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Spectral averaging formula and an absolutely continuous spectrum criteria for block Jacobi operator with random potential

by

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Dedicada a mi familia, a mi profesor guía, y al profesor Gueorgui Raykov Q.E.P.D.

Contents

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Summary

This thesis is about the work on the topics of Jacobi and random Schrödinger operators, which I started to study two years ago. The main model comes from an adaptation of the Anderson model [1] written in 1958, which describes the possibles random impurities of a solid, explaining mathematically in particular the phenomena of the conductivity of electrons in semiconductors in absence of diffusion.

This work is divided in three principal parts: First, we just introduce some preliminaries to keep in order the line of what results are important to us and also to put our notation. in context.

Second: we give the proof a spectral averaging formula for a Jacobi operator on the line using transfer matrices (appearing in [8]), and then the extension of this result to the strip, following [31]. This gives some criteria for absolutely continuous spectrum.

Finally, we prove the purely absolutely continuous spectrum (apart from possible discrete embedded eigenvalues) for some random Schrödinger operator on the strip with decaying matrix potentials. Here, we apply the criterion studied in the last part, using transfer matrices, Schur complements and probabilistic estimates.

Chapter 1

Mathematical preliminaries

We will introduce some very common mathematical notions which are important for the results of this thesis. Let us start with some aspects of possibly unbounded self-adjoint operators on Hilbert spaces. We recall that Hilbert spaces are vector spaces with an inner product $\langle \cdot, \cdot \rangle$ which are complete with respect to the induced norm $\|x\|^2 = \langle x, x \rangle$.

1.1 Operators on Hilbert spaces

Definition 1.1.1. [8] Let be given two (not necessarily different) Hilbert spaces $\mathbb{H}_1$, $\mathbb{H}_2$.

a) A **linear operator** from $\mathbb{H}_1$ to $\mathbb{H}_2$ is a linear map $T : \mathcal{D}(T) \to \mathbb{H}_2$, where $\mathcal{D}(T) \subset \mathbb{H}_1$ is a vector subspace which is called **the domain** of T.

b) $\mathrm{gr}(T) \subset \mathbb{H}_1 \times \mathbb{H}_2$ denotes **the graph** of T defined as: $gr(T) := \{(x, Tx); x \in \mathcal{D}(T)\}$

c) T_1 is an **extension** of T if $gr(T) \subset gr(T_1)$, that means, if $\mathcal{D}(T) \subset \mathcal{D}(T_1)$ and $Tx = T_1 x, \forall x \in \mathcal{D}(T)$. We use the notation $T \subset T_1$ for the extension of T in T_1, and $T_1\big|_{\mathcal{D}(T)} = T$ the restriction of T_1 in $\mathcal{D}(T)$.

d) If $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{H}$ we call T a linear operator on $\mathbb{H}$.

e) The **graph norm** on $\mathcal{D}(T)$ is defined by:

$$\|x\|_T = \|x\|_{\mathbb{H}_1} + \|Tx\|_{\mathbb{H}_2} \tag{1.1.1}$$

$\forall x \in \mathcal{D}(T)$

f) The range of T is defined as $\operatorname{Ran} T := \{Tx; x \in \mathcal{D}(T)\} \subset \mathbb{H}_2$

g) The kernel of T is defined as $\operatorname{Ker} T := \{x \in \mathcal{D}(T); Tx = 0_{\mathbb{H}_2}\} \subset \mathbb{H}_1$

An important notion is the one of a closed operator.

Definition 1.1.2. [8] Let T be a linear operator from $\mathbb{H}_1$ to $\mathbb{H}_2$, both Hilbert spaces.

a) T is called **closed** if $gr(T) = \overline{gr(T)} \subset \mathbb{H}_1 \times \mathbb{H}_2$

b) T is called closable if it has a closed extension

(or $T \subset T_1 : \mathcal{D}(T_1) \to \mathbb{H}$, or $gr(\overline{T}) = \overline{gr(T)} \subset \mathbb{H}_1 \mathrm{x} \mathbb{H}_2$)

c) If T is closable, then we define **the closure** of T, denoted by $\overline{T}$, as the smallest closed extension of T.

d) If T is closed ($T = \overline{T}$), then $\mathcal{D}$ is said to be **a core** of T if $\mathcal{D} \subset \mathcal{D}(T)$ so that $\overline{T\big|_{\mathcal{D}}} = T$.

A simple fact to note is the following:

Proposition 1.1.3. *T is closable if and only if the closure of the graph $\overline{gr(T)}$ is itself a graph of an operator, in which case we have:*

$$gr(\overline{T}) \;=\; \overline{gr(T)}\;.$$

In this case, $\overline{T}$ is an extension of T and $\mathcal{D}(T)$ is a core of $\overline{T}$.

Definition 1.1.4. [33] Let $T : \mathcal{D}(T) \subset \mathbb{H}_1 \to \mathbb{H}_2$ be a linear operator

a) T is called a **bounded operator** if there is a constant $c > 0$ such that

$$\|Tx\|_{\mathbb{H}_2} \le c\|x\|_{\mathbb{H}_1} \tag{1.1.2}$$

Otherwise, it is called an **unbounded operator**.

b) T is called **symmetric** if $\mathbb{H}_2 = \mathbb{H}_1 = \mathbb{H}$ and:

$$\langle Tx, y\rangle_{\mathbb{H}} = \overline{\langle x, Ty\rangle}_{\mathbb{H}} \tag{1.1.3}$$

$\forall x, y \in \mathcal{D}(T) \subset \mathbb{H}$

Definition 1.1.5. [33] Let $T : \mathcal{D}(T) \subset \mathbb{H}_1 \to \mathbb{H}_2$, on $\mathbb{H}_1, \mathbb{H}_2$ Hilbert Spaces, with $\mathcal{D}(T)$ being a dense domain.

a) The **adjoint** $T^* : \mathcal{D}(T^*) \subset \mathbb{H}_2 \to \mathbb{H}_1$ is defined by

$$\mathcal{D}(T^*) := \{y \in \mathbb{H}_2; \exists! z \in \mathbb{H}_1, \forall x \in D(T), \langle Tx, y\rangle_{\mathbb{H}_2} = \langle x, z\rangle_{\mathbb{H}_1}\} \tag{1.1.4}$$

and $T^* y = z$. Or alternatively:

$$\langle Tx, y\rangle_{\mathbb{H}_2} = \langle x, T^* y\rangle_{\mathbb{H}_1} \tag{1.1.5}$$

$\forall x \in \mathcal{D}(T), y \in \mathcal{D}(T^*)$

b) T is called **self-adjoint**, if $T^* = T$, which means in particular that $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{H}$ and $\mathcal{D}(T^*) = \mathcal{D}(T)$.

If $\mathcal{D}(T)$ is not dense, then the domain of T^* would be empty, as z would never be unique, one could always add some vector in $\mathcal{D}(T)^{\perp}$. By the density of $\mathcal{D}(T)$ and the Riesz representation theorem one has

$$\mathcal{D}(T^*) := \{y \in \mathbb{H}_2 \ : \ x \mapsto \langle Tx, y\rangle \quad \text{is bounded}\}$$

Notice, if T is self-adjoint then T is also symmetric.

Let us state some important facts connecting adjoints to closable operators.

Proposition 1.1.6. [7] Let $T : \mathcal{D}(T) \subset \mathbb{H}_1 \to \mathbb{H}_2$ be a linear operator, $\mathcal{D}(T)$ dense. We have:

a) T^* is a closed operator

b) $\mathcal{D}(T^*)$ is dense if and only if T is closable. In this case $\overline{T} = T^{**}$

Definition 1.1.7. [7] Let $W \subset \mathbb{H}$ be a vector subspace on $\mathbb{H}$ Hilbert Space. The **orthogonal Complement** of W is defined by

$$W^{\perp} := \{x \in \mathbb{H} \colon \langle x, w \rangle_{\mathbb{H}} = 0 \ \forall w \in W\} \tag{1.1.6}$$

In particular, one has:

$$\mathbb{H} = \overline{W} \oplus W^{\perp}. \tag{1.1.7}$$

Notice $W^{\perp}$ is a closed vector subspace, but W not necessarily.

Proposition 1.1.8. [7] Let $T : \mathcal{D}(T) \subset \mathbb{H}_1 \to \mathbb{H}_2$ be an unbounded operator, $\mathbb{H}_1, \mathbb{H}_2$ Hilbert Spaces, with $\mathcal{D}(T)$ being a dense domain, then:

$$Ker(T) = Ran(T^*)^{\perp} \tag{1.1.8a}$$

$$Ker(T^*) = Ran(T)^{\perp} \tag{1.1.8b}$$

$$\overline{Ran(T^*)} \subset Ker(T)^{\perp} \tag{1.1.8c}$$

$$\overline{Ran(T)} = Ker(T^*)^{\perp} \tag{1.1.8d}$$

Definition 1.1.9. Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces. We call $U : \mathbb{H}_1 \to \mathbb{H}_2$ a **unitary transformation**, if U is an isomorphism, this means that U is linear, bijective with

$$\langle Ux, Uy \rangle_{\mathbb{H}_2} = \langle x, y \rangle_{\mathbb{H}_1} \tag{1.1.9}$$

If the equation 1.1.9 holds, we say that U perserves the inner product.

Notice U is also an isometry. Moreover, if $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{H}$, U is an automorphism of $\mathbb{H}$, and in that case U is called a **unitary operator**. Also notice, U is a bounded operator with constant $c = 1$.

Proposition 1.1.10. [7] The following are equivalent

$$U \text{ is unitary operator} \tag{1.1.10a}$$

$$U^*U = UU^* = I \tag{1.1.10b}$$

$$U \text{ is surjective and } \mathbf{U} \text{ perserves the inner product} \tag{1.1.10c}$$

$$Ran(U) \text{ is dense in } \mathbb{H} \text{ and perserves the inner product} \tag{1.1.10d}$$

Definition 1.1.11. Let T be an self-adjoint on $\mathbb{H}$ Hilbert. The **Caley Transform** $U : Ker(T^* - iI) \to Ker(T^* + iI)$ is defined as

$$U := (T^* - iI)(T^* + iI)^{-1} \tag{1.1.11}$$

Notice U is well defined bounded $T^* + iI$ and $T^* - iI$ have bounded inverse in $\mathbb{H}$.

Lemma 1.1.12. [33]: Let T a self-adjoint on $\mathbb{H}$ Hilbert. The Caley transform U of T is an unitary transform, $I - U$ is inyective and $T = i(I + U)(I - U)^{-1}$

1.2 Essentially self-adjoint operators

Definition 1.2.1. [33] : Let T be a symmetric operator in a Hilbert Space $\mathbb{H}$. T is called an **essentially self-adjoint operator** if $\overline{T} = T^*$ in which case $T^{**} = T^*$, i.e. $\overline{T} = T^*$ is self-adjoint.

Definition 1.2.2. [33] Let T be a symmetric operator on a Hilbert Space $\mathbb{H}$. For $z \in \mathbb{C}/\mathbb{R}$ (or $\mathbb{C}/\mathbb{R}^+$), we define the **deficiency index** $d(z), d_+, d_-$ as:

$$d(z) = dimRan(T - zI)^{\perp} = dimKer(T^* - \overline{z}I) \tag{1.2.1a}$$

$$d_+ = dimKer(T^* - iI) \tag{1.2.1b}$$

$$d_- = dimKer(T^* + iI) \tag{1.2.1c}$$

Theorem 1.2.3. [33] Let T be a closed symmetric operator on $\mathbb{H}$ Hilbert Space, then

$$\mathcal{D}(T^*) = \mathcal{D}(T) \oplus Ker(T^* - iI) \oplus Ker(T^* + iI) \tag{1.2.2}$$

Lemma 1.2.4. [33] Let T be a symmetric operator on $\mathbb{H}$ Hilbert Space. If $\pm Imz > 0$ so $d(z) = d_{\pm}$.

Theorem 1.2.5. [33] Let T be a closed symmetric operator on $\mathbb{H}$ Hilbert Space. Then $T^* = T$ if and only if $d_+ = d_- = 0$.

Corollary 1.2.6. [33] : Let T be a symmetric operator on $\mathbb{H}$ Hilbert Space. T is essentially self-adjoint if and only if $d_+ = d_- = 0$.

Theorem 1.2.7. [33] Let T be operator on $\mathbb{H}$ Hilbert Space. T is essentially self-adjoint if and only if there exists a unique self-adjoint extension of T which is $\overline{T}$.

1.3 Spectral Theory

Definition 1.3.1. [7]: Let T be a closed operator on $\mathbb{H}$,

a) The resolvent set is defined as:

$$\rho(T) := \{\lambda \in \mathbb{C} : Ker(T - \lambda I) = \{0\}, \mathrm{Ran}(T - \lambda I) = \mathbb{H}\} \tag{1.3.1}$$

b) The spectrum is defined as:

$$\sigma(T) := \mathbb{C}/\rho(T) \tag{1.3.2}$$

We need to extend this definition for non-closed operator.

Definition 1.3.2. [7]: Let T be an operator on $\mathbb{H}$. The resolvent is set also defined by $\rho(T)$ as equation 1.3.1 , if $T - \lambda I : \mathcal{D}(T) \subseteq \mathbb{H} \to \mathbb{H}$ satisfies all of the below:

$$Ker(T - \lambda I) = \{0\} \tag{1.3.3a}$$

$$(T - \lambda I)(\mathcal{D}(T)) \text{ is dense in } \mathbb{H} \tag{1.3.3b}$$

$$(T - \lambda I)^{-1} \text{ is bounded} \tag{1.3.3c}$$

Definition 1.3.3. [8] Let T be an operator with $\lambda \in \rho(T)$ we define the **resolvent operator** $R_\lambda : \mathbb{H} \to \mathbb{H}$ as:

$$R_\lambda x := (T - \lambda I)^{-1} x, \; x \in \mathbb{H}$$

Proposition 1.3.4. [8] **(First resolvent identity)** For an operator T with $\lambda_1, \; \lambda_2 \in \rho(T)$

$$R_{\lambda_1} - R_{\lambda_2} = (\lambda_1 - \lambda_2) R_{\lambda_1} R_{\lambda_2}$$

Proof. As $R_{\lambda_1}(T - \lambda_1 I) = (T - \lambda_1 I) R_{\lambda_1} = I = R_{\lambda_2}(T - \lambda_2 I) = (T - \lambda_2 I) R_{\lambda_2}$

$$\begin{aligned} R_{\lambda_1} - R_{\lambda_2} &= R_{\lambda_1}(T - \lambda_2 I) R_{\lambda_2} - R_{\lambda_1}(T - \lambda_1 I) R_{\lambda_2} \\ &= R_{\lambda_1}((T - \lambda_2 I) - (T - \lambda_1 I)) R_{\lambda_2} \\ &= (\lambda_1 - \lambda_2) R_{\lambda_1} R_{\lambda_2} \end{aligned}$$

$\square$

Definition 1.3.5. [6]: Let $W \subset \mathbb{H}$ be a closed vector subspace of a Hilbert Space $\mathbb{H}$, as $\mathbb{H} = W \oplus W^{\perp}$. Let $P : \mathbb{H} \to W$ be an operator. We call P an **orthogonal projection** if

$$Px = y \ , \ x = y + z \qquad x \in \mathbb{H} \ , \ y \in W \ , \ z \in W^{\perp} \tag{1.3.4}$$

Notice P is a bounded operator with $P^2 = P$

Definition 1.3.6. [8]: Let $\mathbb{H}$ be a Hilbert space, with $\mathcal{P}(\mathbb{H})$ we denote the set of orthogonal projections on $\mathbb{H}$. A **spectral measure** or **projection-valued measure** is a mapping $\mathcal{E} : \mathfrak{B}(\mathbb{R}) \to \mathcal{P}(\mathbb{H})$ (where $\mathfrak{B}(\mathbb{R})$ denotes the Borel-sigma-algebra) such that

$$\mathcal{E}(A) \text{ is a orthogonal projection, } \forall A \in \mathfrak{B}(\mathbb{R}) \tag{1.3.5a}$$

$$\mathcal{E}(\mathbb{R}) = I \text{ (identity operator)} \tag{1.3.5b}$$

$$\mathcal{E}(\emptyset) = 0 \tag{1.3.5c}$$

$$\mathcal{E}(A \cap B) = \mathcal{E}(A)\mathcal{E}(B), \ \forall A, B \subset \mathfrak{B}(\mathbb{R}), \text{for } A \cap B = \emptyset, A \neq B \tag{1.3.5d}$$

$$\sum_{n \geq 1} \mathcal{E}(A_n) = \mathcal{E}(\mathbb{R}), \text{ for } \bigsqcup_{n \geq 1} A_n = \mathbb{R} \tag{1.3.5e}$$

Definition 1.3.7. (Spectral measures) [16] Let be given a spectral resolution $\mathcal{E}$.

For $\phi, \psi \in \mathbb{H}$ we have the associated spectral measures at ψ and ϕ, φ given by

$$\mu_{\psi}(A) : = \langle \psi, \mathcal{E}(A)\psi \rangle \ , \ \forall \, A \in \mathfrak{B}(\mathbb{R}) \tag{1.3.6a}$$

$$\mu_{\phi,\psi}(A) : = \langle \phi, \mathcal{E}(A)\psi \rangle \ , \ \forall \, A \in \mathfrak{B}(\mathbb{R}) \tag{1.3.6b}$$

Note μ_{ψ} is a finite measure, and $\mu_{\phi,\psi}$ is a complex measure.

