedades geométricas de los γ-trasladores en $\mathbb{R}^{n+1}$. Entendemos por propiedades geométricas la aplicación de la teoría

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[16]Esto es para $c > 0$, $\gamma(c\lambda) = c^\alpha \gamma(\lambda)$.
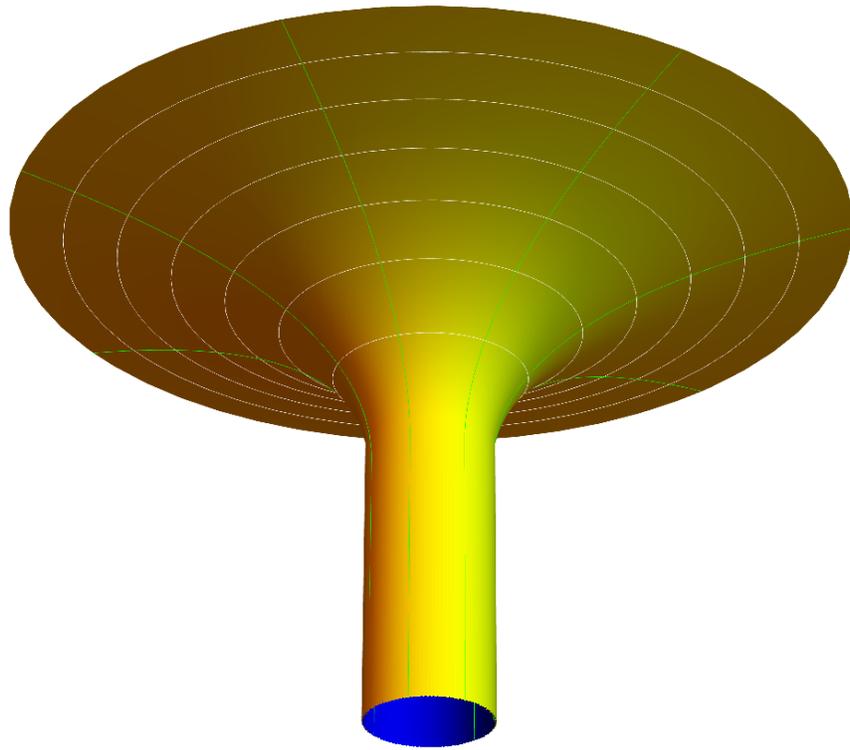
Figure 1.4: A $\sqrt{S_2}$-traslador Σ_a en $\mathbb{R}^3$. Imagen es cortesia de Francisco Martín.

de las EDP aplicada en el contexto de la geometría diferencial: principios tangenciales, inexistencia, unicidad y estimaciones de convexidad. Uno de los ejemplos más importantes de estas aplicaciones son los principios del máximo aplicados a cantidades geométricas de una hipersuperficie.

Recordemos que los H-trasladores son hipersuperficies mínimas en $\left(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1}\rangle} dx^2\right)$. Por lo tanto, el principio tangencial, que permite comparar H-trasladores en función de cómo se tocan, es un resultado bien conocido de la teoría de superficies mínimas. Sin embargo, dado que los γ-trasladores no son hipersuperficies mínimas para $\gamma \neq H$, desarrollamos un principio de tangencia que dice lo siguiente.

Teorema 1.2.8. *Sean $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ dos γ-trasladores embebidos, conexos y completos tales que*

1. *$\gamma : \Gamma \to [0, \infty)$ satisface las propiedades 1-4 y 7 dadas en la sección 5.1.*

2. *Σ_1 es estrictamente convexo.*

3. *Σ_2 es convexo.*

Entonces,

a) *(**Principio de tangencia interior**) Supongamos que existe un punto interior $p \in \Sigma_1 \cap \Sigma_2$ tal que los espacios tangentes coinciden en p. Si Σ_1 se encuentra en un lado de Σ_2, entonces ambas hipersuperficies coinciden.*

b) *(**Principio tangencial de frontera**) Supongamos que los bordes $\partial\Sigma_i$ están contenidos en el mismo hiperplano Π, y la intersección de Σ_i con Π es transversal. Si Σ_1 se encuentra localmente en uno de los lados de Σ_2 y existe $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ tales que los espacios tangentes a las hipersuperficies y a las fronteras coinciden, entonces ambas hipersuperficies coinciden.*

Este teorema se enunciará de nuevo en Thm. 2.2.8 y Thm. 5.0.3, y se probará en la sección 5.2 a partir de la pág. 170.

Notamos que sin la hipótesis que γ sea real analítica, entonces Σ_1 y Σ_2 coincidirían solo localmente en p. Por otro lado, sin la hipótesis de completitud, el principio de tangencia implicaría que $\Sigma_1 \cup \Sigma_2$ es tambien un soliton por traslación.

Además, gracias a los resultados expuestos en [Ren] (véanse los teoremas 1.3-1.4) también pudimos demostrar un resultado de no existencia para el γ-trasladores tal que el γ-traslador de tipo "bowl" está definido en una bola.

Corolario 1.2.1. *Supongamos que $\gamma : \Gamma \to [0, \infty)$ satisface las propiedades 1-4 y 7. Entonces, si el γ-traslador de tipo "bowl" está definido en una bola, entonces no existen γ-trasladores enteros convexos en $\mathbb{R}^{n+1}$.*

Este corolario se enunciará nuevamente en 2.2.9 adn 5.0.4, y se probará en la sección 5.3 a partir de la pág 174.

A continuación, aplicamos también el método de planos móviles de Alexandrov para obtener que un γ-traslador que es un gráfo convexo sobre una bola hereda las propiedades definidas en la frontera de su dominio.

Teorema 1.2.9. *Sea $\Sigma \subset \mathbb{R}^{n+1}$ un γ-traslador completo tal que*

1. *$\gamma : \Gamma \to [0, \infty)$ satisface las propiedades 1-4 y 7.*

2. *Σ es un grafo estrictamente convexo sobre una bola $B_r^n(0) \subset \mathbb{R}^n$.*

3. *Σ posee un único final $\mathcal{C}^2$-asintótico al cilindro $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, es decir: las curvaturas principales de Σ satisfacen*

$$\min \{\lambda_i(p) : i = 1, \ldots, n\} = \lambda_1(p) \to 0,$$
$$\forall i \in \{2, \ldots, n\}, \lambda_i(p) \to \frac{1}{r},$$

cuando $|p| \to \infty$.

Entonces Σ es rotacionalmente simétrico con respecto al eje x_{n+1}.

Este teorema se enunciará nuevamente en Thm. 2.2.10 y Thm. 5.0.5, y se probará en la sección 5.4 a partir de la pág 177.

Además, como corolario del teorema anterior y del resultado de unicidad expuesto en [Ren], obtenemos la siguiente propiedad de unicidad.

Corolario 1.2.2. *Sea Σ un γ-traslador como en el teorema anterior tal que el γ-traslador de tipo "bowl" está definido en una bola de radio $\gamma(1, \ldots, 1)$. Entonces, Σ coincide con el γ-traslador de tipo "bowl".*

Este corolario se enunciará nuevamente en Thm. 2.2.11 y Thm. 5.0.6, y se probará en la sección 5.4 a partir de la pág 187.

Finalmente, el resultado principal de esta tesis es una estimación de convexidad para los γ-trasladores en $\mathbb{R}^{n+1}$. Para dar un contexto de por qué este es el resultado principal de esta tesis, empezamos anunciando algunos hechos relacionados con H-trasladores con $H > 0$. En primer lugar, en [SX], los autores demostraron que los H-trasladores con $H > 0$ en $\mathbb{R}^3$ son convexos (véase [HIMW3] para una demostración más breve). Este resultado tuvo una notable implicación en la clasificación de los H-trasladores gráficos en $\mathbb{R}^3$. Además, en [Has], el autor muestra que un H-traslador α-no colapsado[17] y convexo debe ser el solitón "bowl". En resumen, puesto que H-trasladores con $H > 0$ son convexos, se deduce que el único de ellos que es entero debe ser el solitón "bowl". Remitimos al lector a [HIMW2] para la clasificación de H-trasladores que son grafos.

Por lo tanto, en el espíritu de mostrar que un γ-traslador que es un grafo uniforme 2-convexos es convexo, probamos la siguiente estimación de convexidad.

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[17]Si el traslador acota una región abierta $\Omega \subset \mathbb{R}^{n+1}$ y posee $H > 0$ en cada punto, entonces diremos que es α-no colapsado si para cada punto p del traslador hay una bola abierta B de radio $\dfrac{\alpha}{H(p)}$ contenida en Ω con $p \in \partial B$.

Teorema 1.2.10. *Sean $n \geq 3$, $\alpha, \delta > 0$ y $\Sigma \subset \mathbb{R}^{n+1}$ un γ-traslador inmerso completo conexo tal que*

a) *$\gamma : \Gamma \to \mathbb{R}$ satisface las propiedades 1-3 and 5-7.*

b) *$\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : \alpha H \leq (\delta + 1)\gamma\}$ está compactamente soportado en $\Gamma \setminus Cyl_{n-1}$[18], donde*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\}$$

y e_i es la base canónica de $\mathbb{R}^n$.

c) *Existe una constante $\beta \in (0, 1)$ tal que $\lambda_i + \lambda_j \geq \beta H$, para todo $1 \leq i < j \leq n$.*

Entonces, $\lambda_1 \geq H - \alpha\gamma$ en Σ, donde $\lambda_1(p) = \min\{\lambda_i(p) : i = 1, \ldots, n\}$.

Este teorema se enunciará nuevamente en Thm. 2.2.12 y Thm. 5.0.7, y se probara en la sección 5.5 a partir de la página 191.

Notamos que la hipótesis 1.2.10 implica que solamente el mínimo de la curvaturas principales puede ser negativa. Además, notamos que $\lambda \in \Gamma_{\alpha,\delta}$ no da necesariamente un signo fijo a $H - \alpha\gamma$ en Σ, pero al menos da una manera de controlar por abajo λ_1. Por otra parte, al aplicar el γ-flujo la condición $\lambda \in \Gamma_{\alpha,\delta}$ se preserva durante el flujo.

La demostración del Teorema 2.2.12 está inspirada en [SS], donde los autores muestran que un H-traslador con $H > 0$ y uniformemente 2-convexo es convexo. Observamos que nuestra demostración no funciona para H-trasladores. La razón principal es que el cono $\Gamma_{\alpha,\delta}$ necesita estar soportado de forma compacta en $\Gamma_1 \setminus C_{n-1}$, pero para $\gamma = H$, $\Gamma_{\alpha,\delta} = \Gamma_1$. Por otro lado, la demostración dada en [SS] no puede adaptarse directamente a una función de curvatura cóncava γ, ya que los autores utilizan una aproximación cóncava para λ_1, y esta no se comporta de la misma manera para funciones de curvatura cóncavas.

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[18]Esto significa que $\Gamma_{\alpha,\delta} \cap \partial B_1(0)$ es compacto en $\Gamma \setminus Cyl_{n-1}$.

Observación 1.2.1. Una estimación similar se demostró en [Lyn2] para una familia de funciones de curvatura de la forma

$$\gamma = (1 - c)H - c \left(\sum_{1 = i_1 < \ldots < i_k = n} \frac{1}{\lambda_{i_1} + \ldots + \lambda_{i_k}} \right)^{-1}, \; c \in (0, 1).$$

En contraste con nuestro resultado, su estimación es en el contexto parabólico compacto.

En consecuencia, como una aplicación de la estimación de convexidad, demostramos que bajo las mismas hipótesis del Teorema 1.2.10, los γ-trasladores son asintóticamente convexos.

Corolario 1.2.3. *Sea $\Sigma \subset \mathbb{R}^{n+1}$ el γ-traslador donde $\gamma : \Gamma \to \mathbb{R}$ satisface la hipótesis del Teorema 1.2.10, entonces Σ es asintóticamente convexo.*

Este corolario será enunciado nuevamente en Cor. 2.2.13 y Cor. 5.0.8, y se demostrará en la sección 5.5 a partir de la página 197.

Chapter 2

Introduction

A geometric flow is a time dependent partial differential equation (PDE) which involves geometric quantities from a given manifold. In the last decades of the last century, this topic gained a lot of interest in the mathematical community from its applications in Differential Geometry, Topology, Physics, and, in a more applied context, Image Processing.

From the geometric perspective, one can distinguish two types of geometric flows: intrinsic and extrinsic. An intrinsic geometric flow is understood to be a PDE which considers the evolution of intrinsic geometric quantities from a given riemannian manifold.

An important example of these types of flows consists of modifying a riemannian metric g_0 into one with some constant curvature tensor. More precisely, find a 1-parameter family of metrics $\{g(t)\}_{0 \leq t < T}$ for a manifold M such that

$$\begin{cases} \partial_t g = F(g) \quad , \, t \in (0, T) \\ g(0) = g_0. \end{cases},$$

where F is a tensor which depends on the metric $g(t)$ and its derivatives. Finally, one studies the limit metric $g(T)$, where $T \in (0, \infty]$ is the maximum existence time of the flow, with the

expectation that it satisfies the prescribe curvature condition.

In this setting, the most remarkable and studied intrinsic flow is $F(g) = -2\text{Ric}(g)$, where $\text{Ric}(g)$ denote the Ricci curvature tensor of $g(t)$, and it is known as the Ricci Flow.

The Ricci Flow was introduced by R. Hamilton in the 1980's as a tool to prove the Thurston's geometrization conjecture for closed 3-manifolds. This problem states that any closed 3-manifold can be canonically decomposed into pieces, where each piece possesses one of eight different types of geometric structures. In 2003, thanks to the work of G. Perelman, this conjecture was finally proved and, as a consequence, the Poincaré conjecture, which was an open problem since 1904. We refer to [CZ] for an excellent self contained book about this topic.

Figure 2.1: Dumbbel's evolution under the Ricci flow. Image courtesy of Ignacio Mcmanus.

On the other hand, an extrinsic geometric flow corresponds to a one-parameter family of immersions whose deformations depend on the extrinsic curvatures of a submanifolds in a ambient riemannian manifold.

More precisely, in the codimension one case, if $F_0 : M^n \to N^{n+1}$ is an isometric immersion from a manifold to a riemannian manifold (N, g), one wants to find a 1-parameter family of

isometric immersions $F : M \times (0, T) \to N$ such that

$$\begin{cases} \partial_t F = \gamma(\lambda)\nu, \ (x, t) \in M \times (0, T), \\ F(x, 0) = F_0(x), \end{cases} \tag{2.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and ν are the principal curvature vector and the inward unit normal vector of $M_t = F(M, t)$ in N, respectively. In this thesis we will refer to Equation 2.1 as a γ-flow for short.

The most studied γ-flow in the literature is the Mean Curvature Flow, where

$$\gamma = H = \lambda_1 + \ldots + \lambda_n.$$

The H-flow also appears as the negative gradient flow of the area functional of M in N, and consequently, it has been used to deform hypersurfaces into minimal hypersurfaces in different ambient spaces.

Next, we want to give a brief description of the analysis of singularities related to the H-flow in $\mathbb{R}^{n+1}$ that are related to the results obtained in this thesis.

Firstly, to contextualize the analysis of the singularities that appear under the evolution through the H-flow, we want to mention a pioneering result given by G. Huisken in [Hui2], which states that any closed and convex[1] hypersurface will collapse to a point in finite time. In addition, this phenomenon was characterized by the blow up of the quantity

$$\sup_{M_t} |A|,$$

where $|A| = \sqrt{\lambda_1^2 + \ldots + \lambda_n^2}$ as $t \to T$, where T is the maximal time of existence of the flow. To exemplify this, the evolution of a sphere of radius R_0 under the H-flow in $\mathbb{R}^{n+1}$ is given

---

[1]The principal curvatures of the hypersurface are non-negative.

by

$$F(p, t) = R(t)p,$$

where $R(t) = \sqrt{R_0^2 - 2nt}$ and $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. We note that the maximal time of existence is given by $T = \dfrac{R_0^2}{2n}$, and therefore, it holds that

$$|A|(t) = \frac{\sqrt{n}}{R(t)} = \frac{1}{\sqrt{2}\sqrt{T-t}}.$$
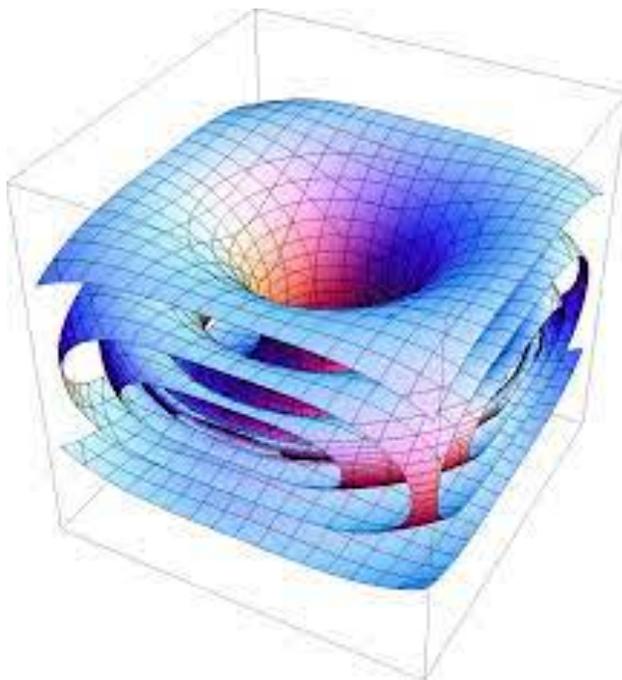
Figure 2.2: The evolution of a surface under the H-flow in $\mathbb{R}^3$. Image courtesy of Francisco Martín.

In fact, the singularities under the H-flow in $\mathbb{R}^{n+1}$ are classified into two types related to the blow up rate of $\sup_{M_t} |A|$:

a) Type I: There exists a constant $C > 0$ such that $\sup_{M_t} |A| \leq \dfrac{C}{\sqrt{T-t}}$.

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b) Type II: It is not of type I, i.e: $\displaystyle\limsup_{t \to T}{}_{M_t} |A|\sqrt{T - t} = \infty$.

The classification of these singularities is an open problem in all its generality. However, when the initial data is mean convex, i.e: $H > 0$, the authors in [HS1] showed that after a suitable parabolic rescaling[2], the singularity can be modeled by a self-similar solution[3] of the H-flow. Indeed, these solutions are:

a) Self-shrinkers: They appear under a parabolic rescaling around a type I singularity. After passing to a limit they converge to an ancient[4] self-similar solution which evolves only by dilation, ie: $F(x, t) = \sqrt{-t}F_0(x)$ with $t \in (-\infty, 0]$.

   In addition, each time slice is convex and satisfies the equation $H = -\langle p, \nu \rangle$ for $p \in M_t$.

b) Translating solitons: They appear under a parabolic rescalling around a type II singularity. After passing to a limit they converge to an eternal[5] self-similar solution which evolves by translation in a unitary direction, i.e: $F(x, t) = F_0(x) + vt$ for $t \in \mathbb{R}$ and some unit vector $v \in \mathbb{S}^n$.

   In addition, each time slice can be written as $M_t = \mathbb{R}^{n-k} \times S^k$ where S^k is strictly convex hypersurface, and the equation $H = \langle \nu, v \rangle$ holds in M_t.

The main tool used in the proof of this classification was the Huisken's monotonicity formula

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[2]If $(x_n, t_n) \in M \times (0, T)$ is a sequence that satisfy $a_n = |A|(x_n, t_n) \to \infty$, then under the parabolic rescaling of the form

$$M_n = a_n \left(M_{\frac{t_n}{a_n^2}} - x_n \right),$$

we will see that M_n converge to a self-similar solution of the H-flow.

[3]A solution to the H-flow of the form $F(x, t) = r(t)O(t)F_0(x) + V(t)$, where $r(t) \geq 0$, $V(t) \in \mathbb{R}^{n+1}$ and $O(t) \in \mathcal{O}(n+1)$ is an orthogonal matrix. Visually this means that the initial hypersurface F_0 evolves under a dilation $r(t)$, a linear isometry $O(t)$, and a translation $V(t)$ in $\mathbb{R}^{n+1}$.

[4]Solutions to Equation (2.1) which are defined for $(-\infty, a]$ for some $a \in \mathbb{R}$.

[5]Solutions to Equation (2.1) which are defined for $(-\infty, \infty)$.

(see [Hui1]), related to the evolution of a heat kernel functional,

$$\frac{d}{dt} \int\limits_{M} \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}} dx = -\int\limits_{M} \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}} \left| H + \frac{\langle x - p, \nu \rangle}{2|t_0-t|} \right|^2 dx \le 0.$$

On the other hand, we want to mention that there is an additional type of self-similar solution of the H-flow called self-expanders. In [EH1], the authors showed that the evolution of an entire Lipschitz graph exists for all $t > 0$ and, after a suitable rescaling, the evolution converges to a self-expander as $t \to \infty$, ie: the limit flow converge to $F(x,t) = \sqrt{t}F_0(x)$ and each time slice satisfies $H = \langle p, \nu \rangle$. For a recent study of self-expanders we the refer to [Smo].

Finally, the classification of self-similar solutions of the H-flow is an important task to understand the behavior under the evolution of this flow. Actually, this problem has been completed for some particular cases.

In the case of convex self-shrinker the possible solutions, up to isometries and tangential diffeomorphisms, are spheres, cylinders or Abresch-Langer hypersurfaces.

On the other hand, the classification of translating solitons have been recently completed for graphs in $\mathbb{R}^3$, see [HIMW2] for details. The possible solutions are (see Figure 2.3 below):

1. The family of tilted "grim reapers" given by the graph of the functions $g_\theta(x,y) = \csc^2(\theta) \ln(\cos(y\cos(\theta))) + x\tan(\theta)$ defined in the strip $\left\{ (x,y) \in \mathbb{R}^2 : |y| < \dfrac{\pi}{2\cos(\theta)} \right\}$ where $\theta \in [0, \pi/2)$. We note that for $\theta = 0$, the graph of $g_0(x,y)$ is know as the grim reaper cylinder.

2. The "bowl" soliton which is the unique (modulo traslation) entire rotationally symmetric graph which behave at infinity like a paraboloid. More precisely, if $u : \mathbb{R}^2 \to \mathbb{R}$ denotes the function the define the "bowl" soliton, then $u(r) = \dfrac{r^2}{2} - \ln(r) + O(r^{-2})$ as $r := \sqrt{x^2 + y^2} \to \infty$.

(a) Bowl soliton in $\mathbb{R}^3$.

(b) Grim reaper and the tilted grim reaper for $\theta = \dfrac{\pi}{6}$ in $\mathbb{R}^3$.

(c) Delta wing in $\mathbb{R}^3$.
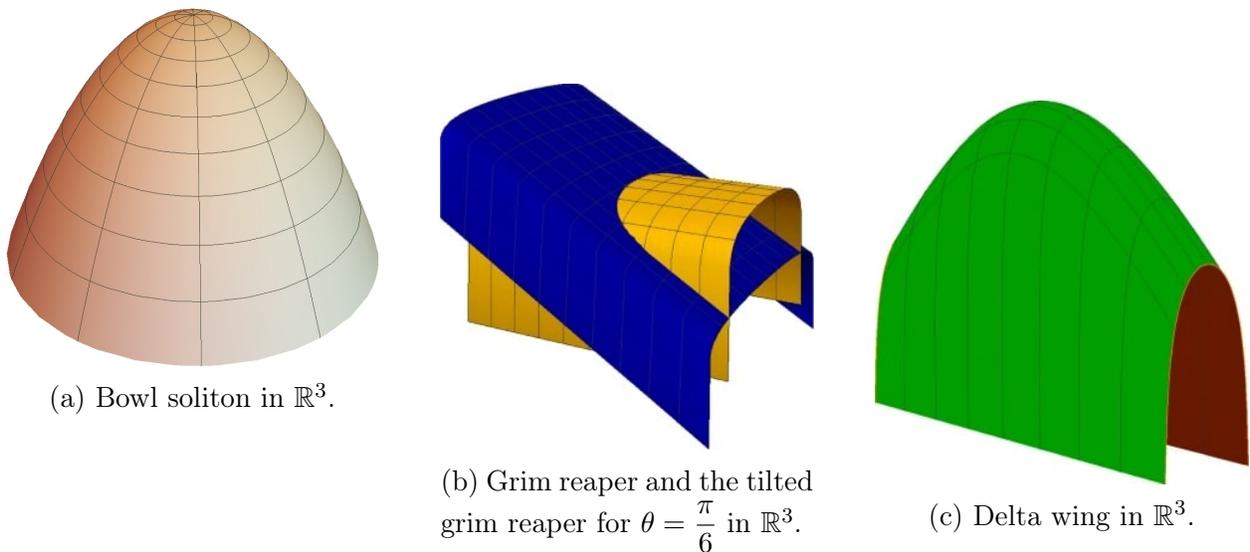
Figure 2.3: These surfaces are taken as solutions to $H = \langle \nu, -e_3 \rangle$. Images courtesy of Francisco Martín.

3. The family of delta-wings, given by the graph of a function $u_b : \{(x, y) \in \mathbb{R}^2 : |y| < b\} \to \mathbb{R}$ with $b > \dfrac{\pi}{2}$. These translating solitons satisfy $u_b(x, y) = u_b(-x, y) = u_b(x, -y)$ and reach its maximum at the origin.

It is important to mention that in the classification of these self-similar solutions of the H-flow, the theory of minimal surfaces was used.

More precisely, in [Ilm], T. Ilmanen showed that these solutions can also be seen as minimal hypersurfaces in $(\mathbb{R}^{n+1}, g)$, where g corresponds to $e^{-|p|^2} dx^2$ for self-shrinkers, and $e^{-\langle p,v \rangle} dx^2$ for translating solitons. We also note that these metrics are not complete, which means that the global theory of minimal surfaces does not apply for these self-similar solutions, but the local theory does.

2.1 Thesis Framework

This thesis is about problems related to extrinsic geometric flow (2.1) for which the curvature function $\gamma : \Gamma \to \mathbb{R}$ is smooth, symmetric and 1-homogeneous function defined in an open cone $\Gamma \subset \mathbb{R}^n$ that contains the positive cone $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0, \ i = 1, \ldots, n\}$.

We chose to study this type of curvature functions because they are a natural geometrical analogue from the theory of PDEs. In fact, we want to remark the theory developed by Nirenberg, Caffarelli and Spruck in [CNS1] and [CNS2] for prescribing fully nonlinear[6] curvature equations in bounded domains in $\mathbb{R}^n$ with Dirichlet boundary conditions. This theory served as inspiration for the work developed in this thesis.

Moreover, an important class of 1-homogeneous curvature function considered in this thesis are of quotient Hessian type[7], i.e:

$$\sqrt[k]{S_k} \text{ and } Q_k = \frac{S_{k+1}}{S_k}. \tag{2.2}$$

Recall that these functions are evaluated in the principal curvatures of a given hypersurface, and by definition they are the eigenvalues of the shape operator, i.e:

$$\mathcal{W}_p : \ T_p M \to T_p M$$

$$\vec{v} \to \nabla_{\vec{v}} \nu,$$

where ∇ and ν are the Levi-Civita connection of $\mathbb{R}^{n+1}$ and the inward[8] unit normal vector of M in $\mathbb{R}^{n+1}$. In local coordinates, the shape operator can be written by $h_j^i = g^{ik} h_{kj}$, where g^{ij} denotes the coefficients of the inverse of the metric tensor g_{ij} of the hypersurface, and h_{ij}

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[6]This means that in coordinates $(x, u(x))$, the prescribed curvature equation is a fully nonlinear elliptic PDE.

[7]The reason of the name comes from the theory of prescribing the elemental symmetric polynomials evaluated in the eigenvalues of $\mathrm{Hess}(u)$. They are also known as of Monge–Ampère type since the elementary symmetric polynomials appear in expression of the form $\det(\mathrm{Hess}(u) - \lambda\mathrm{Id})$.

[8]Orientation imposed such that the principal curvature are positive.

denotes the coefficients of the second fundamental form of hypersurface, respectively.

In addition, the coefficients of the second fundamental form correspond to the tangent components of the covariant derivative of the unit normal map (modulo orientation), i.e.: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ where $\{e_i\}$ denotes an orthonormal basis of the tangent space at some point of the hypersurface.

Furthermore, the symbol

$$S_k(\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k} \tag{2.3}$$

denotes the elementary symmetric polynomial of degree k in n-variables, and we use the convention $S_0 = 1$ and $S_j = 0$ for $j > n$.

Note that the mean curvature $H = S_1(\lambda)$, the scalar curvature $R_g = \dfrac{2}{n(n-1)} S_2(\lambda)$ and the Gauss-Kronecker curvature $K = S_n(\lambda)$.

Finally, is important to remark that these flows are parabolic when the principal curvatures of the evolution M_t belong to the Gärding cones

$$\Gamma_k := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : S_j(\lambda) > 0 \text{ with } j = 1, \ldots, k\}. \tag{2.4}$$

We also mention that for us Γ_k always will denote the connected component that contains the positive cone Γ_+.

Now we want to discuss some previous works and results related to γ-flows in the literature.

Firstly, when the initial data is compact and strictly convex, the evolution under a γ-flow presents a similar behavior to the H-flow. In fact, in [And2] the author showed that under some natural assumptions on γ (see for instance Properties 1-7 in Chapter 4) the evolution will collapse to a point in finite time, and after a suitable parabolic rescaling, the solution

converges to a unit sphere in $\mathbb{R}^{n+1}$.

However, up to the present, the analysis of singularities related to the evolution under γ-flows is not as well understood as in the H-flow case. In fact, one of the main problems is the nonlinearity related to the geometry involved. For example, Huisken's monotonicity formula is available to describe singularities of the H-flow, while a similar formula is not known for other curvature functions.

Encouragingly, there are few results related to the study of singularities under γ-flows that we briefly describe. In [Lyn3] the author classified uniformly parabolic convex ancient solutions to γ-flows which present a Type I singularity.

Namely, these solutions are shrinkers spheres or cylinders. One important aspect of his result is that includes convex and concave curvature functions which satisfy extra concavity properties and admits a splitting theorem.

Furthermore, if the curvature function is convex (and satisfies properties (1)-(3) and 5 described in Chapter 5), more results are known related to Type II singularities. In [ALM], the authors showed that eternal solutions which present a Type II singularities split as $M = \Sigma^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and evolve under translation by a fixed unitary direction. It is still an open question if this phenomena occurs for concave functions.

Finally, there is another type of singularity which arise in fully nonlinear curvature flows which is related to the loss of convexity of the initial data under a γ-flow. To describe this phenomena we firstly mention that if the initial hypersurface satisfies that the principal curvatures belong to the symmetric cone

$$\left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \frac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0, \text{ for } i = 1, \ldots, n \right\}^{[9]},$$

---

[9]Note that this is an open cone when γ is a symmetric 1-homogeneous continuous function. In addition, if γ is concave, then this cone is also convex. This can be seen by the fact under the 1-homogeneous hypothesis, concave is equivalent to being sub-additive (i.e: $\gamma(a\lambda + b\mu) \geq a\gamma(\lambda) + b\gamma(\mu)$).

then under the γ-flow this property is also satisfied by the evolution M_t. In fact, this can be seen as an easy application of the strong maximum principle applied to the equation

$$(\partial_t - \Delta_\gamma)\gamma = |A|^2_\gamma \gamma,$$

where $\Delta_\gamma = \dfrac{\partial \gamma}{\partial h_{ij}} \nabla_i \nabla_j$ and $|A|^2_\gamma = \dfrac{\partial \gamma}{\partial h_{ij}} h_{ia} h_{aj}$. Note that for this argument we used that the matrix $\left(\dfrac{\partial \gamma}{\partial h_{ij}}\right)_{i,j}$ is a locally uniformly positive definite matrix.

Nevertheless, there are examples of curvature functions that do not preserve the positive cones Γ_+ under the evolution of the flow.

In [AMZ], the authors constructed a convex hypersurface in $\mathbb{R}^{n+1}$ such that the evolution under the γ-flow given by

$$\gamma(\lambda) = \sum_{1 \leq i < j \leq n} \frac{\lambda_i \lambda_j}{\sqrt{\lambda_i^2 + \lambda_j^2}},$$

looses it convexity for small $t > 0$. Note that $\gamma(\lambda)$ is positive, strictly increasing in each variable, and concave in the cone $\{\lambda \in \mathbb{R}^n : \forall i \neq j, \lambda_i + \lambda_j > 0\}$.

Moreover, the authors characterized the loss of convexity under the evolution of concave curvature functions if the hypothesis of being inverse-concave at the restriction of the boundary of Γ_+ was not satisfied. This means that if $\gamma(1, ..., 1, 0) > 0$, then the function

$$\tilde{\gamma}(\lambda_1, \ldots, \lambda_{n-1}) = \gamma(\lambda_1^{-1}, \ldots, \lambda_{n-1}^{-1}, 0)^{-1}$$

is concave in the $(n-1)$-dimensional positive cone $\Gamma_+^{n-1} = \{\lambda \in \mathbb{R}^{n-1} : \lambda_i > 0\}$.

Finally, we want to mention that these type of singularities do not occur under the evolution of the H-flow. In fact, in [HS2] the authors showed that all the Garlin cones Γ_k are preserved under H-flow, given in particular that the initial data will not lose its convexity under the

flow. Consequently, the study of γ-flows for nonlinear curvature functions can be used to evolve hypersurfaces such that only some convexity cones are preserved under the evolution of the flow.

2.2 Curvature Problems and Results

Firstly, in this thesis we are also interested in translating solitons of the γ-flow in $\mathbb{R}^{n+1}$. As we mentioned for the H-flow, these are eternal self-similar solutions of (2.1) which evolve under translations by vector field $V : \mathbb{R} \to \mathbb{S}^n$, i.e:

$$F(x,t) = F_0(x) + V(t).$$

We can check that $V(t) = \vec{v}t$ for some constant vector $\vec{v} \in \mathbb{S}^n$.
Indeed, we note that

$$\partial_t V = \partial_t F(x,t) = -\gamma(\lambda)\nu.$$

Then, since the shape operator $\mathcal{W}_p$ is invariant under translation, the principal curvatures also are invariant. In particular $\partial_t\gamma(\lambda) = 0$, and this implies that $\partial_t^2 V(t) = 0$, or equivalently, $V(t) = \vec{v}t$ for some constant vector $v \in \mathbb{S}^n$. However, in this thesis we will only consider translating solitons of the γ-flow with translation vector $\vec{v} = e_{n+1}$ for simplicity.

Furthermore, we note that translating solitons of the γ-flow, or γ-translators for short, satisfy in each time slice M_t the equation

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle . \tag{2.5}$$

This means that γ-translators can be studied with the theory of parabolic equations, or with

the theory of prescribing elliptic curvature equations.

The problems that we study in this thesis are more related with the elliptic setting of γ-translators. In fact, the assumptions regarding the curvature function $\gamma : \Gamma \to \mathbb{R}$ (see Section 5.1) are in the spirit of the classical differential geometry problems:

- γ is evaluated in the cone $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$.

- For every $i = 1, \ldots, n$, $\dfrac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0$ holds in Γ.

In local coordinates, this property gives that Equation (2.5) is locally uniformly elliptic and every connected simply connected γ-translator is a vertical graph over some open connected set $\Omega \subset \mathbb{R}^n$.

On the other hand, when we consider curvature functions that have a continuous extension at the boundary of $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$, such as the family $\sqrt[k]{S_k}$ supported in Γ_k, we have as trivial solutions to Equation (2.5) vertical hyperplanes and round cylinders (the unit normal vector is orthogonal to the hyperplane which contains e_{n+1}). Note that there are curvature functions that do not have this property, such as Q_k supported in Γ_k, given that the trivial solutions are vertical round cylinders.

Finally, we want to mention that the theory of degenerated elliptic PDE's is useful for the study of curvature functions that vanish in the boundary of Γ. We refer the reader to see [CNS1], [CNS2], [Ivo], [SUW] for nice introductory papers on this subject.

It is worth to mention that if we consider γ-translators in the parabolic setting, there are studies that one can initiate the γ-flow for curvature functions γ that are supported in $\{\lambda \in \mathbb{R}^n : \gamma(\lambda) \geq 0\} \cap \overline{\Gamma}_+$.

In fact, this was done by the author in [Die] for the Q_k-flow. She showed that the Q_k-flow can be initiated for closed convex initial data such that $Q_k \geq 0$ and the principal curvatures belong to Γ_{k-1}. Her main contribution was the use of cylindrical barriers that allow the H-flow to be initiated, giving that the evolution becomes strictly convex for positive times,

allowing the Q_k-flow to be started. In particular, the evolution possesses the same behavior as the exposed in [And1].

It remains an open problem if this procedure can be applied to different curvature functions, or if it can be extended to different open cones rather than Γ_{k-1}. We refer the reader to [ACGL] for an introductory book about γ-flows in different settings.

2.2.1 Non-variational approach for Q_k-translators in $\mathbb{R}^{n+1}$

In this subsection we are going to show that Q_k-translators in $\mathbb{R}^{n+1}$ are not critical points of an area functional. In particular, we have little hope for γ-translator to be minimal hypersurface in conformal euclidean spaces.

Recall Q_k-translators satisfy

$$Q_k = \langle \nu, e_{n+1} \rangle \Leftrightarrow S_{k+1} - S_k \langle \nu, e_{n+1} \rangle = 0, \tag{2.6}$$

note that the if, and only if, occurs when the principal curvatures belong to Γ_k.
Therefore, to check if we can find a functional such that the equation (2.6) appears as an Euler-Lagrange equation, we will use the formulas obtained in Chapter 4 of [Ros].

Firstly, for a given smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we consider the functional over closed manifolds $M \subset \mathbb{R}^{n+1}$ given by

$$\mathcal{F}(M) = \int_M f(S_1, \ldots, S_n, \langle X, e_{n+1} \rangle) dA,$$

where S_i are the symmetric elemental polynomials evaluated in the principal curvatures of $M \subset \mathbb{R}^{n+1}$ and X is the position vector in $\mathbb{R}^{n+1}$.

Then, by choosing a normal variation $\varphi : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ with $\varphi_0 = \mathrm{Id}$ and

$\left. \dfrac{d\varphi_t}{dt} \right|_{t=0} = g\nu$, where $\varphi_t(x) = \varphi(x, t)$ and $g \in \mathcal{C}_c^\infty(M)$, we can see that

$$\left. \frac{d}{dt}\mathcal{F}(M_t) \right|_{t=0} = \int_M \left. \frac{d}{dt}f \right|_{t=0} dA + \int_M f \left. \frac{d}{dt}dA \right|_{t=0}, \tag{2.7}$$

where $M_t = \varphi_t(M)$.

Next, we calculate each term in Equation 2.7 separately.

We use Theorem 4.1 in [Ros] to calculate the first term in the above equation

$$
\begin{aligned}
\left. \frac{d}{dt}f \right|_{t=0} &= \sum_{i=1}^{n} D_i f \left. \frac{d}{dt}S_i \right|_{t=0} + D_{n+1}f \left. \frac{d}{dt}\langle X, e_{n+1}\rangle \right|_{t=0} \\
&= \sum_{i=1}^{n} D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1)S_{i+1}) \right) + g D_{n+1}f \langle \nu, e_{n+1}\rangle,
\end{aligned}
$$

where $L_{i-1}(g) = \mathrm{div}_M (T_{i-1}\nabla g)$ and T_i is the ith-Newton transformation of the shape operator of M in $\mathbb{R}^{n+1}$ defined inductively by the relation

$$
\begin{cases}
T_i(\mathcal{W}_p) = S_i Id - \mathcal{W}_p T_{i-1}(\mathcal{W}_p), \\[2mm]
T_0 = Id.
\end{cases}
$$

On the other hand, it is a well know result that the variation formula of the volume form under normal variations is given by

$$\left. \frac{d}{dt}dA \right|_{t=0} = -gH dA.$$

Consequently, we obtain the following formula for the first variation of $\mathcal{F}$ given by

$$\frac{d}{dt}\mathcal{F}(M_t)\bigg|_{t=0} \tag{2.8}$$
$$= \int_M \sum_{i=1}^n D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1)S_{i+1})\right) + g D_{n+1} f \langle \nu, e_{n+1}\rangle - gHf dA.$$

As we mentioned before, H-translators are critical points of the functional

$$F_0(M) = \int_M e^{\langle X, e_{n+1}\rangle} dA.$$

Therefore, it makes sense to consider the following function for Q_k-translator given by

$$\mathcal{F}_k(M) = \int_M S_k e^{(k+1)\langle X, e_{n+1}\rangle} dA, \ k = 1, \ldots, n.$$

Then, since $f = x_k e^{(k+1)x_{n+1}}$ in $\mathcal{F}_k(M)$, the first variation is easy to compute. Indeed, by Formula (2.8), it follows

$$\frac{d}{dt}\mathcal{F}_k(M_t)\bigg|_{t=0} \tag{2.9}$$
$$= \int_M -(k+1)g \left(S_{k+1} - S_k \langle \nu, e_{n+1}\rangle\right) e^{(k+1)\langle X, e_{n+1}\rangle} + L_{k-1}(g)e^{(k+1)\langle X, e_{n+1}\rangle} dA.$$

Remark 2.2.1. Note that the term $L_{k-1}(g)$ in (2.9) only vanishes when $k = 0$, since $T_{-1} = 0$ by definition, and this agree with the fact that H-translators are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle X, e_{n+1}\rangle} dx^2)$.

Moreover, for $k = 1$, $L_0(g) = \text{div}_M (T_0 \nabla g) = \Delta g$, where Δ is the Laplace-Beltrami operator of M. Therefore, by the divergence theorem, this term in the formula (2.9) can be written

by

$$\int_M L_0(g)e^{2\langle X,e_{n+1}\rangle}dA = \int_M g\Delta e^{2\langle X,e_{n+1}\rangle}dA$$
$$= \int_M 2g\left(2|e_{n+1}^\top|^2 - H\langle\nu,e_{n+1}\rangle\right)e^{2\langle X,e_{n+1}\rangle}dA.$$

where $e_{n+1}^\top$ is the orthogonal projection onto T_pM in $T_p\mathbb{R}^{n+1}$.

Finally, as the reader could suspect, Equation (2.6) is not the Euler-Lagrange for $\mathcal{F}_k$ for $k \geq 1$, since more curvature terms will appears in the Euler-Lagrange equation. Consequently, we do not expect the existence of any functional such that a Q_k-translator appears as a minimal hypersurface in a weighted Euclidean space.

2.2.2 Gradient and second order estimates for Q_k-equations

From the PDE perspective, gradient and second-order estimates are important tools for constructing solutions, and analyzing their properties and behavior.

As mentioned above, in [EH1] and [EH2] the authors obtain local gradient and curvature estimates in order to construct noncompact solutions of the H-flow in $\mathbb{R}^{n+1}$. These results and techniques motivated the development of a similar theory to be applied for the γ-flows. In this sense, we want to mention the work of [CD], where the authors constructed complete non-compact convex graphs evolving under the Q_k-flow in $\mathbb{R}^{n+1}$, see Equation (2.2) for the definition. In their work the use of convexity was an important fact.

In this thesis we are concerned in relaxing the convexity assumption used in [CD] to obtain local gradient and second order estimates for the Q_k-flow, and also for Q_k-translators, such that the principal curvatures belong to the cone Γ_{k+1}.

More precisely, if $F(x,t) = (x,u(x,t))$ where $u : \Omega \times [0,T) \to \mathbb{R}$ then Equation (2.1) with

$\gamma = Q_k$, can be written by

$$u_t = Q_k(Du, D^2 u)\sqrt{1 + |Du|^2}, \tag{2.10}$$

and for Q_k-translators, if $F(x) = (x, u(x))$ with $u : \Omega \to \mathbb{R}$, Equation (2.5) can be written by

$$Q_k(Du, D^2 u) = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{2.11}$$

Therefore, to obtain gradient and second-order estimates without the hypothesis of convexity imposed in the graphs, we had to get a better understanding of the term $Q_k(Du, D^2 u)$[10] when the principal curvatures of the graphs belong to the cone Γ_{k+1}.

In fact, thanks to the work realized in [Hol] for the $\sqrt[k]{S_k}$-flow when the principal curvatures of M_t belong to the cone Γ_k, we could obtain the following estimate for the Q_k-flow.

