ON FINDING SURFACES VIA $\mathbb{Q}$-GORENSTEIN DEFORMATIONS AND ITS AUTOMATIZATION

by

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Contents

Introduction

For quite some time now we know that algebraic curves have "parameter" spaces, for each $g \geq 0$ there is a moduli space $\mathcal{M}_g$ that parametrizes every possible smooth curve of that genus up to isomorphism. This space is in general not compact, and so Pierre Deligne and David Mumford devised a way to naturally compactify it into the now called Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ [DM69]. The extra points in this moduli space represent curves with singularities, and these curves along with the smooth ones are called stable curves.

There is a similar story for the case of algebraic surfaces. This time if we fix two invariants: K^2, the self-intersection of the canonical class, and χ, the holomorphic Euler characteristic of the structure sheaf, then Gieseker [G77] proved the existence of a moduli space $\mathcal{M}_{K^2,\chi}$ whose points correspond to minimal surfaces of general type S with $K_S^2 = K^2$ and $\chi(S) = \chi$. In a manner analogous to the case of curves, it is possible to compactify this space into $\overline{\mathcal{M}_{K^2,\chi}}$, called now the KSBA compactification, due to Kollár, Shepherd-Barron and Alexeev (see [KSB88] and [A94]). The surfaces that can appear in $\overline{\mathcal{M}_{K^2,\chi}}$ are also called stable surfaces, and consist of those surfaces with the given invariants that have semi-log-canonical singularities and ample dualizing sheaf. Up to this day, very little is known about these moduli spaces, however we know that moduli spaces can be very complex. For example, Vakil's Murphy's law [V06] states that certain moduli spaces have arbitrary singularities. Manetti [M01] tells us that the moduli spaces of diffeomorphic surfaces of general type have arbitrarily many connected components. Recent examples ([R21]) of extra components added to $\mathcal{M}_{K^2,\chi}$, which consist of points parameterizing non-smoothable surfaces, show us that the compactification of the moduli space increases the complexity of the space even more.

One aim of this thesis is to find examples of singular surfaces in the boundary of the moduli space $\mathcal{M}_{K^2,1}$, for $K^2 = 1, 2, 3, 4$ by fixing $p_g = q = 0$ (although the methods can be extended to other invariants such as $p_g = 1$) while answering relevant questions, such as:

- Is this surface in the same component as one with smooth surfaces?
- What is the fundamental group of this surface?

The first question can be answered via $\mathbb{Q}$-Gorenstein smoothing in the following way: if there exists a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X} \to \mathbb{D}$ of a singular surface X_0, where $\mathcal{X}$ is a threefold and $\mathbb{D}$ is a disc or smooth curve, such that the general member X_t is smooth, then $\mathbb{D}^\times$ corresponds (in a vague sense) to a piece of the moduli space $\mathcal{M}_{K^2,\chi}$ that parametrizes smooth surfaces, so X_0 is in the actual boundary of this component. This method can also be seen as a proof of non-vacuity of $\mathcal{M}_{K^2,\chi}$, since at least some smooth surfaces do exist. This is the technique developed by Lee and Park (and later Shin) in [LP07, PPS09a, PPS09b].

In [R78], Reid asked if there existed simply connected surfaces with $p_g = q = 0$ and $K^2 = 1$, and conjectured that the space $\mathcal{M}_{1,1}$ should have one component for each fundamental group, where π_1 must be $\mathbb{Z}/n\mathbb{Z}$, for $n = 1, 2, 3, 4, 5$. This last conjecture is still an open problem, but the first question has a positive answer. The first example of a simply connected surface of general type with

$p_g = q = 0$ was given by Barlow in 1985 ([B85]) and had $K^2 = 1$ but had no ample canonical class, and the method used was via quotients of surfaces. The construction was actually motivated by the conjecture that asked "Are all simply connected surfaces with $p_g = q = 0$ rational?", which is answered negatively. Afterwards, Dolgachev and Werner [DW99] showed that a certain surface from Craighero and Gattazzo with $K^2 = 1$ was algebraically simply connected and had ample canonical class, and a proof of simply connectedness can be found at [RTU17]. In fact, this and the Barlow surface are the only concrete simply connected surface known in any of the $\mathcal{M}_{K^2,1}$.

The method of $\mathbb{Q}$-Gorenstein smoothings was first used for this problem by Lee and Park in [LP07], where they proved the existence of simply connected surfaces of general type with $p_g = q = 0$ and $K^2 = 2$. It was followed by [PPS09a] and [PPS09b], where Park, Park and Shin did the same with surfaces having $K^2 = 3$ and 4 respectively. Further examples where the technique is used are [PSU13] and [PPS13], among others. In [SU16], Stern and Urzúa give a (possibly) exhaustive list of every Wahl singularity known to be in the KSBA moduli spaces. In [BCP11], Bauer, Catanese and Pignatelli give a survey of what was known at the time for other surfaces with $p_g = q = 0$.

As said before, the process consists of smoothing certain singular surfaces. The singular surfaces that Lee, Park and Shin looked for were rational and had only Wahl singularities (see [W81, Example 5.9.1]). Via resolution of singularities, the problem of finding these singular surfaces is equivalent to finding smooth rational surfaces with special configurations of curves called Wahl chains. The conditions that the chains must hold are very restrictive, and carry a very discrete and combinatorial structure.

For this thesis, we wrote a computer program able to tackle this problem: we feed this program with information about a certain rational surface and configurations of curves inside of it, and it searches for ways of using these curves in many different manners in hopes of finding a surface with Wahl chains. It can be proved then that these surfaces admit a $\mathbb{Q}$-Gorenstein smoothing if certain criteria are met.

As an example of a given input and output of this program, consider the Figures 1 and 2 respectively.

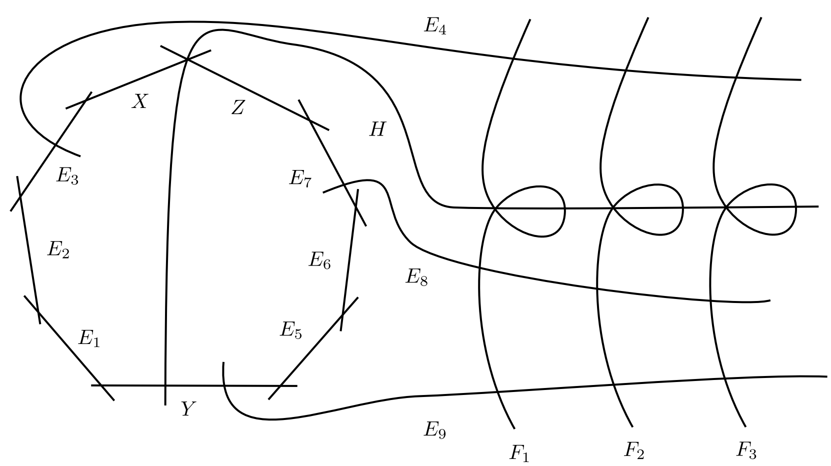


Figure 1: Input configuration

Figure 1 represents a configuration of curves in a rational elliptic surface. The program searches among many different ways to *blow up* intersections in this configuration in order to obtain a new configuration of curves, namely Figure 2. One may recognize two disjoint chains of curves in this new configuration, and it turns out that they are Wahl chains. If one contracts these chains in order

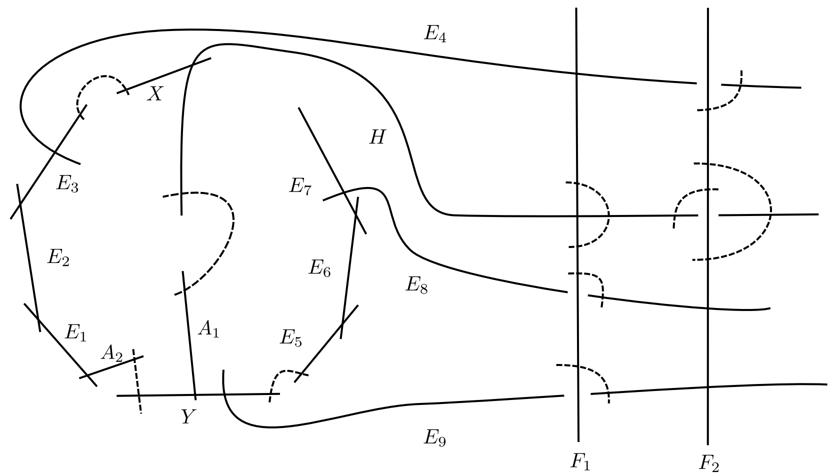


Figure 2: Output configuration

to obtain a singular rational surface, it can be verified that it admits a $\mathbb{Q}$-Gorenstein smoothing, and that it lives in the boundary of $\mathcal{M}_{3,1}$. All details regarding this example are explained in Section 5.2.

Thanks to this computer program we have been able to obtain several tens of thousands of new examples, which helped us deduce the following theorem, that gives more insight in the complexity of the moduli spaces.

Theorem 1.

- Among unobstructed singular surfaces in the boundary of $\mathcal{M}_{1,1}$ there exist at least 44 types of Wahl singularities arranged in at least 214 combinations.
- Among unobstructed singular surfaces in the boundary of $\mathcal{M}_{2,1}$ there exist at least 508 types of Wahl singularities arranged in at least 3595 combinations.
- Among unobstructed singular surfaces in the boundary of $\mathcal{M}_{3,1}$ there exist at least 2104 types of Wahl singularities arranged in at least 10169 combinations.
- Among unobstructed singular surfaces in the boundary of $\mathcal{M}_{4,1}$ there exist at least 1246 types of Wahl singularities arranged in at least 2454 combinations.

As a stable surface may have more than one Wahl singularity, there may arise many different combinations of types of singularities. The complexity can be understood in a more precise way: each unobstructed singularity that appears corresponds to a divisor in the given component of the moduli space.

One may notice that the amount of examples roughly increases with the K^2, but surprisingly decreases at the $K^2 = 4$ point. This may be explained in the following way. The complexity of the configuration of curves tends to increase strictly with the K^2. For low K^2, since this complexity must be low, there is less room for wilder of bigger configurations, so not many examples arise. For high K^2, although there is more room for possibilities, the amount of "coincidences" that need to happen also increases, which start to limit the amount of examples.

We note that in the literature there were essentially only two examples with $K^2 = 4$ ([PPS09b]), with some deformations of them appearing in [HTU17], and now there are thousands. We also got very unexpected results. As we hinted before, our objective was to find surfaces with $K^2 = 1, 2, 3, 4$, because by this method, the expected dimension of the moduli space around the singular surface is $10 - K^2$, so for $K^2 \geq 5$ it is impossible to find examples with no obstructions to deformations. However we have indeed found (obstructed) surfaces with $K^2 = 5$. This leads to the next aim of this thesis, which is much more topological in nature.

Thanks to Freedman [F82], we know very well the homeomorphism type of simply connected surfaces, and as it turns out, many of them are homeomorphic, but not diffeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$, the projective plane blown up n times. Such surfaces are said to be exotic. It is well known that under some conditions, if X is the rational blow down of some Wahl chain in a smooth surface Y with $p_g = q = 0$, then X is an exotic $\mathbb{CP}^2\#(9 - c_1^2(X))\overline{\mathbb{CP}^2}$. Here, $c_1^2(X)$ coincides with K_Z^2 if Z is the contraction of the Wahl chains in Y.

With this we are able to still use these unexpected surfaces to answer questions of existence of exotic structures. Although there are already examples in the literature of exotic structures with n as low as 2 (see for instance [AP10]), and although we have not found yet surfaces with $K_Z^2 \geq 6$, we do have the first example of such an exotic surface obtained via rational blow down surgery with $n = 4$.

This partially answers the following question of existence asked in [BKS22, Question 2]: *"Is there an exotic $\mathbb{CP}\#m\overline{\mathbb{CP}^2}$ with $m < 5$ that can be obtained from a standard rational surface via rational blow downs? If so, what is the smallest such m?"*. We leave open the question for $m < 4$, perhaps it is only a matter of time given the program we have.

The program can also be used in cases when $\chi > 1$, as with the case $p_g = 1, q = 0$. We have already used it in [RU21] by using suitable K3 surfaces instead of rational elliptic fibrations. There we deduced the following theorem:

Theorem 2. There exist complex algebraic simply connected surfaces with $p_g = 1$, $q = 0$ with $K^2 = 1, 2, 3, 4, 5, 6, 7, 8, 9$.

We note that in the literature ([PPS13]) there only existed examples with $K^2 = 1, 2, 3, 4, 5, 6, 8$, so the cases $K^2 = 7, 9$ are new.

If now X is the rational blow down of a surface Y with $p_g = 1$, then, in a similar way s before, X is an exotic $3\mathbb{CP}^2\#(19 - K_Z^2)\overline{\mathbb{CP}^2}$. We also found unexpected surfaces with $K^2 = 10, 11, 12$, so that we have:

Theorem 3. There exist exotic $3\mathbb{CP}\#n\overline{\mathbb{CP}^2}$ with $n \in \{7, 8, 9\}$ obtained via rational blow down surgery.

As before, in the literature there existed examples with n as low as 4 (for instance, see [AP10]), but the minimum obtained via rational blow down surgery up to this day was 11 in [PPS13].

One final application of this program was to find examples of algebraically simply connected surfaces of general type in positive characteristic. We were able to complete the main theorem in [LN12], which now reads:

Theorem 4. For any algebraically closed field $\Bbbk$ and integer $1 \leq K^2 \leq 4$, there exists an algebraically simply connected minimal surface S of general type over $\Bbbk$ with $p_g(S) = q(S) = H^2(S, \mathcal{T}_{S/\Bbbk}) = 0$, $K_S^2 = K^2$ and K_S ample.

The problem of this proof lied in the absence of examples of simply connected surfaces with $p_g = q = 0$ and $K^2 = 1, 2, 4$, so for the pairs $(p, K^2) = (2, 1), (2, 2), (2, 4)$ they could not prove the existence of such surfaces in characteristic p with $K_S^2 = K^2$. Additionally, the example they gave for the pair $(p, K^2) = (3, 4)$ had a small inaccuracy, so in order to complete the theorem, we also found an example that fills this gap.

The last three theorems will not be treated on this paper, but can be deduced from the list of examples we provide along with the program. They will appear in future papers.

We now outline the structure of this thesis. The second chapter of this thesis introduces our main object of study, the Wahl singularities and Wahl chains, and we give several properties that will be useful in the computer program. The third chapter gives a small review of deformation theory, and a strategy to determine when a given example has no obstructions to deformations, so that singularities can be smoothed out. The fourth chapter deals with the topology of our examples. We explain the rational blow down surgery, first introduced in [FS95], and a strategy to determine when the fundamental group of our example is trivial. We also deal with exotic structures and prove the before mentioned theorem. The fifth chapter introduces us with the rational surfaces we will be using as a base to find all our examples. We give an example of a surface constructed with this method, and go into detail explaining how all the tools we developed work. We conclude the chapter with further combinatorial properties which are fundamental to the program. The sixth chapter gives an overview of what the program does. Since the complete source code and all the examples we found are far too big for this thesis, they can be found at an online repository, given in the Appendix. We finalize this thesis with three interesting examples, including one of the unexpected surfaces with $K^2 = 5$ we have found.

Chapter 1

Preliminaries

1.1 Basic Definitions

In this section we will go though some definitions and basic properties of algebraic varieties and surfaces in particular. These definitions will be given for varieties over $\mathbb{C}$ where most properties are well behaved, and where we have some intuition from complex analysis. They, however, are also valid over other algebraically closed fields. Most of these definitions are standard and no proof of some claims will be given.

Definition 1.1.1. A subset $X \subseteq \mathbb{A}^n$ is algebraic if it is the zero set of a set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. A subset $X \subseteq \mathbb{P}^n$ is (projective) algebraic if it is the zero set of a set of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. The family of all algebraic sets in $\mathbb{A}^n$ or $\mathbb{P}^n$ form the closed sets of the *Zariski topology*. A set is quasi-affine (resp. quasi-projective) if it is an open set of a closed algebraic subset of $\mathbb{A}^n$ (resp. $\mathbb{P}^n$), thus all quasi-affine sets are also quasi-projective. A quasi-projective set is irreducible if every nonempty open set in the Zariski topology is dense. Irreducible quasi-projective sets are also called *varieties*. A projective variety is a variety isomorphic to a closed set in some projective space.

As it turns out, all quasi-projective sets have a *smooth locus*, that is, an open set of points that is locally bi-holomorphic to an open set of $\mathbb{C}^n$ for some n.

Definition 1.1.2. Let X be a variety. Then the number n defined above is constant for every point in the smooth locus, and is called the *dimension* of X. A variety of dimension 1 is called a *curve*, a variety of dimension 2 is called a *surface*, a variety of dimension 3 is called a *threefold*, and so on.

The dimension defined above does not correspond to the usual notion we have. For instance, a curve, which locally looks like an open set in $\mathbb{C}$, must be two-dimensional as a real manifold. In fact, smooth projective curves are precisely the closed Riemann Surfaces.

Definition 1.1.3. Let X, Y be two varieties. A morphism $f: X \to Y$ is a function that locally looks like polynomial functions in the coordinates of X. A rational map $f: X \dashrightarrow Y$ is a partial function defined on a dense open subset $X' \subseteq X$ that is a morphism restricted to X'. X' is usually taken to be maximal among all open subsets where f can be defined. A rational map (or morphism) is *dominant* if its image is a dense subset of Y. It is called birrational if there is a rational map $g: Y \dashrightarrow X$ such that $f \circ g$ and $g \circ f$ are the identities in the open sets where the are defined (and so, they are actually the identity function thanks to the maximality of these open sets).

Definition 1.1.4. A rational function f on a variety is a meromorphic function, that is, a partial function defined locally as fractions of polynomials, where the denominator must not be zero everywhere. The set of rational functions is denoted by $K(X)$.

Definition 1.1.5. A variety is said to be regular in codimension n if the singular locus has codimension at least n.

Definition 1.1.6 ([H70, II.6])**.** Let X be a variety, regular in codimension 1. A *prime Weil divisor* $D \subseteq X$ is a subvariety of codimension 1 in X. An example of a Weil divisor is an irreducible curve inside a surface.

A *Weil divisor* is an element of the formal direct sum of all prime divisors, and the set of Weil divisors is called $\operatorname{Div} X$. A non-zero rational function f on X gives rise to a Weil divisor $\operatorname{div} f$, which is called a *principal divisor* given by

$$\operatorname{div} f = \sum_{D \subseteq X} v_f(D) D$$

where $v_f(D)$ is the order of zero or pole of f along D. This notion turns out to be well defined and the sum is finite. The principal divisors form a subgroup of $\operatorname{Div} X$, and we say that D is rationally equivalent do D', or $D \sim D'$ if their difference is a rational divisor. Denote by $\operatorname{Cl} X$ -the Class group of X- the quotient of $\operatorname{Div} X$ by the principal divisors.

The support of a divisor D, denoted by $\operatorname{Supp} D$ is the union of the subvarieties which have non-zero coefficient in D.

A divisor D is effective if every prime divisor in D appears with non-negative multiplicity. It is reduced if every prime divisor in D has multiplicity 1. Given an effective divisor $D = \sum n_i D_i$, with $n_i > 0$, we can define the reduced divisor $D_{red} = \sum D_i$.

Definition 1.1.7. Let X be a variety regular in codimension 1. A *Cartier divisor* is a Weil divisor that is locally the $\operatorname{div}(f)$ for some f. A Weil divisor D is said to be $\mathbb{Q}$-Cartier if nD is a Cartier divisor for some $n > 0$.

In fact, the notion of Cartier divisor can be extended to varieties not necessarily regular in codimension 1. The way to do this is forgetting that there is a Weil divisor and only remember the information of the $f \in K(X)$ in an open cover of X, which should be compatible so that they define the same divisor everywhere.

Definition 1.1.8. Let X be a variety. A Cartier divisor is determined by the following data: $\{(U_i, f_i)\}$ where $U_i \subseteq X$ be an open cover of X and $f_i \in K(U_i) = K(X)$ rational functions such that $\frac{f_i}{f_j}$ is an invertible holomorphic function on $U_i \cap U_j$, that is, it has no zeros nor poles.

A Cartier divisor is principal if it is determined by $\{(X, f)\}$ for some $f \in K(X)$. The set of classes of Cartier divisors modulo principal divisors is denoted $\operatorname{CaCl} X$.

Of course, there could be different covers and different functions that define the same divisor, we leave the details to [H70, II.6] where the definitions are made in terms of sheaves. If the variety X is non-singular, then Cartier divisors and Weil divisors coincide. There is a final notion strongly related to divisors that is important for our purposes.

Definition 1.1.9. Let X be a variety not necessarily regular in codimension 1. A line bundle on X is a fibration $\pi \colon L \to X$ that locally looks like the projection $U \times \mathbb{C} \to U$. Let $\operatorname{Pic} X$ be set of isomorphism classes of line bundles of X together with tensor product as an operation. As it turns out, $\operatorname{Pic} X$ is a group (see [H70, Proposition II.6.12]).

A Cartier divisor D determines a line bundle $\mathcal{O}(D)$ by glueing $U_i \times \mathbb{C}$ and $U_j \times \mathbb{C}$ through multiplication by $\frac{f_j}{f_i}$.

For an open set $U \subseteq X$, the set of algebraic sections of π restricted to U is $\Gamma(U, L)$. A section is a function $\sigma\colon U \to L$ such that $\pi \circ \sigma = \mathrm{id}_U$. For the trivial line bundle $X \times \mathbb{C}$ we simply say $\Gamma(U)$. Its sections are precisely the algebraic holomorphic functions on X.

Proposition 1.1.10. Let D_1,D_2 Cartier divisors on X. Then $D_1 \sim D_2$ if and only if $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ as line bundles.

Proof. See [H70, Proposition II.6.13]. □

Proposition 1.1.11. If X is a variety, the morphism $\operatorname{CaCl} X \to \operatorname{Pic} X$ given by $D \mapsto \mathcal{O}(D)$ is an isomorphism

Proof. See [H70, Proposition II.6.15]. □

To see how this works, let $s \in \Gamma(U, L)$ be a non-zero local section, which must exist for some U that trivializes π. Then s extends uniquely as a meromorphic function $X \to L$. We let D be the divisor of zeros minus the divisor of poles of s.

An important line bundle (and divisor) is the canonical line bundle of a variety.

Definition 1.1.12. Let X be a variety of dimension n regular in codimension 1. Let $X_{smooth} \subseteq X$ the smooth locus of X. Then $\pi\colon \bigwedge^n \Omega_X = \omega_X \to X_{smooth}$ is the line bundle of n-forms over X_{smooth}. We let K_X be a Weil divisor defined by ω_X. K_X is called a canonical divisor, and if X is smooth,ω_X is called the canonical bundle.

When X is singular, then ω_X may not extend to a line bundle on the whole X, but sometimes $\omega_X^{\otimes m}$ does. This gets translated into that mK_X is a Cartier divisor, so K_X is $\mathbb{Q}$-Cartier. When this happens, we say that X is $\mathbb{Q}$-Gorenstein.

The notion of a canonical line bundle can be generalized to singular varieties thanks to the dualizing sheaf ([H70, pp 241]), which is also called ω_X.

Whenever we have a morphism $f\colon Y \to X$ and a line bundle $\pi\colon L \to X$, we can define the pullback line bundle $f^*L \to Y$ where the fiber over every point $y \in Y$ coincides with the fiber $\pi^{-1}(f(y))$. When f is an inclusion of a subvariety, we say $f^*L = L|_Y$. We can also define the pullback of a Cartier divisor, where $f^*(\{U_i, f_i\}) = \{f^{-1}(U_i), f_i \circ f\}$, as long as $\operatorname{Supp} D$ does not contain the image of f. The contention $\operatorname{Supp} f^*(D) \subseteq f^{-1}(\operatorname{Supp} D)$ is clear. This pullback defined in $\operatorname{Div} X$ (or $\operatorname{Pic} X$) has a natural extension to $\mathbb{Q}$-divisors in $\operatorname{Div} X \otimes \mathbb{Q}$ (or $\operatorname{Pic} X \otimes \mathbb{Q}$).

Definition 1.1.13. Let X be a smooth complete curve and $D = \sum_i n_i P_i$, where $P_i \in X$ are points. We define the degree of D, $\deg D = \sum_i n_i$.

Equivalently, if $\pi\colon L \to C$ is a line bundle, $\deg E$ is defined as the number of zeros minus the number of poles of a non-zero meromorphic section s of π. This number is independent of the choice of the s.

Proposition 1.1.14. Let X be a complete smooth curve. Then a principal divisor has degree 0. In particular, deg is defined in $\operatorname{Cl} X$

Proof. This is [H70, Corollary II.6.10]. □

An important example of the above is the degree of the canonical divisor (or canonical bundle) on a smooth curve.

Proposition 1.1.15. Let X be a smooth complete curve of genus g. Then $\deg K_X = 2g-2$.

Proof. This is a consequence of the Riemann-Roch Theorem. See [H70, Example IV.1.3.3]. $\square$

Definition 1.1.16. Let X be a smooth surface. A configuration of curves is an (effective) reduced divisor $D = C_1 + \ldots + C_n$. It has normal crossings if has transversal intersections, that is, every singularity in D is locally analytically isomorphic to the singularity $\{(x, y) \mid xy = 0\}$. It is simple normal crossing (snc) if in addition, every curve C_i is smooth.

Definition 1.1.17. Let $\mathcal{C}_0$ be a possibly non-snc divisor in a smooth surface X. A log resolution of $\mathcal{C}_0$ is a sequence of blow ups $\pi \colon Y \to X$ such that $\mathcal{C} = (\pi^*\mathcal{C}_0)_{\mathrm{red}}$ is a snc configuration. π is a minimal log resolution if whenever a π-exceptional (-1)-curve is contracted, the image of $\mathcal{C}$ is not snc.

We conclude this section with a very useful construction that we will use extensively, the blow up of a non-singular point.

Definition 1.1.18. Let X be a variety of dimension n and $x \in X$ be a non-singular point. The blow up of x is a morphism $\pi \colon \mathrm{Bl}_{x,X} \to X$ such that $\pi|_{\pi^{-1}(X-x)}$ is an isomorphism and $E = \pi^{-1}(x)$ parametrizes the tangent directions of X at x, so that $\pi^{-1}(x) \cong \mathbb{P}^{n-1}$. E is called the exceptional divisor of the blow up.

For an explicit construction of the blow up (as in [B96, II.1]), let $x_1, \ldots, x_n \in \Gamma(U)$ be local parameters of x such that x is the only common zero in a neighborhood U of x. Then define $\pi|_{\pi^{-1}(U)} \colon \mathrm{Bl}_{x,X} \cap \pi^{-1}(U) \to U$ as the restriction of the projection $U \times \mathbb{P}^{n-1} \to U$ to the subvariety defined by the equations $\{x_i y_j = x_j y_i\}$, where $y_1, \ldots, y_n$ are the homogeneous coordinates of $\mathbb{P}^{n-1}$. Note that π is an isomorphism outside x since at least one parameter is invertible, so we can glue π with the identity of the open set $X - x$. For a more general approach, refer to [H70, II.7].

Sheaves over an algebraic variety, (and in particular line bundles) have a cohomology theory, where the cohomology groups of a sheaf (or line bundle) L over X are denoted by $H^i(X, L)$, $i \geq 0$. In particular, $H^0(X, L) = \Gamma(X, L)$. When X is a projective variety of dimension n, all cohomology groups are finite dimensional vector spaces over $\mathbb{C}$, and all groups $H^i(X, L)$ vanish for $i > n$ (this is [H70, Theorem III.2.7]). The dimension of $H^i(X, L)$ is denoted by $h^i(X, L)$. The Euler characteristic of a sheaf is defined as $\chi(L) = \sum_{i \geq 0} (-1)^i h^i(X, L)$, in particular, the Euler characteristic of an irreducible surface X is

$$\chi(X) = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - q(X) + p_g(X),$$

where $q(X) = h^1(\mathcal{O}_X)$ is the irregularity of X and $p_g(X) = h^2(\mathcal{O}_X)$ is the geometric genus of X.

Theorem 1.1.19 (Serre Duality)**.** Let X be a smooth variety of dimension n and L a locally free sheaf on X. Then there are natural isomorphisms

$$H^i(X, L) \cong H^{n-i}(X, \omega_X \otimes L^\vee)^\vee$$

Proof. This is a particular case of the duality for projective schemes, [H70, Theorem III.7.6 and Corollary III.7.7]. $\square$

In particular, if D is a divisor on a smooth surface, we have $h^2(D) = h^0(K_X - D)$.