Definition 1.3.8. For a measurable function $f : \mathbb{R} \to \mathbb{C}$ we define the operator

$$T_f \ := \int_{\mathbb{R}} f(\lambda) \, d\mathcal{E}(\lambda) \tag{1.3.7}$$

by its domain $D(T_f) := \{\psi \in H : \int_{\mathbb{R}} |f(\lambda)|^2 d\mu_\psi(\lambda) < \infty\}$ and

$$\langle \phi, T_f \psi \rangle := \int f(\lambda)\, d\mu_{\phi,\psi}(\lambda) \quad \text{for} \quad \psi \in D(T_f),\ \phi \in H \tag{1.3.8}$$

Theorem 1.3.9. [33] (**Spectral theorem for self-adjoint operator**): Let T be a (possibly unbounded) self-adjoint operator on a Hilbert space $\mathbb{H}$. Then, there exists a unique orthogonal projection valued measure $\mathcal{E}$ on $\mathbb{R}$, supported in $\sigma(T)$, such that

$$T = \int_{\sigma(T)} \lambda\, d\mathcal{E}(\lambda) \; = \int_{\mathbb{R}} \lambda\, d\mathcal{E}(\lambda) \tag{1.3.9}$$

$\forall f \in C_b(\mathbb{R})$ (and in fact for all measurable f) we find

$$f(T) = \int_{\sigma(T)} f(\lambda)\, d\mathcal{E}(\lambda) \; = \int_{\mathbb{R}} f(\lambda)\, d\mathcal{E}(\lambda) \tag{1.3.10}$$

For the associated spectral measures at vectors we have

$$\langle \psi, f(T)\psi \rangle \; = \int_{\mathbb{R}} f(\lambda)\, d\mu_\psi(\lambda)\,.$$

In particular, one finds

$$\operatorname{Ran} \mathcal{E}(A) = \{\psi \,:\, \mu_\psi(\mathbb{R} \setminus A) = 0\}\,.$$

That is, $\mathcal{E}(A)$ for $A \in \mathfrak{B}(\mathbb{R})$ is the projection on those vectors where the measure μ_ψ is essentially supported within A.

1.4 Convergence of measures

Definition 1.4.1. [33]: Let $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu)$ be a measure space with a finite Borel measure μ on $\mathbb{R}$. Let $\{\mu_n\}_{n\geq 1}$ be finite Borel measures on $\mathbb{R}$.

a) We say $\mu_n \xrightarrow{w} \mu$ **converges weakly** if for $n \to \infty$

$$\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu, \quad \forall f \in C_b(\mathbb{R}) \tag{1.4.1}$$

b) We say $\mu_n \xrightarrow{v} \mu$ **converges vaguely** if for $n \to \infty$

$$\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu, \quad \forall f \in C_0(\mathbb{R}) \tag{1.4.2}$$

Definition 1.4.2. [33]: Let μ be a finite Borel measure on $\mathbb{R}$. The **Stieltjes transformation** of μ, S_μ is given by:

$$S_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}, \qquad z \in \mathbb{C}^+ \tag{1.4.3}$$

Definition 1.4.3. We have the **Cauchy distribution** with parameteres $\lambda \in \mathbb{R}, \eta > 0$ as, $\nu(x;\lambda,\eta)dx$ where dx is the Lebesgue measure and:

$$\nu(x;\lambda,\eta) := \frac{1}{\pi\eta\left[1+\left(\frac{x-\lambda}{\eta}\right)^2\right]} = \frac{1}{\pi}\left[\frac{\eta}{(x-\lambda)^2+\eta^2}\right] \tag{1.4.4}$$

If we put $\psi = \lambda + i\eta$ we can write the density as:

$$\nu(x;\psi) = \frac{1}{\pi}\Im m\left(\frac{1}{x-\psi}\right)$$

Definition 1.4.4. Given a finite measure μ on $\mathfrak{B}(\mathbb{R})$ and $\eta > 0$ we define the absolutely continuous measure μ^η by:

$$\frac{d\mu^{(\eta)}(\lambda)}{d\lambda} := \frac{1}{\pi}\Im m(S(\lambda+i\eta)) = \int_R \frac{1}{\pi}\frac{\eta}{(x-\lambda)^2+\eta^2}d\mu(x) \tag{1.4.5}$$

And for $f \in \mathbf{C}_b(\mathbb{R})$

$$f^{(\eta)}(x) := \int_R \frac{1}{\pi}\frac{\eta}{(x-\lambda)^2+\eta^2}f(\lambda)d\lambda, \qquad f \in C_b(\mathbb{R}) \tag{1.4.6}$$

Lemma 1.4.5. *If $f \in C_b(\mathbb{R})$, $\mu^{(\eta)}(f) = \mu(f^{(\eta)})$*

Proof. Using Fubini:

$$\mu^{(\eta)}(f) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\pi}\frac{f(\lambda)\eta}{(x-\lambda)^2+\eta^2}d\mu(x)d\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\pi}\frac{\eta}{(x-\lambda)^2+\eta^2}f(\lambda)d\lambda d\mu(x) = \mu(f^{(\eta)})$$

□

Lemma 1.4.6. *If $f \in C_0(\mathbb{R}), \|f^{(\eta)} - f\|_\infty \to 0$ for $\eta \searrow 0$*

Proof. Let $f \in C_0(\mathbb{R})$ and let $\epsilon > 0$. Using the uniform continuity of f, $\exists \delta > 0$ such that $|x - \lambda| < \delta \implies |f(x) - f(\lambda)| < \epsilon$

In fact, by the Cauchy distribution:

$$\int_{\mathbb{R}} \frac{1}{\pi} \frac{\eta}{(x-\lambda)^2 + (\eta)^2} d\lambda = 1$$

. Now, with $y = \lambda - x$

$$\begin{aligned}
|f^{(n)}(\lambda) - f(\lambda)| \leq \int_{\mathbb{R}} \frac{1}{\pi} \frac{|f(x) - f(\lambda)|n}{(x-\lambda)^2 + n^2} d\lambda = \\
\int_{\{|x-\lambda|<\delta\}} \frac{1}{\pi} \frac{|f(x) - f(\lambda)|\eta}{(x-\lambda)^2 + \eta^2} d\lambda + \int_{\{|x-\lambda|\geq\delta\}} \frac{1}{\pi} \frac{|f(x) - f(\lambda)|\eta}{(x-\lambda)^2 + \eta^2} d\lambda \leq \\
\epsilon \int_{\{|x-\lambda|<\delta\}} \frac{1}{\pi} \frac{\eta}{(x-\lambda)^2 + \eta^2} d\lambda + \int_{\{y\geq\delta\}} \frac{2}{\pi} \frac{|f(x) - f(\lambda)|\eta}{y^2 + \eta^2} dy \leq \\
\epsilon + \int_{\delta}^{\infty} \frac{2}{\pi} \frac{2\|f\|_\infty \eta}{y^2 + \eta^2} dy \leq \epsilon + \frac{4}{\pi}\|f\|_\infty \int_{\delta}^{\infty} \frac{\eta}{y^2 + \eta^2} dy = \\
\epsilon + \frac{4}{\pi}\|f\|_\infty (\frac{\pi}{2} - \arctan(\frac{\delta}{\eta})) < \epsilon(1 + \frac{4}{\pi}\|f\|_\infty)
\end{aligned}$$

and last line is given by $\eta < \dfrac{\delta}{\tan(\frac{\pi}{2} - \epsilon)}$

□

Lemma 1.4.7. *For $\eta > 0$ and a finite measure μ one finds $\mu^{(\eta)}(\mathbb{R}) = \mu(\mathbb{R})$.*

Proof. Using Fubini, dominated convergence theorem and Cauchy distribution:

$$\begin{aligned}
\mu^{(\eta)}(\mathbb{R}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\pi} \frac{\eta}{(x-\lambda)^2 + \eta^2} d\mu(x) d\lambda = \\
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\pi} \frac{\eta}{(x-\lambda)^2 + \eta^2} d\lambda d\mu(x) = \\
\int_{\mathbb{R}} \frac{1}{\pi} \arctan\left(\frac{\lambda - x}{\eta}\right) \Big|_{-\infty}^{\infty} d\mu(x) = \int_{\mathbb{R}} 1 d\mu(x) = \mu(\mathbb{R})
\end{aligned}$$

□

Lemma 1.4.8. *Let μ_n, μ be finite measures. If $\mu_n \xrightarrow{v} \mu$ vaguely and $\mu_n(\mathbb{R}) \to \mu(\mathbb{R})$, then $\mu_n \xrightarrow{w} \mu$ weakly.*

Proof. Let $f \in C_b(\mathbb{R})$ and take

$$g(x) = \begin{cases} (N+\epsilon+x)/\epsilon & \text{if } -N-\epsilon < x < -N \\ 1 & \text{,if } -N \leq x \leq -N \\ (N+\epsilon-x)/\epsilon & \text{if } N < x < N+\epsilon \\ 0 & \text{else} \end{cases}$$

where we will choose N sufficiently large such that $\mu(1-g) < \epsilon$ for some given $\epsilon > 0$. As $g \in C_0(\mathbb{R})$ we note that $\mu_n(g) \to \mu(g)$, and as $\mu_n(1) = \mu_n(\mathbb{R}) \to \mu(1) = \mu(\mathbb{R})$ we have $\mu_n(1-g) \to \mu(1-g)$. Thus, we find,

$$\exists N > 0\,, \forall\, n > N \,:\, \mu_n(1-g) \leq 2\epsilon\,.$$

Hence, for $n > N$, we find

$$|\mu_n(f(1-g))| \;\leq\; \|f\|_\infty\, \mu_n(1-g) \;\leq\; 2\epsilon\, \|f\|_\infty \quad \text{and} \quad |\mu(f(1-g))| \;<\; \epsilon \|f\|_\infty$$

Moreover, $fg \in C_0(\mathbb{R})$, hence $\mu_n(fg) \to \mu(fg)$.

$$\begin{aligned} |\mu_n(f) - \mu(f)| &\leq |\mu_n(f(1-g)) - \mu(f(1-g))| + |\mu_n(fg)) - \mu(fg))| \\ &\leq 3\epsilon\|f\|_\infty \,+\, |\mu_n(fg) - \mu(fg)| \;\to\; 3\epsilon\|f\|_\infty \end{aligned}$$

for $n \to \infty$. As $\epsilon > 0$ was arbitrarily small, we get $|\mu_n(f) - \mu(f)| \to 0$, giving the result. □

Theorem 1.4.9. *: Let μ be a finite Borel measure on $\mathbb{R}$. Then $\mu^{(\eta)} \xrightarrow{w} \mu$ for $\eta \searrow 0$*

Proof. First, let $f \in C_0(\mathbb{R})$, and by Lemma 1.4.6

$$|\mu^{(\eta)}(f) - \mu(f)| = |\mu(f^{(\eta)} - f)| \leq \mu(\mathbb{R})\|f^{(\eta)} - f\|_\infty \to 0$$

showing vague convergence. Now by Lemma 1.4.7 and Lemma 1.4.8 we get the result. ☐

Proposition 1.4.10. *If* $\sup_n \mu_n(\mathbb{R}) < \infty$, $\mu(\mathbb{R}) < \infty$, *and* $S_{\mu_n}(z)$ *converges pointwise to* $S_\mu(z)$, $\forall z \in \mathbb{C}^+$, *then* $\mu_n \xrightarrow{v} \mu$ *vaguely.*

Proof. Let $f \in C_0(\mathbb{R})$, let $\epsilon > 0$. There is $\eta > 0$ such that $\|f^{(\eta)} - f\|_\infty < \epsilon$.

$$\begin{aligned}|\mu_n(f) - \mu(f)| &\leq |\mu_n(f) - \mu_n(f^{(\eta)})| + |\mu_n(f^{(\eta)}) - \mu(f^{(\eta)})| + |\mu(f^{(\eta)}) - \mu(f)| \\ &\leq \|f^{(\eta)} - f\|_\infty \mu_n(\mathbb{R}) + |\mu_n^{(\eta)}(f) - \mu^{(\eta)}(f)| + \|f^{(\eta)} - f\|_\infty \mu(\mathbb{R}) \\ &\leq \epsilon\mu_n(\mathbb{R}) + \int_{\mathrm{supp}(f)} |S_{\mu_n}(\lambda + i\eta) - S_\mu(\lambda + i\eta)||f(\lambda)| d\lambda + \epsilon\mu(\mathbb{R})\end{aligned}$$

Now, as $|S_{\mu_n}(z) - S_\mu(z)| \to 0$ we find by dominated convergence that the integral above converges to 0 for $n \to \infty$. Thus, choosing n large enough we have $|\mu_n(f) - \mu(f)| \leq \epsilon(\sup_n \mu_n(\mathbb{R}) + \mu(\mathbb{R}) + 1)$. As ϵ was arbitrary small, we find $\mu_n(f) \to \mu(f)$ for $n \to \infty$. As $f \in C_0(\mathbb{R})$ was arbitrary, this implies $\mu_n(f) \xrightarrow{v} \mu(f)$ ☐

Proposition 1.4.11. *If* $\mu_n(\mathbb{R}) \to \mu(\mathbb{R}) < \infty$ *and* $S_{\mu_n}(z)$ *converges pointwise to* $S_\mu(z)$, $\forall z \in \mathbb{C}^+$ *then* $\mu_n \xrightarrow{w} \mu$ *weakly.*

Proof. This follows by Proposition 1.4.10 and Lemma 1.4.8. ☐

Definition 1.4.12. [8]: Let $\{T_n\}_{n\in\mathbb{N}}$ be self-adjoint operators on H. We say T_n converge to a self-adjoint operator T **in the sense of strong resolvent convergence** if the resolvent operator $(T_n - \lambda I)^{-1}(x)$ goes to $(T - \lambda I)^{-1}(x)$ for $\lambda \in \mathbb{C}/\mathbb{R}$, $\forall x \in H$

$$\lim_{n\to\infty} \|(T_n - \lambda I)^{-1}(x) - (T - \lambda I)^{-1}(x)\| = 0 \tag{1.4.7}$$

We say in a short way $(T_n - \lambda I)^{-1} \xrightarrow{s} (T - \lambda I)^{-1}$

Proposition 1.4.13. [8]: Let $T, \{T_n\}_{n\in\mathbb{N}}$ be self-adjoint operators of H and let $x \in \mathcal{D}(T_n)$ for all n and $x \in \mathcal{D}(T)$. Then, the spectral measures at x for T_n converge to the spectral measure at x for T, weakly.

Proof. We define:

$$\mu_x(f) := \langle x, f(T)x\rangle,\ \mu_{x,n}(f) := \langle x, f(T_n)x\rangle$$

That means, need to prove $\mu_{x,n}(f) \xrightarrow{w} \mu_x(f)$. Notice by definition

$$\mu_{x,n}(\mathbb{R}) = \langle x, x\rangle = \|x\|^2 = \mu_x(\mathbb{R})$$

. Now using the Stieltjes transform and by hypotesis:

$$\lim_{n\to\infty} S_{\mu_n}(z) = \lim_{n\to\infty} \langle x, \frac{1}{T_n - z} x\rangle = \langle x, \frac{1}{T - z} x\rangle = S_\mu(z)$$

Finally, using Proposition 1.4.11, we have the result. □

Chapter 2

Jacobi operators and transfer matrices

2.1 Jacobi operators

Definition 2.1.1. [8] [35] [36] : A **Jacobi-operator** J on $\ell^2(\mathbb{Z}_+)$ is given by

$$(J\psi)_n = \begin{cases} -a_1\psi_1 + b_0\psi_0 & , \text{if } n = 0 \\ -a_{n+1}\psi_{n+1} - a_n\psi_{n-1} + b_n\psi_n & , \text{if } n \geq 1 \end{cases} \tag{2.1.1}$$

with $a_n \in \mathbb{R} \setminus \{0\}, b_n \in \mathbb{R}, n \in \mathbb{Z}$. Note that formaly $J\psi$ can be defined for $\psi \in \mathbb{C}^{\mathbb{Z}_+}$ giving a sequence in $\mathbb{C}^{\mathbb{Z}_+}$.

The **minimal domain** of J is given by the compactly supported vectors:

$$D_{\min} = \{\psi \in \ell^2(\mathbb{Z}_+) \, : \, \#\{n \, : \, \psi_n \neq 0\} < \infty\}$$

The **maximal domain** or **natural domain** is given by

$$D_{\max} = \{\psi \in \ell^2(\mathbb{Z}_+) \, : \, J\psi \in \ell^2(\mathbb{Z}_+)\} \, .$$

By $J_{\min}$ we denote the operator J with the domain $D_{\min}$. If we use just the symbol J we typically mean the operator with its natural domain $D_{\max}$.

Clearly, J is an extension of $J_{\min}$ and $J_{\min}$ is symmetric. However, J is not necessarily symmetric on its maximal domain.

Theorem 2.1.2. [8] [30] [21] Let J be a Jacobi operator as above. Then, $J^*_{\min} = J$.