Theorem 2.2.1. *Let $u \in \mathcal{C}^{3,2}(\Omega)$[11] be a solution to Equation (2.10), where $\Omega = B_r(0) \times (0, T] \subset \mathbb{R}^{n-1} \times \mathbb{R}$ with $r > 0$. Assume that the principal curvatures of the graph of u lies in the cone Γ_{k+1} defined in (2.4) for each t. Then, there exist a constant $K = K(k, n) > 0$ independent of u such that for each $t > 0$, it holds*

$$|Du(0,t)| \leq \exp\left(K + \frac{KmT}{rt}\frac{mrT}{t} + \frac{m^2}{t} + \frac{(T+1)r^2}{t} + \frac{Tm^2}{tr^2} + \frac{Tm}{tr}\right),$$

where $m = \sup\limits_{\Omega} u$.

This theorem will be stated again in Thm. 3.0.1, and it will be proven in section 3.2 from page 78 onwards.

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[10]We note that we are abusing with the notation in writing $Q_k(Du, D^2 u)$ instead of $Q_k(\lambda)$. The reason why we choose to wrote in this way is to emphasize the dependence in the variables of the operator Q_k when is evaluated at the principal curvatures of M_t or M.
In addition, we used a convenient way to work with this term, see Section 3.2 in Chapter 3 for more details.

[11]This means that $u(x,t)$ is of class $\mathcal{C}^3$ in $x \in B_r(0)$ and is of class $\mathcal{C}^2$ in $t \in (0, T)$.

Moreover, since we were able to adapt the technique used in the previous theorem for Q_k-translators, we obtained the following estimate of the gradient for that equation.

Theorem 2.2.2. *Let $r > 0$ and $u \in \mathcal{C}^3 \left(B_r(0) \right)$ be a solution (2.11) such that the principal curvatures of graph(u) lie in Γ_{k+1}. Then, there exist a constant $C = C(k, n) > 0$ independent of u such that*

$$|Du(0)| \leq \exp \left(\frac{Cm}{r} + \frac{Cm^2}{r^2} + C \right),$$

where $m = \sup\limits_{B_R(0)} u$.

This theorem will be stated again in Thm. 3.0.2, and it will be proven in section 3.2 from page 90 onwards.

On the other hand, by adapting the proof of the previous theorem, we obtained a Liouville type theorem for Q_k-translators which reads as follows.

Theorem 2.2.3. *There are no solutions $u \in \mathcal{C}^3(\mathbb{R}^n)$ such that*

1. *The principal curvatures of the graph of u lie in Γ_{k+1}.*

2. *$u(x) = o(|x|)$ as $|x| \to \infty$.*

3. *The graph of u is a Q_k-translator in $\mathbb{R}^{n+1}$.*

This theorem will be stated again in Thm. 3.0.3, and it will be proven in section 3.2 from page 94 onwards.

The second order estimates that we obtained below depend on the assumption that the principal curvatures belong to the cone Γ_{k+2} for $k \geq 1$. The reason is that any hypersurface with their principal curvatures lying in Γ_{k+2} with $k \geq 1$, satisfies $|A|^2 \leq H^2$ and $|A|_k^2 \leq \tilde{C}(k,n)H^2$ where $|A|_k^2 = \sum\limits_{i=1}^{n} \frac{\partial Q_k}{\partial \lambda_i} \lambda_i^2$ and $\tilde{C}(k, n)$ is positive constant. Therefore, since it is

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easier to estimate H than $|A|$ or $|A|_k^2$, we apply techniques similar to those in [EH2] to obtain second order estimate for solutions to equations (2.10) and (2.11), respectively.

In addition, as in [EH2], we will assume that the evolution M_t is given by the graph of a function over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$, and there is a non-negative function $h(x,t)$ such that

$$|(\partial_t - \Box_k)h| \leq C(k,n) \text{ and } ||\nabla h||_k^2, |\nabla h|^2 \leq C(k,n)h, \tag{2.12}$$

where $\Box_k f = \dfrac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$ and $||X||_k^2 = \dfrac{\partial Q_k}{\partial h_{ij}} X_i X_j$.

Usually, in Ecker-Huisken estimations, the function $h(x,t) = |x|^2 - \langle x, \vec{w} \rangle + \dfrac{2(n-k)}{k+1} t$ is used. Note that this $h(x,t)$ measure a time depending square distance from the hyperplane orthogonal to $\vec{w}$. See Remark 3.3.2 in Section 3.3 for the details.

Theorem 2.2.4. *Let $R > 0$ such that $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ is a graph over a ball of radius R in the hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$ for all $[0,T)$ and $h(x,t)$ satisfies (2.12). Assume that M_t evolves under the Q_k-flow and the principal curvatures of M_t belong to the cone Γ_{k+2} with $k \geq 1$. Then, for any $t_0 \in (0,T)$ and $\theta \in [0,1)$, the following estimate holds*

$$\sup_{M_t} H^2(x,t) \leq \frac{c(k,n)}{(1-\theta)^2} \left(\frac{1}{t} + \frac{1}{R^2} \right) \sup_{[0,t_0]} \sup_{M_t} v^4,$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$.

This theorem will be stated again in Thm. 3.0.4, and it will be proven in section 3.3 from page 104 onwards.

Analogously, for Q_k-translators, we will assume that the Q_k-translator is a graph over a ball in $\mathbb{R}^n$ and the existence of a non-negative function $h(x)$ with the properties in (2.12)

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but with the change in the assumption $|(\partial_t - \square_k)h| \leq C(k, n)$ by

$$|\square_k h + \langle h, e_{n+1} \rangle\,| \leq C(k, n). \tag{2.13}$$

We note that this assumption is natural for translators since e_{n+1} is the direction of the translation under the flow. Unfortunately, the function $h(x) = |x|^2 - \langle x, e_{n+1} \rangle^2$ do not satisfies (2.13), we refer the reader to Remark 3.3.5 in the section 3.3 for details.

Theorem 2.2.5. *Let $R > 0$ and $h(x)$ be a non-negative function that satisfies* (2.12) *and* (2.13). *Assume that that $M = \{x \in M : h(x) \leq R^2\}$ is a graph over a ball of radius $R > 0$ in the hyperplane orthogonal to e_{n+1}.*

Moreover, assume that M is a Q_k-translator such that the principal curvatures of M belong to the Γ_{k+2} with $k \geq 1$. Then, for any $\theta \in [0, 1)$, the following estimate holds

$$\sup_{B_R} H^2(x) \leq \frac{c(k, n)}{(1 - \theta)^2} \left(1 + \frac{1}{R^2}\right) \sup_{\{h \leq \theta R^2\}} v^4,$$

where $v = \langle \nu, e_{n+1} \rangle^{-1}$.

This theorem will be stated again in Thm. 3.0.5, and it will be proven in section 3.3 from page 110 onwards.

2.2.3 Rotationally symmetric γ-translators in $\mathbb{R}^{n+1}$

In this subsection we want to discuss about the cunstruction of rotationally symmetric γ-translators in $\mathbb{R}^{n+1}$ for some particular curvature functions.

Firstly, a graphical rotationally symmetric γ-translator in $\mathbb{R}^{n+1}$ with $\gamma > 0$ corresponds to the set $\{(x, u(x)) \in \mathbb{R}^{n+1} : |x| < R\}$ where $u(x) = \mathbf{u}(r)$, $\mathbf{u} : [0, R) \to \mathbb{R}$ and $r = |x|$.

Then, Equation (2.5) turns into an ordinary differential equation (ODE) of the form

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2) f\left(\frac{\dot{\mathbf{u}}}{r}\right),$$

where the dot notation means derivatives with respect to r and f is a function which appears by applying the implicit function theorem to the connected components of the level set $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$. We note that in this case, the principal curvatures are two, and since $\gamma(\lambda)$ is increasing in each variable, the implicit function theorem guarantees a local expression of $f(x)$ in each connected component of $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$.

In the case of the H-translators, the authos in [AW] constructed a strictly convex solution defined for $R = \infty$ called the "bowl" soliton (see Fig. 2.3a) which behaves like a paraboild at infinity.

In this thesis, we construct "bowl"-type solution for $\gamma = Q_{n-1}$ and $\gamma = \sqrt[n]{S_n}$ in $\mathbb{R}^{n+1}$. These solutions are strictly convex rotationally symmetric graphs over the $\{x_{n+1} = 0\}$ hyperplane, which are defined in balls or in the whole hyperplane.

Particularly, for $n = 2$, these solutions are explicit and they are given by

$$-\ln(1 - r^2) \quad \text{and} \quad \int_0^r \sqrt{e^{s^2} - 1}\, ds$$

for Q_1 and $\sqrt{S_2}$, respectively. These results will be proven in Thm. 4.2.1 and Thm 4.2.2. In addition, we note that the "bowl"-type $\sqrt{S_2}$-translator is not $\mathcal{C}^2$ at the origin, and the "bowl"-type Q_1-translator is $\mathcal{C}^2$-asymptotic[12] to the cylinder $\mathbb{S}^1 \times \mathbb{R}$.

Theorem 2.2.6. *For $n \geq 3$ there exists a unique[13] smooth strictly convex rotationally symmetric translator in $\mathbb{R}^{n+1}$ for the curvature functions Q_{n-1} and $\sqrt[n]{S_n}$.*

In addition, it holds

---

[12]This means that the principal curvatures $\lambda_1 \leq \ldots \leq \lambda_n$ satisfy $\lambda_1 \to 0$ and $\lambda_i \to 1$ for $i \neq 1$ as $|p| \to \infty$.
[13]The uniqueness is among rotationally symmetric translators.
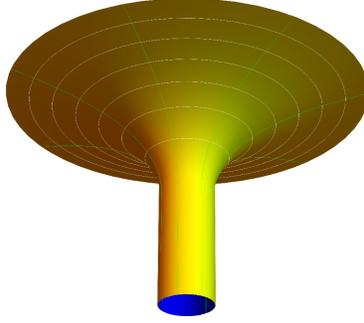
Figure 2.4: A $\sqrt{S_2}$-translator Σ_a in $\mathbb{R}^3$. Image cortesy of Francisco Martín.

1. For $n \geq 3$, the Q_{n-1}-translator is defined in a ball of radius $Q_{n-1}(1, \ldots, 1) = \dfrac{1}{n}$ and is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}\left(\frac{1}{n}\right) \times \mathbb{R}$.

2. For $n \geq 3$, the $\sqrt[n]{S_n}$-translator is entire.

This theorem will be stated again in Thm. 4.0.1, and the existence part for the $\sqrt[n]{S_n}$ curvature function will be proven in Thm. 4.4.2 (page. 132) and for Q_{n-1} in Thm. 4.5.2 (page. 138), and the uniqueness will be proven in Thm. 4.6.3 (page 146).

Furthermore, we will also find a family of rotationally symmetric $\sqrt{S_2}$-translators in $\mathbb{R}^3$ which the generating curve are graphs over the $\{x_1 = 0\}$ plane in $\mathbb{R}^3$ which satisfy $H < 0$ and $K > 0$. The shape of these translators resemble a catenoid minimal surface (see Fig. 2.4).

Theorem 2.2.7. *Let $a \in [0, 1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{ (r_a(z) \cos(\theta), r_a(z) \sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R} \right\},$$

where $r_a(z)$ is implicitly define by the equation

$$z = \int\limits_{1}^{r_a(z)} \sqrt{e^{s^2 - a} - 1}\, ds,\quad \text{with } r_a(0) = 1,$$

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are complete $\sqrt{S_2}$-translators in $\mathbb{R}^3$ which satisfy $H < 0$ and $K > 0$.

This theorem will be stated again in Thm. 4.0.2, and it will be proven in Thm. 4.7.1.

More recently, in [Ren], the author showed the existence of "bowl"-type solutions for a general class of α-homogeneous[14] symmetric curvature functions, without any assumption about the convexity or concavity. In addition he characterized when the solution is defined in a ball or is entire in terms of the curvature function, see the introduction of Chapter 4 below.

2.2.4 Geometric Properties of γ-translators in $\mathbb{R}^{n+1}$

Another important aspect that we studied in this thesis are geometric properties of γ-translators in $\mathbb{R}^{n+1}$. Here we understand geometric properties as the application of PDE theory in the context of differential geometry: tangential principles, non-existence, uniqueness , and convexity estimates. One of the most important examples of such applications is the maximum principles over geometric quantities of a hypersurface.

Recall that H-translators are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1} \rangle} dx^2)$. Therefore, the tangential principle, which allows comparing H-translators according to how they touch each other, is a well-known result from the minimal surface theory. Nevertheless, since γ-translators are not minimal hypersurfaces for $\gamma \neq H$, we develop a tangency principle which reads as follows.

Theorem 2.2.8. *Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ be two embedded connected complete γ-translators such that*

1. $\gamma : \Gamma \to [0, \infty)$ *satisfies properties 1–4 and 7 given in Section 5.1.*

2. Σ_1 *is strictly convex.*

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[14]This means $\gamma(c\lambda) = c^\alpha \gamma(\lambda)$ for every $c > 0$.

3. Σ_2 is convex.

Then,

a) (**Interior tangential principle**) *Assume that there exists an interior point $p \in \Sigma_1 \cap \Sigma_2$ such that the tangent spaces coincide at p. If Σ_1 locally lies at one side of Σ_2, then both hypersurfaces coincide.*

b) (**Boundary tangential principle**) *Assume that the boundaries $\partial \Sigma_i$ lie in the same hyperplane Π and the intersection of Σ_i with Π is transversal. If Σ_1 lies at one side of Σ_2 and there exist $p \in \partial \Sigma_1 \cap \partial \Sigma_2$ such that the tangent spaces to Σ_i and $\partial \Sigma_i$ coincide, then both hypersurfaces coincide.*

This theorem will be stated again in Thm 5.0.3, and it will be proven in section 5.2 from page 170.

We note that without the hypothesis of γ being real analytic, the tangential principles will imply that Σ_1 and Σ_2 would coincide only locally at p. On the other hand, without the completeness hypothesis, the tangential principle would state that $\Sigma_1 \cup \Sigma_2$ is also a γ-translator.

In addition, thanks to the results exposed in [Ren] (see Theorems 1.3-1.4) we also could show a non-existence result for γ-translator such that the "bowl"-type γ-translator is defined in a round ball.

Corollary 2.2.9. *Assume that $\gamma : \Gamma \to [0, \infty)$ satisfies Properties 1-4 and 7. Then, if the "bowl"-type γ-translator is defined in a ball, then there cannot exist a convex entire γ-translators in $\mathbb{R}^{n+1}$.*

This corollary will be stated again in Cor. 5.0.4, and it will be proven in section 5.3 from page 174.

Next, we also apply the method of moving planes of Alexandrov to obtain that any graphical γ-translator which is convex and is defined in a ball inherits the properties of the boundary of its domain.

Theorem 2.2.10. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a complete γ-translator such that*

1. *$\gamma : \Gamma \to [0, \infty)$ satisfies properties 1-4 and 7.*

2. *Σ is a strictly convex graph over a ball $B_r^n(0) \subset \mathbb{R}^n$.*

3. *Σ possesses a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, i.e: the principal curvatures of Σ satisfy*

$$\min \left\{ \lambda_i(p) : i = 1, \ldots, n \right\} = \lambda_1(p) \to 0,$$
$$\forall i \in \{2, \ldots, n\}, \lambda_i(p) \to \frac{1}{r},$$

as $|p| \to \infty$.

Then, Σ is rotationally symmetric with respect to the x_{n+1}-axis.

This theorem will be stated again in Thm. 5.0.5, and it will be proven in section 5.3 from page 177.

Moreover, as a corollary of the above theorem and the uniqueness result exposed in [Ren], we obtain the following uniqueness property.

Corollary 2.2.11. *Let Σ be a γ-translator as in the previous theorem such that the "bowl"-type γ-translator is defined in a ball of radius $\gamma(1, \ldots, 1)$. Then, Σ coincides with the "bowl"-type γ-translator.*

This corollary will be stated again in Cor. 5.0.6, and it will be proven in section 5.4 from page 187.

Finally, the main result of this thesis is a convexity estimate for γ-translators in $\mathbb{R}^{n+1}$. To give a context of why this is the main result of this thesis, we start by announcing some facts related to mean convex H-translators.

Firstly, in [Wan] the author showed that any convex entire H-translator in $\mathbb{R}^3$ must be roationally symmetric. Then, in [SX], the authors showed that any mean convex H-translator in $\mathbb{R}^3$ is convex (see [HIMW3] for a shorter proof). This result had a remarkable implication in the classification of graphical H-translators in $\mathbb{R}^3$, since it follows that the "bowl" soliton is the unique (module vertical translation) entire mean convex translator in $\mathbb{R}^3$.

Moreover, in [Has], the author shows that an α-noncollapsed[15] and convex H-translator must be the bowl soliton. Therefore, the "bowl" soliton is the unique α-noncollapsed H-translator among the mean convex family.

In summary, the convexity estimate showed by Spruck and Xiao is a powerful tool to classify mean convex H-translators in $\mathbb{R}^3$. In addition, we refer the reader to [HIMW2] for the classification of H-translators which are graphs.

Therefore, in the spirit of showing that γ-translators are convex we proved the following convexity estimate.

Theorem 2.2.12. *Let $n \geq 3$, $\alpha, \delta > 0$ and $\Sigma \subset \mathbb{R}^{n+1}$ be a connected complete immersed γ-translator such that*

a) $\gamma : \Gamma \to \mathbb{R}$ *satisfies Properties 1-3 and 5-7.*

b) $\lambda \in \Gamma_{\alpha, \delta} = \{\lambda \in \Gamma : \alpha H \leq (\delta + 1)\gamma\}$ *which is compactly supported[16] in $\Gamma \setminus Cyl_{n-1}$, where*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\},$$

---

[15]Let Σ be a mean convex H-translator which bounds an open region $\Omega \subset \mathbb{R}^{n+1}$, then we say Σ is α-noncollapsed if for all $p \in \Sigma$ there exist an open ball $B \subset \Omega$ of radius $\dfrac{\alpha}{H(p)}$ with $p \in \partial B$.

[16]This means that $\Gamma_{\alpha, \delta} \cap \partial B_1(0)$ is compact in $\Gamma \setminus Cyl_{n-1}$.

and $\{e_i\}$ is the canonical bases of $\mathbb{R}^n$.

c) There exists a constant $\beta \in (0,1)$ such that $\lambda_i + \lambda_j \geq \beta H$, for every $1 \leq i < j \leq n$.

Then, $\lambda_1 \geq H - \alpha\gamma$ in Σ, where $\lambda_1(p) = \min\{\lambda_i(p) : i = 1, \ldots, n\}$.

This theorem will be announced again in Thm. 5.0.7, and it will be proven in section 5.5 from page 191.

We note that the hypothesis c) implies that only the minimum of the principal curvatures can be negative. In addition, we note that $\lambda \in \Gamma_{\alpha,\delta}$ does not necessarily give a fixed sign to $H - \alpha\gamma$ on Σ, but at least it gives a way to control λ_1 from below.

On the other hand, by a maximum principle argument, the hypothesis b) is preserved under the γ-flow, and it can be used to study the limits under suitable parabolic resclaing under the γ-flow. Finally, property b) can be seeing as a non-splinting hypothesis in the sense that the set $\Gamma_{\alpha,\delta}$ is away from having $\lambda_1 = 0$.

The proof of Theorem 2.2.12 was inspired by [SS], where the authors show that a mean convex and uniform 2-convex translating soliton of the MCF is convex. We remark that our proof does not work for translators of the MCF. The main reason is that the cone $\Gamma_{\alpha,\delta}$ needs to be compactly supported in $\Gamma_1 \setminus C_{n-1}$, but for $\gamma = H$, $\Gamma_{\alpha,\delta} = \Gamma_1$.

On the other hand, the proof given in [SS] cannot be directly adapted to a concave curvature function γ, since the authors use a concave approximation of λ_1 that cannot be used for concave curvature functions γ.

Remark 2.2.2. A similar estimate was proved in [Lyn2] for a family curvature functions of the form

$$\gamma = (1-c)H - c\left(\sum_{1=i_1<\ldots<i_k=n} \frac{1}{\lambda_{i_1} + \ldots + \lambda_{i_k}}\right)^{-1}, \ c \in (0,1).$$

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In contrast with our result, his estimate is in the compact parabolic context[17].

As a consequence, as an application of the convexity estimate, we were able to show that γ-translators are asymptotically convex.

Corollary 2.2.13. *Let* $\Sigma \subset \mathbb{R}^{n+1}$ *be a* γ-*translator where* $\gamma : \Gamma \to \mathbb{R}$ *satisfies the hypothesis of Theorem 2.2.12. Then,* Σ *is asymptotically convex.*

This corollary will be stated again in Cor. 5.0.8, and it will be proven in section 5.5 from page 197 onwards.

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[17]The author showed the similar estimate for compact initial data that evolve under the γ-flow.

Chapter 3

Estimates for Q_k-flow and Q_k-translators

In this chapter, we are interested in a particular class of degenerated geometric extrinsic evolution equations, where the speed of the flow is given by a 1-homogeneous function in the principal curvatures of the given initial hypersurface and along the flow.

More precisely, given a manifold M^n and an immersion $F_0 : M^n \to \mathbb{R}^{n+1}$, we say that hypersurface $M_0 = F_0(M)$ evolves under the Q_k-flow if there exists a 1-parameter family of immersions $F : M \times [0,T) \to \mathbb{R}^{n+1}$ such that

$$\begin{cases} \left\langle \dfrac{\partial F}{\partial t}, \nu \right\rangle = Q_k(\lambda), \text{ in } M \times (0,T), \\ F(\cdot, 0) = F_0(\cdot), \end{cases} \tag{3.1}$$

where ν is inward[1] unit normal vector of $M_t = F(M,t)$ in $\mathbb{R}^{n+1}$,

$$Q_k = \frac{S_{k+1}}{S_k},$$

---

[1]The orientation imposed on the unit normal vector is so that the principal curvatures are defined in an open symmetric cone such that it contains the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$.

and $S_k(\lambda)$ denotes the elementary symmetric polynomial in n variables of degree k evaluated at the principal curvatures of $M_t = F(M, t)$, i.e:

$$S_{k+1}(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots i_{k+1} \leq n} \lambda_{i_1} \ldots \lambda_{i_{k+1}}. \tag{3.2}$$

Recall that the principal curvatures of a hypersurface M^n in $\mathbb{R}^{n+1}$ are the eigenvalues of the Weingarten map $h_j^i = g^{ik} h_{kj}$, where g^{ij} denotes the coefficients of the inverse of the metric tensor g_{ij}, and h_{ij} denotes the coefficients of the second fundamental form of M, respectively. The coefficients of the second fundamental form correspond to the tangent components of the covariant derivative of the unit normal map (modulo orientation), i.e.: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ where $\{e_i\}$ denotes an orthonormal basis of $T_p M$.

In addition, by definition $S_0 = 1$ and $S_i = 0$ for $i > n$. We remark that $Q_0 = H$ is the scalar mean curvature, which has been widely studied in this century, see [CMP] for an introductory survey.

From the PDE perspective, Equation 3.1 is geometrically equivalent[2] to a locally uniformly parabolic equation when the principal curvature vector λ lies in the cone

$$\Gamma_{k+1} = \{\lambda \in \mathbb{R}^n : S_l(\lambda) > 0, \ l = 0, \ldots, k + 1\}.$$

In particular, we have $\Gamma_0 = \mathbb{R}^n$. Furthermore, $\lambda \in \Gamma_1$ if, and only if, M is strictly mean convex. Finally, $\lambda \in \Gamma_n$ if, and only if, M is strictly convex.

The property of being locally uniformly parabolic for the Q_k-flow, when $\lambda \in \Gamma_{k+1}$, motivates the study of the Q_k-flow in papers [And1], [Die] and [CD], which we describe in what follows. In [And1], the author shows classical results about existence, uniqueness and collapsing for closed strictly convex initial hypersurfaces. In [Die], the author uses a weaker

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[2]This means that Equation (3.1) could have many different solutions by adding tangential components in the hypersurface parameterization, but from a geometric point of view these solutions are reparametrizations of each other.

condition on the convexity of the initial hypersurface to show similar results to [And1]. Namely, the author showed that one can start the Q_k-flow when $\lambda \in \Gamma_k \cap \{\lambda \geq 0\}$ instead of being strictly convex. This allows the function Q_k to vanish at some points of M_0. Finally, in [CD], the authors show an existence result of non-compact complete convex solutions in the spirit of [EH1] for the H-flow under the same assumptions of [Die].

In this chapter we focus on eternal solutions[3] of the Q_k-flow which evolve by translation on a fixed constant unitary direction, usually known as translating solitons, or Q_k-translators for short. These type of solutions are immersions of the form

$$F(x,t) = F_0(x) + tv,$$

where $v \in \mathbb{S}^n$ is a fixed direction. Since we are interested in the normal part, Q_k-translators can also be seen as hypersurfaces $M_0 \subset \mathbb{R}^{n+1}$ which satisfy the fully non-linear degenerated equation

$$Q_k(\lambda) = \langle \nu, v \rangle. \tag{3.3}$$

These type of solitons have been studied by many authors in the case $k = 0$, we recommend the reader to see the survey [HIMW3] as a guide in this area.

The interest on H-translators arises mainly due to two reasons: they appear as a model for type II singularities of the Mean Curvature Flow, and they are also minimal surfaces in weighted Euclidean spaces. See the Introduction 2 of this thesis for a brief review about this topic.

Finally, the work of this chapter is inspired by the seminal work of Ecker and Huisken [EH1] and [EH2], in obtaining gradient and second order local estimates for the study of the evolution of Lipschitz entire graphs under the H-flow. We obtain several results for the

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[3]Solutions which exist for all times.

Q_k-flow and Q_k-translators in $\mathbb{R}^{n+1}$ that we summarize below.

Indeed, by adapting the gradient estimates given in [SUW] and [Hol] for the Q_k functions, we obtained local gradient estimates for these equations.

Theorem 3.0.1. *Let $r > 0$ and $u \in \mathcal{C}^{3,2}(B_r(0) \times (0,T])$[4] such that the graph of u evolves under the Q_k-flow and the principal curvatures of the graph of u lie in the cone Γ_{k+1} for each t. Then, there exist a positive constant $K = K(k,n)$ independent of u such that for each $t > 0$, it holds*

$$|Du(0,t)| \le \exp\left(K + \frac{KmT}{rt} + \frac{m^2}{t} + \frac{(T+1)r^2}{t} + \frac{Tm^2}{tr^2} + \frac{Tm}{tr}\right), \qquad (3.4)$$

where $m = \sup\limits_{B_r(0) \times \{t\}} u$.

This theorem will be proven in 3.2 from page 78 onwards.

Theorem 3.0.2. *Let $r > 0$ and $u \in \mathcal{C}^3(B_r(0))$ such that the graph(u) is a Q_k-translator and the principal curvatures lie in the cone Γ_{k+1}. Then, there exist a positive constant $C = C(k,n)$ independent of u such that*

$$|Du(0)| \le \exp\left(C + \frac{Cm}{r} + \frac{Cm^2}{r^2}\right),$$

where $m = \sup\limits_{B(0,r)} u$.

This theorem will be proven in 3.2 from page 90 onwards.

Next, by adapting the proof of the gradient estimate, we also obtained a Liouville's type result for Q_k-translators which reads as follow.

Theorem 3.0.3. *There are no functions $u \in \mathcal{C}^3(\mathbb{R}^n)$ such that*

1. The principal curvatures of the graph of u lie in Γ_{k+1}.

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[4]This means that $u(x,t)$ is of class $\mathcal{C}^3$ in $x \in B_r(0)$ and is of class $\mathcal{C}^2$ in $t \in (0,T)$.

2. $u(x) = o(|x|)$ as $|x| \to \infty$.

3. The graph of u is a Q_k-translator in $\mathbb{R}^{n+1}$.

This theorem will be proven in 3.2 from page 94 onwards.

Moreover, by assuming $k \geq 1$, any hypersurface M_0 whose principal curvatures lies in Γ_{k+2}, satisfies $|A|^2 \leq H^2$ and $|A|_k^2 \leq \tilde{C}(k,n)H^2$ where $|A|_k^2 = \sum_{k=1}^{n} \dfrac{\partial Q_k}{\partial \lambda_i} \lambda_i^2$ and $C(k,n)$ is a positive constant defined in Equation (3.58) below.

Therefore, by applying similar techniques as in [EH2], we obtain local second order estimates for solutions of Q_k-flows and Q_k-translators in $\mathbb{R}^{n+1}$, respectively.

In the statement of the following theorems we will assume that the evolution M_t is given by the graph of a function defined over a ball in a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$, and also the existence of a non-negative function $h(x,t)$ such that $M_t = \{h(x,t) \leq R^2\}$ and the following estimates holds

$$|(\partial_t - \square_k)h| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h, \tag{3.5}$$

where $\square_k f = \dfrac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$ and $|X|_k = \dfrac{\partial Q_k}{\partial h_{ij}} X_i X_j$. We will see below that the function $h(x,t)$ can be taken as a time depended distance between the evolution M_t and the hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$.

On the other hand, for Q_k-translators, we will assume that the translator is a graph over a ball in a hyperplane orthogonal to e_{n+1}, and also the existence of a non-negative function $h(x)$ such that

$$|\square_k h + \langle \nabla h, e_{n+1} \rangle| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h. \tag{3.6}$$

In addition, as we will note later in this chapter, when the Q_k-flow is convex, the function

$$h(x,t) = |x|^2 - \langle x, \vec{w} \rangle^2 - \frac{2(n-k)}{k+1} t,$$

which measures the time dependent distance from the tangent space orthogonal to the vector $\vec{w} \in \mathbb{S}^n$, satisfies the assumptions in (3.5). Meanwhile, for Q_k-translators, we only work with the abstract function $h(x)$ since, as we will show later, $h(x) = |x|^2 - \langle x, e_{n+1} \rangle^2$ does not satisfy Equation (3.6).

Theorem 3.0.4. *Let $R > 0$ and $\vec{w} \in \mathbb{S}^n$ be such that there exist a non-negative function $h(x,t)$ which satisfies Equation (3.5) and $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ is the graph over a ball of radius R in the hyperplane orthogonal to $\vec{w}$ for all $[0,T)$.*
Then, if the principal curvatures of M_t belong to the cone Γ_{k+2} with $k \geq 1$, for any $t_0 \in (0,T)$ and $\theta \in [0,1)$, the following estimate holds

$$\sup_{M_t} H^2(x,t) \leq \frac{c(k,n)}{(1-\theta)^2} \left(\frac{1}{t} + \frac{1}{R^2} \right) \sup_{[0,t_0]} \sup_{M_t} v^4,$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$.

This theorem will be proven in section 3.0.4 from page 104 onwards.

Theorem 3.0.5. *Let $R > 0$ be such that there exist a non-negative function $h(x)$ which satisfies Equation (3.6) and $\{x \in M : h(x) \leq R^2\}$ is a graph over a ball of radius R in a hyperplane orthogonal to e_{n+1}.*
Then, if M is a Q_k-translator and the principal curvatures of M belong to the cone Γ_{k+2} with $k \geq 1$ for any $\theta \in [0,1)$, the following estimate holds

$$H^2(x) \leq \frac{c(k,n)}{(1-\theta)^2} \left(1 + \frac{1}{R^2} \right) \sup_{\{h \leq \theta R^2\}} v^4,$$

where $v = \langle \nu, e_{n+1} \rangle^{-1}$.

This theorem will be proven in section 3.0.4 from page 110 onwards.

All of these results are in the spirit of showing existence of entire graph solutions to the Q_k-flow and Q_k-translators in $\mathbb{R}^{n+1}$. Until now, the non-trivial examples for these geometric problems are of "bowl"-type, see [Ren] or Chapter 4 below for more details. These solutions are called of "bowl type" because they resemble the "bowl" soliton of the H-flow. Indeed, they are complete strictly convex rotationally symmetric graphs defined in $\mathbb{R}^n$ or in a ball of radius $Q_k(1, \ldots, 1)$, and are asymptotic to paraboloids or round cylinders. Consequently, these estimates could be applied as in [CD] to find more examples of graph solutions of the Q_k-flow.

On the other hand, these techniques can be applied to other type of 1-homogeneous curvature functions. More precisely, for any $\gamma : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ that satisfies

(a) $\Gamma \subset \mathbb{R}^n$ is a symmetric open convex cone and γ is symmetric[5], smooth and positive in Γ.

(b) γ is strictly increasing in each variable, i.e: $\dfrac{\partial \gamma}{\partial \lambda_i} > 0$ in Γ for every i.

(c) γ is 1-homogeneous, i.e: for every $c > 0$, $\gamma(c\lambda) = c\gamma(\lambda)$ in Γ.

The organization of this chapter goes as follows: In Section 3.1 we give some properties of the S_k and Q_k functions that we will use along the chapter. In Section 3.2 we develop the gradient estimates for graph solutions to the Q_k-flow and the Q_k-translator equation and the proof of Theorem (3.0.3). In Section 3.3 we prove second order estimates for both settings.

3.1 Properties of Q_k functions

In this section we list some properties of the functions S_k and Q_k in $\mathbb{R}^n$.

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[5]Invariant under the n-permutation group in its variables.

Definition 3.1.1. The Gärdin cones defined by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \; S_i(\lambda) > 0, \; \text{for } i = 1, \ldots, k\},$$

where $S_0 := 1$, $S_k := 0$ for $k > n$ and S_k as given in formula (3.2) for $k = 1, \ldots, n$, are symmetric open convex cones in $\mathbb{R}^n$.

Example 1. The most common examples of Q_k functions coresspond to

$$Q_0 = \lambda_1 + \ldots + \lambda_n \text{ and } Q_{n-1} = \left(\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n}\right)^{-1},$$

when we consider these expressions in (3.1) we have the mean and the inverse mean harmonic curvature flows, respectively.

Along this chapter we will denote by $S_{k,i}(\lambda)$ and $S_{k,i;j}(\lambda)$ the sum of all terms of $S_k(\lambda)$ which do not contain the factor λ_i and the factors λ_i and λ_j, respectively.

Lemma 3.1.1. *For any $k \in \{0, \ldots, n\}$, $i \in \{1, \ldots, n\}$ and $\lambda \in \mathbb{R}^n$ we have*

$$\frac{\partial S_{k+1}}{\partial \lambda_i} = S_{k,i}, \tag{3.7}$$

$$S_{k+1} = S_{k+1,i} + \lambda_i S_{k,i}, \tag{3.8}$$

$$\sum_{i=1}^{n} S_{k,i} = (n-k)S_k, \tag{3.9}$$

$$\sum_{i=1}^{n} \lambda_i S_{k,i} = (k+1)S_{k+1}, \tag{3.10}$$

$$\sum_{i=1}^{n} \lambda_i^2 S_{k,i} = S_1 S_{k+1} - (k+2)S_{k+2}, \tag{3.11}$$

$$\sum_{i=1}^{n} S_{k,i;j} = (n-k-1)S_{k,j}. \tag{3.12}$$

Proof. For a proof of (3.7)-(3.11) we refer to [HS1]. Note that Equation (3.12) follows by

taking derivatives with respect λ_j on (3.9). $\qquad\qquad\qquad\qquad\qquad\qquad$ □

The next lemma contains similar formulas for the Q_k rational functions.

Lemma 3.1.2. *For any $k \in \{0, \ldots, n\}$, $i \in \{1, \ldots, n\}$ and $\mu \in \mathbb{R}^n$ we have*

$$\sum_{i=1}^{n} \frac{\partial Q_k}{\partial \lambda_i} = (n-k) - (n-k+1)\frac{Q_k}{Q_{k-1}}, \tag{3.13}$$

$$\sum_{i=1}^{n} \lambda_i \frac{\partial Q_k}{\partial \lambda_i} = Q_k, \tag{3.14}$$

$$\sum_{i=1}^{n} \lambda_i^2 \frac{\partial Q_k}{\partial \lambda_i} = (k+1)Q_k^2 - (k+2)Q_{k+1}Q_k. \tag{3.15}$$

Proof. For Equation (3.13) we have,

$$\frac{\partial Q_k}{\partial \lambda_i} = \frac{1}{S_k^2} \left(S_k S_{k,i} - S_{k+1} S_{k-1,i} \right).$$

Then, by (3.9), it follows

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i} = \frac{1}{S_k^2} \left((n-k)S_k^2 - (n-k+1)S_{k+1}S_{k-1} \right).$$

For Equation (3.14), we use (3.10) to show that

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i} \lambda_i = \frac{1}{S_k^2} \left((k+1)S_{k+1}S_k - kS_{k+1}S_k \right).$$

Finally, for Equation (3.15) we use (3.11) to see that

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i} \lambda_i^2 = \frac{1}{S_k^2} \left((k+1)S_{k+1}^2 - (k+2)S_{k+2}S_k \right).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

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In addition, we have a Newton's inequality for the polynomials S_k:

Lemma 3.1.3. *For any $k \in \{1, \ldots, n-1\}$ and $\lambda \in \mathbb{R}^n$ we have*

$$(k+1)\left(n-(k-1)\right) S_{k-1}S_{k+1} \le k(n-k)S_k^2.$$

Equality holds if, and only if, all λ_i are equal.

Proof. For a proof we refer to [HLP]. $\square$

In addition, by iterating Lemma 3.1.3, it can be shown a Newton Inequality for Q_k:

Corollary 3.1.4. *For any $l, k \in \{0, \ldots, n-1\}$ such that $l \le k$ and $\lambda \in \Gamma_{k+1}$, we have*

$$Q_k \le \frac{(l+1)(n-k)}{(k+1)(n-l)}Q_l. \tag{3.16}$$

Proof. First we note that if $\lambda \in \Gamma_{k+1}$, then it follows

$$Q_k \le \frac{k(n-k)}{(k+1)(n-k+1)}Q_{k-1}.$$

Then, by iterating the above formula the result easily follows. $\square$

3.2 Gradient Estimates and applications

In this section we derive local gradient estimates for solutions to equations (3.1) and (3.3). We first note that a smooth hypersurface in $\mathbb{R}^{n+1}$ can be locally written, as a graph, which after a rotation is of the form $(x, u(x,t))$. In this setting, Equation (3.1) becomes

$$u_t = Q_k(Du, D^2u)\sqrt{1+|Du|^2}, \tag{3.17}$$

and Equation (3.3) becomes

$$Q_k(Du, D^2u) = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{3.18}$$

It is important to mention that we are abusing with the notation on the term $Q_k(Du, D^2u)$, since Q_k is a function in n-variables. In fact, we oftently will write

$$Q_k(\lambda) = Q_k(\lambda(g^{-1}II)) = Q_k(Du, D^2u)$$

when the hypersurface is given by the graph of a function. Note that $\lambda(g^{-1}II)$ denotes the principal curvature (or eigenvalues) vector of the shape operator $g^{-1}II$, where g^{-1} is the inverse of the metric of the hypersurface, and II is the second fundamental form of the hypersurface in $\mathbb{R}^{n+1}$.

Moreover, we will use the Einstein summation convention of upper and lower index to write any tensor equation. In particular, in graphical local coordinates, the shape operator is written by $h^i_j = g^{ik}h_{kj}$, where

$$h_{ij} = \frac{u_{ij}}{w}, \ g^{ij} = \delta_{ij} - \frac{u_i u_j}{w^2},$$

and $w = \sqrt{1 + |Du|^2}$[6], are the coefficients of the second fundamental form and the inverse of the metric tensor $g_{ij} = \delta_{ij} + u_i u_j$, respectively. Here the subindex denotes derivatives with respect to the corresponding variable.

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[6]Note that for graphs we have $w = \left(\frac{1}{\sqrt{1 + |Du|^2}}\right)^{-1} = \langle \nu, e_{n+1} \rangle^{-1} = v^{-1}$, where v is defined in Theorem 3.0.5.

However, the principal curvatures can also be obtained as the eigenvalues of the matrix

$$A_{ij} = w^{-1}\left(u_{ij} - \frac{u_i u_k u_{kj}}{w(1+w)} - \frac{u_j u_k u_{ki}}{w(1+w)} + \frac{u_i u_j u_k u_l u_{kl}}{w^2(1+w)^2}\right). \tag{3.19}$$

In fact, $A_{ij} = \sqrt{g^{ik}} h_{kl} \sqrt{g^{lj}}$, where $\sqrt{g^{ij}}$ is the positive square root of g^{ij}. Then, since the determinant is a multiplicative function, it follows that

$$\det\left(h^i_j - \lambda Id\right) = 0 \Leftrightarrow \det\left(A_{ij} - \lambda Id\right) = 0,$$

which implies that the eigenvalues of h^i_j and A_{ij} coincide.

Finally, the reason to consider the matrix A_{ij} instead of h^i_j is only technical. The A_{ij} matrix is symmetric, and therefore, facilitates all calculations related to it. We refer the reader to Section 1 of [CNS1] for more details related to the matrix A_{ij}.

The method that we employ to obtain the estimate in Theorems 3.0.1 and 3.0.2 was first used in [CNS2] for deriving gradient estimates for Curvature PDEs with Dirichlet boundary condition. In [Hol], the author also used this method for deriving gradient estimates to the $\sqrt[k]{S_k}$-flow in $\mathbb{R}^{n+1}$.

Remark 3.2.1. The functions S_k can be evaluated in any symmetric matrix B by the relation

$$\det(tI + B) = t^n + S_1(B)t^{n-1} + \ldots + S_n(B).$$

Indeed, we choose $\alpha \subset \{1, \ldots, n\}$ and denote $|\alpha|$ its cardinality. Then, by setting $B[\alpha]$ to

be the principal submatrix of B[7], it follows that

$$S_k(B) = \sum_{\alpha \subset \{1,\ldots,n\}, |\alpha|=k} \det(B[\alpha]).$$

As an example to see this relation we choose

$$S_1(B) = \sum_{\alpha \subset \{1,\ldots,n\}, |\alpha|=1} \det(B[\alpha]) = \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(B),$$

$$S_n(B) = \sum_{\alpha \subset \{1,\ldots,n\}, |\alpha|=n} \det(B[\alpha]) = \det(B).$$

Recall that we are considering the symmetric elementary polynomial without being normalized. It is relevant to remark that the properties stated in Section 3.1 are valid for the eigenvalues of a given matrix (or equivalently, for diagonal matrices).

The following lemma was inspired by [Sae] and it is due to the shape of matrix $A := (A_{ij})$ after a change of coordinates for which $S_k(A)$ is easier to compute.

Remark 3.2.2. We will also use the following notation along this section: $S_{l,ij} = \dfrac{\partial S_l}{\partial u_{ij}}$, $\tilde{\lambda} = \mathrm{diag}(A_{22}, \ldots, A_{nn}) = A|1$, $S_k(B|i)$ or $S_k(B|ij)$ means that the i-th row and i-th column resp. i,j-th row and i,j-th column are omitted from a matrix B.

Lemma 3.2.1. *If the matrix A has the form,*

$$A_{11} = \frac{u_{11}}{w^3} < 0, \ A_{1j} = \frac{u_{1j}}{w^2} = A_{j1}, \ A_{jj} = \frac{u_{jj}}{w} \ \text{and} \ A_{ij} = A_{ji} = 0 \ (\text{for } 2 \le i \ne j \le n),$$

where the principal curvatures of the graph of u lie in Γ_{k+1}, the following equations hold:

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[7]The principal submatrix $B[\alpha]$ of a symmetric matrix B is given by the partition

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$$

where B_{11} is of order $|\alpha|$ is the principal submatrix and B_{22} is of order $n - |\alpha|$.