1.2 Review of Intersection Theory for Smooth Surfaces

Let us suppose now that X is a non-singular complete surface, and let Y, Z be two different prime divisors. We will define their intersection $Y \cdot Z \in \mathbb{Z}$ and then extend it to $\operatorname{Div} X$. We first give a naive definition that requires a strong assumption on Y and Z.

Definition 1.2.1. Suppose that Y and Z intersect transversely, that is, at each point of intersection it is locally isomorphic (in the analytic topology) with the crossing $\{xy = 0\} \subseteq \mathbb{C}^2$. We define $Y \cdot Z$ to be the number of intersection points of Y and Z.

Of course we would like to extend our definition for every pair of divisors. As it turns out this is possible.

Theorem 1.2.2. There exists a unique symmetric bilinear pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ that extends $Y \cdot Z$ when Y and Z are smooth and transversal such that whenever $C_1 \sim C_2$ and $D_1 \sim D_2$, then $C_1 \cdot D_1 = C_2 \cdot D_2$.

Proof. This is [H70, Theorem V.1.1]. □

This is thanks to the Moving Lemma, which states that any divisor in a smooth surface is linearly equivalent to the difference of two smooth curves, and these curves can be chosen transversal to any other finite set of curves. This lemma is found implicitly in the proof of [H70, Theorem V.1.1]. The fact that this intersection form is defined for every pair of curves means that we can talk about the self-intersection of a curve C, denoted by $C^2 = C \cdot C$. For example, an important invariant of a surface X is K_X^2.

As an example of self-intersection of a curve, consider a line $L \subseteq \mathbb{P}^2$. This line is linearly equivalent to any other line $L' \subseteq \mathbb{P}^2$, and any two different lines in $\mathbb{P}^2$ intersect each other at one point transversally. Therefore $L^2 = L.L' = 1$. Also, it is well known that $K_{\mathbb{P}^2} \sim -3L$, so $K_{\mathbb{P}^2}^2 = 9$.

Definition 1.2.3. By a $(-n)$-curve, we mean a smooth rational curve E with $E^2 = -n$.

There is also intrinsic way of defining the intersection between two curves:

Proposition 1.2.4. Let X be a smooth surface and $C, D \subseteq X$ be two irreducible curves, C smooth and projective. Then

$$C \cdot D = \deg(\mathcal{O}(D)|_C) = \deg(\mathcal{O}_C(D))$$

Theorem 1.2.5 (Adjunction Formula)**.** Let X be a smooth surface and $C \subseteq X$ a smooth complete curve. Then

$$\omega_C = (\omega_X \otimes \mathcal{O}(C))|_C.$$

This translates to the numerical equality

$$2g(C) - 2 = \deg K_C = (K_X + C) \cdot C = K_X \cdot C + C^2$$

This equality is also true for singular complete curves, where $g(C)$ is replaced by the arithmetic genus of C.

Proof. This is [H70, Proposition II.8.20] and [H70, Proposition V.1.5]. □

We now compare the intersection theories of a surface and the blow up at one of its points.

Definition 1.2.6. Let $x \in X$ and $\pi \colon Y \to X$ the blow up at x. Let $C \subseteq X$ be a curve. The strict transform of C is $\hat{C} = \overline{f^{-1}(C - x)}$. The strict or proper transform of a divisor is defined by linearity.

Proposition 1.2.7. Let X be a smooth surface and $x \in X$. Let $\pi\colon Y \to X$ be the blow up at x, and let E be its exceptional divisor. Then the following holds

1. If $C \subseteq X$ has multiplicity m at x, then $\pi^*C = \hat{C} + mE$.
2. If D is a divisor on X, then $\pi^*D \cdot E = 0$.
3. If D, D' are divisors in X, then $(\pi^*D) \cdot (\pi^*D') = D \cdot D'$.
4. $E^2 = -1$.
5. $K_Y = \pi^*K_X + E$.
6. $\operatorname{Pic} Y \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Proof. This is [B96, Lemma II.2 and Proposition II.3] or [H70, Propositions V.3.2 and V.3.3]. For the sake of having some calculations, we may assume 1. and prove the rest. Let $D \subseteq X$. By choosing a divisor D' in the rational equivalence class of D that does not pass through x, we see that $\pi^*D \cdot E = \pi^*D' \cdot E = 0$ since π^*D' and E do not intersect. This proves item 2. We can prove item 3 in the same way, choosing D'' and D''' in the rational equivalence classes of D and D' that are transversal to each other and do not pass through x. Then since π is an isomorphism outside x, $\pi^*D'' \cdot \pi^*D''' = D'' \cdot D'''$.

Let C be a curve with multiplicity 1 at x. Then $\pi^*C = \hat{C} + E$. $\hat{C}$ intersects E transversely, since E parametrizes tangent directions at x and C has a single tangent direction at x. Therefore, $0 = \pi^*C \cdot E = \hat{C} \cdot E + E^2 = 1 + E^2$. This proves item 4.

Since π is an isomorphism outside x, then ω_X and ω_Y are isomorphic outside x and E. by choosing a representative of K_X that do not pass though x, we must have $K_Y = \hat{K}_X + kE = \pi^*K_X + kE$ for some k. By intersecting with E we get $K_Y \cdot E = kE^2 = -k$, but since E is a smooth curve of genus 0, by adjunction formula we have $K_Y \cdot E = -2 - E^2 = -1$. This proves $k = 1$. This is item 5.

To prove item 6 we note that the morphism $\operatorname{Pic} X \oplus \mathbb{Z} \to \operatorname{Pic} Y$ given by $(D, n) \mapsto \pi^*D + nE$ is surjective, since every curve in Y is either E or of the form $\hat{C}$. We also use item 1. To see it is injective, suppose that $\pi^*D + nE = 0$. Then $0 = E \cdot (\pi^*D + nE) = -n$ so $n = 0$. We note that the image of π^*D by π is precisely D, so if $\pi^*D = 0$, then $D = 0$. □

Corollary 1.2.8. Suppose C has multiplicity n and D has multiplicity m at x. Then

$$\hat{C} \cdot \hat{D} = C \cdot D - nm \quad \text{and} \quad \hat{C} \cdot E = n.$$

In particular, $\hat{C}^2 = C^2 - n^2$. Also,

$$K_Y^2 = K_X^2 - 1.$$

Proof. Since $\pi^*C = \hat{C} + nE$ and $\pi^*D = \hat{D} + mE$, then

$$0 = \pi^*C \cdot E = \hat{C} \cdot E + nE^2 = \hat{C} \cdot E - n, \quad \text{so} \quad \hat{C} \cdot E = n.$$

Also,

$$\hat{C} \cdot \hat{D} = (\pi^*C - nE)(\pi^*D - mE) = \pi^*C \cdot \pi^*D - m\pi^*C \cdot E - n\pi^*D \cdot E + nmE^2 = C \cdot D - nm.$$

Since $K_Y = \pi^*K_X + E$, then $K_Y^2 = (\pi^*K_X + E)^2 = K_X^2 + E^2 = K_X^2 - 1$. □

Now that we know that the exceptional curve of a blow up has self-intersection -1, we can state the inverse operation.

Theorem 1.2.9 (Castelnuovo's Contraction Theorem [B96, Theorem II.17])**.** Let $E \in X$ be a smooth rational curve with $E^2 = -1$. Then there is a smooth surface Y, $P \in Y$ and a morphism $\pi\colon X \to Y$ such that π is the blow up at P and the exceptional curve is E. This operation that contracts E is called a blow down.

1.3 Intersection Theory for Normal Surfaces

The question of existence of an intersection theory naturally extends to the case of singular surfaces. An intersection theory always exists in the case of Cartier Divisors, however it is not obvious to extend it to Weil divisors. Mumford [M61, II (b)] proposes the following solution for normal surfaces.

Let X be a normal surface and $\pi\colon Y \to X$ a resolution of singularities. Such a resolution always exists, and has smooth exceptional curves $E_1, \ldots, E_n$, intersecting each other transversally. Here Y is a smooth surface, so we aim to translate the problem of intersecting curves in X to intersecting them in Y.

Theorem 1.3.1. The matrix $(E_i \cdot E_j)_{ij}$ is negative definite.

Proof. This is [M61] page 230. At least the case with a single singularity. The general case is a formal consequence. □

Definition 1.3.2. Let $A \subseteq X$ be a divisor, and let $A_0 \subseteq Y$ its proper transform. Define the pullback $\pi^*(A)$ by $A_0 + \sum r_i E_i$, where the r_i satisfy the relations

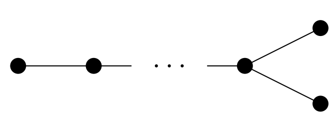
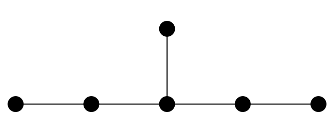
$$0 = E_j \cdot \pi^*(A) = E_j \cdot A_0 + \sum r_i (E_i \cdot E_j)$$

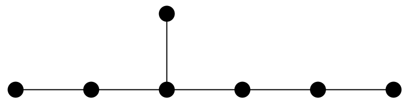
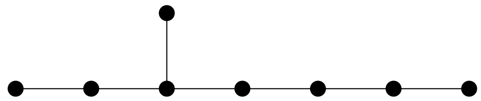
By Theorem 1.3.1, these r_i are well defined and unique rational numbers. We can finally define $A \cdot B = \pi^*(A) \cdot \pi^*(B)$. As it turns out, this definition does not depend on the resolution $Y \to X$ and has the property that $\pi^*(A_1) \sim \pi^*(A_2)$ whenever $A_1 \sim A_2$. So in particular, if $A_1 \sim A_2$ and $B_1 \sim B_2$, then $A_1 \cdot B_1 = A_2 \cdot B_2$. It also has the property that every r_i is non-negative if A is effective.

Note that since the E_i are exceptional, $A \cdot B = \pi^*(A) \cdot \pi^*(B) = A_0 \cdot \pi^*(B)$.

1.4 Du Val Singularities and ADE Classification

There are certain families of singularities that may appear in surfaces. These are the so-called *rational double points* or *Du Val singularities*, and for various reasons these are regarded as the simplest among all surface singularities (see [I18, §7.5]). As with any singularity, they are uniquely determined by the configuration of curves arising in a minimal resolution. The following is classification of all such possible exceptional divisors:

- A_n : 
- D_n : 
- E_6 : 

- E_7 : 
- E_8 : 

Here are denoted the *dual graphs* of the respective configuration of curves. Nodes in the graph correspond to curves and two nodes are connected if and only if the curves intersect, and if two curves intersect, they only do it once. All intersections are transversal, and all curves have self-intersection -2.

These graphs are precisely the simply laced Dynkin diagrams, which in turn are classified as one of the families A-D-E.

Chapter 2

T-Singularities

Before dealing with T-singularities, we define a broad class of singularities that may appear in surfaces.

Definition 2.0.1 ([KM98])**.** Let (X, Δ) be a log pair, that is, X is a variety and Δ is a $\mathbb{Q}$-divisor. Assume that X is normal and that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then (X, Δ) is said to be log-canonical if the following situation holds:

Let $\nu \colon Y \to X$ be a log-resolution of (X, Δ), that is, Y is smooth, relatively minimal and the divisor $f^{-1}(\operatorname{Supp}\Delta) + \operatorname{exc}\nu$ is simple normal crossing. Let $\tilde{\Delta}$ the strict birrational transform of Δ, and write $K_Y + \tilde{\Delta}$ as

$$K_Y + \tilde{\Delta} = \nu^*(K_X + \Delta) + \sum_{E_i \subseteq \operatorname{exc}\nu} a_i E_i$$

Then for every $E_i \subseteq \operatorname{exc}\nu$, $a_i \geq -1$.

A normal surface X is said to have log-canonical singularities if $(X, 0)$ is a log-canonical pair.

In this thesis we will only deal with log-canonical (in fact log-terminal, which require $a_i > -1$) singularities, but in order to fully describe the KSBA moduli space we need a slightly more general class of singular surfaces, whose singularities are called semi-log-canonical (slc). For a rigorous definition, refer to [KSB88]. Surfaces with slc singularities and ample dualizing sheaf are called stable surfaces, and they are the kind of singularities that appear in the compactification of $\mathcal{M}_{K^2,\chi}$.

2.1 Cyclic Quotient Singularities

Definition 2.1.1. Let m, q be coprime integers with $1 \leq q < m$. Let $\mu_m \in \mathbb{C}$ be a primitive m-th root of unity and consider the cyclic group G of automorphisms of $\mathbb{C}^2$ generated by $(x, y) \mapsto (\mu_m x, \mu_m^q y)$. The point $0 \in \mathbb{C}^2/G$ is a rational singularity ([I18, Definition 6.2.10 and Corollary 7.4.10]) called *cyclic quotient singularity*, and any surface singularity analytically isomorphic to it is said to be of type $\frac{1}{m}(1, q)$.

In order to give an algebraic-geometric description of the quotient $\mathbb{C}^2/G$, we note that G acts naturally on the coordinate ring $\mathbb{C}[x, y]$ of $\mathbb{A}^2$: if $g \in G$ and $f \in \mathbb{C}[x, y]$ is a polynomial, then

$$g \cdot f = f \circ g.$$

With this in mind, the quotient space is realized as

$$\mathbb{C}^2/G = \operatorname{Spec}(\mathbb{C}[x, y]^G)$$

where $\mathbb{C}[x,y]^G \subseteq \mathbb{C}[x,y]$ is the $\mathbb{C}$-sub algebra of polynomials that are G-invariant. Note that G is a linear group acting on $\mathbb{C}$, and so, $\mathbb{C}^2/G$ and its resolution can be interpreted as a toric variety ([F93, §2.2]).

It is known thanks to Kawamata [I18, Theorem 7.4.9 and 7.4.11] that in characteristic 0 and dimension 2, the slightly more general class of quotient singularities are precisely the log-terminal singularities. They are in particular semi-log-canonical singularities, and so, working with surfaces with only cyclic quotient singularities provide a nice, controlled framework to find examples of surfaces that may be deformed. This is the reason that the technique due to Lee and Park is applied almost exclusively with these types of singularities. See [LP07], [PPS09a], [PPS09b] for constructions with $p_g = 0$ and [PPS13] and [RU21] for constructions for $p_g = 1$.

Cyclic quotient singularities have long since been very well understood in the literature. We will now present some relevant definitions and facts related to them.

Definition 2.1.2 ([H53])**.** Let m, q be coprime integers with $1 \leq q < m$. The Hirzebruch-Jung continued fraction of $\frac{m}{q}$ is given by

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_l}}},$$

where the $e_i \geq 2$ are unique. We also denote

$$\frac{m}{q} = [e_1, \ldots, e_l].$$

Theorem 2.1.3. [I18, Theorem 7.4.16] Let $p \in U$ be a cyclic quotient singularity of type $\frac{1}{m}(1,q)$ in some neighborhood U, and let $\pi \colon V \to U$ the minimal resolution of the point p. Then $\pi^{-1}(p)$ is a divisor consisting of $l > 0$ smooth rational curves $E_1, \ldots, E_l$ such that

1. $E_i \cdot E_{i+1} = 1$ for $i = 1, \ldots, l-1$,
2. $E_i \cdot E_j = 0$ if $j \notin \{i+1, i, i-1\}$,
3. $E_i^2 = -e_i$, where e_i is the corresponding value of the Hirzebruch-Jung continued fraction $\frac{m}{q} = [e_1, \ldots, e_l]$.

From now on, we will now denote by $[e_1, \ldots, e_l]$ both the fraction $\frac{m}{q}$ and the chain of $\mathbb{P}^1$ that are the exceptional divisor of the minimal resolution of a $\frac{1}{m}(1,q)$ singularity. Note that both $[e_1, \ldots, e_l]$ and $[e_l, \ldots, e_1]$ define the same singularity. The following lemma is a classic result that relates both continued fractions (another proof can be seen in [V20, Corolario 1.19]).

Lemma 2.1.4. Let $\frac{m}{q} = [e_1, \ldots, e_l]$ and $\frac{m'}{q'} = [e_l, \ldots, e_1]$. Then $m = m'$ and $qq' \equiv 1 \mod m$.

Proof. Let a_i be defined by

$$a_0 = m, \quad a_1 = q, \quad \text{and} \quad a_{i+1} = e_i a_i - a_{i-1}.$$

By solving for $\frac{a_{i-1}}{a_i}$, we note that this is exactly the partial steps in the construction of its continued fraction, so this means that $\frac{a_{i-1}}{a_i} = [e_i, \ldots, e_l]$. Also, this construction ends when $a_l = 1$ and $a_{l+1} = 0$. Define also

$$b_0 = 0, \quad b_1 = 1, \quad \text{and} \quad b_{i+1} = e_i b_i - b_{i-1}.$$

This also coincides with the construction of a continued fraction, but from the other end, that is, $\frac{b_{i+1}}{b_i} = [e_i, \dots, e_1]$, in particular, $b_l = q'$ and $b_{l+1} = m'$. We prove by induction that for $i \geq 1$, $qb_i \equiv a_i \mod m$. This is true by definition for $i = 0, 1$, and

$$qb_{i+1} = qe_ib_i - qb_{i-1} \equiv e_ia_i - a_{i-1} = a_{i+1}.$$

This means in particular that $qq' \equiv 1 \mod m$ and $qm' \equiv 0 \mod m$. By coprimality of m and q, this means that $m \mid m'$. A symmetric arguments proves that $m' \mid m$, so $m' = m$. □

Whenever m and q are clear by context, q' will denote $q^{-1} \mod m$, and a_i, b_i will be as in the construction above.

Even though the singularities $\frac{1}{m}(1, q)$ and $\frac{1}{m}(1, q')$ are isomorphic, sometimes the order in which we consider the curves in the resolution is important, and we will choose either q or q' accordingly.

Example 2.1.5. The simplest nontrivial cyclic quotient singularity is of type $\frac{1}{2}(1, 1)$, where the group action is given by $(x, y) \mapsto (-x, -y)$.

A polynomial $f \in \mathbb{C}[x, y]$ is invariant with respect to this action if and only if every monomial has an even degree, which means that $\mathbb{C}[x, y]^G$ is generated by the elements x^2, xy, y^2. Naming them u, v, w respectively, we obtain the relation $uw = v^2$, which can be verified to be essentially the only one.

This means that

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[u, v, w]/(uw - v^2)$$

and so, the quotient space $\mathbb{C}^2/G$ can be realized as

$$Z(uw - v^2) \subseteq \mathbb{C}^3_{u,v,w}$$

which can be readily recognized as a simple cone with a node at the origin. It is a usual exercise to verify that the resolution of that singularity (which can be obtained blowing up the origin in the ambient space $\mathbb{C}^3$) has a single $\mathbb{P}^1$ over the singular point with self-intersection -2.

Indeed, the Hirzebruch-Jung continued fraction of this singularity is

$$\frac{2}{1} = 2$$

as expected.

The converse of the resolution also holds, that is, given a surface V with curves $E_1, \dots, E_l$ satisfying the three conditions of Theorem 2.1.3, where V is smooth along $E_1, \dots, E_l$, then there exists a surface U and a morphism $\pi \colon V \to U$ such that π is an isomorphism outside $E_1 \cup \ldots \cup E_l$ and $\pi(E_1 \cup \ldots \cup E_l) = p$. This is a consequence of a well known numerical criteria for contractibility due to Artin ([A62]), which requires that the matrix $(E_i \cdot E_j)_{ij}$ is negative definite.

Since cyclic quotient singularities are rational singularities, a surface X with only cyclic quotient singularities is $\mathbb{Q}$-Gorenstein ([I18, Theorem 7.3.2]), that is, mK_X is a Cartier divisor for some $m > 0$. Because of this, the canonical divisor K_X is well defined in $\operatorname{Div} X \otimes \mathbb{Q}$. Consider $\pi \colon Y \to X$ a resolution of singularities of X with exceptional divisors E_i, $i = 1, \dots, l$. Since π is an isomorphism outside the total exceptional divisor, one must have the equality

$$K_Y = \pi^* K_X + \sum_{j=1}^{l} d_j E_j. \tag{2.1}$$

for some rational numbers d_j called *discrepancies*. $\sum_{j=1}^{l} d_j E_j \in \operatorname{Div} Y \otimes \mathbb{Q}$ is called the discrepancy $\mathbb{Q}$-divisor. In order to find them, we intersect K_Y with every E_i. By adjunction formula (Theorem 1.2.5), since each E_i is a smooth rational curve, we have $K_Y \cdot E_i = -2 - E_i^2$, and since E_i is exceptional, we have $\pi^* K_X \cdot E_i = 0$. Therefore we get the linear system

$$-2 - E_i^2 = \sum_{j=1}^{l} d_j (E_j \cdot E_i)$$

Since π is a resolution of singularities, by Theorem 1.3.1 we know that the matrix $(E_i \cdot E_j)_{ij}$ is negative definite, so there must be a unique solution to this system. In the case of cyclic quotient singularities we can determine the discrepancies in terms of the continued fraction of $\frac{m}{q} = [e_1, \ldots, e_l]$. Note that in this case the system becomes

$$e_i - 2 = d_{i-1} - e_i d_i + d_{i+1}$$

where for convenience $d_0 = d_{l+1} = 0$. The following propositon is a classic result that relates the discrepancies of a cyclic quotient singularity and a_i, b_i.

Proposition 2.1.6. Let $x \in X$ be a cyclic quotient singularity of type $\frac{m}{q} = [e_1, \ldots, e_l]$ and $\pi \colon Y \to X$ its minimal resolution with exceptional curves $E_1, \ldots, E_l$ (in order as in Theorem 2.1.3). Let a_i and b_i be as in lemma 2.1.4. Then the i-th discrepancy has

$$d_i = -\left(1 - \frac{a_i + b_i}{m}\right).$$

for $i = 0, \ldots, l+1$, where we define $d_0 = d_{l+1} = 0$.

Proof. It is enough to show that those numbers are a solution to the system. For $i = 0$ and $i = l+1$ this is clear. For $i = 1, \ldots, l$,

$$\begin{aligned} d_{i-1} - e_i d_i + d_{i+1} &= -\left(1 - \frac{a_{i-1} + b_{i-1}}{m}\right) + e_i \left(1 - \frac{a_i + b_i}{m}\right) - \left(1 - \frac{a_{i+1} + b_{i+1}}{m}\right) \\ &= e_i - 2 + \frac{(a_{i+1} - e_i a_i + a_{i-1}) + (b_{i+1} - e_i b_i + b_{i-1})}{m} \\ &= e_i - 2. \end{aligned}$$

□

Example 2.1.7. Going back to our example of $\frac{1}{2}(1, 1)$, the discrepancy of the single exceptional over the point must satisfy the equation

$$e_1 d_1 = e_1 - 2 = 0$$

and therefore $d_1 = 0$. In fact as long as every $e_i = 2$, every discrepancy must be zero as the system is trivial. These are precisely the A_n type of double point singularities (see Section 1.4)

2.2 T-Singularities and Wahl Chains

One of the main reasons for using $\mathbb{Q}$-Gorenstein smoothings is that they preserve several invariants of surfaces, in particular they preserve K^2, even if the surface is singular. This immediately gives us a restriction on the kind of singularities that can appear on X: They must ensure that K_X^2 is an integer.

By squaring equation 2.1 and noting that each E_i is π-exceptional, we obtain

$$K_Y^2 = K_X^2 + \left(\sum_{i=1}^{l} d_i E_i\right)^2.$$

As K_Y^2 is an integer because Y is smooth, K_X^2 being an integer is equivalent to $\left(\sum_{i=1}^{l} d_i E_i\right)^2$ being an integer. The following result is classical.

Proposition 2.2.1. Let $1 \leq q < m$ be coprime integers as before, and let $x \in X$ be an isolated singularity of type $\frac{1}{m}(1, q)$. The following are equivalent:

1. K_X^2 is an integer.
2. $m = dn^2$, $q = dna - 1$ for a triple of integers (d, n, a) where $1 \leq a \leq n$ and $\gcd(n, a) = 1$.

Proof. We have

$$\begin{aligned}
\left(\sum_{i=1}^{l} d_i E_i\right)^2 &= \sum_{i=1}^{l} d_i(d_{i-1} - e_i d_i + d_{i+1}) \\
&= \sum_{i=1}^{l} d_i(e_i - 2) \\
&= \sum_{i=1}^{l} -\left(1 - \frac{a_i + b_i}{m}\right)(e_i - 2) \\
&= \sum_{i=1}^{l}(2 - e_i) + \sum_{i=1}^{l}\left(\frac{a_i + b_i}{m}\right)(e_i - 2) \\
&= \sum_{i=1}^{l}(2 - e_i) + \sum_{i=1}^{l} e_i\left(\frac{a_i + b_i}{m}\right) - \sum_{i=2}^{l+1}\frac{a_{i-1} + b_{i-1}}{m} - \sum_{i=0}^{l-1}\frac{a_{i+1} + b_{i+1}}{m} \\
&= \sum_{i=1}^{l}(2 - e_i) + \sum_{i=1}^{l}\left(\frac{e_i a_i - a_{i-1} - a_{i+1} + e_i b_i - b_{i-1} - b_{i+1}}{m}\right) \\
&\quad + \frac{a_0 + b_0 - a_l - b_l}{m} + \frac{a_{l+1} + b_{l+1} - a_1 - b_1}{m} \\
&= \sum_{i=1}^{l}(2 - e_i) + \frac{m + 0 - 1 - q' + 0 + m - q - 1}{m} \\
&= \sum_{i=1}^{l}(2 - e_i) + 2 - \frac{q + q' + 2}{m}
\end{aligned}$$

therefore, K_X^2 is an integer if and only if $q + q' + 2 \equiv 0 \mod m$. It is clear that 2. implies 1., as the inverse of $dna - 1$ modulo m is $dn(n - a) - 1$.

For 1. implies 2. note that by multiplying by q we obtain $q^2 + 2q + 1 = (q + 1)^2 \equiv 0 \mod m$, therefore $m \mid (q + 1)^2$. Let $g = \gcd(m, q + 1)$, $n = \frac{m}{g}$, $d = \frac{g^2}{m}$ and $a = \frac{q+1}{g}$, so n and a are coprime. Since m divides $(q+1)^2$, then it also divides g^2, so d is an integer. Clearly $m = dn^2$ and $q = dna - 1$. $\square$

We will now differentiate two cases. The first is when $n = a = 1$ and $d \geq 2$. The corresponding singularity is of type $\frac{1}{d}(1, d-1)$, and it is not hard to verify that

$$\frac{d}{d-1} = [\underbrace{2, \ldots, 2}_{d-1 \text{ times}}],$$

so it is exactly a singularity of type A_{d-1}, from the ADE classification. As we saw, each discrepancy is zero, and in particular $K_X^2 = K_Y^2$, an integer. The truly interesting case is when $n > 1$.

Definition 2.2.2. Let $1 \leq a < n$ and $d \geq 1$ be integers with $\gcd(n, a) = 1$. Then the singularity $\frac{1}{dn^2}(1, dna - 1)$ is called a T-singularity, and is denoted by $T(d, n, a)$, and the corresponding chain of $\mathbb{P}^1$'s in its resolution is called a T-chain. When $d = 1$, it is also called a Wahl singularity, and the corresponding chain is called a Wahl chain.

Note however, that classically, as in [KSB88], T-singularities are defined as the ones above plus ADE singularities.

We will now provide a complete description of all T-singularities.

Proposition 2.2.3 (T-chain algorithm, [KSB88, Proposition 3.11])**.** The following are 'initial' T-chains:

- For $d = 1$, $[4]$ is of type $T(1, 2, 1)$.
- For $d > 1$, $[3, 2, \ldots, 2, 3]$ is of type $T(d, 2, 1)$, where the amount of twos that appear in the expansion is $d - 2$.

Now, if $[e_1, \ldots, e_l]$ is of type $T(d, n, a)$, then the following are also T-chains

- $[2, e_1, \ldots, e_{l-1}, e_l + 1]$ is of type $T(d, 2n - a, n)$.
- $[e_1 + 1, e_2, \ldots, e_l, 2]$ is of type $T(d, n + a, a)$.

Every T-chain can be obtained starting from an initial T-chain and applying the above operations repeated times.