Proof. First, we show for $\phi \in D_{\min}, \psi \in D_{\max}$ it is easy to verify that

$$\langle J_{\min}\phi, \psi\rangle = \langle \phi, J\psi\rangle$$

and as $J\psi \in \ell^2(\mathbb{Z}_+)$ this implies $\psi \in D(J^*_{\min})$ and $J^*_{\min}\psi = J\psi$. We need to show that for $\psi \in D(J^*_{\min})$ we have $\psi \in D_{\max}$. Let $\psi \in D(J^*_{\min})$ and let $\phi^n_m = (J\psi)_m$ for $m \leq n$ and $\phi^n_m = 0$ for $m > n$. Moreover let $\psi^n_m = \psi_n$ for $m \leq n$ and $\psi^n_m = 0$ for $m > n$. Then

$$\infty > \|J^*_{\min}\psi\| \geq \frac{|\langle J_{\min}\phi^n, \psi\rangle|}{\|\phi^n\|} = \frac{\langle J_{\min}\phi^n, \psi^{n+1}\rangle}{\|\phi^n\|} = \frac{\langle \phi^n, J\psi^{n+1}\rangle}{\|\phi^n\|} = \|\phi^n\|$$

So we have $\|J\psi\| = \lim_{n\to\infty} \|\phi^n\| \leq \|J^*_{\min}\psi\| < \infty$ □

Corollary 2.1.3. *If J is symmetric (with domain $D_{\max}$), then J is self-adjoint.*

In particular, in this case, there is a unique spectral measure for J at the root $\delta_0 \in \ell^2(\mathbb{Z}_+)$ given from spectral theory.

Proof. $J_{\min} \subset J \Rightarrow J^* \subset J^*_{\min} = J$, and as J is symmetric, $J \subset J^*$ and therefore $J = J^*$. □

Theorem 2.1.4. *Let $J_{\min}$ a Jacobi operator on $\ell^2(\mathbb{Z}_+)$. If*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \tag{2.1.2}$$

holds, then $J_{\min}$ is essentially self-adjoint and $J = J^*_{\min}$ is self-adjoint.

Proof. We have to prove that $J = J^*_{\min}$ is symmetric, so let $\phi, \psi \in D_{\max}$.

$$\begin{aligned}\langle \phi, J\psi\rangle - \langle J\phi, \psi\rangle &= \lim_{n\to\infty} \left(\langle \phi^n, J\psi\rangle - \langle J\phi, \psi^n\rangle\right) \\ &= \lim_{n\to\infty} \sum_{m=0}^{n} \left(\bar{\phi}_m (J\psi)_m - \overline{(J\phi)_m}\psi_m\right) = \lim_{n\to\infty} a_{n+1}(\bar{\phi}_n\psi_{n+1} - \bar{\phi}_{n+1}\psi_n)\end{aligned} \tag{2.1.3}$$

Notice, that $x_n = \bar{\phi}_n\psi_{n+1} - \bar{\phi}_{n+1}\psi_n$ is a sequence in $\ell^1(\mathbb{Z}_+)$. and the limit $\lim\limits_{n\to\infty} a_{n+1}x_n$ exists. We want to show that it is zero, therefore it is sufficient to show that $\liminf\limits_{n\to\infty} |a_{n+1}x_n|$ is zero. Suppose $\liminf\limits_{n\to\infty} |a_{n+1}x_n| = c \neq 0$, then for some $N \in \mathbb{Z}, \forall n \geq N$ implies

$$\frac{c}{2} < |a_{n+1}x_n| \leq |a_{n+1}||x_n|$$

Now dividing and taken the sum in both sides.

$$\frac{2}{c}|x_n| > \frac{1}{|a_{n+1}|} \implies \sum_{n\geq N} \frac{1}{|a_{n+1}|} < \sum_{n\geq N} \frac{2}{c}|x_n| < \infty$$

Then, by contrapositive we have the result. □

2.2 Transfer matrix

Definition 2.2.1. [36] : The **transfer matrix** at site $n \in \mathbb{Z}_+$ and spectral parameter $z \in \mathbb{C}$ is given by

$$T_n^z := \begin{pmatrix} (b_n - z)/a_n & -a_n \\ 1/a_n & 0 \end{pmatrix} \tag{2.2.1}$$

where we use $a_0 = 1$ (note that the operator J on $\ell^2(\mathbb{Z}_+)$ des not define a_0)

Note, that for any $\psi \in \mathbb{C}^{\mathbb{Z}_+}$, $z \in \mathbb{C}$, the equation $(J\psi)_n = z\psi_n$ is equivalent to

$$\begin{pmatrix} a_{n+1}\psi_{n+1} \\ \psi_n \end{pmatrix} = T_n^z \begin{pmatrix} a_n\psi_n \\ \psi_{n-1} \end{pmatrix}.$$

Let us also define the products for $m > n \in \mathbb{Z}_+$

$$T^z_{m,n} := T^z_n T^z_{n-1} \cdots T^z_{m+1} T^z_m$$

so that $J\psi = z\psi$ implies

$$\begin{pmatrix} a_{n+1}\psi_{n+1} \\ \psi_n \end{pmatrix} = T^z_{m,n} \begin{pmatrix} a_m \psi_m \\ \psi_{m-1} \end{pmatrix}.$$

Definition 2.2.2. We denote the restriction of J to $\ell^2(\{m, m+1, \ldots, n\}) \equiv \mathbb{C}^{n-m+1}$ by $J_{m,n}$ $(n \geq m)$. This means, $J_{m,n}$ is given by the matrix

$$J_{m,n} = \begin{pmatrix} b_m & -a_{m+1} & & \\ -a_{m+1} & \ddots & \ddots & \\ & \ddots & \ddots & -a_n \\ & & -a_n & b_n \end{pmatrix}$$

if we identify $\delta_m, \delta_{m+1}, \ldots, \delta_n$ with the standard basis in $\mathbb{C}^{n-m+1}$.
Furthermore, we define the boundary resolvent data for $J_{m,n}$ at spectral parameter $z \notin \sigma(J_{m,n})$ by

$$\begin{aligned} \alpha^z_{m,n} &:= \langle a_m \delta_m, (J_{m,n} - z)^{-1} a_m \delta_m \rangle \\ \beta^z_{m,n} &:= \langle a_m \delta_m, (J_{m,n} - z)^{-1} \delta_n \rangle \\ \gamma^z_{m,n} &:= \langle \delta_n, (J_{m,n} - z)^{-1} a_m \delta_m \rangle \\ \delta^z_{m,n} &:= \langle \delta_n, (J_{m,n} - z)^{-1} \delta_n \rangle \end{aligned}$$

Note that δ_m is a cyclic vector for $J_{m,n}$ and therefore, the meromorphic function $z \mapsto \beta^z_{m,n}$ is not identically zero. By Kramer's rule it is a rational function in z, hence it has at most finitely many zeroes and finitely many poles.

Proposition 2.2.3. *Let be given $n \geq m \in \mathbb{Z}_+$ and let $z \notin \sigma(J_{m,n})$. Then*

$$T^z_{m,n} = \begin{pmatrix} (\beta^z_{m,n})^{-1} & -(\beta^z_{m,n})^{-1}\alpha^z_{m,n} \\ \delta^z_{m,n}(\beta^z_{m,n})^{-1} & \gamma^z_{m,n} - \delta^z_{m,n}(\beta^z_{m,n})^{-1}\alpha^z_{m,n} \end{pmatrix}$$

Proof. First, for $m = n$ this follows by a simple calculation. Now, let z be such that $\beta^z_{m,k} \neq 0$, $b_k - z \neq 0$ and $z \notin \sigma(J_{m,k})$ for any $m \leq k \leq n$. The more general case follows from analytic extension. We will show the statement for these z and the transfer matrix $T^z_{m,k}$ by induction. As stated, $k = m$ is trivial. As long as $k < n$ we need the step from $k - 1$ to k.

We define:

$$\hat{\psi}_k := \psi_k, \quad k < m. \qquad \hat{\psi}_m := \begin{pmatrix} \psi_m \\ \psi_{m+1} \\ \vdots \\ \psi_n \end{pmatrix}$$

$$\hat{\psi}_{m+1} := \psi_{n+1}, \qquad \xi := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \qquad \phi := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

then we have

$$\psi_m = \xi^* \hat{\psi}_m \ , \quad \psi_n := \phi^* \hat{\psi}_m$$

and we get

$$\widehat{(J\hat{\psi})}_m := J_{m,n}\hat{\psi}_m - \xi\hat{\psi}_{m-1} - \phi\hat{\psi}_{m+1} \, .$$

Moreover, we define

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} \xi^* \\ \phi^* \end{pmatrix} (J_{m,n} - z)^{-1} \begin{pmatrix} \xi & \phi \end{pmatrix} \, .$$

With z being the spectral parameter, $\widehat{(J\hat{\psi})}_m = z\hat{\psi}_m$ leads to

$$z\hat{\psi}_m = J_{m,n}\hat{\psi}_m - \xi\hat{\psi}_{m-1} - \phi\hat{\psi}_{m+1} \implies$$

$$\phi\hat{\psi}_{m+1} = (J_{m,n} - z)\hat{\psi}_m - \xi\hat{\psi}_{m-1} \implies$$

$$\xi^*(J_{m,n}-z)^{-1}\phi\hat{\psi}_{m+1} = \xi^*\hat{\psi}_m - \xi^*(J_{m,n}-z)^{-1}\xi\hat{\psi}_{m-1} \implies$$

$$\beta\psi_{n+1} = \psi_m - \alpha\psi_{m-1} \implies \psi_{n+1} = \beta^{-1}\psi_m - \beta^{-1}\alpha\psi_{m-1}$$

Multiplying from the left with ϕ^* instead of ξ^* leads to

$$\phi^*(J_{m,n}-z)^{-1}\phi\hat{\psi}_{m+1} = \phi^*\hat{\psi}_m - \phi^*(J_{m,n}-z)^{-1}\xi\hat{\psi}_{m-1} \implies$$

$$\delta\psi_{n+1} = \psi_n - \gamma\psi_{m-1} \implies$$

$$\delta(\beta^{-1}\psi_m - \beta^{-1}\alpha\psi_{m-1}) = \psi_n - \gamma\psi_{m-1} \implies$$

$$\psi_n = \delta\beta^{-1}\psi_m + (\gamma - \delta\beta^{-1}\alpha)\psi_{m-1}$$

Finally, we have:

$$\begin{pmatrix}\psi_{n+1}\\ \psi_n\end{pmatrix} = \begin{pmatrix}\beta^{-1} & -\beta^{-1}\alpha\\ \delta\beta^{-1} & \gamma-\delta\beta^{-1}\alpha\end{pmatrix}\begin{pmatrix}\psi_m\\ \psi_{m-1}\end{pmatrix}$$

As ψ_m, ψ_{m-1} determine the solution to $J\psi = z\psi$ uniquely, the matrix must be $T^z_{m,n}$. □

2.3 Spectral averaging formulas

In this section let J be a self-adjoint Jacobi operator with maximal domain.

Definition 2.3.1. Let χ_n be the operator

$$(\chi_n\psi)_m := \begin{cases} 0 & \text{if } m \neq n \\ \psi_n & \text{if } m = n \end{cases}$$

Theorem 2.3.2. *Let J be a self-adjoint Jacobi operator (self-adjoint on its maximal domain). For any sequence V_n of real numbers and complex number $z \in \mathbb{C}\setminus\mathbb{R}$ we find*

$$\lim_{n\to\infty}\langle\delta_0, (J_{0,n} + V_n\chi_n - z)^{-1}\delta_0\rangle = \langle\delta_0, (J-z)^{-1}\delta_0\rangle$$

Also, for any probability measure ν on $\mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ we find that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \langle \delta_0, (J_{0,n} + V\chi_n - z)^{-1}\delta_0\rangle d\nu(V) = \langle \delta_0, (J-z)^{-1}\delta_0\rangle$$

Proof. The operators $J_{m,n}$ can be extended to operators on $\ell^2(\mathbb{Z}_+)$ by a direct sum with zero. Then, for any sequence V_n of reals we have that $J_{0,n} + V_n\chi_n$ converges strongly to J on $D_{\min}$. By Proposition 1.4.13 we have strong resolvent convergence

$$(J_{0,n} + V_n\chi_n - z)^{-1} \quad \xrightarrow{s} \quad (J-z)^{-1}$$

for any $z \notin \mathbb{R}$. This implies the first statement. Using dominated convergence, we have the second statement. □

Lemma 2.3.3. *We find*

$$\langle \delta_0, (J_{0,n} - V\chi_n - z)^{-1}\delta_0\rangle \;=\; \alpha^z_{0,n} + V\beta^z_{0,n}(1 - V\delta^z_{0,n})^{-1}\gamma^z_{0,n}$$

Proof. Using the first resolvent identity Theorem 1.3.4 with the definition of inner product we have the result. □

Let ν be the Cauchy distribution and define the **spectral average measure** at the root,

$$\mu_n(g) \;=\; \int_{\mathbb{R}} \langle \delta_0, g(J_{0,n} - V\chi_n), \delta_0\rangle \, d\nu(V)\ .$$

Then one has the following formula for the spectral average:

Theorem 2.3.4. [8] *(**Average formula**) The measure μ_n is absolutely continuous and has the density*

$$\frac{d\mu_n(\lambda)}{d\lambda} \;=\; \frac{1}{\pi}\frac{(\beta^\lambda_{0,n})^2}{1+(\delta^\lambda_{0,n})^2} \;=\; \frac{1}{\pi}\frac{1}{\|T^\lambda_{0,n}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|^2}$$

Proof. Using the last definition with $g(V) = \dfrac{1}{V-z}$ and the Stieltjes transform with $z = \lambda + i\eta$

$$\mu_n(g(z)) \;=\; \int_{\mathbb{R}} \langle \delta_0, (J_{0,n} - V\chi_n - z)^{-1}\delta_0\rangle d\nu(V) \;=\; \int_{\mathbb{R}} (\alpha^z_{0,n} + V\beta^z_{0,n}(1 - V\delta^z_{0,n})^{-1}\delta^z_{0,n}) d\nu(V) =$$

$$= \alpha^z_{0,n} + i\beta^z_{0,n}(1 - i\delta^z_{0,n})^{-1}\gamma^z_{0,n}$$

Using rank-one perturbation arguments it is known that μ_n has to be an absolutely continuous measure, its density is given by $\frac{1}{\pi}\lim_{\eta\downarrow 0}\Im m(g(\lambda + i\eta))$. We note that $\alpha^\lambda_{0,n}$ is real, and $\gamma^\lambda_{0,n} = \beta^\lambda_{0,n}$ for $\lambda \in \mathbb{R}$. Thus,

$$\frac{d\mu_n}{d\lambda} = \Im m \frac{i(\beta^\lambda_{0,n})^2}{\pi(1 - i\delta^\lambda_{0,n})} = \frac{\Re e(\beta^\lambda_{0,n})^2(1 + i\delta^\lambda_{0,n})}{\pi(1 + (\delta^\lambda_{0,n})^2)} = \frac{1}{\pi}\,\frac{(\beta^\lambda_{0,n})^2}{1 + (\delta^\lambda_{0,n})^2}$$

For the last part:

$$\frac{(\beta^\lambda_{0,n})^2}{1 + (\delta^\lambda_{0,n})} = \frac{1}{\left\| \begin{pmatrix} (\beta^\lambda_{0,n})^{-1} \\ \delta^\lambda_{0,n}(\beta^\lambda_{0,n})^{-1} \end{pmatrix} \right\|^2} = \frac{1}{\|T^\lambda_{0,n}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\|^2}$$

□

Corollary 2.3.5. *Let J be self-adjoint on $D_{\max}$, and let μ be the spectral measure at the origin δ_0, then $\mu_n \xrightarrow{w} \mu$*

Proof. Using the last theorem 2.3.4 with proposition 1.4.11 we have the result. □

2.4 Spectral averaging formula on the strip

The formula of Theorem 2.3.4 can be extended to strips, i.e. block-Jacobi operators, and even more generally to self-adjoint operators with locally finite hopping [31].

Definition 2.4.1. A block-Jacobi operator J on $\ell^2(\mathbb{Z}_+, \mathbb{C}^l) \cong \ell^2(\mathbb{Z}) \otimes \mathbb{C}^l$ is given by

$$(J\psi)_n = \begin{cases} -A_1^*\psi_1 + B_0\Psi_0 & , \text{if } n = 0 \\ -A_{n+1}^*\psi_{n+1} + B_n\psi_n - A_n\psi_{n-1} & \text{if } n \geq 1\,. \end{cases} \tag{2.4.1}$$

where $A_n \in \mathrm{GL}(l)$ are invertible $l \times l$ matrices, $B_n \in \mathrm{Her}(l)$ are Hermitian $l \times l$ matrices, and $\psi = (\psi_n)_{n\in\mathbb{Z}_+}$ is an ℓ^2 sequence of vectors in $\mathbb{C}^l$, $\psi_n \in \mathbb{C}^l$. Note that formally $J\psi$ can be defined for $\psi \in (\mathbb{C}^l)^{\mathbb{Z}_+}$.

Like above, we will assume that J is self-adjoint on its natural domain, which implies that J_{min} is essentially self-adjoint, were the domain of J_{min} are the compactly supported functions.

If $A_n = I$ are the unit matrices and $B_n = B$ is some fixed Hermatian matrix denoting the adjacency matrix of a finite graph $\mathbb{G}$, then J can be seen as a discrete Laplacian on the product $\mathbb{Z}_+ \times \mathbb{G}$. In fact in this case,

As before, solving $J\psi = z\psi$ leads to the equation

$$\begin{pmatrix} A_{n+1}^* \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n^z \begin{pmatrix} A_n^* \psi_n \\ \psi_{n-1} \end{pmatrix} \quad \text{where} \quad T_n^z = \begin{pmatrix} (B_n - z)(A_n^*)^{-1} & -A_n \\ (A_n^*)^{-1} & \mathbf{0} \end{pmatrix} \tag{2.4.2}$$

Note that the 4 blocks of the transfer matrices are $l \times l$ matrices and T_n^z is a $2l \times 2l$ matrix.