(a) $\sum\limits_{i=1}^{n} \dfrac{\partial Q_k}{\partial A_{ij}} A_{ij} = Q_k.$

(b) $S_l(\tilde{\lambda}) > 0, \dfrac{\partial S_l}{\partial A_{jj}}(\tilde{\lambda}) = S_{l-1}(\tilde{\lambda}|j) > 0$ *for* $j = 2, \ldots n$ *and* $l = 1, \ldots, k+1.$

(c) $S_l(\tilde{\lambda}) = S_l(\tilde{\lambda}|j) + A_{jj} S_{l-1}(\tilde{\lambda}|j).$

(d) $S_l(\tilde{\lambda}) > S_l.$

(e) $S_l(A) = S_l(\tilde{\lambda}) + A_{11} S_{l-1}(\tilde{\lambda}) - \sum\limits_{j>1} A_{1j}^2 S_{l-2}(\tilde{\lambda}|j).$

(f) $S_l(A) = S_l(A|i) + A_{ii} S_{l-1}(A|i) - A_{1i}^2 S_{l-2}(A|1i).$

(g) $\dfrac{\partial Q_k}{\partial A_{11}} \geq \dfrac{n}{(k+1)(n-k)}, \dfrac{\partial Q_k}{\partial A_{ii}} > 0$ *for* $i > 1.$

(h) $\dfrac{\partial Q_k}{\partial A_{11}} \geq \dfrac{n}{(n-k)^2(k+1)} \sum\limits_{i>1} \dfrac{\partial Q_k}{\partial A_{ii}}.$

Proof. We denote $Q_{k,ij} = \dfrac{\partial Q_k}{\partial u_{ij}}$. Then under this setting we have

$$Q_{k,ij} = \left(\dfrac{S_{k+1}}{S_k}\right)_{ij} = \dfrac{S_{k+1,ij} S_k - S_{k+1} S_{k,ij}}{S_k^2}.$$

Then, by combining the chain rule with the derivatives of the coefficients of A_{ij} given in (3.19), we have

$$Q_{k,11} = \dfrac{\partial Q_k}{\partial A_{11}} \dfrac{\partial A_{11}}{\partial u_{11}} = \dfrac{\partial Q_k}{\partial A_{11}} \dfrac{1}{w^3},$$
$$Q_{k,1i} = \dfrac{\partial Q_k}{\partial A_{1i}} \dfrac{\partial A_{1i}}{\partial u_{1i}} = \dfrac{\partial Q_k}{\partial A_{1i}} \dfrac{1}{w^2} = Q_{k,i1}, \; i \neq 1$$
$$Q_{k,ii} = \dfrac{\partial Q_k}{\partial A_{ii}} \dfrac{\partial A_{ii}}{\partial u_{ii}} = \dfrac{\partial Q_k}{\partial A_{ii}} \dfrac{1}{w}$$
$$Q_{k,ij} = \dfrac{\partial Q_k}{\partial A_{ij}} \dfrac{\partial A_{ij}}{\partial u_{ij}} = \dfrac{\partial Q_k}{\partial A_{ij}} \dfrac{1}{w} = 0 = Q_{k,ji} \text{ (for } 2 \leq i \neq j \leq n).$$

Firstly, we note that Equation (a) follows from Equation (3.14) together with the formula
$\sum\limits_{i,j} \dfrac{\partial S_k}{\partial A_{ij}} A_{ij} = (k+1)S_{k+1}.$

Furthermore, since $A_{11} < 0$, it follows that $S_l(\tilde{\lambda}) > 0$ for $l = 1$. In particular, this fact implies that $\frac{\partial S_l}{\partial A_{jj}}(\tilde{\lambda}) = S_{l-1}(\tilde{\lambda}|j) > 0$.

Then, by iterating this process, we can get the same result for $j = 2, \ldots n$ and $l = 1, \ldots, k+1$.

Consequently, by the shape of the matrix A_{ij} and the properties given in Lemma 3.1.1, Equations (c)-(f) hold.

Secondly, we analyze the diagonal terms $\frac{\partial Q_k}{\partial A_{ii}}$. We start with $i = 1$,

$$\frac{\partial Q_k}{\partial A_{11}} = \frac{1}{S_k^2}\left(\frac{\partial S_{k+1}}{\partial A_{11}}S_k - S_{k+1}\frac{\partial S_k}{\partial A_{11}}\right) = \frac{1}{S_k^2}\left(S_k(\tilde{\lambda})S_k - S_{k+1}S_{k-1}(\tilde{\lambda})\right).$$

Then, by combining the above equation with Equation (c), we obtain

$$\frac{\partial Q_k}{\partial A_{11}} = \frac{1}{S_k^2}\left(S_k(\tilde{\lambda})^2 - S_{k-1}(\tilde{\lambda})S_{k+1}(\tilde{\lambda}) + \sum_{j>1} A_{1j}^2(S_{k-1}(\tilde{\lambda}|j)^2 - S_k(\tilde{\lambda}|j)S_{k-2}(\tilde{\lambda}|j))\right).$$

Note that by Lemma 3.1.3, the term in the sum is non-negative, and therefore we drop it from the inequality.

Consequently, by Equation (d), it follows that

$$\frac{\partial Q_k}{\partial A_{11}} \geq \frac{n}{(k+1)(n-k)}\left(\frac{S_k(\tilde{\lambda})}{S_k}\right)^2 \geq \frac{n}{(k+1)(n-k)}.$$

Next, we are going to show that $\frac{\partial Q_k}{\partial A_{ii}} > 0$ for $i > 1$. Indeed, by differentiating Equation (f) with respect to A_{ii} with $i > 1$, we have

$$\begin{aligned}
\frac{\partial Q_k}{\partial A_{ii}} &= \frac{1}{S_k^2}\left(S_k\frac{\partial S_{k+1}}{\partial A_{ii}} - S_{k+1}\frac{\partial S_k}{\partial A_{ii}}\right)\\
&= \frac{1}{S_k^2}\left(S_kS_k(A|i) - S_{k+1}S_{k-1}(A|i)\right)\\
&= \frac{1}{S_k^2}\left(S_k(A|i)^2 - S_{k+1}(A|i)S_{k-1}(A|i)\right) + \frac{A_{1i}^2}{S_k^2}\left[S_{k-1}(A|i)S_{k-1}(A|1i) - S_k(A|i)S_{k-2}(A|1i)\right].
\end{aligned}$$

Now, we use Equation (e) in the second term from the above line to obtain

$$\frac{\partial Q_k}{\partial A_{ii}} = \frac{1}{S_k^2} \left(S_k(A|i)^2 - S_{k+1}(A|i)S_{k-1}(A|i) \right) + \frac{A_{1i}^2}{S_k^2} \left(S_{k-1}(\tilde{\lambda}|i)^2 - S_k(\tilde{\lambda}|i)S_{k-2}(\tilde{\lambda}|i) \right)$$
$$+ \frac{A_{1i}^2}{S_k^2} \sum_{j>1, j\neq i} A_{1j}^2 \left[S_{k-2}^2(\tilde{\lambda}|ij) - S_{k-3}(\tilde{\lambda}|ij)S_{k-1}(\tilde{\lambda}|ij) \right].$$

Finally, since Lemma 3.1.3 works for every symmetric matrix evaluated in the S_k polynomials, it follows that the three terms in the right hand side of the above line are positive, which means that $\dfrac{\partial Q_k}{\partial A_{ii}} > 0$.

Next, we study the sum the diagonal terms $\dfrac{\partial Q_k}{\partial A_{ii}}$.

Firstly, we write

$$\frac{\partial Q_k}{\partial A_{11}} = \frac{S_k(\tilde{\lambda})}{S_k} + \frac{Q_k}{S_k}S_{k-1}(\tilde{\lambda}),$$
$$\frac{\partial Q_k}{\partial A_{ii}} = \frac{S_k(A|i)}{S_k} + \frac{Q_k}{S_k}S_{k-1}(A|i)$$
$$= \frac{1}{S_k} \left(S_k(\tilde{\lambda}|i) + A_{11}S_{k-1}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-2}(\tilde{\lambda}|ij) \right)$$
$$+ \frac{Q_k}{S_k} \left(S_{k-1}(\tilde{\lambda}|i) + A_{11}S_{k-2}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-3}(\tilde{\lambda}|ij) \right).$$

Then, it follows that

$$\sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} = \frac{1}{S_k} \left(S_k(\tilde{\lambda}) + \sum_{i>1} \left[S_k(\tilde{\lambda}|i) + A_{11}S_{k-1}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-2}(\tilde{\lambda}|ij) \right] \right) \quad (3.20)$$
$$- \frac{Q_k}{S_k} \left(S_{k-1}(\tilde{\lambda}) + \sum_{i>1} \left[S_{k-1}(\tilde{\lambda}|i) + A_{11}S_{k-2}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-3}(\tilde{\lambda}|ij) \right] \right).$$

Next, by applying Equation (3.9) on the terms $\sum\limits_{i>1} S_l(\tilde{\lambda}|i)$ with $l = k - 2, k - 1$, we have

$$\sum_{i>1} S_l(\tilde{\lambda}|i) = (n - 1 - l)S_l(\tilde{\lambda}). \tag{3.21}$$

Moreover, applying Equation (3.12) on the terms $\sum\limits_{i>1} S_l(\tilde{\lambda}|ij)$ with $l = k - 3, k - 2$, we have

$$\sum_{i>1} S_l(\tilde{\lambda}|ij) = (n - 1 - l - 1)S_l(\tilde{\lambda}|j). \tag{3.22}$$

Consequently, by replacing Equations (3.21) and (3.22) in (3.20) we get

$$\sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} = \frac{(n - k)}{S_k}\left(S_k(\tilde{\lambda}) + A_{11}S_{k-1}(\tilde{\lambda}) - \sum_{j>1} A_{1j}^2 S_{k-2}(\tilde{\lambda}|j) \right)$$
$$- (n - k + 1)\frac{Q_k}{S_k}\left(S_{k-1}(\tilde{\lambda}) + A_{11}S_{k-2}(\tilde{\lambda}) - \sum_{j>1} A_{1j}^2 S_{k-3}(\tilde{\lambda}|j) \right).$$

Finally, by applying Equation (e) to the terms in parenthesis of the above line we have

$$\sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} = (n - k) - (n - k + 1)\frac{Q_k}{Q_{k-1}} \leq (n - k),$$

where in the last inequality we use that $\dfrac{Q_k}{Q_{k-1}} > 0$. Then, by combining this equation with Equation (g), Equation (h) holds. $\qquad\square$

Now we prove the gradient estimate for the Q_k-flow.

Proof of Theorem 3.0.1. The proof is very similar to the given for Theorem 1.1 in [Hol]. For this reason we will use the same notation.

Moreover, by a translation along the x_{n+1}-axis, we may assume that $u > 0$. We also take the matrix A given in (3.19), and by a slight abuse of notation we set, $Q_k(A) = \dfrac{S_{k+1}(A)}{S_k(A)}$, where each function is evaluated in the eigenvalues of A.

Consequently, Equation (3.17) becomes

$$Q_k(A) = \frac{u_t}{w}, \text{ where } w = \sqrt{1 + |Du|^2}. \tag{3.23}$$

Now we define the test function $G : \Omega \times \mathbb{S}^{n-1} \to \mathbb{R}$ given by

$$G(x, t, \xi) = t\rho(x)\varphi(u)\ln(u_\xi), \tag{3.24}$$

here ξ denotes the direction derivative vector, $\rho(x) = 1 - \frac{|x|^2}{r^2}$ and $\varphi(u) = 1 + \frac{u}{m}$, where $m = \sup_\Omega u$. Furthermore, since ρ vanishes at $\partial B_r(0)$, we may assume that G attains its maximum at some point (x_0, t_0) with $t_0 > 0$ and $|x_0| < r$. Moreover, after a rotation, we may choose $\xi = \varepsilon_1$ where ε_i denotes the canonical euclidean basis of $\mathbb{R}^{n+1}$.

Next, since $G(x, t, \xi)$ reaches a maximum at (x_0, t_0) we have the following equations

$$0 = (\ln G)_i = \frac{\rho_i}{\rho} + \frac{\varphi'}{\varphi}u_i + \frac{u_{1i}}{u_1 \ln u_1}, \tag{3.25}$$

where the second derivatives satisfies

$$0 \geq (\ln G)_{ij}$$
$$= \left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2}\right) + \left(\frac{\varphi''}{\varphi} - \left[\frac{\varphi'}{\varphi}\right]^2\right)u_iu_j + \frac{\varphi'}{\varphi}u_{ij} + \frac{u_{1ij}}{u_1 \ln u_1} - \left(1 + \frac{1}{\ln u_1}\right)\frac{u_{1i}u_{1j}}{u_1^2 \ln u_1}$$
$$= \frac{\rho_{ij}}{\rho} + \frac{\varphi'}{\varphi}u_{ij} + \frac{\varphi'}{\varphi\rho}(\rho_iu_j + \rho_ju_i) + \frac{u_{ij1}}{u_1 \ln u_1} - \left(1 + \frac{2}{\ln u_1}\right)\frac{u_{1i}u_{1j}}{u_1^2 \ln u_1}. \tag{3.26}$$

In the third line we use Equation (3.25), $\varphi'' = 0$ and that u is of class $\mathcal{C}^3$. Furthermore, the time derivative satisfy

$$0 \leq (\ln G)_t = \frac{\varphi'}{\varphi}u_t + \frac{u_{1t}}{u_1 \ln u_1} + \frac{1}{t}. \tag{3.27}$$

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Finally, we make another change of coordinates in Ω such that the following equations hold

$$u_i(x_0, t_0) = 0, \text{ for } i \neq 1; \; u_{ij}(x_0, t_0) = 0, \text{ for } i \neq j \text{ and } i, j \geq 2; \quad (3.28)$$

$$u_{22}(x_0, t_0) \geq \ldots \geq u_{nn}(x_0, t_0).$$

Before using the equations in Lemma 3.2.1, we need to show $u_{11} < 0$ at (x_0, t_0). From now on all the quantities will be evaluated at (x_0, t_0).

Recall that $Du = (u_1, 0, \ldots, 0,)$, then

$$1 - \frac{u_1^2}{w(w+1)} = \frac{w^2 + w - u_1^2}{w(w+1)} = \frac{1}{w}.$$

In addition, we have that

$$A_{11} = \frac{u_{11}}{w^3}, \; A_{1j} = \frac{u_{1j}}{w^2} = A_{j1}, \; A_{jj} = \frac{u_{jj}}{w} \text{ and } A_{ij} = A_{ji} = 0 \text{ (for } 2 \leq i \neq j \leq n). \quad (3.29)$$

Furthermore, by Equation (3.25), it follows that

$$\frac{u_{11}}{u_1 \ln u_1} = -\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1, \quad (3.30)$$

$$\frac{u_{1i}}{u_1 \ln u_1} = -\frac{\rho_i}{\rho} \text{ (for } i \geq 2). \quad (3.31)$$

Next, we will assume that $G(x_0, t_0)$ is large enough such that $G = t\rho\varphi \ln u_1 \geq 16\frac{mT}{r}$. Consequently, we obtain

$$u_1 \geq \frac{8m}{r\rho} \text{ and } 16\frac{mT}{r} \geq \frac{8\varphi t}{\varphi' r}. \quad (3.32)$$

Then, for all j, we can estimate

$$\left|\frac{\rho_j}{\rho}\right| < \frac{1}{4}\frac{\varphi'}{\varphi}u_1, \tag{3.33}$$

Finally, by combining (3.30) and (3.33), we obtain

$$u_{11} = u_1 \ln u_1 \left(-\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi}u_1\right) \leq -u_1^2 \frac{\varphi'}{2\varphi}\ln u_1 < 0. \tag{3.34}$$

Hence, $A_{11} < 0$ and we can use the equations from Lemma 3.2.1.

Then, by Equation (3.26), it follows

$$0 \geq Q_{k,ij}(\ln G)_{ij} = \underbrace{Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} + \frac{\varphi'}{\varphi}u_{ij} + \frac{\varphi'}{\varphi\rho}(\rho_i u_j + \rho_j u_i)\right)}_{B} \tag{3.35}$$
$$+ \underbrace{Q_{k,ij}\left(\frac{u_{ij1}}{u_1 \ln u_1} - \left(1 + \frac{2}{\ln u_1}\right)\frac{u_{1i}u_{1j}}{u_1^2 \ln u_1}\right)}_{C}.$$

We start with the term B in (3.35). Firstly, by Equation (3.28), we can split B in the following way

$$B = Q_{k,11}\left(\frac{\rho_{11}}{\rho} + \frac{\varphi'}{\varphi}u_{11} + \frac{2\varphi'}{\varphi\rho}\rho_1 u_1\right) + \sum_{i>1}Q_{k,ii}\left(\frac{\rho_{ii}}{\rho} + \frac{\varphi'}{\varphi}u_{ii}\right) \tag{3.36}$$
$$+ 2\sum_{j>1}Q_{k,1j}\left(\frac{\rho_{1j}}{\rho} + \frac{\varphi'}{\varphi}u_{1j} + \frac{\varphi'}{\varphi\rho}\rho_j u_1\right).$$

Secondly, by equations (a) and (3.23), the term $\sum_{i,j}Q_{k,ij}\frac{\varphi'}{\varphi}u_{ij}$ in (3.36) satisfies

$$\frac{\varphi'}{\varphi}\sum_{i,j}Q_{k,ij}u_{ij} = \frac{\varphi'}{\varphi}\left(Q_{k,11}u_{11} + 2\sum_{i>1}Q_{k,1i}u_{1i} + Q_{k,ii}u_{ii}\right) = \frac{\varphi'}{\varphi}\frac{u_t}{w}. \tag{3.37}$$

Moreover, by equation (3.31), the last term in (3.36) can be written as

$$2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 = -2\frac{u_1^2\ln u_1}{\rho}\frac{\varphi'}{\varphi}\sum_{j>1}\frac{\partial Q_k}{\partial A_{1j}}A_{1j} \tag{3.38}$$

$$= 2\frac{u_1^2\ln u_1}{\rho}\frac{\varphi'}{\varphi}\sum_{j>1}\frac{A_{1j}^2}{S_k^2}\left(S_{k-1}(A|1j)S_k - S_{k+1}S_{k-2}(A|1j)\right).$$

Then, by using (e), we can write the sum in the right hand side by the elementary symmetric polynomials evaluated in $\tilde{\lambda}$ as follows

$$2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 = 2\frac{u_1^2\ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_k(S_{k+1}(\tilde{\lambda}) + A_{11}S_k(\tilde{\lambda}) - S_{k+1}) - S_{k+1}(S_k(\tilde{\lambda}) + A_{11}S_{k-1}(\tilde{\lambda}) - S_k)}{S_k^2}$$

$$= 2\frac{u_1^2\ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_k S_{k+1}(\lambda) - S_{k+1}S_k(\lambda)}{S_k^2},$$

and in the last line we used $S_l(\tilde{\lambda}) + A_{11}S_{l-1}(\tilde{\lambda}) = S_l(\lambda)$ with $l = k+1, k$. Recall that $S_l(\lambda)$ denotes the l-elemental symmetric polynomial evaluated at the diagonal matrix $\lambda = (A_{ii})_{i\geq 1}$. Then, it follows that

$$2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 = 2\frac{u_1^2\ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_k(\lambda)}{S_k}(Q_k(\lambda) - Q_k).$$

Following an idea of [Sae], we are going to use a first order Taylor expansion on Q_k, viewed as an $\mathbb{R}^{n^2}$ real function, around the "vector" $\lambda = (A_{ii})_{i\geq 1}$. Indeed, we have

$$Q_k = Q_k(\lambda) + \sum_{i,j\geq 1}\frac{\partial Q_k}{\partial A_{ij}}(\lambda)(A_{ij} - A_{ii}) + \frac{1}{2}\sum_{i,j,k,l\geq 1}\frac{\partial^2 Q_k}{\partial A_{ij}\partial A_{kl}}(\eta)(A_{ij} - A_{ii})(A_{kl} - A_{kk}),$$

where $\eta \in \mathbb{R}^{n^2}$ is close to λ. Then, by the shape of the matrix A_{ij} (3.29), we have

$$Q_k = Q_k(\lambda) + \sum_{j>1}\frac{\partial Q_k}{\partial A_{1j}}(\lambda)A_{1j} + \frac{1}{2}\sum_{i,j,k,l\geq 1}\frac{\partial^2 Q_k}{\partial A_{ij}\partial A_{kl}}(\eta)(A_{ij} - A_{ii})(A_{kl} - A_{kk}).$$

Finally, since Q_k is a concave function and for $j > 1$

$$\frac{\partial S_l}{\partial A_{1j}}(\lambda) = 0,$$

it follows that $\dfrac{\partial Q_k}{\partial A_{1j}}(\lambda) = 0$ for $j > 1$, and therefore $Q_k \leq Q_k(\lambda)$. Consequently, the whole term is non-negative and we may drop it from (3.36).

Thirdly, to estimate the term $2Q_{k,11}\dfrac{\varphi'\rho_1}{\varphi\rho}u_1$, we use Equations (g) and (3.30), to see that

$$2Q_{k,11}\frac{\varphi'\rho_1}{\varphi\rho}u_1 = -2Q_{k,11}\frac{\varphi'}{\varphi}\left(\frac{u_{11}}{u_1\ln u_1} + \frac{\varphi'}{\varphi}u_1\right)u_1 \geq -\frac{\varphi'^2 u_1^2}{2\varphi^2 w^3}\frac{\partial Q_k}{\partial A_{11}}$$

$$\geq -\frac{u_1^2}{2m^2 w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.$$

Finally, for the term $\sum_{i,j}Q_{k,ij}\dfrac{\rho_{ij}}{\rho}$ it follows that

$$\sum_{i,j}Q_{k,ij}\frac{\rho_{ij}}{\rho} = -\frac{2}{r^2\rho}\sum_{i\geq 1}Q_{k,ii} \tag{3.39}$$

$$= -\frac{2}{r^2\rho}\left(\frac{1}{w^3}\frac{\partial Q_k}{\partial A_{11}} + \frac{1}{w}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\right) \geq -\frac{2}{wr^2\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.$$

Consequently we obtain,

$$B \geq \frac{\varphi'}{\varphi}\frac{u_t}{w} - \frac{2}{wr^2\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}} - \frac{u_1^2}{2m^2 w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}. \tag{3.40}$$

Now we estimate the term C in Equation (3.35).

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Firstly, by differentiating Equation (3.23) in the ε_1-direction, we obtain

$$\frac{u_{t1}}{w} - \frac{u_t}{w^3}u_1u_{11} = \frac{\partial Q_k}{\partial A_{ij}}\frac{\partial A_{ij}}{\partial x_1} \tag{3.41}$$

$$= \frac{\partial Q_k}{\partial A_{11}}\frac{\partial A_{11}}{\partial x_1} + 2\frac{\partial Q_k}{\partial A_{i1}}\frac{\partial A_{i1}}{\partial x_1} + \frac{\partial Q_k}{\partial A_{ii}}\frac{\partial A_{ii}}{\partial x_1}.$$

Next, we estimate the right hand side of the above line explicitly. For this purpose, we may write

$$A_{11} = \frac{1}{w}\left(u_{11} - \frac{2u_1u_ku_{1k}}{w(1+w)} + \frac{u_1^2u_ku_lu_{kl}}{w^2(1+w)^2}\right)$$

$$= \frac{1}{w}\left(\frac{u_{11}}{w^2} - \frac{2u_1}{w(1+w)}\sum_{k>1}u_ku_{1k} + \frac{u_1^2}{w^2(1+w)^2}\sum_{k,l>1}u_ku_lu_{kl}\right),$$

$$A_{1i} = \frac{1}{w}\left(u_{1i} - \frac{u_iu_ku_{1k}}{w(1+w)} - \frac{u_1u_ku_{ki}}{w(1+w)} + \frac{u_1u_iu_ku_lu_{kl}}{w^2(1+w)^2}\right)$$

$$= \frac{1}{w}\left(\frac{u_{1i}}{w} - \frac{u_iu_1u_{11}}{w(1+w)} - \frac{u_iu_1u_{ii}}{w(1+w)} + \frac{u_iu_1^3u_{11}}{w^2(1+w)^2}\right.$$

$$\left. - \frac{u_i}{w(1+w)}\sum_{k>1}u_ku_{1k} - \frac{u_1}{w(1+w)}\sum_{k>1,k\neq i}u_ku_{ki} + \sum_{k,l>1}\frac{u_1u_iu_ku_lu_{kl}}{w^2(1+w)^2}\right),$$

$$A_{ii} = \frac{1}{w}\left(u_{ii} - \frac{2u_iu_ku_{1k}}{w(1+w)} + \frac{u_i^2u_ku_lu_{kl}}{w^2(1+w)^2}\right).$$

Then, by denoting $A_{ij,1} = \dfrac{\partial A_{ij}}{\partial x_1}$, the following equations hold

$$A_{11,1} = \frac{u_{111}}{w^3} - \frac{3u_1}{w^5}u_{11}^2 - \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2,$$

$$A_{1i,1} = \frac{u_{1i1}}{w^2} - \frac{2u_1}{w^4}u_{11}u_{1i} - \frac{u_1}{w^2(1+w)}u_{1i}u_{ii} - \frac{u_1}{w^3(w+1)}u_{11}u_{1i},$$

$$A_{ii,1} = \frac{u_{ii1}}{w} - \frac{u_1}{w^3}u_{11}u_{ii} - \frac{2u_1}{w^2(1+w)}u_{1i}^2.$$

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Here we used that $\dfrac{\partial w}{\partial x_1} = \dfrac{u_1 u_{11}}{w}$. Then, the following equation holds

$$Q_{k,ij}u_{ij1}$$

$$= \frac{\partial Q_k}{\partial A_{11}}\frac{u_{111}}{w^3} + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\frac{u_{1i1}}{w^2} + \sum_{i>1}\frac{\partial Q_k}{\partial A_{11}}\frac{u_{ii1}}{w}$$

$$= \frac{\partial Q_k}{\partial A_{11}}\left(A_{11,1} + \frac{3u_1 u_{11}^2}{w^5} + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right)$$

$$+ 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(A_{1i,1} + \frac{2u_1 u_{11}u_{1i}}{w^4} + \frac{u_1 u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1 u_{11}u_{1i}}{w^3(1+w)}\right)$$

$$+ \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\left(A_{ii,1} + \frac{u_1 u_{11}u_{ii}}{w^3} + \frac{2u_1 u_{1i}^2}{w^2(1+w)}\right)$$

$$= \frac{u_{t1}}{w} - \frac{u_t}{w^3}u_1 u_{11} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{3u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right)$$

$$+ 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{2u_1 u_{11}u_{1i}}{w^4} + \frac{u_1 u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1 u_{11}u_{1i}}{w^3(1+w)}\right)$$

$$+ \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{u_1 u_{11}u_{ii}}{w^3} + \frac{2u_1 u_{1i}^2}{w^2(1+w)}\right)$$

$$= \frac{u_{t1}}{w} - \frac{u_1 u_{11}}{w^2}Q_k + \left(\frac{\partial Q_k}{\partial A_{11}}A_{11} + 2\frac{\partial Q_k}{\partial A_{1i}}A_{1i} + \frac{\partial Q_k}{\partial A_{ii}A_{ii}}\right)\frac{u_1 u_{11}}{w^2}$$

$$+ \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1 u_{11}u_{1i}}{w^4} + \frac{u_1 u_{1i}u_{ii}}{w^2(1+w)}\right)$$

$$+ \frac{u_1 u_{11}u_{1i}}{w^3(w+1)}\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1 u_{1i}^2}{w^2(1+w)}$$

$$= \frac{u_{t1}}{w} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1 u_{1i}^2}{w^2(1+w)}$$

$$+ 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1 u_{11}u_{1i}}{w^4} + \frac{u_1 u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1 u_{11}u_{1i}}{w^3(w+1)}\right).$$

In the third equality we used Equation (3.41), in the fourth equality we used Equation (3.23), and in the last equality we used Equation (a).

Now we replace the above equations in C to obtain

$$C = Q_{k,ij} \left(\frac{u_{ij1}}{u_1 \ln u_1} - \left(1 + \frac{2}{\ln u_1}\right) \frac{u_{1i}u_{1j}}{u_1^2 \ln u_1} \right)$$

$$= \frac{u_{t1}}{w u_1 \ln u_1}$$

$$+ \frac{\partial Q_k}{\partial A_{11}} \left(\frac{2u_1^2 u_{11}^2}{w^5} - \left(1 + \frac{2}{\ln u_1}\right) \frac{u_{11}^2}{w^3} + \frac{2u_1^2}{w^3(1+w)} \sum_{k>1} u_{1k}^2 \right) \frac{1}{u_1^2 \ln u_1}$$

$$+ 2 \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} \left(\left[\frac{(2w+1)u_1^2}{w^2(1+w)} - \left(1 + \frac{2}{\ln u_1}\right) \right] \frac{u_{11}u_{1i}}{w^2} + \frac{u_1^2 u_{1i}u_{ii}}{w^2(1+w)} \right) \frac{1}{u_1^2 \ln u_1}$$

$$+ \sum_{i>1} \frac{\partial Q_k}{\partial A_{ii}} \left(\frac{2u_1^2}{w^2(1+w)} - \left(1 + \frac{2}{\ln u_1}\right) \frac{1}{w} \right) \frac{u_{1i}^2}{u_1^2 \ln u_1}.$$

Furthermore, we note that if u_1 is large enough, then the following inequalities hold

$$\frac{(2w+1)u_1^2}{w^2(1+w)} = 2 - \frac{w+1}{w^2} \geq 1 + \frac{2}{\ln u_1} = 1 + \frac{2}{\ln(\sqrt{w^2-1})}, \tag{3.42}$$

$$\frac{2u_1^2}{w^2(1+w)} = \frac{2}{w} - \frac{2}{w^2} \geq \left(1 + \frac{2}{\ln u_1}\right)\frac{1}{w}, \tag{3.43}$$

Therefore, by removing the positives terms in C, we finally obtain

$$C \geq \frac{u_{t1}}{w u_1 \ln u_1} + \frac{\partial Q_k}{\partial A_{11}} \left(\frac{2u_1^2}{w^2} - 1 - \frac{2}{\ln u_1} \right) \frac{u_{11}^2}{w^3 u_1^2 \ln u_1} + \frac{2}{w^2(1+w)\ln u_1} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i}u_{ii}.$$

$$\tag{3.44}$$

Note that the term $u_{11} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} \frac{u_{1i}}{w^2}$ is non-negative.

Next, we analyze the third term in Equation (3.44).

We note that by differentiating Equation (f) we have

$$\frac{\partial S_l}{\partial A_{1i}} = -2A_{1i}S_{l-2}(A|1i).$$

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Then, it follows that

$$\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} = -2 \frac{S_{k-1}(A|1i)S_k - S_{k+1}S_{k-2}(A|1i)}{S_k^2} A_{1i} u_{1i} u_{ii}.$$

Moreover, by using the shape of the coefficients of the matrix A, Equation (3.29), we have

$$\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} = 2 \frac{u_{1i}^2}{w S_k^2} \left(S_{k+1} A_{ii} S_{k-2}(A|1i) - S_k A_{ii} S_{k-1}(A|1i) \right)$$

$$= 2 \frac{u_{1i}^2}{w S_k^2} \left(S_{k+1}(S_{k-1}(\tilde{\lambda}) - S_{k-1}(\tilde{\lambda}|i)) - S_k(S_k(\tilde{\lambda}) - S_k(\tilde{\lambda}|i)) \right)$$

$$= 2 \frac{u_{1i}^2}{w S_k^2} \left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_k S_k(\tilde{\lambda}|i) - S_{k+1}S_{k-1}(\tilde{\lambda}|i) \right),$$

where in the second line we used Equation (c) with the term $A_{ii}S_l(A|1i) = A_{ii}S_l(\tilde{\lambda}|i)$ with $l = k-1, k-2$. In the last line we used that $\dfrac{\partial Q_k}{\partial A_{11}} = \dfrac{S_k(\tilde{\lambda})S_k - S_{k+1}S_{k-1}(\tilde{\lambda})}{S_k^2}$ where the functions are evaluated in the matrix A_{ij}.

Now, we are going to show that the difference of the last two terms in the last line is non-negative. For this purpose we consider

$$\frac{\partial Q_k}{\partial A_{ii}}(\tilde{\lambda}) = \frac{S_k(\tilde{\lambda}|i)S_k(\tilde{\lambda}) - S_{k+1}(\tilde{\lambda})S_{k-1}(\tilde{\lambda}|i)}{S_k^2(\tilde{\lambda})}.$$

Then, we note that we can write the term $S_k S_k(\tilde{\lambda}|i) - S_{k+1}S_{k-1}(\tilde{\lambda}|i)$ in the following manner

$$S_k S_k(\tilde{\lambda}|i) - S_{k+1}S_{k-1}(\tilde{\lambda}|i) = \frac{\partial Q_k}{\partial A_{ii}}(\tilde{\lambda})S_k(\tilde{\lambda})S_k + \frac{S_{k-1}(\tilde{\lambda}|i)S_{k+1}(\tilde{\lambda})S_k}{S_k(\tilde{\lambda})} - S_{k+1}S_{k-1}(\tilde{\lambda}|i)$$

$$\geq S_{k-1}(\tilde{\lambda}|j)S_k(Q_k(\tilde{\lambda}) - Q_k),$$

in the last line we dropped the first term because is non-negative.

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Finally, putting all the above calculation together we obtain

$$
\begin{aligned}
\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &\geq 2 \frac{u_{1i}^2}{w S_k^2} \left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k (Q_k(\tilde{\lambda}) - Q_k) \right) \\
&= 2 \frac{u_{1i}^2}{w S_k^2} \left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k \left[Q_k(\tilde{\lambda}) + \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11} - \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11} - Q_k \right] \right) \\
&= 2 \frac{u_{1i}^2}{w S_k^2} \left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k \left[Q_k(\lambda) - Q_k - \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11} \right] \right) \\
&\geq -2 \frac{\partial Q_k}{\partial A_{11}} \frac{u_{1i}^2}{w},
\end{aligned}
\tag{3.45}
$$

where in the third line we used the 1-homogeneity of $Q_k(\lambda)$, and in the last line we used that $A_{11} < 0$ and $Q_k(\lambda) - Q_k \geq 0$.

Consequently, for the whole third term in (3.44) we have

$$
\begin{aligned}
\frac{2}{w^2(1+w)\ln u_1} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &\geq -\frac{4}{w^3(1+w)} \frac{\partial Q_k}{\partial A_{11}} \sum_{i>1} \frac{u_{1i}^2}{\ln u_1} \\
&\geq -4 \frac{u_1^2 \ln u_1}{w^3(1+w)} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}} \sum_{i>1} \frac{\rho_i^2}{\rho^2} \\
&\geq -16 \frac{u_1^2 \ln u_1}{w^3(1+w)\rho^2} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}},
\end{aligned}
\tag{3.46}
$$

where in the second line we used the estimate (3.45), in the third line we used Equation (3.31) and that $\sum_{i>1} \rho_i^2 \leq 4$.

Next, we study the second term in (3.44). We note that the term in parenthesis satisfies

$$
\frac{2u_1^2}{w^2} - \left(1 + \frac{2}{\ln u_1} \right) = 1 - \frac{2}{w^2} - 1 - \frac{2}{\ln u_1} > \frac{1}{2},
\tag{3.47}
$$

if u_1 is large enough.

Therefore, by combining the last inequality with Equation (3.34), we have

$$\frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2}{w^2}-\left(1+\frac{2}{\ln u_1}\right)\right)\frac{u_{11}^2}{w^3 u_1^2 \ln u_1} \geq \frac{\ln u_1}{2w^3}\frac{\partial Q_k}{\partial A_{11}}\frac{u_{11}^2}{u_1^2 \ln^2(u_1)}$$

$$\geq \frac{u_1^2 \ln u_1}{8w^3}\frac{\partial Q_k}{\partial A_{11}}\frac{\varphi'^2}{\varphi^2}$$

$$\geq \frac{u_1^2 \ln u_1}{32m^2 w^3}\frac{\partial Q_k}{\partial A_{11}}$$

$$\geq \frac{C(k,n)}{32m^2}\frac{u_1^2 \ln u_1}{w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}, \qquad (3.48)$$

where $C(k,n)$ is the constant given in (h). Recall that we have chosen c_0 large enough such that $u_1 > c_0$ and the inequalities (3.32), (3.42), (3.43) and (3.47) hold.

Finally, the term C can be bounded from below in the following way

$$C \geq \sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{C(k,n)}{32m^2}\frac{u_1^2 \ln u_1}{w^3}-C\frac{u_1^2 \ln u_1}{w^3(w+1)}\frac{16}{\rho^2}\right)+\frac{u_{t1}}{wu_1 \ln u_1}. \qquad (3.49)$$

Moreover, by combining the lower bounds from (3.40) and (3.49) for B and C defined in (3.35), we finally obtain

$$0 \geq \frac{u_{t1}}{wu_1 \ln u_1}+u_t\frac{\varphi'}{w\varphi}$$

$$+\sum_i\frac{\partial Q_k}{\partial A_{ii}}\left(-\frac{4r}{\rho w^3}\frac{u_1^2}{m}-\frac{2}{wr^2\rho}+\frac{C_2(k,n)}{64m^2}\frac{u_1^2 \ln u_1}{w^3}-C\frac{u_1^2 \ln u_1}{w^3(w+1)}\frac{16}{\rho^2}\right).$$

On the other hand, by Equation (3.27) we have

$$\frac{\varphi'}{\varphi}u_t+\frac{u_{t1}}{u_1 \ln u_1}\geq -\frac{1}{t}. \qquad (3.50)$$

Then, after using Equation (3.50) and multiplying by $w^2 \left(\sum_{i \geq 1} \frac{\partial Q_k}{\partial A_{ii}} \right)^{-1}$, we obtain

$$m^2 \left(C \frac{\ln u_1}{(w+1)} \frac{16}{\rho} + \frac{k+1}{n-k} \frac{w^2 \rho}{u_1^2 t} + \frac{4r}{m} + \frac{2w^2}{u_1^2 r^2} \right) \geq \frac{C_2}{64} \rho \ln u_1.$$

Consequently, we obtain

$$m^2 \left(C \frac{16r}{m} + \left(1 + \frac{r^2}{m^2} \right) \frac{1}{c_3 t} + \frac{4r^2}{m^2} + \frac{1}{r^2} \left(1 + \frac{r}{m} \right) \right) \geq \frac{c_2}{64} \rho \ln u_1,$$

here we use the assumption (3.32). Note that this implies,

$$\rho \ln u_1 \leq K \left(mr + \frac{1}{t}(m^2 + r^2) + r^2 + \frac{m^2}{r^2} + \frac{m}{r} \right),$$

for a universal constant $K = K(k,n)$. Then, we have

$$\ln |Du(0,t)| \leq \frac{t_0}{t} \frac{\varphi(u(x_0, t_0))}{\varphi(u(0,t))} \frac{\rho(x_0) \ln u_1(x_0, t_0)}{\rho(0)}$$
$$\leq K \left(\frac{mrT}{t} + \frac{m^2}{t} + \frac{r^2}{t} + \frac{Tr^2}{t} + \frac{Tm^2}{tr^2} + \frac{Tm}{tr} \right).$$

Since we assumed that $u_1 \geq c_0$ and $G(x_0, t_0) \geq 16 \frac{mT}{r}$, we finally obtain

$$|Du(0,t)| \leq \exp \left(K + \frac{KmT}{rt} + \frac{m^2}{t} + \frac{(T+1)r^2}{t} + \frac{Tm^2}{tr^2} + \frac{Tm}{tr} \right).$$

$\square$

Recall that a Q_k-translator is a surface that evolves by translations with unit speed, hence we may use the same method to obtain a local gradient estimate for graphical solutions to Equation (3.18).

Proof of Theorem 3.0.2. This proof is very similar to the one given in Theorem 3.0.1, for

this reason we only point out the main differences from it.

First, we note that equation (3.18) can be written as

$$Q_k(A) = \frac{1}{w},$$ (3.51)

where A is the matrix given in (3.19).

Secondly, we use the same test function $G(x, \xi)$ given in (3.24) without the time factor. We also change the cut off function ρ by

$$\rho(x) = r^2 - |x|^2.$$

As in the proof of Theorem 3.0.1 we may assume that the maximum of G is attached at some point $x_0 \in B_r(0)$. We also apply the same change of coordinates as we did before.

Now, if we want to use the equations from Lemma 3.2.1, we need to ensure that $u_{11} < 0$ at x_0. Recall, that the function $\varphi(u) = 1 + \dfrac{u}{m}$ where $m = \sup\limits_{B(0,r)} u$.

Then, we will assume that

$$G = \rho\varphi \ln u_1 \geq 16rm.$$ (3.52)

Then, it follows that $u_1 \geq \dfrac{8rm}{\rho}$ and $\dfrac{\varphi'}{\varphi} \geq \dfrac{1}{2m}$, which also implies

$$\left|\frac{\rho_j}{\rho}\right| < \frac{2r}{\rho} \leq \frac{\varphi'}{2\varphi} u_1.$$

Finally we get,

$$u_{11} = u_1 \ln u_1 \left(-\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1\right) \leq -u_1^2 \frac{\varphi'}{2\varphi} \ln u_1 < 0.$$

Thirdly, by the same calculations as in the proof of Theorem 3.0.1, we will obtain the same B and C terms of Equation (3.35), which we will analyze in this setup.

We start with B and note that the only terms that change are equations (3.37) and (3.39). In this case, we have

$$\sum_{i,j\geq 1} Q_{k,ij}\frac{\varphi'}{\varphi}u_{ij} = \frac{\varphi'}{\varphi}Q_k = \frac{\varphi'}{w\varphi},$$

and

$$\sum_{i,j\geq 1} Q_{k,ij}\frac{\rho_{ij}}{\rho} = -\frac{2}{\rho}\sum_{i\geq 1}Q_{k,ii} = -\frac{2}{\rho}\left(\frac{1}{w^3}\frac{\partial Q_k}{\partial A_{11}} + \frac{1}{w}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\right) \geq -\frac{2}{w\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.$$

Recall that the cut off function $\rho(x) = r^2 - |x|^2$ so that is why does not appear a r^{-1} factor in the last inequality.

Therefore,

$$B \geq \frac{\varphi'}{\varphi}\frac{1}{w} - \frac{2}{w\rho}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}} - \frac{u_1^2}{2m^2w^3}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}.$$

For the term C, we only need to estimate the term $Q_{k,ij}u_{ij1}$. We observe that

$$\begin{aligned}
Q_{k,ij}u_{ij1} &= \frac{\partial Q_k}{\partial A_{11}}\frac{u_{111}}{w^3} + 2\sum_i\frac{\partial Q_k}{\partial A_{1i}}\frac{u_{1i1}}{w^2}\sum_i\frac{\partial Q_k}{\partial A_{11}}\frac{u_{ii1}}{w} \\
&= \underbrace{-\frac{u_1u_{11}}{w^3} + \frac{u_1u_{11}}{w^2}Q_k}_{=0} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1}{w^2(1+w)}u_{1i}^2 \\
&\quad + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1}{w^4}u_{11}u_{1i} + \frac{u_1}{w^2(1+w)}u_{1i}u_{ii} + \frac{u_1}{w^3(w+1)}u_{11}u_{i1}\right).
\end{aligned}$$

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Then, by using the same bounds given in Equation (3.44), it follows that

$$C \geq \frac{\partial Q_k}{\partial A_{11}} \left(\frac{2u_1^2}{w^2} - \left(1 + \frac{2}{\ln u_1} \right) \right) \frac{u_{11}^2}{w^3 u_1^2 \ln u_1} + \frac{2}{w^2(1+w)\ln u_1} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i}u_{ii}. \qquad (3.53)$$

Now, from Equation (3.46) we have

$$\frac{2}{w^2(1+w)\ln u_1} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i}u_{ii} \geq -\frac{4u_1^2 \ln u_1}{w^3(1+w)} \sum_{i>1} \frac{\rho_i^2}{\rho^2} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}} \geq -\frac{16r^2 u_1^2 \ln u_1}{w^3(1+w)\rho^2} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}}.$$

In the last inequality we used that $\sum_{i>1} \frac{\rho_i^2}{\rho^2} \leq 4\frac{r^2}{\rho^2}$.

Furthermore, by Equation (3.48), it follows that

$$\frac{\partial Q_k}{\partial A_{11}} \left(\frac{2u_1^2}{w^2} - \left(1 + \frac{2}{\ln u_1} \right) \right) \frac{u_{11}^2}{w^3 u_1^2 \ln u_1} \geq \frac{C(k,n)}{32m^2} \frac{u_1^2 \ln u_1}{w^3} \sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}},$$

where $C(k,n)$ is the constant in (h).

Hence, it follows that

$$C \geq \sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} \left(\frac{C(k,n)}{32m^2} \frac{u_1^2 \ln u_1}{w^3} - \frac{16u_1^2 \ln u_1 r^2}{w^3(w+1)\rho^2} \right).$$

Finally, by adding the bounds from the estimates of B and C, we see that

$$0 \geq B + C$$
$$\geq \sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} \left(\frac{C(k,n)}{32m^2} \frac{u_1^2 \ln u_1}{w^3} - 16\frac{u_1^2 \ln u_1 r^2}{w^3(w+1)\rho^2} - \frac{2}{w\rho} - \frac{u_1^2}{2m^2 w^3} \right),$$

or equivalently,

$$\frac{C(k,n)}{32m^2} \ln u_1 \rho \leq 16\frac{r^2 \ln u_1}{\rho(w+1)} + \frac{2w^2}{u_1^2} + \frac{\rho}{2m^2}.$$

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Then, by using Equation (3.52) we obtain

$$\rho \ln u_1 \leq C \left(m^2 r^2 + r^2 + m^2 \right).$$

Consequently, we have

$$\ln |Du(0)| \leq \frac{\varphi(u(x_0))\rho(x_0)\ln u_1(x_0)}{\varphi(u(0))\rho(0)} \leq C(k,n) \left(\frac{m^2}{r^2} + \frac{m}{r} + 1 \right).$$

$\square$

As a consequence of the proof given for Theorem 3.0.2 we also obtain a non-existence result for graphical Q_k-translator which have linear growth at infinity.