Proof. The cases for $d = 1$ and $d = 2$ can be readily checked. For $d > 2$, we know that

$$[\underbrace{2, \ldots, 2}_{d-2 \text{ times}}] = \frac{d-1}{d-2}$$

Then

$$[3, 2, \ldots, 2] = 3 - \frac{d-2}{d-1} = \frac{2d-1}{d-1}$$

The inverse of $d - 1$ modulo $2d - 1$ is $2d - 3$, therefore, by Lemma 2.1.4,

$$[2, \ldots, 2, 3] = \frac{2d-1}{2d-3}$$

and finally,

$$[3, 2, \ldots, 2, 3] = 3 - \frac{2d-3}{2d-1} = \frac{4d}{2d-1}$$

which is precisely the fraction associated to $T(d,2,1)$. In the case of the operations, we know that

$$[e_1,\dots,e_l] = \frac{dn^2}{dna-1}$$

The inverse of $dna-1$ modulo dn^2 is $dn(n-a)-1$, so

$$[e_l,\dots,e_1] = \frac{dn^2}{dn(n-a)-1}$$

Therefore

$$[e_l+1,\dots,e_1] = 1 + \frac{dn^2}{dn(n-a)-1} = \frac{dn(2n-a)-1}{dn(n-a)-1}$$

The inverse of $dn(n-a)-1$ modulo $dn(2n-a)-1$ is $da(2n-a)-2$, so

$$[e_1,\dots,e_l+1] = \frac{dn(2n-a)-1}{da(2n-a)-2}$$

Finally,

$$\begin{aligned}
[2,e_1,\dots,e_l+1] &= 2 - \frac{da(2n-a)-2}{dn(2n-a)-1} \\
&= \frac{2dn(2n-a)-2-da(2n-a)+2}{dn(2n-a)-1} \\
&= \frac{d(2n-a)^2}{d(2n-a)n-1}
\end{aligned}$$

which is precisely the fraction associated to $T(d,2n-a,n)$. For the last case, note that if $[e_1,\dots,e_l]$ is of type $T(d,n,a)$, then $[e_l,\dots,e_1]$ is of type $T(d,n,n-a)$. By the previous part, $[2,e_l,\dots,e_1+1]$ is of type $T(d,2n-(n-a),n) = T(d,n+a,n)$. Again, by reversing the chain we get that $[e_1+1,\dots,e_l,2]$ is of type $T(d,n+a,a)$.

We can use a simple argument of infinite descent to prove that every T-chain can be obtained with this algorithm: Let (n,a) be a pair of coprime integers with $n > a \geq 1$, $n > 2$. If $a < \frac{n}{2}$, then $(n',a') = (n-a,a)$ also satisfies $n' > a' \geq 1$, and $n' < n$. In the other case, if $a > \frac{n}{2}$ (equality cannot happen due to coprimality), then $(n',a') = (a,2a-n)$ also satisfies $n' > a' \geq 1$ and $n' < n$. This process cannot go infinitely, and must stop when $n=2$ and $a=1$. □

Example 2.2.4. The simplest Wahl chains are of type $T(1,n,1)$. In the algorithm above, since a is never increased, it means that only operation 2 is applied, and it is done $n-2$ times. The chain we obtain is thus

$$[n+2, \underbrace{2,\dots,2}_{n-2 \text{ times}}] = \frac{n^2}{n-1}.$$

Its reversed chain is of type $T(1,n,n-1)$,

$$[\underbrace{2,\dots,2}_{n-2 \text{ times}}, n+2] = \frac{n^2}{n(n-1)-1}.$$

These will be called "linear chains". Note that if $[e_1,\dots,e_l]$ is not a linear chain, i.e. both operations are applied at least once, then two things happen. First, there must exist an index $i \neq 1,l$ such that $e_i \neq 2$. Second, let i be the greatest index such that $e_i > 2$. Then the chain is of the form

$$[e_1,\dots,e_i, \underbrace{2,\dots,2}_{e_1-2 \text{ times}}].$$

since there must have been an operation of the first kind followed by $e_1 - 2$ operations of the second kind. In particular, if $e_1 = 2$, then $e_l > 2$, since we must have finished with an operation of the first kind. Of course, this also works the other way around: if i is the least index such that $e_i > 2$, then the chain is of the form

$$[\underbrace{2,\ldots,2}_{e_l - 2 \text{ times}}, e_i, \ldots, e_l].$$

By using this construction of T-chains, we can calculate the discrepancies of each of the exceptional curves by introducing a little bit more information.

Lemma 2.2.5 ([V20, Proposición 2.20])**.** Denote by $D(d,2,1)$ the tuple $(1,\ldots,1)$ with d ones. If $D(d,n,a)$ is defined as the tuple $(\delta_1,\ldots,\delta_l)$, then

- $D(d, 2n-a, n)$ is the tuple $(\delta_1 + \delta_l, \delta_1, \ldots, \delta_l)$.
- $D(d, n+a, a)$ is the tuple $(\delta_1, \ldots, \delta_l, \delta_1 + \delta_l)$.

If $T(d,n,a)$ is the T-singularity $[e_1,\ldots,e_l]$ and $D(d,n,a)$ is $(\delta_1,\ldots,\delta_l)$, then the following holds

1. $n = \delta_1 + \delta_l$,
2. $a = \delta_1$,
3. $d_i = -\left(1 - \frac{\delta_i}{n}\right)$ for $i = 1,\ldots,l$.

Proof. By Proposition 2.1.6, 3. holds if and only if $a_i + b_i = dn\delta_i$ for every $i = 1,\ldots,l$, and this easily implies 1. and 2. For $T(d,n,1)$ we only have to prove that every discrepancy is $-\frac{1}{2}$. This can be verified immediately for $T(1,2,1)$ and $T(2,2,1)$. For $[3,2,\ldots,2,3]$, for $i = 2,\ldots,l-1$ we have the equation

$$0 = e_i - 2 = d_{i-1} - 2d_i + d_{i+1}$$

therefore

$$d_i - d_{i-1} = d_{i+1} - d_i$$

so discrepancies increase or decrease linearly with the constant $C = d_i - d_{i-1}$. However, $[3,2,\ldots,2,3]$ is symmetric, so if the discrepancies were increase to the right, they would also increase to the left and vice versa. This forces $C = 0$, that is, all discrepancies are equal. We finally evaluate at $i = 1$:

$$1 = e_1 - 2 = d_1 - 3d_2 = -2d_1$$

therefore $d_i = d_1 = -\frac{1}{2}$ for every i.

We will prove that if the discrepancies of $T(d,n,a)$ can be calculated with $D(d,n,a)$ via 3. then the discrepancies of $T(d,\tilde{n},\tilde{a}) = T(d,n+a,a)$ can be calculated with $D(d,\tilde{n},\tilde{a})$. We note that since $\tilde{a} = a$, then $\tilde{a} = \delta_1$ so 2. holds. This means that $\tilde{a}_1 + \tilde{b}_1 = \tilde{q} + 1 = d\tilde{n}\tilde{a} = d\tilde{n}\delta_1$ so the condition holds for $i = 1$.

Since the discrepancies of $T(d,n,a)$ can be calculated with $D(d,n,a)$, then $n = \delta_1 + \delta_l$, and since $\tilde{a} = a = \delta_1$, then $\tilde{n} = n + a = 2\delta_1 + \delta_l = \delta_1 + \delta_{l+1}$, so $\delta_{l+1} = \tilde{n} - \tilde{a}$, and

$$\tilde{a}_{l+1} + \tilde{b}_{l+1} = 1 + \tilde{q}' = 1 + (d\tilde{n}(\tilde{n} - \tilde{a}) - 1) = d\tilde{n}\delta_{l+1}.$$

So the condition also holds for $i = l+1$. This also solves the problem when $l = 1$.

Now, if $l > 1$, recall that if $T(d,n,a) = [e_1,\ldots,e_l]$ then $T(d,\tilde{n},\tilde{a}) = [e_1+1,\ldots,e_l,2]$. We have

$$dn\delta_2 = a_2 + b_2 = e_1(a_1+b_1) - (a_0+b_0) = e_1(a_1+b_1) - m = e_1 dna - dn^2 = dn(e_1 a - n)$$

so $\delta_2 = e_1 a - n$, then

$$\tilde{a}_2 + \tilde{b}_2 = (e_1+1)(\tilde{a}_1 + \tilde{b}_1) - \tilde{m} = (e_1+1)d(n+a)a - d(n+a)^2 = d(n+a)(e_1 a - n) = d(n+a)\delta_2$$

So the condition holds for $i = 2$. By induction, for $i = 2, \dots, l-1$,

$$dn\delta_{i+1} = a_{i+1} + b_{i+1} = e_i(a_i + b_i) - (a_{i-1} + b_{i-1}) = dn(e_i\delta_i - \delta_{i-1})$$

so $\delta_{i+1} = e_i\delta_i - \delta_{i-1}$, and again by induction, for $i = 2, \dots, l-1$,

$$\tilde{a}_{i+1} + \tilde{b}_{i+1} = e_i(\tilde{a}_i + \tilde{b}_i) - (\tilde{a}_{i-1} + \tilde{b}_{i-1}) = d\tilde{n}(e_i\delta_i - \delta_{i-1}) = d\tilde{n}\delta_{i+1}$$

We can conclude that $\tilde{a}_i + \tilde{b}_i = d\tilde{n}\delta_i$ for every $i = 3, \dots, l$. The case for $T(d, 2n-a, n)$ is clear by symmetry. □

Example 2.2.6. By applying this construction to linear chains of type $T(1, n, 1)$ we see that discrepancies increase linearly. Thus the name.

Remark 2.2.7. Note that unless the T-chain is initial, if we write it as $[e_1, \dots, 2]$ for $e_1 > 2$, then the discrepancy of the first curve is always $< -\frac{1}{2}$, and the discrepancy of the last curve is always $> -\frac{1}{2}$. Moreover the curves with discrepancy $\geq -\frac{1}{2}$ are always at the "tail" of (-2)-curves of a Wahl chain.

Lemma 2.2.8. Let $x \in X$ be a $T(d, n, a)$ singularity, and let $Y \to X$ be its resolution. If l is the length of the chain over the point, then

$$K_Y^2 - K_X^2 = d - 1 - l.$$

In particular, if X has a Wahl singularity, then

$$K_X^2 = K_Y^2 + l.$$

Proof. Recall that

$$K_Y^2 - K_X^2 = \left(\sum_{i=1}^{l} d_i E_i\right)^2 = \sum_{i=1}^{l}(2 - e_i) + 2 - \frac{dna - 1 + dn(n-a) - 1 + 2}{dn^2} = \sum_{i=1}^{l}(2 - e_i) + 1.$$

Note that for a $T(d, 2, 1)$ chain we have $l = d$ and

$$\sum_{i=1}^{l}(2 - e_i) = -2,$$

so for initial T-chains we have

$$\sum_{i=1}^{l}(2 - e_i) + 1 = -1.$$

By applying one operation of Proposition 2.2.3, we have that $\sum_i(2 - e_i)$ decreases by one. In order to get a chain of length l we must apply operations $l - d$ times. Since each operation decreases the value by one, we obtain

$$\sum_{i=1}^{l}(2 - e_i) + 1 = -1 + d - l.$$

□

We conclude this section by giving an example of how we can calculate the invariants of a given chain.

Example 2.2.9. Suppose we are given a chain of $\mathbb{P}^1$'s that corresponds to

$$[3,2,3,2,3,5,4,2]$$

Is it a T-singularity? if so, what are its invariants (d,n,a) and its discrepancies?

For these questions, we may naively calculate the fraction using the Hirzebruch-Jung coefficients, but it will quickly become bothersome doing it by hand.

Alternatively we can use the algorithm in Proposition 2.2.3 backwards. If we arrive to a chain of type $T(d,2,1)$ that meant that our chain was a T-singularity. We can then use the algorithm in Lemma 2.2.5 to calculate its invariants.

Applying 2.2.3 backwards successive times we obtain:

$$\begin{aligned}
[3,2,3,2,3,5,4,2] &\to [2,2,3,2,3,5,4] \\
[2,2,3,2,3,5,4] &\to [2,3,2,3,5,3] \\
[2,3,2,3,5,3] &\to [3,2,3,5,2] \\
[3,2,3,5,2] &\to [2,2,3,5] \\
[2,2,3,5] &\to [2,3,4] \\
[2,3,4] &\to [3,3]
\end{aligned}$$

This means that we indeed are looking at a T-singularity with $d=2$. Now that we also know how to construct it starting from an 'initial' T-singularity, we may apply algorithm in lemma 2.2.5:

$$\begin{array}{r} [3,\ 3] \\ (1,1) \end{array} \to \begin{array}{l} [2,\ 3,\ 4] \\ (2,1,1) \end{array}$$

$$\begin{array}{r} [2,\ 3,\ 4] \\ (2,1,1) \end{array} \to \begin{array}{l} [2,\ 2,\ 3,\ 5] \\ (3,2,1,1) \end{array}$$

$$\begin{array}{r} [2,\ 2,\ 3,\ 5] \\ (3,2,1,1) \end{array} \to \begin{array}{l} [3,\ 2,\ 3,\ 5,\ 2] \\ (3,2,1,1,4) \end{array}$$

$$\begin{array}{r} [3,\ 2,\ 3,\ 5,\ 2] \\ (3,2,1,1,4) \end{array} \to \begin{array}{l} [2,\ 3,\ 2,\ 3,\ 5,\ 3] \\ (7,3,2,1,1,4) \end{array}$$

$$\begin{array}{r} [2,\ 3,\ 2,\ 3,\ 5,\ 3] \\ (7,3,2,1,1,4) \end{array} \to \begin{array}{l} [2,\ \ 2,\ 3,\ 2,\ 3,\ 5,\ 4] \\ (11,7,3,2,1,1,4) \end{array}$$

$$\begin{array}{r} [2,\ \ 2,\ 3,\ 2,\ 3,\ 5,\ 4] \\ (11,7,3,2,1,1,4) \end{array} \to \begin{array}{l} [3,\ \ 2,\ 3,\ 2,\ 3,\ 5,\ 4,\ \ 2] \\ (11,7,3,2,1,1,4,15) \end{array}$$

With this, we conclude that $n=26$ and $a=11$, so the singularity is of type $\frac{1}{1352}(1,571)$. The discrepancies are respectively

$$\left(-\frac{15}{26},-\frac{19}{26},-\frac{23}{26},-\frac{24}{26},-\frac{25}{26},-\frac{25}{26},-\frac{22}{26},-\frac{11}{26}\right).$$

Since $l=8$ and $d=2$, we can also know that $K_Y^2-K_X^2=-7$.

2.3 Extremal P-resolutions

P-resolutions ([KSB88, Definition 3.8]) appear in some deep parts of the theory of singularities, including for example as byproducts of applying MMP to certain families of surfaces. They are also related to general smoothings of cyclic quotient singularities. However we are only interested in a concrete algorithm encountered in the case of extremal P-resolutions ([HTU17, §4]). To deal with the algorithm, we must first extend our language of continued fractions a bit.

In this section we deal with 'generalized' Hirzebruch-Jung continued fractions, that is, negative continued fractions that allows ones to appear, and also the special value [0]. Now the continued fraction of a rational number is not unique, and not every possible combination of numbers allows for a valid continued fraction. We will now see some properties of these new kind of chains.

Definition 2.3.1. A generalized continued fraction $[e_1, \dots, e_n]$ is admissible if in the partial calculations $\frac{p}{q} = [e_i, \dots, e_n]$ are positive for $i > 1$ and non-negative for $i = 1$. We say the chain $[e_1, \dots, e_n]$ represents $\frac{p}{q}$.

Thus, we disallow partial calculations which imply divisions by zero or negative numbers.

Example 2.3.2. $[1, 1, 1]$ is not admissible, since there is a division by zero in $1 - \frac{1}{1-\frac{1}{1}}$, however $[1, 1]$ is admissible since $1 - \frac{1}{1} = 0$, so $[1, 1]$ *represents zero.*

Example 2.3.3. Both $[2, 2]$ and $[3, 1, 3]$ represent $\frac{3}{2}$. In fact there are infinitely many chains that represent any non-negative number. This will be seen just below.

Definition 2.3.4. Let $[e_1, \dots, e_l]$ be a generalized chain. A blow up is an operation on the chain modifies it in one of the three following ways

$$[e_1, \dots, e_l] \mapsto [1, e_1 + 1, \dots, e_l],$$

$$[e_1, \dots, e_i, e_{i+1}, \dots, e_l] \mapsto [e_1, \dots, e_i + 1, 1, e_{i+1} + 1, \dots, e_l],$$

$$[e_1, \dots, e_l] \mapsto [e_1, \dots, e_l + 1, 1].$$

A blow down operation is the inverse process.

Of course, blow ups and blow downs of chains mimic the respective operations in linear configurations of curves.

Proposition 2.3.5 ([V20, Proposición 1.21])**.** $[e_1, \dots, e_l]$ is admissible if and only if any of its blow up is admissible. Moreover, if $\frac{m}{q} = [e_1, \dots, e_l]$, then

$$[1, e_1 + 1, \dots, e_l] = \frac{m}{m+q},$$

$$[e_1, \dots, e_i + 1, 1, e_{i+1} + 1, \dots, e_l] = \frac{m}{q},$$

$$[e_1, \dots, e_l + 1, 1] = \frac{m}{q}.$$

Every admissible chain is obtained from blow ups from a minimal chain which is either [0] or a classical Hirzebruch-Jung chain.

The above proposition provides us with an algorithm to determine if a chain is admissible, where we just have to blow down until we arrive to a valid minimal chain. If in any moment there appears a 0 and we are not at [0], the chain would be invalid.

Definition 2.3.6 ([KSB88, Definition 3.8])**.** Let $0 < \Omega < \Delta$ coprime integers, and $P \in X$ be a cyclic quotient singularity of type $\frac{1}{\Delta}(1, \Omega)$. A P-resolution (positive resolution) is a partial resolution $\pi\colon Y \to X$ of P such that $\pi^{-1}(P)$ consists of rational curves, all positive for K_Y, and contains only T-singularities and A_n singularities, for possibly many different n.

To build a P-resolution one may start from the minimal resolution, blow up enough times so that only Wahl chains remain with curves connecting them, then contract said Wahl chains into Wahl singularities, all while satisfying that these curves connecting them are positive for the canonical class.

Example 2.3.7. Let $P \in X$ be the singularity given by $\frac{\Delta}{\Omega} = \frac{129}{76} = [2, 4, 2, 2, 5, 2]$. We can blow up this chain twice to obtain the chain $[2, 5, 1, 4, 1, 3, 5, 2]$, which still admits a contraction $Y \to X$, where we recognize the Wahl chains $[2, 5]$, $[4]$ and $[3, 5, 2]$ connected by two (-1)-curves Γ_1 and Γ_2. Let $\pi\colon Y \to Z$ be their contraction, and $\tilde{\Gamma}_i$ the image of Γ_i. The discrepancy of the first (-5)-curve is $-\frac{2}{3}$, for the (-4)-curve is $-\frac{1}{2}$, and for the (-3)-curve is $-\frac{3}{5}$, so $K_Z \cdot \tilde{\Gamma}_1 = -1 + \frac{2}{3} + \frac{1}{2} > 0$ and $K_Z \cdot \tilde{\Gamma}_2 = -1 + \frac{1}{2} + \frac{3}{5} > 0$, so $Z \to X$ is a P-resolution of the singularity $\frac{1}{129}(1, 76)$.

Example 2.3.8. There may be many ways to partially solve a singularity that are not P-resolutions. For example the chain $[2, 6, 2, 2]$ admits the partial resolution $[6, 2, 2, 1, 4, 2, 2, 7, 2, 2]$, but it can be checked that the (-1)-curve in the middle is negative for the canonical class.

For the theory on general P-resolutions, the reader may refer to [KSB88, HTU17]. We will be interested in a particular kind, called extremal P-resolutions.

Definition 2.3.9 ([HTU17, §4])**.** Let $0 < \Omega < \Delta$ be coprime integers, and let $P \in X$ be the cyclic quotient singularity $\frac{1}{\Delta}(1, \Omega)$. An extremal P-resolution is a P-resolution $Y \to X$ which contains a single rational curve C over P. In particular, Y has at most 2 Wahl singularities.

Now we explain why extremal P-resolutions are important for our purposes. Suppose we are trying to search for two Wahl chains from a given configuration of curves by blowing up several times on intersections, all while ensuring that every non contracted exceptional curve is positive for K_X. Suppose that after some blow ups we arrive to a linear chain of curves with some invariants that cannot be modified (cf. 5.3). Now the problem reduces to finding extremal P-resolutions of this chain. This can become a priori really hard to do, since one might be forced to do several infinitely near blow ups.

Notation 2.3.10. Even though in the literature, an extremal P-resolution is a configuration of curves with some singularities, we are mainly interested in chains of curves, so we say that a chain $[f_1, \ldots, f_n]$ is an extremal P-resolution of a chain $[e_1, \ldots, e_n]$ if it corresponds to the resolution of an extremal P-resolution of the singularity given by $[e_1, \ldots, e_n]$. In other words, if $[f_1, \ldots, f_n]$ is obtained by blowing up $[e_1, \ldots, e_n]$ and contains at most two Wahl chains (or A_n chains) connected by a curve.

Example 2.3.11. Consider

$$\frac{1984909}{452505} = [5, 2, 3, 4, 2, 3, 2, 2, 4, 5, 4, 2, 3, 4, 2, 2, 2].$$

This chain has exactly one extremal P-resolution, namely

$$[5, 2, 3, 4, 2, 3, 2, 2, 7, \overset{3}{4}, \overset{6}{2}, \overset{7}{3}, \overset{9}{4}, \overset{12}{2}, \overset{13}{2}, \overset{14}{2}, \overset{15}{1}, \overset{11}{5}, \overset{10}{2}, \overset{8}{3}, \overset{5}{4}, \overset{4}{2}, \overset{2}{3}, \overset{1}{2}, 6, 4, 2, 3, 4, 2, 2, 2]$$

where the number above shows the order in which blow ups where done. Here,

$$[5,2,3,4,2,3,2,2,7,4,2,3,4,2,2,2] = \frac{824464}{187955} = T(1,908,207), \text{ and}$$

$$[5,2,3,4,2,3,2,6,4,2,3,4,2,2,2] = \frac{511225}{116544} = T(1,715,163).$$

As it can be seen, it is not practical to just blow up blindly with the hopes to find an extremal P-resolution. In [HTU17] an algorithm is explained that is used to find every extremal P-resolution of a singularity, if they exist.

Definition 2.3.12. Let $[e_1,\dots,e_l]$ be the (classic) Hirzebruch-Jung continued fraction of $\frac{m}{q}$. The dual continued fraction of $\frac{m}{q}$ is $\frac{m}{m-q} = [f_1,\dots,f_s]$

Proposition 2.3.13 ([HTU17, §4.1])**.** Let $0 < \Omega < \Delta$ be coprime integers. Let $\frac{\Delta}{\Delta-\Omega} = [f_1,\dots,f_s]$ be the dual fraction of $\frac{\Delta}{\Omega}$. Then the extremal P-resolutions of $\frac{\Delta}{\Omega}$ are in 1-1 correspondence with pairs $1 \le \alpha < \beta \le s$ such that the chain

$$[f_1,\dots,f_{\alpha-1}, f_\alpha - 1, f_{\alpha+1},\dots,f_{\beta-1}, f_\beta - 1, f_{\beta+1},\dots,f_s]$$

represents zero. They are labeled as

$$[f_1,\dots,\bar{f}_\alpha,\dots,\bar{f}_\beta,\dots,f_s].$$

Additionally, a given singularity may have 0, 1 or 2 different extremal P-resolutions ([HTU17, Theorem 4.3]).

With this proposition the problem of finding P-resolutions of a given chain is reduced to a simple algorithm, cubic in the length of the chain. Also from [HTU17] we obtain the following

Proposition 2.3.14 ([HTU17, Proposition 4.1])**.** In the situation above, if $[f_1,\dots,\bar{f}_\alpha,\dots,\bar{f}_\beta,\dots,f_s]$ represents zero, for $1 < \alpha < \beta < s$ then defining

$$\frac{n_1}{a_1} = [f_1,\dots,f_{\alpha-1}], \quad \text{and} \quad \frac{n_2}{a_2} = [f_s,\dots,f_{\beta+1}],$$

and letting $\frac{n_1^2}{n_1a_1-1} = [g_1,\dots,g_r]$ and $\frac{n_2^2}{n_2a_2-1} = [h_1,\dots,h_t]$, the corresponding extremal P-resolution for $[e_1,\dots,e_l]$ is

$$[g_r,\dots,g_1,c,h_1,\dots,h_t].$$

where c is given by the formula $\delta = cn_1n_2 - n_1a_2 - n_2a_1$, and δ is given by $\frac{\delta}{\epsilon} = [f_{\alpha+1},\dots,f_{\beta-1}]$. It corresponds to the self intersection of the curve connecting both T-singularities.

Example 2.3.15. Consider $\frac{196}{141} = [2,2,3,5,2,4]$. Then the dual chain is

$$\frac{196}{196-141} = \frac{196}{55} = [4,3,2,2,4,2,2].$$

Here we can find two different extremal P-resolutions, since

$$[4,2,2,1,4,2,2] = 0, \quad \text{and} \quad [4,3,1,2,3,2,2] = 0.$$

In the first case we have $\frac{n_1}{a_1} = [4] = \frac{4}{1}$, so the rightmost chain must be $T(1,4,1) = [6,2,2]$, and also $\frac{n_2}{a_2} = [2,2,4] = \frac{10}{7}$, so the leftmost chain must be $T(1,10,7) = [2,2,6,2,4]$. The sub-chain between the selected indices is $[2] = \frac{2}{1}$, so $\delta = 2$. Solving for c gives us 1, so the extremal P-resolution is then

$$[2, 2, 6, 1, 2, 2, 6, 2, 4].$$

In the second case we have $\frac{n_1}{a_1} = [4, 3] = \frac{11}{3}$, so the rightmost chain must be $T(1, 11, 3) = [4, 5, 3, 2, 2]$, and also $\frac{n_2}{a_2} = [2, 2] = \frac{3}{2}$, so the leftmost chain must be $T(1, 3, 2) = [2, 5]$. Again, $c = 1$, so the extremal P-resolution is then

$$[2, 2, 3, 5, 4, 1, 2, 5].$$

Example 2.3.16. The chain $[4, 2, 4]$ defines its own extremal P-resolution, with $c = 2$. However, for our purposes, we will always require the middle curve have $c = 1$.

Remark 2.3.17. It is possible to choose $\alpha = \beta$, that is, to subtract 2 from the index at α and check if the new chain represents zero. If it does, the result is not a P-resolution, but a resolution in which the exceptional curve satisfies $\Gamma \cdot K_Z = 0$. Since discrepancies in the middle must add up to -1, both singularities are the same, and actually these are partial resolutions of singularities of type $T(2, d, n)$.

Example 2.3.18. For an example of the previous remark, consider $T(2, 3, 1) = \frac{18}{5} = [4, 3, 2]$. Its dual chain is $\frac{18}{13} = [2, 2, 3, 3]$, and choosing the third position, we note that $[2, 2, 1, 3]$ represents zero. For the first chain we have $\frac{n_1}{a_1} = [2, 2] = \frac{3}{2}$, that is, $T(1, 2, 2) = [2, 5]$. For the second we have $\frac{n_2}{a_2} = [3]$, that is $T(1, 3, 1) = [5, 2]$, so the resolution becomes $[5, 2, 1, 5, 2]$. This happens in general for $T(d, n, a)$, which can be 'separated' into d copies of $T(1, n, a)$. This is called an M-resolution of $T(d, n, a)$ (see [BC94]).

There is a final proposition that we will use later in the algorithm, but first we require a lemma.

Lemma 2.3.19. Let $T(1, n_1, a_1) = [e_1, \dots, e_l]$ and $T(1, n_2, a_2) = [f_1, \dots, f_s]$ be Wahl chains. The following are equivalent

- $[e_1, \dots, e_l, 1, f_1, \dots, f_s]$ is an extremal P-resolution of some chain.
- $[e_1 + 1, \dots, e_l, 2, 1, f_1 + 1, \dots, f_s, 2]$ is an extremal P-resolution of some chain.
- $[2, e_1, \dots, e_l + 1, 1, 2, f_1, \dots, f_s + 1]$ is an extremal P-resolution of some chain.
- $\dfrac{n_2}{a_2} > \dfrac{n_1}{a_1}$.