As above, (case $l = 1$) we introduce the products

$$T_{m,n}^z = T_m^z T_{m-1}^z \cdots T_{m+1}^z T_m^z \ ,$$

and the restrictions of J to $\ell^2(\{m, m+1, \ldots, n\}) \otimes \mathbb{C}^l \cong \mathbb{C}^{l(n-m+1)}$ given by

$$J_{m,n} = \begin{pmatrix} B_m & -A_{m+1}^* & & \\ -A_{m+1} & \ddots & \ddots & \\ & \ddots & \ddots & -A_n^* \\ & & -A_n & B_n \end{pmatrix} . \tag{2.4.3}$$

Now we identify δ_j for $j = m, \ldots, n$ with the canonical injection of $\ell^2(\{j\}) \otimes \mathbb{C}^l \cong \mathbb{C}^l$ into $\ell^2(\{m, \ldots, n\} \otimes \mathbb{C}^l)$. For a vector $v \in \mathbb{C}^l$ we thus have $\delta_j v = \delta_j \otimes v$ where on the left hand side δ_j is the classical δ symbol given one of the canonical basis vectors in $\ell^2(\{m, \ldots, n\})$. Thus, δ_j is a $l(n - m + 1) \times l$ matrix consisting of $(n - m + j)$ blocks in a column of size $l \times l$, where the $j + 1 - m$ block is the unit matrix and all other blocks are the zero matrix.

Then, similar as above, for $z \notin \sigma(J_{m,n})$ we define the boundary resolvent data for $J_{m,n}$ by

$$\alpha^z_{m,n} := A^*_m \delta^*_m \, (J_{m,n} - z)^{-1} \, \delta_m A_m \tag{2.4.4}$$

$$\beta^z_{m,n} := A^*_m \delta^*_m \, (J_{m,n} - z)^{-1} \, \delta_n \tag{2.4.5}$$

$$\gamma^z_{m,n} := \delta^*_n \, (J_{m,n} - z)^{-1} \, \delta_m A_m \tag{2.4.6}$$

$$\delta^z_{m,n} := \delta^*_n \, (J_{m,n} - z)^{-1} \, \delta_n \; . \tag{2.4.7}$$

$$\tag{2.4.8}$$

Now, all these are $l \times l$ matrices.

A consequence of the more general calculations in [31] is the following

Theorem 2.4.2. [31] *For given $m \leq n$ one finds that for all but finitely many values of $z \in \mathbb{C} \setminus \sigma(J_{m,n})$, the matrix $\beta^z_{m,n}$ is invertible, and*

$$T^z_{m,n} = \begin{pmatrix} (\beta^z_{m,n})^{-1} & -(\beta^z_{m,n})^{-1} \alpha^z_{m,n} \\ \delta^z_{m,n} (\beta^z_{m,n})^{-1} & \gamma^z_{m,n} - \delta^z_{m,n} (\beta^z_{m,n})^{-1} \alpha^z_{m,n} \end{pmatrix} .$$

Proof. The proof is exactly the same as for Proposition 2.2.3 . □

Now in order to get an equivalent of the spectral averaging formula and the limit expresion of a spectral measure at a vector, we need to first fix a vector in the root-shell. Thus, we choose some $\vec{A} \in \mathbb{C}^l$ which we identify with $\delta_0 \otimes \vec{A} \in \ell^2\{\mathbf{Z}_+\} \otimes \mathbb{C}^l$. (Note that there is no A_0 matrix defined formally. For the transfer matrices, one can choose any invertible one. In some sense, the vector $\vec{A}$ is now replacing the matrix A_0.)

Let us assume that $\|\vec{A}\| = 1$, so that $\vec{A}^* \vec{A} = 1$. Furthermore, identifying $\vec{A}^*$ with a linear map from $\mathbb{C}^l$ to $\mathbb{C}$, we have a $l-1$ dimensional kernel consisting of the vectors orthogonal to $\vec{A}$.

$$\mathbb{K} := \ker(\vec{A}^*) = \{v \in \mathbb{C}^l \, : \, \vec{A}^* v = 0\} \; = \; \{v \in \mathbb{C}^l \, : \, \vec{A} \cdot v = 0\}.$$

Then, in this special case, the work of [31] simply replaces T_0^z by the set of $2l \times 2$ matrices

$$\mathbb{T}_0^z = \left\{ \begin{pmatrix} (B_n - z)(\vec{A} + v) & -\vec{A} + (B_n - z)w \\ \vec{A} + v & w \end{pmatrix} \, : \, v, w \in \mathbb{K} \right\} \subset \mathbb{C}^{2l \times 2} . \tag{2.4.9}$$

Choosing $A_0 = I$ to be the identity matrix, one has

$$T_0^z = \begin{pmatrix} B_n - z & -I \\ I & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbb{T}_0^z = T_0^z \left\{ \begin{pmatrix} \vec{A} + v & w \\ \mathbf{0} & \vec{A} \end{pmatrix} \, : \, v, w \in \mathbb{K} \right\} \tag{2.4.10}$$

where we adopt the notation that $T\mathbb{A} = \{TA \, : \, A \in \mathbb{A}\}$ for sets of matrices $\mathbb{A}$.

Moreover we consider the spectral measure $\mu_{\vec{A}}$ at the vector $\vec{A} \equiv \delta_0 \otimes \vec{A}$, that means

$$\int f d\mu_{\vec{A}} \, = \, \langle \delta_0 \otimes \vec{A}, \, f(J) \, (\delta_0 \otimes \vec{A}) \, \rangle$$

measure

Now, using that the operator J can not have compactly supported eigenfunctions, Theorem 1 in [31] implies the following:

Theorem 2.4.3. [31] *In the sense of a weak limit for finite measures one finds that*

$$\mathrm{d}\mu_{\vec{A}}(\lambda) \, = \, \lim_{n \to \infty} \frac{1}{\pi} \frac{\mathrm{d}\lambda}{\min\limits_{T^\lambda \in \mathbb{T}_0^\lambda} \|T_{1,n}^\lambda \, T^\lambda \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \|^2} \, = \, \lim_{n \to \infty} \frac{1}{\pi} \frac{\mathrm{d}\lambda}{\min\limits_{\vec{v} \in \mathbb{K}} \left\| T_{0,n}^\lambda \left(\begin{smallmatrix} \vec{A} + \vec{v} \\ 0 \end{smallmatrix} \right) \right\|^2}$$

Using the symplectic structure of the transfermatrices and the Banach-Alaoglu theorem one can obtain a criterion for absolute continuity (see [31]).

Theorem 2.4.4. *If one finds $\vec{u}_{\lambda,n} \in \mathbb{C}^m$ for $\lambda \in (a,b)$, $n \in \mathbb{N}$, such that*

$$\liminf_{n \to \infty} \int_a^b \left\| T_{0,n}^\lambda \left(\begin{smallmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{smallmatrix} \right) \right\|^4 \mathrm{d}\lambda \, < \, \infty$$

then, the measure $\mu_{\vec{A}}$ is absolutely continuous in the interval (a,b).

Proof. First, in [31] it was shown that the minimum $\min\limits_{\vec{v} \in \mathbb{K}} \left\| T_{0,n}^\lambda \left(\begin{smallmatrix} \vec{A} + \vec{v} \\ 0 \end{smallmatrix} \right) \right\|$ is achieved at a very specific vector which we call $\vec{v}_{\lambda,n} \in \mathbb{K}$. Defining

$$f_n(\lambda) := \pi^{-1} \left\| T^{\lambda}_{0,n} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} \right\|^{-2}$$

we see from Theorem 2.4.3 that $\mu_{\vec{A}}$ is the weak limit of $f_n(\lambda)\mathrm{d}\lambda$ in the interval (a,b). Note that

$$\begin{aligned}
&\left(T^{\lambda}_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix}\right)^* \begin{pmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{pmatrix} T^{\lambda}_{0,n} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} = \\
&\quad = \begin{pmatrix} \vec{u}^*_{\lambda,n} & \vec{A}^* \end{pmatrix} \left(T^{\lambda}_{0,n}\right)^* \begin{pmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{pmatrix} T^{\lambda}_{0,n} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} = \\
&\quad = \begin{pmatrix} \vec{u}^*_{\lambda,n} & \vec{A}^* \end{pmatrix} \begin{pmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{u}^*_{\lambda,n} & \vec{A}^* \end{pmatrix} \begin{pmatrix} 0 \\ \vec{A}+\vec{v}_{\lambda,n} \end{pmatrix} = 1
\end{aligned}$$

where we use $\|\vec{A}\| = 1$ and $\vec{A}^*\vec{v}_{\lambda,n} = 0$ as $\vec{v}_{\lambda,n} \in \mathbb{K}$. Now, using the Cauchy Schwartz inequality, this gives

$$1 \le \left\| T^{\lambda}_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \right\| \cdot \left\| T^{\lambda}_{0,n} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} \right\|$$

and hence

$$\pi^2 |f_n(\lambda)|^2 = \frac{1}{\left\| T^{\lambda}_{0,n} \begin{pmatrix} \vec{A}+\vec{v}_{\lambda,n} \\ 0 \end{pmatrix} \right\|^4} \le \left\| T^{\lambda}_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \right\|^4 .$$

Thus, the estimate given implies that

$$\liminf_{n\to\infty} \int_a^b |f_n(\lambda)|^2 \,\mathrm{d}\lambda < \infty .$$

This means, along a suitable sub-sequence, the norm of f_n in $L^2(a,b)$ is bounded. By Banach-Alaaoglu, there is a sub-sequence (o better, a sub-sub-sequence of the suitable sub-sequence) f_{n_k} which converges weakly in $L^2(a,b)$ to a limit $f \in L^2(a,b)$. Noting that bounded continuous functions $g \in C_b(a,b)$ are also in $L^2(a,b)$ one has

$$\lim_{k\to\infty} \int_a^b g(\lambda)\, f_{n_k}(\lambda)\,\mathrm{d}\lambda = \int_a^b g(\lambda)\, f(\lambda)\,\mathrm{d}\lambda .$$

for all $g \in C_b(a,b)$. But since $f_{n_k}(\lambda)\mathrm{d}\lambda$ converges weakly to the measure $\mu_{\vec{A}}$ this means that in the interval (a, b) we have

$$\mathrm{d}\mu_{\vec{A}}(\lambda) = f(\lambda)\,\mathrm{d}\lambda$$

which is an absolutely continuous measure in (a, b) with a density in $L^2(a, b)$. $\square$

Chapter 3

Absolutely continuous spectrum for block Jacobi operators with random ℓ^2 matrix-potential

3.1 Model and main result

We consider a random family of block-Jacobi operators on $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^l$ given by

$$(H_\omega \Psi)_n = -\Psi_{n-1} - \Psi_{n+1} + A\Psi_n + V_n \Psi_n \tag{3.1.1}$$

where $V_n = V_n(\omega) \in \mathrm{Her}(l)$ are independent random Hermitian $l \times l$ matrices satisfying

$$\sum_{n \geq 0} \left(\|\mathbb{E}(V_n)\| + \mathbb{E}(||V_n||^2) \right) \,<\, \infty\;, \tag{3.1.2}$$

where $\mathbb{E}$ denotes the expectation value. This means, we have some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and $\mathrm{Her}(l)$ valued random variables $V_n \,:\, \Omega \to \mathrm{Her}(l)$. $\Psi = (\Psi_n)_{n\geq 0} \in \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^l$ means $\Psi_n \in \mathbb{C}^l$, $\forall n \in \mathbb{Z}_+$ with $\sum_{n\geq 0} ||\Psi_n||^2 < \infty$, A is a vertical Laplacian, which is a fixed Hermitian matrix ($A = A^*$) and finally we have $V_n = V_n(\omega)$ a random matrix potential.

We also de fine the 'unperturbed' operator H_0 by

$$(H_0\Psi)_n = -\Psi_{n-1} - \Psi_{n+1} + A\Psi_n \tag{3.1.3}$$

H_0 and H_ω can be seen as quasi-one dimensional discrete Schrödinger operators on a strip of width l. The matrix A maybe the adjacency matrix of a finite graph $\mathbb{G}$, in which case H_0 would be like a discrete Laplace operator on the product graph $\mathbb{Z}_+ \times \mathbb{G}$. H is then a random perturbation of H_0 adding the matrix potentials V_n at each horizontal level n. This way, H_ω falls into the class of operators describing randomly perturbed quantum systems. The study of such systems was initiated by Anderson [1] with the today called Anderson model where one studies operators on $\mathbb{Z}^d$ with independent identically distributed potentials on each lattice site. In general for such models one finds Anderson localization at large disorder (large variance of the potential) and at the edges of the spectrum. Anderson localization means one has pure point spectrum and exponentially decaying eigenfunctions. There are two general methods to prove this, the fractional moment method [3] and multi-scale analysis [13, 14, 15]. The fractional moment method at high disorder works fine in any graphs with a finite upper bound on the connectivity of one point [34]. For a long time the high disorder Bernoulli Anderson model on $\mathbb{Z}^d$ for $d > 1$ could not be handled. However, recently the localization has also been shown in this case in 2 and 3 dimensions [23, 24]. Existence of absolutely continuous spectrum for Anderson models at low disorder has first been proved for infinite dimensional hyperbolic type graphs like regular trees and tree-like structures [2, 4, 10, 11, 17, 19, 20, 27, 28]. Absolutely continuous spectrum has also been shown for the Anderson model on special graphs with a finite-dimensional growth, so called anti-trees and partial antitrees [29, 30].

As a mean to study critical transitions from absolutely continuous to pure point spectrum, random decaying potentials in one dimension were also investigated [12, 18, 22]. Here, we extend and improve on the result in [12] using methods similar to [22]. The key point for the absolutely continuous spectrum result in [22] has been the spectral average formula by Carmona-Lacroix as stated in Theorem 2.3.4. The key point here is its generalization to strips Theorem 2.4.3 which is a special case of the broader generalizaion in [31].

3.1.1 Spectrum and spectral bands

Without loss of generality, we may assume that A in (3.1.1) is a diagonal matrix: If this is not the case, then, as $A^* = A$, there is a unitary matrix U such that U^*AU is diagonal.

Then, define the unitary operator $\mathcal{U} : \ell^2(\mathbb{Z}_+, \mathbb{C}^l) \to \ell^2(\mathbb{Z}_+, \mathbb{C}^l)$ by

$$(\mathcal{U}\Psi)_n := U\Psi_n$$

and one finds:

$$(\mathcal{U}^* H_\omega \mathcal{U}\Psi)_n = U^*(H\mathcal{U}\Psi)_n =$$

$$U^*(-(\mathcal{U}\Psi)_{n+1} - (\mathcal{U}\Psi)_{n-1} + A(\mathcal{U}\Psi)_n + V_n(\mathcal{U}\Psi)_n) =$$

$$U^*(-U\Psi_{n+1} - U\Psi_{n-1} + AU\Psi_n + V_nU\Psi_n) =$$

$$-\Psi_{n+1} - \Psi_{n-1} + U^*AU\Psi_n + U^*V_nU\Psi_n$$

Now U^*AU is diagonal and U^*V_nU are random Hermitian matrices satisfying an inequality as (3.1.2). Thus, using this unitary conjugation, we may assume that A is diagonal, hence

$$A = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_l \end{pmatrix} \tag{3.1.4}$$

with $\alpha_j \in \mathbb{R}$, $j = 1, \dots, l$ being the eigenvalues of A.

Proposition 3.1.1. *Let $(Ju)_n = -u_{n+1} - u_{n-1}, u \in \ell^2(\mathbb{Z}_+)$ then:*

$$H_0 \cong \bigoplus_{j=1}^{l} (J + a_j I), \quad \textit{and, hence,} \quad \sigma(H_0) = \bigcup_{j=1}^{l} [a_j - 2, a_j + 2]$$

with pure absolutely continuous spectrum.