Proof of Theorem 3.0.3. Let $u \in \mathcal{C}^3(\mathbb{R}^n)$ be a solution to Equation (3.18) such that satisfies Properties (1) and (2) in the statement of Theorem 3.0.3.

Then, by property 2, it follows that $\max_{B_r} |u| \leq Cr$, for all $r > 1$. This fact together with Theorem 3.0.2 implies that $|Du(x)| \leq C_1$ for all $x \in \mathbb{R}^n$.

Claim 3.2.1. *The function u is constant in $\mathbb{R}^n$.*

Proof. We argue this by contradiction, and without loss of generality we may assume that there is a $\delta > 0$ such that $|Du(0)| \geq \delta$.

In addition, we want to mention that the proof is very similar to the one given for Theorem (3.0.2). The reason of this is that Property 2 combined with Theorem 3.0.2 give us a control from below of the solution. In particular, by minor modifications in the test function, we were able to show a suitable estimate to conclude the proof of this theorem.

Let $r > 1$ and we consider the test function given by

$$G(x) = \rho(x)\varphi(u)|Du|,$$

where $\rho(x) = r^2 - |x|^2$, $\varphi(u) = \left(1 - \dfrac{u - \inf\limits_{B(0,r)} u}{m}\right)^{\beta}$, $m = 2(\sup\limits_{B(0,r)} u - \inf\limits_{B(0,r)} u)$ and the constant

$\beta < 0$ is still to be fixed.

Moreover, we note that

$$m \leq 4\,||u||_{\infty}, \quad \frac{1}{2} \leq 1 - \frac{u - \inf\limits_{B_r} u}{m} \leq 1,$$

and $G : \overline{B_r} \to \mathbb{R}$ attains its maximum at an interior point $x_0 \in B_r$.

In addition we will choose a coordinate system such that $u_{ij}(x_0)$ is a diagonal matrix for $2 \leq i, j \leq n$, $u_1(x_0) = |Du(x_0)|$ and $u_i(x_0) = 0$, $i \geq 2$.

Now let $\delta_1 \in [\delta, 1)$ such that $u_1(x_0) \geq \delta_1$, and since x_0 is an interior point of $B(0, r)$, we may shrink δ_1 to assume that

$$\rho(x_0) \geq \delta_1 r^2. \tag{3.54}$$

Note that Equation (3.54) is equivalent to write $(1 - \delta_1) \geq \dfrac{|x_0|^2}{r^2}$.

Next, we have the following equations at x_0.

First, for the gradient we have

$$0 = (\ln G)_i = \frac{\rho_i}{\rho} + \frac{\varphi'}{\varphi} u_i + \frac{u_{1i}}{u_1}. \tag{3.55}$$

Note that, by Equation (3.55), it follows that

$$\frac{u_{11}}{u_1} = -\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1 \quad \text{and} \quad \frac{u_{1i}}{u_1} = -\frac{\rho_i}{\rho}. \tag{3.56}$$

Consequently, since $\beta < 0$ and $\dfrac{\varphi'}{\varphi} = -\dfrac{\beta}{m}\left(1 - \dfrac{u - \inf\limits_{B_r} u}{m}\right)^{-1} > \dfrac{|\beta|}{4\,||u||_{\infty}}$, we may enlarge r

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such that $r > \dfrac{8\,||u||_\infty}{|\beta|\,\delta^2}$ and

$$\frac{u_{11}}{u_1} = \frac{2x_1}{\rho} - \frac{\varphi'}{\varphi}u_1 \le \frac{2}{\delta_1 r} - \frac{|\beta|}{4\,||u||_\infty}\delta_1 < 0.$$

This fact allows us to use the equations from Lemma 3.2.1.

Moreover, the second derivatives satisfy

$$0 \ge Q_{k,ij}(\ln G)_{ij}$$
$$= Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2} + \frac{\varphi'}{\varphi}u_{ij} + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2 \right)u_i u_j + \frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2} \right).$$

We note that by Equation (3.56) the last term in the above equation satisfies

$$-\sum_{i,j} Q_{k,ij}\frac{u_{1i}u_{1j}}{u_1^2} = -Q_{k,11}\frac{u_{11}^2}{u_1^2} - \sum_{i,j>1} Q_{k,ij}\frac{u_{1i}u_{1j}}{u_1^2}$$

$$= -Q_{k,11}\left(\frac{\rho_1}{\rho} + \frac{\varphi'}{\varphi}u_1 \right)^2 - \sum_{i,j>1} Q_{k,ij}\frac{\rho_i \rho_j}{\rho^2}$$

$$= -2Q_{k,11}\frac{\rho_1}{\rho}\frac{\varphi'}{\varphi}u_1 - Q_{k,ij}\frac{\rho_i \rho_j}{\rho^2} - Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j$$

$$\ge -Q_{k,11}\left(\frac{\rho_1^2}{\rho^2} + u_1^2\left(\frac{\varphi'}{\varphi}\right)^2 \right) - Q_{k,ij}\frac{\rho_i \rho_j}{\rho^2} - Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j$$

$$\ge -2Q_{k,ij}\frac{\rho_i \rho_j}{\rho^2} - 2Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j.$$

In the last line we use that $u_i(x_0) = 0$ for $i > 1$ and $Q_{k,ij}$ is a definite positive matrix.

Therefore, it follows that

$$0 \ge Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} - 3\frac{\rho_i \rho_j}{\rho^2} + \frac{\varphi'}{\varphi}u_{ij} + \left(\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2 \right)u_i u_j + \frac{u_{1ij}}{u_1} \right). \qquad (3.57)$$

Now we are going to analyze the terms in (3.57).

Indeed, by Equation (3.29), the first and second term in (3.55) satisfy

$$\sum_{i,j\geq1} Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} - 3\frac{\rho_i\rho_j}{\rho^2}\right) = -2\sum_{i\geq1}\frac{Q_{k,ii}}{\rho} - 8\sum_{i,j\geq1} Q_{k,ij}\frac{x_ix_j}{\rho^2}$$

$$\geq -\frac{2}{\rho w}\sum_{i\geq1}\frac{\partial Q_k}{\partial A_{ii}} - 12\sum_{i,j\geq1} Q_{k,ii}\frac{|x_0|^2}{\rho^2}$$

$$\geq \frac{-1}{wr^2}\left(\frac{2}{\delta_1} + \frac{12}{\delta_1^2}\right)\sum_{i\geq1}\frac{\partial Q_k}{\partial A_{ii}}.$$

In the second line we used that $Q_{k,ij}$ is a definite positive matrix, which allows to estimate $\langle x_0, Q_{k,ij}x_0\rangle \leq \text{Tr}(Q_{k,ij})|x_0|^2$. In the last line we used $|x|^2 \leq r^2$ and Equation (3.54).

Next, we estimate the third and fourth term from (3.57).

Indeed, if we choose $\beta = \dfrac{-1}{4}$, it follows that

$$Q_{k,ij}\left(u_{ij}\frac{\varphi'}{\varphi} + \left(\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2\right)u_iu_j\right) = \frac{\varphi'}{\varphi}\sum_{i,j\geq1}\frac{\partial Q_k}{\partial A_{ij}}A_{ij} + Q_{k,11}\left(\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2\right)u_1^2$$

$$\geq \frac{1}{16m^2w^3}\frac{\partial Q_{11}}{\partial A_{11}}u_1^2$$

$$\geq \frac{u_1^2}{16m^2w^3}\sum_{i\geq1}\frac{\partial Q_k}{\partial A_{ii}}.$$

In the second inequality we used (a) and

$$\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2 = -\left(1 - \frac{u - \inf_{B_r} u}{m}\right)^{-2}\frac{\beta(3\beta+1)}{m^2} \geq \frac{1}{16m^2}.$$

Finally, we estimate the fifth term from (3.57).

We note that Equation (3.53) applies in the following way

$$Q_{k,ij}\frac{u_{ij1}}{u_1} \geq \frac{2}{w^2(1+w)}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii},$$

where as in the proof of Theorem 3.0.2 we omitted all the non-negative terms.

In addition, Equation (3.46) can be applied in a similar manner

$$
\begin{aligned}
\frac{2}{w^2(1+w)} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &\geq -\frac{4}{w^3(1+w)} \frac{\partial Q_k}{\partial A_{11}} \sum_{i>1} u_{1i}^2 \\
&= -\frac{4u_1^2}{w^3(1+w)} \frac{\partial Q_k}{\partial A_{11}} \sum_{i>1} \frac{\rho_i^2}{\rho^2} \\
&\geq -\frac{16u_1^2 r^2}{w^3(1+w)\rho^2} \sum_{i\geq1} \frac{\partial Q_k}{\partial A_{ii}} \\
&\geq -\frac{16u_1^2}{w^4 r^2 \delta_1^2} \sum_{i\geq1} \frac{\partial Q_k}{\partial A_{ii}}.
\end{aligned}
$$

Consequently, by combining the above estimates, it holds

$$
u_1^2 \left(\frac{1}{16m^2} - \frac{16}{wr^2\delta_1^2} \right) \leq \frac{w^2}{r^2} \left(\frac{2}{\delta_1} + \frac{12}{\delta_1^2} \right).
$$

Then, by enlarging r so that $r^2 \geq 16m^2 \left(\dfrac{16}{\delta^2\sqrt{1+\delta^2}} + 1 \right)$, it follows that

$$
\frac{1}{r^2} \leq \frac{1}{16m^2} - \frac{16}{\delta^2\sqrt{1+\delta^2}r^2} \leq \frac{1}{16m^2} - \frac{16}{wr^2\delta^2} \leq \left(\frac{1}{16m^2} - \frac{16}{wr^2\delta_1^2} \right).
$$

Here we used that $\dfrac{1}{w} \leq \dfrac{1}{\sqrt{1+\delta^2}}$.

Therefore, this assumption implies that

$$
u_1^2(x_0) \leq C\frac{m^2 w^2}{r^2} \leq C'\frac{C_1^2 \|u\|_\infty^2}{r^2},
$$

where C and C' are constants that only depend on δ, and recall that C_1 is the constant such that $|Du| \leq C_1$.

Finally, since $G(x_0)$ is a maximum, it follows that

$$|Du(0)| \leq \frac{\rho(x_0)}{\rho(0)} \frac{\varphi(u(x_0))}{\varphi(u(0))} u_1(x_0) \leq C''(\delta) \frac{||u||_\infty}{r}$$

which implies that $|Du(0)| \to 0$ as $r \to \infty$, a contradiction with $|Du(0)| \geq \delta$. $\qquad \square$

Therefore, u is constant but this fact contradicts property 1 from the statement of Theorem 3.0.3. $\qquad \square$

3.3   Second Order Estimates

In this section we derive interior second order estimates for solutions to the Q_k-flow and Q_k-translator equation. For this purpose, we derive local uniform estimates for H^2 in both settings for cases $k \geq 1$.

The reason of this is that if M_0 satisfies $\lambda \in \Gamma_{k+2}$ for $k \geq 1$, then by Equations (3.15) and Corollary (3.1.4) it follows that

$$|A|_k^2 = (k+1)Q_k^2 - (k+2)Q_{k+1}Q_k \leq (k+1)Q_k^2 \leq \underbrace{\frac{(n-k)^2}{n^2(k+1)}}_{:=\tilde{C}(k,n)} H^2 \qquad (3.58)$$

and $|A|^2 \leq H^2$ holds on M_0. In addition, we will assume that M_0 is locally the graph of a function over a hyperplane orthogonal to the direction $\vec{w} \in \mathbb{S}^n$.

Lemma 3.3.1. *Let M_t be a solution to the Q_k-flow (3.1) and $\vec{w} \in \mathbb{S}^n$. Then, we have the*

following equations at $p \in M_t$:

$$(\partial_t - \Box_k)\, u = 0, \quad \text{where } u = \langle p, \vec{w} \rangle, \tag{3.59}$$

$$(\partial_t - \Box_k)\, h_{ii} = \frac{\partial^2 Q_k}{\partial h_{cd} \partial h_{ab}} \nabla_i h_{cd} \nabla_i h_{ab} + |A|_k^2 h_{ij} - Q_k h_{il} h_j^l, \tag{3.60}$$

$$(\partial_t - \Box_k)\, \langle \nu, \vec{w} \rangle = |A|_k^2 \langle \nu, \vec{w} \rangle, \tag{3.61}$$

$$(\partial_t - \Box_k)\, v = -v|A|_k^2 - 2\,||\nabla v||_k^2\, v^{-1}, \quad \text{where } v = \langle \nu, \vec{w} \rangle^{-1}, \tag{3.62}$$

$$(\partial_t - \Box_k)\, H = \frac{\partial^2 Q_k}{\partial h_c^d \partial h_a^b} \nabla^i h_c^d \nabla_i h_a^b + |A|_k^2 H, \tag{3.63}$$

$$(\partial_t - \Box_k)\, Q_k = |A|_k^2 Q_k \tag{3.64}$$

$$(\partial_t - \Box_k)\, r^2 = 0, \quad \text{where } r^2 = |p|^2 + 2\frac{n-k}{k+1}t, \tag{3.65}$$

where $Q_{k,ij} = \dfrac{\partial Q_k}{\partial h_{ij}}$, $\Box_k f = \sum\limits_{i,j} Q_{k,ij} \nabla_i \nabla_j f$, $||X||_k^2 = \langle X, X \rangle_k$, $\langle X, Y \rangle_k = \sum_{i,j} Q_{k,ij} X^i Y^j$ and $|A|_k^2 = \sum\limits_{i,j,l} Q_{k,ij} h_{il} h_{lj}$.

Proof. The proofs of Equations (3.59)-(3.62) can be found in [CD], and for Equations (3.63) and (3.64) the proofs can be found in [Die].

On the other hand, for Equation (3.65), we use a normal frame at $p \in M_t$ given by $e_i \in T_p M_t$. This means that $\{e_i\}_{i=1}^n \subset T_p M$ is an orthogonal basis of eigenvalues of the second fundamental form (i.e: $h_i^j(p) = h_{ij}(p) = \lambda_i(p)\delta_j^i$ and λ_i is the principal curvature) at p.

Then, it follows that

$$\partial_t r^2 = 2\,\langle Q_k \nu, p \rangle + 2\frac{n-k}{k+1},$$

$$\Box_k r^2 = 2 Q_{k,ij} \delta_j^i + 2\,\langle Q_k \nu, p \rangle.$$

Therefore, by Equation (3.13) together with Corollary 3.1.4, it follows that $(\partial_t - \Box_k) r = 0$ since $Q_{k,ij} \delta_j^i = \dfrac{n-k}{k+1}$. $\qquad\square$

Remark 3.3.1. Note that when $\vec{w} = e_{n+1}$, then $v = \langle \nu, \vec{w} \rangle^{-1} = \sqrt{1 + |Du|^2} = w$ from the last section, and also we choose v rather than w since w is for an specific vector in $\mathbb{S}^n$. In addition, Equation (3.62) implies that the assumption imposed on M_0, being locally a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^{n-1}$, is still valid under the Q_k-flow if the domain of v lies in the support of $R^2 - r^2$, where R depends on the domain of definition of u.

Remark 3.3.2. Let assume that M_t evolves by the Q_k-flow such that each M_t is convex and it can be written as a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$ for all $t \in [0, T)$. Then, the function

$$h(x, t) = r^2 - u^2$$

satisfies $|(\partial_t - \Box_k)h| \leq C(k, n)$, $\|\nabla h\|_k^2 \leq Ch$ and $|\nabla h| \leq Ch$. To see this, we choose normal coordinates at $p \in M_t$, then by Equations (3.59) and (3.65), we have

$$(\partial_t - \Box_k)h = (\partial_t - \Box_k)r^2 - (\partial_t - \Box_k)u^2 = -2u(\partial_t - \Box_k)u + 2 \|\nabla u\|_k^2 = 2 \|\nabla u\|_k^2 .$$

In addition, since M_t is convex, it follows that $Q_{k,ij} = \dfrac{\partial Q_k}{\partial \lambda_i} \leq 1$. We refer to Proposition 2.2 in [CD] for a proof of this.

Furthermore, we have $\nabla u = \vec{w}^\top$, which implies that $\|\nabla u\|_k \leq 1$. In particular, it follows that $|(\partial_t - \Box_k)h| \leq 2$.

On the other hand,

$$|\nabla h|^2 = 4 \left| (p - u\vec{w})^\top \right|^2 \leq 4 |p - u\vec{w}|^2 = 4(|p|^2 - u^2) \leq 4h.$$

Finally, by the same argument, we also have $\|\nabla h\|_k^2 \leq Ch$.

In [CD] Theorem 2.4, the authors give a gradient estimate for the Q_k-flow of the form

$$\phi v \leq \sup_{M_0} \phi(p,0)v(p,0),$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$ and $\phi = R^2 - r^2$, for a convex M_0 initial data such that $\lambda \in \Gamma_{k+1}$. Instead, we give a similar estimate without the convexity assumption.

Proposition 3.3.2. *Let M_t be a solution to (3.1) such that $\lambda \in \Gamma_{k+1}$. Then, for $R > 0$ and $x_0 \in \mathbb{R}^{n+1}$, the following estimate holds*

$$\phi_+(x,t)v(x,t) \leq \sup_{x \in M_0} \{\phi_+(x,0)v(x,0)\},$$

where $\phi_+ = \max\{\phi, 0\}$, $\phi(x,t) = R^2 - |x - x_0|^2 - 2\dfrac{n-k}{k+1}t$ and v is defined in the support of ϕ_+.

Proof. Firstly, the proof of this result is very similar to the one give in [EH2] for the mean curvature flow.

In addition, we note that without loosing generality its enough to consider $x_0 = 0$.

Let $\eta(r) = (R^2 - r^2)^2$, where $r^2 = |x|^2 + 2\dfrac{n-k}{k+1}t$ as in Equation (3.65). We note that $\dfrac{(\eta')^2}{\eta} = 4$ and $\eta'' = 2$, where $(\cdot)'$ denotes the derivative with respect r.

Next, we study the equation which satisfies ηv^2. In fact, we have

$$(\partial_t - \Box_k)(\eta v^2) = v^2(\partial_t - \Box_k)\eta + 2v\eta(\partial_t - \Box_k)v - 2\eta\,||\nabla v||_k^2 - 4v\,\langle \nabla \eta, \nabla v \rangle_k. \qquad (3.66)$$

Note that by Equation (3.65) the term $(\partial_t - \Box_k)\eta$ satisfies

$$(\partial_t - \Box_k)\eta = -2(R^2 - r^2)(\partial_t - \Box_k)r^2 - 2\,||\nabla r^2||_k^2 \leq -2\,||\nabla |x|^2||_k^2.$$

Then, by applying this equation together with Equation (3.62) onto Equation (3.66), we

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obtain

$$(\partial_t - \Box_k)\,(\eta v^2) \leq -2v^2 \left|\left|\nabla|x|^2\right|\right|_k^2 - 6\eta\,||\nabla v||_k^2 - 2|A|_k^2\eta v^2 - 4v\,\langle\nabla\eta, \nabla v\rangle_k.\qquad(3.67)$$

On the other hand, we have the term

$$-4v\,\langle\nabla\eta, \nabla v\rangle_k = -6v\,\langle\nabla\eta, \nabla v\rangle_k + \eta^{-1}\,\langle\nabla\eta, \nabla(\eta v^2)\rangle_k - \eta^{-1}v^2\,||\nabla\eta||_k^2.$$

Note that $\eta^{-1}v^2\,||\nabla\eta||_k^2 = 4v^2\,||\nabla|x|^2||_k^2$.

Then, by combining the above equation with Equation (3.67), it holds

$$(\partial_t - \Box_k)\,(\eta v^2) \leq -6v^2\left|\left|\nabla|x|^2\right|\right|_k^2 - 6\,\langle\nabla\eta, \nabla v\rangle_k - 6\eta\,||\nabla v||_k^2 + \eta^{-1}\,\langle\nabla\eta, \nabla(\eta v^2)\rangle_k,\quad(3.68)$$

note that we dropped the term $-2|A|_k^2\eta v^2$.

Now, let us estimate the following term

$$\begin{aligned}
-6v\,\langle\nabla\eta, \nabla v\rangle_k &= 6Q_{k,ij}v\frac{|\eta'|}{\sqrt{\eta}}\nabla_i|x|^2\sqrt{\eta}\nabla_j v\\
&\leq 6Q_{k,ij}\left((\eta')^2\eta^{-1}\varepsilon\frac{\nabla_i|x|^2\nabla_j|x|^2}{2}v^2 + \eta\frac{\nabla_i v\nabla_j v}{2\varepsilon}\right)\\
&= 12\varepsilon v^2\left|\left|\nabla|x^2|\right|\right|_k + \frac{3}{\varepsilon}\eta\,||\nabla v||_k^2.
\end{aligned}$$

where in the second line we use Young's Inequality.

Then, by plugging this estimate onto (3.68) we get

$$(\partial_t - \Box_k)\,(\eta v^2) \leq (12\varepsilon - 6)v^2\left|\left|\nabla|x|^2\right|\right|_k^2 + \left(\frac{3}{\varepsilon} - 6\right)\eta\,||\nabla v||_k^2 + \eta^{-1}\,\langle\nabla\eta, \nabla(\eta v^2)\rangle_k.$$

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Therefore, if we choose $\varepsilon = \frac{1}{2}$ it follows that

$$(\partial_t - \Box_k)(\eta v^2) \le \eta^{-1} \left\langle \nabla \eta, \nabla(\eta v^2) \right\rangle_k.$$

Then, since $\lambda \in \Gamma_{k+1}$, the operator $\Box_k$ is locally uniformly elliptic. In particular, the weak parabolic maximum principle applied to ηv^2 yields the result. Moreover, since changing η by ϕ_+ does not modify the estimate as long as v is defined in the support of ϕ_+. $\qquad \square$

Remark 3.3.3. Note that this estimate as $R \to \infty$ gives a gradient global bound for entire graphs of the Q_k-flow whose principal curvatures belong at Γ_{k+1}.

Now we give the proof for the second order estimate for the Q_k-flow.

Proof of Theorem 3.0.4. The proof is very similar to the one given in [EH2] for the mean curvature flow. For completeness we give all the details here.

Let $\varphi : I \to \mathbb{R}$ be a smooth function defined in a open interval $I \subset \mathbb{R}$. This function will be chosen later, and its derivatives will be denoted by φ' and φ''. Then, we consider

$$(\partial_t - \Box_k)\left(H^2 \varphi(v^2)\right) = 2H\varphi(\partial_t - \Box_k)H - 2\varphi \|\nabla H\|_k^2 + 2vH^2 \varphi'(\partial_t - \Box_k)v$$
$$- 2H^2 \varphi' \|\nabla v\|_k^2 - 2\left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \|\nabla v\|_k^2.$$

Then, by replacing the above equation with Equations (3.62) and (3.63) together with the concavity of Q_k, it follows

$$(\partial_t - \Box_k)(H^2 \varphi) \le 2H^2 \varphi |A|_k^2 - 2\varphi \|\nabla H\|_k^2 - 2v^2 H^2 \varphi' |A|_k^2 \qquad (3.69)$$
$$- 6H^2 \varphi' \|\nabla v\|_k^2 - 2\left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \|\nabla v\|_k^2.$$

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Next, we estimate the term $-2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k$. Indeed, we have

$$
\begin{aligned}
-2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k &= -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + H^2\varphi^{-1}\left\|\nabla\varphi\right\|_k^2 - 4H\varphi'v\left\langle\nabla H, \nabla v\right\rangle_k \\
&= -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + 4H^2\varphi^{-1}(\varphi')^2v^2\left\|\nabla v\right\|_k^2 - 4H\varphi'v\left\langle\nabla H, \nabla v\right\rangle_k \\
&\leq -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + 6H^2\varphi^{-1}(\varphi')^2v^2\left\|\nabla v\right\|_k^2 + 2\varphi\left\|\nabla H\right\|_k^2.
\end{aligned}
$$

In the last line we used $4H\varphi'v\left\langle\nabla H, \nabla v\right\rangle_k \leq 2\varphi\left\|\nabla H\right\|_k^2 + 2\varphi^{-1}(\varphi')^2v^2\left\|\nabla v\right\|_k$.
Then, by substituting the last equation onto (3.69), we obtain

$$
\begin{aligned}
\left(\partial_t - \Box_k\right)H^2\varphi(v^2) \leq{}&2|A|_k^2H^2(\varphi - v\varphi') - \varphi^{-1}\left\langle\nabla(H^2\varphi), \nabla\varphi\right\rangle_k \\
&- \left\|\nabla v\right\|_k^2 H^2\left(6\varphi'(1 - \varphi'\varphi^{-1}v^2) + 4\varphi''v^2\right).
\end{aligned} \tag{3.70}
$$

Now we let $\phi = H^2\varphi(v^2)$ and choose $\varphi(x) = \dfrac{x}{1 - ax}$, where a is still to be chosen. In addition, since $\lambda \in \Gamma_{k+2}$, it follows that $|A|_k^2 \leq \tilde{C}(k,n)H^2$ (see Equation (3.58)). Therefore, we have

$$
\begin{aligned}
\left(\partial_t - \Box_k\right)\phi \leq{}& 2\tilde{C}(k,n)\varphi^{-2}(\varphi - v^2\varphi')\phi^2 - \varphi^{-1}\left\langle\nabla\phi, \nabla\varphi\right\rangle_k \\
&- \phi\left\|\nabla v\right\|_k^2\left(6\varphi'(1 - \varphi\varphi'v^2) + 4\varphi''v^2\right),
\end{aligned} \tag{3.71}
$$

and

$$
\begin{cases}
\varphi - v^2\varphi' = -a\varphi^2, \\[2mm]
\varphi^{-1}\left\langle\nabla\phi, \nabla\varphi\right\rangle_k = 2\varphi v^{-3}\nabla v, \\[2mm]
\left(6\varphi'(1 - \varphi'\varphi^{-1}v^2) + 4\varphi''v^2\right) = \dfrac{2a}{(1 - ax)^2}.
\end{cases}
$$

Consequently, by substituting these equations into Equation (3.71), we obtain

$$(\partial_t - \Box_k)\,\phi \leq -2\tilde{C}(k,n)a\phi^2 - 2\varphi v^{-3}\langle\nabla\phi,\nabla\varphi\rangle_k - \frac{2a\phi}{(1-av^2)^2}\,||\nabla v||_k^2\,. \tag{3.72}$$

On the other hand, we consider $\eta(x,t) = (R^2 - h(x,t))^2$ where $h(x,t)$ is the function given by hypothesis and which satisfies Equation (3.5).

Then, the evolution equation for η satisfies

$$(\partial_t - \Box_k)\,\eta \leq 2C(k,n)R^2 - 2\,||\nabla h||_k^2\,. \tag{3.73}$$

Furthermore, the evolution equation of $\phi\eta$ is given by

$$(\partial_t - \Box_k)\,(\phi\eta) = \eta\,(\partial_t - \Box_k)\,\phi + \phi\,(\partial_t - \Box_k)\,\eta - 2\,\langle\nabla\eta,\nabla\phi\rangle_k\,. \tag{3.74}$$

Note that the term

$$-2\,\langle\nabla\eta,\nabla\phi\rangle_k = -2\eta^{-1}\,\langle\nabla\eta,\nabla\phi\eta\rangle_k + 8\phi\,||\nabla h||_k^2\,.$$

Then, by substituting Equations (3.72) and (3.73) with the above equation into Equation (3.74), we obtain

$$
\begin{aligned}
(\partial_t - \Box_k)\,(\phi\eta) &\tag{3.75}\\
\leq -2a\tilde{C}(k,n)\phi^2\eta &- \frac{2a\phi\eta}{(1-av^2)^2}\,||\nabla v||_k^2 + 2C(k,n)\phi R^2\\
&- 2\eta^{-1}\,\langle\nabla(\phi\eta),\nabla\eta\rangle + 6\phi\,||\nabla h||_k^2 - 2\varphi\eta v^{-3}\,\langle\nabla\phi,\nabla v\rangle_k\,.
\end{aligned}
$$

Note that the term $-2\varphi\eta v^{-3}\langle\nabla\phi,\nabla v\rangle_k$ satisfies

$$-2\varphi\eta v^{-3}\langle\nabla\phi,\nabla v\rangle_k = -2v^{-3}\varphi\langle\nabla(\phi\eta),\nabla v\rangle_k + 4\phi\langle\eta^{1/2}\nabla v, v^{-3}\varphi\nabla h\rangle_k$$

$$\leq -2v^{-3}\varphi\langle\nabla(\phi\eta),\nabla v\rangle_k + 2\phi\left(\eta\frac{||\nabla v||_k^2}{\varepsilon} + \varphi^2 v^{-6}\varepsilon\,||\nabla h||_k^2\right)$$

$$= -2v^{-3}\varphi\langle\nabla v,\nabla(\phi\eta)\rangle_k + 2(\phi\eta)\frac{a}{(1-av^2)^2}\,||\nabla v||_k^2 + \frac{\phi}{v^2 a}\,||\nabla h||_k^2.$$

In second line we use Young's inequality, in third line we use $\varepsilon = \dfrac{(1-av^2)^2}{a}$ and $\varphi^2 v^{-6}\varepsilon = \dfrac{v^{-2}}{a}$.

Then, by applying the above equation onto Equation (3.75), it follows that

$$(\partial_t - \Box_k)(\phi\eta) \leq -2a\tilde{C}(k,n)\eta\phi^2 + C(k,n)\phi\left(h\left(\frac{1}{v^2 a}+1\right)+R^2\right) \tag{3.76}$$

$$-2\langle\nabla(\phi\eta), v^{-3}\varphi\nabla v - 2\eta^{-1}\nabla\eta\rangle_k.$$

Finally, we consider the test function $G = t\phi\eta$. Note that G reaches it maximum at $t_0 > 0$ and in a interior point of M_{t_0}. In particular, at this point we have that $\nabla G = t\nabla(\phi\eta) = 0$ and

$$0 \leq (\partial_t - \Box_k)\,G \leq -2a\tilde{C}(k,n)\phi^2\eta t_0 + C(k,n)\phi t_0\left(\frac{h}{v^2 a}+h+R^2\right) + \phi\eta. \tag{3.77}$$

Then, by multiplying $\dfrac{\eta t_0}{2a\tilde{C}(k,n)}$ to Equation (3.77), we see that

$$m(t_0)^2 \leq m(t_0)\frac{C(k,n)}{2a\tilde{C}(k,n)}\left(t_0\left(\frac{h}{v^2 a}+h+R^2\right)+\eta\right),$$

where $m(t_0) = \sup_{[0,t_0]}\sup_{M_t} G$.

Consequently, since $\eta \leq R^4$ and $M_t = \{h(x, t) \leq R^2\}$, it follows that

$$m(t_0) \leq c(k, n) \left(t_0 R^2 + R^4\right) \sup_{s \in [0, t_0]} \sup_{M_s} v^2,$$

provided that $a = \dfrac{1}{2} \inf_{s \in [0, t_0]} \inf_{M_s} v^{-2}$.

Finally, since $\varphi(v^2) \geq 1$ and $\eta > (1 - \theta)^2 R^4$ in $\{x \in M_t : h(x, t) \leq \theta R^2\}$ for $\theta \in [0, 1)$, the estimate from Theorem 3.0.4 follows since

$$H^2 \leq \frac{m(t_0)}{R^4 t(1 - \theta)^2}, \quad \text{holds on } M_t \text{ with } t \leq t_0.$$

$\square$

Remark 3.3.4. By Remark 3.3.2 and Theorem 3.0.4, an analogous result to [EH1] and [EH2] can be proven for the Q_k-flow for convex initial data. But it is worth mentioning that these results were proven in [CD] using different techniques.

Now we develop a similar estimate for Equation (3.18). Recall that we are assuming that $M_0 = M$ is locally a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$.

Lemma 3.3.3. *Let M be a Q_k-translator which is locally a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$. Then, we have the following equations at $p \in M$:*

$$\Box_k u = -Q_k \langle \nu, \vec{w} \rangle \quad \text{and} \quad \nabla u = \vec{w}^\top := \langle e_i, \vec{w} \rangle e_i, \quad \text{where } u = \langle p, \vec{w} \rangle, \tag{3.78}$$

$$\Box_k r^2 = 2 \left(-Q_k \langle \nu, p \rangle + Q_{k,ij} \delta_j^i\right), \quad \text{where } r^2 = |p|^2, \tag{3.79}$$

$$\Box_k H + \sum_{i=1}^{n} Q_{k,ab;cd} \nabla^i h_a^b \nabla_i h_c^d + |A|_k^2 H + \langle \nabla H, e_{n+1} \rangle = 0, \tag{3.80}$$

$$\Box_k Q_k + |A|_k^2 Q_k + \langle \nabla Q_k, e_{n+1} \rangle = 0, \tag{3.81}$$

$$\Box_k v - v^2 \langle \nabla Q_k, \vec{w} \rangle - |A|_k^2 v - 2v^{-1} ||\nabla v||_k^2 = 0, \quad \text{where } v = \langle \nu, w \rangle^{-1}. \tag{3.82}$$

Proof. Firstly, we use normal coordinates centered at $p \in M$. This means that $\{e_i\}_{i=1}^n \subset T_pM$ is an orthogonal basis of eigenvalues of the second fundamental form (i.e: $h_i^j(p) = h_{ij}(p) = \lambda_i(p)\delta_j^i$ and λ_i is the principal curvature) at p.

We note that by the calculation given in the proof of Lemma 3.3.1 we only need to show Equations 3.78 (3.80) (3.81) (3.82).

Then, the height function $u = \langle p, \vec{w} \rangle$ satisfies

$$\nabla_i u = \langle e_i, \vec{w} \rangle,$$

$$\nabla_j \nabla_i u = \langle \nabla_j e_i, \vec{w} \rangle = -h_{ij} \langle \nu, \vec{w} \rangle,$$

$$\Box_k u = -Q_k \langle \nu, \vec{w} \rangle.$$

This proves Equation (3.78).

On the other hand, for the coefficient of the second fundamental form of M we have

$$\nabla_j \nabla_i Q_k = \frac{\partial^2 Q_k}{\partial h_{cd} \partial h_{ab}} \nabla_j h_{cd} \nabla_i h_{ab} + \Box_k h_{ij} + |A|_k^2 h_{ij}$$
$$- Q_k h_{il} h_{lj} + \frac{\partial Q_k}{\partial h_{ab}} (h_{ib} h_{am} h_{mj} - h_{im} h_{aj} h_{mb}).$$

Here we used the equations from Theorem 2.1 given in [HP].

Moreover, since $Q_k = \langle \nu, e_{n+1} \rangle$, it follows that

$$\nabla_j \nabla_i Q_k = -\langle \nabla h_{ij}, e_{n+1} \rangle - h_{il} h_{jl} Q_k,$$

where $e_{n+1} = (\underbrace{0, \ldots, 0}_{n-\text{times}}, 1)$. Then, after combining both equations for $\nabla_j \nabla_i Q_k$, we obtain

$$
\begin{aligned}
\square_k h_{ij} = & - \frac{\partial^2 Q_k}{\partial h_{cd} \partial h_{ab}} \nabla_j h_{cd} \nabla_i h_{ab} \\
& - |A|_k^2 h_{ij} - \langle \nabla h_{ij}, e_{n+1} \rangle - \frac{\partial Q_k}{\partial h_{ab}} \left(h_{ib} h_{am} h_{mj} - h_{im} h_{aj} h_{mb} \right).
\end{aligned}
$$

Consequently, Equation (3.80) follows by taking trace in the above equation.

For the function $\langle \nu, \vec{w} \rangle$, it follows that

$$
\nabla_i \langle \nu, \vec{w} \rangle = -h_{il} \langle e_l, \vec{w} \rangle \text{ and } \nabla_j \nabla_i \langle \nu, \vec{w} \rangle = -\nabla_l h_{ij} \langle e_l, \vec{w} \rangle - h_{il} h_{lj} \langle \nu, \vec{w} \rangle,
$$

note that we also use the Codazzi equations in the last line. Therefore, it holds

$$
\square_k \langle \nu, \vec{w} \rangle + |A|_k^2 \langle \nu, \vec{w} \rangle + \langle \nabla Q_k, \vec{w} \rangle = 0. \tag{3.83}
$$

Finally, for $v = \langle \nu, \vec{w} \rangle^{-1}$, we have

$$
\nabla_i v = -v^2 \nabla_i \langle \nu, \vec{w} \rangle, \text{ and } \square_k v = -v^2 \square_k \langle \nu, \vec{w} \rangle + 2v^{-1} \left\| \nabla v \right\|_k^2.
$$

Then, by substituting the above equations with Equation (3.83), Equation (3.82) holds. Note that Equation (3.81) follows by taking $\vec{w} = e_{n+1}$. $\qquad \square$

Now we derive an Ecker-Huisken type interior estimate for Q_k-translators with principal curvature vector $\lambda \in \Gamma_{k+2}$. As in the parabolic case, we assume that M is given by the graph of a function defined in a ball of radius R in a hyperplane orthogonal to e_{n+1} and there exist a non-negative function $h(x)$ such that $M = \{x \in M : h(x) \leq R^2\}$ and satisfies the estimates in Equation (3.6). See Remark 3.3.5 below for an example of this function.

Proof of Theorem 3.0.5. The proof is very similar to the given in 3.0.4. Therefore, we only

point out the main differences. We consider the test function

$$G(x) = H^2 \varphi(v^2) \eta(h),$$

where $\varphi(x) = \dfrac{x}{1 - ax}$, $a = \dfrac{1}{2} \inf v^{-2}$, $\eta = (R^2 - h)^2$ and h satisfies (3.6).

Let $\phi = H^2 \phi(v^2)$, then it follows that

$$-\Box_k \phi = -2H\varphi \Box_k H - 2\varphi \left\| \nabla H \right\|_k^2 - 2vH^2 \varphi' \Box_k v$$
$$- 2H^2 \varphi' \left\| \nabla v \right\|_k^2 - 2 \left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \left\| \nabla v \right\|_k^2.$$

Then, by replacing the above equation by Equations (3.80) and (3.82) together with the concavity of Q_k, it follows

$$-\Box_k \phi \leq 2H^2 \varphi |A|_k^2 (\varphi - \varphi' v^2) - 2\varphi \left\| \nabla H \right\|_k^2$$
$$- 6H^2 \varphi' \left\| \nabla v \right\|_k^2 - 2 \left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \left\| \nabla v \right\|_k^2$$
$$- 2v^3 H^2 \varphi' \left\langle \nabla Q_k, e_{n+1} \right\rangle + 2H\varphi \left\langle \nabla H, e_{n+1} \right\rangle.$$

We note that since $Q_k = \langle \nu, e_{n+1} \rangle$, it follows that $\nabla Q_k = -v^{-2} \nabla v$. Therefore, we have

$$-\Box_k \phi \leq 2H^2 \varphi |A|_k^2 (\varphi - \varphi' v^2) - 2\varphi \left\| \nabla H \right\|_k^2 + \left\langle \nabla \phi, e_{n+1} \right\rangle$$
$$- 6H^2 \varphi' \left\| \nabla v \right\|_k^2 - 2 \left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \left\| \nabla v \right\|_k^2.$$

In the first line we used that $\nabla \phi = 2H\varphi \nabla H + 2H^2 \varphi' v \nabla v$.

Therefore, by following the steps given in the proof of Theorem 3.0.4 it follows that

$$-\Box_k \phi \leq -2\tilde{C}(k,n) a\phi^2 - 2\varphi v^{-3} \left\langle \nabla \phi, \nabla \varphi \right\rangle_k - \frac{2a\phi}{(1 - av^2)^2} \left\| \nabla v \right\|_k^2 + \left\langle \nabla \phi, e_{n+1} \right\rangle.$$

On the other hand, $\eta(x) = (R^2 - h(x))^2$ satisfies the following equation

$$-\Box_k \eta = 2\eta^{\frac{1}{2}}\Box_k h - 2\,\|\nabla h\|_k^2 \leq 2C(k,n)R^2 + \langle \nabla \eta, e_{n+1}\rangle - 2\,\|\nabla h\|_k^2\,.$$

Therefore, for $G = \phi\eta$ holds the following estimate

$$-\Box_k G = -\,\eta\Box_k\phi - \phi\Box_k\eta - 2\,\langle \nabla\eta, \nabla\phi\rangle_k\,.$$

$$\leq -\,2\tilde{C}(k,n)aG\phi - 2\eta\varphi v^{-3}\,\langle \nabla\phi, \nabla\varphi\rangle_k - \frac{2aG}{(1-av^2)^2}\,\|\nabla v\|_k^2 + \langle \nabla\phi, e_{n+1}\rangle$$

$$+\,2C(k,n)R^2\phi + \langle \nabla G, e_{n+1}\rangle - 2\phi\,\|\nabla h\|_k^2 - 2\,\langle \nabla\eta, \nabla\phi\rangle_k\,.$$

Note that we can follow the same steps from the proof given in the parabolic case, this will yields to

$$-\Box_k G \leq -\,2a\tilde{C}(k,n)G\phi + C(k,n)\phi\left(h\left(\frac{1}{v^2 a}+1\right)+R^2\right)$$

$$-\,2\,\langle \nabla G, v^{-3}\varphi\nabla v - 2\eta^{-1}\nabla\eta\rangle_k + \langle \nabla G, e_{n+1}\rangle\,.$$

Then, as in the parabolic case, G reaches it maximum at an interior point of M. In particular, at this point we have that $\nabla G = 0$ and

$$0 \leq -\Box_k G \leq -\,2a\tilde{C}(k,n)G\phi + C(k,n)\phi\left(h\left(\frac{1}{v^2 a}+1\right)+R^2\right)$$

Then, by multiplying $\dfrac{\eta}{2a(k+1)}$ in the last line, we see that

$$m^2 \leq m\frac{C(k,n)}{2a\tilde{C}(k,n)}\left(t_0\left(\frac{h}{v^2 a}+h+R^2\right)+\eta\right),$$

where $m = \sup_{M} G$.

Consequently, since $a = \inf_M v^{-2}$, $\eta \leq R^4$ and $M = \{h(x) \leq R^2\}$, it follows that there is a constant $c(k,n)$ such that

$$m \leq c(k,n) \left(R^2 + R^4\right) \sup_M v^2.$$

Finally, since $\varphi(v^2) \geq 1$ and $\eta > (1-\theta)^2 R^4$ in $\{x \in M_t : h(x) \leq \theta R^2\}$ for $\theta \in [0,1)$, the estimate from Theorem 3.0.4 follows since

$$H^2 \leq \frac{m}{R^4(1-\theta)^2}, \text{ holds on } M.$$

$\square$

Remark 3.3.5. We note that the function $h = r^2 - u^2$, where $r = |p|^2$ and $u = \langle p, e_{n+1} \rangle$, satisfies the following equations

$$\nabla_i h = \nabla_i r^2 - 2u\nabla_i u = 2\langle e_i, p - ue_{n+1}\rangle \Rightarrow \langle \nabla h, e_{n+1}\rangle = 2\langle e_i, p - ue_{n+1}\rangle\langle e_i, e_{n+1}\rangle,$$

$$\square_k h = 2(-Q_k\langle \nu, p\rangle + Q_{k,ij}\delta_j^i) + 2uQ_k\langle \nu, e_{n+1}\rangle - 2\,||\nabla u||_k^2$$

$$= 2Q_k\langle \nu, p - ue_{n+1}\rangle + 2Q_{k,ij}\delta_j^i - 2\,||\nabla u||_k^2.$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM of the second fundamental form of M and $e_{n+1} = (0,\ldots,0,1)$. Then, since $Q_k = \langle \nu, e_{n+1}\rangle$, it follows that

$$|\square_k h + \langle \nabla h, e_{n+1}\rangle| \leq 2(n+1)|p - ue_{n+1}| + Q_{k,ij}\delta_j^i + 2\,||\nabla u||_k^2$$

$$= 2(n+1)h + 2\frac{n-k}{k+1} + \left|\left|e_{n+1}^\top\right|\right|_k^2.$$

Therefore, this function cannot be used in the statement of Theorem 3.0.5 without an estimate on h depending only on k and n.

Chapter 4

Rotationally Symmetric γ-Translators

From a historical point of view, rotationally symmetric hypersurfaces are often the first constructions made when one wants to prescribe a curvature function.

The most famous examples of these surfaces in $\mathbb{R}^3$ are the plane and the catenoid, which are the unique rotationally symmetric minimal surfaces, and spheres, which are the unique closed surface of constant mean curvature. Note that in these examples the mean curvature H is 0 in the former case and equal to a positive constant in the latter.