Proof. The discrepancy for e_l in $T(1, n_1, a_1)$ is $-\left(1 - \frac{\delta_{l,1}}{n_1}\right) = -\frac{a_1}{n_1}$. The discrepancy for f_1 in $T(1, n_2, a_2)$ is $-\left(1 - \frac{\delta_{1,2}}{n_2}\right) = -\frac{n_2 - a_2}{n_2}$. The condition for being a P-resolution is then

$$\begin{aligned} \frac{a_1}{n_1} + \frac{n_2 - a_2}{n_2} > 1 &\iff a_1 n_2 + n_1 n_2 - n_1 a_2 > n_1 n_2 \\ &\iff \frac{n_2}{a_2} > \frac{n_1}{a_1} \end{aligned}$$

For the second equivalence (the third is analogous), note that the first chain is of type $T(1, n_1 + a_1, a_1)$ and the second one is of type $T(1, n_2 + a_2, a_2)$. Therefore the relevant discrepancies are $-\frac{a_1}{n_1 + a_1}$ and $-\frac{n_2}{n_2 + a_2}$. The condition for being a P-resolution is then

$$\begin{aligned} \frac{a_1}{n_1 + a_1} + \frac{n_2}{n_2 + a_2} > 1 &\iff a_1 n_2 + a_1 a_2 + n_1 n_2 + a_1 n_2 > n_1 n_2 + n_1 a_2 + a_1 n_2 + a_1 a_2 \\ &\iff \frac{n_2}{a_2} > \frac{n_1}{a_1} \end{aligned}$$

□

Definition 2.3.20. During the construction of a Wahl chain with the algorithm 2.2.3, call the starting [4] the center of the chain, which remains unchanged during the process. That is, if we denote the center by $[e_1, \ldots, \tilde{e}_i, \ldots, e_l]$, then the centers of the successive chains are $[2, e_1, \ldots, \tilde{e}_i, \ldots, e_l + 1]$ and $[e_1 + 1, \ldots, \tilde{e}_i, \ldots, e_l, 2]$.

By Lemma 2.2.5, for any Wahl chain, its center is always the curve with more negative discrepancy.

Corollary 2.3.21. Let $[e_1, \ldots, e_l, 1, f_1, \ldots, f_s]$ be an extremal P-resolution of some chain. Then after blowing down all ones that appear, no center is blown down. Moreover, each center will end up as a 3 or greater.

Proof. The proof is by induction in the amount of blow downs to do.

If $e_l, f_1 > 2$, then after one blow down we are done, and in particular no members in the Wahl chains where contracted. If e_l or f_1 was a center, then it was at least a 4, then after blowing down it will be at least a 3.

The case $e_l = f_1 = 2$ cannot happen by remark 2.2.7.

Suppose without loss of generality that $e_l = 2$ and $f_1 > 2$. Then by 2.2.7 $s \geq 2$ and also $f_s = 2$ (since the discrepancy of $e_l > -\frac{1}{2}$, so $[f_1, \ldots, f_s]$ cannot be initial). Note that neither e_l or f_s are centers, so that if the centers of

$$[e_1, \ldots, e_{l-1}, 2, 1, f_1, \ldots, f_{s-1}, 2]$$

are contracted, the same happens for the centers of

$$[e_1 - 1, \ldots, e_{l-1}, 1, f_1 - 1, \ldots, f_{s-1}]$$

since they are shared by the construction algorithm. But by induction, those centers cannot be contracted and will end up as a 3 or greater, and all members of $[e_1 - 1, \ldots, e_{l-1}, 1, f_1 - 1, \ldots, f_{s-1}]$ are greater than or equal to their counterparts in $[e_1, \ldots, e_{l-1}, 2, 1, f_1, \ldots, f_{s-1}, 2]$. $\square$

Definition 2.3.22. Let $[e_1, \ldots, (e_k), \ldots, e_n]$ be a chain with a mark at position k. An extension operation on the chain is to replace it with

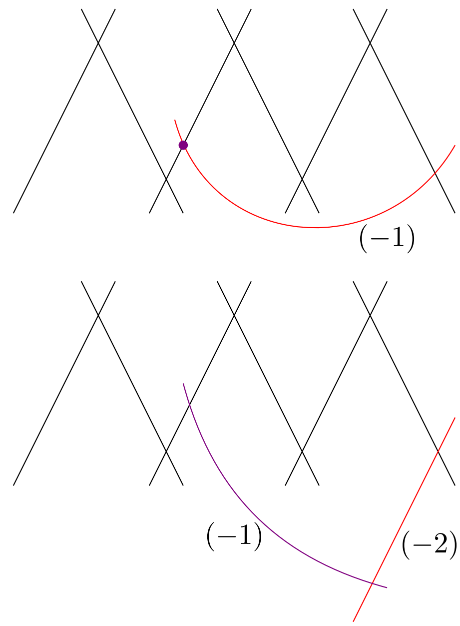
$$[e_1, \ldots, e_{k-1}, (e_k + 1), e_{k+1}, \ldots, e_n, 2].$$

A blow up operation on the chain preserves the marked position, that is,

$$[e_1, \ldots, e_i, e_{i+1}, \ldots, (e_k), \ldots, e_l] \mapsto [e_1, \ldots, e_i + 1, 1, e_{i+1} + 1, \ldots, (e_k), \ldots, e_l], \text{ and}$$

$$[e_1, \ldots, (e_k), \ldots, e_i, e_{i+1}, \ldots, e_l] \mapsto [e_1, \ldots, (e_k), \ldots, e_i + 1, 1, e_{i+1} + 1, \ldots, e_l].$$

This definition recreates the following situation: Suppose we have a chain of smooth rational curves $\mathcal{C} = C_1 \cup \ldots \cup C_n$ in a surface, and a (-1)-curve Γ intersecting C_k and C_n. The operation consists in blowing up $C_k \cap \Gamma$ and appending (the strict transform of) Γ to (the strict transform of) $\mathcal{C}$. The exceptional curve from this blow up becomes the new Γ for the new chain. The following diagram illustrates the process.



Proposition 2.3.23. Let $[e_1, \dots, (e_k), \dots, e_n]$ be a marked chain with $k \neq 1$, $e_i \geq 2$ for all i. Extend the chain any positive amount of times. Then blow up the chain to obtain an extremal P-resolution. If the first member of the second chain is not the marked position, then its value is bounded by $e_1 + \max_{i \neq k} e_i - 1$. In particular, there can be at most $e_1 + \max_{i \neq k} e_i - 3$ extensions.

To give an example of the process in the proposition, consider the marked chain $[e_1, (e_2), e_3] = [3, (2), 4]$. Extend twice to obtain $[3, (4), 4, 2, 2]$, then blow up twice to obtain its extremal P-resolution $[3, (5), 2, 1, 6, 2, 2]$. The first member of the second chain is 6, and it is indeed bounded by $e_1 + \max\{e_1, e_3\} - 1 = 3 + \max\{3, 4\} - 1 = 6$.

Proof of Proposition 2.3.23. Let $[f_1, \dots, (f_k), \dots, f_s]$ be the chain after all $s - n$ extensions. Since at least one extension was done, $f_s = 2$ and $f_k \geq 3$, and since $k \neq 1$, $f_1 = e_1$.

In order to obtain an extremal P-resolution, let the first blow up happen between indices i and $i + 1$, so we are in one of the following scenarios

1. $[e_1, \dots, f_{s-1} + 1, 1, 3]$, (where possibly $f_{s-1} + 1$ is marked), or
2. $[e_1 + 1, 1, f_2 + 1, \dots, 2]$, (where possibly $f_2 + 1$ is marked), or
3. $[e_1, \dots, f_i + 1, 1, f_{i+1} + 1, \dots, 2]$, (where possibly $f_i + 1$ or $f_{i+1} + 1$ is marked).

The first scenario is impossible by Corollary 2.3.21. For the second scenario, thanks to Corollary 2.3.21 we know that $e_1 \geq 3$ and also e_1 must be the center of the first chain, so the first chain must end up as a linear Wahl chain. Then for some $m \geq 0$, we must blow up m times to the left of each of the ones that appear, and then $e_1 + m - 3$ times to the right, so we would end up with either

$$[e_1 + 1, \underbrace{2, \dots, 2}_{e_1 - 3}, 1, f_2 + e_1 - 2, \dots, 2],$$

if $m = 0$, or

$$[e_1 + m + 1, \underbrace{2, \dots, 2}_{e_1 + m - 3}, 1, e_1 + m - 1, \underbrace{2, \dots, 2}_{m - 1}, f_2 + 1, \dots, 2],$$

if $m > 0$.

In the first case, $f_2 + e_1 - 2$ is the first member of the second chain, so it is not marked, so $f_2 = e_2$ and we see that $f_2 + e_1 - 2 \leq e_1 + \max_{i \neq k} e_i - 1$.

In the second case note that the second chain is not a linear Wahl chain, so there must be exactly $e_1 + m - 3$ twos at the end. By Lemma 2.3.19 this is a P-resolution if and only if

$$[4, 1, 2, \underbrace{2, \ldots, 2}_{m-1}, f_2 + 1, \ldots, f_{s-e_1-m+3}]$$

is also a P-resolution. This is not true by Remark 2.2.7, so this case is discarded.

For the third scenario, one possibility is to blow up successively m times to the right of each of the ones that appear, thus appending m twos to the first of the chain and increasing by m at the first index of the second chain, obtaining

$$[e_1, \ldots, f_i + 1, \underbrace{2, \ldots, 2}_{m \text{ times}}, 1, f_{i+1} + m + 1, \ldots, 2].$$

But $[e_1, \ldots, f_i + 1, 2, \ldots, 2]$ is not a linear Wahl chain, so $m = e_1 - 2$. Since $f_{i+1} + m + 1$ is the first member of the second chain, it is not marked, so either $f_{i+1} = e_{i+1}$ if $i + 1 \leq n$ or 2 if $i + 1 > n$. Here we have $f_{i+1} + m + 1 = e_{i+1} + e_1 - 1 \leq e_1 + \max_{j \neq k} e_j - 1$.

The other possibility is to alternate blow ups to the left and right of the ones that appear. However by the same argument above, we must finish with $m = e_1 - 2$ blow ups to the right. Thus after the last blow up to the left we go from

$$[*, 1, 2, *] \quad \text{to} \quad [*, \underbrace{2, \ldots, 2}_{e_1 - 2 \text{ times}}, 1, e_1, *]$$

And obviously $e_1 \leq e_1 + \max_{j \neq k} e_j - 1$. Note that if $e_1 = 2$, then there cannot be alternated blow ups, since the last blow up would need to be at the left, thus adding a 2 to the beginning of the second chain. This chain would have twos at both ends, which cannot happen for Wahl chains.

To bound the number of extensions, the tail of (-2)-curves at the end ot the second chain must be of length $< e_1 + \max_{i \neq k} e_i - 3$ by Example 2.2.4. Finally, every curve appended due to an extension operation must be part of the same tail of (-2)-curves, since no blow up can occur between them thanks to Corollary 2.3.21. $\square$

Chapter 3

Deformations

In this chapter we will introduce the concept of deformation, and how local deformations can be extended to global deformations. We will not go into much details as the studies of deformations are quite extensive. We will follow Hartshorne's Deformation Theory [H09], and we will be using the language of schemes and sheaves.

3.1 Review of Deformation Theory

In general, given a scheme X_0, a deformation of X_0 is a scheme $\mathcal{X}$ together with a flat morphism to a parameter space $f\colon \mathcal{X} \to B$, with a special point $0 \in B$ such that the scheme-theoretic fiber $f^{-1}(0)$ is isomorphic to $\mathcal{X} \times_B 0 \cong X_0$. A deformation $\mathcal{X} \to B$ is said to be $\mathbb{Q}$-Gorenstein if the relative canonical class $K_{\mathcal{X}/B} = K_{\mathcal{X}} - f^*K_B$ is $\mathbb{Q}$-Cartier. As a consequence of this, a $\mathbb{Q}$-Gorenstein deformation preserves K^2 and χ along its fibers.

We can easily determine when a morphism over a curve is flat, given by the following criteria:

Theorem 3.1.1 ([H70, Proposition III.9.7])**.** Let $f\colon X \to Y$ be a morphism of schemes, with Y an irreducible smooth curve. Then f is flat if and only if it maps every associated point of X to the generic point of Y.

As an example of deformation, consider $\mathcal{X} = \{xy = t\} \subseteq \mathbb{A}^3$ and its morphism $f\colon \mathcal{X} \to \mathbb{A}^1$ given by $(x, y, t) \mapsto t$. This is a flat morphism by Theorem 3.1.1, and its special member $\mathcal{X}_0$ is the union of two lines at a point. However, for a $t \neq 0$, $\mathcal{X}_t = f^{-1}(t)$ is a smooth conic in the affine plane, so $\{xy = 0\}$ can be deformed to smooth curves. This deformation is $\mathbb{Q}$-Gorenstein since $\mathcal{X}$ is smooth. It can be checked that this deformation is essentially unique, that is, all deformations of $\{xy = 0\}$ look like this one.

As a motivation for infinitesimal deformations, as in [H70, III.9], whenever we have a deformation $f\colon \mathcal{X} \to B$, the tangent space of $0 \in B$ is described by morphisms $D = \operatorname{Spec}(k[t]/t^2) \to B$ where the image of the unique point of D is 0, and so we get a new morphism by base extension $f'\colon \mathcal{X}' \to D$, where $\mathcal{X}' = \mathcal{X} \times_B D$. This way, the single point $0' \in D$ has $\mathcal{X}' \times_D 0' \cong X_0$, so f' is again a deformation which encodes infinitesimal information of the deformation f.

Then our objective is to study or classify every possible infinitesimal deformation of X. Let $\mathcal{C}$ be the category of local Artinian $\mathbb{C}$-algebras with quotient field $\mathbb{C}$, and $\hat{\mathcal{C}}$ be the category of complete local $\mathbb{C}$-algebras with quotient field $\mathbb{C}$. We define the (covariant) deformation functor $\operatorname{Def}_X\colon \mathcal{C} \to \mathrm{Set}$

the as the one that to each ring A associates the set of equivalence classes of diagrams

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} A \end{tikzcd} \tag{3.1}$$

where $X_0 = \mathcal{X} \times_A \mathbb{C}$ and $\mathcal{X} \to \operatorname{Spec} A$ is flat. This is an infinitesimal deformation. For instance, $\mathcal{X}' \to \operatorname{Spec}(\mathbb{C}[t]/t^2)$ of the motivation above is an element of $\operatorname{Def}_{X_0}(\mathbb{C}[t]/t^2)$. As $\operatorname{Def}_{X_0}(\mathbb{C}[t]/t^2)$ classifies all possible tangent spaces of deformations of X_0, we call it the tangent space to the functor Def_{X_0}. The functor acts on morphisms by pullback: if $A \to B$ is a morphism of $\mathbb{C}$-algebras, for any diagram as in 3.1 in $\operatorname{Def}_{X_0}(A)$ we can extend it as

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \mathcal{X} \times_A \operatorname{Spec} B \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd} \tag{3.2}$$

which induces a diagram in $\operatorname{Def}_{X_0}(B)$. We say Def_{X_0} is pro-representable if there exists a complete local algebra $R \in \hat{\mathcal{C}}$ and an isomorphism of functors, or natural equivalence, $h_R \to \operatorname{Def}_{X_0}$, where $h_R \colon \mathcal{C} \to \mathrm{Set}$ associates A to $\operatorname{Hom}(R, A)$. Being pro-representable would strongly suggests the existence of a scheme and a flat morphism $\xi \colon \mathfrak{X} \to \operatorname{Spec} R$ that fits in the pullback diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

so that it exactly recovers every possible infinitesimal deformation, so we may think of $\operatorname{Spec} R$ as a local moduli space for X.

As it turns out, it will not always be possible for Def_X to be pro-representable ([H09, Examples 18.1.1 and 18.1.2]), so we need some weaker notions of representability. Suppose we have a complete local ring R with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{C}$. First note that $R/\mathfrak{m}^n \in \mathcal{C}$ for every n, and suppose that we have a compatible family of flat morphisms $\xi_n \colon \mathcal{X}_n \to \operatorname{Spec} R/\mathfrak{m}^n$, that is, $\mathcal{X}_n = \mathcal{X}_{n+1} \times_{R/\mathfrak{m}^{n+1}} \operatorname{Spec} R/\mathfrak{m}^n$. This defines a morphism of functors, or natural transformation, $h_R \to \operatorname{Def}_{X_0}$ in the following way: For each $A \in \mathcal{C}$ and $f \colon R \to A$, let $n \gg 0$ be large enough so we can factor $f \colon R \to R/\mathfrak{m}^n \to A$. Then this morphism assigns f to the diagram associated with the pullback $\mathcal{X}_n \times_{R/\mathfrak{m}^n} \operatorname{Spec} A \to \operatorname{Spec} A$. Due to the compatibility condition of the ξ_n, this map is well defined, and it can be seen that every functor $h_R \to \operatorname{Def}_{X_0}$ arises uniquely from this way, so we also identify the functor with $\xi = \{\xi_n\}$.

In the case above, we say (R, ξ) is a formal family. (R, ξ) is said to be a versal family if $\xi \colon h_R \to \operatorname{Def}_X$ is surjective and smooth. The surjective condition means that for every A, every flat morphism $\mathcal{X} \to \operatorname{Spec} A$ can be obtained by pullback from $\mathfrak{X} \to \operatorname{Spec} R$. The smoothness condition states that for every surjective morphism $B \to A$, the natural morphism $h_R(B) \to h_R(A) \times_{\operatorname{Def}_{X_0}(A)} \operatorname{Def}_{X_0}(B)$ is also surjective. This means that given A and B in $\mathcal{C}$, whenever we have a diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \begin{array}{c}\mathcal{X}_n \times_{R/\mathfrak{m}^n} \operatorname{Spec} A \\ \| \\ \mathcal{Y} \times_B \operatorname{Spec} A\end{array} \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathcal{Y} \arrow[d] & \mathcal{X}_n \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} A \arrow[r, hook] \arrow[rr, bend right] & \operatorname{Spec} B & \operatorname{Spec} R/\mathfrak{m}^n
\end{tikzcd}$$

then after possibly increasing n we may fill the gaps: we can extend the morphism $\operatorname{Spec} A \to \operatorname{Spec} R/\mathfrak{m}^n$ to a morphism $\operatorname{Spec} B \to \operatorname{Spec} R/\mathfrak{m}^n$ such that $\mathcal{Y} = \mathcal{X}_n \times_{R/\mathfrak{m}^n} \operatorname{Spec} B$.

We also say that (R, ξ) is miniversal or a pro-representable hull if $\xi(D) \colon h_R(D) \to \operatorname{Def}_{X_0}(D)$ is a bijection, where $D = \mathbb{C}[t]/t^2$. A miniversal family is unique up to non-unique isomorphism. If $\xi \colon h_R \to \operatorname{Def}_{X_0}$ is an isomorphism, then we say (R, ξ) is universal. A universal family is unique up to unique isomorphism.

The existence of versal families is granted once Schlessinger's conditions are met ([H09, Theorem 6.2]), and this in turn strongly suggests the existence of a local 'coarse' moduli space for X_0. The Schlessinger's conditions are satisfied whenever X_0 is projective, or affine with isolated singularities ([H09, Theorem 18.1]).

Once a versal family is found, we can take the limit over $\mathcal{X}_n \to \operatorname{Spec} R/\mathfrak{m}^n$ to get a local deformation $\mathcal{X} \to \operatorname{Spf} R$ ([H70, §II.9]). The question now is, given this deformation over the formal spectrum, does there exist some $\overline{X} \to \operatorname{Spec} R$ flat such that the completion along the closed fiber is $\mathcal{X}$. In this case we say that $\mathcal{X}$ is *effective*.

Theorem 3.1.2 (Grothendieck, [H09, Theorem 21.2])**.** Let $\mathcal{X}$ be a formal scheme, proper over $\operatorname{Spf} R$, and suppose there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L} \otimes_R \mathbb{C}$ is ample on $X_0 = \mathcal{X} \times \mathbb{C}$. Then $\mathcal{X}$ is effective.

One can verify the existence of such line bundle when $H^2(\mathcal{O}_{X_0}) = p_g(X_0) = 0$, and this will happen in our case.[1]

Now comes Artin's Algebraization theorem, which states that if X_0 is a projective scheme and $\mathcal{X} \to \operatorname{Spf} R$ is effective, then there exists a scheme S of finite type over k, $s_0 \in S$ and a flat and finite type $X \to S$ such that $X \times s_0 = X_0$, and the formal completion of X along s_0 is $\mathcal{X}$. This S is unique locally around s_0 up to étale covering. This is the "moduli" space we were looking for, however it is not a real moduli space since it does not necessarily parametrizes different surfaces up to isomorphism.

Let U be a neighborhood of a singularity of type T or ADE. Then U has a $\mathbb{Q}$-Gorenstein smoothing ([KSB88, Proposition 3.10]). In particular, its versal deformation space, also called Def_U has non-trivial deformations. In particular, it contains a component consisting of $\mathbb{Q}$-Gorenstein deformations denoted by $\operatorname{Def}_{QG,U}$.

Now let X be a projective slc (semi log canonical) surface. Let $\mathcal{T}_X^i = \mathcal{E}xt^i(\Omega_X, \mathcal{O}_X)$, so that $\mathcal{T}_X^0 = \mathcal{T}_X = \operatorname{Hom}(\Omega_X, \mathcal{O}_X)$ the usual tangent sheaf of X. Let $\mathbb{T}_X^1 = \operatorname{Ext}_X^1(\Omega_X, \mathcal{O}_X)$ the set of first infinitesimal deformation classes of X. Then there is an exact sequence ([H09, Exercise 5.7])

$$0 \to H^1(X, \mathcal{T}_X) \to \mathbb{T}_X^1 \to H^0(X, \mathcal{T}_X^1) \to H^2(X, \mathcal{T}_X),$$

where $H^0(X, \mathcal{T}_X^1)$ represents the local infinitesimal deformations of first order. The image of an infinitesimal deformation of $H^0(X, \mathcal{T}_X^1)$ in $H^2(X, \mathcal{T}_X)$ is called the local-to-global obstruction to

[1] In characteristic zero, this might not be needed. See [M91, Lemma 1]

this deformation, and so a global deformation exists if and only this element is zero. This means that if we prove that $H^2(X, \mathcal{T}_X)$ vanishes for a surface X, then every local first order deformation of X, including $\mathbb{Q}$-Gorenstein deformations, can be globalized. However to lift this deformation to bigger Artinian rings, it is necessary for some other obstructions to vanish. These obstructions lie now in $\mathbb{T}^2_X = \operatorname{Ext}^2(\Omega_X, \mathcal{O}_X)$. It can be further verified that this group vanishes if $H^2(X, \mathcal{T}_X) = 0$ (see [H12, §3]).

3.2 Local to Global Obstruction

Let X be a $p_g = 0$ variety with quotient singularities. Then to guarantee that there are no local-to-global obstructions to deformations of X it is sufficient that $H^2(X, \mathcal{T}_X)$ vanishes. Here we will explain some situations where this happens.

Definition 3.2.1. Let D be a SNC divisor on a smooth surface X. The sheaf of differentials with simple poles along D is denoted by $\Omega^1_X(\log D)$ [EV92, Definition 2.1]. The dual of this sheaf is the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$, whose sections are vector fields tangent to D.

Proposition 3.2.2 ([LP07, Theorem 2])**.** Let X be a normal surface with singularities of class T or rational double points, and $Y \to X$ be the minimal resolution of X and E the reduced exceptional divisor. Then $h^2(Y, \mathcal{T}_Y(-\log E)) = h^2(X, \mathcal{T}_X)$.

Proposition 3.2.3 ([FZ94, Proposition 1.5])**.** Let X be a smooth surface and $\pi\colon Y \to X$ be the blow up at $p \in X$. Let D be a reduced simple normal crossing divisor on X and $\tilde{D} = \pi^*(D)_{\text{red}}$. Then

$$h^2(X, \mathcal{T}_X(-\log D)) = h^2(Y, \mathcal{T}_Y(-\log \tilde{D})).$$

Proposition 3.2.4 ([FZ94, Proposition 1.7])**.** Let X be a smooth surface and D be a reduced simple normal crossing divisor containing a curve E with $E^2 \geq -1$.Then

$$h^2(X, \mathcal{T}_X(-\log(D - E))) = h^2(X, \mathcal{T}_X(-\log D)).$$

With 3.2.2 we translate the problem in X to Y. The general strategy is as follows. Start with a divisor D on a surface S for which we know has $h^2(\mathcal{T}_S(-\log D)) = 0$. Then blow up as much as we want while possibly adding or subtracting curves (as long as $E^2 \geq -1$) until we arrive to a divisor $\tilde{D}$ with only disjoint Wahl chains, while preserving $h^2(\mathcal{T}_{\tilde{S}}(-\log \tilde{D})) = 0$. We have an additional tool to add (-2)-curves in some cases.

Proposition 3.2.5 ([PSU13, Theorem 4.4])**.** Let D be a reduced simple normal crossing divisor and B a reduced divisor of (-2)-curves in some of the ADE configurations, such that $D \cap B = \varnothing$. Then

$$h^2(X, \mathcal{T}_X(-\log D)) = h^2(X, \mathcal{T}_X(-\log(D + B))).$$

Recall that the ADE configurations are precisely those whose dual graph is one of the simply laced Dynkin diagrams (see Section 1.4).

Example 3.2.6. Consider a rational elliptic fibration S with two nodal rational fibers (cf. Section 5.1), and let $S' \to S$ be the blow up at the two nodes, and let $D = F_1 + F_2$, where F_1 and F_2 are the strict transforms of the rational fibers. This situation is shown in Figure 3.1.

Let $S' \to Z$ be the contraction of the curves F_1 and F_2. Since both their images are Wahl singularities, they have a local $\mathbb{Q}$-Gorenstein smoothing. We will later see that $H^2(S', \mathcal{T}_{S'}(-\log D)) = 0$ (cf. Proposition 5.1.4), so we obtain $H^2(Z, \mathcal{T}_Z) = 0$. This means that the local smoothings can be globalized. It can be verified that the general smooth member of a deformation is an Enriques surface ([U16, Theorem 4.2]).

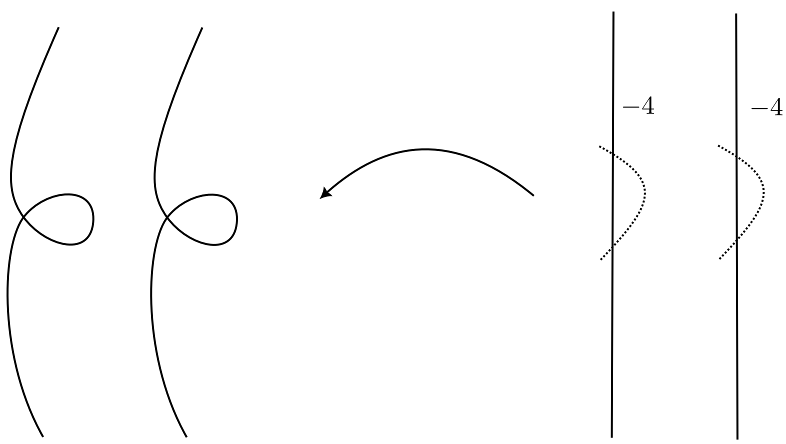


Figure 3.1: The surface S and its blow up

Chapter 4

Topology of the Smoothing

In this chapter, we will deal with the analytic topology of varieties, and so we will deal with the topological fundamental group. We will give a criterion for the fundamental group of the general fiber of a $\mathbb{Q}$-Gorenstein smoothing to be trivial. This is useful since surfaces with different fundamental groups belong to different components of the moduli space of stable surfaces with fixed K^2 and χ (although different components may have the same fundamental group), so by this criterion we can identify to which component the surface we are constructing belongs. Unfortunately, this criterion only works with simply connectedness, and if a particular example fails some of the hypotheses, it seems to be very difficult to actually determine the fundamental group.

We will also talk about exotic structures on complex surfaces. This is because surfaces obtained via $\mathbb{Q}$-Gorenstein deformations are sometimes homeomorphic but not diffeomorphic to simpler complex manifolds.

4.1 Symplectic Rational Blow Down

We start with a construction from symplectic geometry, namely the (generalized) rational blow-down ([FS95]). They are analogous to a $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity in the symplectic category. A symplectic manifold M is a smooth manifold equipped with a closed non-degenerate 2-form. A smooth algebraic variety or complex manifold is naturally also a symplectic manifold.