Proof. Let

$$\Psi^j := (\Psi_{n,j})_{n=1}^{\infty}, \forall j \in \{1, ..., l\}$$

then

$$U : \ell^2(\mathbb{Z}_+, \mathbb{C}^l) :\to \bigoplus_{j=1}^{l} \ell^2(\mathbb{Z}_+), U\Psi = \bigoplus_{j=1}^{l} \Psi^j$$

is unitary and

$$UH_0U^* \bigoplus_{j=1}^{l} \Psi^j = \bigoplus_{j=1}^{l} (J + a_j I)$$

□

We call $[\alpha_j - 2, \alpha_j + 2]$ the j-th band of the spectrum of H_0, $\{\alpha_j - 2, \alpha_j + 2\}$ are the band-edges of this band. Each band-edge can be internal, meaning inside of another band, or external, meaning an edge (boundary point) of the spectrum of H_0. We consider the spectrum of H_0 without all the (external and internal) band-edges and define

$$\Sigma = \left[\bigcup_{j=1}^{l} (\alpha_j - 2, \alpha_j + 2) \right] \setminus \left[\bigcup_{j=1}^{l} \{\alpha_j - 2, \alpha_j + 2\} \right] \tag{3.1.5}$$

Note Σ is open and $\overline{\Sigma} = \sigma(H_0)$. We also define the intersection of all open bands,

$$\Sigma_0 = \bigcap_{j=1}^{l} (\alpha_j - 2, \alpha_j + 2) \tag{3.1.6}$$

which might be empty. For the essential spectrum we note:

Proposition 3.1.2. $\sigma_{\text{ess}}(H_\omega) = \overline{\Sigma} = \sigma(H_0)$

Proof. As $\sum_{n=1}^{\infty} \mathbb{E}(||V_n||)^2 < \infty$ that implies $\sum_{n=1}^{\infty} ||V_n||^2 < \infty$ almost surely. Then

$$\left\| \left(\bigoplus_{n=1}^{\infty} V_n \right) \psi - \left(\bigoplus_{n=1}^{m} V_n \right) \psi \right\|^2 \leq \sum_{n=m+1}^{\infty} ||V_n \psi_n||^2 \leq \sum_{n=m+1}^{\infty} ||V_n||^2 ||\psi_n||^2 \leq \sum_{n=m+1}^{\infty} ||V_n||^2 ||\psi||^2$$

Thus

$$\left\| \bigoplus_{n=1}^{\infty} V_n - \bigoplus_{n=1}^{m} V_n \right\|^2 \leq \sum_{n=m+1}^{\infty} ||V_n||^2 \to 0$$

and $\bigoplus_{n=1}^{\infty} V_n$ is a compact operator (limit of of operators with finite dimensional range) and, hence,

$$\sigma_{\text{ess}}(H_\omega) = \sigma_{\text{ess}}\left(H_0 + \bigoplus_{n=1}^{\infty} V_n\right) = \sigma_{\text{ess}}(H_0) = \overline{\Sigma}$$

□

3.1.2 The main result

The main theorem of the whole thesis is the following:

Theorem I. *Apart from discrete spectrum, (embedded isolated eigenvalues) the spectrum of H_ω is almost surely purely absolutely continuous in Σ. Moreover, there are no embedded eigenvalues in the intersection of the bands, Σ_0. That means, there may be embedded eigenvalues in $\Sigma \setminus \Sigma_0$ which may only accumulate at the boundary $\partial\Sigma$, that is, the internal and external band-edges.*
In technical terms, this means, there is a set $\hat{\Omega} \subset \Omega$ of probability one, $\mathbf{P}(\hat{\Omega}) = 1$, such that for all $\omega \in \hat{\Omega}$ and all compact subsets $\mathfrak{C} \subset \Sigma$, there is a finite subset of eigenvalues $\mathfrak{E} \subset \mathfrak{C} \setminus \Sigma_0$, such that the spectrum of H_ω is purely absolutely continuous in $\mathfrak{C} \setminus \mathfrak{E}$.

Under the stronger assumption that $\mathbb{E}(V_n) = 0$ and $\sum_n \mathbb{E}(\|V_n\|^2 + \|V_n\|^4) < \infty$ it was aready shown in [12] that the spectrum is purely absolutely continuous in Σ_0, and that there is absolutely continuous spectrum in all of Σ. However the proof method used there for the set Σ does not exclude any other type of singular spectrum.

Note in the line case, $l = 1$ we have $\Sigma = \Sigma_0$ and thus purely absolutely continuous spectrum as already shown in [22]. On the line case it is also known that for any, also non-random ℓ^2-potential, one has absolutely continuous spectrum in Σ [9], but again, any other type of embedded singular spectrum is possible (not excluded in the proof).

The general operator H_ω investigated here allows the case, were the operator (almost surely) splits into the direct sum of two strip operators $H^1 \oplus H^2$ (two separated strips). Then, adjusting one of the V_n one may create an eigenvalue for H^1, lying outside of its essential spectrum, but lying inside the essential spectrum of H^2. In fact, one may have the part of V_n belonging to H^1 non-random and create some fixed embedded eigenvalue (for all of ω). Thus, without further 'channel-mixing' assumptions, one can not expect to obtain pure absolutely continuous spectrum within Σ. But under sufficient 'mixing' created by the V_n, this should be true.

3.2 Transfer matrices, elliptic and hyperbolic channels

Recall that H_ω is given by

$$(H_\omega \psi)_n = -\psi_{n+1} - \psi_{n-1} + (A + V_n)\psi_n$$

The eigenvalue equation $H_\omega \psi = \lambda \psi$ is a recursion that can be written in the matrix form as follows.

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n^\lambda \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \quad \text{where} \quad T_n^\lambda = \begin{pmatrix} V_n + A - \lambda I & -I \\ I & 0 \end{pmatrix} .$$

We may write

$$T_n^\lambda = T_{H_0}^\lambda + \begin{pmatrix} V_n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad T_{H_0}^\lambda = \begin{pmatrix} A - \lambda I & -I \\ I & 0 \end{pmatrix}$$

where is basically the transfer matrix of the unperturbed operator H_0. We will now write the transfer matrix in some basis which diagonalises $T_{H_0}^\lambda$.

Recall, A is assumed diagonal and its eigenvalues are $\alpha_1, \ldots, \alpha_l$. Adopting the notions of [26] we define:

Definition 3.2.1. Let $\lambda \in \mathbb{R}$. We call the j-th channel

1. **Elliptic** at λ if $|\alpha_j - \lambda| < 2$
2. **Hyperbolic** at λ if $|\alpha_j - \lambda| > 2$
3. **Parabolic** at λ if $|\alpha_j - \lambda| = 2$

Now fix some $\lambda \in \Sigma$. Note that by the definition of Σ, there are no parabolic channels and there is at least one elliptic channel. at λ We assume the channels to be ordered such that:

$$|\alpha_j - \lambda| < 2 \quad \forall j \in \{1, ..., l_e\}$$

$$|\alpha_j - \lambda| > 2 \quad \forall j \in \{l_e + 1, ..., l\}$$

Note that the set of all λ satisfying these inequalities is some open interval $(\lambda_0, \lambda_1) \subset \Sigma$. We later vary λ slightly within this interval. For $\lambda \in (\lambda_0, \lambda_1)$, and $j \in \{1, ..., l_e\}$ we define $k_j = k_j(\lambda) \in (0, \pi)$ by

$$2\cos(k_j) = \alpha_j - \lambda \, .$$

For $j \in \{1, ..., l_h\}$ with $l_h = l - l_e$ we define $\gamma_j = \gamma_j(\lambda) \in \mathbb{R}$, $|\gamma_j| > 1$, by

$$\gamma_j + \frac{1}{\gamma_j} = \alpha_{j+l_e} - \lambda$$

We define the diagonal matrices

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \gamma_{l_h} \end{pmatrix} , \quad K = \begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & k_{l_e} \end{pmatrix}$$

such that

$$A - \lambda I = \begin{pmatrix} 2\cos(K) & 0 \\ 0 & \Gamma + \Gamma^{-1} \end{pmatrix} .$$

Thus, for the transfer matrices of H_0 we find

$$T^{\lambda}_{H_0} = \begin{pmatrix} 2\cos(K) & 0 & -I & 0 \\ 0 & \Gamma + \Gamma^{-1} & 0 & -I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} .$$

Also we note that

$$T^\lambda_{H_0}\begin{pmatrix} e^{\pm iK} \\ 0 \\ I \\ 0 \end{pmatrix} = \begin{pmatrix} e^{\pm iK} \\ 0 \\ I \\ 0 \end{pmatrix} e^{\pm iK}, \qquad T^\lambda_{H_0}\begin{pmatrix} 0 \\ \Gamma^{\pm 1} \\ 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma^{\pm 1} \\ 0 \\ I \end{pmatrix} \Gamma^{\pm 1}$$

so that $e^{\pm ik_j}$, γ_j and γ_j^{-1} are the eigenvalues of $T^\lambda_{H_0}$. In order to diagonlize $T^\lambda H_0$ we introduce

$$Q_\lambda = \begin{pmatrix} e^{iK} & e^{-iK} & 0 & 0 \\ 0 & 0 & \Gamma^{-1} & \Gamma \\ I_{l_e} & I_{l_e} & 0 & 0 \\ 0 & 0 & I_{l_h} & I_{l_h} \end{pmatrix},$$

where I_d is the unit matrix of size $d \times d$, then

$$Q_\lambda^{-1} T_n^\lambda Q_\lambda = T^\lambda + \mathcal{V}_n^\lambda \tag{3.2.1}$$

with T^λ being diagonal, more precisely,

$$T^\lambda = \begin{pmatrix} e^{iK} & 0 & 0 & 0 \\ 0 & e^{-iK} & 0 & 0 \\ 0 & 0 & \Gamma^{-1} & 0 \\ 0 & 0 & 0 & \Gamma \end{pmatrix}, \qquad \mathcal{V}_n^\lambda = Q_\lambda^{-1} \begin{pmatrix} V_n & 0 \\ 0 & 0 \end{pmatrix} Q_\lambda\,. \tag{3.2.2}$$

We note that Q_λ is indeed invertible for $\lambda \in (\lambda_0, \lambda_1)$ as $e^{ik_j} \neq e^{-ik_j}$ and $\gamma_j \neq 1/\gamma_j$ in this case. Defining

$$\mathcal{Q}_K = (e^{iK} - e^{-iK})^{-1}\,, \quad \mathcal{Q}_\Gamma = (\Gamma^{-1} - \Gamma)^{-1} \tag{3.2.3}$$

we find

$$Q_\lambda^{-1} = \begin{pmatrix} \mathcal{Q}_K & 0 & -e^{-iK}\mathcal{Q}_K & 0 \\ -\mathcal{Q}_K & 0 & \mathcal{Q}_K e^{iK} & 0 \\ 0 & \mathcal{Q}_\Gamma & 0 & -\Gamma\mathcal{Q}_\Gamma \\ 0 & -\mathcal{Q}_\Gamma & 0 & \Gamma^{-1}\mathcal{Q}_\Gamma \end{pmatrix} \tag{3.2.4}$$

Now, in order to work with uniform estimates we will restrict our consideration to a compact interval $[a,b] \subset (\lambda_0, \lambda_1) \subset \Sigma$. Chosen such a compact interval and allowing complex values for k_j and γ_j, we can extend the definitions of $K = K(\lambda), \Gamma = \Gamma(\lambda), Q_\lambda, Q_\lambda^{-1}, T^\lambda$ analytically to spectral parameters z in the complex plane, $z = \lambda + i\eta \in [a,b] + i[-c,c] \subset \mathbb{C}$, for c small enough. This means $\lambda \in [a,b],\ \eta \in [-c,c]$, and the equations (3.2.1) and (3.2.2) still hold with λ replaced by z. We will need this extension in some part to use analyticity arguments.

Choosing $c > 0$ small enough, one can guarantee by compactness and analyticity arguments, that there is some $\gamma > 0$ such that

$$\left\|\Gamma^{-1}(\lambda+i\eta)\right\| \le e^{-2\gamma}\ , \quad \text{and} \quad \left\|e^{\pm iK(\lambda+i\eta)}\right\| \le e^{\gamma} \quad \forall\, \lambda \in [a,b],\ \forall\, |\eta| \le c\ . \tag{3.2.5}$$

Note for $\lambda \in [a,b]$ we have $\|e^{\pm iK}\| = 1$.

3.3 The key estimate

We consider the following general situation:

Let be given independent random $(l_0+l_1)\times(l_0+l_1)$ matrices given by

$$T_n = T + W_n \quad \text{where} \quad T = \begin{pmatrix} S & \\ & \Gamma \end{pmatrix}\ , \quad S \in \mathbb{C}^{l_0\times l_0}, \quad \Gamma \in \mathbb{C}^{l_1\times l_1} \tag{3.3.1}$$

where for some fixed $\gamma > 0$ we have

$$\|S\| \le e^{\gamma} \quad , \quad \|\Gamma^{-1}\| \le e^{-2\gamma}\ . \tag{3.3.2}$$

Note that the second condition implies $\|\Gamma v\| \ge e^{2\gamma}\|v\|$ for any vector $v \in \mathbb{C}^{l_0+l_1}$. Moreover,

W_n are independent random $(l_0+l_1)\times(l_0+l_1)$ matrices satisfying, with some fixed constant $C_W > 0$,

$$\|W_n\| \le \frac{e^{2\gamma}-e^{\gamma}}{4} \quad \text{and} \quad \sum_{n=1}^{\infty} \|\mathbb{E}(W_n)\| + \mathbb{E}(\|W_n\|^2) \le \mathcal{K} < \infty \tag{3.3.3}$$

For certain parts we will also assume the stricter bound $\|S\| \le 1$.

Now let us consider the Markov process of $(l_0+l_1)\times(l_0+l_1)$ matrices given by

$$\mathcal{X}_0 = I, \quad \mathcal{X}_{n+1} = T_n\,\mathcal{X}_n\ .$$

Using the splitting into blocks of sizes d_0 and d_1 like above we write

$$\mathcal{X}_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \quad \text{and} \quad W_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{3.3.4}$$

From the process $\mathcal{X}_n$ we will define the process of pairs of matrices (X_n, Z_n) given by

$$X_n = A_n - B_n D_n^{-1} C_n\ , \quad Z_n = B_n D_n^{-1}\ . \tag{3.3.5}$$

X_n is a so called Schur-complement. Some standard calculations , see for instance [32], show that (X_n, Z_n) can be seen as the process of equivalence classes of $\mathcal{X}_n$ defining

$$\mathcal{X}_1 \sim \mathcal{X}_2 \quad \Leftrightarrow \quad \mathcal{X}_1 = \mathcal{X}_2 \begin{pmatrix} I & 0 \\ M & G \end{pmatrix}$$

with I being the $l_0 \times l_0$ identity matrix, $G \in \mathrm{GL}(l_1)$ and M being any $l_1 \times l_0$ matrix. Note that the set of matrices of the form $\left(\begin{smallmatrix} I & 0 \\ M & G \end{smallmatrix}\right)$ is a group.

In that sense if D is invertible we get

$$\mathcal{X}_{n+1} = \begin{pmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & D_{n+1} \end{pmatrix} \sim \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ -C_n D_n^{-1} & D_n^{-1} \end{pmatrix} = \begin{pmatrix} X_n & Z_n \\ \mathbf{0} & I \end{pmatrix}$$

and we find

$$\mathcal{X}_{n+1} \sim \begin{pmatrix} S+a_n & b_n \\ c_n & \Gamma + d_n \end{pmatrix} \begin{pmatrix} X_n & Z_n \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} (S+a_n)X_n & (S+a_n)Z_n + b_n \\ c_n X_n & c_n Z_n + \Gamma + d_n \end{pmatrix}$$

which leads to the identities

$$Z_{n+1} = \big((S+a_n)Z_n + b_n\big)\big(c_n Z_n + \Gamma + d_n\big)^{-1} \tag{3.3.6}$$

$$X_{n+1} = (S+a_n)X_n - Z_{n+1}\, c_n X_n \tag{3.3.7}$$

provided that D_n^{-1} and D_{n+1}^{-1} exist.