In relation with γ-translators, the "bowl" soliton is the unique H-translator which is a strictly convex entire graph in $\mathbb{R}^3$. That solution was found by Altschuler and Wu in [AW] in $\mathbb{R}^3$, and they noticed later that the same construction works in $\mathbb{R}^{n+1}$. In addition, this solution behaves at infinity like a paraboloid. More precisely, the authors in [CSS] showed that the "bowl" soliton in $\mathbb{R}^{n+1}$, given by the graph of $u(x) = u(|x|)$, satisfies

$$u(|x|) = \frac{|x|^2}{2n} - \ln(|x|) + O\left(\frac{1}{|x|}\right), \text{ as } |x| \to \infty.$$

Regarding to the classification of H-translators in $\mathbb{R}^{n+1}$, the study of the "bowl" soliton led to important results in this area. For instance, in [Has], the author showed that a α-

noncollapse[1] and convex H-translator must be the bowl soliton. Moreover, in [MSHS], the authors showed that a convex H-translator which is asymptotic to the "bowl" soliton, is in fact the "bowl" soliton.

Furthermore, the authors in [SX] showed that mean convex H-translators in $\mathbb{R}^3$ are in fact convex. Consequently, the "bowl" soliton is the unique mean convex entire H-translator in $\mathbb{R}^3$. We refer the reader to [HIMW2] and [HMW] for the classification of H-translators which are graphs and semigraphs[2], respectively.

Therefore, "bowl"-type solutions[3] can lead to important results in the classification of γ-translators in $\mathbb{R}^{n+1}$ as we will show in Chapter 5.

For this reason in this chapter we construct "bowl"-type γ-translators in $\mathbb{R}^{n+1}$ for the following families of curvature functions $\sqrt[r]{S_n}$ and Q_{n-1}, where $Q_k = \dfrac{S_{k+1}}{S_k}$ and $S_k(\lambda)$ denotes the elementary symmetric polynomial in n variables of degree k evaluated in the principal curvatures of the γ-translator, see Equation 2.3 in the Introduction for the definition of S_k.

Recall from the introduction that a γ-translator is a hypersurface $M \subset \mathbb{R}^{n+1}$ which satisfies the equation

$$\gamma(\lambda_1, \ldots, \lambda_n) = \langle \nu, e_{n+1} \rangle,$$

where ν is the unit normal vector and λ_i is the principal curvature of M in $\mathbb{R}^{n+1}$, respectively. Note that these hypersurfaces evolve under translation in direction e_{n+1} when the γ-flow (2.1) is applied to them.

As we mentioned in the Introduction, the author in [Ren] constructed "bowl"-type γ-translators for a general class of curvature functions. Moreover, he showed they are asymp-

---

[1]Let Σ be a mean convex H-translator which bounds an open region $\Omega \subset \mathbb{R}^{n+1}$, then we say Σ is α-noncollapsed if for all $p \in \Sigma$ there exist an open ball $B \subset \Omega$ of radius $\dfrac{\alpha}{H(p)}$ with $p \in \partial B$.

[2]A properly embedded translator M (without boundary) that contains a discrete, nonempty collection L of vertical lines such that $M \setminus L$ is a graph.

[3]Strictly convex rotationally symmetric graphs in $\mathbb{R}^{n+1}$ which are entire or defined in a round ball.

totic to paraboloids up to first order at infinity, and characterized when the graph will be entire or defined in round ball of radius $\gamma(1, \ldots, 1)$ (see Remark 4.0.1 below) in terms of γ. A summary of this dichotomy is characterized by the following hypotheses on γ (see Theorems 1.3-1.4 in [Ren]):

- If $\gamma(0, 1, \ldots, 1) > 0$, then the "bowl"-type solution is entire.

- If $\gamma(0, 1, \ldots, 1) = 0$, and, under the constraint $\gamma(x, y, \ldots, y) = 1$, $x \to L > 0$ as $y \to \infty$, then the "bowl"-type solution is defined in round ball.

  In addition, if $L = 0$, then we have the following extra conditions:

  - If $x = O(y^{-1})$[4], then the "bowl"-type solution is entire.

  - If there are positive constants C and $k \in (0, 1)$ such that $x \geq Cy^{-k}$ for large enough y, then the "bowl"-type solution is defined in a round ball.

On the other hand, we want to mention that in $\mathbb{R}^3$ the "bowl"-type solutions are explicit for the curvature functions Q_{n-1} and $\sqrt[n]{S_n}$ (see Fig. 4.1 below), and implicit for $n \geq 3$.

In addition, these curvature functions agree with the result exposed in [Ren] above: $\sqrt[n]{S_n}$ and Q_{n-1} both vanish at $(0, 1, \ldots, 1)$ and satisfy

$$\sqrt[n]{S_n}(x, y, \ldots, y) = 1 \Leftrightarrow x = \frac{1}{y^{n-1}} \to 0 \text{ as } y \to \infty,$$

$$Q_{n-1}(x, y, \ldots, y) = 1 \Leftrightarrow x = \frac{y}{y - \binom{n-1}{n-2}} \to 1 \text{ as } y \to \infty.$$

We note that this description for the "bowl"-type $\sqrt[n]{S_n}$-translator in $\mathbb{R}^{n+1}$ only works for $n = 2$ since $x = O(y^{n-1})$.

---

[4]The big O notation as $|y| \to \infty$, i.e: $f(y) = O(g(y))$ if there exist a constant $C > 0$ and $y_0 \gg 1$ such that $|f(y)| \leq Cg(y)$ for all $y \geq y_0$.

The summary of the results of this chapter reads as follows.

Theorem 4.0.1. *For $n \geq 3$ there exists a unique[5] smooth strictly convex rotationally symmetric translator in $\mathbb{R}^{n+1}$ for the curvature functions Q_{n-1} and $\sqrt[r]{S_n}$.*
In addition, it holds

1. *For $n \geq 3$, the Q_{n-1}-translator is defined in a ball of radius $Q_{n-1}(1,\ldots,1) = \dfrac{1}{n}$ and is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}\left(\frac{1}{n}\right) \times \mathbb{R}$, i.e: If Σ denotes the Q_{n-1}-translator, then $p \in \Sigma$ we have*

$$\min\left\{\lambda_i(p) : i = 1,\ldots,n\right\} = \lambda_n(p) \to 0,$$
$$\forall i \in \{1,\ldots,n-1\}, \lambda_i(p) \to \frac{1}{Q_{n-1}(1,\ldots,1)},$$

   as $|p| \to \infty$.

2. *For $n \geq 3$, the $\sqrt[r]{S_n}$-translator is entire.*

This existence part of this theorem will be proven in Thm 4.4.2 and Thm 4.5.2, and the uniqueness in Thm 4.6.3.

Now we briefly describe the strategy used in the proof of Theorem 4.0.1.
Firstly, rotationally symmetric translators are hypersurface of the form $\{(x, u(x)) : x \in B_r(0)\}$ such that $u(x) = u(|x|) = u(r)$ and satisfy Equation (2.5), see Remark 4.0.1 below for details. Consequently, Equation (2.5) can be written as ODE of the form

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2)f\left(\frac{\dot{\mathbf{u}}}{r}\right), \tag{4.1}$$

where the derivatives are taken with respect the variable $r = \sqrt{x_1^2 + \ldots + x_n^2}$, $f(x,y)$ is defined in an open cone in $\mathbb{R}^2$. We remark that the above equation can be obtained because the curvature functions are 1-homogeneous and strictly increasing in the positive cone

---

[5]The uniqueness is among rotationally symmetric translators.
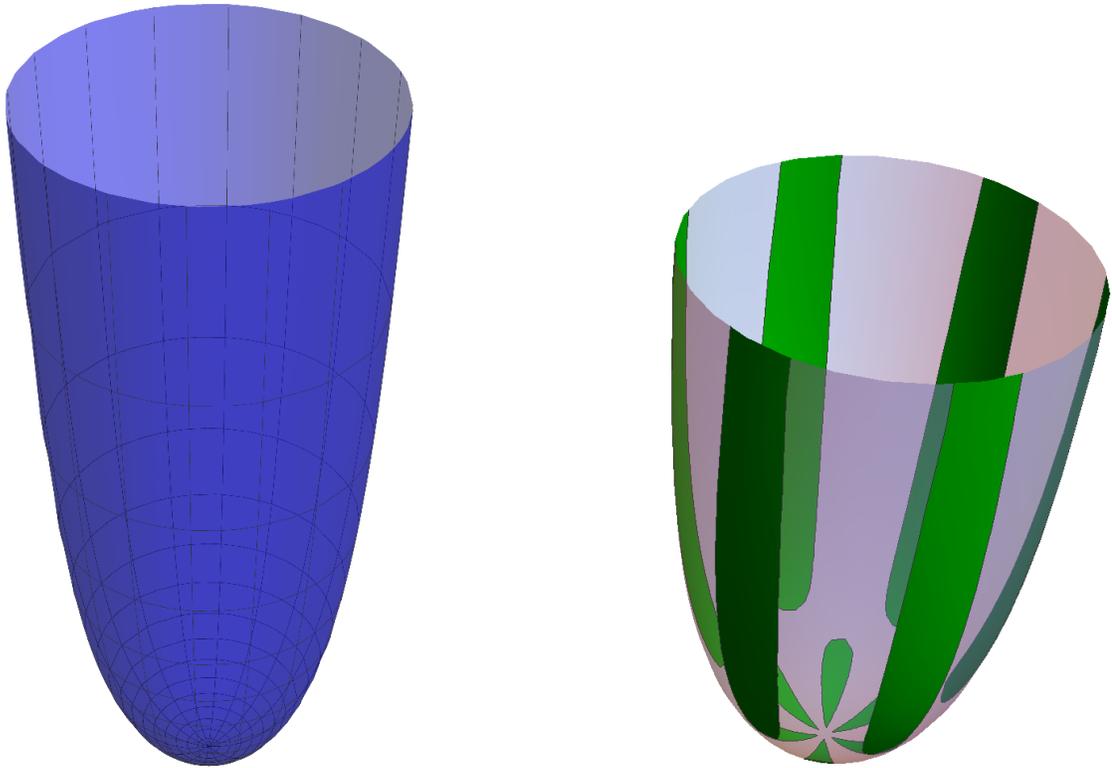
Figure 4.1: The $\sqrt{S_2}$-translator given by $z = \int_0^{\sqrt{x^2+y^2}} \sqrt{e^{s^2} - 1}\, ds$ (left) and the Q_1-translator given by $z = -\ln(1 - x^2 - y^2)$ (right), both in $\mathbb{R}^3$. Image courtesy of Francisco Martín.

$\{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$[6]. Secondly, to find the domain of $\mathbf{u}$, we find barriers solutions to a first order reduction of Equation (4.1) by setting $\mathbf{v} = \dot{\mathbf{u}}$ and $\mathbf{v}(0) = 0$. Finally, we use standard ODE theory (implicit function theorem, Arzela-Ascoli arguments, solution extension results, etc...) for first order ODEs to show that these unique solutions exist for $r \in [0, r_n)$ and they are smooth.

The last result of this chapter is an implicit "catenoid"-type solution constructed in the spirit of the "wing like" H-translators found in [CSS], see Fig. 4.2 above.

Theorem 4.0.2. *Let $a \in [0, 1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi), \ z \in \mathbb{R} \right\},$$

where $r_a(z)$ is defined implicitly by the Equation

$$z = \int_1^{r_a(z)} \sqrt{e^{s^2 - a} - 1}\, ds, \ \text{with} \ r_a(0) = 1,$$

is a complete $\sqrt{S_2}$-translators in $\mathbb{R}^3$ which satisfies $H < 0$ and $K > 0$.

Remark 4.0.1. In Proposition 4.1.1 below, we show that the principal curvatures of a rotationally symmetric graph $M = \{(x, \mathbf{u}(|x|)), x \in [0, r_n)\} \subset \mathbb{R}^{n+1}$ are given by

$$\lambda_1 = \ldots = \lambda_{n-1} = \frac{\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}} \ \text{and} \ \lambda_n = \frac{\ddot{\mathbf{u}}}{(1 + \dot{\mathbf{u}}^2)^{\frac{3}{2}}},$$

where the dot derivative is taken with respect the variable $r = |x| = \sqrt{x_1^2 + \ldots + x_n^2}$.

---

[6]An important fact is that we are not using the implicit function theorem for writing f. For general curvature functions, f can be written in terms of the connected components of the level set $\{\sqrt{1 + \dot{\mathbf{u}}^2}\gamma(\lambda) = 1\}$.
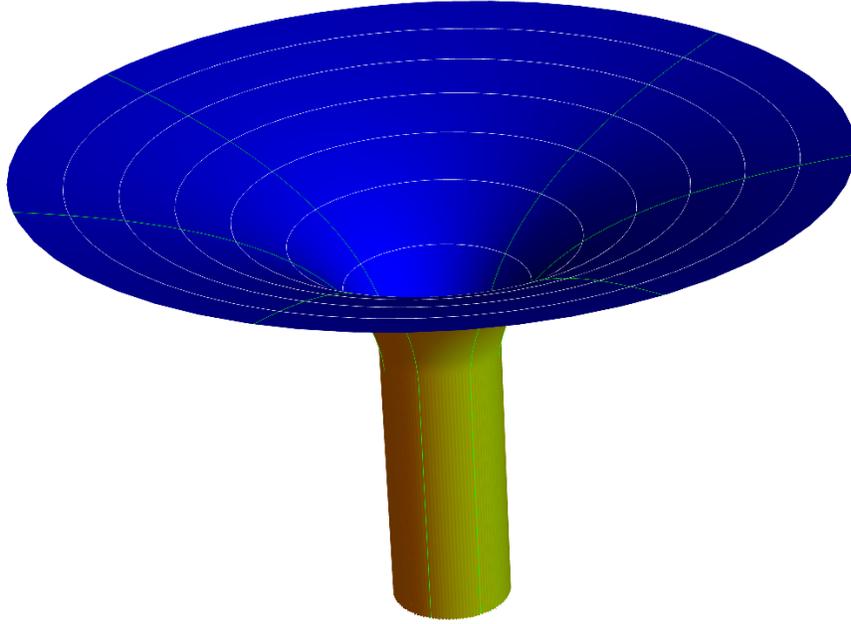
Figure 4.2: A $\sqrt{S_2}$-translator Σ_a in $\mathbb{R}^3$. Image courtesy of Francisco Martín.

In particular, the γ-translator equation has the following form

$$\gamma(\lambda_1, \ldots, \lambda_n) = \langle \nu, e_{n+1} \rangle = \frac{1}{\sqrt{1 + \dot{\mathbf{u}}^2}}. \tag{4.2}$$

Consequently, by assuming continuity in the second derivative of $\mathbf{u}$ together with the initial data $\dot{\mathbf{u}}(0) = 0$, we can see after taking limits when $r \to 0$ that

$$\gamma(\ddot{\mathbf{u}}(0), \ldots, \ddot{\mathbf{u}}(0)) = 1 \Leftrightarrow \ddot{\mathbf{u}}(0) = \frac{1}{\gamma(1, \ldots, 1)}.$$

In particular, with this information, we were able to find explicit barriers to describe our solutions.

The organization of this chapter goes as follows: In Section 4.1 we develop all the equations related to the geometry of rotationally symmetric γ-translators for $\gamma = Q_{n-1}$ and $\gamma = \sqrt[n]{S_n}$ in $\mathbb{R}^{n+1}$. In Section 4.2, we show the explicit γ-translators in $\mathbb{R}^3$ for $\gamma = \sqrt{S_2}$ and

$\gamma = Q_1$. In section 4.3, we show the barriers and results related to the extension of the solution domain of these ODEs. In Section 4.4, we prove the existence part of Theorem 4.0.1 for the "bowl"-type $\sqrt[n]{S_n}$-translator. In Section 4.5 we prove the existence part of Theorem 4.0.1 for the "bowl"-type Q_{n-1}-translator. In Section 4.6, we show the uniqueness and regularity part of Theorem 4.0.1 for these curvature functions. In Section 4.7, we prove Theorem 4.0.2.

4.1 Definitions and Properties

A graphical rotationally symmetric hypersurface $M \subset \mathbb{R}^{n+1}$ can be written as

$$M = \left\{ (r\theta, \mathbf{u}(r)) : \theta \in \mathbb{S}^{n-1}, 0 \le r < R \right\}, \tag{4.3}$$

where $\mathbf{u}(r) = u(|x|)$ is a smooth function $u : B_R(0) \to \mathbb{R}$ and $R \in [0, \infty]$. Note that we are using cylindrical coordinates $\theta = (\theta_1, \dots, \theta_{n-1})$, $r = |x|$ and $s = x_{n+1}$, where θ_i are the canonical coordinates of $\mathbb{S}^{n-1}$ and $|\cdot|$ denotes the euclidean norm.

Remark 4.1.1. We will use the dot notation to denote radial derivatives $\dfrac{\partial}{\partial r}$ and bold letters for rotationally symmetric functions.

The following proposition contains geometric properties about these hypersurfaces in $\mathbb{R}^{n+1}$.

Proposition 4.1.1. *Let $M \in \mathbb{R}^{n+1}$ be a rotationally symmetric graph. Then, in the frame $\left\{ \partial_{\theta_1}, \dots, \partial_{\theta_{n-1}}, \partial_r, \partial_s \right\} \subset T_p \mathbb{R}^{n+1}$, we have the following quantities at any $p \in M$:*

(a) *Unit normal vector:* $\nu = \dfrac{\partial_s - \dot{\mathbf{u}}\partial_r}{\sqrt{1 + \dot{\mathbf{u}}^2}}.$

(b) *Principal directions:* ∂_{θ_i}, $\dot{\mathbf{u}}\partial_s + \partial_r$.

(c) *Principal curvatures:* $\lambda_i = \dfrac{\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}}$, $\lambda_n = \dfrac{\ddot{\mathbf{u}}}{(1 + \dot{\mathbf{u}}^2)^{\frac{3}{2}}}.$

Proof. (a) Let $p \in M$, $p = (r\theta, \mathbf{u}(r))$. Then, a basis for T_pM given by the cylindrical coordinates (θ, r, s) is $e_i = r\partial_{\theta_i}$ for $i = 1, \ldots, n-1$ and $e_n = \dot{\mathbf{u}}\partial_s + \partial_r$.

Indeed, it is not hard to check that the vector $\nu = \dfrac{\partial_s - \dot{\mathbf{u}}\partial_r}{\sqrt{1 + \dot{\mathbf{u}}^2}}$ is unitary and $\nu \perp T_pM$.

Indeed, we have

$$\langle e_i, \nu \rangle = \frac{\langle r\partial_{\theta_i}, \partial_s - \dot{\mathbf{u}}\partial_r \rangle}{\sqrt{1 + \dot{\mathbf{u}}^2}} = 0, \text{ for } i = 1, \ldots, n-1$$

$$\langle e_n, \nu \rangle = \frac{\langle \partial_r + \dot{\mathbf{u}}\partial_s, \partial_s - \dot{\mathbf{u}}\partial_r \rangle}{\sqrt{1 + \dot{\mathbf{u}}^2}} = \frac{\dot{\mathbf{u}} - \dot{\mathbf{u}}}{\sqrt{1 + \dot{\mathbf{u}}^2}} = 0.$$

(b,c) Firstly, we note that the Christoffel symbols at p satisfy

$$\Gamma_{ks}^k = \Gamma_{kk}^s = \Gamma_{rr}^r = \Gamma_{\theta_i r}^r = 0 \text{ and } \Gamma_{\theta_i \theta_j}^r = -\delta_{ij}r,$$

where δ_{ij} denotes the Kronecker delta function and the index k with respect $\varepsilon_k \in \{\partial_{\theta_1}, \ldots, \partial_{\theta_{n-1}}, \partial_r, \partial_s\}$.

Therefore, the covariant derivatives in this coordinates are given by

$$\nabla_{e_i} e_j = \begin{cases} r^2 \left(\Gamma_{\theta_i \theta_j}^{\theta_k} \partial_{\theta_k} - \delta_{ij}r\partial_r \right), \text{ for } i, j = 1 \ldots, n-1, \\ r \left(\dot{\mathbf{u}} \left(\Gamma_{\theta_i s}^{\theta_k} \partial_{\theta_k} + \Gamma_{\theta_i s}^s \partial_s + \Gamma_{\theta_i s}^r \partial_r \right) + \left(\Gamma_{\theta_i r}^{\theta_k} \partial_{\theta_k} + \Gamma_{\theta_i r}^s \partial_s + \Gamma_{\theta_i r}^r \partial_r \right) \right), i \leq n-1 \text{ and } j = n, \\ \dot{\mathbf{u}}^2 \Gamma_{ss}^k \varepsilon_k + \dot{\mathbf{u}}\Gamma_{sr}^k \varepsilon_k + \ddot{\mathbf{u}}\partial_s + \Gamma_{rr}^k \varepsilon_k, \text{ for } i = j = n. \end{cases}$$

In addition, the metric in these coordinates is given by

$$g = \begin{pmatrix} r^2 g_{\mathbb{S}^{n-1}} & 0 \\ 0 & 1 + \dot{\mathbf{u}}^2 \end{pmatrix},$$

where $g_{\mathbb{S}^{n-1}}$ denotes the round metric of $\mathbb{S}^{n-1}$.

Consequently, since the principal curvatures are the eigenvalues of the shape operator

$A = g^{-1}II$, we calculate the terms of the second fundamental form of M

$$II_{ij} = \langle \nabla_{e_i} e_j, \nu \rangle = \begin{cases} \dfrac{\delta_{ij} r \dot{\mathbf{u}}}{\sqrt{1 + \dot{\mathbf{u}}^2}} & , i, j = 1, \ldots, n-1, \\ 0 & , i \leq n-1, j = n, \\ \dfrac{\ddot{\mathbf{u}}}{\sqrt{1 + \dot{\mathbf{u}}^2}} & , i = j = n. \end{cases}$$

Finally, the shape operator in this coordinates is given by

$$A(e_k, e_j) = g^{ik} II(e_k, e_j) = \begin{cases} 0 & , \text{ if } i \neq j, \\ \dfrac{\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}} & , \text{ if } i = j, \, i = 1, \ldots, n-1, \, . \\ \dfrac{\ddot{\mathbf{u}}}{(1 + \dot{\mathbf{u}}^2)^{\frac{3}{2}}} & , \text{ if } i = j = n. \end{cases}$$

We note that the shape operator A is diagonal, and the eigenvalues of this matrix correspond to the terms in the diagonal with eigenvectors given by $\{e_i\}_{i=1}^n$.

$\square$

In the following proposition we calculate the curvature functions in terms of r, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$.

Proposition 4.1.2. *The functions S_k and Q_k evaluated in the principal curvatures of M are given by*

$$S_k = \frac{1}{k} \binom{n-1}{k-1} \left(\frac{\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}} \right)^{k-1} \left[\frac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}} + \frac{k\ddot{\mathbf{u}}}{(1 + \dot{\mathbf{u}}^2)^{\frac{3}{2}}} \right];$$

$$Q_k = \frac{(n-k)}{(k+1)} \frac{\dot{\mathbf{u}}}{r\sqrt{1 + \dot{\mathbf{u}}^2}} \frac{(n-k-1)(1 + \dot{\mathbf{u}}^2)\dot{\mathbf{u}} + (k+1)r\ddot{\mathbf{u}}}{(n-k)(1 + \dot{\mathbf{u}}^2)\dot{\mathbf{u}} + kr\ddot{\mathbf{u}}};$$

Proof. The proof is a direct computation from the last proposition. In fact, by Proposition

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, the principal curvatures of M are given by

$$\lambda_1 = \ldots = \lambda_{n-1} = \frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} \quad \text{and} \quad \lambda_n = \frac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}.$$

Then, the function S_k evaluated at the principal curvatures is given by

$$S_k(\lambda) = S_{k,n}(\lambda) + \lambda_n S_{k-1,n}(\lambda)$$

$$= \binom{n-1}{k}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^k + \binom{n-1}{k-1}\frac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}}}\right)^{k-1}$$

$$= \frac{1}{k}\binom{n-1}{k-1}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^{k-1}\left[\frac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \frac{k\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right].$$

Finally, the function Q_k evaluated in the principal curvatures is given by

$$Q_k(\lambda) = \frac{S_{k+1}(\lambda)}{S_k(\lambda)}$$

$$= \frac{\dfrac{1}{k+1}\binom{n-1}{k}\left(\dfrac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^k\left[\dfrac{(n-k-1)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \dfrac{(k+1)\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right]}{\dfrac{1}{k}\binom{n-1}{k-1}\left(\dfrac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^{k-1}\left[\dfrac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \dfrac{k\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right]}$$

$$= \frac{(n-k)}{(k+1)}\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\frac{(n-k-1)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + (k+1)r\ddot{\mathbf{u}}}{(n-k)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + kr\ddot{\mathbf{u}}}.$$

$\square$

As a corollary we obtain an ODE satisfied by these γ- translators.

Corollary 4.1.3. *1. Rotationally symmetric Q_k-translators satisfy the ODE*

$$\begin{cases} \ddot{\mathbf{u}} = F_{k,n}(r,\dot{\mathbf{u}}) \\ \\ \mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0. \end{cases}, \tag{4.4}$$

where $F_{k,n}(x, y) = \dfrac{n-k}{k+1}(1 + y^2)\dfrac{y}{x}\left(\dfrac{x(k+1) - (n-k-1)y}{(n-k)y - kx}\right)$.

2. *Rotationally symmetric $\sqrt[k]{S_k}$-translators satisfy the ODE*

$$\begin{cases} \ddot{\mathbf{u}} = G_{k,n}(r, \dot{\mathbf{u}}) \\ \mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0. \end{cases}, \tag{4.5}$$

where $G_{k,n}(x, y) = \dfrac{y}{x}(1 + y^2)\left(\dfrac{1}{\binom{n-1}{k-1}}\left(\dfrac{x}{y}\right)^k - \dfrac{(n-k)}{k}\right)$.

Proof. It is a straightforward calculation since the equations come from $\gamma(\lambda) = \dfrac{1}{\sqrt{1 + \dot{\mathbf{u}}^2}}$. $\square$

Remark 4.1.2. As we mentioned in the introduction of this chapter, all of the equations above can be seen as a first order ODE by considering $\mathbf{v} = \dot{\mathbf{u}}$. In particular, this fact simplifies finding barriers and the comparison methods that we show in Section 4.3.

4.2 Explicit Q_1 and $\sqrt{S_2}$ translators in $\mathbb{R}^3$

In this section we construct explicit examples of rotationally symmetric strictly convex translators for the curvature function Q_1 and $\sqrt{S_2}$ in $\mathbb{R}^3$. For this purpose, we set $\mathbf{v} = \dot{\mathbf{u}}$, and therefore these γ-translators in $\mathbb{R}^{n+1}$ satisfy

$$\begin{cases} \dot{\mathbf{v}}(r) = F_{n-1,n}(r, \mathbf{v}) = \dfrac{\mathbf{v}(1 + \mathbf{v}^2)}{\mathbf{v} - (n-1)r}, \\ \mathbf{v}(0) = 0, \end{cases} \tag{4.6}$$

for $\gamma = Q_{n-1}$, and for the $\gamma = \sqrt[n]{S_n}$-translators we have

$$\begin{cases} \dot{\mathbf{v}}(r) = G_{n,n}(r, \mathbf{v}) = (1 + \mathbf{v}^2)\left(\dfrac{r}{\mathbf{v}}\right)^{n-1}, \\ \mathbf{v}(0) = 0, \end{cases} \tag{4.7}$$

In the following theorems we find explicit solutions for these ODEs.

Theorem 4.2.1. *The surface*

$$M = \left\{ (r\theta, -\ln(1-r^2)) : \theta \in \mathbb{S}^1, r \in [0,1) \right\}$$

is a complete and strictly convex Q_1-translator C^2-asymptotic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{R}^3$, i.e: $\lambda_2(p) \to 0$ and $\lambda_1(p) \to 1$ as $|p| \to \infty$, where $p \in M$.

Proof. The function $\mathbf{v}(r) = \dfrac{2r}{1-r^2}$ with $r \in (0,1)$ satisfies Equation (4.6). In fact, it is not hard to see that $\dot{\mathbf{v}} = \dfrac{2(1+r^2)}{(1-r^2)^2}$ and

$$F_{1,2}(r,\mathbf{v}) = \frac{2r}{1-r^2} \frac{\dfrac{(1+r^2)^2}{(1-r^2)^2}}{\dfrac{r(1+r^2)}{1-r^2}} = \frac{2(1+r^2)}{(1-r^2)^2} = \dot{\mathbf{v}}.$$

Then, by simple integration, the function $\mathbf{u} = \int_0^r \mathbf{v}(s)ds = -\ln(1-r^2)$ solves equation (4.4) for $k = 1$ and $n = 2$.

Moreover, this surface is complete but not entire since $\lim\limits_{r \to 1} \mathbf{u}(r) = \infty$.

Finally, the strictly convexity follows from

$$\lambda_1 = \frac{\mathbf{v}}{r\sqrt{1+\mathbf{v}^2}} = \frac{2}{(1-r^2)\sqrt{1 + \dfrac{4r^2}{(1-r^2)^2}}} = \frac{2}{\sqrt{(1-r^2)^2 + 4r^2}},$$

$$\lambda_2 = \frac{\dot{\mathbf{v}}}{(1+\mathbf{v}^2)^{\frac{3}{2}}} = \frac{2(1+r^2)}{(1-r^2)^2} \frac{1}{\left(1 + \dfrac{4r^2}{(1-r^2)^2}\right)^{\frac{3}{2}}} = \frac{2(1+r^2)(1-r^2)}{((1-r^2)^2 + 4r^2)^{\frac{3}{2}}}.$$

In addition, by the above equations, we see that $\lambda_1 \to 1$ and $\lambda_2 \to 0$ as $r \to 1$. $\quad\square$

Theorem 4.2.2. *The surface*

$$M = \left\{ \left(r\theta, \int_0^r \sqrt{e^{s^2} - 1} ds \right) : \theta \in \mathbb{S}^1, r \in [0, \infty) \right\}$$

is a complete strictly convex $\sqrt{S_2}$-translator in $\mathbb{R}^3$ which is smooth for $r \neq 0$ and 1-differentiable at $r = 0$.

Proof. First, we show that the function $\mathbf{v}(r) = \sqrt{e^{r^2} - 1}$ satisfies Equation (4.7) n=2. In fact, we have that

$$\dot{\mathbf{v}} = \frac{r e^{r^2}}{\sqrt{e^{r^2} - 1}} = (1 + \mathbf{v}^2)\frac{r}{\mathbf{v}} = G_{2,2}(r, \mathbf{v}), \ r \geq 0.$$

Finally, the result follows by taking $\mathbf{u} = \int_0^r \mathbf{v}(s) ds$. In particular, the fundamental theorem of calculus implies that $\mathbf{u} = \int_0^r \mathbf{v}(s) ds$ is of class $\mathcal{C}^1$ but not $\mathcal{C}^2$ in the origin because the square root in $\mathbf{v}$. $\qquad \square$

4.3 Barriers and Extension of solutions

In this section we show explicit barriers solutions to Equations (4.6) and (4.7). Then, as an application of ODE classic theory, we are able to extend the interval of existence of a solution for both equations.

Definition 4.3.1. Firstly, we say that a function $\mathbf{w}(r)$ is a supersolution (subsolution respectively) to an initial value problem

$$\begin{cases} \dot{\mathbf{v}} = F(r, \mathbf{v}), \ \text{in} \ (r_0, R) \\ \\ \mathbf{v}(r_0) = v_0, \end{cases} \tag{4.8}$$

if for all $r \in (r_0, R)$ we have $\dot{\mathbf{w}} \geq F(r, \mathbf{w})$ with $\mathbf{w}(r_0) \geq v_0$ ($\dot{\mathbf{w}} \leq F(r, \mathbf{w})$ with $\mathbf{w}(r_0) \leq v_0$, respectively)

In addition, we say that the slope function F has a comparison principle if given a supersolution (subsolution respectively) $\mathbf{w}$ to the initial value problem (4.8) it holds that for all $r \in (r_0, R)$ it holds $\mathbf{v}(r) \leq \mathbf{w}(r)$ ($\mathbf{w}(r) \leq \mathbf{v}(r)$, respectively) for any solution $\mathbf{v}$ to Equation (4.8).

In the next proposition we show that Equations (4.6) and (4.7) posses comparison principles for any positive initial condition.

Proposition 4.3.1. *Let $r_0, v_0 > 0$, and $f : (0, \infty) \to (0, \infty)$ be a continuous decreasing function. Assume that $\mathbf{v}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ are a solution and a supersolution (resp. subsolution), respectively, to the initial value problem*

$$\begin{cases} \dot{\mathbf{v}} = (1 + \mathbf{v}^2) f \left(\dfrac{\mathbf{v}}{r} \right) \\ \mathbf{v}(r_0) = v_0. \end{cases} \tag{4.9}$$

Then, $\mathbf{v}(r) \leq \mathbf{w}(r)$ (resp. $\mathbf{v}(r) \geq \mathbf{w}(r)$) holds in $[r_0, R)$.

Proof. We only show the case when $\mathbf{w}$ is a supersolution, since the subsolution case is analogous.

Let $\mathbf{v}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ be a solution and a supersolution to the initial value problem (4.9), respectively. Then, since $\mathbf{w}$ is a supersolution to Equation (4.9), we have $\mathbf{v}(r_0) = v_0 \leq w_0 = \mathbf{w}(r_0)$ and at any r,

$$\dot{\mathbf{w}} \geq (1 + \mathbf{w}^2) f \left(\frac{\mathbf{w}}{r} \right) \Leftrightarrow \frac{d}{dr} \arctan(\mathbf{w}) \geq f \left(\frac{\mathbf{w}}{r} \right).$$

Now we assume the contrary of the result of this proposition. This means there exists $r_1 \in (r_0, R)$ such that $\mathbf{w}(r_1) < \mathbf{v}(r_1)$.

Next, we consider

$$d(r) = \arctan(\mathbf{w}) - \arctan(\mathbf{v}),$$

and we note that $d(r_0) \geq 0$, $d(r_1) < 0$. Then, by continuity, we have that $d(c) = 0$ where $c = \inf \left\{ r \in (r_0, r_1) : \left. d \right|_{[r,r_1]} < 0 \right\}$. Moreover, since $\tan(\cdot)$ is an increasing function, it holds that

$$0 \geq d(r) = \arctan(\mathbf{w}) - \arctan(\mathbf{v}) \Leftrightarrow \mathbf{v}(r) \geq \mathbf{w}(r) \text{ for } r \in [c, r_1].$$

In particular, it follows that

$$\dot{d} \geq f\left(\frac{\mathbf{w}}{r}\right) - f\left(\frac{\mathbf{v}}{r}\right) \geq 0,$$

since f is an decreasing function. Therefore, d is an increasing function in $[c, r_1]$. This means that $0 = d(r_2) \leq d(r_1) < 0$, a contradiction.

Consequently, $d \geq 0$ holds at $[r_0, R)$, and by composing with $\tan(\cdot)$, it holds that $\mathbf{v} \leq \mathbf{w}$ in $[r_0, R)$. $\qquad \square$

Remark 4.3.1. We note that Proposition 4.3.1 applies to Equations (4.6) and (4.7) since the functions $f(x)$ are given by

$$\frac{x}{x - (n-1)} \text{ and } \frac{1}{x^{n-1}}$$

respectively.

Moreover, as we will see in the following sections, Proposition 4.3.1 can also be applied to the initial condition $r_0 = v_0 = w_0 = 0$, since $\left. f\left(\frac{\mathbf{v}}{r}\right) \right|_{r=0} = f(\dot{\mathbf{v}}(0))$. In fact, the solutions to Equations (4.6) and (4.7) satisfy $\dot{\mathbf{v}}(0) = n$ and $\dot{\mathbf{v}}(0) = 1$, respectively, which means that the

respective functions of both ODE are well-defined continuous decreasing functions.

The following propositions provide explicit barrier solutions to Equations (4.6) and (4.7).

Proposition 4.3.2. *Let $\alpha \in (n-1, n]$, $\beta \geq n$ such that $\alpha r_0 \leq v_0 \leq \dfrac{\beta r_0}{\sqrt{1 - \beta^2 r_0^2}}$ for some $v_0 \in \mathbb{R}$. Then, the functions*

$$\mathbf{w}_{0,\alpha}(r) = \alpha r \ \text{ and } \ \mathbf{w}_{1,\beta}(r) = \frac{\beta r}{\sqrt{1 - \beta^2 r^2}}$$

defined for $r \in \mathbb{R}$ and $r \in \left[0, \dfrac{1}{\beta}\right)$, respectively, are a subsolution and supersolution, respectively, to the initial value problem

$$\begin{cases} \dot{\mathbf{v}} = (1 + \mathbf{v}^2) \dfrac{\dfrac{\mathbf{v}}{r}}{\dfrac{\mathbf{v}}{r} - (n-1)}, \\[4mm] \mathbf{v}(r_0) = v_0. \end{cases} \tag{4.10}$$

Proof. Firstly, we will show that $\mathbf{w}_{0,\alpha}(r)$ is a subsolution to Equation (4.10). We note $\mathbf{w}_{0,\alpha}(r_0) = r_0 \leq v_0 = \mathbf{v}(r_0)$ so the initial condition is satisfied.

Moreover, since $\alpha \in (n-1, n]$, it holds that

$$\frac{\dot{\mathbf{w}}_{0,\alpha}}{1 + \mathbf{w}_{0,\alpha}^2} - \frac{\dfrac{\mathbf{w}_{0,\alpha}}{r}}{\dfrac{\mathbf{w}_{0,\alpha}}{r} - (n-1)} = \frac{\alpha}{1 + \alpha^2 r^2} - \frac{\alpha}{\alpha - (n-1)}$$

$$= \frac{\alpha}{(\alpha - (n-1))(1 + \alpha^2 r^2)}(\alpha - n - \alpha^2 r^2)$$

$$\leq 0.$$

This proves that $\mathbf{w}_{0,\alpha}$ with $\alpha \in (n-1, n]$ is a subsolution to Equation (4.10) for all $r \in \mathbb{R}$.

On the other hand, for $\mathbf{w}_{1,\beta}$ we note that $\mathbf{w}_{1,\beta}(r_0) = \dfrac{\beta r_0}{\sqrt{1 - \beta^2 r_0^2}} \geq v_0 = \mathbf{v}(0)$.

In addition,

$$\dot{\mathbf{w}}_{1,\beta} = \frac{\mathbf{w}_{1,\beta}}{r}(1 + \mathbf{w}_{1,\beta}^2)$$

$$= \frac{\dfrac{\mathbf{w}_{1,\beta}}{r}}{n - (n-1)\dfrac{\mathbf{w}_{1,\beta}}{r}}(1 + \mathbf{w}_{1,\beta}^2)$$

$$\geq \frac{\dfrac{\mathbf{w}_{1,\beta}}{r}}{\dfrac{\mathbf{w}_{1,\beta}}{r} - (n-1)}(1 + \mathbf{w}_{1,\beta}^2),$$

in the last inequality we use that $\dfrac{\mathbf{w}_{1,\beta}}{r} = \dfrac{\beta}{\sqrt{1 - \beta^2 r^2}} \geq \beta \geq n$ in $\left[0, \frac{1}{\beta}\right]$. This shows that $\mathbf{w}_{1,\beta}$ with $\beta \geq n$ is a supersolution to Equation (4.10) in $\left[0, \frac{1}{\beta}\right]$. $\qquad\square$

Proposition 4.3.3. *Let $n \geq 3$ and r_0, v_0 such that $r_0 \leq v_0 \leq e^{r_0} - 1$. Then, the functions*

$$\mathbf{w}_2(r) = r \ \text{ and } \ \mathbf{w}_3(r) = e^r - 1,$$

are a subsolution and supersolution to the initial value problem

$$\begin{cases} \dot{\mathbf{v}} = (1 + \mathbf{v}^2)\left(\dfrac{r}{\mathbf{v}}\right)^{n-1}, \\ \mathbf{v}(r_0) = v_0. \end{cases} \tag{4.11}$$

for all $r \in \mathbb{R}$, respectively.

Proof. Firstly, we start showing that $\mathbf{w}_2$ is a subsolution to Equation (4.11) for all $r \in \mathbb{R}$. Then, by definition we have $\mathbf{w}_2(r_0) = r_0 \leq v_0 = \mathbf{v}(0)$, and

$$\frac{\dot{\mathbf{w}}_2}{1 + \mathbf{w}_2^2} - \left(\frac{r}{\mathbf{w}_2}\right)^{n-1} = \frac{1}{1 + r^2} - 1 \leq 0.$$

This shows that $\mathbf{w}_2$ is a subsolution to Equation (4.11).

On the other hand, for $\mathbf{w}_3$, we have by assumption that $\mathbf{w}_3(r_0) = e^{r_0} - 1 \geq v_0 = \mathbf{v}(0)$,

and

$$\frac{\dot{\mathbf{w}}_3}{1 + \mathbf{w}_3^2} = \frac{e^r}{2 - 2e^r + e^{2r}} \geq \frac{r^2}{(e^r - 1)^2} \geq \left(\frac{r}{e^r - 1}\right)^{n-1} = \left(\frac{r}{\mathbf{w}_3}\right)^{n-1},$$

which holds for all $r \geq 0$ and $n \geq 3$. To see that the first inequality holds, we note that

$$\frac{e^r}{2 - 2e^r + e^{2r}} \geq \frac{r^2}{(e^r - 1)^2} \Leftrightarrow \frac{e^r}{r^2} \geq \frac{1}{(e^r - 1)^2} + 1 \Leftrightarrow e^r - r^2 \geq \left(\frac{r}{e^r - 1}\right)^2.$$

Then, $(e^r - r^2)|_{r=0} = 1 = \lim_{r \to 0} \left(\frac{r}{e^r - 1}\right)^2$, but $e^r - r^2$ is increasing and $\left(\frac{r}{e^r - 1}\right)^2$ is decreasing. This completes the proof that $\mathbf{w}_3$ is a supersolution of the Equation (4.11). $\quad\square$

We now present a classical result from the theory of ODEs that allows us to extend solutions of initial value problems up to their maximal intervals of existence.

Lemma 4.3.4. *Let $U \subset \mathbb{R}^2$ be an open set that contains the curve $\{(r, \mathbf{v}(r)) : r \in (r_-, r_+)\}$. Assume that $\mathbf{v}$ is a solution to a initial value problem (4.8). Furthermore, assume that there is a compact set $[r_0, r_+] \times C \subset U$ such that $\mathbf{v}(r_m) \in C$ for some sequence $r_m \in [r_0, r_+)$ converging to r_+ .*

Then, there exist $\varepsilon > 0$ and a function $\mathbf{v}_\varepsilon \in \mathcal{C}^1((r_-, r_+ + \varepsilon))$ such that $\mathbf{v}_\varepsilon|_{(r_-, r_+)} = \mathbf{v}$ and $\mathbf{v}_\varepsilon$ is a solution to the initial value problem (4.8) with initial condition $\mathbf{v}_\varepsilon(r_+)$.

Moreover, if there is such compact set C for every $r_+ > r_0$ (C might depend on r_+), then there exists a solution $\tilde{\mathbf{v}} \in \mathcal{C}^1((r_-, \infty))$ of the initial value problem (4.8) with initial condition $\tilde{\mathbf{v}}(r_+)$ such that $\tilde{\mathbf{v}}|_{(r_-, r_+)} = \mathbf{v}$. The same result holds for r_-.

Proof. We refer the reader to see the proof in [Tes] Corollary 2.15. $\quad\square$

Remark 4.3.2. The Lemma 4.3.4 is a powerful tool to find the maximal interval of existence of a solution to an initial value problem of first order. Usually, the ingredients for using this result are:

- An ODE which has a comparison principle. This will ensure that the extension of the solution will be unique. We will see this fact in the section 4.6.

- Upper and lower barriers. This is also related to the previous point, but the important point is that the barriers will give an explicit construction of the compact set C to apply the Lemma 4.3.4.

In summary these assumptions are for producing unique solutions whose domain is the maximum existence interval associated with that of the initial value problem.

On the other hand, the negation of the Lemma statement 4.3.4 provides a method to find the maximum existence interval of the solution. In fact, if the supersolution and the subsolution tend to $\pm\infty$ at the extremes of the interval where they are defined, then the solution also will have this property. We emphasize that the existence interval of the solution must be contained in that of the barriers for this to take effect.

The following corollaries reveal the maximum existence interval for the solutions of our initial value problems.