Let $1 \leq a < n$ be coprime integers, and μ_n a primitive n-th root of unity. Let $D = \{p \in \mathbb{C}^2 \mid |p| \leq 1\}$ be the unitary ball in $\mathbb{C}^2$ and $P \in D/G$ be the quotient singularity of type $\frac{1}{n^2}(1, na-1)$, where the group $G \cong \mathbb{Z}/n^2\mathbb{Z}$ acts on D by $(x, y) \mapsto (\mu_{n^2}x, \mu_{n^2}^{na-1}y)$. The lens space $L(n^2, na-1)$ is defined as $\partial D/G = S^3/G$. As in [P97], $L(n^2, na-1)$ not only bounds the quotient singularity $\frac{1}{n^2}(1, na-1)$ but also bounds a (smooth) rational homology ball $B_{n,a}$.

Definition 4.1.1 (Generalized Rational Blow Down, [FS95, P97])**.** Let Y be a surface (therefore a symplectic 4-fold) with a Wahl chain $C_{n,a}$ The rational blow down of $C_{n,a}$ is as follows. Consider a small neighborhood V of $C_{n,a}$ such that $\partial V = L(n^2, na-1)$ (for example, under the contraction of $C_{n,a}$, $\pi\colon Y \to X$ such that $\pi^{-1}(P) = C_{n,a}$, there exists a small neighborhood $U \subseteq X$ of P such that $\partial U = L(n^2, na-1)$, so let $V = \pi^{-1}(U)$). The rational blow down is the surface $\tilde{X}$ obtained by replacing V with the rational homology ball $B_{n,a}$ (more precisely, $\pi_1(B_{n,a})$ is finite and cyclic, and $H_i(B_{n,a}, \mathbb{Q}) = 0$ for all $i > 0$). $\tilde{X}$ is uniquely determined up to diffeomorphism and also admits a symplectic structure.

In this generality, the rational blow down of an algebraic surface need not to be algebraic. However, as in [LP07, Proposition 8], if $\mathcal{X} \to \Delta$ is a $\mathbb{Q}$-Gorenstein smoothing of a surface X_0 with only Wahl singularities, then the general fiber is diffeomorphic to the rational blow down of the Wahl chains of the minimal resolution $Y \to X$. Therefore it suffices to find a condition for $\tilde{X}$ to be simply connected.

4.2 Criterion For Simply Connectedness

In what follows of the chapter, we will recap of the strategy applied in [LP07, Theorem 3].

Lemma 4.2.1. Let X be a surface with only Wahl singularities $P_1, \ldots, P_m$, $Y \to X$ its minimal resolution, and $\tilde{X}$ the rational blow down of the corresponding Wahl chains. If $X_0 = X - \{P_1, \ldots, P_m\}$ is simply connected, then so is $\tilde{X}$.

Proof. Suppose that the points P_i are of type $\frac{1}{n_i^2}(1, n_i a_i - 1)$, and let $U_1, \ldots, U_m$ small disjoint neighborhoods of $P_1, \ldots, P_m$ such that $\partial U_i = L(n_i^2, n_i a_i - 1)$. This way we have

$$X = X_0 \cup U_1 \cup \ldots \cup U_m$$

and

$$\tilde{X} = X_0 \cup B_{n_1,a_1} \cup \ldots \cup B_{n_m,a_m}$$

Define $X_i = X_0 \cup B_{n_1,a_1} \cup \ldots \cup B_{n_i,a_i}$, $i = 0, \ldots, m$. We will now prove by induction that X_i is simply connected. This is true by hypothesis for X_0. Suppose now that X_{i-1} is simply connected.

It is known that the inclusion induced homomorphism

$$\iota^* \colon \pi_1(L(n_i^2, n_i a_i - 1)) \cong \mathbb{Z}/n_i^2\mathbb{Z} \to \pi_1(B_{n_i,a_i}) \cong \mathbb{Z}/n_i\mathbb{Z}$$

is surjective (see for example [FS95] for the case of linear Wahl chains). By the theorem of Seifert-Van Kampen, since $X_{i-1} \cap B_{n_i,a_i}$ has the homotopy type of $\partial B_{n_i,a_i} = L(n_i^2, n_i a_i - 1)$, we know that

$$\pi_1(X_i) = \pi_1(X_{i-1}) *_{\pi_1(L(n_i^2, n_i a_i - 1))} \pi_1(B_{n_i,a_i}) = (\mathbb{Z}/n_i\mathbb{Z}) \ / \ N(\iota^*(\mathbb{Z}/n_i^2\mathbb{Z})) = \{e\}$$

Where $N(G)$ is the normal subgroup generated by G. In particular, $\tilde{X} = X_n$ is simply connected. $\square$

From the above proof we can see that if $\pi_1(X_0)$ is not simply connected, then it can be very difficult to determine the amalgamated free product of $\pi_1(X_0)$ and $\mathbb{Z}/n_1\mathbb{Z}$. This complexity increases along with the amount of singularities.

Let $\pi \colon Y \to X$ be the minimal resolution of X, and let $\mathcal{C}_i = \pi^{-1}(P_i) = C_{i,1} \cup \ldots \cup C_{i,l_i}$ be the corresponding Wahl chains. Let V_i be a small neighborhood of the chain $\mathcal{C}_i$, so that $Y = X_0 \cup V_1 \cup \ldots \cup V_m$. Note that $\mathcal{C}_i$ is a wedge of spheres, so $\pi_1(\mathcal{C}_i) = 1$, and there is a deformation retract $V_i \to \mathcal{C}_i$ (see the construction of φ in [M61]), so $\pi_1(V_i) = 1$. We know that $M_i = \partial V_i \cong L(n_i^2, n_i a_i - 1)$. By [M61], the group $\pi_1(M_i) \cong \mathbb{Z}/n_i^2\mathbb{Z}$ is generated by a loop $\alpha_{i,1} \subseteq M_i$ around a non-intersection point of $C_{i,1}$. Again by [M61], it is possible to choose orientations of loops $\alpha_{i,j} \subseteq M_i$ around non-intersection points of $C_{i,j}$ so that we obtain the relations in $H_1(M_i)$ ($\cong \pi_1(M_i)$)

$$\left.\begin{array}{llllll}
+e_{i,1}\alpha_{i,1} & -\alpha_{i,2} & & & & = 0 \\
-\alpha_{i,1} & +e_{i,2}\alpha_{i,2} & -\alpha_{i,3} & & & = 0 \\
 & -\alpha_{i,2} & +e_{i,3}\alpha_{i,3} & -\alpha_{i,4} & & = 0 \\
 & & & \cdots & & \\
 & & & -\alpha_{i,l_i-1} & +e_{i,l_i}\alpha_{i,l_i} & = 0
\end{array}\right\}, \tag{4.1}$$

where $\frac{n_i^2}{n_i a_i - 1} = [e_{i,1}, \dots, e_{i,l_i}]$. Note that the associated matrix has determinant n_i^2, and the system has a unique solution in terms of $\alpha_{i,1}$ or α_{i,l_i} modulo n_i^2. It is not immediate from the result in homology, but if $\tau_{i,j}$ is an appropriate path from a basepoint to the start of $\alpha_{i,j}$, then, for example, $\tau_{i,1}\alpha_{i,1}^{e_{i,1}}\tau_{i,1}^{-1} = \tau_{i,2}\alpha_{i,2}\tau_{i,2}^{-1}$. Note that solving for $\alpha_{i,j}$ in terms of $\alpha_{i,1}$ follows the same formulas as the construction of the b_i's. This means that $\alpha_{i,j}$ is homotopic to $\alpha_{i,1}^{b_j}$.

Proposition 4.2.2. Suppose that Y is simply connected and that $\alpha_{i,1}$ can be contracted in X_0 for every $i = 1, \dots, m$. Then X_0 is also simply connected.

Proof. Let $Y_i = X_0 \cup V_{i+1} \cup \dots \cup V_m$ so that $Y_0 = Y$ and $Y_m = X_0$. We will prove by induction that Y_i is simply connected. This is true by hypothesis for Y_0. Suppose that Y_{i-1} is simply connected. By Seifert-Van Kampen's theorem, we have

$$1 = \pi_1(Y_{i-1}) = \pi_1(V_i) *_{\pi_1(M_i)} \pi_1(Y_i) = \pi_1(Y_i) \ / \ N(\iota_i^*(\pi_1(M_i))),$$

because V_i is simply connected. Therefore $\pi_1(Y_i) = N(\iota_i^*(\pi_1(M_i))) = N(\iota_i^*(\alpha_{i,1}))$, where $\iota_i \colon M_i \to Y_i$ is the inclusion. But since $\alpha_{i,1}$ can be contracted in $X_0 \subseteq Y_i$, we obtain $\pi_1(Y_i) = N(e) = \{e\}$. $\square$

Thus the two hypothesis we require is that the surface Y is simply connected (which will always be true in our case) and that the $\alpha_{i,1}$ can be contracted in $Y - \{P_1, \dots, P_m\}$. We will now present two situations which gives us very good information for determining that $\alpha_{i,1}$ can be contracted in X_0.

1. Suppose that there exists a 2-sphere $S \subseteq Y$ (for example, an algebraically embedded $\mathbb{P}^1$) that intersects $\bigcup_i \mathcal{C}_i$ transversally in a single point $Q \in C_{i,j}$. Then $M_i \cap S$ consists of a loop homotopy equivalent to a single loop γ around Q, which by the system 4.1 must be homotopic to $\alpha_{i,1}^{\pm k}$ for some k, depending on the orientation of γ. But $S \cap X_0$ is a punctured sphere, so we can deform γ along $S \cap X_0$ until it contracts. This shows that $\alpha_{i,1}^{\pm k} = e$ in X_0. In particular, if $j = 1$ or l_i or if k is coprime with n_i, then immediately $\alpha_{i,1}$ contracts in X_0.

2. Suppose that there exists a 2-sphere $S \subseteq Y$ that intersects $\bigcup_i \mathcal{C}_i$ transversally at two points, $Q_1 \in C_{i_1,j_i}$ and $Q_2 \in C_{i_2,j_2}$. Then as before they define loops γ_1 and γ_2 loops in $S \cap X_0$ around Q_1 and Q_2. But $S \cap X_0$ is a twice punctured sphere with two loops around each hole. Thus γ_1 and γ_2 are homotopic up to perhaps changing orientation. If γ_1 is homotopic to $\alpha_{i_1,1}^{k_1}$ and γ_2 is homotopic to $\alpha_{i_2,1}^{k_2}$ and if τ is a path from the start of γ_1 to the start of γ_2 along S and ρ_1 (resp. ρ_2) a path from the start of $\alpha_{i_1,1}$ to γ_1 (resp. from $\alpha_{i_2,i}$ to γ_2), we have $\rho_2^{-1}\alpha_{i_1,1}^{k_1}\rho_2 = \tau\rho_1^{-1}\alpha_{i_2,1}^{\pm k_2}\rho_1\tau^{-1}$.

   For example if $\gcd(k_1, n_{i_1}) = \gcd(n_{i_1}, n_{i_2}) = 1$, then immediately $\alpha_{i_1,1}$ is contractible in X_0 since we can annihilate the loop. This situation happens very frequently.

In the Example 3.2.6, none of these conditions hold, so we cannot verify that the rational blow down of Z, or equivalently, the general smooth member W of a $\mathbb{Q}$-Gorenstein smoothing, is simply connected. In fact, as these members are Enriques surfaces, we know that $\pi_1(W) \cong \mathbb{Z}/2\mathbb{Z}$.

4.3 Exotic Structures on $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$

In this section, the name $\mathbb{CP}^n$ is used for the n-th dimensional complex projective space. Many statements will be given without proof.

We start by roughly defining the connected sum of two manifolds. Let X, Y be two oriented, connected n-dimensional differential manifolds. The connected sum $X\#Y$ is the surface obtained by removing small discs from X and Y and glueing along the borders in a smooth manner while preserving orientation. This surgery operation is unique up to diffeomorphism, and if X, Y are compact manifolds, so is $X\#Y$. This way, by $pX\#qY$ we mean

$$\underbrace{X\#\ldots\#X}_{p}\#\underbrace{Y\#\ldots\#Y}_{q}$$

By $\overline{X}$ we mean the manifold obtained from X by reversing its orientation. In this way, the blow up of X at one point is actually $X\#\overline{\mathbb{CP}^2}$ ([H04, Proposition 2.5.8]). Our objective is show that there exist 4-manifolds homeomorphic to $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$, but not diffeomorphic to it. There are, however, examples in the literature with n as low as 2 (see for instance [AP10]).

The cornerstone of our approach is a theorem of Freedman, [F82, Theorem 1.5], one of whose consequence is the following

Theorem 4.3.1 ([BHPV03, Corollary IX.1.2])**.** The oriented homeomorphism type of any simply connected differentiable 4-manifold V is completely determined by its intersection form S_V, where $S_V\colon H^2(V,\mathbb{Z})\times H^2(V,\mathbb{Z})\to\mathbb{Z}$ is given by $S_V(\alpha,\beta)=\langle\alpha\cup\beta,[V]\rangle$.

And intersection forms are in turn determined by their rank $r=b^+ + b^-$, index $\tau = b^+ - b^-$ and parity, where (b^+, b^-) is the signature of S_V. In the notation of [BHPV03, Section IX.2], if H is the intersection form of $\mathbb{CP}^1\times\mathbb{CP}^1$ (that is, given by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$), there exist two families of intersection forms, namely

$$F^{p,q} = p\langle\mathbb{1}\rangle\oplus q\langle-\mathbb{1}\rangle,\quad \tau = p-q \text{ (for odd forms)}$$

$$E^{p,\pm q} = pH\oplus q(\pm E_8),\quad \tau = \pm 8q \text{ (for even forms)}$$

Here we note that $F^{p,q}$ is the intersection form of $p\mathbb{CP}^2\#q\overline{\mathbb{CP}^2}$, so we need a tool to discard the even case. This tool is given in the form of a theorem of Rochlin.

Theorem 4.3.2 ([BHPV03, Theorem IX.2.1])**.** For differentiable 4-manifolds with even intersection form, τ is divisible by 16.

From here, let X be a symplectic manifold, and $c_1(X)$, $c_2(X)$ be the Chern classes of X. We know that $c_2(X)=e(X)$, the topological Euler characteristic of X, and $c_1(X)=-K_X$. b_1 and b_2 denote the Betti numbers of X. We have $b_2 = b^+ + b^-$, and as a consequence of Poincaré duality, $e(X) = 2-2b_1+b_2$. If X is simply connected, then $b_1=0$, since this is just the rank of $H_1(X)$.

Theorem 4.3.3 (Index theorem of Thom-Hirzebruch, [BHPV03, Theorem I.3.1])**.** If X is a compact, connected, oriented 4-manifold that admits an almost-complex structure (in particular, a symplectic structure), then

$$\tau(X) = b^+(X) - b^-(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X)).$$

Theorem 4.3.4 (Noether Formula)**.** Let X be a projective algebraic surface. Then

$$12\chi(X) = c_1^2(X) + c_2(X).$$

By this formula, we may define the invariant $\chi(X) = \frac{1}{12}(c_1^2(X)+c_2(X))$ for any compact, connected, oriented 4-manifold. With these formulae, we can deduce the following:

Theorem 4.3.5. Let X be a compact, simply connected, oriented 4-manifold that admits an almost-complex structure. Then

$$b^+(X) = 2\chi(X) - 1,$$
$$b^-(X) = 10\chi(X) - 1 - c_1(X)^2, \text{ and}$$
$$\tau(X) = c_1(X)^2 - 8\chi(X)$$

Since $\tau = c_1^2 - 8\chi$, unless $c_1^2 = 8\chi + 16k$, for $k \in \mathbb{Z}$, then by Rochlin's theorem it would be impossible that X had an even intersection form. Therefore, it has the same intersection form as $(2\chi - 1)\mathbb{CP}^2\#(10\chi - 1 - c_1^2)\overline{\mathbb{CP}^2}$, so it is homeomorphic to it due to Friedman's work.

We will now give a criterion for determining that X is not diffeomorphic to $(2\chi-1)\mathbb{CP}^2\#(10\chi-1-c_1^2)\overline{\mathbb{CP}^2}$. The following is a proof of a well known theorem, which was implicitly used in [PPS09a] and [PPS09b].

Theorem 4.3.6. Let Y be a smooth $\chi(Y) = \chi$ algebraic surface with disjoint Wahl chains and let $Y \to Z$ be the contraction of the chains, so that Z has only Wahl singularities. Let X be the symplectic rational blow down of the Wahl chains. If X is simply connected and K_Z is ample, with $K_Z^2 \not\equiv 8\chi \mod 16$ and $K_Z^2 \neq 10\chi - 1$, then X is an exotic $(2\chi - 1)\mathbb{CP}^2\#(10\chi - 1 - K_Z^2)\overline{\mathbb{CP}^2}$.

Proof. Let the i-th Wahl chain $\mathcal{C}_i$ be of type $T(1, n_i, a_i)$, $i = 1, \ldots, k$, so that its image via $Y \to Z$ is the singularity P_i. Choose small disjoint neighborhoods D_i of P_i such that $\partial D_i = L(n_i^2, n_i a_i - 1)$. Let $V \subseteq Z$ be the smooth locus of Z, so that $Z = V \cup \bigcup_i D_i$. The rational blow down surgery replaces D_i with a rational homology ball B_i, so that $X = V \cup \bigcup_i B_i$. We will now see that up to a multiple, every symplectically embedded sphere is in the same homology class to some other surface contained just in V. Let $E \subseteq X$ be the embedded sphere, and denote by $V_j = V \cup \bigcup_{i>j} B_i$, so that $V_0 = X$ and $V_k = V$. The Mayer-Vietoris sequence tells us that for $i = 1, \ldots, k$

$$H_2(B_i) \oplus H_2(V_i) \to H_2(V_{i-1}) \to H_1(\partial B_i)$$

is exact. Since B_i is a rational homology ball, $H_2(B_i)$ is finite, and $H_1(\partial B_1) = \mathbb{Z}/n_i^2\mathbb{Z}$ is also finite. Therefore, any 2-manifold in $H_2(V_{i-1})$ can be "moved" to $H_2(V_i)$ up to multiplying by some large integer. Repeating the process k times, we get that $[mE] \in H_2(V)$ for some large m.

Now, since K_Z is ample, K_V must also be ample, so $F \cdot c_1(X) < 0$ for any symplectically embedded 2-manifold F. So, given E, choose F such that $[F] \in H_2(V)$ and $[F] = [mE]$. Then we have

$$nE \cdot c_1(X) = F \cdot c_1(X) = F \cdot c_1(V) < 0$$

By applying adjunction formula, we obtain $0 > E \cdot c_1(X) = 2 + E^2$, so $E^2 \neq -1$. This means that X is minimal.

Since $\chi(X) = \chi(Y) = \chi$, X is simply connected and $c_1(X)^2 = K_Z^2 \not\equiv 8\chi \mod 16$, X is homeomorphic to $(2\chi - 1)\mathbb{CP}^2\#(10\chi - 1 - K_Z^2)\overline{\mathbb{CP}^2}$. To finalize, we note that a minimal surface cannot be diffeomorphic to a blow up of $(2\chi - 1)\mathbb{CP}^2$ (here we use $K_Z^2 \neq 10\chi - 1$), so this is an exotic example. $\square$

From this result we also know that it would be impossible to find such a surface Y with $K_Z^2 > 10\chi(Y) - 1$. We will see later (cf. §5.3), that in fact $K_Z^2 \leq 9\chi(Y) - 1$.

In our case, with $\chi = 1$, as long as $K_Z \neq 8$ (since $K_Z \neq 9$ by the above restriction), X is an exotic $\mathbb{CP}^2\#(9 - K_Z^2)\overline{\mathbb{CP}^2}$. If $K_Z^2 = 8$, then X might instead be homeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Chapter 5

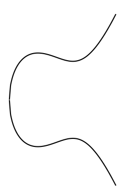
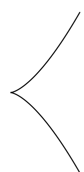
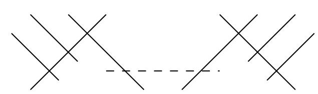
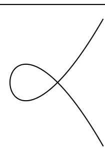
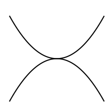
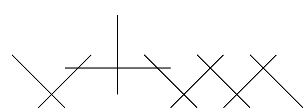
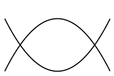
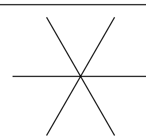
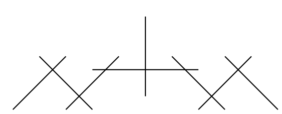
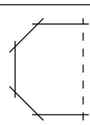
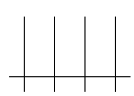
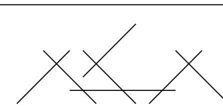
The Problem of Finding Surfaces with Wahl Chains

5.1 Elliptic Fibrations

Definition 5.1.1. An elliptic fibration is a surface S together with a morphism $\varphi\colon S \to C$ where C is a smooth curve such that every fiber is connected and almost every fiber is a smooth curve of genus 1. It is minimal if no fiber contains (-1)-curves.

A trivial example of an elliptic fibration is the product of an elliptic curve with any other curve. As a less trivial example, it is known that any surface S with Kodaira dimension $\kappa(S) = 1$ admits an elliptic fibration ([B96, Proposition IX.2]). We will be dealing with rational elliptic fibrations arising from pencils of cubics in $\mathbb{P}^2$. But before we explain that construction, we will talk about the singular fibers of an elliptic fibration. The following classification and notation is due to Kodaira.

Theorem 5.1.2 (Kodaira's classification of singular fibers [K63, Theorem 6.2])**.** Let $S \to C$ be a minimal elliptic fibration. The following table classifies every possible fiber of S.

Symbol	Configuration	Symbol	Configuration	Symbol	Configuration
I_0		II		I_n^*	
I_1		III		II^*	
I_2		IV		III^*	
I_n		I_0^*		IV^*	

Here, the I_n fiber has n components, and I_n^* has $n+5$ components. Note that for any singular fiber that is not of type I_1 or II, any strict subset of its curves consists of disjoint ADE configurations. The multiplicities of curves in the fibers are not included, but they are always 1 for singular fibers of types I_n, II, III and IV in an elliptic surface with a section. The reader may refer to [P90] for an exhaustive list of every possible combination of fibers that exist for rational elliptic fibrations in characteristic zero.

We now proceed with the construction of an elliptic surface given a cubic pencil in $\mathbb{P}^2$. Let $f, g \in \mathbb{C}[x,y,z]$ be two coprime homogeneous polynomials of degree 3 that have no singularities in common. They define the pencil of curves in $\mathbb{P}^2$

$$\Phi_{[s,t]} = \{sf + tg = 0\}, \quad [s,t] \in \mathbb{P}^1$$

where the conditions of coprimality and no common singularities guarantee that the general member of the family is smooth. These polynomials also define a rational function $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ by

$$\varphi(P) = [-g(P), f(P)]$$

This rational function is undefined precisely in the 9 points of intersection between $\{f = 0\}$ and $\{g = 0\}$, counted with multiplicity. This means that after a sequence of 9 (possibly infinitely near) blow ups $\psi\colon S \to \mathbb{P}^2$ we obtain a morphism $\pi\colon S \to \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[dr, "\pi"'] & \mathbb{P}^2 \arrow[d, dashed, "\varphi"] \\ & \mathbb{P}^1 \end{tikzcd}$$

is commutative. Note that for a general $[s,t] \in \mathbb{P}^1$,

$$\varphi^{-1}([s,t]) = \left\{ P \in \mathbb{P}^2 \mid \frac{s}{t} = \frac{-g(P)}{f(P)} \right\} = \left\{ P \in \mathbb{P}^2 \mid sf(P) + tg(P) = 0 \right\} = \Phi_{[s,t]}$$

is a smooth curve of genus one, since it is a smooth curve of degree 3 in the plane ([H70, Exercise II.8.4]). Since there are finitely many exceptional curves of ψ, a general fiber of $\pi\colon S \to \mathbb{P}^1$ contains none of those exceptional curves, and since $\Phi_{[s,t]}$ is a smooth curve, it is isomorphic to its strict transform $\widehat{\Phi_{[s,t]}} = \pi^{-1}([s,t])$, and thus, $\pi\colon S \to \mathbb{P}^1$ is an elliptic fibration. Since ψ is the minimal blow up to solve the indeterminacies of φ, then there can be no (-1)-curves in any fiber, so the fibration is actually minimal.

Definition 5.1.3. An n-multi-section of S is a curve which intersects the general fiber F n times. A rational 1-multi-section is simply a section. A rational 2-multi-section (resp. 3-multi-section) may also be called double section (resp. triple section).

Note that a section admits a (true) section $\mathbb{P}^1 \to S$. Also, $K_S \sim -F$ for any fiber F. By adjunction formula, E is a smooth rational n-multi-section if and only if $E^2 = -2 - K_S \cdot E = n - 2$. In particular, a (-1)-curve is a section. Note that S contains sections, as the last blow up in the resolution $S \to \mathbb{P}^2$ is a (-1)-curve. In the same way, a smooth curve E is a fiber component if and only if $E^2 = -2$. As a consequence of $K_S \sim -F$, we also have $K_S^2 = 0$. This can also be seen as S being the blow up at 9 points of $\mathbb{P}^2$, because $K_{\mathbb{P}^2}^2 = 9$ and each blow up decreases it by one.

Another important property of rational elliptic fibrations is that they are simply connected. This is because $\mathbb{P}^2$ is simply connected and blowing up preserves fundamental group. This is because the blow up of X is homeomorphic to $X \# \overline{\mathbb{CP}^2}$: Extracting a disc from both 4-manifolds preserves fundamental group since their topological dimension is greater than 2, and the connected sum also preserves fundamental group thanks to Seifert-Van Kampen's theorem, due to $\mathbb{CP}^2$ being simply connected.

We will now deal with cohomological invariants of elliptic surface that will be useful for proving that certain surfaces will have no obstruction for deformations. The following is a generalization of [PSU13, Theorem 2.1].

Proposition 5.1.4. Let F_1, F_2 two singular fibers of type I_{n_1}, I_{n_2} in the rational elliptic surface S, with $n_1, n_2 \geq 1$, and let $W \to S$ the blow up at the nodes of F_1 or F_2 if they were of type I_1. Call $\tilde{F}_1$ and $\tilde{F}_2$ their strict transforms. Then

$$h^2(W, \mathcal{T}_W(-\log(\tilde{F}_1 + \tilde{F}_2))) = 0.$$

The original version of this theorem dealt with the case when both F_1 and F_2 are of type I_1. The proof we will give is a slight modification of the one given in [PSU13]. We must first recall a property of sheaves of differentials with poles.