Lemma 3.3.1. *If M is a $d \times d$ square matrix we find for $c > 0$ fixed*

$$\|Mv\| \geq c\|v\| \quad \textit{for all } v \in \mathbb{C}^d \quad \Leftrightarrow \quad M \;\textit{ is invertible and }\; \|M^{-1}\| \leq \frac{1}{c}$$

Proof. First we show "$\Rightarrow$". Given v one finds The left condition clearly gives that $v \mapsto Mv$ is injective, hence also surjective (as a linear map from a finite dimensional vector space to itself). And using $M^{-1}v$ for v in the inequality

$$\|M^{-1}v\| \leq \frac{1}{c}\|MM^{-1}v\| = \frac{1}{c}\,\|v\|\;.$$

For the other direction "$\Leftarrow$" note

$$\|v\| = \|M^{-1}Mv\| \leq \|M^{-1}\|\,\|Mv\| \;\leq\; \frac{1}{c}\,\|Mv\|\;.$$

$\square$

Lemma 3.3.2. *Assume $\|Z_n\| \leq 1$ and that the bounds conditions* (3.3.2), (3.3.3) *hold. Then*

$$\|(c_n Z_n + \Gamma + d_n)^{-1}\| \;\leq\; \frac{1}{e^{2\gamma} - 2\|W_n\|} \;\leq\; \frac{2}{e^{2\gamma} + e^{\gamma}}$$

Proof. We use Lemma 3.3.1 noting

$$\begin{aligned}\|(c_nZ_n+(\Gamma+d_n))v\| &\geq \|\Gamma v\|-\|(c_nZ_n+d_n)v\|\geq e^{2\gamma}\|v\|-(\|c_n\|\|Z_n\|+\|d_n\|)\|v\| \\ &\geq \left(e^{2\gamma}-2\|W_n\|\right)\|v\| \;\geq\; \left(e^{2\gamma}-\frac{e^{2\gamma}-e^{\gamma}}{2}\right)\|v\|=\frac{e^{2\gamma}+e^{\gamma}}{2}\;\|v\|\;.\end{aligned}$$

□

Now we are ready to make the first main step for the needed estimates:

Proposition 3.3.3. *Under the given assumptions* (3.3.2) *and* (3.3.3) *one finds: D_n is invertible for all $n\in\mathbb{N}$, hence, X_n and Z_n are well defined for all n and*

$$\sup_{n\in\mathbb{N}}\|Z_n\|\leq 1$$

Proof. The proof will be by induction. First, we notice $D_0=I$ is invertible and $\|Z_0\|=\|0\|=0\leq 1$. Now assume $\|Z_n\|\leq 1$ and D_n being invertible. We find

$$\begin{pmatrix}A_{n+1} & B_{n+1}\\ C_{n+1} & D_{n+1}\end{pmatrix}=\begin{pmatrix}S+a_n & b_n\\ c_n & \Gamma+d_n\end{pmatrix}\begin{pmatrix}A_n & B_n\\ C_n & D_n\end{pmatrix}$$

Now by the lower right block

$$\begin{aligned}D_{n+1}&=c_nB_n+(\Gamma+d_n)D_n\\ D_{n+1}D_n^{-1}&=c_nB_nD_n^{-1}+(\Gamma+d_n)\;=\;c_nZ_n+(\Gamma+d_n)\end{aligned}$$

Lemma 3.3.2 now shows invertibility of $D_{n+1}D_n^{-1}$ and hence of D_{n+1}. Using (3.3.6) we find

$$\|Z_{n+1}\|=\|((S+a_n)Z_n+b_n)(c_nZ_n+(\Gamma+d_n))^{-1}\|\leq\;\left(e^{\gamma}+\frac{e^{2\gamma}-e^{\gamma}}{2}\right)\;\frac{2}{e^{2\gamma}+e^{\gamma}}\;=\;1\;.$$

This finishes the induction. □

Proposition 3.3.4. *Under the given assumptions* (3.3.2) *and* (3.3.3) *and the aditional* *property* $\lim_{n\to\infty} W_n = 0$ *we find*

$$\lim_{n\to\infty} Z_n = 0\,, \quad \lim_{n\to\infty} D_n^{-1} = 0 \quad \textit{and} \quad \lim_{n\to\infty} D_n^{-1} C_n \,=\, Y \quad \textit{exists}\,.$$

Proof. We will prove by induction that there exists N_k such that for all $n > N_k$ we have $\|Z_n\| \le e^{-k\gamma/2}$ for all $n > N_k$. The induction start for $k = 0$ is given by Proposition 3.3.3. Assume the statement is true for k. Then we find N such that for all $n > N$

$$\|W_n\| \,<\, e^{-k\gamma/2}\, \frac{e^{3\gamma/2} - e^{\gamma}}{4} \quad \text{and} \quad \|Z_n\| \le e^{-k\gamma/2}\,.$$

Using (3.3.6), Lemma 3.3.1 we find for $n > N$ that

$$\begin{aligned} \|Z_{n+1}\| \,&\le\, \frac{(e^{\gamma} + \|W_n\|)\|Z_n\| + \|W_n\|}{e^{2\gamma} - 2\|W_n\|} \,\le \\ &\le\, e^{-k\gamma/2}\, \frac{e^{\gamma} + \frac{1}{4}(e^{3\gamma/2} - e^{\gamma})(1 + e^{-k\gamma/2})}{e^{2\gamma} - \frac{1}{2}(e^{3/2\gamma} - e^{\gamma})e^{-k\gamma/2}} \\ &\le\, e^{-k\gamma/2}\, \frac{e^{\gamma} + \frac{1}{2}(e^{3\gamma/2} - e^{\gamma})}{e^{2\gamma} - \frac{1}{2}(e^{2\gamma} - e^{3\gamma/2})} \,\le\, e^{-(k+1)\gamma/2} \end{aligned}$$

This finishes the induction and the first statement.

For the second statement note

$$\begin{aligned} D_{n+1}^{-1} C_{n+1} \,&=\, [(\Gamma + d_n) D_n + c_n B_n]^{-1} [c_n A_n + (\Gamma + d_n) C_n] \,= \\ &=\, D_n^{-1} \left[\Gamma + d_n + c_n Z_n\right]^{-1} \left[c_n A_n + (\Gamma + d_n) C_n\right] \end{aligned}$$

Now, using

$$\left[\Gamma + d_n + c_n Z_n\right]^{-1} (\Gamma + d_n) =\, I - \left[\Gamma + d_n + c_n Z_n\right]^{-1} c_n Z_n$$

we obtain

$$\begin{aligned} D_{n+1}^{-1}C_{n+1} &= D_n^{-1}C_n + D_n^{-1}\left[\Gamma + d_n + c_n Z_n\right]^{-1} c_n(A_n - Z_n C_n) \\ &= D_n^{-1}C_n \; + \; D_{n+1}^{-1}c_n X_n \; . \end{aligned}$$

Therefore, using $D_0 = I, C_0 = 0$,

$$D_{n+1}^{-1}C_{n+1} = \sum_{k=0}^{n} D_{k+1}^{-1}c_k X_k \; . \tag{3.3.8}$$

One obtains

$$\|D_{n+1}^{-1}\| \leq \|D_n^{-1}\| \, \|[\Gamma + d_n + c_n Z]^{-1}\| \; \leq \; \frac{2}{e^{2\gamma} + e^{\gamma}}\|D_n\|^{-1} \; .$$

which gives

$$\|D_n^{-1}\| \; \leq \; \left(\frac{e^{2\gamma} + e^{\gamma}}{2}\right)^{-n} \; \to 0 \tag{3.3.9}$$

where we use that $D_0 = I$ as $\mathcal{X}_0 = I$.

Using (3.3.7) and $\|Z_n\| \leq 1$ and the condition (3.3.2) we obtain

$$\|X_{n+1}\| \; \leq \; (e^{\gamma} + 2\|W_n\|)\|X_n\| \; .$$

We note that for $\varepsilon > 0$ sufficiently small we have $e^{\gamma} + 2\varepsilon < \frac{e^{2\gamma}+e^{\gamma}}{2}$. Now there exists $N > 0$ such that for $n > N$ we have $\|W_n\| < \varepsilon$ and thus for $n > N$ we find for $\mathcal{C}_0 = \|X_N\|$ that

$$\|D_{n+1}^{-1}c_n X_n\| \leq \mathcal{C}_0 \underbrace{\left(\frac{2(e^{\gamma} + 2\varepsilon)}{e^{2\gamma} + e^{\gamma}}\right)}_{<1}^{n} \varepsilon \; .$$

Therefore,

$$\sum_{n=0}^{\infty} D_{n+1}^{-1}c_n X_n$$

converges absolutely and

$$\lim_{n\to\infty} D_n^{-1}C_n = \sum_{k=0}^{\infty} D_{k+1}^{-1}c_k X_k$$

exists. $\square$

The main point of this section is the following proposition.

Proposition 3.3.5. *Under the given assumptions* (3.3.2) *and* (3.3.3) *and the additional condition* $\|S\| \le 1$*, one has*

$$\sup_n \mathbb{E}(\|X_n\|^4) \le \mathcal{C}_{\mathcal{K},\gamma} < \infty$$

where $\mathcal{C}_{\mathcal{K},\gamma}$ *is a continuous function in* $\gamma > 0$*, and* $\mathcal{K} > 0$ *as they appear in* (3.3.2), (3.3.3).

Proof. Given a starting vector $v_0 \in \mathbb{C}^{l_0}$ we define $(v_n)_n$ inductively by $v_n = X_n v_0$. Using (3.3.7) we find

$$\begin{aligned}
\|v_{n+1}\|^2 &= \langle v_0^* X_{n+1}^*, X_{n+1} v_0 \rangle \\
&= v_0^* X_n^* [S^* + a_n^* - c_n^* Z_{n+1}^*][S + a_n - Z_{n+1} c_n] X_n v_0 \\
&= \underbrace{v_n^* S^* S v_n}_{\chi_1} + \underbrace{2\Re e(v_n^* S^* a_n v_n)}_{\chi_2} + \underbrace{-2\Re e[v_n^* S^* Z_{n+1} c_n v_n]}_{\chi_3} \\
&\quad + \underbrace{v_n^* (a_n^* - c_n^* Z_{n+1}^*)(a_n - Z_{n+1} c_n) v_n}_{\chi_4}
\end{aligned}$$

Now:

$$\|v_{n+1}\|^4 = \chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2 + 2 \sum_{j=2}^{4} \sum_{i=1}^{j-1} \chi_i \chi_j$$

As $||S|| \le 1$ and $\|Z_n\| \le 1$, we first note

$$\begin{aligned}
|\chi_1| &\le \|v_n\|^2 \\
|\chi_2| &\le 2\|W_n\| \|v_n\|^2 \\
|\chi_3| &\le \|W_n\| \|v_n\|^2 \\
|\chi_4| &\le 4\|W_n\|^2 \, \|v_n\|^2
\end{aligned}$$

The problematic terms, were we can not use the expectation outside the norm are $\chi 1 \chi_2$

and $\chi_1\chi_3$. For the other terms, we remark

$$\begin{aligned}
\mathbb{E}(\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2 + 2\chi_1\chi_4 + 2\chi_2\chi_3 + 2\chi_2\chi_4 + 2\chi_3\chi_4) \;\leq& \\
\leq\; & \mathbb{E}\left(\|v_n\|^4 \left(1 + 17\|W_n\|^2 + 24\|W_n\|^3 + 16\|W_n\|^4\right)\right) \\
\leq\; & \mathbb{E}(\|v_n\|^4)\left(1 + \mathbb{E}(\|W_n\|^2)[17 + 6(e^{2\gamma} - e^{\gamma}) + (e^{2\gamma} - e^{\gamma})^2]\right) \qquad (3.3.10)
\end{aligned}$$

For the last step we use the bound (3.3.3) and the fact that W_n is independent of X_n and thus v_n.

For the frst problematic term, note that

$$\mathbb{E}(\chi_1\chi_2) \;=\; \mathbb{E}(\mathbb{E}(\chi_1\chi_2|X_n)) = \mathbb{E}\big(\chi_1 2\Re e(v_n^* S^* \mathbb{E}(a_n) v_n)\big)$$

using the fact that v_n is X_n measurable and a_n is independent of X_n. Thus

$$|\mathbb{E}(\chi_1\chi_2)| \;\leq\; 2\,\mathbb{E}(\|v_n\|^4)\,\|\mathbb{E}(W_n)\|\;. \qquad (3.3.11)$$

Now for the term $\chi_1\chi_3$ we want to use a similar estiamte. However, one of hte problem is now that Z_{n+1} actually depends on W_n and X_n. However, W_n is independent of (X_n, Z_n). Thus we want to condition on (Z_n, X_n). Furthermore, before that, in order to handle some the inverse, we use we use a resolvent identity together with (3.3.6) to find

$$Z_{n+1} \;=\; ((S + a_n)Z_n + b_n)\left(\Gamma^{-1} - (\Gamma + d_n + c_n Z_n)^{-1}(c_n Z_n + d_n)\Gamma^{-1}\right)$$

giving

$$Z_{n+1} = ((S + a_n)Z_n + b_n)\Gamma^{-1} - Z_{n+1}(c_n Z_n + d_n)\Gamma^{-1}\;.$$

Thus,

$$Z_{n+1} = S Z_n \Gamma^{-1} \;+\; M_n$$

where

$$\|M_n\| \;\leq\; 4\,\|W_n\|\,\|\Gamma^{-1}\| \;\leq\; 4\,e^{-2\gamma}\,\|W_n\|\;.$$

Splitting up Z_{n+1} this way gives

$$\mathbb{E}(\chi_1\chi_3) \;=\; -2\Re e\,\mathbb{E}(\chi_1 v_n^* S^* S Z_n \Gamma^{-1} c_n v_n) \;-\; 2\Re e\mathbb{E}(\chi_1 v_n^* S^* M_n c_n v_n)\;.$$

Using the bonds from above, we see

$$|2\Re e\mathbb{E}(\chi_1 v_n^* S^* M_n c_n v_n)| \;\le\; 8e^{-2\gamma}\mathbb{E}(\|W_n\|^2\|v_n\|^4) = 8e^{-2\gamma}\,\mathbb{E}(\|W_n\|^2)\mathbb{E}(\|v_n\|^4)\;.$$

and

$$\begin{aligned}\big|\mathbb{E}(\chi_1 v_n^* S^* S Z_n \Gamma^{-1} c_n v_n)\big| &= \big|\mathbb{E}\big(\mathbb{E}(\chi_1 v_n^* S^* S Z_n \Gamma^{-1} c_n v_n|(X_n, Z_n))\big)\big| \\ &= \big|\mathbb{E}(\chi_1 v_n^* S^* S Z_n \Gamma^{-1}\mathbb{E}(c_n) v_n)\big| \;\le\; e^{-2\gamma}\,\|\mathbb{E}(W_n)\|\,\mathbb{E}(\|v_n\|^4)\end{aligned}$$

Thus, we have in total the bound

$$|\mathbb{E}(\chi_1\chi_3)| \;\le\; \mathbb{E}(\|v_n\|^4)\;\big(2e^{-2\gamma}\|\mathbb{E}(W_n)\| + 8e^{-2\gamma}\mathbb{E}(\|W_n\|^2)\big) \tag{3.3.12}$$

In summary we find

$$\mathbb{E}(\|v_{n+1}\|^4) \le \mathbb{E}(\|v_n\|^4)(1+\alpha(\gamma)||\mathbb{E}(W_n)|| + \beta(\gamma)\mathbb{E}||(W_n)||^2)$$

where $\alpha(\gamma)$ and $\beta(\gamma)$ are some positive conitnuous functions in γ. Taking $\mathcal{C}_\gamma = \max(\alpha(\gamma),\beta(\gamma))$ we find

$$\begin{aligned}\mathbb{E}(\|v_{n+1}\|^4) &\le \|v_0\|^4 \prod_{n\ge 1}(1+\alpha(\gamma)||\mathbb{E}(W_n)|| + \beta(\gamma)\mathbb{E}||(W_n)||^2)) \\ &\le \|v_0\|^4 \exp[\mathcal{C}_\gamma(\sum_{n\ge 1}||\mathbb{E}(W_n)|| + \mathbb{E}||(W_n)||^2)] \le \|v_0\|^4\exp(\mathcal{C}_\gamma\mathcal{K})\end{aligned}$$

Use $\|X\| \le \sum_k \|Xw_k\|$ for $(w_k)_k$ being some orthogonal basis to get the result.

□

3.4 Applying the key estimates to the transfer matrices

The main point of this section will be to apply the estimates from Section 3.3 to the conjugated transfer matrices as developed in Section 3.2.

Like indicated at the end of Section 3.2 we choose some compact interval $[a, b] \subset \Sigma$ such that for $\lambda \in [a, b]$ the first l_e channels are elliptic and the other $l - l_e = l_h$ channels are hyperbolic.

In the notations of the previous sections, we have $l_0 = 2l_e + l_h$, $l_1 = l_h$ and the matrices T and S as defined in (3.3.1) are given by

$$T = \begin{pmatrix} S & \\ & \Gamma \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} e^{-iK} & & \\ & e^{iK} & \\ & & \Gamma^{-1} \end{pmatrix} .$$

where K and Γ depend analytically on $z = \lambda + i\eta \in [a, b] + i[-c, c]$. Using continuity and compactness arguments we have uniform estimates like

$$\|\Gamma^{-1}\| < e^{-2\gamma} , \quad \|Q_z\| < \mathcal{C}_Q , \quad \|Q_z^{-1}\| < \mathcal{C}_Q$$

for all $z = \lambda + i\eta$ with $\lambda \in [a, b]$ and $\eta \in [-c, c]$. This leads to

$$\|\mathcal{V}_n^z\| = \left\| Q_z^{-1} \begin{pmatrix} V_n & 0 \\ 0 & 0 \end{pmatrix} Q_z \right\| \le \mathcal{C}_Q^2 \, \|V_n\| = \mathcal{C}_Q^2 \, \|V_n(\omega)\| . \tag{3.4.1}$$

for all $z = \lambda + i\eta \in [a, b] + i[-c, c] \subset \mathbb{C}$.

In order to apply the results of Section 3.3 we need $\|\mathcal{V}_n^z\| < \frac{e^{2\gamma} - e^{\gamma}}{4}$. We therefore will replace $\mathcal{V}_n^\lambda$ by

$$W_n^z = W_n^z(\omega) = \mathcal{V}_n^z(\omega) \cdot 1_{\|V_n\| < (e^{2\gamma} - e^{\gamma})/(4\mathcal{C}_Q^2)}(\omega) \tag{3.4.2}$$

where the latter expression is the indicator function on the event that $\|V_n(\omega)\| < \frac{e^{2\gamma} - e^{\gamma}}{4\mathcal{C}_Q^2}$ on the probability space Ω. This means essentially to replace the potential V_n by

$$\widehat{V}_n = V_n \cdot 1_{\|V_n\| < \frac{e^{2\gamma} - e^{\gamma}}{4\mathcal{C}_Q^2}} .$$

Note that by the estimates above

$$\|W_n^z\| < \frac{e^{2\gamma} - e^{\gamma}}{4} \quad \text{for all} \quad z = \lambda + i\eta \in [a,b] + i[-c,c] \tag{3.4.3}$$

We modify the transfer matrices accordingly and let

$$\widehat{T}_n^z = \begin{pmatrix} A + \widehat{V}_n \,-\, z\,I & -I \\ I & \mathbf{0} \end{pmatrix} . \tag{3.4.4}$$

With these definitions we note that

$$Q_z^{-1} \widehat{T}_n^z Q_z \;=\; T^z \;+\; W_n^z \;.$$

Similarly to the products $T^z_{m,n}$ we define

$$\widehat{T}^z_{m,n} \;=\; \widehat{T}^z_n \widehat{T}^z_{n-1} \cdots \widehat{T}^z_{m+1} \widehat{T}^z_m \;.$$

and

$$\mathcal{X}^z_{m,n} \;=\; Q_z^{-1} \widehat{T}^z_{m,n} Q_z \tag{3.4.5}$$

Using the splitting into blocks of sizes $l_0 = 2l_e + l_h$ and $l_1 = l_h$ we write

$$\mathcal{X}^z_{m,n} \;=\; \begin{pmatrix} A^z_{m,n} & B^z_{m,n} \\ C^z_{m,n} & D^z_{m,n} \end{pmatrix} \tag{3.4.6}$$

and we define the Schur complements

$$X^z_{m,n} \;=\; A^z_{m,n} \;-\; B^z_{m,n} \left(D^z_{m,n}\right)^{-1} C^z_{m,n} \quad \text{and} \quad Z^z_{m,n} \;=\; B^z_{m,n} \left(D^z_{m,n}\right)^{-1} \;. \tag{3.4.7}$$

The reason that we will work with the products from some m on is that for large $n \geq m$ and some random m, we will have that $V_n = \widehat{V}_n$. More precise probabilistic arguments will be given later.