Corollary 4.3.5. *Let $n \geq 3$, $0 < r_0 \leq v_0 \leq e^{r_0} - 1$ and $\mathbf{v} \in \mathcal{C}^1([r_0, R])$ be a solution to Equation (4.11). Then, the solution $\mathbf{v}(r)$ exists for all $r \geq 0$.*

Proof. The proof consists of applying Lemma 4.3.4 with the barriers found in Proposition 4.3.3. Firstly, we note that the solution $\mathbf{v}$ verifies

$$\mathbf{w}_2(r_0) = r_0 \leq \mathbf{v}(r_0) \leq e^{r_0} - 1 = \mathbf{w}_3(r_0).$$

This implies that the functions $\mathbf{w}_2$ and $\mathbf{w}_3$ are a subsolution and supersolution, respectively, to Equation (4.11). Then, by Proposition 4.3.1, $\mathbf{w}_2 \leq \mathbf{v} \leq \mathbf{w}_3$ holds in $[r_0, R]$.

On the other hand, we set $r_+ \geq R$ and $C = [v_0, \mathbf{w}_3(r_+)]$. Then, it follows that $\mathbf{v}(r_m) \in C$ for every sequence $r_m \to r_+$. In particular, by Lemma 4.3.4, the solution $\mathbf{v}(r)$ to Equation

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(4.11) exists for all $r \geq r_0$. Finally, the same argument holds by replacing C with $[\mathbf{w}_2(r_-), v_0]$ where $0 \leq r_- \leq r_0$. Therefore, $\mathbf{v}(r)$ can be extended to $[0, \infty)$ as a solution to Equation (4.11). $\qquad\square$

Theorem 4.3.6. *Let $n \geq 3$, $\varepsilon \in [0, 1)$, R, r_0 and v_0 such that*

$$R \in \left(0, \frac{1}{n}\right), \ r_0 > 0 \ \text{and} \ (n - \varepsilon)r_0 \leq v_0 \leq \frac{(n + \varepsilon)r_0}{\sqrt{1 - (n + \varepsilon)^2 r_0^2}}.$$

Assume that $\mathbf{v} \in \mathcal{C}^1([r_0, R])$ is a solution to the initial value problem (4.10) with $v(r_0) = v_0$. Then, $\mathbf{v}(r)$ can be extended to $r \in \left[0, \frac{1}{n}\right)$ and $\lim\limits_{r \to \frac{1}{n}} \mathbf{v}(r) = \infty$.

Proof. Firstly, we will show that any solution $\mathbf{v}(r)$ to Equation (4.10) can be extended to $\left[0, \frac{1}{n}\right)$. For this purpose we set $r_+ \in \left[R, \frac{1}{n}\right)$ and $C = [0, \mathbf{w}_{1,n+\varepsilon}(r_+)]$. Note that by hypothesis $\mathbf{w}_{0,n-\varepsilon}(r_0) \leq \mathbf{v}(r_0) \leq \mathbf{w}_{1,n+\varepsilon}(r_0)$.

Then, by Proposition 4.3.1 and Proposition 4.3.2, it follows that $\mathbf{w}_{0,n-\varepsilon}$ and $\mathbf{w}_{1,n+\varepsilon}$ are a subsolution and a supersolution, respectively, to Equation (4.10). This implies that

$$(n - \varepsilon)r = \mathbf{w}_{0,n-\varepsilon}(r) \leq \mathbf{v} \leq \mathbf{w}_{n+\varepsilon}(r) = \frac{(n + \varepsilon)r}{\sqrt{1 - (n + \varepsilon)^2 r^2}}, \ \text{for } r \in [r_0, R].$$

Therefore, for any sequence $r_m \to r_+$, we have $\mathbf{v}(r_m) \in [0, \mathbf{w}_{1,n+\varepsilon}(r_+)]$.

Next, by Lemma 4.3.4, any solution $\mathbf{v}(r)$ to Equation (4.10) exists up to $r < \dfrac{1}{n + \varepsilon}$. In particular, by taking $\varepsilon \to 0$, we can extended $\mathbf{v}(r)$ as a solution to Equation (4.10) up to $r < \dfrac{1}{n}$.

In addition, the same argument will hold by replacing C with $[\mathbf{w}_{0,n-\varepsilon}(r_-), R]$, where $0 \leq r_- \leq r_0$. This means that we can extend $\mathbf{v}$ to $r = 0$ as a solution to Equation (4.10).

On the other hand, since $v(r) \geq \mathbf{w}_{0,n} = nr$, the following inequalities hold

$$\dot{\mathbf{v}} = (1 + \mathbf{v}^2) \frac{\dfrac{\mathbf{v}}{r}}{\dfrac{\mathbf{v}}{r} - (n-1)}$$

$$\geq (1 + \mathbf{v}^2)$$

$$\geq \mathbf{v}^2.$$

In the second line we use that $\dfrac{x}{x - (n-1)} \geq 1$ for $x = \dfrac{\mathbf{v}}{r} \geq 0$.

Consequently, since $\mathbf{w}_4(r) = \dfrac{n}{1 - nr}$ satisfies $\dot{\mathbf{w}}_4 = \mathbf{w}_4^2$, it follows that for r close to $\dfrac{1}{n}$, $\mathbf{v} \geq \dfrac{n}{1 - nr}$. This gives that $\lim\limits_{r \to \frac{1}{n}} \mathbf{v}(r) = \infty$ finalizing the proof. $\qquad\square$

4.4 A $\sqrt[n]{S_n}$-translator in $\mathbb{R}^{n+1}$ for $n \geq 3$

In this section we construct a solution to Equation (4.7) for $n \geq 3$.

The method that we employ is constructing a solution in a small interval containing $r = 0$. To accomplish this we use the implicit function theorem centered at a sequence of points $(a_m, b_m) \in \mathbb{R}^2 \to (0, 0)$ as $m \to \infty$ and an Arzela-Ascoli type argument for constructing this solution. Finally, by the result of the previous section, we are able to extend this solution for $r \geq 0$.

Firstly, we note that Equation (4.7),

$$\dot{\mathbf{v}} = (1 + \mathbf{v}^2) \left(\frac{r}{\mathbf{v}} \right)^{n-1},$$

can be solved implicitly. To see this we take $\mathbf{w} = \arctan(\mathbf{v})$, then $\mathbf{w}$ satisfies

$$\dot{\mathbf{w}} \tan^{n-1}(\mathbf{w}) = r^{n-1}.$$

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Then, by integration we obtain

$$\int_0^r \tan^{n-1}(\mathbf{w})d\mathbf{w} = \frac{r^n}{n}.$$

We note that the right hand side of the above equation can be written recursively by

$$\begin{aligned}
I_{n-1}(\mathbf{w}) :&= \int_0^r \tan^{n-1}(\mathbf{w})d\mathbf{w} \\
&= \int_0^r \tan^{n-3}(\mathbf{w})(\sec^2(\mathbf{w}) - 1)d\mathbf{w} \\
&= \frac{\tan^{n-2}(\mathbf{w})}{n-2} - \int_0^r \tan^{n-3}(\mathbf{w})d\mathbf{w} \\
&= \frac{\tan^{n-2}(\mathbf{w})}{n-2} - I_{n-3}(\mathbf{w}).
\end{aligned}$$

Consequently, a solution to Equation (4.7) can be written implicitly by the relation

$$\mathcal{G}(r, \mathbf{v}) = \frac{\mathbf{v}^{n-2}}{n-2} - \int_0^{\mathbf{v}} \frac{t^{n-3}}{1+t^2}dt - \frac{r^n}{n} = 0, \ (r, \mathbf{v}) \in [0, \infty)^2. \tag{4.12}$$

Moreover, we note that the implicit curve (4.12) can be computed recursively by

$$I_0(\arctan(\mathbf{v})) = \arctan(\mathbf{v}) \text{ and } I_1(\arctan(\mathbf{v})) = \frac{1}{2}\ln(1 + \mathbf{v}^2).$$

On the other hand, we already know that a solution $\mathbf{v}$ to Equation (4.7) can be written as an implicit curve $\mathcal{G}(r, \mathbf{v}) = 0$ such that $\mathbf{v}(0) = 0$. Unfortunately, for $n \geq 2$, we cannot apply directly the implicit function theorem to Equation (4.12) at $(0,0)$ since

$$\partial_y \mathcal{G}(x, y)|_{(0,0)} = \frac{y^{n-1}}{1+y^2}\bigg|_{(0,0)} = 0.$$

Nevertheless, we may apply an approximation procedure for constructing a solution to Equation (4.7).

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Proposition 4.4.1. *For $n \geq 3$, let $b_m \leq \sqrt{\dfrac{n+2}{n}}$ be a sequence of positive numbers converging to 0 as $m \to \infty$. Then, by setting*

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt,$$

we have the following properties:

1. *The points $(a_m, b_m) \in \mathbb{R}^2$ satisfy $\mathcal{G}(a_m, b_m) = 0$.*

2. *$0 < a_m \leq b_m$. Moreover, we have $\sqrt[n]{1 - \dfrac{n}{n+2} b_m^2} \leq \dfrac{a_m}{b_m} \leq 1$. In particular, it follows*

$$\lim_{m \to \infty} \frac{b_m}{a_m} \to 1 \quad and \quad \lim_{m \to \infty} a_m = 0.$$

Proof. Firstly, we note that Property (1) follows by definition.

Secondly, since $b_m > 0$, we have

$$t^{n-3} - t^{n-1} \leq \frac{t^{n-3}}{1+t^2} \leq t^{n-3} - t^{n-1} + t^{n+1} \text{ for } t \in [0, b_m].$$

Then, it follows

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt \leq \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \left(t^{n-3} - t^{n-1} \right) dt = \frac{b_m^n}{n}.$$

For the other inequality, we have

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt \geq \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} t^{n-3} - t^{n-1} + t^{n+1} dt = \frac{b_m^n}{n} - \frac{b_m^{n+2}}{n+2} \geq 0.$$

Finally, since the function $\dfrac{(\cdot)^n}{n}$ is 1-1 for positive numbers, it follows

$$b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2} \leq a_m \leq b_m.$$

$\square$

In the following theorem we construct a solution to Equation (4.7) for $n \geq 3$ by the implicit function theorem and a Arzela-Ascoli's argument.

Theorem 4.4.2. *For $n \geq 3$ there exists a non-constant solution $\mathbf{v} \in \mathcal{C}^1([0,\infty))$ to Equation (4.7).*

Proof. Firstly, we choose a sequence $b_m \leq \sqrt{\dfrac{n+2}{n}}$ of positive numbers converging to 0 as $m \to \infty$ such that $b_m \leq \exp\left(b_m \sqrt[n]{1 - \dfrac{n}{n+2}b_m^2} \right) - 1$ holds for all $m \gg 1$.

In fact, the function

$$T(x) = \exp\left(x \sqrt[n]{1 - \frac{n}{n+2}x^2} \right) - x - 1$$

satisfies $T(0) = 0$,

$$T'(0) = \exp\left(x \sqrt[n]{1 - \frac{n}{n+2}x^2} \right) \left[\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} - \frac{2x^2}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} \right] - 1 \Bigg|_{x=0} = 0,$$

$$T''(0) = \exp\left(x \sqrt[n]{1 - \frac{n}{n+2}x^2} \right) \left[\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} - \frac{2x^2}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} \right]^2$$
$$+ \exp\left(x \sqrt[n]{1 - \frac{n}{n+2}x^2} \right) \left[\frac{4nx^3}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}-2} - \frac{6x}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}-1} \right] \Bigg|_{x=0}$$
$$= 1.$$

This means that $T'(x)$ is increasing for $x > 0$ close to 0, or $T'(x) > 0$ for $x > 0$. In particular, this means that $T(x)$ is increasing for $x > 0$ close to 0, and therefore $T(x) \geq 0$ for $x > 0$ close to 0.

Consequently, since $b_m \to 0$ as $m \to \infty$, there exists $M \gg 1$ such that

$$b_m \leq \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2}b_m^2} \right) - 1$$

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for $m \geq M$.

Next, by setting a_m as in Proposition 4.4.1, we note that

$$\partial_y \mathcal{G}(x,y)|_{(a_m, b_m)} = \frac{b_m^{n-1}}{1 + b_m^2} > 0.$$

Then, by the implicit function theorem there exists $\mathbf{v}_m(r) \in \mathcal{C}^\infty([a_m, R_m])$ such that $\mathcal{G}(r, \mathbf{v}_m(r)) = 0$ at $[a_m, R_m]$ with $\mathbf{v}_m(a_m) = b_m$ for some $R_m > 0$ and for all $m \geq M$.

Moreover, since $\mathbf{v}_m$ is smooth, it also satisfies Equation (4.11) with initial data $\mathbf{v}_m(a_m) = b_m$.

In addition, by Property 2 from Proposition 4.4.1, it follows that

$$\mathbf{w}_2(a_m) = a_m \leq b_m = \mathbf{v}_m(a_m) \leq \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2} \right) - 1 \leq e^{a_m} - 1 = \mathbf{w}_3(a_m),$$

where $\mathbf{w}_2$ and $\mathbf{w}_3$ are defined in Proposition 4.3.3. Therefore, by Proposition 4.3.3, the functions $\mathbf{w}_2$ and $\mathbf{w}_3$ are a subsolution and supersolution, respectively, to the intial value problem (4.11) with initial data $\mathbf{v}_m(a_m) = b_m$.

Consequently, by Theorem 4.3.5, we can extend $\mathbf{v}_m$ up to $[0, R]$ for some $0 < R < \frac{\pi}{6}$ fixed.

Secondly, we are going to show that the sequence $\{\mathbf{v}_m\}_m$ is uniformly bounded in $\mathcal{C}^2([0, R])$. We start by showing the uniform bound in $\mathcal{C}([0, R])$. In fact, since

$$\dot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2) \left(\frac{r}{\mathbf{v}_m} \right)^{n-1} \geq (1 + \mathbf{v}_m^2) \left(\frac{r}{\mathbf{w}_{3m}} \right)^{n-1} = (1 + \mathbf{v}_m^2) \left(\frac{r}{e^r - 1} \right)^{n-1} > 0,$$

it follows by the fundamental theorem of calculus that

$$\arctan\left(\mathbf{v}_m(r)\right) = \arctan(b_m) + \int_{a_m}^{r} \left(\frac{s}{\mathbf{v}_m(s)}\right)^{n-1} ds$$

$$\leq \arctan(b_m) + \int_{a_m}^{r} \left(\frac{s}{\mathbf{w}_2(s)}\right)^{n-1} ds$$

$$\leq \arctan(b_m) + r - a_m$$

$$\leq \arctan(b_m) + R + a_m.$$

In the second line we use $\mathbf{w}_2 \leq \mathbf{v}_m$ for $r \geq 0$.

Moreover, since b_m and a_m tend to 0 as $m \to \infty$, it follows that

$$\mathbf{v}_m \leq \tan(3R). \tag{4.13}$$

for $m \gg M$. In particular, there exists C_1 such that $\sup_m ||\mathbf{v}_m||_\infty \leq C_1$.

Next, we are going to show a uniform bound for $\dot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by Equation (4.7), Property (2) and the uniform bound for $||\mathbf{v}_m||_\infty$, we have

$$0 < \dot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2)\left(\frac{r}{\mathbf{v}_m}\right)^{n-1}$$

$$\leq (1 + C_1^2)\left(\frac{r}{\mathbf{w}_2}\right)^{n-1}$$

$$= (1 + C_1^2).$$

Therefore, we can find a constant $C_2 > 0$ such that $\sup_m ||\dot{\mathbf{v}}_m||_\infty \leq C_2$.

Finally, we are going to show an uniform bound for $\ddot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by differentiating Equations (4.7), it follows

$$\ddot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2)(n - 1)\left(\frac{r}{\mathbf{v}_m}\right)^{n-2}\left(\frac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2}\right) + 2\mathbf{v}_m\dot{\mathbf{v}}_m\left(\frac{r}{\mathbf{v}_m}\right)^{n-1}.$$

First, we are going to estimate the term $\dfrac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2}$ in the above equation. In fact, we have

$$
\left| \frac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2} \right| = \frac{\left| \mathbf{v}_m - r(1 + \mathbf{v}_m^2)\left(\dfrac{r}{\mathbf{v}_m} \right)^{n-1} \right|}{\mathbf{v}_m^2}
$$

$$
= \frac{1}{\mathbf{v}_m^2} \left| \frac{(\mathbf{v}_m - r)(\mathbf{v}_m^{n-1} + \ldots + r^{n-1})}{\mathbf{v}_m^{n-1}} - \frac{r^n}{\mathbf{v}_m^{n-3}} \right|
$$

$$
\leq n\frac{\mathbf{v}_m - r}{\mathbf{v}_m^2} + \frac{r^n}{\mathbf{v}_m^{n-1}}
$$

$$
\leq n\frac{\mathbf{w}_3 - r}{\mathbf{w}_2^2} + \frac{r^n}{\mathbf{w}_2^{n-1}}
$$

$$
= n\frac{e^r - 1 - r}{r^2} + R
$$

$$
\leq n\frac{e^R - 1 - R}{R^2} + R.
$$

In the last line we use that the function $\dfrac{e^r - 1 - r}{r^2} = \sum\limits_{k \geq 2} \dfrac{r^{k-2}}{k!}$ is increasing for all $r \geq 0$. Therefore, by the previous calculation, we obtain

$$
||\ddot{\mathbf{v}}_m||_\infty \leq (1 + C_1^2)(n-1)\left(n\frac{e^R - 1 - R}{R^2} + R \right) + 2C_1 C_2 = C_3.
$$

Consequently, we have shown that $\sup\limits_{m} ||\mathbf{v}||_{\mathcal{C}^2([a_m, \varepsilon'_m])} \leq C$ where $C = C_1 + C_2 + C_3$.

Finally, we have that $\{\mathbf{v}_m\}_m$ is uniformly bounded and equicontinuous in $\mathcal{C}^1([0, R])$ since $\mathbf{v}_m$ and $\dot{\mathbf{v}}_m$ are Lipschitz functions with Lipschitz constant less that C. Therefore, by Arzela-Ascoli's theorem, there exists a subconvergent sequence in the $\mathcal{C}^1$ topology of $\{\mathbf{v}_m\}$ to a function $\mathbf{v} \in \mathcal{C}^1([0, R])$. In addition, by continuity we have

$$
\mathbf{v}(0) = \mathbf{v}\left(\lim_{m \to \infty} a_m \right) = \lim_{m \to \infty} b_m = 0,
$$

$$
\dot{\mathbf{v}}(0) = \lim_{m \to \infty} (1 + b_m^2)\left(\frac{a_m}{b_m} \right)^{n-1} = 1.
$$

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Moreover, $\mathbf{v}$ satisfies $\mathbf{w}_2 \leq \mathbf{v} \leq \mathbf{w}_3$ at $[0, R]$. Then, by Theorem 4.3.5, $\mathbf{v}$ exists for all $r \geq 0$, finalizing the proof of this theorem. $\qquad\square$

4.5   A Q_{n-1}-translator in $\mathbb{R}^{n+1}$ for $n \geq 3$

In this section we apply the same technique as the above section to construct a solution to Equation (4.6) in a small neighborhood containing $r = 0$. For this purpose we note that Equation (4.6),

$$\dot{\mathbf{v}} = (1 + \mathbf{v}^2) \frac{\dfrac{\mathbf{v}}{r}}{\dfrac{\mathbf{v}}{r} - (n-1)},$$

can be solved implicitly. In fact, after some algebraic arrangements and multiplying by $\dfrac{\mathbf{v}^{n-2}}{(1 + \mathbf{v}^2)^{\frac{n+1}{2}}}$, we see that Equation (4.6) is equivalent to

$$\frac{\mathbf{v}^{n-2}\dot{\mathbf{v}}}{(1 + \mathbf{v}^2)^{\frac{n+1}{2}}}(\mathbf{v} - (n-1)r) - \left(\frac{\mathbf{v}}{\sqrt{1 + \mathbf{v}^2}}\right)^{n-1} = 0.$$

Then, by the method of separation of variables, we look for $\mathcal{F}(r, v)$ such that

$$\frac{\partial \mathcal{F}}{\partial r} = -\left(\frac{\mathbf{v}}{\sqrt{1 + \mathbf{v}^2}}\right)^{n-1} \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = \frac{\mathbf{v}^{n-2}}{(1 + \mathbf{v}^2)^{\frac{n+1}{2}}}(\mathbf{v} - (n-1)r).$$

Indeed, by an integration/differentiation procedure, we obtain an implicit curve given by

$$\mathcal{F}(r, \mathbf{v}) = -r\left(\frac{\mathbf{v}}{\sqrt{1 + \mathbf{v}^2}}\right)^{n-1} + \int_0^{\mathbf{v}} \frac{t^{n-1}}{(1 + t^2)^{\frac{n+1}{2}}} dt = 0. \qquad (4.14)$$

Unfortunately, as in the previous section, we cannot apply directly the implicit function

theorem to Equation (4.14) at $(0,0)$ since

$$\partial_y \mathcal{F}(x,y)|_{(0,0)} = \left. \frac{y^{n-2}}{(1+y^2)^{\frac{n+1}{2}}}(y-(n-1)x)\right|_{(0,0)} = 0.$$

Nevertheless, we may apply an approximation procedure for constructing a solution to Equation (4.6) which passes through $(0,0)$.

Proposition 4.5.1. *For $n \geq 3$, let $b_m \in (0,1)$ be a sequence of positive numbers converging to 0 as $m \to \infty$. Then, by setting*

$$a_m = \left(\frac{\sqrt{1+b_m^2}}{b_m}\right)^{n-1} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}}dt,$$

we have the following properties:

1. *The points $(a_m, b_m) \in \mathbb{R}^2$ satisfy $\mathcal{F}(a_m, b_m) = 0$.*

2. *$a_m > 0$ and $\lim\limits_{m \to \infty} a_m = 0$.*

3. *$\lim\limits_{m \to \infty} \dfrac{a_m}{b_m} = \dfrac{1}{n}.$*

Proof. Firstly, we note that Property (1) and $a_m > 0$ follow by definition. Moreover, the limit of $\dfrac{a_m}{b_m}$ can be calculated by L'Hospital's rule. Indeed, we have

$$\begin{aligned}
\lim_{m \to \infty} \frac{a_m}{b_m} &= \lim_{m \to \infty} \frac{(1+b_m^2)^{\frac{n-1}{2}} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}}dt}{b_m^n} \\
&= \lim_{m \to \infty} \frac{b_m(n-1)(1+b_m^2)^{\frac{n-3}{2}} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}}dt + \frac{b_m^{n-1}}{(1+b_m^2)}}{nb_m^{n-1}} \\
&= \frac{1}{n}.
\end{aligned}$$

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Finally, since $\lim\limits_{m\to\infty} b_m = 0$ and $\lim\limits_{m\to\infty} \dfrac{a_m}{b_m} = \dfrac{1}{n}$, it follows that $\lim\limits_{m\to\infty} a_m = 0$. $\qquad\square$

In the following theorem we construct a solution to Equation (4.7) for $n \geq 3$ by the implicit function theorem and an Arzela-Ascoli's argument.

Theorem 4.5.2. *For $n \geq 3$ there exists a non-constant solution $\mathbf{v} \in \mathcal{C}^1\left(\left[0, \frac{1}{n}\right)\right)$ to Equation (4.6) such that $\lim\limits_{r\to\frac{1}{n}} \mathbf{v}(r) = \infty$.*

Proof. Firstly, since $\dfrac{b_m}{a_m} \to n$ as $m \to \infty$, it holds

$$\partial_y \mathcal{F}(x,y)\big|_{(a_m, b_m)} = a_m \frac{b_m^{n-2}}{(1 + b_m^2)^{\frac{n+1}{2}}} \left(\frac{b_m}{a_m} - (n-1)\right) > 0,$$

for $m \geq M$. Then, by the implicit function theorem there exists $\mathbf{v}_m(r) \in \mathcal{C}^\infty([a_m, R_m])$ for some $R_m > 0$ such that $\mathcal{F}(r, \mathbf{v}_m(r)) = 0$ at $[a_m, R_m]$ with $\mathbf{v}_m(a_m) = b_m$.

Moreover, since $\mathbf{v}_m$ is smooth, it also satisfies Equation (4.6) with initial data $\mathbf{v}_m(a_m) = b_m$. Therefore, since $\dfrac{b_m}{a_m} \to n$ as $m \to \infty$, it follows that the functions $\mathbf{w}_{0,n-\varepsilon}$ and $\mathbf{w}_{1,n+\varepsilon}$ defined in Proposition 4.3.2 satisfy

$$\mathbf{w}_{0,n-\varepsilon}(a_m) = (n-\varepsilon)a_m \leq b_m \leq a_m(n+\varepsilon) \leq \frac{(n+\varepsilon)a_m}{\sqrt{1 - (n+\varepsilon)^2 a_m^2}} = \mathbf{w}_{1,n+\varepsilon}(a_m),$$

for some $\varepsilon \in (0,1)$ and $m \geq M$. In particular, by Theorem 4.3.6, we can extend $\mathbf{v}_m$ to $[0, R]$ for some $R < \dfrac{1}{n}$.

Secondly, we are going to show that the sequence $\{\mathbf{v}_m\}_m$ is uniformly bounded in $\mathcal{C}^2([0, R])$. For this purpose, we start by showing the uniform bound in $\mathcal{C}([0, R])$. Note that each $\mathbf{v}_m$ satisfies

$$\dot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2)\frac{\dfrac{\mathbf{v}_m}{r}}{\dfrac{\mathbf{v}_m}{r} - (n-1)} > 0.$$

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Then, by the fundamental theorem of calculus together with that

$$\mathbf{w}_{0,n}(r) = nr \text{ and } \mathbf{w}_{1,n}(r) = \frac{nr}{\sqrt{1 - n^2 r^2}} \tag{4.15}$$

are a subsolution and supersolution, respectively, to Equation (4.6), we have

$$\arctan(\mathbf{v}_m) = \arctan(b_m) + \int_{a_m}^{r} \frac{\frac{\mathbf{v}_m}{s}}{\frac{\mathbf{v}_m}{s} - (n-1)} ds$$

$$\leq \arctan(b_m) + \int_{a_m}^{r} \frac{\frac{\mathbf{w}_{0,n}}{s}}{\frac{\mathbf{w}_{0,n}}{s} - (n-1)} ds$$

$$= \arctan(b_m) + n(r - a_m)$$

$$\leq \arctan(b_m) + n(R - a_m).$$

In the second line we use that the function $\dfrac{x}{x - (n-1)}$ is decreasing in $x = \dfrac{\mathbf{v}(r)}{r}$ for $r \geq a_m$ and $n = \dfrac{\mathbf{w}_{0,n}}{r} \leq \dfrac{\mathbf{v}_m}{r} \leq \dfrac{\mathbf{w}_{1,n}}{r} = \dfrac{n}{\sqrt{1 - n^2 r^2}}$.

Then, by taking limits as $r \to R$ on the right hand side of the last inequality we see that it converges to nR. Therefore, by composing with $\tan(\cdot)$, we can find a constant C_1 such that $\sup_m ||\mathbf{v}_m||_\infty \leq C_1$.

Next, we are going to show by the same methods a uniform bound for $\dot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by Equation (4.6), Property (3) in Proposition 4.5.1 and the estimate from above,

it follows that

$$0 < \dot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2) \frac{\frac{\mathbf{v}_m}{r}}{\frac{\mathbf{v}_m}{r} - (n-1)}$$

$$\leq (1 + C_1^2) \frac{\frac{\mathbf{w}_{0,n}}{r}}{\frac{\mathbf{w}_{0,n}}{r} - (n-1)}$$

$$= n(1 + C_1^2). \tag{4.16}$$

Recall that $\mathbf{w}_{0,n}$ and $\mathbf{w}_{1,n}$ are the functions in (4.15).Therefore, we can find a constant $C_2 > 0$ such that $\sup_m ||\dot{\mathbf{v}}_m||_\infty \leq C_2$.

Finally, we are going to show a uniform bound for $\ddot{\mathbf{v}}_m$ in $\mathcal{C}(0, R])$. In fact, by differentiating Equations (4.7), and Property (2), it follows

$$\ddot{\mathbf{v}}_m = -(1 + \mathbf{v}_m^2) \frac{n-1}{\left(\frac{\mathbf{v}_m}{r} - (n-1)\right)^2} \left(\frac{\dot{\mathbf{v}}_m r - \mathbf{v}_m}{r^2}\right) + 2\mathbf{v}_m \dot{\mathbf{v}}_m \frac{\frac{\mathbf{v}_m}{r}}{\frac{\mathbf{v}_m}{r} - (n-1)}$$

From our previous bounds, we only need to estimate the term $\frac{|\dot{\mathbf{v}}_m r - \mathbf{v}_m|}{r^2}$ to control $|\ddot{\mathbf{v}}_m|$.
We note that

$$\frac{|\dot{\mathbf{v}}_m r - \mathbf{v}_m|}{r^2} = \frac{\left| (1 + \mathbf{v}_m^2) \frac{\frac{\mathbf{v}_m}{r}}{\frac{\mathbf{v}_m}{r} - (n-1)} r - \mathbf{v}_m \right|}{r^2}$$

$$= \frac{\frac{\mathbf{v}_m}{r}}{r^2 \left(\frac{\mathbf{v}_m}{r} - (n-1)\right)} \left| n + \mathbf{v}_m^2 - \frac{\mathbf{v}_m}{r} \right|.$$

On the other hand, the term

$$n + \mathbf{v}_m^2 - \frac{\mathbf{v}_m}{r} \leq n + \mathbf{v}_m^2 - \frac{\mathbf{w}_{0,n}}{r} = \mathbf{v}_m^2 \leq \mathbf{w}_{1,n}^2(r),$$

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and

$$\frac{\mathbf{v}_m}{r} - n - \mathbf{v}_m^2 \leq \frac{\mathbf{w}_{1,n}}{r} - n - \mathbf{w}_{0,n}^2$$

$$= \frac{n}{\sqrt{1 - n^2 r^2}} - n - n^2 r^2$$

$$= n^2 r^2 \left(\frac{n}{\sqrt{1 - n^2 r^2}(1 + \sqrt{1 - n^2 r^2})} - 1 \right)$$

$$\leq n \mathbf{w}_{1,n}^2(r).$$

Therefore,

$$\frac{|\dot{\mathbf{v}}_m r - \mathbf{v}_m|}{r^2} \leq n \frac{\mathbf{w}_{1,n}^3}{r^2} \leq \frac{n^3 R}{\sqrt{1 - n^2 R^2}}.$$

Consequently,

$$|\ddot{\mathbf{v}}_m| \leq 2(n-1)(1 + C_1^2)\frac{n^3 R}{\sqrt{1 - n^2 R^2}} + 2nC_1 C_2 = C_3.$$

This implies, $\sup_m ||\ddot{\mathbf{v}}_m|| \leq C_3$. In particular we have show that $\sup_m ||\mathbf{v}||_{\mathcal{C}^2([0,R])} \leq C$ where $C = C_1 + C_2 + C_3$.

Finally, since $\mathbf{v}_m$ and $\dot{\mathbf{v}}_m$ are Lipschitz functions for every $m \geq M$, we obtain an uniformly bounded equicontinuous sequence in $\mathcal{C}^1([0,R])$. Therefore, by Arzela-Ascoli's theorem, there exists a subconvergent sequence in the $\mathcal{C}^1$ topology of $\{\mathbf{v}_m\}$ to a function $\mathbf{v} \in \mathcal{C}^1([0,R])$. In

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addition, by continuity we have

$$\mathbf{v}(0) = \mathbf{v}\left(\lim_{m\to\infty} a_m\right) = \lim_{m\to\infty} b_m = 0,$$

$$\dot{\mathbf{v}}(0) = \lim_{m\to\infty} (1 + b_m^2)\frac{\dfrac{b_m}{a_m}}{\dfrac{b_m}{a_m} - (n-1)} = n,$$

$$\mathbf{w}_{0,n}(r) \le \mathbf{v} \le \mathbf{w}_{1,n}(r), \text{ for all } r \in [0, R],$$

Therefore, by Theorem 4.3.6 the function $\mathbf{v}(r)$ can be extended to $r \in \left[0, \frac{1}{n}\right)$ as a solution of Equation (4.6) such that $\lim_{r\to\frac{1}{n}} \mathbf{v}(r) = \infty$. □

We finish this section by showing that the "bowl"-type Q_{n-1}-translator is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(Q_{n-1}(1,\ldots,1)) \times \mathbb{R}$.

Theorem 4.5.3. *The "bowl"-type Q_{n-1}-translator $\Sigma \subset \mathbb{R}^{n+1}$ constructed above satisfies*

$$\min\{\lambda_i(p) : i = 1,\ldots,n\} = \lambda_n(p) \to 0,$$

$$\forall\, i \in \{1,\ldots,n-1\}, \lambda_i(p) \to \frac{1}{Q_{n-1}(1,\ldots,1)},$$

as $|p| \to \infty$ in Σ.

Proof. The proof is a consequence of the barriers of a "bowl"-type Q_{n-1}-translator. Indeed, we already know that $\lim_{r\to\frac{1}{n}} \mathbf{v}(p) = \infty$. Then, for the minimum of the principal curvatures of Σ we have

$$\lambda_n(p) = \frac{\dot{\mathbf{v}}}{(1 + \mathbf{v}^2)^{\frac{3}{2}}} = \frac{f\left(\dfrac{\mathbf{v}}{r}\right)}{\sqrt{1 + \mathbf{v}^2}},$$

where $f(x) = \dfrac{x}{x - (n-1)}$. Therefore, by the barrier $\mathbf{w}_{0,n}(r) \le \mathbf{v}(r)$ it follows that $1 \le f\left(\dfrac{\mathbf{v}}{r}\right) \le n$, this means that $\lambda_n(p) \to 0$ as $r \to \frac{1}{n}$.

On the other hand, for the rest of the principal curvatures, we can use that $\mathbf{v}(r) \leq \mathbf{w}_4(r) = \frac{n}{1 - nr}$ when r is close to $\frac{1}{n}$ to see that

$$\frac{n}{r\sqrt{(1-nr)^2+n^2}} = \frac{\mathbf{w}_4}{r\sqrt{1+\mathbf{w}_4^2}} \leq \lambda_i(p) = \frac{\mathbf{v}}{r\sqrt{1+\mathbf{v}^2}} \leq \frac{1}{r}.$$

Therefore, by taking limits in the last expresion as $r \to \frac{1}{n}$, we obtain $\lambda_i(p) \to n$. Recall that $\frac{1}{Q_{n-1}(1,\ldots,1)} = n.$ $\qquad\square$

4.6 Uniqueness and Regularity

In this section we show uniqueness and regularity results of solutions to Equations (4.6) and (4.7).

We start by showing Gronawall's inequalities for solutions to the initial value problems 4.11 and 4.10.

Lemma 4.6.1. *Let $n \geq 3$, $0 \leq r_0 < R \leq \infty$, $r_0 \leq w_0, u_0 \leq e^{r_0} - 1$ and $\mathbf{u}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ be solutions to Equation (4.7) with initial conditions $\mathbf{u}(r_0) = u_0$ and $\mathbf{w}(r_0) = w_0$, respectively. Then, the following inequality holds at (r_0, R)*

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(-\int_{r_0}^r \alpha(s)ds\right),$$

where $\alpha(r) = (n-3)\dfrac{r^{2n-5}}{(e^r-1)^{2n-6}}.$

Proof. Then, by setting $d = \mathbf{u} - \mathbf{w}$, we see that

$$\dot{d} = (1 + \mathbf{u}^2)\left(\frac{r}{\mathbf{u}}\right)^{n-1} - (1 + \mathbf{w}^2)\left(\frac{r}{\mathbf{w}}\right)^{n-1}$$

$$= r^{n-1}\left(\frac{\mathbf{w}^{n-1} - \mathbf{u}^{n-1}}{(\mathbf{uw})^{n-1}} + \frac{\mathbf{w}^{n-3} - \mathbf{u}^{n-3}}{(\mathbf{uw})^{n-3}}\right)$$

$$= -\left(\frac{r^{n-1}(\mathbf{w}^{n-2} + \ldots + \mathbf{u}^{n-2})}{(\mathbf{uw})^{n-1}} + \frac{r^{n-1}(\mathbf{w}^{n-4} + \ldots + \mathbf{u}^{n-4})}{(\mathbf{uw})^{n-3}}\right)d.$$

Next, we want to estimate the term in parenthesis in the last line of the above equation. For this purpose we use the hypothesis $r_0 \leq w_0, u_0 \leq e^{r_0} - 1$, then by Proposition (4.3.3), we have

$$\mathbf{w}_2(r) = r \leq \mathbf{w}(r), \mathbf{u}(r) \leq \mathbf{w}_3(r) = e^r - 1.$$

Then, we may estimate the term in parenthesis in the following way

$$\frac{r^{n-1}(\mathbf{w}^{n-2} + \ldots + \mathbf{u}^{n-2})}{(\mathbf{uw})^{n-1}} + \frac{r^{n-1}(\mathbf{w}^{n-4} + \ldots + \mathbf{u}^{n-4})}{(\mathbf{uw})^{n-3}}$$

$$\geq \frac{r^{n-1}(n-3)\mathbf{w}_2^{n-4}}{\mathbf{w}_3^{2(n-3)}}$$

$$= (n-3)\frac{r^{2n-5}}{(e^r - 1)^{2n-6}} = \alpha(r).$$

In the second line we discard the first term and for the second term we applied the barriers $\mathbf{w}_2$ in the numerator and $\mathbf{w}_3$ in the denominator.

Therefore, we obtain that $\dot{d} \leq -\alpha(r)d$. This yields, by integration, that

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0|\exp\left(-\int_{r_0}^{r}\alpha(s)ds\right),$$

where $|\cdot|$ appears by interchanging $\mathbf{u}$ with $\mathbf{w}$ in the proof. $\qquad\square$

Lemma 4.6.2. *Let $0 \leq r_0 < R \leq \infty$, $nr_0 \leq w_0, u_0 \leq \dfrac{nr_0}{\sqrt{1 - n^2 r_0^2}}$ and $\mathbf{u}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ be solutions to Equation (4.6) with initial conditions $\mathbf{u}(r_0) = u_0$ and $\mathbf{w}(r_0) = w_0$, respectively. Then, the following inequality holds at (r_0, R)*

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(\int_{r_0}^r \alpha(s)ds\right),$$

where $\alpha(r) = \dfrac{2n^3 r}{\sqrt{(1 - n^2 r^2)^3}}$.

Proof. Then, by setting $d = \mathbf{u} - \mathbf{w}$, we see that

$$\dot{d} = (1 + \mathbf{u}^2)\frac{\dfrac{\mathbf{u}}{r}}{\dfrac{\mathbf{u}}{r} - (n-1)} - (1 + \mathbf{w}^2)\frac{\dfrac{\mathbf{w}}{r}}{\dfrac{\mathbf{w}}{r} - (n-1)}$$

$$= \frac{(n-1)\left(\dfrac{\mathbf{w}}{r} - \dfrac{\mathbf{u}}{r}\right)}{\left(\dfrac{\mathbf{u}}{r} - (n-1)\right)\left(\dfrac{\mathbf{w}}{r} - (n-1)\right)} + \frac{\left(\dfrac{\mathbf{u}^3\mathbf{w}}{r^2} - \dfrac{\mathbf{u}\mathbf{w}^3}{r^2}\right) + (n-1)\left(\dfrac{\mathbf{w}^3}{r} - \dfrac{\mathbf{u}^3}{r}\right)}{\left(\dfrac{\mathbf{u}}{r} - (n-1)\right)\left(\dfrac{\mathbf{w}}{r} - (n-1)\right)}$$

$$= \frac{\left(\dfrac{\mathbf{u}\mathbf{w}}{r^2}(\mathbf{u} + \mathbf{w}) - \dfrac{(n-1)}{r}(1 + \mathbf{w}^2 + \mathbf{u}\mathbf{w} + \mathbf{u}^2)\right)}{\left(\dfrac{\mathbf{u}}{r} - (n-1)\right)\left(\dfrac{\mathbf{w}}{r} - (n-1)\right)}d$$

Next, we want to estimate the last term in the above equation. We already know by Proposition 4.3.1 that Equation (4.6) satisfies a comparison principle. Then, by Proposition 4.3.2, we may use the barriers $\mathbf{w}_{0,n} \leq \mathbf{w}, \mathbf{u} \leq \mathbf{w}_{1,n}$ to estimate the term in parenthesis in the following way

$$\frac{\left(\dfrac{\mathbf{u}\mathbf{w}}{r^2}(\mathbf{u} + \mathbf{w}) - \dfrac{(n-1)}{r}(1 + \mathbf{w}^2 + \mathbf{u}\mathbf{w} + \mathbf{u}^2)\right)}{\left(\dfrac{\mathbf{u}}{r} - (n-1)\right)\left(\dfrac{\mathbf{w}}{r} - (n-1)\right)} \leq \frac{\dfrac{2\mathbf{w}_{1,n}^3}{r^2}}{\left(\dfrac{\mathbf{w}_{0,n}}{r} - (n-1)\right)^2} = \alpha(r).$$

Therefore, since $\dot{d} \leq \alpha(r)d$, it follows by integration that

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(\int_{r_0}^{r} \alpha(s)ds\right),$$

where $|\cdot|$ appears by interchanging $\mathbf{u}$ with $\mathbf{w}$ in the proof. $\qquad\square$

Corollary 4.6.3. *For $n \geq 3$ the solutions to Equations (4.6) and (4.7) are unique in their respective domains. In addition, for $n = 2$, the solution $\mathbf{v} = \dfrac{2r}{1 - r^2}$ to Equation (4.6) is unique.*

Proof. The proof follows by the estimates from Lemmas 4.6.1 and 4.6.2, since the boundary condition for two different solutions coincide. $\qquad\square$

The last part of this section is devoted to the regularity of the solutions found in Theorems 4.4.2 and 4.5.2. We note that in the construction of these solutions, the regularity can be obtained by an inductive procedure on the estimates of the derivatives of the solutions $\mathbf{v}_m$, since each derivative can be obtained by a regular function of the previous derivatives of $\mathbf{v}_m$. On the other hand, we prefer to use a well-known regularity result related to the prescription of curvature functions. We refer the reader to [CNS2], [Tru], [SUW] and [AMZ] for a better knowledge on this theory.

First, we use a well-know result about the regularity of solutions to an elliptic PDE.

Lemma 4.6.4. *Assume that $u \in \mathcal{C}^2(\Omega)$ satisfies*

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega,$$

where $F : \Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n)_+ \to \mathbb{R}$, is monotone increasing with respect the matrix variable in $S_+(n)$[7]. If $F \in \mathcal{C}^{k,\alpha}(\Gamma)$ for some $k \geq 1$ and $\alpha \in (0,1)$, then $u \in \mathcal{C}^{k+2,\alpha}(\Omega)$. In

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[7]The set $S_+(n)$ denotes the vector space of real $n \times n$ symmetric positive definite matrices. Moreover, this condition is equivalent to the operator F to be elliptic.

particular, if F is smooth, then so is u.

Proof. We refer the reader to [GT] Lemma 17.16. $\qquad\square$

As an application of Lemma 4.6.4 we show that these γ-translators are smooth.

Corollary 4.6.5. *The γ-translators found in Theorems 4.4.2 and 4.5.2 are smooth.*

Proof. First, we note that the solutions to Equation (4.4) with $k = n-1$ and Equation (4.5) with $k = n$, respectively, can be written as $\mathbf{u}(r) = \int_0^r \mathbf{v}(s)ds$, where $\mathbf{v}$ is the unique solution of class $\mathcal{C}^1$ to the Equations (4.6) and (4.7), respectively. In particular, $\mathbf{u}$ is of class $\mathcal{C}^2$. Then, since the translators are graph of the form $M = \{(x, u(x)) : x \in B_R(0)\}$ where $R \in (0, \infty]$ and $u(x) = \mathbf{u}(|x|)$, it follows that the principal curvatures can be calculated as the eigenvalues of the symmetric matrix (3.19) used in Chapter 2. Recall that this matrix is defined by

$$A = \sqrt{g^{-1}} II \sqrt{g^{-1}},$$

where $g = (\delta_{ij} + u_i u_j)dx^i dx^j$ is the metric of M, $g^{-1} = I - \dfrac{Du \otimes Du}{1 + |Du^2|}$ is the inverse of the metric g,

$$\sqrt{g^{-1}} = I - \frac{Du \otimes Du}{(1 + \sqrt{1 + |Du|^2})\sqrt{1 + |Du|^2}}$$

is the square root matrix of g^{-1}, and $II = \dfrac{D^2 u}{\sqrt{1 + |Du|^2}}$ is the second fundamental form of M. In addition, by the multiplicative properties of the determinant of a matrix, the eigenvalues of A coincides with the eigenvalues of $g^{-1} II$, which are the principal curvatures of M.