Proposition 5.1.5 ([EV92, Properties 2.3])**.** Let X be a smooth surface and $D = \sum_i D_i$ be a SNC divisor. One has the following exact sequences

1. $0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_i \mathcal{O}_{D_i} \to 0$
2. $0 \to \Omega^1_X(\log D) \to \Omega^1_X(\log(D - D_1))(D_1) \to \Omega^1_{D_1}(D|_{D_1}) \to 0$

Proof of Proposition 5.1.4. Let $Z \to S$ be the blow up at every node of F_1 and F_2 and denote by $\hat{F}_1 = \sum_{i=1}^{n_1} C_i$ and $\hat{F}_2 = \sum_{j=1}^{n_2} D_j$ their strict transforms. Note that $Z \to S$ factors through $W \to S$. Let $\sum_{i=1}^{n_1} E_i$ and $\sum_{j=1}^{n_2} G_j$ be the exceptional curves. Then $\sum_i (C_i + E_i)$ and $\sum_i (D_j + G_j)$ are cycles of alternating (-4) and (-1)-curves. Here we see that $K_Z \sim -\sum_j D_j + \sum_i E_j - \sum_j G_j$. By Serre's duality, we have

$$h^2(Z, \mathcal{T}_Z(-\log(\hat{F}_1 + \hat{F}_2))) = h^0(Z, \Omega^1_Z(\log(\hat{F}_1 + \hat{F}_2))(K_Z))$$

Applying n_2 times the second property of Proposition 5.1.5, we obtain

$$\begin{aligned} H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i + \sum_j D_j\big)\big)(K_Z)\big) &\subseteq H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i\big)\big)\big(K_Z + \sum_j D_j\big)\big) \\ &= H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i\big)\big)\big(\sum_i E_i - \sum_j D_j\big)\big) \\ &\subseteq H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i\big)\big)\big(\sum_i E_i\big)\big) \end{aligned}$$

The following step is to show that

$$H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i\big)\big)\big(\sum_i E_i\big)\big) = H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i + \sum_i E_i\big)\big)\big)$$

Applying the second property of Proposition 5.1.5 n_1 times, it suffices to show that

$$H^0\big(E_s, \Omega^1_{E_s}\big(\big(\sum_i C_i + \sum_{i=1}^{s} E_s\big)|_{E_s}\big)\big(\sum_{i=s+1}^{n_1} E_i|_{E_s}\big)\big) = 0$$

However we know that $\Omega^1_{E_s} = \mathcal{O}_{E_s}(-2)$ and since E_s intersects $\sum_i C_i$ twice and is disjoint with the other $E_{s'}$, we have

$$\deg \mathcal{O}\big(\sum_i C_i + \sum_{i=1}^{s} E_s\big)|_{E_s} = \sum_i C_i \cdot E_s + \sum_{i=1}^{s} E_i \cdot E_s = 2 - 1 = 1,$$

and

$$\deg \mathcal{O}\big(\sum_{i=s+1}^{n_1} E_i\big)|_{E_s} = \sum_{i=s+1}^{n_1} E_i \cdot E_s = 0.$$

Therefore we can conclude that

$$H^0\big(E_s, \Omega^1_{E_s}\big(\big(\sum_i C_i + \sum_{i=1}^{s} E_s\big)|_{E_s}\big)\big(\sum_{i=s+1}^{n_1} E_i|_{E_s}\big)\big) = H^0(E_s, \mathcal{O}_{E_s}(-1)) = 0.$$

The next step is the same as [PSU13, proof of Theorem 2.1], namely constructing the long exact sequence from the first property of Proposition 5.1.5 (and using $H^0(Z, \Omega^1_Z) = 0$ since Z is rational):

$$0 \to H^0\big(Z, \Omega^1_Z\big(\log\big(\sum_i C_i + \sum_i E_i\big)\big)\big) \to \bigoplus_i H^0(C_i, \mathcal{O}_{C_i}) \oplus \bigoplus_i H^0(E_i, \mathcal{O}_{E_i}) \xrightarrow{\delta} H^1(Z, \Omega^1_Z)$$

and noting that the morphism δ is injective since the C_i's and E_i's are linearly independent in the Picard group of Z.

We have proven that $h^2(Z, \mathcal{T}_Z(-\log(\hat{F}_1 + \hat{F}_2))) = 0$. All that is left is to "add" exceptionals E_i or G_i to the cohomology using Proposition 3.2.4 and blow down until we arrive to W using Proposition 3.2.3. □

With this we have all the tools we need to finally start constructing surfaces.

5.2 Worked Example

We will use all the machinery we developed throughout this thesis to construct an example of a surface Z with Wahl singularities, $K_Z^2 = 4$ and $p_g(Z) = q(Z) = 0$, K_Z ample with no obstruction for deformations, where the general member Y of a smoothing is a simply connected surface of general type, $K_Y^2 = 4$, $p_g(Y) = q(Y) = 0$.

Consider the pencil of cubics in $\mathbb{P}^2$ given by

$$\Phi_t = (y^3 - zx^2 + z^2x) + 3txyz.$$

This pencil has 4 singular members, with $t = \infty$, $t = 1$, $t = \omega$, and $t = \omega^2$, where ω is a primitive third root of 1. Φ_{ω^i} is a nodal cubic with a node at $[-1, -\omega^{2i}, 1]$, for $i = 0, 1, 2$. The base points of this pencil are $[0, 0, 1]$ with multiplicity 4, $[1, 0, 0]$ with multiplicity 4, and $[1, 0, 1]$ with multiplicity 1. The configuration is as in Figure 5.1

To resolve the indeterminacies of the fibration, blow up four times at $[0, 0, 1]$ calling the successive exceptionals E_1, E_2, E_3, E_4, blow up another four times at $[1, 0, 0]$ calling the exceptionals E_5, E_6, E_7, E_8, and blow up $[1, 0, 1]$ and call the exceptional E_9 Let X, Y, Z, F_1, F_2, F_3 be the strict transforms of $\{x = 0\}, \{y = 0\}, \{z = 0\}, \Phi_1, \Phi_\omega, \Phi_{\omega^2}$. Also call H the strict transform of $\{x+1 = 0\}$. This is a very special triple section through $X \cap Z$ and the nodes of all F_i's. The result is a $I_9 + 3I_1$ with three sections and a triple section as in Figure 5.2

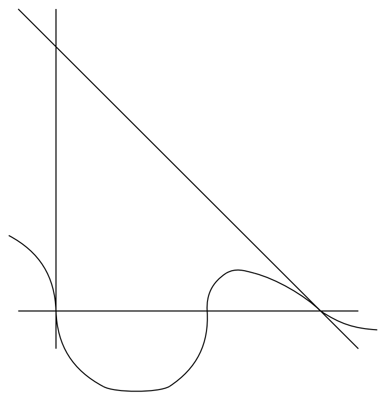


Figure 5.1: Φ_∞ and Φ_0.

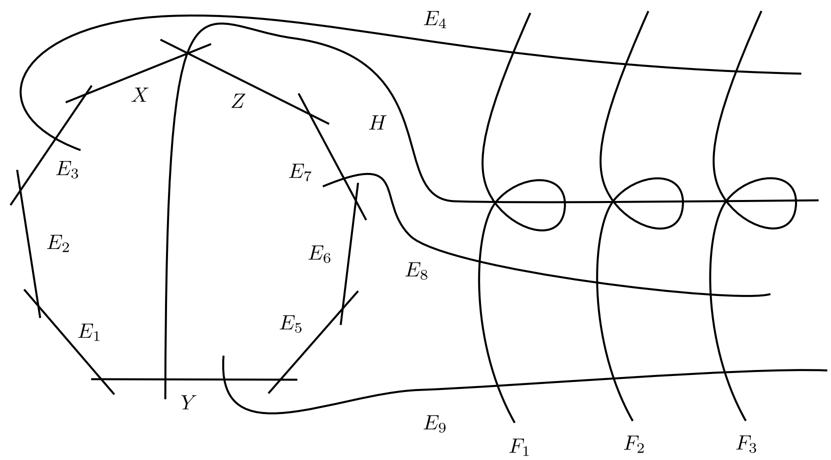


Figure 5.2: $I_9 + 3I_1$ fibration.

From now on we ignore Z and F_3, and whenever we blow up, we still call the strict transform of curves by the same name. Blow up at the nodes of F_1 and F_2, resulting in a surface S_1 as in Figure 5.3. Call the exceptionals G_1 and G_2 respectively.

We apply Propositions 5.1.4, 3.2.5 four times 3.2.4, and again two times 3.2.4 to remove the exceptionals from the nodes of F_1 and F_2. If $F = F_1 + F_2$, $A = X + E_3 + E_2 + E_1 + Y + E_5 + E_6 + E_7$ and $E = E_4 + E_8 + E_9 + H$ (note that here $H^2 = -1$)

$$H^2(S_1, \mathcal{T}_{S_1}(-\log(F + A + E))) = 0$$

We do eight further blow ups calling the exceptionals respectively:

$$\begin{array}{rrr} A_1 \mapsto Y \cap E_1 & A_2 \mapsto Y \cap H & G_3 \mapsto Y \cap E_5 \\ G_4 \mapsto X \cap E_3 & G_5 \mapsto F_1 \cap E_8 & G_6 \mapsto F_1 \cap E_9 \\ G_7 \mapsto F_1 \cap E_4 & G_8 \mapsto F_2 \cap H & \end{array}$$

Call the resulting surface S_2. Let $D = A_1 + A_2$. They are (-1)-curves intersecting transversally, so we may "add" them to the cohomology so that

$$H^2(S_2, \mathcal{T}_{S_2}(-\log(F + A + E + D))) = 0$$

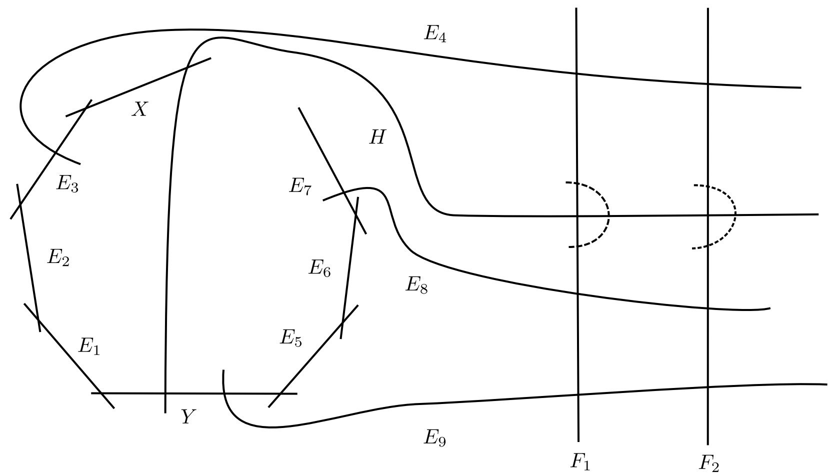


Figure 5.3: The surface S_1.

Blow up two more times at $A_1 \cap H$ and $A_2 \cap Y$, and call the exceptionals G_9 and G_{10}. the resulting surface S_3. This looks as in Figure 5.4.

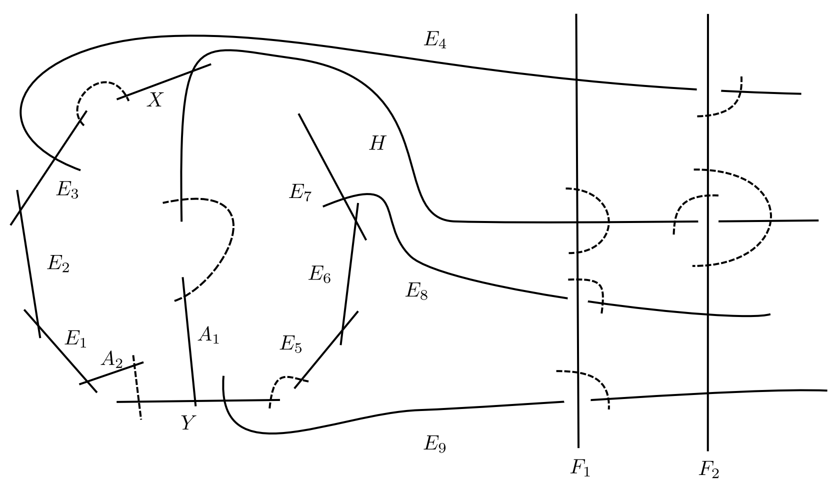


Figure 5.4: The surface S_3.

Here we identify two chains of curves, in order

$$\mathcal{C}_1 : E_5, E_6, E_7, E_8, F_2, E_9, Y, A_1,$$

and

$$\mathcal{C}_2 : X, H, F_1, E_4, E_3, E_2, E_1, A_2.$$

Their self-intersections are given respectively by $[3, 2, 2, 2, 6, 2, 6, 2] = T(1, 29, 13)$ and $[3, 4, 6, 2, 3, 2, 3, 2] = T(1, 41, 15)$. Their discrepancies are

$$d(\mathcal{C}_1) = \left(-\frac{16}{29}, -\frac{19}{29}, -\frac{22}{29}, -\frac{25}{29}, -\frac{28}{29}, -\frac{27}{29}, -\frac{26}{29}, -\frac{13}{29}\right),$$

and

$$d(\mathcal{C}_2) = \left(-\frac{26}{41}, -\frac{37}{41}, -\frac{40}{41}, -\frac{39}{41}, -\frac{38}{41}, -\frac{34}{41}, -\frac{30}{41}, -\frac{15}{41}\right).$$

Let $\pi\colon S_3 \to Z$ be the contraction of these two Wahl chains. We will see now that $\pi^* K_Z = K_{S_3} - \sum_i d_i C_i$ is $\mathbb{Q}$-effective, i.e. a multiple of it is an effective divisor. Recall that $K_S = -F = -F_1 = -F_2$ where F is any fiber, therefore we can write $2K_S = -F_1 - F_2$, or better, $K_S = -\frac{1}{2}F_1 - \frac{1}{2}F_2$. Since the node of each F_i is a double point, if $\psi_1\colon S_1 \to S$ is their blow up, then $\psi_1^*(-\frac{1}{2}F_1) = -\frac{1}{2}F_1 - G_1$ and similarly, $\psi_1^*(-\frac{1}{2}F_2) = -\frac{1}{2}F_2 - G_2$. This means that $K_{S_1} = -\frac{1}{2}F_1 - \frac{1}{2}F_2$. After blowing up the eight points at the following step $\psi_2\colon S_2 \to S_1$, we must have

$$K_{S_2} = -\frac{1}{2}F_1 - \frac{1}{2}F_2 + A_1 + A_2 + G_3 + G_4 + \frac{1}{2}G_5 + \frac{1}{2}G_6 + \frac{1}{2}G_7 + \frac{1}{2}G_8$$

After the blow up $\psi_3\colon S_3 \to S_2$, we have

$$K_{S_3} = -\frac{1}{2}F_1 - \frac{1}{2}F_2 + A_1 + A_2 + G_3 + G_4 + \frac{1}{2}G_5 + \frac{1}{2}G_6 + \frac{1}{2}G_7 + \frac{1}{2}G_8 + 2G_9 + 2G_{10}$$

We finally subtract the discrepancies, obtaining

$$\begin{aligned}\pi^* K_Z = {} & \frac{39}{82}F_1 + \frac{27}{58}F_2 + \frac{30}{41}E_1 + \frac{34}{41}E_2 + \frac{38}{41}E_3 + \frac{39}{41}E_4 + \frac{16}{29}E_5 + \frac{19}{29}E_6 + \frac{22}{29}E_7 + \frac{25}{29}E_8 \\ & + \frac{27}{29}E_9 + \frac{26}{41}X + \frac{26}{29}Y + \frac{37}{41}H + \frac{42}{29}A_1 + \frac{55}{41}A_2 + G_3 + G_4 + \frac{1}{2}G_5 + \frac{1}{2}G_6 + \frac{1}{2}G_7 + \frac{1}{2}G_8 \\ & + 2G_9 + 2G_{10}\end{aligned}$$

We now will see that K_Z is ample. Let $\Gamma \subseteq Z$ be any curve. $K_Z \cdot \Gamma = \pi^* K_Z \cdot \pi^* \Gamma = \pi^* K_Z \cdot \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the strict transform of Γ. If $\pi^* K_Z \cdot \Gamma < 0$, then since $\pi^* K_Z$ is $\mathbb{Q}$-effective, $\tilde{\Gamma}$ must be one of the curves of $\pi^* K_Z$. The only non-exceptional curves in this representation of $\pi^* K_Z$ are the $G_3, \ldots, G_{10}$, and they all have positive intersection:

$$\begin{aligned} \pi^*K_Z \cdot G_3 &= -1 + \frac{16}{29} + \frac{26}{29} = \frac{13}{29} & \pi^*K_Z \cdot G_4 &= -1 + \frac{26}{41} + \frac{38}{41} = \frac{23}{41} \\ \pi^*K_Z \cdot G_5 &= -\frac{1}{2} + \frac{39}{82} + \frac{25}{29} = \frac{996}{1189} & \pi^*K_Z \cdot G_6 &= -\frac{1}{2} + \frac{39}{82} + \frac{27}{29} = \frac{1078}{1189} \\ \pi^*K_Z \cdot G_7 &= -\frac{1}{2} + \frac{27}{58} + \frac{39}{41} = \frac{1090}{1189} & \pi^*K_Z \cdot G_8 &= -\frac{1}{2} + \frac{27}{58} + \frac{37}{41} = \frac{1032}{1189} \\ \pi^*K_Z \cdot G_9 &= -2 + \frac{37}{41} + \frac{42}{29} = \frac{417}{1189} & \pi^*K_Z \cdot G_{10} &= -2 + \frac{26}{29} + \frac{55}{41} = \frac{283}{1189} \end{aligned}$$

So there cannot be a Γ such that $K_Z \cdot \Gamma < 0$. The above calculation also tells us that if $K_Z \cdot \Gamma = 0$, then $\tilde{\Gamma}$ must not be a component of K_Z. If $\tilde{\Gamma}$ is part of a fiber of $S_3 \to \mathbb{P}^1$, then it would be either Z, G_1, G_2, or any fiber different than those over $\infty, 1, \omega$. But all those curves intersect $\pi^* K_Z$, a contradiction. This means Γ must be a n-multiple-section. It cannot intersect $E_1, \ldots, E_4$, G_4 or G_7, so its image in $\mathbb{P}^2$ would be a curve that intersects the line $\{x = 0\}$ 0 times. This is a contradiction.

Therefore $K_Z \cdot \Gamma > 0$ for any curve Γ. Also $K_{S_3}^2 = -12$, since the total blow ups were 12, and the sum of lengths of the Wahl chains is 16. This means $K_Z^2 = 4$. By Nakai-Moishezon's criterion, K_Z is ample.

For the fundamental group, note that G_9 is a sphere intersecting transversally the end of $\mathcal{C}_1$ and the curve H in $\mathcal{C}_2$. Also $\gcd(29, 41) = 1$, so this is one of the situations studied at the end of Section 4.2. We can conclude that the loop around G_9 over the point $G_9 \cap E_5$ contracts in

$S_3 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$. Similarly G_{10} intersects the curve A_2 (the end of $\mathcal{C}_2$) and the curve Y in $\mathcal{C}_1$. These all imply that, since S_3 is simply connected, the symplectic rational blow down of $\mathcal{C}_1$ and $\mathcal{C}_2$ is also simply connected.

Finally, since $H^2(S_3, \mathcal{T}_{S_3}(-\log(\mathcal{C}_1 + \mathcal{C}_2))) = 0$, we know we can smoothen out these singular points via a $\mathbb{Q}$-Gorenstein deformation, and the general surface has $p_g = q = 0$, $K^2 = 4$, is simply connected, with K ample. Thus it lives in the moduli space $\mathcal{M}_{4,1}$.

Remark 5.2.1. As in the previous example, to verify that $\pi^* K_Z$ is $\mathbb{Q}$-effective, it is enough to select two complete fibers of type I_n F_1 and F_2 and having the discrepancies of each of their curves to be $\leq -\frac{1}{2}$. This is because by writing $K_S \sim -\frac{1}{2}F_1 - \frac{1}{2}F_2$, then after all blow ups we will have $K_{S_3} \sim -\frac{1}{2}F_1 - \frac{1}{2}F_2 + \sum n_i C_i$ for the exceptional curves C_i, and $n_i \geq 0$ (by applying 1.2.7 successive times). Then, after subtracting the discrepancies, every curve will have a non-negative coefficient. This would not work with fibers of type I_n^*, II^*, III^* and IV^* since they have curves with higher multiplicities, and neither with fibers of type II, III, IV, since during their resolutions, there will appear exceptional curves with negative multiplicity. Exceptionally, one may also check nefness when dealing with 3 fibers F_1, F_2, F_3 of type IV, by choosing $K_S = -\frac{1}{3}F_1 - \frac{1}{3}F_2 - \frac{1}{3}F_3$, since the exceptional over the blow up at the triple point would have multiplicity 0. Here the example is most probably obstructed though.

Remark 5.2.2. The condition of discrepancies $\leq -\frac{1}{2}$ in the remark above always work when choosing two fibers of type I_n, provided no multi-section is blown down. This is because in order to obtain chains, one must "separate" the cycles in the fibers by blowing up, and by doing that, it is impossible to obtain tails of (-2)-curves from these curves (cf. 2.2.7).

Remark 5.2.3. Assuming $\pi^* K_Z$ is $\mathbb{Q}$-effective using the remarks above, since $\pi^* K_Z = K_{S_3} - \sum_i d(C_i) C_i$, then when checking for nefness (or ampleness) of K_Z, by adjunction formula, we only have to verify it for non π-exceptional (-1)-curves in the representation of $\pi^* K_Z$ (since this representation can only contain curves of negative self-intersection). For such a curve G_j we have

$$\pi^* K_Z \cdot G_j = K_{S_3} \cdot G_j + \sum_i d(C_i) C_i \cdot G_j = -1 - \sum_i d(C_i)(C_i \cdot G_j)$$

so it is enough to check $-\sum_i d(C_i)(C_i \cdot G_j) \geq 1$ (or > 1 respectively for ampleness). Note that these curves must always intersect at least two π-exceptional curves (so blow ups can only be done at nodes), and in this case, by Remark 2.2.7 the inequality always holds if neither of the curves is a (-2)-curve at the tail of a Wahl chain (or initial chains too for ampleness), so we only truly needed to check for G_9 and G_{10} in the construction above.

Remark 5.2.4. In order to verify $H^2(Z, \mathcal{T}_Z) = H^2(S_3, \mathcal{T}_{S_3}(-\log \mathcal{C})) = 0$, then

- By Propositions 3.2.2, 5.1.4 and 3.2.5, it is enough that the original configuration in the elliptic surface consists of at most two complete fibers F_1 and F_2, since partial fibers can be added as ADE configurations; and
- By Propositions 3.2.3 and 3.2.4, if $\mathcal{D}$ is obtained by log-resolution of the original configuration, then it is enough that every curve S coming from a multi-section has $S^2 \geq -1$.

5.3 Combinatorics

The objective of this section is to define the invariants P and K of a given configuration of curves in a surface that gives us information of the possible T-singularities that could be obtained from this configuration. This will be useful when dealing with many configurations of curves as a way to analyze the viability of obtaining T-singularities before even starting to search for them.

Definition 5.3.1. Let X be a smooth projective surface and D be a reduced divisor with "bouquet" crossings–that is, every singularity of D locally looks like the union of different lines through a single point in $\mathbb{C}^2$–where in particular we allow curves with singular points. Let $\operatorname{Sing} D$ be the set of singular points of D, and define

$$P(X, D) = \sum_{C \in D} \left(C^2 + 5 - \sum_{p \in \operatorname{Sing} D} m_p(C)^2 \right),$$

where $m_p(C)$ is the multiplicity of C at p ([H70, Definition pp. 388]).

In particular, if D is a simple normal crossing divisor,

$$P(X, D) = \sum_{C \in D} C^2 + 5|D| - 2|\operatorname{Sing} D|, \tag{5.1}$$

where $|D|$ is the number of curves in D. We will now prove several properties of P that make it useful for computations.

Proposition 5.3.2. Let X be a smooth projective surface and D a divisor with bouquet crossings. Let $\pi \colon Y \to X$ be the blow up at $p_0 \in \operatorname{Sing} D$, and let $\hat{D}$ be the strict transform of D. Then

$$P(Y, \hat{D}) = P(X, D)$$

Proof. Every branch of D at p_0 has different tangent directions, by blowing up that point we separate every branch, thus, every point of the exceptional $E \cap \hat{D}$ is non-singular in $\hat{D}$, so π induces a correspondence between $\operatorname{Sing} D - \{p_0\}$ and $\operatorname{Sing} \hat{D}$. By corollary 1.2.8 we have

$$\begin{aligned}
P(X, D) &= \sum_{C \in D} \left(C^2 + 5 - \sum_{p \in \operatorname{Sing} D} m_p(C)^2 \right) \\
&= \sum_{\hat{C} \in \hat{D}} \left(\hat{C}^2 + m_{p_0}(C)^2 + 5 - \sum_{p \in \operatorname{Sing} D} m_p(C)^2 \right) \\
&= \sum_{\hat{C} \in \hat{D}} \left(\hat{C}^2 + 5 - \sum_{p \in \operatorname{Sing} D, p \neq p_0} m_p(C)^2 \right) \\
&= \sum_{\hat{C} \in \hat{D}} \left(\hat{C}^2 + 5 - \sum_{p \in \operatorname{Sing} \hat{D}} m_p(\hat{C})^2 \right) \\
&= P(Y, \hat{D})
\end{aligned}$$

□

Proposition 5.3.3. Let $E \subseteq X$ be a smooth curve not in D intersecting D at n smooth points and m singular points. Then

$$P(X, D + E) = P(X, D) + E^2 + 5 - 2n - m$$

In particular, if $\pi \colon Y \to X$ is the blow up at a node of D and $E_0 \subseteq Y$ is the exceptional curve, then

$$P(Y, \hat{D} + E_0) = P(Y, \hat{D}) = P(X, D)$$

Proof. The new singular points in $D+E$ are given by the n intersections with D at smooth points p_i of curves C_i, that is $m_{p_i}(C_i)=1$. Since at singular points the sum for curves in D is unaffected, then curves in D contribute with

$$\sum_{C\in D}\left(C^2+5-\sum_{p\in\operatorname{Sing}D}m_p(C)^2\right)-n=P(X,D)-n$$

to the sum. Since E is smooth, $m_p(E)=1$ for every $p\in E\cap D$, so E contributes to the sum with

$$E^2+5-n-m.$$

□

The final property of P is essentially the reason why it was defined.

Proposition 5.3.4. Let D be a divisor consisting of k disjoint T-singularities (d_i,n_i,a_i), $i=1,\dots,k$, then

$$P(X,D)=\sum_{i=1}^{k}d_i$$

in particular, if every chain is a Wahl chain, $P(X,D)=k$.

Proof. This is a direct calculation using 5.1 for initial T-singularities, noting that it is invariant under the algorithm and that P is additive for disjoint configurations. □

This way, whenever we choose a configuration D of rational curves with only nodes, we can immediately determine the amount of T-singularities we would obtain by consecutively blowing up nodes and either adding or not to our configuration the exceptional curves that appear. The next invariant we will define will do the same thing, but instead of counting the T-singularities, it determines K_Z^2 if $\varphi\colon X\to Z$ is the contraction of the T-singularities obtained from D.

Definition 5.3.5. Let X and D be as in 5.3.1. Define

$$K(X,D)=K_X^2+2|D|-|\operatorname{Sing}D|-P(X,D)$$

In particular, if D is simple normal crossing, then

$$K(X,D)=K_X^2-\sum_{C\in D}C^2-3|D|+|\operatorname{Sing}D|$$

Proposition 5.3.6. Let X and D be as in 5.3.1, and let $\pi\colon Y\to X$ be the blow up at $p_0\in\operatorname{Sing}D$. Let $\hat{D}$ be the strict transform of D in Y. Then

$$K(Y,\hat{D})=K(X,D)$$

Proof. We already know $P(X,D)$ and $|D|$ remain constant. We also know that $|\operatorname{Sing}D|$ decreases by one and by corollary 1.2.8, $K_Y^2=K_X^2-1$. □

Proposition 5.3.7. Let $E\subseteq X$ be a smooth curve not in D intersecting D at n smooth points and m singular points. Then

$$K(X,D+E)=K(X,D)-E^2+n+m-3.$$

In particular, if $\pi\colon Y\to X$ is the blow up at a node of D and $E_0\subseteq Y$ is the exceptional curve, then

$$K(Y,\hat{D}+E_0)=K(Y,\hat{D})=K(X,D)$$

Proof. $|D|$ increases by 1, $|\operatorname{Sing} D|$ increases by n and $P(X, D+E) = P(X,D)+E^2+5-2n-m$. □

Proposition 5.3.8. Let $D \subseteq X$ be a divisor consisting of disjoint T-singularities and let $\varphi\colon X \to Z$ the contraction of D. Then

$$K_Z^2 = K(X, D)$$

Proof. Given a T-chain $T(d, n, a)$ of length l with curves $C_1, \ldots, C_l$, it is easy to see that

$$-\sum_{i=1}^{l} C_i^2 = 3l + 2 - d$$

since $d = P(X, \sum_{i=1}^{l} C_i) = 3l + 2 + \sum_{i=1}^{l} C_i^2$. This way if D consists of k T-chains, $T(d_i, n_i, a_i)$, by using Lemma 2.2.8 k times,

$$\begin{aligned}
K(X, D) &= K_X^2 - \sum_{C \in D} C^2 - 3|D| + |\operatorname{Sing} D| \\
&= K_X^2 + \sum_{i=1}^{k}((3l_i + 2 - d_i) - 3l_i + (l_i - 1)) \\
&= K_X^2 + \sum_{i=1}^{k}(l_i - d_i + 1) \\
&= K_Z^2
\end{aligned}$$

□

To get extra bounds on what we may arrive to, we will define logarithmic pairs, their Chern classes and state the logarithmic Bogomolov-Miyaoka-Yau inequality.

Definition 5.3.9 ([H70, A.3 & A.4])**.** Let X be a smooth surface. Its Chern classes are $c_1(X) = c_1(\mathcal{T}_X)$ and $c_2(X) = c_2(\mathcal{T}_X)$. They coincide with the topological Chern classes so $c_2(X) = e(X)$ is the Euler characteristic, and $c_1(X) = -K_X$.