First, we need to check that the matrices W_n^z do indeed satisfy the bounds we need:

Proposition 3.4.1. *There exists $\mathcal{K} < \infty$ (depending on the chosen compact interval $[a,b] \subset \Sigma$ and the chosen $c > 0$) such that for all $z = \lambda + i\eta \in [a,b] + i[-c,c]$ we have*

$$\sum_{n=0}^{\infty} \left(\|\mathbb{E}\,(W_n^z)\| \,+\, \mathbb{E}\left(\|W_n^z\|^2\right) \right) \,\leq\, \mathcal{K}\,.$$

Proof. First we note

$$\sum_{n=0}^{\infty} \mathbb{E}\left(\|W_n^z\|^2\right) \,\leq\, \sum_{n=0}^{\infty} \mathbb{E}\left(\|\mathcal{V}_n^z\|^2\right) \,\leq\, \mathcal{C}_Q^4 \sum_{n=0}^{\infty} \mathbb{E}\left(\|V_n\|^2\right) \,=\, \mathcal{K}' \,<\, \infty\,.$$

uniformly for $z \in [a,b] + i[-c,c] \subset \mathbb{C}$, which bounds the second term as needed.

Using the Cauchy-Schwartz Inequality in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ we find

$$\begin{aligned} \mathbb{E}\,(\|W_n^z - \mathcal{V}_n^z\|) \,&=\, \mathbb{E}\left(\|\mathcal{V}_n^z\| \cdot 1_{\|V_n\| \geq \frac{e^{2\gamma}-e^{\gamma}}{4\mathcal{C}_Q^2}} \right) \\ &\leq\, \sqrt{\mathbb{E}\left(\|\mathcal{V}_n^z\|^2\right)} \sqrt{\mathbb{E}\left(1_{\|V_n\| \geq \frac{e^{2\gamma}-e^{\gamma}}{4\mathcal{C}_Q^2}}\right)}\,. \end{aligned}$$

For the first term we use (3.4.1), for the second term, we use Chebyshev's inequality

$$\mathbb{E}\left(1_{\|V_n\| \geq \frac{e^{2\gamma}-e^{\gamma}}{4\mathcal{C}_Q^2}}\right) \,=\, \mathbf{P}\left(\|V_n\| \geq \frac{e^{2\gamma}-e^{\gamma}}{4\mathcal{C}_Q^2} \right) \,\leq\, \frac{16\,\mathcal{C}_Q^4}{(e^{2\gamma}-e^{\gamma})^2}\,\mathbb{E}(\|V_n\|^2)$$

in order to get

$$\mathbb{E}\left(\left\| W_n^\lambda - \mathcal{V}_n^\lambda \right\| \right) \,\leq\, \frac{4\,\mathcal{C}_Q^4}{e^{2\gamma}-e^{\gamma}}\,\mathbb{E}(\|V_n\|^2) \tag{3.4.8}$$

for all $z \in [a,b] + i[-c,c] \subset \mathbb{C}$. Thus,

$$\|\mathbb{E}\,(W_n^z)\| \,\leq\, \|\mathbb{E}\,(\mathcal{V}_n^z)\| \,+\, \mathbb{E}\,(\|W_n^z - \mathcal{V}_n^z\|) \,\leq\, \mathcal{C}_Q\,\|\mathbb{E}\,(V_n)\| \,+\, \frac{4\,\mathcal{C}_Q^4}{e^{2\gamma}-e^{\gamma}}\mathbb{E}(\|V_n\|^2)$$

which leads to

$$\sum_{n=0}^{\infty} \left\|\mathbb{E}\left(W_n^\lambda\right)\right\| \,\leq\, \sum_{n=0}^{\infty} \left(\mathcal{C}_Q\,\|\mathbb{E}\,(V_n)\| \,+\, \frac{4\,\mathcal{C}_Q^4}{e^{2\gamma}-e^{\gamma}}\mathbb{E}(\|V_n\|^2) \right) \,=\, \mathcal{K}'' < \infty.$$

Now $\mathcal{K} = \mathcal{K}' + \mathcal{K}''$ does the job. $\square$

Thus, we can apply the results from Section 3.3.

Proposition 3.4.2. *Let $\Omega' = \{\omega \,:\, \lim_{n\to\infty} V_n(\omega) = 0\}$ which satisfies $\mathbf{P}(\Omega') = 1$.*

(i) *For all $\omega \in \Omega'$, all $m \in \mathbb{Z}_+$, and for all $z \in [a,b] + i[-c,c]$ we have,*

$$\lim_{n\to\infty} Z^z_{m,n} = 0 \;, \quad \lim_{n\to\infty} (D^z_{m,n})^{-1} = 0 \;, \quad \textit{and} \quad Y^z_m := \lim_{n\to\infty} (D^z_{m,n})^{-1} C^z_{m,n} \quad \textit{exists.}$$

(ii) *For all $\omega \in \Omega'$, all $m \in \mathbb{Z}_+$*

$$z \mapsto Y^\lambda_m \quad \textit{is analytic for } z = \lambda + i\eta \in (a,b) + i(-c,c)$$

and, uniformly in $z = \lambda + i\eta \in [a,b] + i[-c,c]$ we find

$$\lim_{m\to\infty} Y^z_m \;=\; 0 \;.$$

(iii) *We have for all $\omega \in \Omega$, and $z = \lambda + i\eta \in [a,b] + i[-c,c]$ that $\|Z^z_{m,n}\| \le 1$.*

(iv) *We find uniformly for all $\lambda \in [a,b]$, all $m \in \mathbb{Z}_+$ and all $n \ge m$, $n \in \mathbb{Z}_+$, that*

$$\sup_{n\ge m} \mathbb{E}(\|X^\lambda_{m,n}\|^4) \;\le\; \mathcal{C} < \;\infty \;.$$

for some fixed constant $\mathcal{C}$ (independent of λ, m, n).

Proof. For part (i) note that with probability 1, $\|V_n\| \to 0$ for $n \to \infty$. We let $\Omega' \subset \Omega$ be the set of probability one where $V_n = V_n(\omega) \to 0$. Then we have the same for W^z_n and the limits follow from Proposition 3.3.4.

For part (ii) note first that $(D^z_{m,n})^{-1} C^z_{m,n}$ is analytic in $z \in [a,b] + i[-c,c]$. Now, for $\omega \in \Omega'$ fixed, one sees from the estimates in Proposition 3.3.4 that the convergence of the series $\sum_{n>m} [(D^z_{m,n+1})^{-1} C^z_{m,n+1} - (D^z_{m,n})^{-1} C^z_{m,n}]$ is uniform for $z \in [a,b] + i[-c,c]$. Hence, the limiting function is analytic in z. Moreover, if for all $n > m$ we have $\|V_n\| < \varepsilon$ then one sees that with a uniform constant $\mathcal{C}_Y < \infty$, we have $\|Y_m^z\| < \varepsilon \mathcal{C}_Y$. As $V_n \to 0$ for $\omega \in \Omega'$, we find ε arbitrarily small as $m \to \infty$ and hence $\lim_{m\to\infty} Y_m^z = 0$ uniformly in z.

Part (iii) simply follows from Proposition 3.3.3 and part (iv) from Proposition 3.3.5, noting that all bounds are uniform for $\lambda \in [a,b]$ and $\|S\| \leq 1$ for $\lambda \in [a,b]$. □

Proposition 3.4.3. *There is a set of probability one, $\tilde{\Omega} \subset \Omega$, $\mathbf{P}(\tilde{\Omega}) = 1$, such that for any $\omega \in \tilde{\Omega}$ and any $m \in \mathbb{Z}_+$ we find*

$$\liminf_{n\to\infty} \int_a^b \|X_{m,n}^\lambda\|^4 \, \mathrm{d}\lambda \; < \; \infty \, .$$

Proof. By Proposition 3.4.2 (iii), Fatou lemma and Fubini theorem we find

$$\mathbb{E} \liminf_{n\to\infty} \int_a^b (||X_{m,n}^z||^4) d\lambda \leq \liminf_{n\to\infty} \int_a^b \mathbb{E}(||X_{m,n}^z||^4) d\lambda \leq \mathcal{C}(b-a) \; < \; \infty$$

Hence, $\mathbf{P}(\liminf \int_a^b \|X_{m,n}^z\| \mathrm{d}\lambda = \infty) = 0$. Then we have the result. □

3.5 Absolutely continuous spectrum

In this section we finally prove Theorem I. Recall in Proposition 3.4.2 we defined the set Ω' of probability one, where $V_n \to 0$. For $\omega \in \Omega'$ we find m such that for $n \geq m$ and all $\lambda \in [a,b] + i[-c,c]$ we have $\mathcal{V}_n^z = W_n^z$. However, the m is random. and not uniform in ω. Therefore, we define the events

$$\Omega_m \; = \; \left\{ \omega \in \Omega' \; : \; \left(\forall n \geq m \; : \; \|V_n(\omega)\| < \frac{e^{2\gamma} - e^{\gamma}}{4\mathcal{C}_Q^2} \right) \right\}$$

For $\omega \in \Omega_m$, $z = \lambda + i\eta \in [a,b] + i[-c,c]$ and $n \geq m$ we find $\widehat{T}_n^z = T_n^z$ and, hence,

$$T_{m,n}^z(\omega) \; = \; \widehat{T}_{m,n}^z(\omega) \; = \; Q_z \mathcal{X}_{m,n}^z(\omega) Q_z^{-1} \, .$$

Moreover,

$$\mathbf{P}\left(\bigcup_{m=0}^{\infty} \Omega_m \right) \; = \; \mathbf{P}\left(\Omega'\right) \; = \; 1.$$

We from now on fix some $\omega \in \Omega_{m'}$. We may still vary $m \geq m'$, but note that $\omega \in \Omega_{m'} \subset \Omega_m$ for $m \geq m'$.

The main work left to do now is to use Proposition 3.4.3 to obtain an estimate of the form as needed in Theorem 2.4.4.

Thus, given a vector $\vec{A} \in \mathbb{C}^l$ associated to some vector in the 0-th shell, we need to find vectors $\vec{u}_{\lambda,n} \in \mathbb{C}^l$ such that

$$\liminf_{n\to\infty} \int_a^b \left\| T^\lambda_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \right\|^4 \mathrm{d}\lambda \, < \, \infty \, .$$

Note that one has

$$T^z_{0,n} \begin{pmatrix} \vec{u}_{z,n} \\ \vec{A} \end{pmatrix} \; = \; Q_z \begin{pmatrix} A^z_{m,n} & B^z_{m,n} \\ C^z_{m,n} & D^z_{m,n} \end{pmatrix} Q_z^{-1} T^z_{0,m-1} \begin{pmatrix} \vec{u}_{z,n} \\ \vec{A} \end{pmatrix} \, .$$

Lemma 3.5.1. *For $Y \in \mathbb{C}^{l_h \times (l+l_e)}$ assume*

$$\operatorname{rank} \left[\begin{pmatrix} Y & I_{l_h} \end{pmatrix} Q_\lambda^{-1} T^\lambda_{0,m-1} \begin{pmatrix} I_l \\ 0 \end{pmatrix} \right] \; = \; l_h \, , \tag{3.5.1}$$

then, for any $\vec{A}$ one finds $\vec{u}_{\lambda,Y} \in \mathbb{C}^l$, $\vec{y}_Y \in \mathbb{C}^{l+l_e}$ such that

$$Q_\lambda^{-1} T^\lambda_{0,m-1} \begin{pmatrix} \vec{u}_{\lambda,Y} \\ \vec{A} \end{pmatrix} \; = \; \begin{pmatrix} \vec{y}_{\lambda,Y} \\ -Y \, \vec{y}_{\lambda,Y} \end{pmatrix} \, . \tag{3.5.2}$$

If the condition (3.5.1) is fulfilled for specific $\lambda = \lambda_0$ and $Y = Y_0$, then it is fulfilled in a neighborhood of (λ_0, Y_0). Moreover, given a fixed vector $\vec{A}$, one may get solutions $\vec{u}_Y$ and $\vec{y}_Y$ that depend continuously on (λ, Y) in a neighborhood of (λ_0, Y_0).

Using $Y = (D^\lambda_{m,n})^{-1} C^\lambda_{m,n}$ and denoting $\vec{u}_{\lambda,Y}$, $\vec{y}_{\lambda,Y}$ with $\vec{u}_{\lambda,n}$ and $\vec{y}_{\lambda,n}$ in this case, we find

$$T^\lambda_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \; = \; Q_\lambda \begin{pmatrix} X^\lambda_{m,n} \, \vec{y}_{\lambda,n} \\ 0 \end{pmatrix} \, . \tag{3.5.3}$$

Proof. Using (3.5.2) in the decomposition of $T^z_{0,n}$ above, with $z = \lambda$ and $Y = (D^\lambda_{m,n})^{-1} C^\lambda_{m,n}$, the statement (3.5.3) follows directly. Thus, we need to check that we find $\vec{u}_{\lambda,Y}$ and $\vec{y}_{\lambda,Y}$ such that (3.5.2) is satisfied. Dividing the $2l \times 2l$ matrix $Q_\lambda^{-1} T^\lambda_{0,m-1}$ horizontally in blocks of sizes l and l, and vertically into blocks of sizes $l + l_e$ and l_h we may write

$$Q_\lambda^{-1} T^\lambda_{0,m-1} = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} \quad \text{and} \quad Q_\lambda^{-1} T^\lambda_{0,m-1} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}\vec{u}_{\lambda_n} + \mathfrak{b}\vec{A} \\ \mathfrak{c}\vec{u}_{\lambda,n} + \mathfrak{d}\vec{A} \end{pmatrix}$$

Note $\mathfrak{a}, \mathfrak{b} \in \mathbb{C}^{(l+l_e)\times l}$, $\mathfrak{c}, \mathfrak{d} \in \mathbb{C}^{l_h \times l}$. Then, (3.5.2) is satisfied for $\vec{y}_{\lambda,Y} = \mathfrak{a}\vec{u}_{\lambda,Y} + \mathfrak{b}\vec{A}$ if and only if

$$\mathfrak{c}\vec{u}_{\lambda,Y} + \mathfrak{d}\vec{A} = -Y \left(\mathfrak{a}\vec{u}_{\lambda,Y} + \mathfrak{b}\vec{A} \right)$$

This is equivalent to

$$(Y\,\mathfrak{a} + \mathfrak{c})\, \vec{u}_{\lambda,Y} = (-Y\mathfrak{b} - \mathfrak{d})\ \vec{A}\,.$$

Thus, we find a solution $\vec{u}_{\lambda,Y}$ for any $\vec{A}$, if $Y\,\mathfrak{a} + \mathfrak{c}$ is surjective (as a linear map from $\mathbb{C}^l$ to $\mathbb{C}^{l_h}$), which is exactly the rank condition given in the assumption.

Note, if this is fulfilled for some specific $Y = Y_0$, and some specific spectral parameter $\lambda = \lambda_0$, then we find a matrix $M \in \mathbb{C}^{l \times l_h}$ such that $\det((Y_0\,\mathfrak{a} + \mathfrak{c})M) \neq 0$,. So in a neighborhood of Y_0 and λ_0, this determinant is still not zero and we may use

$$\vec{u}_Y = M[(Y\mathfrak{a} + \mathfrak{c})M]^{-1}\ (-Y\mathfrak{b} - \mathfrak{d})\ \vec{A}$$

and as above, $\vec{y}_{\lambda,Y} = \mathfrak{a}\vec{u}_{\lambda,Y} + \mathfrak{b}\vec{A}$. Thus, both depend continuously on (λ, Y). □

Lemma 3.5.2. *Given $\omega \in \Omega_{m'}$, and $c > \eta > 0$ fixed, there exists $\tilde{m} > m'$ such that $\forall m > \tilde{m}$ and $\forall \lambda \in [a,b]$:*

$$\text{rank} \left[\begin{pmatrix} Y^z_m & I_{l_h} \end{pmatrix} Q^{-1}_{\lambda+i\eta} T^{\lambda+i\eta}_{0,m-1} \begin{pmatrix} I_l \\ 0 \end{pmatrix} \right] = l_h\,.$$

Proof. For notation we let $z = \lambda + i\eta$ and we let $\mathcal{A}^z$ denote de matrix on the left hand side (inside the rank function). Moreover, we let

$$\mathcal{A}^z_m = \begin{pmatrix} Y^z_m & I_{l_h} \end{pmatrix} Q_z^{-1} T^z_{0,m-1} \begin{pmatrix} I_l \\ 0 \end{pmatrix}$$

Moreover, from Proposition 3.4.2 part (ii)we find that Y_m^z is uniformly small for m sufficiently big. This means, for any $\varepsilon > 0$, there exists $\tilde{m} > m'$ such that for any $n > m > \tilde{m}$ and any $z = \lambda + i\eta \in [a, b] + i[-c, c]$ we have

$$\|Y_m^z\| < \varepsilon \,.$$

The ε needed for the statement will be chosen later.