On the other hand, M is a γ-translator for $\gamma = S_{n-1}$ and Q_{n-1}, which means that

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle \text{ holds in } M, \tag{4.17}$$

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where λ are the eigenvalues of A and $\nu = \dfrac{(1, -Du)}{1 + |Du|^2}$ is the inward unit normal.

Now we are going to show that Equation (4.17) can be seen as an elliptic PDE of the form $F(Du, D^2u) = 0$ which satisfies the hypothesis of Lemma 4.6.4. In fact, by setting

$$\tilde{\gamma}(Du, D^2u) - \frac{1}{\sqrt{1 + |Du|^2}} = 0,$$

where $\tilde{\gamma} : \mathbb{R}^n \times S_+(n) \to \mathbb{R}$ is defined by

$\tilde{\gamma}(v, B)$

$$= \gamma \left(\underbrace{\left(I - \frac{v \otimes v}{(1 + \sqrt{1 + |v|^2})\sqrt{1 + |v|^2}} \right) \frac{B}{\sqrt{1 + |v|^2}} \left(I - \frac{v \otimes v}{(1 + \sqrt{1 + |v|^2})\sqrt{1 + |v|^2}} \right)}_{=Z} \right),$$

and γ is treated as a function defined in the eigenvalues $z_1, \ldots, z_n$ of the symmetric matrix Z. Moreover, $\tilde{\gamma}$ is increasing in the entries of B. In fact, we note

$$\frac{\partial \tilde{\gamma}}{\partial B_{ij}} \xi_i \xi_j = \frac{\partial \gamma}{\partial B_{pq}} \sqrt{g^{-1}}_{pi} \sqrt{g^{-1}}_{qj} \xi_i \xi_j > 0$$

holds for all $\xi \in \mathbb{R}^n \setminus \{0\}$, because $\sqrt{g^{-1}}$ is non-degenerate and the eigenvalues of the matrix $\dfrac{\partial \gamma}{\partial X_{pq}}$, which are equal to γ_{z_i} for $i = 1, \ldots, n$, are positive by Lemmas 3.1.1 and 3.1.2 since M is strictly convex.

Finally, by Lemma 4.6.4, the solution u is smooth since S_n and Q_{n-1} are smooth in the positive cone Γ_n. $\square$

4.7 A catenoid-like example

We end this chapter by giving an implicit formulation of a family of rotationally symmetric $\sqrt{S_2}$-translators in $\mathbb{R}^3$ which satisfy $H < 0$ and $K > 0$.

Theorem 4.7.1. *Let $a \in [0, 1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi), \; z \in \mathbb{R} \right\},$$

where $r_a(z)$ is giving implicitly by

$$z = \int_1^{r_a(z)} \sqrt{e^{\frac{s^2}{2} - a} - 1} \, ds \; \text{ with } r_a(0) = 1,$$

are $\sqrt{S_2}$-translators in $\mathbb{R}^3$. In addition, Σ_a are complete and satisfy $H < 0$ and $K > 0$.

Proof. The surface of the form

$$\Sigma = \left\{ (r(z)\cos(\theta), r(z)\sin(\theta), z) : (\theta, z) \in [0, 2\pi] \times \mathbb{R} \right\},$$

where $r(z)$ is a function to be found, satisfies the following formulae for the principal curvatures and unit normal vector

$$\lambda_1 = \frac{r''}{(1 + r'^2)^{\frac{3}{2}}}, \quad \lambda_2 = \frac{-1}{r\sqrt{1 + r'^2}} \text{ and } \nu = \frac{(-\cos(\theta), -\sin(\theta), r')}{\sqrt{1 + r'^2}}.$$

Then, equation $(S_2)^{\frac{1}{2}} = \langle \nu, e_3 \rangle$ reads as

$$\lambda_1 \lambda_2 = \langle \nu, e_3 \rangle^2 \Leftrightarrow r'' = -(1 + r'^2)rr'^2 \Leftrightarrow ff' = -(1 + f^2)rf^2,$$

where $r' = f(r)$ and $r'' = f'(r)r' = ff'$. Note that we are using r as an independent variable. Consequently, $f = 0$ is a trivial solution which gives that $r = C$ for some constant $C \in \mathbb{R}$.

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On the other hand, by assuming that $f \neq 0$ which obtain $f' = -(1 + f^2)rf$. It is easy to check that the solutions to this equation are $f = \dfrac{1}{\sqrt{e^{r^2 - a} - 1}}$ where $a \in \mathbb{R}$ is any constant. Indeed, we have

$$-(1 + f^2)rf = -\left(1 + \frac{1}{e^{r^2 - a} - 1}\right) \frac{r}{\sqrt{e^{r^2 - a} - 1}} = -\frac{re^{r^2 - a}}{(e^{r^2 - a} - 1)^{\frac{3}{2}}} = f'.$$

Finally, after integrating, we obtain the solutions $r(z) = C$ and

$$z = \int_1^{r_a(z)} \sqrt{e^{s^2 - a} - 1}\, ds \text{ with } r_a(0) = 1.$$

In particular, since we choose $r_a(0) =$ with $a \in [0, 1)$, it follows that r_a is well defined as an increasing function.

Moreover, it is not hard to see that Σ_a satisfies $H < 0$ and $K > 0$. To see this we note that

$$r' = \frac{1}{\sqrt{e^{r_a^2 - a} - 1}} > 0 \text{ and } r'' = -\frac{e^{r_a^2 - a} r_a' r_a}{(e^{r_a^2 - a} - 1)^{\frac{3}{2}}} = -\frac{e^{r_a^2 - a} r_a}{(e^{r_a^2 - a} - 1)^2} < 0$$

Therefore, r is an increasing concave function. Then, by substituting in the principal curvatures, we see that λ_1 and λ_2 are negative, and therefore, $H < 0$ and $K > 0$. $\qquad \square$

4.8 Appendix: "bowl"-type solution $\mathcal{C}^2$-asymptotics to cylinders

Proposition 4.8.1. *The "bowl"-type γ-translator which is defined in the ball $B_{r_0}(0) \subset \mathbb{R}^n$ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$ where $r_0 = \gamma(1, \ldots, 1)$.*

Proof. Firstly, we note that the profile curve of the "bowl"-type γ-translator is given by

$$\left\{ (r, \mathbf{u}(r)) \in \mathbb{R}^2 : r \in [0, r_0) \right\}.$$

Then, by setting $\mathbf{v} = \dot{\mathbf{u}}$, we have that

$$\dot{\mathbf{v}} = (1 + \mathbf{v}^2) f\left(\frac{\mathbf{v}}{r}\right), \tag{4.18}$$

where $f(x)$ appears in (4.18) by the implicit function theorem applied on the level set

$$\left\{ (x, y) \in \mathbb{R}^2 : \gamma(x, y, \dots, y) = 1 \right\}.$$

Consequently, since the "bowl"-type solution is defined in a ball, it follows that $\lim\limits_{x \to \infty} f(x) = L \geq 0$.

Next, the principal curvatures of the "bowl"-type solution satisfy

$$0 < \lambda_1 = \frac{\dot{\mathbf{v}}}{(1 + \mathbf{v}^2)^{\frac{3}{2}}} = \frac{f\left(\frac{\mathbf{v}}{r}\right)}{\sqrt{1 + \mathbf{v}^2}} \leq \frac{C}{\sqrt{1 + \mathbf{v}^2}},$$

$$\lambda_i = \frac{\mathbf{v}}{r\sqrt{1 + \mathbf{v}^2}}, \text{ for } i \in \{2, \dots, n\},$$

In the first line we used that f is bounded as r approaches to r_0.

Therefore, since $\lim\limits_{r \to r_0} \mathbf{v}(r) = \infty$, it follows that $\lambda_1 \to 0$ and $\lambda_i = \dfrac{1}{r_0}$ as $|p| \to \infty$. $\qquad \square$

Chapter 5

Geometric Properties of γ-Translators

Geometric properties of hypersurfaces in a $\mathbb{R}^{n+1}$ are one of the most important aspects for studying their classification. Roughly speaking, these properties appear when we apply PDE techniques over geometric quantities of a hypersurface.

The most remarkable PDE technique used in the literature is the maximum principle. Usually, the maximum principle is applied to obtain inequalities related to geometric quantities of the hypersurface. More precisely, a maximum principle will state that if a geometric quantity attains its maximum or minimum at a interior point of a hypersurface and it satisfies an elliptic linear PDE, then the quantity will be constant.

In differential geometry, one of the famous applications of maximum principles are the tangential or avoidance principle. This result states that if two connected hypersurfaces which satisfy a geometric equation, touch in a certain way, then both hypersurfaces must coincide locally. In particular, one of the oldest applications of this type of result is that spheres are the only closed embedded constant mean curvature surfaces in $\mathbb{R}^3$. That proof used the method of moving planes of Alexandrov, which implies that a closed embedded constant mean curvature surface will be rotationally symmetric, by comparing it with its reflections over "moving" planes. Consequently, one can check that the only rotationally

symmetric constant mean curvature surface is a sphere.

Finally, we also want to mention that maximum principle arguments have been generalized in many directions in recent years. We refer the reader to [AMR] for an excellent self contained book about this topic.

In this chapter we study related geometric properties of γ-translators in $\mathbb{R}^{n+1}$ for 1-homogeneous symmetric concave functions $\gamma : \Gamma \to [0, \infty)$, where $\Gamma \subset \mathbb{R}^n$ is an open convex cone[1].

For this purpose we use maximum principles related to the translator equation

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the principal curvature vector and ν is the outward unit normal of the hypersurface in $\mathbb{R}^{n+1}$, respectively.

Remark 5.0.1. The adjective "outward" on the unit normal vector ν only makes sense when the hypersurface is compact and embedded. As the reader can check in Proposition 5.0.1, there are no compact and embedded γ-translators in $\mathbb{R}^{n+1}$ whose principal curvatures belong to Γ. However, we refer to this adjective in the sense that the orientation imposed on ν agrees with the sign of being positive as in the closed case.

Moreover, a γ-translator $M_0 = F_0(M) \subset \mathbb{R}^{n+1}$ is also an eternal solution[2] to the γ-flow Equation (2.1) which evolves by translations in direction e_{n+1}. In fact, by setting

$$F(x, t) = F_0 + te_{n+1},$$

and taking time derivative and inner product with the unit normal vector ν of $M_t = F(M, t)$,

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[1]See section 5.2 below for the precise hypothesis that we consider over $\gamma : \Gamma \to [0, \infty)$.
[2]A solution to Equation (2.1) which is defined for $t \in (-\infty, \infty)$.

we see that

$$\langle \partial_t F(x,t), \nu \rangle = \langle e_{n+1}, \nu \rangle = \gamma(\lambda).$$

Finally, by taking a normal reparametrization of $F(x,t)$, the γ-flow (2.1) holds for all $t \in \mathbb{R}$.

The first application of the maximum principle that we give is about the nonexistence of closed γ-translators in $\mathbb{R}^{n+1}$ whose principal curvatures belong to Γ.

Proposition 5.0.1. *Let $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$ and assume that $\gamma : \Gamma \to (0, \infty)$ satisfies properties 1-3. Then, there is no closed immersed γ-translator in $\mathbb{R}^{n+1}$ such that its principal curvatures belong to the cone Γ.*

This proposition will be proven in section 5.1 from page 169 onwards.

On the other hand, this result is a well-know fact for H-translators, since they are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1}\rangle} dx^2)$. The main idea in our proof is the use of the maximum principle on the height function of the γ-translator.

Moreover, as an easy consequence of Proposition 5.0.1, we show that there is no totally umbilical strictly convex γ-translator in $\mathbb{R}^{n+1}$ without boundary.

Corollary 5.0.2. *Under the hypothesis of Theorem 5.0.1, there is no complete totally umbilical strictly convex γ-translator in $\mathbb{R}^{n+1}$ without boundary.*

Next, we consider the extra hypothesis on the curvature function γ to be also real analytic, locally uniformly elliptic in Γ_+. This will imply that in local coordinates of the form $(x, u(x))$ of a γ-translator, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the equation

$$F(x, Du, D^2u) = \gamma(\lambda) - \langle \nu, e_{n+1} \rangle = 0 \tag{5.1}$$

is uniformly elliptic and real analytic, in the sense that the function $F : \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is analytic, where $S(n)$ denotes the space of symmetric matrices of size n. We refer the

reader to classic regularity results from PDE theory given in [Mor1] and [Mor2] (see [Bla] for a shorter proof) that a solution $u \in \mathcal{C}^\infty(\Omega)$ to Equation (5.1) is real analytic. In particular, under this assumption, we have that a smooth γ-translator in $\mathbb{R}^{n+1}$ is real analytic.

Remark 5.0.2. We note that the part $\langle \nu, e_{n+1} \rangle = \dfrac{1}{\sqrt{1 + |Du|^2}}$ in Equation (5.1) is always analytic in local coordinates.

Consequently, under the assumptions of γ being locally uniformly elliptic and real analytic in Γ_+ with $\gamma(0) = 0$, we develop an interior and a boundary tangential principle for γ-translators in $\mathbb{R}^{n+1}$ which reads as follows.

Theorem 5.0.3. *Let* $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ *be two embedded connected complete γ-translators such that*

1. $\gamma : \Gamma \to [0, \infty)$ *satisfies properties 1-4 and 7 given below.*

2. Σ_1 *is strictly convex.*

3. Σ_2 *is convex.*

Then,

a) *(**Interior tangential principle**) Assume that there exists an interior point $p \in \Sigma_1 \cap \Sigma_2$ such that the tangent spaces coincide at p. If Σ_1 lies at one side of Σ_2, then both hypersurfaces coincide.*

b) *(**Boundary tangential principle**) Assume that the boundaries $\partial\Sigma_i$ lie in the same hyperplane Π and the intersection of Σ_i with Π is transversal. If Σ_1 lies at one side of Σ_2 and there exist $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that the tangent spaces to Σ_i and $\partial\Sigma_i$ coincide, then both hypersurfaces coincide.*

This theorem will be proven in section 5.2 from page 170 onwards. In addition, as a corollary of the tangential principle, we show a non-existence result for entire γ-translatora in $\mathbb{R}^{n+1}$ such that the "bowl"-type[3] γ-translators is defined in a round ball. We refer the reader to [Ren] Theorems 1.3-1.4 for the characterization of 'bowl"-type γ-translators in $\mathbb{R}^{n+1}$.

Corollary 5.0.4. *Assume that $\gamma : \Gamma \to [0, \infty)$ satisfies Properties 1-4 and 7. Then, if the "bowl"-type γ-translator is defined in a ball, then there cannot exist a complete convex entire γ-translators in $\mathbb{R}^{n+1}$.*

This corollary will be proven in section 5.2 from page 174 onwards.

We also include in this chapter a uniqueness result for γ-translators in $\mathbb{R}^{n+1}$ which are strictly convex, defined over ball with a single end asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$.

Theorem 5.0.5. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a complete γ-translator such that*

1. *$\gamma : \Gamma \to [0, \infty)$ satisfies properties 1-4 and 7.*

2. *Σ is strictly convex graph over a ball $B_r^n(0) \subset \mathbb{R}^n$.*

3. *Σ has a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, i.e: the principal curvatures of Σ satisfy*

$$\min \{\lambda_i(p) : i = 1, \ldots, n\} = \lambda_1(p) \to 0,$$
$$\forall i \in \{2, \ldots, n\}, \lambda_i(p) \to \frac{1}{r},$$

as $|p| \to \infty$, where $p \in \Sigma$.

Then Σ is rotationally symmetric with respect the x_{n+1}-axis.

---

[3]A strictly convex rotationally symmetric complete γ-translator which is a graph defined in a ball or in the whole $\mathbb{R}^n$.

This theorem will be proven in section 5.4 from pages 177 onwards.

As a corollary, by the uniqueness result exposed in [Ren], we will obtain the following.

Corollary 5.0.6. *Let Σ be a γ-translator as in Theorem 5.0.5 such that the "bowl"-type γ-translator is defined in a ball of radius $\gamma(1, \ldots, 1)$. Then, Σ coincide with the "bowl"-type γ-translator.*

This corollary will be proven in section 5.4 from pages 187 onwards.

Finally, the main result of this chapter will be a convexity estimate for γ-translators in $\mathbb{R}^{n+1}$.

Theorem 5.0.7 (Convexity Estimate). *Let $n \geq 3$, $\alpha, \delta > 0$ and $\Sigma \subset \mathbb{R}^{n+1}$ be a immersed, connected, complete, γ-translator such that*

a) *$\gamma : \Gamma \to \mathbb{R}$ satisfies Properties 1-3 and 5-7.*

b) *$\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : \alpha H \leq (\delta + 1)\gamma\}$ which is compactly supported[4] in $\Gamma \setminus Cyl_{n-1}$, where*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\},$$

and $\{e_i\}$ is the canonical bases of $\mathbb{R}^n$.

c) *There exists a constant $\beta \in (0,1)$ such that $\lambda_i + \lambda_j \geq \beta H$, for every $1 \leq i < j \leq n$.*

Then, $\lambda_1 \geq H - \alpha\gamma$ in Σ, where $\lambda_1(p) = \min \{\lambda_i(p) : i = 1, \ldots, n\}$.

We note that the hypothesis c) of the above theorem implies that only the minimum of the principal curvatures can be negative. In addition, we note that $\lambda \in \Gamma_{\alpha,\delta}$ does not necessarily give a fixed sign to $H - \alpha\gamma$ on Σ, but at least it gives a way to control λ_1 from below, given the name of convexity estimate.

---

[4]This means that $\Gamma_{\alpha,\delta} \cap \partial B_1(0)$ is compact in $\Gamma \setminus Cyl_{n-1}$.

Furthermore, by a maximum principle argument, the hypothesis b) is preserved under the γ-flow, and it can be used to study the limits under suitable parabolic resclaing under the γ-flow. Finally, property b) can be seeing as a non-splinting hypothesis in the sense that the set $\Gamma_{\alpha,\delta}$ is away from having $\lambda_1 = 0$.

About the proof of the convexity estimate, we will use maximum principle arguments over the function $f = \dfrac{\gamma}{H - \lambda_1}$. In particular, we show under the assumptions of Theorem 5.0.7 that $f \geq \alpha^{-1}$ in Σ, which is equivalent to the convexity estimate. The main part of the proof will use the Omori-Yau maximum principle (see [AMR] Chapter 3), which gives suitable information to formulate a maximum principle at "infinity" of Σ.

The proof of this result is given in section 5.5 from page 1333 onwards, and it was inspired by [SS], where the authors showed that a mean convex and uniform 2-convex H-translator is in fact convex.

Remark 5.0.3. In general, Omori-Yau maximum principles require regularity on the solution to be applied. We note that this is possible since γ is concave. In fact, the Evans-Krylov theory for concave functions (see [Kry] or [GT] Theorem 17.14) can bee applied to give the suitable regularity.

Finally, as a consequence of Theorem 5.0.7, we will be able to show that these γ-translators are asymptotically convex.

Corollary 5.0.8. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be γ-translator for which $\gamma : \Gamma \to \mathbb{R}$ satisfies the hypothesis of Theorem 5.0.7, then Σ is asymptotically convex.*

The organization of this chapter goes as follows: In Section 5.1, we give equations and estimates for γ-translators in $\mathbb{R}^{n+1}$ and the proofs of Proposition 5.0.1 and Corollary 5.0.2. In Section 5.2, we show the tangential principle, Theorem 5.0.3. In Section 5.2, we prove the non-existence theorems. In Section 5.4, we show the uniqueness results rewarding the "bowl"-type γ-translators. In Section 5.5, we give the proof of the convexity estimate Theorem 5.0.7.

5.1 Preliminaries

In this section we give some preliminaries with respect 1-homogeneous symmetric concave functions and the proofs of Proposition 5.0.1 and Corollary 5.0.2.

As we mentioned in the introduction of this chapter we require the following properties for $\gamma : \Gamma \to \mathbb{R}$.

1. $\Gamma \subset \mathbb{R}^n$ is a symmetric open convex cone which contains the positive cone $\Gamma_+ :=$ $\{\lambda \in \mathbb{R}^n : \lambda_i > 0 \ \forall i = 1, \ldots, n\}$ and $\gamma : \Gamma \to \mathbb{R}$ is symmetric, smooth and positive.

2. γ is strictly increasing in each variable, i.e: $\frac{\partial \gamma}{\partial \lambda_i} > 0$ in Γ for every i.

3. γ is 1-homogeneous, i.e: for every $c > 0$, $\gamma(c\lambda) = c\gamma(\lambda)$ in Γ.

4. γ is real analytic, locally uniformly elliptic in Γ_+.

5. γ is strictly concave in off-radial directions, i.e: for every $\lambda \in \Gamma$ and $\xi \in \mathbb{R}^n$ it holds

$$\frac{\partial^2 \gamma}{\partial \lambda_i \partial \lambda_j}(\lambda)\xi_i\xi_j \leq 0,$$

and equality holds if, and only if, ξ is a scalar multiple of λ.

6. Let $G : \{A \in S(n) : \lambda(A) \in \Gamma\} \to \mathbb{R}$ be a smooth symmetric function such that $G(A) = \gamma(\lambda)$, when A is a diagonal matrix with entries $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then, there exists a constant $C > 0$ such that

$$\frac{d}{ds}G(A + sB) \leq C\mathrm{Tr}(B),$$

whenever B is a symmetric 2-nonnegative matrix[5] with $\lambda(B) \in \Gamma$. Moreover, the inequality is strict unless $B = 0$.

---

[5]The eigenvalues of B satisfy $\lambda_i + \lambda_j \geq 0$ for every $i \neq j$.

7. γ vanishes at the boundary of Γ, i.e there exists a continuous extension of γ to $\overline{\Gamma}$ which vanishes identically at $\partial\Gamma$.

The reason why we impose these conditions for γ are the following:

- Property 1 is to simplify certain computations, since it is difficult to calculate derivatives on the eigenvalues rather than the entries of a matrix. Indeed, it is a well-know result (see for instance [Gla], that there exists a smooth symmetric function

$$G : \{A \in \mathrm{Sym}(n) : \lambda(A) \in \Gamma\} \to \mathbb{R}$$

such that $G(A) = \gamma(\lambda)$ where A is a diagonal matrix. See section 5.1 below for more details.

- Property 2 implies that the γ-flow is weakly parabolic, i.e:. when we write Equation (2.1) in local coordinates as a graph of a function under the exponential map defined in a ball in T_pM,

$$\partial_t u = \gamma(\nabla u, \nabla^2 u)\sqrt{1 + |\nabla u|^2},$$

weakly parabolic means that the second order term $\gamma(\nabla u, \nabla^2 u)$ is elliptic, i. e: $\sum\limits_{i,j=1}^{n} \gamma(\nabla u, \nabla^2 u)_{ij}\xi_i\xi_j > 0$ for every $\xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$.

- Property 3 implies that the set of solutions of (2.1) is closed under parabolic scaling. This property is important for studying singularities that appear under these flows.

- Property 4 implies that in local coordinates of the form $(x, u(x))$ of a γ-translator, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the equation

$$F(x, Du, D^2u) = \gamma(\lambda) - \langle \nu, e_{n+1} \rangle = 0 \tag{5.2}$$

is uniformly elliptic and real analytic, in the sense that the function $F : \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is analytic, where $S(n)$ denotes the space of symmetric matrices of order n. We refer the reader to classic regularity results from PDE theory given in [Mor1] and [Mor2] (see [Bla] for a shorter proof) that a solution $u \in \mathcal{C}^\infty(\Omega)$ to Equation (5.2) is real analytic. In particular, under this assumption, we have that a smooth γ-translator in $\mathbb{R}^{n+1}$ is real analytic.

Remark 5.1.1. We note that the part $\langle \nu, e_{n+1} \rangle = \dfrac{1}{\sqrt{1 + |Du|^2}}$ in Equation (5.2) is always analytic in local coordinates.

- Properties 5 -7 are related to convexity estimates under the flow. Since the function γ is concave, a non-positive sign of the second order derivatives of γ (which appears frequently in the evolution equations under the γ-flow) would not give suitable information.

  Property 7 has been used in [Lyn2] and [LL] to preserve other convex cones.

  Furthermore, Property 6 ensures that 2-convexity[6] is preserved under the γ-flow.

  Finally, Property 7 is used to preserve the convexity of the initial hypersurface under the γ-flow.

Regarding the notation that we use for the derivatives of symmetric functions in the principal curvatures of γ-translators along this chapter, we first mention that we abuse on the notation by setting $\varphi(A) = \varphi(\lambda(A))$ for a symmetric function φ, instead of using another function as in Remark 1. Here $A = (h_{ij})$ is a symmetric matrix such that $\lambda(A) \in \Gamma$. Then, we write

$$\dot{\varphi}^{ab}(A) = \frac{\partial \varphi}{\partial h_{ab}}(A), \ \dot{\varphi}^a(\lambda) = \frac{\partial \varphi}{\partial \lambda_a}(\lambda), \ \ddot{\varphi}^{ab,cd}(A) = \frac{\partial^2 \varphi}{\partial h_{ab} \partial h_{cd}}(A), \ \ddot{\varphi}^{ab}(\lambda) = \frac{\partial^2 \varphi}{\partial \lambda_a \partial \lambda_b}(\lambda).$$

---

[6] The principal curvatures of the hypersurface satisfies $\lambda_i + \lambda_j > 0$ for all $i \neq j$.

Furthermore, if φ is 1-homogeneous, then

$$\dot{\varphi}^{ab}(A) = \delta_{ab}\dot{\varphi}^a(\lambda),$$

for any diagonal matrix A.

Finally, if in addition the eigenvalues are simple, $\lambda_1 < \ldots < \lambda_n$, then we can write

$$\ddot{\varphi}^{ab,cd}(A)T_{ab}T_{cd} = \ddot{\varphi}^{ab}(\lambda)T_{aa}T_{bb} + 2\sum_{a<b}\frac{\dot{\varphi}^b(\lambda) - \dot{\varphi}^a(\lambda)}{\lambda_b - \lambda_a}|T_{ab}|^2,$$

for every symmetric matrix T_{ab}.

Lemma 5.1.1. *Let Σ be a γ-translator. Then, we have the following equations at $p \in \Sigma$*

$$\Delta_\gamma h_{ij} + \ddot{\gamma}^{ab,cd}\nabla_i h_{ab}\nabla_j h_{cd} + |A|_\gamma^2 h_{ij} + \langle\nabla h_{ij}, e_{n+1}\rangle = 0, \tag{5.3}$$

$$\Delta_\gamma \varphi + \left(\dot{\varphi}^{ij}\ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij}\ddot{\varphi}^{ab,cd}\right)\nabla_i h_{ab}\nabla_j h_{cd} + |A|_\gamma^2\varphi + \langle\nabla\varphi, e_{n+1}\rangle = 0, \tag{5.4}$$

$$\Delta_\gamma\gamma + \langle\nabla\gamma, e_{n+1}\rangle + |A|_\gamma^2\gamma = 0, \tag{5.5}$$

$$\Delta_\gamma u - |\nabla u|^2 + 1 = 0, \ u(p) = \langle p, e_{n+1}\rangle, \tag{5.6}$$

where $\Delta_\gamma = \dot{\gamma}^{ab}\nabla_a\nabla_b$, $\langle X, Y\rangle_\gamma = \dot{\gamma}^{ab}X_aY_b$ and $|A|_\gamma^2 = \dot{\gamma}^{ab}h_{ai}h_{ib}$.

Proof. The proof of these equations is very similar to that given in Lemma 3.3.3 for $\gamma = Q_k$. We only prove Equation (5.6) which has been written differently. In fact, substituting into Equation (3.78) with Q_k for γ and w for e_{n+1}, we have

$$\Delta_\gamma u = -\gamma\langle\nu, e_{n+1}\rangle = -\gamma^2,$$

here we are using a normal coordinates at $p \in \Sigma$ given by $\{e_1, \ldots, e_n\} \subset T_p\Sigma$ as an ortonormal basis of eigenvectors of the second fundamnetal form of Σ at p.

On the other hand, we note that $\nabla_i u = \langle e_i, e_{n+1} \rangle$ which implies that $\nabla u = e_{n+1}^\top$, where $(\cdot)^\top$ denotes the tangential projection onto $T_p M$. Then, it follows that

$$\gamma^2 = \langle \nu, e_{n+1} \rangle^2 = 1 - \left| e_{n+1}^\top \right|^2 = 1 - |\nabla u|^2 .$$

Therefore, we finally obtain $\Delta_\gamma u - |\nabla u|^2 + 1 = 0$. $\qquad\square$

The next lemma corresponds to a second order estimates of γ that we will use for the convexity estimate.

Lemma 5.1.2. *Let Γ' be a symmetric closed cone compactly supported in $\Gamma \setminus Cyl_{n-1}$, where*

$$Cyl_j = \{ \lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0 \} .$$

Then, there exist a constant $0 < C = C(n, \gamma, \Gamma')$ such that

$$C^{-1} \delta_{ab} \dot{\gamma}^a(\lambda) \leq \dot{\gamma}^{ab}(A) \leq C \delta_{ab} \dot{\gamma}^i(\lambda),$$

$$\sum_{i=1}^n \ddot{\gamma}^{ab,cd}(A) T_{iab} T_{icd} \leq -C \frac{|T|^2}{H},$$

where A is a diagonal matrix with eigenvalues $\lambda \in \Gamma'$ and T_{iab} is a symmetric $(3,0)$-tensor.

Proof. We refer the reader to Lemma 2.5 in [Lyn1]. $\qquad\square$

Now we prove that there is no closed immersed γ-translator in $\mathbb{R}^{n+1}$ such that its principal curvatures belong to Γ.

Proof of Proposition 5.0.1. Let us assume that Σ is a compact γ-translator without boundary in $\mathbb{R}^{n+1}$, and without loss of generality, we will also assume that $0 \in \Sigma$.

Then, by Equation (5.6), the height function $u(p) = \langle p, e_{n+1} \rangle$ satisfies

$$\Delta_\gamma u - |\nabla u|^2 + 1 = 0.$$

We note that property 2 implies that the operator Δ_γ is elliptic when the principal curvatures belong to Γ.

Consequently, by compactness and continuity, u must attains an interior minimum (recall that $\partial \Sigma = \emptyset$). However, this is impossible since $\Delta_\gamma u \geq 0$ and $\nabla u = 0$ at the point where the minimum of u is reached.

Therefore, there cannot exist any closed γ-translator such that its principal curvatures belong to Γ. $\qquad\qquad\square$

To finalize this section, we use the above proposition to show that there is no complete totally umbilical γ-translator in $\mathbb{R}^{n+1}$ without boundary.

Proof of Corollary 5.0.2. Lets assume the opposite, which means there is a strictly convex totally umbilical γ-translator M without boundary in $\mathbb{R}^{n+1}$. It is a well-know result (see for instance Ex. 8.6 in [dC]) that a totally umbilical hypersurface in $\mathbb{R}^{n+1}$ is an open set of a n-hyperplane or a n-sphere. However, since the principal curvatures of M are positive, this implies that M is a open set of a n-sphere, in particular compact. This fact contradicts Proposition 5.0.1. $\qquad\qquad\square$

5.2   Tangential Principle

The proof of Theorem 5.0.3 was inspired by the result in [Mø] for H-translators in $\mathbb{R}^{n+1}$.

Proof of Thoerem 5.0.3. Firstly, we mention that along this proof D_i, D_{ij} will denote the derivatives with respect to a local frame for each hypersurface. Moreover, we will denote by $B_r^i(x)$ a open ball of radius r centered at $x \in \mathbb{R}^i$.

Let $p \in \Sigma_1 \cap \Sigma_2$ be an interior point such that $T_p\Sigma_1 = T_p\Sigma_2$ and Σ_1 lies locally at one side of Σ_2. Then, after a rotation and a translation, there exists $r > 0$ such that each $M_i \cap B_r^{n+1}(p)$ is the graph of a smooth function $u_i : B_r^n(0) \subset \{x_{n+1} = 0\} \to \mathbb{R}$, where $p = (0, u_i(0))$. Therefore, since Σ_1 lies at one side of Σ_2, we may assume that $u_1 > u_2$ in $B_r^n(0) \setminus \{0\}$ and $u_1(0) = u_2(0)$.

Next, we consider a convex combination between u_1 and u_2 given by

$$u_s = (1 - s)u_1 + su_2, \text{ for } s \in [0, 1].$$

It is not hard to see that for each $s \in (0, 1)$, the graph of $u_s : B_r(0) \to \mathbb{R}$ is strictly convex. In fact, this holds since the convex combination of a positive definite matrix with a positive semi-definite matrix is positive definite. Here the involved matrices are the shape operators of the graphs of u_1 and u_2, respectively.

In particular, since for each $s \in (0, 1)$, the principal curvatures of the graph u_s lie in the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$, it follows that u_s can be evaluated in the functional

$$E(s) = \gamma(h_j^i(s)) - \langle \nu(s), e_{n+1} \rangle.$$

Recall that the s dependence in $E(s)$ is related to the coefficients of the shape operator and the unit normal vector of u_s.

Furthermore, by the mean value theorem together with the fact that $E(1) = E(0) = 0$, it follows that

$$0 = E(1) - E(0) = \frac{\partial E}{\partial s}(s_0),$$

for some $s_0 \in (0, 1)$.

On the other hand, we can explicitly calculate the term $\dfrac{\partial E}{\partial s}(s)$.

Indeed, by denoting $v = u_2 - u_1$ and $\dot{\gamma}^{ij}(s) = \dfrac{\partial \gamma}{\partial h_j^i}(h_j^i(s))$, we have

$$
\begin{aligned}
\frac{\partial E}{\partial s}(s) =& \dot{\gamma}^{ij}(s)\frac{\partial h_j^i}{\partial s}(s) - \frac{\partial}{\partial s}\frac{1}{\sqrt{1+|Du_s|^2}} \\
=& \dot{\gamma}^{ij}(s)\frac{\partial}{\partial s}\left[\left(\frac{\delta_{ik}}{\sqrt{1+|Du_s|^2}} - \frac{D_i u_s D_k u_s}{(1+|Du_s|^2)^{3/2}}\right)D_{kj}u_s\right] + \frac{\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}} \\
=& \dot{\gamma}^{ij}(s)\left[\left(-\frac{\delta_{ik}\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}} - \frac{D_i v D_k u_s + D_k v D_i u_s}{(1+|Du_s|^2)^{\frac{3}{2}}} + 3\frac{D_i u_s D_k u_s \langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{5}{2}}}\right)D_{kj}u_s\right. \\
& \left. + \left(\frac{\delta_{ik}}{\sqrt{1+|Du_s|^2}} + \frac{D_i u_s D_k u_s}{(1+|Du_s|^2)^{\frac{3}{2}}}\right)D_{kj}v\right] + \frac{\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}}.
\end{aligned}
$$

We claim that $v = 0$ in $B_r(0)$, and we will argue this by contradiction, i.e: we will assume that v is not a constant function such that $v(0) = 0$ and $v(x) \leq 0$ in $B_r^n(0)$.

Firstly, we note that by continuity, v reaches a maximum in $\overline{B_r^n(0)}$.

Let us assume first that the maximum is reached at 0. Recall that v satisfies the linear elliptic PDE given by

$$
\begin{aligned}
0 =& \dot{\gamma}^{ij}(s_0)\left(\frac{\delta_{ik}}{\sqrt{1+|Du_{s_0}|^2}} - \frac{D_i u_{s_0} D_k u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right)D_{kj}v \\
& + \left\langle\left[\dot{\gamma}^{ij}(s_0)\left(-\frac{\delta_{ik}D_{kj}u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}} + 3\frac{D_i u_{s_0} D_k u_{s_0} D_{kj}u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{5}{2}}}\right) + \frac{1}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right]Du_{s_0}, Dv\right\rangle \\
& - \dot{\gamma}^{ij}(s_0)D_{kj}u_{s_0}\frac{D_i u_{s_0} D_k v + D_k u_{s_0} D_i v}{(1+|Du_{s_0}|)^{\frac{3}{2}}}.
\end{aligned}
$$

Then, since γ is a locally uniformly elliptic operator when the principal curvatures of the graph of u_{s_0} belong to Γ_+, it follows that the second order term in the above equation

$$
\dot{\gamma}^{ij}(h_j^i(s_0))\left(\frac{\delta_{ik}}{\sqrt{1+|Du_{s_0}|^2}} - \frac{D_i u_{s_0} D_k u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right)D_{kj}v
$$

is uniformly elliptic in $B_r^n(0)$.

Consequently, the hypotheses of the strong maximum principle (Theorem 3.5 in [GT]) are satisfied, which means that v is constant in $B_r^n(0)$, giving a contradiction with our assumption.

On the other hand, let us assume that the maximum of v is reached at some $x_0 \in \partial B_r^n(0)$. Then, since v cannot reach an interior maximum, it follows that $v(x_0) > 0$.

In particular, the hypotheses of Hopf's Lemma (Lemma 3.4 in [GT]) hold, which implies that $\frac{\partial v}{\partial N}(x_0) > 0$ where $N = \frac{x}{r}$ is the outward unit normal of $\partial B_r^n(0)$. This means that $\alpha(t) = v\left(\frac{x_0}{r}t\right)$ is an increasing function when t is close to r.

However, by continuity together with $v \leq 0$ in $B_r^n(0)$, we can find a t_0 close to r such that $\alpha(t_0) = 0$. This contradicts the fact that v only vanishes at $x = 0$. This finishes the proof of the claim $v = 0$ in $B_r^n(0)$.

Therefore, we have that $\Sigma_1 \cap B^{n+1}(p,r) = \Sigma_2 \cap B^{n+1}(p,r)$. Finally, since both hypersurfaces are connected, we may apply the weak uniqueness continuation principle since each Σ_i is real analytic, to obtain that $\Sigma_1 = \Sigma_2$.

For the boundary tangency principle, we only need to change $B(p,r)$ with a hemisphere $B(p,r) \cap \{x_{n+1} \geq 0\}$, here we consider $\Pi = \{x_{n+1} = 0\}$.

Then, since the intersection $\Sigma_i \cap \Pi$ is transversal, the function $v = u_2 - u_1$ satisfies $\frac{\partial v}{\partial N}(p) = 0$ where $N = e_{n+1}$ is the normal unit vector of $\partial \Sigma_i$ at p.

On the other hand, the hypothesis of the Hopf's Lemma (Lemma 3.4 in [GT]) holds, $\frac{\partial v}{\partial N}(p) > 0$ which is a contradiction. Therefore, the same argument holds in that case to conclude the result. $\qquad \square$

Remark 5.2.1. Note that without the hypothesis of γ being real analytic in Γ_+, we only would obtain that Σ_1 agrees with Σ_2 locally at p.

Remark 5.2.2. A possible generalization of Theorem 5.0.3 could be made by showing an analogous result related to the preservation of the local uniform ellipticity of γ when the

principal curvatures of the graph of u_s belong to other cones instead of Γ_+.

5.3 A non-existence result

In this section we prove a non-existence result for γ-translators in $\mathbb{R}^{n+1}$ which are convex and entire graphs.

Proof of Corollary 5.0.4. We prove this by contradiction. Assume that there is an entire γ-translator Σ given by a function $u : \mathbb{R}^n \to \mathbb{R}$ such that its principal curvatures belong to the cone $\{\lambda \in \mathbb{R}^n : \lambda_i \geq 0\}$. This last property is equivalent to Σ being convex.

Let C_n be the "bowl"-type γ-translator in $\mathbb{R}^{n+1}$. Recall that C_n is the unique[7] strictly convex rotationally symmetric graph defined in $B(0, r_0)$ which is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$ (see the Appendix 4.8 in Chapter 4 for a proof of this fact), where $r_0 = \gamma(1, \ldots, 1)$.

Then, translating suitably C_n over Σ, we can find a $t_0 > 0$ such that $C_n + te_{n+1}$ lies strictly above from Σ for $t \geq t_0$. Note that this can be done since C_n is not an entire graph. Now, we may translate $C_n + t\varepsilon_{n+1}$ downward until it touches Σ for the first time.

Finally, by the interior tangential principle Theorem 5.0.3, we obtain that $\Sigma = C_n$ which contradicts that Σ is entire. $\qquad\square$

5.4 Uniqueness

In this section we proof the uniqueness result Theorem 5.0.5 and Corollary 5.0.6.

Firstly, since the shape operator of a given hypersurface in $\mathbb{R}^{n+1}$ is invariant under isometries of $\mathbb{R}^{n+1}$, it follows that $\gamma(\tilde{\lambda}) = \gamma(\lambda)$ where $\tilde{\Sigma}$ is the image of any isometry applied to a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$.

---

[7]Unique among rotationally symmetric strictly convex graphs.

In particular, if R is a rotational field which fixes the x_{n+1}-axis in $\mathbb{R}^{n+1}$, then

$$\langle \tilde{\nu}, e_{n+1} \rangle = \langle \nu, R^{-1} e_{n+1} \rangle = \langle \nu, e_{n+1} \rangle = \gamma(\lambda) = \gamma(\tilde{\lambda}).$$

This means that γ-translators remain as γ-translators after applying rotational fields which fix the direction of translation under the γ-flow.

Consequently, under the hypothesis of Theorem 5.0.4, it is enough to show that Σ is symmetric along the plane $\{x_1 = 0\}$ to obtain that Σ is rotationally symmetric. For this purpose we will adopt the following notations and definition used in [MSHS] for the case when $\gamma = H$.

Let $A \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, then we set:

- The 1-parameter family of vertical hyperplanes $\Pi_t = \{x \in \mathbb{R}^{n+1} : \mathbf{p}(x) = t\}$, where

$$\mathbf{p}(x_1, \ldots, x_{n+1}) = x_1.$$

In addition, we denote $\Pi = \Pi_0$.

- The 1-parameter family of horizontal spaces $Z_t = \{x_{n+1} > t\}$.

- The 1-parameter family of subsets of A given by

$$A_+(t) = \{x \in A : \mathbf{p}(x) \geq t\},$$
$$A_-(t) = \{x \in A : \mathbf{p}(x) \leq t\},$$
$$\delta_t(A) = A \cap \Pi_t.$$

Note that $A_+(t)$ and $A_-(t)$ are the right hand side and the left hand side, respectively, along Π_t of A (see Definition 5.4.1 given below).

- The 1-parameter families of right and left reflections of A, respectively, along the hyperplane Π_t given by

$$A_+^*(t) = \left\{ (2t - x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \in A_+(t) \right\},$$
$$A_-^*(t) = \left\{ (2t - x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \in A_-(t) \right\},$$

respectively.

- The orthogonal projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to the hyperplane Π given by

$$\pi(x_1, \ldots, x_{n+1}) = (0, x_2, \ldots, x_{n+1}).$$

Definition 5.4.1. Let A, B be two subsets of $\mathbb{R}^{n+1}$. We say that $B \leq A$ and it is read as "A is on the right side of B" if, and only if, for every $x \in \Pi$ such that

$$\pi^{-1}(\{x\}) \cap A \neq \emptyset \text{ and } \pi^{-1}(\{x\}) \cap B \neq \emptyset,$$

we have that

$$\sup \left\{ \mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap B \right\} \leq \inf \left\{ \mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap A \right\}. \tag{5.7}$$

Note that for arbitrary sets, the relation $B \leq A$ is not a partial order, but for sets given by the graph of an entire function over the plane Π, it works as a partial order.

Remark 5.4.1. The method of moving planes requires specifically two things to be applied:

The first one is that the hyperplanes to be solutions to Equation (2.5). To accomplish this, we add the Property 7 on γ, which means that vertical hyperplanes (whose unit normal vector is different from $\pm e_{n+1}$) are γ-translators.

On the other hand, we need tangential interior and boundary principles, respectively, to

decide what happens when the hypersurface intersects tangentially the reflection of the same hypersurface along a moving plane. In our case, Theorem 5.0.3 gives us these results. Recall that vertical hyperplanes are real analytic convex hypersurfaces.

Proof of Theorem 5.0.5. Recall the hypotheses of this theorem are: $\Sigma \subset \mathbb{R}^{n+1}$ is a complete γ-translator such that

1. $\gamma : \Gamma \to [0, \infty)$ satisfies properties 1-4 and 7.

2. Σ is strictly convex graph over a ball $B_r^n(0) \subset \mathbb{R}^n$.

3. Σ has a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$.