Definition 5.3.10 ([U10, Definition 2.3])**.** Let X be a smooth surface. A log pair is a tuple (X, D), where D is a simple normal crossing divisor in X. The log Chern classes are $\overline{c_1}(X, D) = c_1(\mathcal{T}_X(-\log D))$ and $\overline{c_2}(X, D) = c_2(\mathcal{T}_X(-\log D))$.

Proposition 5.3.11 ([U10, Proposition 2.4])**.** Consider a log pair (X, D). Then

- $\overline{c_1}(X, D) = -K_X - D$.
- $\overline{c_2}(X, D) = c_2(X) + \frac{1}{2}D.(K_X + D) + \sum_{C \in D}(g(C) - 1)$.

where $g(C)$ is the genus of C.

If $|\operatorname{Sing}(D)| = t_2 = \frac{1}{2}(D^2 - \sum_{C \in D} C^2) = \sum_{C \neq C' \in D} C.C'$ is the number of nodes of D, and D consists of only rational curves as in our case, then, by using Noether's formula $12\chi(X) = c_1^2(X) + c_2(X)$, the log Chern numbers of (X, D) may be written as

- $\overline{c_1}^2(X, D) = K_X^2 + 2|\operatorname{Sing}(D)| - \sum_i D_i^2 - 4|D| = K_X^2 - P(X, D) + |D|$.
- $\overline{c_2}(X, D) = c_2(X) + |\operatorname{Sing}(D)| - 2|D| = 12\chi(X) - P(X, D) - K(X, D)$.

Theorem 5.3.12 (Log-Bogomolov-Miyaoka-Yau inequality [K85, Theorem 2])**.** Let (X, D) be a log pair. Then the inequality $\overline{c_1}^2(X, D) \leq 3\overline{c_2}(X, D)$ holds.

Theorem 5.3.13. Let Z be a surface with only P Wahl singularities. Then $K_Z^2 \leq 9\chi(Z) - \frac{1}{2}P$.

Proof. Suppose that D is a configuration of disjoint Wahl chains in a smooth surface X. Then by Proposition 5.3.8 and Lemma 2.2.8, $K_X^2 + |D| = K(X, D)$, so we obtain the inequality

$$K(X, D) \leq 9\chi(X) - \frac{1}{2}P(X, D).$$

□

In particular, if Z is a surface with only Wahl singularities then $K_Z^2 \leq 9\chi(Z) - 1$. This tells us that with our method we will never be able to construct an exotic $\mathbb{CP}^2$.

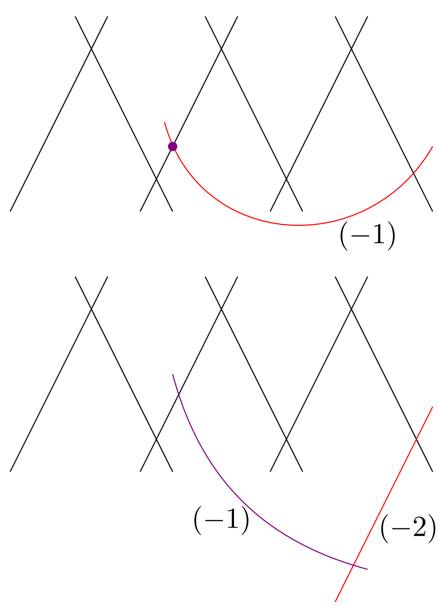
Chapter 6

Automatization

In this chapter we will describe the program we wrote and used to search for surfaces with Wahl configurations. For complexity reasons, this algorithm only works for finding configurations with one or two Wahl chains. In principle, the program searches for Wahl chains arising from configurations of curves in a rational elliptic fibration, and so we will describe the program assuming we are in that situation. The program may very well be used in other situations, such as using as a starting point a $K3$ surface (as with [RU21]) or any configuration of rational curves in arbitrary surfaces, but some parts of the analysis over the examples would provide wrong results, such as the obstruction test. The program can also search for more general $\mathbb{Q}HD$ singularities, but they will be omitted from this explanation. There are several optimizations and fine details that will also be omitted.

The process can be summarized as follows. Start with a configuration $\mathcal{C}$ of rational curves in a surface X. Consider every sub-configuration of curves $D \subseteq \mathcal{C}$, and search every possible way to blow up nodes in D (with possibly infinitely near blow ups) to obtain surfaces $Y \to X$ that hopefully contain configurations of Wahl chain. If we denote the contraction of said Wahl chains by $Y \to Z$, then the algorithm will also check that K_Z is nef and big ($K_Z^2 > 0$), and discard the examples that violate these rules.

The main way in which the algorithm works is by blowing up enough times D in order to get chain "candidates" which can be modified using (-1)-curves. A "candidate" and its modification can be visualized as follows.



This operation creates a new (-1)-curve, so can be applied recursively. The purple point need not to be in the same chain. Let us formalize this

Definition 6.0.1. Let $D = C_1 + \ldots + C_n$ be a chain in a surface X, where C_i intersects C_{i-1} and C_{i+1}. Let G be a (-1)-curve that intersects only C_n and some rational curve B, which may be any of the C_i except C_1 (if $B = C_n$, this means that C_n intersects G twice at different points). Suppose that $D + G + B$ is snc, and let p be a point of intersection of G and B (which is unique if $B \neq C_n$ and one of two if $B = C_n$). A single extension operation on (X, D, G, B) is the the tuple $(Y, D', G', \widehat{B})$, where we blow up at p, $\pi \colon Y \to X$, we define the chain $D' = \widehat{D} + \widehat{G}$, and G' is the π-exceptional curve.

Note that $(Y, D', G', \widehat{B})$ also satisfies the hypotheses of this definition. An n-extension operation is applying a single extension operation n times.

Combinatorially, an n-extension can be seen as extending the chain D with a sequence of n (-2)-curves at one end, and decreasing B^2 by n.

Definition 6.0.2. Let $\mathcal{C}_0$ be a configuration of rational curves in a rational elliptic surface S_0. Then we can write $\mathcal{C}_0 = \mathcal{F}_0 + \mathcal{S}_0$, where $\mathcal{F}_0$ consists of all curves contained in fibers, and $\mathcal{S}_0$ consists of all other curves, in other words, multi-sections.

If $\pi \colon Y \to X$ is the minimal log resolution of $\mathcal{C}_0$, we may write $\mathcal{C} = \mathcal{F} + \mathcal{S} + \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{S}$ are the strict transforms of $\mathcal{F}_0$ and $\mathcal{S}_0$, and $\mathcal{E}$ is the reduced exceptional divisor of π. Such a pair $(Y, \mathcal{C})$ together with the subdivision into $\mathcal{F}, \mathcal{S}, \mathcal{E}$ is called a starting configuration.

Note that it is equivalent to give a starting configuration than to give a configuration $\mathcal{C}_0$ of curves. We do it this way in order to simplify the program so it can only accept simple normal crossing configurations.

Definition 6.0.3. Let D be a snc configuration of curves.

- C a curve in D. Define $\delta_D(C) = (D - C).C$, that is, the number of intersections of C with the other curves in D. Define $\Delta(D) = \max_{C \leq D} \delta_D(C)$.
- D is a chain-cycle forest if it is the disjoint union of chains and cycles. Equivalently, D is a chain-cycle forest if $\Delta(D) \leq 2$.

Definition 6.0.4. Let D be a snc configuration of curves in a surface X. A separation of D is a sequence of blow ups $\pi \colon Y \to X$ such that $\Delta(\widehat{D}) \leq 2$. A minimal separation is a separation that is minimal among all separations.

Note that D might have many different minimal separations. Also, since all blow ups must be done at nodes of D, then $P(Y, \widehat{D}) = P(X, D)$ and $K(Y, \widehat{D}) = K(X, D)$.

Definition 6.0.5. Let $D = C_1 + \ldots + C_n$ be a chain in a surface X. define

$$\Lambda(D) = 1 - P(X, D) = -3n - 1 - \sum_{i=1}^{n} C_i^2.$$

Proposition 6.0.6. Let D_1, D_2 be disjoint chains in a surface X, $D_1 = C_{1,1} + \ldots + C_{1,n_1}$, $D_2 = C_{2,1} + \ldots + C_{2,n_2}$, where $C_{i,j}$ intersects $C_{i,j-1}$ and $C_{i,j+1}$. If D_i is a Wahl chain, then $\Lambda(D_i) = 0$.

Let G_1 be a (-1)-curve and suppose $(X, D_1, G_1, A_{1,j})$ satisfies the hypotheses of Definition 6.0.1, Let $(Y, D_1', G_1', \widehat{A_{1,j}})$ be its n-extension. Then $\Lambda(D_1') = \Lambda(D_1)$.

Suppose now that G_2 is a (-1)-curve such that $(X, D_1, G_2, A_{2,j})$ satisfies the hypotheses of Definition 6.0.1, and G_2 intersects only $A_{2,j}$ in D_2. Let $(Y, D_1', G_1', \widehat{A_{2,j}})$ be its n-extension and $D_2' = \widehat{D_2}$. Then

$$\Lambda(D_1') = \Lambda(D_1) - n, \quad \Lambda(D_2') = \Lambda(D_2) + n.$$

Proof. The first statement is just 5.3.4.

For the others, note that Λ only depends on the combinatorics of a chain. If we append a (-2)-curve to a chain, Λ decreases by one, and if we decrease the self intersection of a curve in a chain by one, Λ increases by one. □

Remark 6.0.7. The algorithm also handles blowing down in the middle of chains. Testing for obstructions and nefness works essentially the same as with the case with only blow ups, though the implementation does get a bit complicated. The only essential difference is that before applying extensions, any (-1)-curve in the middle of chains gets contracted (unless that curve is the one where an extension happen). For this reason, and because the notation would get too heavy, we will assume no blow downs in the middle of chains occur. Cases like these are rare but not extremely so.

The algorithm is divided into several "functions", which are not entirely in order. The parameter they take are shown in parenthesis. The first function we execute is Init().

Init()

Before starting the algorithm, we first describe the input to the program. The input consists essentially of two things: On the one hand, options indicating what to search for. This includes

- Whether we want to search for configurations consisting of one Wahl chain, two Wahl chains, or search for both.
- The target K^2's we are aiming for K_Z^2.
- Whether to test for obstructions or not.
- Whether to test for K_Z nef or not.
- Whether to test if $K_Y - D$ is $\mathbb{Q}$-effective or not, where D is the discrepancy $\mathbb{Q}$-divisor.

On the other hand, the input also describes a starting configuration of rational curves. $\mathcal{C} = \mathcal{F} + \mathcal{S} + \mathcal{E}$ in a surface S, obtained by log resolution $\pi\colon S \to S_0$ of $\mathcal{C}_0 = \mathcal{F}_0 + \mathcal{S}_0$ a configuration in an elliptic surface.

Every piece of data of the input is regarded as a global invariant. That means that every Step has access to these values without needing them to be parameters.

Call Sub-Configuration-Selection().

Sub-Configuration-Selection()

In this step we want to select iterate through all sub-configurations of $D \leq \mathcal{C}$. The problem of simply selecting a sub-configuration is that S might have been obtained by more than necessary blow ups to make D snc, so we need to blow down some curves.

For every sub-configuration $D \leq \mathcal{C}$ do the following. Blow down curves in $\mathcal{E}$, $\varphi\colon S \to S'$ successively until no more curve can be blown down without making $\varphi(D)$ a non-snc divisor. Note that $S' \to S_0$ might not be the log resolution of $\pi(D)$ (where $\pi\colon S \to S_0$), since the log resolution requires also that every exceptional curve has also simple normal crossings with $\pi(D)$. Algorithmically, this construction is standard: for every subset indexed by a binary mask, blow down vertices of $\mathcal{E}$ whenever possible.

Call Calculate-Invariants(S',$\varphi(D)$).

Calculate-Invariants(a surface X, a snc configuration D in X)

Now calculate the invariants P and K from our configuration D. If $P(X,D) \neq 1,2$ (also depending on the input) or $K(X,D)$ is not part of the target K^2 given, we discard this sub configuration and return from this function. This is because it will be impossible to obtain desired configurations of Wahl chains as described in Section 5.3.

We also calculate obstructions of (X,D). If this test was enabled in the input, call Calculate-Obstruction(X,D). If (X,D) did not pass the "test", discard it and return from this function.

If (X,D) passes all these "tests", call Separate(X,D).

Separate(a surface X, a snc configuration D in X)

In this step we start blowing up. Since a configuration of chains W in a surface Y has always $\Delta(W) \leq 2$, our first approach is to blow up enough times D in order to obtain this condition.

Construct every minimal separation $\varphi\colon (Y,\widehat{D}) \to (X,D)$. This step is done with the obvious backtracking algorithm, which iterates through curves and blows up intersections whenever $\delta_D(C) > 2$. The decision tree may be huge, so instead of constructing every possible $(Y,\widehat{D})$, the configuration (X,D) is modified via blow ups and blow downs internally. Also note that this algorithm, although is guaranteed to give every minimal separation, may also give non-minimal ones. It is possible to trim them out afterwards.

For every minimal separation $(Y,\widehat{D})$ do the following:

Let $\mathcal{G}$ be the reduced exceptional divisor of φ. Then $\mathcal{G}$ consists of (-1)-curves that intersect only two curves in $\widehat{D}$. At this point, $\widehat{D}$ is a chain-cycle forest.

If $P(X,D)=1$, call Single-Chain-Init(Y, $\widehat{D}$, $\mathcal{G}$)

Otherwise, if $P(X,D)=2$, call Double-Chain-Init(Y, $\widehat{D}$, $\mathcal{G}$).

Single-Chain-Init(a surface X, a chain-cycle forest D, a configuration of (-1)-curves $\mathcal{G}$)

At this step we have a chain-cycle forest D inside X with a set of (-1)-curves $\mathcal{G}$. We wish to construct a single Wahl chain from this forest.

If D is not connected, then it will be impossible to obtain a Wahl chain from this configuration. This is because blowing up will preserve the number of connected components. Therefore in this case discard D and return from this function.

If D is a chain, call Single-Chain-Check-Wahl (X, D, $\mathcal{G}$). Afterwards, call Single-Chain-Extend(X, D, $\mathcal{G}$).

If instead, D is a cycle, for every node $p \in \operatorname{Sing} D$ do the following. Construct the blow up $Y = \mathrm{Bl}_p(X) \to X$. Let $\widehat{D}$ be the strict transform of D and let E be the exceptional curve. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E$. Now $\widehat{D}$ is a chain in Y. Call Single-Chain-Check-Wahl(Y, $\widehat{D}$, $\mathcal{G}'$). Afterwards, call Single-Chain-Extend(Y, $\widehat{D}$, $\mathcal{G}'$).

Single-Chain-Check-Wahl (a surface X, a chain D, a configuration of (-1)-curves $\mathcal{G}$)

At this step we have a chain D in X with a set of (-1)-curves $\mathcal{G}$, and we want to check if D is already a Wahl chain. This can be quickly checked if all its curves A_i in D have $A_i^2 \leq -2$ and verifying that $[-A_1^2, \ldots, -A_l^2]$ is a fraction of the form $\frac{n^2}{na-1}$, where $D = A_1 + \ldots + A_l$ and A_i intersects A_{i-1} and A_{i+1}. If D is not a Wahl chain, return from this function.

If the $\mathbb{Q}$-effective test was enabled in the input, call Q-Effective-Test(X, D, $\mathcal{G}$). If it did not pass the test, discard this example and return from this function.

If the nef test was enabled in the input, call Nef-Test(X, D, $\mathcal{G}$). If it did not pass the test, discard this example and return from this function.

At this point, D is a Wahl chain that passed all needed tests. Add this to the list of found examples.

Single-Chain-Extend(a surface X, a chain D,configuration of (-1)-curves $\mathcal{G}$)

Here we will want to extend one of the ends of D via extension operations. We do not want a (-1)-curve intersecting only both ends of a chain, as this means there will be a 0 curve for K_Z, thus it would not be ample (this also means that there could be infinitely many Wahl chain configuration arising from D, all of them with K_Z non ample, so we we wish to avoid this situation).

Since any Wahl chain of length > 1 must have a (-3)-curve or lower at one end, and a (-2)-curve at the other, we do not want to extend an end of the chain if the other end is not (-3) or lower.

Write $D = A_1 + \ldots + A_l$, where A_i intersects A_{i-1} and A_{i+1}. There are two ways of writing D in this manner (the other is $A_l + \ldots + A_1$), so for each of them do the following:

If $A_1^2 \geq -2$, ignore this case.

For every exceptional $G \leq \mathcal{G}$ that intersects A_l and does not intersect A_1 do the following: Let A_i be the other curve where G intersects (which may be A_l too).

If $A_i^2 > -2$, apply $2+A_i^2$ extension operations to (X, D, G, A_i), obtaining $(Y_0, D_0', G_0', \widehat{A_i})$, where $\widehat{A_i}^2 = -2$. Let $\mathcal{G}_0' = \widehat{\mathcal{G} - G} + G_0'$ and call Single-Chain-Check-Wahl(Y_0, D_0', $\mathcal{G}_0'$).

In any case, we will try to extend al least enough times to get $\widehat{A_i}^2 \leq -3$. Let $(Y, D', G', \widehat{A_i})$ be the m-extension of (X, D, G, A_i) where $m > 0$ is any integer such that $\widehat{A_i}^2 \leq -3$, and write $D' = B_1 + \ldots + B_{l+m}$, where $B_i = \widehat{A_i}$ for $i \in \{1, \ldots, l\}$. Since $\widehat{A_i}^2 \leq -3$, this means there is a single $j \geq i > 1$ independent of m, such that $B_k^2 = -2$ for every $k > j$, and $B_j = \widehat{A_j}$ (which could be $\widehat{A_i}$) would be a (-3)-curve or lower. Since $j \geq 2$, this cannot be a linear chain (see Example 2.2.4), so if D' were to be a Wahl chain, the amount of (-2)-curves at the end of the chain must be $-2 - \widehat{A_1}^2$, which is independent of m and equal to $-2 - A_1^2$ since $i \neq 1$.

This means that the only m we should check is $m = -2 - A_1^2 - l + j$ (if $m \leq 0$, ignore this and return from this function). Let $\mathcal{G}' = \widehat{\mathcal{G} - G} + G'$ and call Single-Chain-Check-Wahl(Y, D', $\mathcal{G}'$).

Double-Chain-Init(a surface X, a chain-cycle forest D, a configuration of (-1)-curves $\mathcal{G}$)

At this step we have a chain-cycle forest D inside X with a set of (-1)-curves $\mathcal{G}$. We wish to construct two Wahl chains from this forest.

If D has more than 2 connected components, then it will be impossible to obtain two Wahl chains from this configuration so discard this example and return.

If D consists of two chains, let them be D_1 and D_2. Call Double-Chain-Standard(X, D_1, D_2, $\mathcal{G}$).

If instead, D consists of a chain and a cycle, let them be D_1 and D_2 respectively. For every node $p \in \operatorname{Sing} D_2$ do the following. Construct the blow up $Y = \mathrm{Bl}_p(X) \to X$. Let $\widehat{D_i}$ be the strict transform of D_i and let E be the exceptional curve. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E$. Now $\widehat{D_i}$ are both chains in Y. Call Double-Chain-Standard(Y, $\widehat{D_1}$, $\widehat{D_2}$, $\mathcal{G}'$).

If instead, D consists of two cycles, let them be D_1 and D_2 respectively. For every node $p_1 \in \operatorname{Sing} D_1$ and every node $p_2 \in \operatorname{Sing} D_2$ do the following. Construct the blow up $Y = \mathrm{Bl}_{p_2}(\mathrm{Bl}_{p_1}(X)) \to X$ at p_1 and p_2. Let $\widehat{D_i}$ be the strict transform of D_i and let E_1, E_2 be the exceptional curves. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E_1 + E_2$. Now $\widehat{D_i}$ are both chains in Y. Call Double-Chain-Standard(Y, $\widehat{D_1}$, $\widehat{D_2}$, $\mathcal{G}'$).

If instead, D is a single chain, call Extremal-P-Resolution-Check(X, D, $\mathcal{G}$) and Extremal-P-Resolution-Extend(X, D, $\mathcal{G}$). Afterwards, for every node $p \in \operatorname{Sing} D$ do the following. Construct the blow up $Y = \mathrm{Bl}_p(X) \to X$. Let $\widehat{D}$ be the strict transform of D and let E be the exceptional curve. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E$. Now $\widehat{D}$ consists of two chains D_1 and D_2 in Y. Call Double-Chain-Standard(Y, D_1, D_2, $\mathcal{G}'$).

If instead, D is a single cycle, for every node $p \in \operatorname{Sing} D$ do the following. Construct the blow up $Y = \mathrm{Bl}_p(X) \to X$. Let $\widehat{D}$ be the strict transform of D and let E be the exceptional curve. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E$. Now $\widehat{D}$ is a chain in Y. Call Extremal-P-Resolution-Check(Y, $\widehat{D}$, $\mathcal{G}'$) and Extremal-P-Resolution-Extend(Y, $\widehat{D}$, $\mathcal{G}'$).

Again, if D is a single cycle, for every pair of different nodes $p_1, p_2 \in \operatorname{Sing} D$ do the following. Construct the blow up $Y = \mathrm{Bl}_{p_2}(\mathrm{Bl}_{p_1}(X)) \to X$ at p_1 and p_2. Let $\widehat{D}$ be the strict transform of D and let E_1, E_2 be the exceptional curves. Let $\mathcal{G}' = \widehat{\mathcal{G}} + E_1 + E_2$. Now $\widehat{D}$ consists of two chains D_1 and D_2 in Y. Call Double-Chain-Standard(Y, $\widehat{D_1}$, $\widehat{D_2}$, $\mathcal{G}'$).

Double-Chain-Standard(a surface X, a chain D_1, a chain D_2, a configuration of (-1)-curves $\mathcal{G}$)

Here we have two disjoint chains $D_1 = A_{1,1} + \ldots + A_{l_1}$, and $D_2 = A_{2,1} + \ldots + A_{2,l_2}$. Calculate

$$\Lambda_i = \Lambda(D_i) = -3l_i - 1 - \sum_{j=1}^{l_i} A_{i,j}^2, \quad i = 1, 2$$

Since $P(X, D_1 + D_2) = 2$, then $\Lambda_1 + \Lambda_2 = 0$. By Proposition 6.0.6, we want to modify this chains in order to obtain $\Lambda(D_1') = \Lambda(D_2') = 0$. Again, by Proposition 6.0.6, we know that Λ_1 measures the number of extensions done at $(X, D_1, G, A_{2,i})$ for some G that intersects A_{1,n_1} (or $A_{1,1}$) and some $A_{2,i}$, minus the number of extensions done at $(X, D_2, G', A_{1,j})$ for some G' that intersects A_{2,n_2} (or $A_{2,1}$) and some $A_{1,j}$. Since $\Lambda_2 = -\Lambda_1$, the property is symmetric for D_2.

There are several non-exclusive cases:

- If $\Lambda_1 = 0$ call Double-Chain-Independent(X, D_1, D_2, $\mathcal{G}$).

- If $\Lambda_1 \neq 0$ call Double-Chain-Semi-Independent(X, D_1, D_2, $\mathcal{G}$).
- In any case call Double-Chain-Dependent(X, D_1, D_2, $\mathcal{G}$).

Double-Chain-Check-Wahl(a surface X, a chain D_1, a chain D_2, a configuration of (-1)-curves $\mathcal{G}$)

At this step we have chains D_1 and D_2 in X with a set of (-1)-curves $\mathcal{G}$, and we want to check if the D_i's are already Wahl chains. This can be quickly checked if all the curves $A_{i,j}$ in D_i have $A_{i,j}^2 \leq -2$ and verifying that $[-A_{i,1}^2, \ldots, -A_{i,l_i}^2]$ is a fraction of the form $\frac{n_i^2}{n_i a_i - 1}$, where $D_i = A_{i,1} + \ldots + A_{i,l_i}$ and $A_{i,j}$ intersects $A_{i,j-1}$ and $A_{i,j+1}$. If either D_i is not a Wahl chain, return from this function.

If the $\mathbb{Q}$-effective test was enabled in the input, call Q-Effective-Test(X, $D_1 + D_2$, $\mathcal{G}$). If it did not pass the test, discard this example and return from this function.

If the nef test was enabled in the input, call Nef-Test(X, $D_1 + D_2$, $\mathcal{G}$). If it did not pass the test, discard this example and return from this function.

At this point, $D_1 + D_2$ is a pair of Wahl chains that passed all needed tests. Add this to the list of found examples.

Double-Chain-Independent(a surface X, a chain D_1, a chain D_2, a configuration of (-1)-curves $\mathcal{G}$)

Here $\Lambda(D_1) = \Lambda(D_2) = 0$, so both chains D_1 and D_2 have the numerical invariants to be Wahl chains. We test modifications to D_1 and D_2 independently of one another in a way analogous to Single-Chain-Check-Wahl and Single-Chain-Extend. First extend (or not) D_1 with a G_1 intersecting twice D_1 obtaining Y, D_1', D_2' and $\mathcal{G}'$, and then extend (or not) D_2' with a G_2 intersecting twice D_2' obtaining Y', D_1'', D_2'' and $\mathcal{G}''$. Call Double-Chain-Check-Wahl(Y', D_1'', D_2'', $\mathcal{G}''$).

Double-Chain-Semi-Independent(a surface X, a chain D_1, a chain D_2, a configuration of (-1)-curves $\mathcal{G}$)

Here, up to renaming the D_1 and D_2, we may assume $\Lambda(D_1) > 0$, so we will try to extend D_1 with exactly $\Lambda(D_1)$ extensions while decreasing the self intersection of a curve in D_2 by $\Lambda(D_1)$, and then independently extending D_2.

For this, write $D_1 = A_{1,1} + \ldots + A_{1,l_1}$ where $A_{1,i}$ intersects $A_{1,i-1}$ and $A_{1,i+1}$. There are two ways of writing D in this manner (the other is $A_{1,l_1} + \ldots + A_{1,1}$), so for each do the following:

If $A_{1,1} \geq -2$, ignore this case.

For every (-1)-curve $G_1 \leq \mathcal{G}$ that intersects A_{1,l_1} and some $A_{2,i}$ do the following. Let $(Y, D_1', G_1', \widehat{A_{2,i}})$ be the $\Lambda(D_1)$-extension of $(X, D_1, G_1, A_{2,i})$, and let D_2' be the strict transform of D_2. In a way analogous to Single-Chain-Check-Wahl and Single-Chain-Extend, test modifications of D_2' independently of D_1', so extend (or not) D_2 with a G_2 intersecting D_2' twice obtaining Y', D_1'', D_2'' and $\mathcal{G}''$. Call Double-Chain-Check-Wahl(Y', D_1'', D_2'', $\mathcal{G}''$).

Remark 6.0.8. We can easily optimize a step here, since the combinatorics of D_1' do not depend on the choice of G. We first extend with any G, and if we do not obtain a Wahl chain immediately, discard this case.

Double-Chain-Dependent(a surface X, a chain D_1, a chain D_2, a configuration of (-1)-curves $\mathcal{G}$)

Here we will try to extend simultaneously both chains with two curves in $\mathcal{G}$ that intersect them both.

Write $D_i = A_{i,1} + \ldots + A_{i,l_1}$ where $A_{i,j}$ intersects $A_{i,j-1}$ and $A_{i,j+1}$. For each i there are two ways of writing D_i in this manner (the other is $A_{i,l_i} + \ldots + A_{i,1}$), so for each of the 4 possibilities, do the following:

If either $A_{i,1}^2 \geq -2$, ignore this case.

For every $G_1 \leq \mathcal{G}$ that intersects A_{1,l_1} and $A_{2,i}$ for some i, and every $G_2 \leq \mathcal{G}$ different from G_1 that intersects A_{2,l_2} and $A_{1,j}$ for some j, do the following:

If $i = 1$ or $j = 1$, ignore the pair (G_1, G_2), since these cases are taken into account in Extremal-P-Resolution-Extend in some other separation of the original configuration.

Consider the inequality $A_{2,i}^2 - \Lambda(D_1) \geq A_{1,j}^2$. It is equivalent to $A_{1,j}^2 - \Lambda(D_2) \leq A_{2,i}^2$, so up to renaming the chains, we may assume that $A_{2,i}^2 - \Lambda(D_1) \geq A_{1,j}^2$.