Letting $J_{0,m} = J_{0,m}(\omega)$ be the restriction of H_ω to $\ell^2(\{0, \ldots, m\}) \otimes \mathbb{C}^l$ as in (2.4.3) and defining $\beta_{0,m}^z$, $\delta_{0,m}^z$ as in (2.4.5) and (2.4.7) we find using Theorem 2.4.2

$$T_{0,m-1}^z \begin{pmatrix} I_l \\ 0 \end{pmatrix} = \begin{pmatrix} (\beta_{0,m-1}^z)^{-1} \\ \delta_{0,m-1}^z (\beta_{0,m-1}^z)^{-1} \end{pmatrix}$$

By the other ways of writing the transfer matrix, we see that $(\beta_{0,m-1}^z)^{-1}$ exists for any z, at least after analytic continuation. We also note that $\beta_{0,m-1}^z$ exists for any value z except for the eigenvalues of $J_{0,m-1}$. Thus, it exists for any $z = \lambda + i\eta$ with $\eta > 0$.

In order to prove that $\mathcal{A}^z$ is of full rank l_h, it is sufficient to prove that $\mathcal{Q}_\Gamma^{-1} \Gamma \mathcal{A}^z \mathcal{B}$ is invertible, where $\mathcal{B} \in \mathbb{C}^{l \times l_h}$. In particular, we consider

$$\mathcal{B}' = \beta_{0,m-1}^z \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} \quad \text{giving} \quad \mathcal{A}_m^z \mathcal{B}' = \begin{pmatrix} Y_m^z & I_{l_h} \end{pmatrix} Q_z^{-1} \begin{pmatrix} I_l \\ \delta_{0,m-1}^z \end{pmatrix} \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix}$$

First, take the 'limit case' and with (3.2.4) we find

$$\begin{aligned} \begin{pmatrix} 0 & I_{l_h} \end{pmatrix} Q_z^{-1} \begin{pmatrix} I_l \\ \delta_{0,m-1}^z \end{pmatrix} \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} &= \begin{pmatrix} 0 & -\mathcal{Q}_\Gamma & 0 & \Gamma^{-1} \mathcal{Q}_\Gamma \end{pmatrix} \begin{pmatrix} 0 \\ I_{l_h} \\ \delta_{0,m-1}^z \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} \end{pmatrix} \\ &= \mathcal{Q}_\Gamma \Gamma^{-1} \left(-\Gamma + \begin{pmatrix} 0 & I_{l_h} \end{pmatrix} \delta_{0,m-1}^z \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} \right) . \end{aligned}$$

were we note that by their definition, $\mathcal{Q}_\Gamma = (\Gamma^{-1} - \Gamma)^{-1}$ and Γ commute. Thus, we find

$$\mathcal{Q}_\Gamma^{-1} \Gamma \mathcal{A}_m^z \mathcal{B} = \begin{pmatrix} 0 & I_{l_h} \end{pmatrix} \delta_{0,m-1}^z \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} - \Gamma + \mathcal{R}^z$$

where

$$\mathcal{R}^z = \mathcal{Q}_\Gamma^{-1} \Gamma \begin{pmatrix} Y_m^z & 0 \end{pmatrix} Q_z^{-1} \begin{pmatrix} I_{l_h} \\ \delta_{0,m-1}^z \end{pmatrix} \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} .$$

Using $\|\delta_{m,n}^z\| \leq \frac{1}{\eta}$, where $z = \lambda + i\eta$, and compactness, we get with some uniform constant $\mathcal{C} > 0$ that

$$\|\mathcal{Q}_\Gamma^{-1}\Gamma\| \|Q_z^{-1}\| < \mathcal{C} \quad \text{and} \quad \left\| \begin{pmatrix} I_l \\ \delta_{0,m-1}^z \end{pmatrix} \right\| < 1 + \frac{1}{\eta}$$

for all $z \in [a, b] + i[-c, c]$ and all $m > \tilde{m}$. This gives

$$\|\mathcal{R}^z\| \leq \mathcal{C}\, \varepsilon\, (1 + \frac{1}{\eta})$$

for any $z = \lambda + i\eta \in [a, b] + i[-c, c]$ and any $m > \tilde{m} = \tilde{m}(\varepsilon)$. Note, $\Gamma = \Gamma(z)$ is a diagonal matrix, such that

$$\Gamma + \Gamma^{-1} = \begin{pmatrix} \alpha_{l_e+1} - z & & \\ & \ddots & \\ & & \alpha_l - z \end{pmatrix}$$

Moreover, as set above, all diagonal entries of $\Gamma(z)$ are bigger than $e^{2\gamma} > 1$. We note, that the imaginary parts of Γ^{-1} have opposite sign and an absolute value smaller than for the corresponding values of Γ. Thus, we find for $\eta > 0$ that

$$\Im(-\Gamma - \Gamma^{-1}) = \eta\, I \quad \text{implying} \quad \Im(-\Gamma) > \eta\, I \,.$$

In general, we will define the "imaginary" part in C^* algebra sense, that is $\Im(\mathcal{A}) = (\mathcal{A} - \mathcal{A}^*)/(2i)$, then

$$\Im \left[\begin{pmatrix} 0 & I_{l_h} \end{pmatrix} \delta_{0,m-1}^z \begin{pmatrix} 0 \\ I_{l_h} \end{pmatrix} \right] > 0$$

for $\eta > 0$ and $z = \lambda + i\eta$ Hence, we finally obtain

$$\Im\left(\mathcal{Q}_\Gamma^{-1}\Gamma\mathcal{A}_m^z\mathcal{B} - \mathcal{R}^z\right) \,>\, \eta\, I_{l_h}\,.$$

Thus, if

$$\varepsilon < \frac{\eta^2}{\mathcal{C}(1+\eta)} \quad \text{implying} \quad \|\mathcal{R}^z\| \,<\, \eta$$

then, the $l_h \times l_h$ matrix $\mathcal{Q}_\Gamma^{-1}\Gamma\mathcal{A}_m^z\mathcal{B}$ is invertible and we have $\operatorname{rank}(\mathcal{A}_m^z) \geq l_h$ (for any $m > \tilde{m}$). By the dimensions of $\mathcal{A}_m^z \in \mathbb{C}^{l_h \times l}$ we also have $\operatorname{rank}(\mathcal{A}_m^z) \leq l_h$. □

Theorem 3.5.3. *Let $\omega \in \Omega' \cap \tilde{\Omega}$., where $\tilde{\Omega}$ is the set as in Proposition 3.4.3. Then, there is a finite set $\{\lambda_1, \ldots, \lambda_k\}$ such that the spectrum of H_ω is purely absolutely continuous in $(a, b) \setminus \{\lambda_1, \ldots, \lambda_k\}$.*
If there is no hyperbolic channel, that is $l_h = 0$, then the spectrum is purely absolutely continuous in (a, b).

Proof. For some m' we find $\omega \in \Omega_{m'}$. Choose η with $c > \eta > 0$, take $\tilde{m} > m'$ as in Lemma 3.5.2 and consider some fixed $m > \tilde{m}$. We note that we also have $\omega \in \Omega_m$. Now, using the notation as above, $\mathcal{A}_m^z$ has full rank for $\Im m(z) = \eta$. By analyticity, the rank of $\mathcal{A}_m^z$ is full for all but finitely many values of $z = \lambda + i\eta \in [a, b] + i[-c, c]$. We may now restrict to the real line again and let $\{\lambda_1, \ldots, \lambda_k\} \subset [a, b]$ be the finite set of energies, where $\operatorname{rank}(\mathcal{A}_m^\lambda) < l_h$.

We consider now a compact interval $[a', b'] \subset [a, b] \setminus \{\lambda_1, \ldots, \lambda_k\}$. For all $\lambda \in [a', b']$ we find that $\mathcal{A}_m^\lambda$ has full rank l_h. By compactness, the set $\{\mathcal{A}_m^\lambda \,:\, \lambda \in [a', b']\}$ has some positive distance, say $\varepsilon > 0$, to the set of $l_h \times l$ matrices of non full rank.

In order to get to the point of Lemma 3.5.2, let us introduce the notations

$$\mathcal{A}_Y^z = \begin{pmatrix} Y & I_{l_h} \end{pmatrix} Q_z^{-1} T_{0,m-1}^z \begin{pmatrix} I_l \\ 0 \end{pmatrix}$$

$$Y_{m,n}^z = (D_{m,n}^z)^{-1} C_{m,n}^z \quad \text{and} \quad \mathcal{A}_{m,n}^z = \mathcal{A}_{Y_{m,n}^z}^z = \begin{pmatrix} Y_{m,n}^z & I_{l_h} \end{pmatrix} Q_z^{-1} T_{0,m-1}^z \begin{pmatrix} I_l \\ 0 \end{pmatrix}\,.$$

Again by compactness we note that $\|Q_\lambda^{-1} T_{0,m-1}^\lambda\| < \mathcal{C}$ for all $\lambda \in [a,b]$. (Note, that m is fixed now!). Thus we see that

$$\|\mathcal{A}_Y^\lambda - \mathcal{A}_m^\lambda\| \leq \mathcal{C}\|Y - Y_m^\lambda\|$$

for all $\lambda \in [a,b] \supset [a',b']$. Therefore, if

$$\|Y - Y_m^\lambda\| < \frac{\varepsilon}{\mathcal{C}} \quad \text{imlying} \quad \|\mathcal{A}_Y^\lambda - \mathcal{A}_m^\lambda\| < \varepsilon$$

then $\mathcal{A}_Y^\lambda$ is of full rank l_h.

Now, consider the compact set

$$\mathcal{S} = \left\{ \, (\lambda, Y) \, : \, \lambda \in [a', b'] \, , \, \|Y - Y_m^\lambda\| \leq \frac{\varepsilon}{2\mathcal{C}} \right\} \, .$$

By Lemma 3.5.1, for any $(\lambda', Y') \in \mathcal{S}$, we find some neighborhood $\mathcal{U}_{\lambda',Y'}$ and solutions $\vec{u}_{\lambda,Y}$, $\vec{y}_{\lambda,Y}$ to (3.5.2), that depend continuously on $(\lambda, Y) \in \mathcal{U}_{Y',\lambda'}$. Possibly shrinking the neighborhood a bit, we may assume it is compact, and thus, $\|\vec{y}_{\lambda,Y}\|$ attains a maximum in $\mathcal{U}_{\lambda',Y'}$. By compactness, $\mathcal{S}$ can be covered by finitely many such compact neighborhoods $\mathcal{U}'$. Making a specific choice in the overlaps of these finitely many neighborhood, we find piece-wise continuous functions

$$\vec{u} \, : \, \mathcal{S} \to \mathbb{C}^l \, , \, (\lambda, Y) \to \vec{u}_{\lambda,Y} \, , \quad \vec{y} \, : \, \mathcal{S} \to \mathcal{C}^{l+l_e} \, , \, (\lambda, Y) \to \vec{y}_{\lambda,Y}$$

satisfying equation (3.5.2) such that for some constant $\mathcal{C}_{\vec{y}} < \infty$ and all $(\lambda, Y) \in \mathcal{S}$ we have

$$\|\vec{y}_{\lambda,Y}\| \leq \mathcal{C}_{\vec{y}} \, .$$

As mentioned in the proof of Proposition 3.4.2 part (ii), the convergence of $Y_{m,n}^z \to Y_m^z$ for $n \to \infty$ is uniform in z, as such we find $N > 0$ such that $\forall n > N$ and all $\lambda \in [a', b']$ we have

$$\|Y_{m,n}^\lambda - Y_m^\lambda\| \leq \frac{\varepsilon}{2\mathcal{C}} \quad \text{implying} \quad (\lambda, Y_{m,n}^\lambda) \in \mathcal{S} \, .$$

Thus, for all $n > N$, and all $\lambda \in [a', b']$ we may choose

$$\vec{u}_{\lambda,n} = \vec{u}_{Y^\lambda_{m,n},\lambda}\,, \quad \vec{y}_{\lambda,n} = \vec{y}_{Y^\lambda_{m,n},\lambda}\,.$$

By (3.5.3) we obtain that

$$\left\| T^\lambda_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \right\| = \left\| Q_\lambda \begin{pmatrix} X^\lambda_{m,n}\, \vec{y}_{\lambda,n} \\ 0 \end{pmatrix} \right\| \le \|Q_\lambda\|\, \|X^\lambda_{m,n}\|\, \|\vec{y}_{\lambda,n}\| \le \mathcal{C}_Q \mathcal{C}_{\vec{y}} \|X^\lambda_{m,n}\| \tag{3.5.4}$$

for $\lambda \in [a', b']$ and all $n > N$.

Hence, using that $\omega \in \tilde{\Omega}$ we get by Proposition 3.4.3 that

$$\liminf_{n\to\infty} \int_{a'}^{b'} \left\| T^\lambda_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix} \right\|^4 \mathrm{d}\lambda \le \mathcal{C}_Q \mathcal{C}_y \liminf_{n\to\infty} \int_{a'}^{b'} \|X^\lambda_{m,n}\|^4 \,\mathrm{d}\lambda < \infty\,.$$

Hence, Theorem 2.4.4 gives that the spectral measure at $\delta_0 \otimes \vec{A}$ is purely absolutely continuous in $[a', b']$. As $\vec{A} \in \mathbb{C}^l$ was arbitrary, and the closures of $\operatorname{span}(\{(H_\omega)^k \delta_0 \otimes \vec{A} : \vec{A} \in \mathbb{C}^l,\ k \in \mathbb{N}_0\})$ is the whole Hilbert space, we find that the spectrum of H_ω is purely absolutely continuous in (a', b'). Now, the set $\Sigma' = [a, b] \setminus \{\lambda_1, \ldots, \lambda_k\}$ can be written as countable union of intervals (a', b') such that $[a', b'] \subset \Sigma'$. Therefore, the spectrum of H_ω is purely absolutely continuous in Σ'. Note that $\lambda_1, \ldots, \lambda_k$ may be eigenvalues of H_ω, but they do not have to be. If λ_j is not an eigenvalue, then the spectrum of H_ω is also purely absolutely continuous in a neighborhood of λ_j. Thus, we only need to subtract the eigenvalues from the set $[a, b]$.

Note, in the intersection of all the bands, that is, if $l_h = 0$, one has $X^\lambda_{m,n} = Q_\lambda^{-1} T^\lambda_{m,n} Q_\lambda$, $\vec{y}_{\lambda,n} = Q_\lambda^{-1} T^\lambda_{0,n} \begin{pmatrix} \vec{u}_{\lambda,n} \\ \vec{A} \end{pmatrix}$ and one can choose any family of uniformly bounded vectors $\vec{u}_{\lambda,n}$ to get pure absolutely continuous spectrum in (a, b). There is no need to subtract a finite set of values. □

Now Theorem I now essentially follows directly from the previous theorem:

Proof. First note that the set Ω' does not depend on the interval $[a, b]$ analyzed above, but $\tilde{\Omega}$ does. Using compact intervals inside Σ with rational boundary points we may write Σ

as countable union of open intervals, whose closure is inside Σ,

$$\Sigma = \bigcup_{i=1}^{\infty}(a_i, b_i) \quad \text{where} \quad [a_i, b_i] \subset \Sigma \, .$$

As Σ does not contain any band-edges, for each $j = 1 \ldots, l$ the type of the j-th channel does not change in $[a_i, b_i]$. Therefore, one can make the whole analysis as done for the compact interval $[a, b]$ above for the interval $[a_i, b_i]$. In particular, there is a corresponding set $\tilde{\Omega}_j$ of probability one for the set $[a_i, b_i]$. We then let

$$\hat{\Omega} = \Omega' \cap \bigcap_{i=1}^{\infty} \tilde{\Omega}_i$$

and note $\mathbf{P}(\hat{\Omega}) = 1$. Let $\omega \in \hat{\Omega}$ and let $\mathfrak{C} \subset \Sigma$ be compact. Using compactness, there is a finite sub-collection of these intervals, $[a_{i_k}, b_{i_k}]$, $k = 1, \ldots n$, such that

$$\mathfrak{C} \subset \bigcup_{k=1}^{n}(a_{i_k}, b_{i_k}) \, .$$

Theorem 3.5.3 gives that there is a finite set $\mathfrak{E}_k$ of eigenvalues, such that the spectrum of H_ω in $(a_{i_k}, b_{i_k}) \setminus \mathfrak{E}_k$ is purely absolutely continuous. Letting $\mathfrak{E} = \bigcup_{k=1}^{n} \mathfrak{E}_k$, which is finite, we see that the spectrum in $\mathfrak{C} \setminus \mathfrak{E}$ is purely absolutely continuous.

Due to the last comment in Theorem 3.5.3, the spectrum of H_ω is purely absolutely continuous in the intersection of all bands Σ_0 (which might be an empty set). $\square$

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