Firstly, since Σ is a strictly convex vertical graph defined in $B_r(0)$, we have

$$r = \sup \{t > 0 : \Sigma \cap \Pi(t) \neq \emptyset\}.$$

Then, we consider the set

$$\mathcal{A} := \{t \in [0, r) : \Sigma_+(t) \text{ is a graph over } \Pi \text{ and } \Sigma_-(t) \leq \Sigma_+^*(t)\}.$$

The set $\mathcal{A}$ will act as the parameter of the family of moving hyperplanes in the application of the moving plane method (see Fig. 5.1). For this reason, we want to show that $\mathcal{A}$ is the interval $[0, r)$. In fact, this will give us that $\Sigma_-(0) \leq \Sigma_+^*(0)$, and by analogous arguments we will obtain that $\Sigma_-^*(0) \leq \Sigma_+(0)$. We note that the combination of these two properties imply Σ is symmetric about the hyperplane Π.

Therefore, in the following claims we will show that $\mathcal{A} = [0, r)$ by arguing that $\mathcal{A}$ is not empty, and it is an open and closed set of $[0, r)$.

Claim 5.4.1. *The set $\mathcal{A} \neq \emptyset$. In addition, for every $s \in \mathcal{A}$, $[s, r) \subset \mathcal{A}$.*
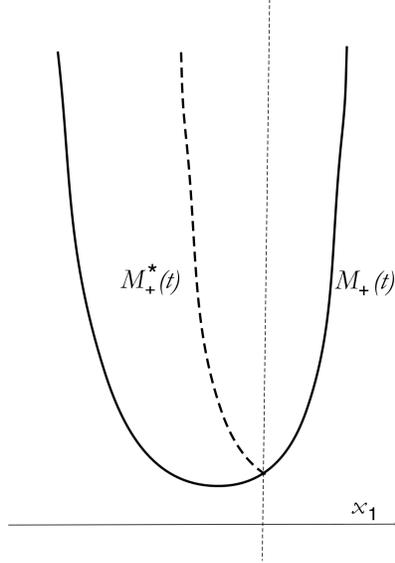
Figure 5.1: The moving plane method acting in a rotationally symmetric strictly convex surface M. Image courtesy of Francisco Martín.

Proof. We will show in the following steps that there exists $\varepsilon \in (0, t_0)$, such that $(t_0 - \varepsilon, t_0) \subset \mathcal{A}$, giving that $\mathcal{A} \neq \emptyset$.

Step 5.4.1. $\Sigma_+(t)$ *is connected for every* $t \in [0, r)$.

Proof. Firstly, lets recall that Σ has a single end. This implies that $\Sigma_+(t)$ has only one unbounded component for every $t \in [0, r)$. Because otherwise, we can choose a compact Σ' component of $\Sigma_+(t)$. Then, we can translate a hyperplane Π_t until it touches Σ' at a first order contact point[8]. Note that the tangential principles Theorem 5.0.3, applied to Σ' with Π_{t_1}, for some $t_1 > 0$, imply that Σ' is totally geodesic, which contradicts that Σ is strictly convex graph.

Therefore, $\Sigma_+(t)$ is connected for every $t \in [0, t_0)$. $\qquad\square$

Step 5.4.2. *There exists* $\varepsilon_0 \in (0, t_0)$ *such that* $\Sigma_+(t)$ *is a graph over* Π *for* $t \in (t_0 - \varepsilon_0, t_0)$.

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[8]This means that both hypersurfaces intersect at a point where the first derivatives coincide. In particular, the tangent planes coincides because the unit normal vectors of each hypersurface concide at this point.

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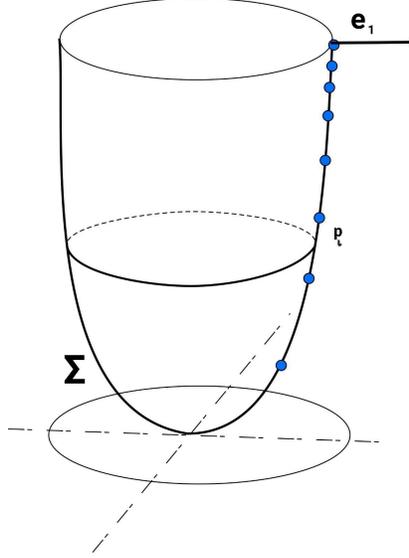
Figure 5.2: Sequence $\nu(p_l) \to e_1$ as $l \to \infty$ where $p_l \in \Sigma_+(t_l)$. Image courtesy of Francisco Martín.

Proof. Firstly, we will show that there exists $\varepsilon_0 \in (0, r)$ such that

$$\langle \nu, e_1 \rangle = \frac{D_1 u}{\sqrt{1 + |Du|^2}} > 0$$

holds in $\Sigma_+(t)$ for all $t \in (r - \varepsilon_0, r)$. Recall that Σ is a vertical graph of a function $u : B_r^n(0) \subset \mathbb{R}^n \to \mathbb{R}$.

For this purpose let us assume the contrary. This means that there exist an increasing sequence t_l converging to t_0 such that $D_1 u(x_1^l, \ldots, x_n^l) \leq 0$ for some $p_l = (x_1^l, \ldots, x_n^l, u(x_1^l, \ldots, x_n^l)) \in \Sigma_+(t_l)$.

Then, since $t_l \leq x_1^l < t_0$, it follows that $|p_l| \to \infty$. Recall that the end of Σ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^n(r) \times \mathbb{R}$, this means that $\nu(p_l) \to (1, 0 \ldots, 0)$ as $l \to \infty$. In fact, this occurs because the intersection of the cylinder $\mathbb{S}^n(r)$ with the hyperplane Π_r occurs at the point with unit normal vector $(1, 0, \ldots, 0)$. Note that this contradicts that $D_1 u(x_1^l, \ldots, x_n^l) = 0$ (see Fig 5.2).

Consequently, there exists $\varepsilon_0 \in (0, r)$ such that $\langle \nu, e_1 \rangle > 0$ holds at $\Sigma_+(t)$ for all $t \in$

$(r - \varepsilon, r)$. In particular, this fact together with Σ being a complete graph defined in $B_r^n(0)$[9] and $\Sigma_+(t)$ being connected, imply that $\Sigma_+(t)$ is a graph over Π for all $t \in (r - \varepsilon_0, r]$. □

Step 5.4.3. *Let* $\varepsilon = \dfrac{\varepsilon_0}{2}$, *then* $\Sigma_-(t) \leq \Sigma_+^*(t)$ *for every* $t \in [r - \varepsilon, r)$. *Finalizing the first part of the proof of Claim* 5.4.1.

Proof. Firstly, since $\Sigma_+(t)$ is a graph over Π for $t \in [r - \varepsilon, r)$, the relation "being at the right side" (see Definition 5.4.1) is a partial order.

Consequently, it suffices to show that for $t_1 = r - \dfrac{\varepsilon_0}{2}$, the condition $\Sigma_-(t_1) \leq \Sigma_+^*(t_1)$ (i.e: the reflection of $\Sigma_+(t_1)$ along Π_{t_1} is at the right hand side of $\Sigma_-(t_1)$) is satisfied.

Let us assume that $\Sigma_-(t_1) \not\leq \Sigma_+^*(t_1)$, then we may find a first order contact point p_0 between $\Sigma_-(t_0)$ and $\Sigma_+^*(t_0)$ for some $0 < t_0 < r - \varepsilon_0$. In fact, $t_0 \notin [r - \varepsilon_0, t_1]$ since $\Sigma_+(t)$ is a graph over Π for $t \in [r - \varepsilon, r)$, and this would imply that $\Sigma_+(t_0)$ is not a graph over Π. Moreover, if $t_0 = 0$, then the proof finishes, since $0 \in \mathcal{A}$ is what we want to prove.

Therefore, the interior and boundary tangential principles in Theorem 5.0.3 imply that $\Sigma_+^*(t_0) = \Sigma_-(t_0)$, which means that Σ is symmetric along Π_{t_0}. But, since the end of Σ is symmetric along the plane Π, it follows that $t_0 = 0$ a contradiction. □

Step 5.4.4. *For every* $s \in \mathcal{A}$, $[s, r) \subset \mathcal{A}$.

Proof. Firstly, we note that if $s \in \mathcal{A}$, then $\Sigma_+(s)$ is a graph over Π. In particular, since $\Sigma_+(t) \subset \Sigma_+(s)$ for every $s < t < r$, it follows that $\Sigma_+(t)$ is a graph over Π.

Now, let $t_1 \in (s, r)$ be such that $\Sigma_-(t_1) \not\leq \Sigma_+^*(t_1)$. Then, we can find $t_0 \in (t_1, r - \varepsilon)$ such that $\Sigma_+^*(t_0) \setminus \delta_{t_0}(\Sigma)$ and $\Sigma_-(t_0) \setminus \delta_{t_0}(\Sigma)$. We note that the interior tangential principle implies that $\Sigma_+^*(t_0) = \Sigma_-(t_0)$, giving that $\Pi_{t_0} \neq \Pi$ is hyperplane of symmetry of Σ, this contradics that the end of sigma is symmetric along Π. □

□

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[9]This means that the Gauss map $\nu : \Sigma \to \mathbb{S}^n$ is a proper local diffeomorphism which maps the boundary to the boundary of the image.

Claim 5.4.2. *The set $\mathcal{A}$ is closed in $[0, t_0)$.*

Proof. Let $s_k \in \mathcal{A}$ be a sequence of points such that $s_k \to s_0$. In the following two steps we are going to show by contradiction that $s_0 \in \mathcal{A}$.

Step 5.4.5. *Assume that $\Sigma_+(s_0)$ is not a graph over Π.*

Proof. The assumption of this step implies that we can find $p, q \in \Sigma_+(s_0)$ such that $p = (p_1, p_2, \ldots, p_{n+1})$ and $q = (q_1, p_2, \ldots, p_{n+1})$ with $p_1 < q_1$.

Then, by Claim 5.4.1, we have that $s > s_0$ for every $s \in \mathcal{A}$. In particular, it follows that follows that $p_1 = s_0$. Because otherwise $q \in \Sigma_+(p)$, and $\Sigma_+(p)$ would be a graph over Π which does not agree with our assumption.

Moreover, we note that

$$q_1 > s = \frac{q_1 + 3s_0}{4} > s_0 = p_1.$$

Then, since $q \in \Sigma_+(s)$, $p \in \Sigma_-(s)$ and

$$2s - q_1 = \frac{3s_0 - q_1}{2} = s_0 + \frac{s_0 - q_1}{2} < p_1,$$

it follows that $\Sigma_-(s) \not\leq \Sigma_+^*(s)$, which contradicts that $s \in \mathcal{A}$. $\square$

Step 5.4.6. *Assume that $\Sigma_+(s_0)$ is a graph over Π and $\Sigma_-(s_0) \not\leq \Sigma_+^*(s_0)$.*

Proof. Firstly, we note that the relation $A \leq B$ is partial order for graphs over Π. Then, if $\Sigma_-(s_0) \not\leq \Sigma_+^*(s_0)$, by continuity we may find a first order contact point in the intersection of these sets which is not in the hyperplane Π_{s_0}, i.e.:

$$p_0 \in (\Sigma_+^*(s_0) \cap \Sigma_-(s_0)) - \delta_{s_0}(\Sigma).$$

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In particular, $\mathbf{p}(p_0) = 2s_0 - x_1$ for some $(x_1, \ldots, x_{n+1}) = q_0 \in \Sigma_+(s_0)$.

Moreover, since $\Sigma_+(s_0)$ is a graph over Π, we may write

$$x_1 = s_0 + f_{s_0}(0, x_2, \ldots, x_{n+1})$$

for some positive continuous function $f_{s_0} : \Pi \to \mathbb{R}$. The reason why f_{s_0} is positive is because $p_0 \notin \delta_{s_0}(\Sigma)$.

Next, we choose $x_1^k = s_k + f_{s_0}(0, x_2, \ldots, x_{n+1})$. Then, $q_k := (x_1^k, x_2, \ldots, x_{n+1}) \in \Sigma_+(s_k)$ and

$$2s_k - x_1^k = s_k - f_{s_0}(0, x_2, \ldots, x_{n+1}).$$

This means that $p_k := (2s_k - x_1^k, x_2, \ldots, x_{n+1}) \in \Sigma_+^*(s_k) \cap \Sigma_-(s_k)$.

Then, since $\Sigma_-(s_k) \leq \Sigma_+^*(s_k)$, it follows that $2s_k - x_1^k = s_k$ (see a proof in (5.10) below). In particular, $f_{s_0}(0, x_2, \ldots, x_{n+1}) = 0$, which means that $\mathbf{p}(p_0) = s_0 = x_1$, or equivalently, $p_0 \in \delta_{s_0}(\Sigma)$ given the desires contradiction. $\qquad \square$

Therefore, we have proven that $\Sigma_+(s_0)$ is a graph over Π and $\Sigma_+^*(s_0)$ is at the right hand side of $\Sigma_-(s_0)$, finalizing the proof of this claim. $\qquad \square$

Claim 5.4.3. *The set $\mathcal{A}$ is open in $[0, r)$.*

Proof. Let $t_1 \in \mathcal{A}$, we want to show that there exists $\varepsilon > 0$ such that $(t_1 - \varepsilon, t_1] \subset \mathcal{A}$. Then, by the Claim 5.4.1, the set $(t_1 - \varepsilon, t_1 + \varepsilon)$ will be an open neighborhod of t_1 contained in $\mathcal{A}$, or equivalently, $\mathcal{A}$ is an open subset of $[0, r)$.

We divide this proof in the following two steps.

Step 5.4.7. *There exist $\varepsilon_2 \in (0, t_1)$ such that $\Sigma_+(t_1 - \varepsilon_2)$ is a graph over Π.*
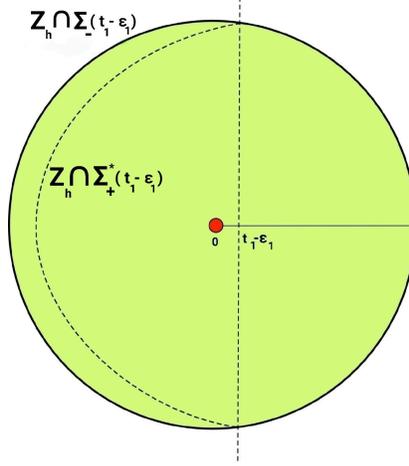
Figure 5.3: View from $x_{n+1} = h \gg 1$ with $\Sigma_+^*(t_1 - \varepsilon_1)$ is drawn in dashed lines. Image courtesy of Francisco Martín.

Proof. Since $t_1 \in \mathcal{A}$, it follows that $\Sigma_+(t_1)$ is a graph over Π. Let $h \gg 1$ such that for every $p \in \Sigma_+(t_1) \cap Z_h$ the unit normal vector at $p \in \Sigma_+(t_1) \cap Z_h$ satisfies $|\nu(p) - \nu_{\mathbb{S}^{n-1} \times \mathbb{R}}(p)| < \varepsilon$. Recall that $Z_h = \{x_{n+1} > h\}$. In particular t

Then, if there is $p \in \Sigma_+(t_1) \cap Z_h$ such that $\nu(p) \in \Pi$ with $\mathbf{p}(p) = \tilde{s} \in [t_1, r)$, we would have by the boundary tangential principle that $\Sigma_+^*(\tilde{s}) = \Sigma_-(\tilde{s})$ (see Fig 5.3.). This means that Σ is symmetric along $\Pi_{\tilde{s}} \neq \Pi$, and also the end of Σ, a contradiction.

In particular, we have

$$\nu(\Sigma_+(t_1)) \cap \Pi = \emptyset.$$

Consequently, there exists an $\varepsilon_0 \in (0, t_1]$ such that

$$\nu(\Sigma_+(t)) \cap \Pi = \emptyset, \text{ holds for all } t \in (t_1 - \varepsilon_0, t_1).$$

Then, by combining this with the fact that Σ is embedded and $\Sigma_+(t)$ is connected for all $t \in [0, t_0)$, we obtain that $\Sigma_+(t) \cap Z_h$ can be written as a graph over its projection in Π for all $t \in (t_1 - \varepsilon_0, t_1)$.

Furthermore, by taking $\varepsilon_1 = \dfrac{\varepsilon_0}{2}$ as we did in Claim 5.4.1, we will get that

$$\Sigma_-(t_1 - \varepsilon_1) \cap Z_h \leq \Sigma_+^*(t_1 - \varepsilon_1) \cap Z_h. \tag{5.8}$$

Next, we consider the compact set

$$K := \Sigma \cap \{x_{n+1} \leq h\}.$$

We note that

$$\langle \nu, e_1 \rangle = \frac{D_1 u}{\sqrt{1 + |Du|^2}} > 0$$

holds in $K_+(t_1)$ since it is a graph over Π.

Then, by compactness there exist a $\varepsilon_2 \in (0, \varepsilon_1)$ such that $D_1 u > 0$ holds in $K_+(t)$ for all $(t_1 - \varepsilon_2, t_1]$. Therefore, by combining this with the fact that Σ is embedded and $\Sigma_+(t)$ is connected, it follows that $K_+(t)$ can be written as a graph over its projection in Π for all $(t_1 - \varepsilon_2, t_1]$.

In summary, we have shown that there exists $\varepsilon_2 \in (0, t_1)$ such that $\Sigma_+(t)$ can be written as a graph over its projection in Π for all $t \in (t_1 - \varepsilon_2, t_1]$. $\qquad\square$

Step 5.4.8. *There exists $\varepsilon_4 \in (0, \varepsilon_2)$, such that $\Sigma_-(t_1 - \varepsilon_4) \leq \Sigma_+^*(t_1 - \varepsilon_4)$.*

Proof. Firstly, since $\Sigma_+(t)$ is a graph over its projection in Π for all $t \in (t_1 - \varepsilon_2, t_1]$, we may find a $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\left(\Sigma_+^*(t) \cap \Sigma_-(t) \cap K\right) - \delta_t(\Sigma) \subset K_-(t - \varepsilon_3). \tag{5.9}$$

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In fact, to see this we note that if

$$x = (x_1, \ldots, x_{n+1}) \in \Sigma_+^*(t) \cap \Sigma_-(t) \cap K \cap \delta_t(\Sigma)^c,$$

we have that $x_1 < t$ and $x_1 = 2t - y_1$ where $y_1 = f(0, x_2, \ldots, x_{n+1}) + t$ for some continuous function $f > 0$ defined over a compact set $\Omega_f \subset \Pi$.

Therefore, by compactness, we have that $y_1 - t \geq \varepsilon_3$ where

$$\varepsilon_3 = \min \left\{ \varepsilon_2, \min_{\Omega_f} f \right\} > 0.$$

This means that for all $t \in (t_1 - \varepsilon_3, t_1]$ it follows that

$$t - x_1 = y_1 - t \geq \varepsilon_3 \Leftrightarrow x_1 \leq t - \varepsilon_3.$$

Moreover, since $\Sigma_-(t_1) \leq \Sigma_+^*(t_1)$, it follows that

$$\Sigma_+^*(t_1) \cap \Sigma_-(t_1) = \delta_{t_1}(\Sigma), \tag{5.10}$$

because otherwise we could find a first order contact point

$$p_0 \in (\Sigma_+^*(t_1) \cap \Sigma_-(t_1)) - \delta_{t_1}(\Sigma).$$

Then, if p_0 is an interior point, the interior tangential principle Theorem 5.0.3 would imply that $\Pi_{\mathbf{p}(p_0)}$ is a hyperplane of symmetry of Σ, since $\Sigma_-(\mathbf{p}(p_0)) = \Sigma_+^*(\mathbf{p}(p_0))$.

Moreover, if p_0 is boundary point the same conclusion will hold by using instead the boundary tangential principle Theorem 5.0.3. Consequently, either Σ is symmetric about the hyperplane Π or $\Sigma_-(t_1) \cap \Sigma_+^*(t_1) = \delta_{t_1}(\Sigma)$.

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Next, we will show that there exists $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\Sigma_+^*(t) \cap \Sigma_-(t) \cap K = \delta_t(\Sigma) \cap K.$$

holds for all $t \in (t_1 - \varepsilon_4, t_1]$.

Indeed, if it not the case, we may find an increasing sequence t_l converging to t_1 such that

$$(\Sigma_+^*(t_l) \cap \Sigma_-(t_l) \cap K) - \delta_t(\Sigma) \neq \emptyset, \ \forall l \in \mathbb{N}.$$

Let $(x_1^l, \ldots, x_{n+1}^l) = p_l \in (\Sigma_+^*(t_l) \cap \Sigma_-(t_l) \cap K) - \delta_t(\Sigma)$. We note that by Equation (5.9) we have

$$x_1^l \leq t_l - \varepsilon_3. \tag{5.11}$$

Then, by compactness, we may assume, after taking a subsequence if it is necessary, that $p_l \to \tilde{p} = (\tilde{x}_1, \ldots, \tilde{x}_{n+1}) \in \Sigma_+^*(t_1) \cap \Sigma_-(t_1) \cap K$.

In particular, Equation (5.10) gives that $\tilde{x}_1 = t_1$. But, after taking limits on Equation (5.11), we see that $\tilde{x}_1 \leq t_1 - \varepsilon_3$, given a contradiction with $\tilde{x}_1 = t_1$.

Therefore, there exists $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\Sigma_+^*(t) \cap \Sigma_-(t) \cap K = \delta_t(\Sigma) \cap K$$

holds for all $t \in (t_1 - \varepsilon_4, t_1]$. This means that

$$\Sigma_-(t) \cap K \leq \Sigma_+^*(t) \cap K \text{ holds for all } t \in (t_1 - \varepsilon_3, t_1].$$

Thus, combining the above line with Equation (5.8), we finally obtain

$$\Sigma_-(t) \leq \Sigma_+^*(t) \text{ holds for all } t \in (t_1 - \varepsilon_4, t_1]$$

$\square$

Therefore, by the Steps 5.4.7 and 5.4.8, it follows that $(t_1 - \varepsilon_4, t_1] \subset \mathcal{A}$, and as we mention in the beginning of the proof of this claim, the set $\mathcal{A}$ is an open subset of $[0, t_0)$. $\square$

The proof finishes by noting that $\mathcal{A}$ is a non-empty open and closed subset of the connected set $[0, t_0)$. Therefore, $\mathcal{A} = [0, t_0)$. In particular, $\Sigma_-(0) \leq \Sigma_+^*(0)$.

An analogous argument will show that $\Sigma_-^*(0) \leq \Sigma_+(0)$. Therefore, Σ is symmetric with respect the hyperplane Π, finalizing the proof of Theorem 5.0.4. $\square$

Proof of Corollary 5.0.6 . Theorem 5.0.4 implies that Σ is a rotationally symmetric graph defined over the ball of radius r. Then, by Theorems 1.3-1.4 in [Ren][10], it follows that $r = \gamma(1, \ldots, 1)$ and Σ is the "bowl"-type γ-translator in $\mathbb{R}^{n+1}$. $\square$

5.5 Convexity Estimate

In this section we prove the convexity estimate for γ-translators Theorem 5.0.7 and, as a consequence, that these γ-translators are asymptotically convex.

Before giving the proof of Theorem 5.0.7, we describe the main tools used:

- Consider a good approximation for estimating $\lambda_1(p) = \{\lambda_i(p) : i = 1, \ldots, n\}$ using a maximum principle argument. For instance, in [SS] the authors used a δ-approximation

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[10]There exist a unique "bowl"-type γ-translator for every curvature function $\gamma : \Gamma \to \mathbb{R}$ that satisfies properties 1-3.

for the minimum between two elements of the form

$$f_\delta(x, y) = \frac{x+y}{2} - \sqrt{\frac{(x-y)^2}{4} + \delta xy}, \ \delta \in \left(0, \frac{1}{2} \right).$$

This δ-approximation is a symmetric 1-homogeneous concave function defined in

$$\left\{ (x, y) \in \mathbb{R}^2 : x + y > 0 \right\},$$

and satisfies nice properties when $\min(x, y) < 0$. In addition, the authors improved this δ-approximation to show that 2-convex and mean convex translators in $\mathbb{R}^{n+1}$ are convex. Nevertheless, the 2-convex hypothesis cannot be removed since it is necessary in the construction of the δ-approximation for the minimum in $n \geq 3$ variables.

- From the previous point, the approximating function will satisfy a maximum principle from which the minimum cannot be negative at an interior point of the γ-translator. A similar argument will be needed to study this behavior at infinity. For this purpose we need to consider an extra hypothesis to ensure a uniformly bounded second fundamental form of the γ-translator when we take limits of sequences of translations of the hypersurface.

In particular, the 2-convexity for $n \geq 3$ gives a natural bound for $|A|$ when mean convex H-translators are considered. Indeed, we have that the only possible negative principal curvature is the minimum[11], and

$$S_2(\lambda_1, \lambda_2, \lambda_3) = \lambda_3 \lambda_2 + \lambda_3 (\lambda_1 + \lambda_2) > 0,$$

$$S_2(\lambda_1, \ldots, \lambda_n) = S_{2;1} + \lambda_1 S_{1;1}$$

$$= S_{2;1,2} + (\lambda_2 + \lambda_1) S_{1;1,2} + \lambda_1 \lambda_2 > 0,$$

---

[11]If the principal curvatures satisfy $\lambda_1 \leq \ldots \leq \lambda_n$, then the 2-convexity implies $\lambda_1 + \lambda_2 > 0$.

here we have arranged $\lambda_1 \leq \ldots \leq \lambda_n$. Note that for $n \geq 4$, it follows that $S_{2;1,2} \geq \lambda_1 \lambda_2$ (see Lemma 3.1.1 for more properties of the polynomials S_k.). Consequently, for a H-translator it follows

$$|A|^2 = H^2 - 2S_2 \leq H^2 = \langle \nu, e_{n+1} \rangle^2 \leq 1.$$

On the other hand, when $\gamma \neq H$, other hypotheses are needed to ensure an uniform bound for $|A|$. In fact, for given $\alpha, \delta > 0$ and a 2-convex γ-translator, the restriction in the domain of γ to the cone

$$\Gamma_{\alpha,\delta} = \left\{ \lambda \in \Gamma : H \leq \frac{\alpha}{\delta+1}\gamma \right\},$$

gives an uniform bound for $|A|$. To see this we note

$$|A|^2 \leq H^2 \leq \gamma^2 \left(\frac{\alpha}{\delta+1} \right)^2 \leq \left(\frac{\alpha}{\delta+1} \right)^2. \tag{5.12}$$

From left to right, the first inequality comes by the 2-convexity, since it also implies $S_2 > 0$. The second inequality follows by $\lambda \in \Gamma_{\alpha,\delta}$. The third inequality follows by noting that $\gamma = \langle \nu, e_{n+1} \rangle \leq 1$.

Moreover, the cone $\Gamma_{\alpha,\delta}$ is preserved under the γ-flow. Indeed, the evolution equation of $\dfrac{H}{\gamma}$ is given by

$$(\partial_t - \Delta_\gamma)\frac{H}{\gamma} = \frac{g^{ij}}{\gamma} \frac{\partial^2 \gamma}{\partial h_{ab} h_{cd}} \nabla_i h_{ab} \nabla_j h_{ab} + \frac{2}{\gamma} \left\langle \nabla \frac{H}{\gamma}, \nabla \gamma \right\rangle_\gamma, \tag{5.13}$$

where $\Delta_\gamma = \dfrac{\partial \gamma}{\partial h_{ij}} \nabla_i \nabla_j$ and $\langle X, Y \rangle = \dfrac{\partial \gamma}{\partial h_{ij}} X_i Y_j$. Then, since γ is concave, it follows that the parabolic maximum principle implies that $\dfrac{H}{\gamma}$ remains bounded under the

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γ-flow.

- Thirdly, a uniform bound on the second fundamental form along the γ-flow implies the existence of sub sequential limits of a sequence of hypersurfaces. Then, by considering a sequence of points such that $\inf \lambda_1$ is reached in the limit, we can translate the γ-translator with respect this sequence and obtain a sub-sequential limit which is a γ-translator where an Omori-Yau maximum principle applies to the limit hypersurface. Thus, the end of the argument is to apply the internal maximum principle to the limiting hypersurface to ensure a contradiction and to obtain the estimate.

- Finally, if $\Gamma_{\alpha,\delta} \neq \emptyset$, then is a convex closed subset of Γ, which is compactly supported in Γ, i.e: the set

$$\overline{\Gamma}_{\alpha,\delta} \cap \partial B(0,1)$$

is compact in Γ. This fact implies an uniform estimate on the second order derivatives of γ, see Lemma 5.1.2.

Remark 5.5.1. It is still an open question which α and δ are optimal. For instance, one can restrict to

$$\frac{\alpha}{\delta+1} = \left.\frac{H}{\gamma}\right|_{\mathrm{Cyl}_j}$$

for some j such that $\mathrm{Cyl}_j \subset \Gamma$.

On the other hand, if there is a Newton-Maclaurin inequality (see Lemma 3.1.3) related to $\gamma : \Gamma \to (0,\infty)$[12], we could obtain that a closed γ-translator is totally umbilical, a contradiction with Corollary 5.0.2.

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[12]Recall that a smooth symmetric functions can be written as smooth functions in the polynomials S_k. Therefore, an inequality of the form $H \leq C\gamma$ could implies the equality case in Lemma 3.1.1 holds.

Proof of Theorem 5.0.7. We consider on Σ the function

$$f(p) = \frac{\gamma(\lambda)}{S_{1,1}(\lambda)},$$

where $S_{1,1} = H - \lambda_1$ and $\lambda_1(p) = \min_{i=1,\ldots,n} \{\lambda_i(p)\}$.

Therefore, the convexity estimate of Theorem 5.0.7 can be written by

$$f(p) \geq \frac{1}{\alpha}, \text{ for every } p \in \Sigma. \tag{5.14}$$

We will argue by contradiction, this means that we are going to assume

$$\inf_{\Sigma} f < \alpha^{-1}.$$

In particular, this assumption together with $\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : (\delta+1)H \leq \alpha\gamma\}$ gives an estimate of the form

$$\lambda_1 < H - \alpha\gamma \leq -\delta H, \tag{5.15}$$

for a sequence $p_N \in \Sigma$ such that $f(p_N) \to \inf_{\Sigma} f$. This is a key fact in our proof, since it permits to control some curvature terms in the equations.

Next, we calculate some equations related to f. For doing this, we use an orthonormal frame of principal directions $\{e_i\} \subset T_p\Sigma$ at any $p \in \Sigma$.

Then, by the equations in Lemma 5.1.1, it follows that

$$\nabla_a f = \frac{S_{1,1}\nabla_a\gamma - \gamma\nabla_a S_{1,1}}{(S_{1,1})^2}, \tag{5.16}$$

$$\Delta_\gamma f = \frac{f}{S_{1,1}}\left(\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\right)\nabla_i h_{ab}\nabla_j h_{cd} - \nabla_{n+1}f - 2\left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}}\right\rangle_\gamma. \tag{5.17}$$

We denote $\nabla_{n+1} f = \langle \nabla f, e_{n+1} \rangle$.

We divide the proof into several claims.

Claim 5.5.1. *Assume that there exists $p_0 \in \Sigma$ such that $f(p_0) = \inf_\Sigma f$. Then, $|A|^2_\gamma \gamma = 0$ in Σ, where $|A|^2_\gamma = \sum\limits_{i=1}^n \dfrac{\partial \gamma}{\partial \lambda_i} \lambda_i^2$. In consequence, f cannot achieve a minimum in an interior point of Σ.*

Proof. Recall that by (5.15), $\lambda_1(p_0) < H(p_0) - \alpha \gamma(p_0) \leq -\delta H(p_0) < 0$.

Then, by rewriting Equation (5.17), we see that

$$\Delta_\gamma f + \nabla_{n+1} f + 2 \left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}} \right\rangle_\gamma = \frac{f}{S_{1,1}} \left(\dot{S}^{ij}_{1,1} \ddot{\gamma}^{\,ab,cd} - \dot{\gamma}^{\,ij} \ddot{S}^{\,ab,cd}_{1,1} \right) \nabla_i h_{ab} \nabla_j h_{cd} \leq 0. \quad (5.18)$$

In the right hand side of the previous equation we have used the concavity of γ, the convexity of $S_{1,1}$, and the fact that $S_{1,1}$ and γ are increasing in each variable.

Furthermore, the left hand side in Equation (5.18) is locally uniformly elliptic as $\lambda \in \Gamma_{\alpha,\delta}$. Therefore, since $f(p_0)$ is a minimum, the strong maximum principle (see for instance Theorem 4.2 in [GP]) implies that the function f is constant in Σ.

Consequently, the right hand side of (5.18) satisfies

$$0 = \left(\dot{S}^{ij}_{1,1} \ddot{\gamma}^{\,ab,cd} - \dot{\gamma}^{\,ij} \ddot{S}^{\,ab,cd}_{1,1} \right) \nabla_i h_{ab} \nabla_j h_{cd} \Leftrightarrow \dot{S}^{ij}_{1,1} \ddot{\gamma}^{\,ab,cd} \nabla_i h_{ab} \nabla_j h_{cd} = \dot{\gamma}^{\,ij} \ddot{S}^{\,ab,cd}_{1,1} \nabla_i h_{ab} \nabla_j h_{cd}.$$

In particular, since $S_{1,1}$ is convex and γ is concave, it follows that the second derivatives of both terms must vanish everywhere at Σ. Furthermore, we note that

$$0 = \dot{S}^{ij}_{1,1} \ddot{\gamma}^{\,ab,cd} \nabla_i h_{ab} \nabla_j h_{cd} = (1 - \delta_{1i}) \ddot{\gamma}^{\,ab,cd} \nabla_i h_{ab} \nabla_i h_{cd}$$

combined with Lemma 5.1.2 implies that $\nabla_i h_{ab} = 0$ holds in Σ for $i > 1$ and all a, b.

Finally, the Codazzi Equations implies that $|\nabla S_{1,1}| = 0$ holds on Σ. Then Equation

(5.16) implies that $|\nabla \gamma| = 0$ holds on Σ. Therefore, γ is constant in Σ.

Consequently, by Equation (5.5), it follows that $|A|_\gamma^2 \gamma = 0$ which cannot occur since at $\lambda(p_0) \in \Gamma_{\alpha,\delta}$ where the functions γ and $|A|_\gamma^2$ are positive. $\qquad\square$

Now we focus on the case when $\inf_\Sigma f(p) \in (0, \alpha^{-1})$ is attained at "infinity". We use the Omori-Yau maximum principle, which gives a sequence $p_N \in \Sigma$ such that $|p_N| \to \infty$, and the following equations hold

$$f(p_N) \to \inf_\Sigma f, \ |\nabla f(p_N)| < \frac{1}{N} \text{ and } \Delta_\gamma f(p_N) \geq -\frac{1}{N}. \qquad (5.19)$$

We also consider the sequence of γ-translators given by

$$\Sigma_N = \Sigma - p_N.$$

Note that the second fundamental form of each Σ_N is uniformly bounded by the same constant since Σ_N is a translation of Σ by p_N. In addition, we note that $0 \in \Sigma_N$ for each N. Therefore, by compactness (see [PR] for details[13]), we can subtract a sub-sequence Σ_N' of Σ_N and a complete immersed γ-translator Σ_∞' such that $\Sigma_N' \to \Sigma_\infty'$ uniformly smoothly on compacts subset of Σ_∞'.

Finally, we denote by Σ_∞ the connected component of Σ_∞' which contains the point 0.

Claim 5.5.2. *At $0 \in \Sigma_\infty'$, the function γ vanishes.*

Proof. If it is not the case, then $\gamma > 0$ at $0 \in \Sigma_\infty'$. But since $f(0) = \inf_\Sigma f$ we get a contradiction with Claim 5.5.1. $\qquad\square$

Claim 5.5.3. *All the principal curvatures vanish at $0 \in \Sigma_\infty'$. Moreover, we have the following*

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[13]This reference is for minimal surfaces. We note the same arguments works for γ-translators that have an uniform bound in $|A|$.

estimates at p_N

$$f \geq \frac{1+\delta}{2\alpha},$$

$$\frac{\lambda_i}{S_{1,1}} \geq \frac{\beta}{2},$$

$$\frac{|\lambda_1|}{S_{1,1}} > \frac{\delta}{2}.$$

Proof. Firstly, recall that at p_N, $\lambda \in \Gamma_{\alpha,\delta}$. This means that $H(p_N) \to 0$ as $N \to \infty$ since $\gamma(p_N) \to 0$ as $N \to \infty$. In particular, by estimate (5.12), it follows that $|A|^2 \to 0$ giving that $\lambda_i(p_n) \to 0$ as $N \to \infty$.

Moreover, by (5.15) we have $\lambda_1(p_N) < 0$ holds for N big enough. Then, since $n \geq 3$, we have at p_N the following

$$H + \lambda_1 \geq (\lambda_3 + \lambda_1) + (\lambda_2 + \lambda_1),$$

which is positive by the 2-convexity. In particular, $H - \lambda_1 \leq 2H$ holds at p_N for N large enough.

Therefore, it follows that

$$f(p_N) = \frac{\gamma}{H - \lambda_1} \geq \frac{\gamma}{2H} \geq \frac{1+\delta}{2\alpha},$$

in the last inequality we use $\lambda \in \Gamma_{\alpha,\delta}$.

On the other hand, by the two uniform convexity, we obtain

$$\frac{\lambda_i}{H - \lambda_1} = \frac{\lambda_i + \lambda_1 - \lambda_1}{H - \lambda_1} \geq \frac{\beta H}{H - \lambda_1} \geq \frac{\beta}{2},$$

at p_N and $i > 1$. In the first inequality we use the 2-uniform convexity $\lambda_i + \lambda_1 \geq \beta H$ and the fact that $\lambda_1(p_N) < 0$.

Finally, for the last term, we have

$$\frac{|\lambda_1|}{H - \lambda_1} = \frac{-\lambda_1}{H - \lambda_1} > \frac{\alpha\gamma - H}{H - \lambda_1} \geq \frac{\delta H}{H - \lambda_1} \geq \frac{\delta}{2},$$

in the second inequality we use Equation (5.15). $\square$

Next, we note that equation (5.16)

$$\begin{aligned} \nabla f &= \frac{\nabla\gamma}{S_{1,1}} - f\frac{\nabla S_{1,1}}{S_{1,1}} \\ &= \frac{\lambda_i \langle e_i, e_{n+1}\rangle}{S_{1,1}} e_i - f\frac{\nabla S_{1,1}}{S_{1,1}}, \end{aligned} \tag{5.20}$$

combined with Claim 5.5.3 gives that the term

$$\frac{\nabla_i \gamma}{S_{1,1}} = \frac{\lambda_i \langle e_i, e_{n+1}\rangle}{S_{1,1}} e_i$$

is uniformly bounded at p_N. In addition, since $\nabla f(p_N) \to 0$ at $N \to \infty$, it follows that the term $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N)$ is also bounded.

Claim 5.5.4. *The term*

$$\frac{\nabla S_{1,1}}{S_{1,1}}(p_N) \to 0, \ \ as \ N \to \infty.$$

Proof. We evaluate (5.17) at p_N. Then, by equation (5.19), we note

$$\frac{-C}{N} \leq \Delta_\gamma f + \nabla_{n+1} f + 2\left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}} \right\rangle_\gamma = f\frac{\left(\dot{S}_{1,1}^{ij} \ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij} \ddot{S}_{1,1}^{ab,cd}\right)}{S_{1,1}} \nabla_i h_{ab} \nabla_j h_{cd} \leq 0.$$

Then, since $\nabla f(p_N) \to 0$, f and the term $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N)$ is bounded from below by a positive

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constant and from above as well, it follows that

$$\frac{1}{S_{1,1}} \left(\dot{S}_{1,1}^{\,ij} \ddot{\gamma}^{\,ab,cd} - \dot{\gamma}^{\,ij} \ddot{S}_{1,1}^{\,ab,cd} \right) \nabla_i h_{ab} \nabla_j h_{cd} \to 0, \text{ as } N \to \infty. \tag{5.21}$$

On the other hand, we claim that if $\lambda_a = \lambda_b$ at p_N for $a \neq b$, then $\nabla h_{ab}(p_N) = 0$. Indeed, to see this we choose normal coordinates for Σ centered at p_N. This means that we have an orthonormal basis of principal directions $\{e_i\}_{i=1}^n$ of $T_{p_N}\Sigma$ such that $h_b^a(p_N) = h_{ab}(p_N) = \lambda_a(p_N)\delta_a^b = 0$ since $a \neq b$. Then, by taking derivatives at p_N we see that

$$h_{ab,i} = h_{ab,i} + \Gamma_{ia}^b \underbrace{(\lambda_a - \lambda_b)}_{=0} = \nabla_i h_{ab} - A(\nabla_i e_a, e_b) - A(e_a, \nabla_i e_b) = e_i(h_{ab}) = e_i(0) = 0.$$

In particular, for those p_N, $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N) = 0$. Therefore, we may assume that $\lambda_a \neq \lambda_b$ at p_N, in order to prove that $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N) \to 0$ as $N \to \infty$.

Moreover, for $i > 1$ and using $\lambda_1(p_N) < 0$ with Lemma 5.1.2, we see that

$$\left(\frac{\nabla_i h_{ab}}{S_{1,1}} \right)^2 \leq \frac{|\nabla_i h_{ab}|^2}{S_{1,1} H} \leq \frac{-1}{C S_{1,1}} \sum_{i>1} \ddot{\gamma}^{\,ab,cd} \nabla_i h_{ab} \nabla_i h_{cd}.$$

Then, by noting that $\dfrac{\partial S_{1,1}}{\partial \lambda_i} = (1 - \delta_{1i})$ and adding the term $\dfrac{\dot{\gamma}^{\,ij} \ddot{S}_{1,1}^{\,ab,cd}}{S_{1,1}} \geq 0$ in the last inequality, it yields

$$\frac{|\nabla_i h_{ab}|^2}{(S_{1,1})^2} \leq \frac{-1}{C S_{1,1}} \left(\dot{S}_{1,1}^{\,ij} \ddot{\gamma}^{\,ab,cd} - \dot{\gamma}^{\,ij} \ddot{S}_{1,1}^{\,ab,cd} \right) \nabla_i h_{ab} \nabla_j h_{cd}.$$

Therefore, by Equation (5.21), the term $\dfrac{\nabla_i h_{ab}}{S_{1,1}}(p_N)$ goes to 0 as $N \to \infty$ for all a, b and $i > 1$. Then, the Codazzi Equations imply that $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N) \to 0$ as $N \to \infty$. $\qquad\square$

We note that the last claim together with Equation (5.20) implies that

$$\left| \frac{\nabla_i \gamma}{S_{1,1}} (p_N) \right| = \left| \frac{\lambda_i}{S_{1,1}} (p_N) \right| \left| \langle e_i, e_{n+1} \rangle (p_N) \right| \to 0 \text{ as } N \to \infty.$$

Finally, by Claim 5.5.3, which states a lower bound for each $\frac{\lambda_i}{S_{1,1}} (p_N)$, we have that $\langle e_i, e_{n+1} \rangle (p_N) \to 0$ [14], as well as $\langle \nu, e_{n+1} \rangle (p_N) = \gamma(p_N) \to 0$ when $N \to \infty$. In particular, we get that $e_{n+1} = \vec{0}$ at $0 \in \Sigma_\infty'$, which is a contradiction, finalizing the proof of Theorem 5.0.7. $\qquad\square$

As an application of the convexity estimate we show the γ-translators are asymptotically convex.

Proof of Corollary 5.0.8. We claim that γ vanishes at infinity, since the Property b) and Property c) imply $0 \leq H \leq \frac{\alpha}{\delta+1}\gamma$. In fact, by the convexity estimate we have

$$\lambda_1 \geq H - \alpha\gamma.$$

Therefore, if γ vanishes at infinity, it follows that $\lambda_1(p) \geq 0$ as $|p - p_0| \to \infty$ for a given point $p_0 \in \Sigma$. $\qquad\square$

Claim 5.5.5. *Let $p_0 \in \Sigma$ a fixed point, then $\gamma(p) \to 0$ when $|p - p_0| \to \infty$.*

Proof. If it is not the case, there exists a sequence $p_i \in \Sigma$ such that $\liminf\limits_{i \to \infty} \gamma(p_i) > 0$ and $|p_i - p_0| \to \infty$. In particular, we can extract a convergent subsequence of γ-translators $\Sigma_i = \Sigma - \{p_i\}$ which will converge to a complete immersed γ-translator Σ_∞. Let $\Sigma_\infty' \subset \Sigma_\infty$ be the connected component containing 0. In particular, γ will attains a positive minimum at $0 \in \Sigma_\infty'$.

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[14] Recall that e_i is a basis of principal directions at $p_N \in \Sigma$.

Nevertheless, by Equation (5.5)

$$\Delta_\gamma \gamma + \langle \nabla \gamma, e_{n+1} \rangle + |A|^2_\gamma \gamma = 0,$$

this minimum cannot be attained unless $|A|^2_\gamma \gamma \big|_0 = 0$ which implies that $\gamma(0) = 0$, given the desire contradiction. $\qquad\square$

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