Let $m_1 > 0$ and $m_2 > 0$ be integers such that $m_1 - m_2 = \Lambda(D_1)$. Consider the m_1-extension $(Y', D_1'', G_1', \widehat{A_{2,i}})$ of $(X, D_1, G_1, A_{2,i})$, and let D_2'' the strict transform of D_2. Now consider the m_2-extension $(Y, D_2', G_2', \widehat{\widehat{A_{1,j}}})$ of $(Y', D_2'', \widehat{G_2}, \widehat{A_{1,j}})$ (all this does is extend D_i with m_i (-2)-curves and decrease $A_{2,i}^2$ by m_1 and $A_{1,j}^2$ by m_2). Write $D_i' = B_{i,1} + \ldots + B_{i,l_i+m_i}$, where $B_{i,j} = \widehat{\widehat{A_{i,j}}}$ if $j \leq l_i$. Finally, let $\mathcal{G}' = (\mathcal{G} - G_1 - G_2)\widehat{\ } + \widehat{G_1'} + G_2'$.

There are two non-exclusive cases.

- $A_{2,i}^2 > -2$. We would like to decrease this self-intersection just enough so that $B_{2,i}^2 = -2$, so now fix $m_1 = A_{2,i}^2 + 2$ (if $m_2 = m_1 - \Lambda(D_1) \leq 0$, then ignore the rest of this item and proceed with the next one). We have

  $$B_{1,j}^2 = A_{1,j}^2 - m_2 = A_{1,j}^2 - m_1 + \Lambda(D_1) \leq A_{2,i}^2 - m_1 = B_{2,i}^2 = -2$$

  so there should be no problems with the restriction $B_{i,j}^2 \leq -2$, at least for $B_{1,j}$ and $B_{2,i}$. Call Double-Chain-Check-Wahl(Y, D_1', D_2', $\mathcal{G}'$). Continue with the following item.

- In any case, we will require m_1 large enough so that $B_{2,i}^2 \leq -3$. This means there is a single $r \geq i > 1$, independent of m, such that every $B_{2,k}$ for $k > r$ would be a (-2)-curve and $B_{2,r}$ (which could be $B_{2,i}$) would be a (-3)-curve or lower. Since $r \geq 2$, this cannot be a linear chain (see Example 2.2.4), so if D_2' were to be a Wahl chain, the amount of (-2)-curves at the end must be $-2 - B_{2,1}^2$, which is independent of m_1 and equal to $-2 - A_{2,1}^2$ since $i \neq 1$.

  This means that the only m_1 we should check is $m_1 = -2 - A_{2,i}^2 - l_2 + r$ (if m_1 or $m_2 = m_1 - \Lambda(D_1) \leq 0$, ignore this and return from this function). Call Double-Chain-Check-Wahl(Y, D_1', D_2', $\mathcal{G}'$).

Extremal-P-Resolution-Check(a surface X, a chain D, a configuration of (-1)-curves $\mathcal{G}$)

Here we have a single chain $D = A_1 + \ldots + A_l$ that we wish to separate in the middle by a sequence of blow ups so that we may obtain two Wahl chains. This is done by extremal P-resolutions of $\frac{\Omega}{\Delta} = [-A_1^2, \ldots, -A_l^2]$ as explained in Section 2.3.

If they exist, let $Y \to X$ be an extremal P-resolution. Write $(\pi^* D)_{red} = D_1 + E + D_2$ so that D_1 and D_2 are the two Wahl chains and E is a (-1)-curve connecting them, and let $\mathcal{G}' = \widehat{\mathcal{G}} + E$. For every such extremal P-resolution do the following:

If the $\mathbb{Q}$-effective test was enabled in the input, call Q-Effective-Test(Y, $D_1 + D_2$, $\mathcal{G}'$). If it did not pass the test, discard this example and return from this function.

If the nef test was enabled in the input, call Nef-Test(Y, $D_1 + D_2$, $\mathcal{G}'$). If it did not pass the test, discard this example and return from this function.

At this point, $D_1 + D_2$ is a pair of Wahl chains that passed all needed tests. Add this to the list of found examples.

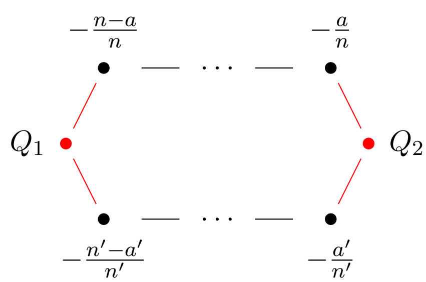
Extremal-P-Resolution-Extend(a surface X, a chain D, a configuration of (-1)-curves $\mathcal{G}$)

Here we will try to extend the chain D before checking for extremal P-resolutions.

Write $D = A_1 + \ldots + A_l$, where A_i intersects A_{i-1} and A_{i+1}. There are two ways of writing D in this manner (the other is $A_l + \ldots + A_1$), so for each of them do the following:

For every $G \in \mathcal{G}$ that intersects A_l and some A_i do the following:

If $i = 1$, ignore this G, since after extending and blowing up in the middle to obtain an extremal P-resolution $Y \to X$, we would have a dual graph with discrepancies (see Lemma 2.2.5):



where Q_i are (-1)-curves. If $\tilde{Q}_1$ and $\tilde{Q}_2$ are the images of Q_1 and Q_2 under the contraction morphism $\pi\colon Y \to Z$, we have by equation 2.1

$$K_Z \cdot \tilde{Q}_1 = \pi^* K_Z \cdot \pi^* \tilde{Q}_1 = \pi^* K_Z \cdot Q_1 = K_Y \cdot Q_1 - \sum d_j E_j \cdot Q_1 = -1 + \frac{n-a}{n} + \frac{n'-a'}{n'}$$

and

$$K_Z \cdot \tilde{Q}_2 = \pi^* K_Z \cdot \pi^* \tilde{Q}_2 = \pi^* K_Z \cdot Q_2 = K_Y \cdot Q_2 - \sum d_j E_j \cdot Q_2 = -1 + \frac{a}{n} + \frac{a'}{n'}$$

and so, since

$$K_Z \cdot \tilde{Q}_1 + K_Z \cdot \tilde{Q}_2 = 0$$

at least one of $K_Z \cdot \tilde{Q}_1$ or $K_Z \cdot \tilde{Q}_2$ is non positive, so K_Z cannot be ample.

Therefore we may assume that $i \neq 1$.

Let $m > 0$ be an integer. Consider the m-extension $(Y, D', G', \widehat{A_i})$ of (X, D, G, A_i). Write $D' = B_1 + \ldots + B_{l+m}$, where $B_i = \widehat{A_i}$ if $i \leq l$. Finally, let $\mathcal{G}' = \widehat{\mathcal{G} - G} + G'$.

There are two non exclusive cases:

- $A_i^2 > -2$. We would like to decrease this self intersection just enough so that $B_i^2 = -2$, so now we fix $m = A_i^2 + 2$. Call Extremal-P-Resolution-Check(Y, D', $\mathcal{G}'$). Continue with the following item.

- In any case, we will require m large enough so that $B_i^2 \leq -3$. Since we do not want a (-1)-curve intersecting both ends of a Wahl chain (because K_Z would not be ample), then after doing all blow ups in the middle, the strict transform of B_i cannot be the first member of the second chain. This is precisely the situation in the hypothesis of Proposition 2.3.23, so if we want to obtain a chain with extremal P-resolutions, we know that $m \leq -A_1^2 - \min_{j\neq i} A_j^2 - 3$. Call Extremal-P-Resolution-Check(Y, D', $\mathcal{G}'$) for every m from $\max(1, A_i^2 + 3)$ to $-A_1^2 - \min_{j\neq i} A_j^2 - 3$.

Q-Effective-Test(a surface X, a chain forest D, a configuration of (-1)-curves $\mathcal{G}$)

In this step D consists of either one or two Wahl chains. Let $\pi\colon X \to Z$ be the contraction of these chains. We want to verify that $\pi^* K_Z$ is $\mathbb{Q}$-effective. By Remark 5.2.1 it is enough to show that D contains (the strict transforms) of at least two fibers of type I_n, and that the discrepancies of each of these curves lesser than or equal to $-\frac{1}{2}$. Discrepancies are easily calculated recursively using the formula

$$e_i - 2 = d_{i-1} - e_i d_i + d_{i+1}$$

while knowing that the first discrepancy for a Wahl chain is always $\frac{a-n}{n}$. A `C++` code for calculating discrepancies is:

```
#include <vector>
// Returns a vector of discrepancies of the Wahl chain associated to (n,a).
// Result is multiplied by n, so they are negative integers.
// Assumes 0 < a < n and gcd(a,n) = 1.
std::vector<long long> get_discrepancies(long long n, long long a)
{
  std::vector<long long> discrepancies;
  long long q = n*n;
  long long m = n*a-1;
  discrepancies.push_back(a-n);
  long long prev_prev_disc = 0;
  long long prev_disc = a-n;
  while (m > 1ll)
  {
    long long disc = n*(q/m-1ll) + (q/m+1ll)*prev_disc - prev_prev_disc;
    discrepancies.push_back(disc);
    prev_prev_disc = prev_disc;
    prev_disc = disc;
    long long temp = m;
    m = m - q%m;
    q = temp;
  }
  return discrepancies;
}
```

Remark 6.0.9. It is important to note that in this step we strongly assume that S_0 is an elliptic fibration where $K_{S_0} \sim -\frac{1}{2}F_1 - \frac{1}{2}F_2$. In other cases, for example when S_0 is a $K3$, so $K_{S_0} \equiv 0$, this test is not required as $\pi^* K_Z$ is always effective.

Remark 6.0.10. Due to Remark 5.2.2, in our case where there are no blow downs, this test will always be passed whenever two complete fibers are used, so it is actually redundant. It might still be necessary in the general case, where curves can be blown down, but the author has never found an example where this fails.

Nef-Test(a surface X, a chain forest D, a configuration of (-1)-curves $\mathcal{G}$)

In this step D consists of either one or two Wahl chains. Let $\pi\colon X \to Z$ be the contraction of these chains. Here we test if $\pi^* K_Z$ is nef. Sadly, we cannot immediately test if it is ample, since there could be curves outside of $\mathcal{C}$ that the program does not know of, and that could intersect 0 with $\pi^* K_Z$, so without more detailed analysis this is the best we could hope for. Let D consist of curves $A_1, \ldots, A_l$ and let d_i be the discrepancy of A_i. Assuming $\pi^* K_Z$ is $\mathbb{Q}$-effective, by remark 5.2.3 if $\varphi\colon X \to S_0$ is the composition of blow ups from the original elliptic fibration, we only need to verify that for every φ-exceptional curve Γ with $\Gamma^2 = -1$ we have $\sum_i d_i A_i \cdot \Gamma \leq -1$. For this we calculate $\sum_i d_i A_i \cdot G \leq -1$ for every $G \in \mathcal{G}$, and that $\sum_i d_i A_i \cdot E \leq -1$ for every curve E coming from $\mathcal{E}$ that has $E^2 = -1$.

Remark 6.0.11. We might want to make this an strict inequality to discard examples that are not ample. However we might also be interested in singularities of type $T(2, n, a)$, whose partial resolution are two $T(1, n, a)$ singularities with a 0 curve connecting them. This translates into two concatenated Wahl chains connected by a (-1)-curve Γ that satisfies $\sum_i d_i C_i \cdot \Gamma = -1$.

Calculate-Obstruction(a surface X, a snc configuration D in X)

Here we test if there are no obstructions for (X, D). Since all we will do to this configuration is add (-1)-curves and blow up at nodes, if $H^2(X, \mathcal{T}_X(-\log D)) = 0$, then $H^2(Y, \mathcal{T}_Y(-\log D')) = 0$ for any configuration obtained by this algorithm.

By Propositions 5.1.4 and 3.2.5 it is enough to verify that at most two singular fibers of type I_n are used completely, and by Propositions 3.2.3 and 3.2.4 that if we write $D = \mathcal{F}' + \mathcal{S}' + \mathcal{E}'$, where $\mathcal{F}'$, $\mathcal{S}'$ and $\mathcal{E}'$ come from $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{E}$ in S respectively, then every curve in $\mathcal{S}'$ has self-intersection ≥ -1.

Remark 6.0.12. As in Q-Effective-Test, this test assumes that the surface S_0 is a rational elliptic surface, since the conditions for non-obstruction in other situations are different.

Remark 6.0.13. In this function, this criterion may give false negatives, as there may exist some order of blow ups such that it could be possible to add some multi sections during the process before their self-intersection gets too low.

Chapter 7

Appendix

7.1 Some Interesting Examples

7.1.1 An example via extremal P-resolution and a blow down

Consider the same $I_9 + 3I_1$ elliptic fibration S as in Section 5.2. Blow up at the nodes of F_1 and F_2 and select curves as in Figure 7.1, later blow up as in Figure 7.2. Note that $G_2 \cap H$ there were five infinitely near blow ups.

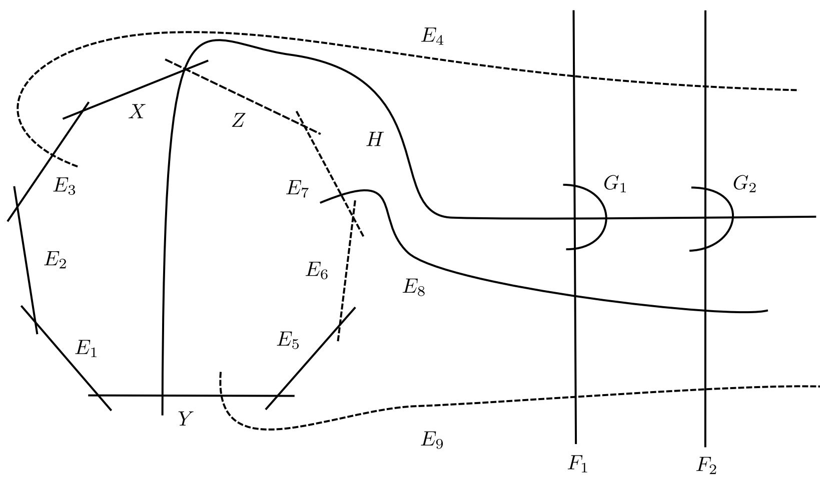


Figure 7.1: The surface S_1.

Call the exceptionals as follows:

$$
\begin{array}{lll}
G_3 \mapsto Y \cap H & G_4 \mapsto F_1 \cap H & G_5 \mapsto F_1 \cap G_1 \\
G_6 \mapsto F_2 \cap H & G_7 \mapsto F_2 \cap G_2 & A_1, A_2, A_3, A_4, G_8 \mapsto G_2 \cap H
\end{array}
$$

Then after blowing down E_8 via $\psi \colon S_2 \to S_3$ we obtain the chain

$$\mathcal{C} : E_5, Y, E_1, E_2, E_3, X, H, G_1, F_1, F_2, G_2, A_1, A_2, A_3, A_4$$

with self-intersections given by

$$[2, 3, 2, 2, 2, 2, 9, 2, 5, 5, 3, 2, 2, 2, 2].$$

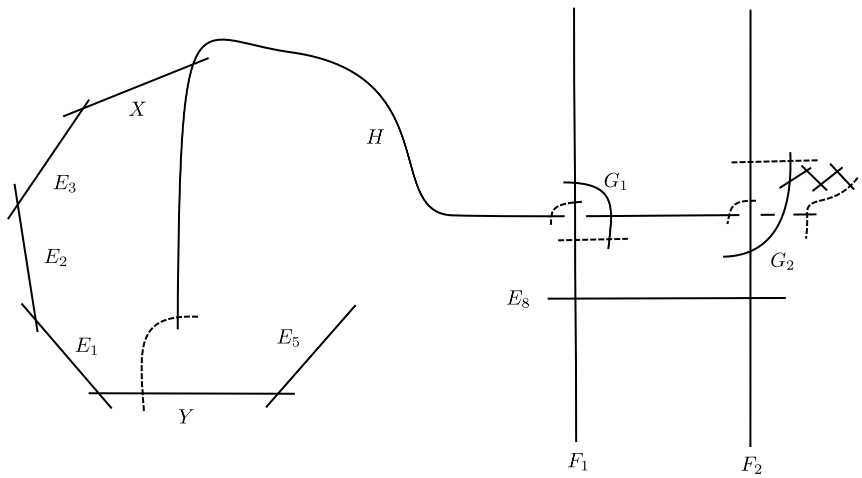


Figure 7.2: The surface S_2.

This chain has exactly one extremal P-resolution, namely

$$[2, 3, 2, 2, 2, 2, 9, 3, 1, 6, 5, 3, 2, 2, 2, 2].$$

Here, $\mathcal{C}_1 : E_5, Y, E_1, E_2, E_3, X, H, G_1$ has self-intersections given by $[2, 3, 2, 2, 2, 2, 9, 3] = T(1, 20, 17)$ and $\mathcal{C}_2 : F_1, F_2, G_2, A_1, A_2, A_3, A_4$ has self-intersections given by $[6, 5, 3, 2, 2, 2, 2] = T(1, 17, 3)$. Their discrepancies are given by

$$d(\mathcal{C}_1) = \left(-\frac{7}{20}, -\frac{14}{20}, -\frac{15}{20}, -\frac{16}{20}, -\frac{17}{20}, -\frac{18}{20}, -\frac{19}{20}, -\frac{13}{20}\right),$$

$$d(\mathcal{C}_2) = \left(-\frac{14}{17}, -\frac{16}{17}, -\frac{15}{17}, -\frac{12}{17}, -\frac{9}{17}, -\frac{6}{17}, -\frac{3}{17}\right)$$

Since $K_{S_3} = -12$ and the sum of lengths of the chains is 15, then $K_Z^2 = 3$. Note that S_1 is the log-resolution of the original curves selected in S, and every curve coming from a multi-section is a (-1)-curve, so by Remark 5.2.4, this example is unobstructed (by Proposition 3.2.3, the blow down preserves the cohomology).

It can easily be verified that $\pi^* K_Z$ is $\mathbb{Q}$-effective by writing $K_S = -\frac{1}{2}F_1 - \frac{1}{2}F_2$. We can also verify that K_Z is nef: since this comes from a P-resolution, the middle curve is positive for $\pi^* K_Z$, so the possibly problematic curves $G_3, \ldots, G_8$, however note that the only one that does not intersect two curves not from some tail of (-2)-curves is G_8 (see Remark 2.2.7), so the others immediately are positive for $\pi^* K_Z$. We have

$$\pi^* K_Z \cdot G_8 = \left(K_{S_3} + \frac{3}{17}A_3 + \frac{19}{20}H\right) \cdot G_8 = -1 + \frac{3}{17} + \frac{19}{20} = \frac{43}{340}.$$

To see that it is ample, by choosing $K_S = -\frac{1}{2}F_1 - \frac{1}{2}F_2$ note that

$$\pi^* K_Z = \frac{11}{34}F_1 + \frac{13}{20}G_1 + \frac{1}{2}G_4 + \frac{1}{2}G_5 + \text{(effective divisor)}$$

so going back to S_2 before blowing down E_8 we have

$$(\psi \circ \pi)^* K_Z = \frac{11}{34}F_1 + \frac{13}{20}G_1 + \frac{1}{2}G_4 + \frac{1}{2}G_5 + \frac{11}{34}E_8 + \text{(effective divisor)}$$

Note that every n-multi-section intersects this divisor at either F_1, G_1, G_4 or G_5, and the same for every fiber component.

To see the resulting surface is simply connected, note that the middle curve of the extremal resolution connects both ends of the Wahl chains, whose indices are coprime.

Theorem 7.1.1. There exists a singular surface Z with $K_Z^2 = 3$ and $\chi(Z) = 1$, which has only the singularities $\frac{1}{20^2}(1, 20\cdot 17-1)$ and $\frac{1}{17^2}(1, 17\cdot 3-1)$. It has no obstructions to deformations, and thus lies in the border of $\mathcal{M}_{3,1}$. A general member of the component in which it lies is smooth, simply connected and an exotic $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$.

7.1.2 An example with $K^2 = 5$

Consider the following pencil in $\mathbb{P}^2$:

$$\Phi_t = 4xyz + t(y-x)(z^2 - xy)$$

It has four singular members at $t = 0, \infty, 1, -1$. This pencil has as base points $[0,0,1]$ with multiplicity 2, $[0,1,0]$ with multiplicity 3, $[1,0,0]$ with multiplicity 3 and $[1,1,0]$ with multiplicity 1. Here, Φ_1 is a nodal cubic curve with a node at $[-1,1,1]$ and Φ_{-1} is a nodal cubic curve with node at $[1,-1,1]$. Also consider the special line $h = \{y - z = 0\}$ which goes through $[-1,1,1]$, $[1,0,0]$ and $[1,1,1]$, a singular point of Φ_∞. The diagram in $\mathbb{P}^2$ is as in Figure 7.3

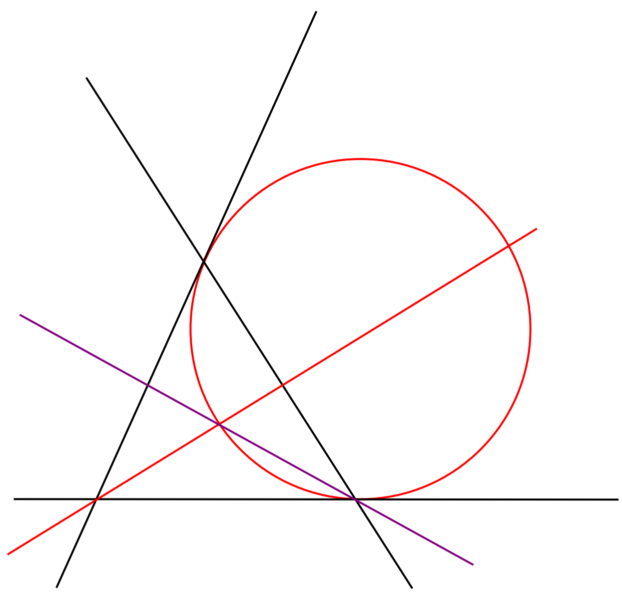


Figure 7.3: In black: Φ_0. In red: Φ_∞. In purple: h

Call $S \to \mathbb{P}^2$ the resulting fibration. Let E_1, E_2 be the exceptionals over $[0,0,1]$, E_3, E_4, E_5 the exceptionals over $[0,1,0]$, E_6, E_7, E_8 the exceptionals over $[1,0,0]$ and E_9 the exceptional over $[1,1,0]$. Call $X, Y, Z, L, C, H, F_1, F_2$ the strict transforms of $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, $\{y - x = 0\}$, $\{z^2 - xy = 0\}$, $\{y - z = 0\}$, Φ_1, Φ_{-1}. The fibration is of type $I_8 + I_2 + 2I_1$, and looks as in Figure 7.4.

Blow up at the node of F_1 and $H \cap C \cap L$, and call them G_1 and G_2 respectively, and call the resulting surface S_1. Select curves as in Figure 7.5. We immediately note that three complete fibers

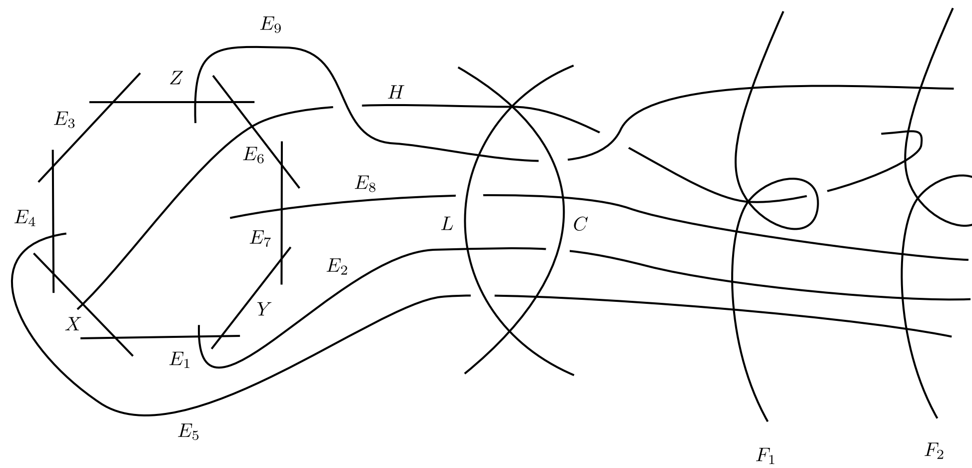


Figure 7.4: The surface S, an $I_8 + I_2 + 2I_1$ fibration

are used, so the strategy for the vanishing of the cohomology does not work (in fact, this example must be obstructed).

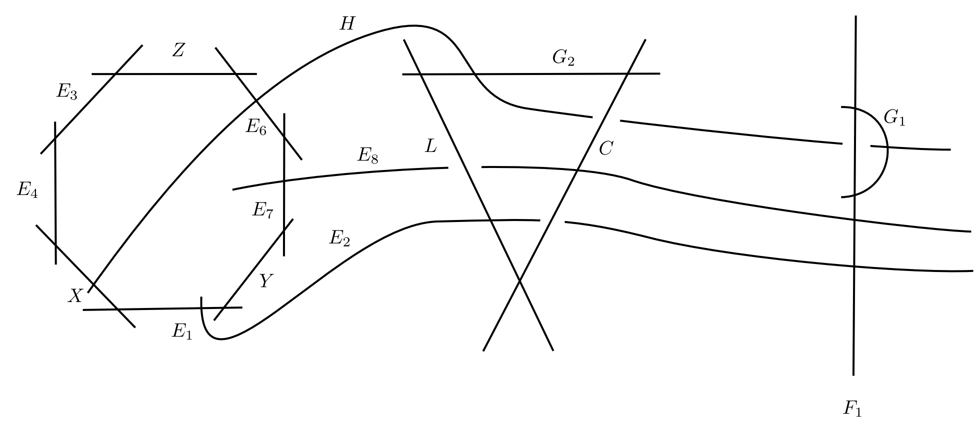


Figure 7.5: The surface S_1

We blow up 11 more times as in Figure 7.6, and call the resulting surface S_2. Call the exceptionals as follows:

$$\begin{array}{rrr} G_3 \mapsto E_2 \cap F_1 & G_4 \mapsto E_6 \cap H & G_5 \mapsto E_7 \cap E_8 \\ G_6 \mapsto \text{(one of) } F_1 \cap G_1 & G_7 \mapsto H \cap G_1 & G_8 \mapsto C \cap L \\ G_9 \mapsto X \cap H & A_1, A_2, A_3, G_{10} \mapsto X \cap E_1 & \end{array}$$

Here we recognize the chain

$$\mathcal{C} : H, G_1, F_1, E_8, C, G_2, L, E_2, E_1, Y, E_7, E_6, Z, E_3, E_4, X, A_1, A_2, A_3$$

with self-intersections given by the chain

$$[5, 2, 6, 2, 4, 2, 4, 2, 6, 2, 3, 3, 2, 2, 2, 4, 2, 2, 2] = \frac{5354298}{1203449}$$

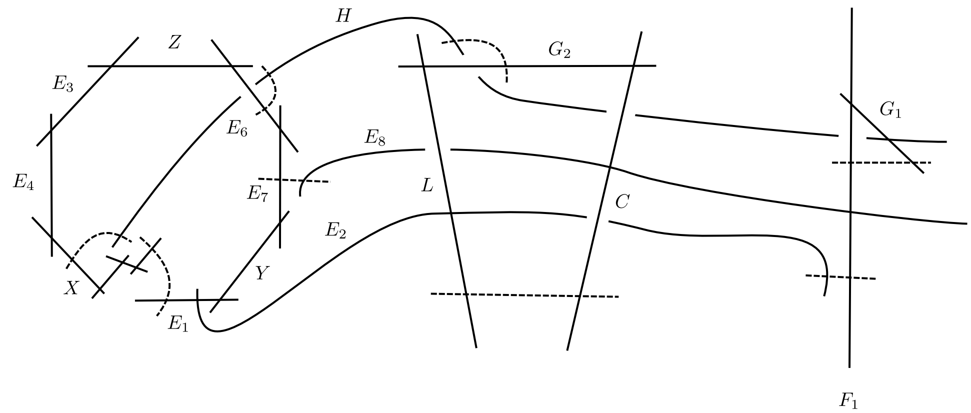


Figure 7.6: The surface S_2

This chain has a single extremal P-resolution after 11 more blow ups at $C \cap G_2$, and call the resulting surface S_3:

$$[5,2,6,2,6,2,2,2,4,2,2,2,1,5,2,6,3,4,2,6,2,3,3,2,2,2,4,2,2,2]$$

Call the exceptional curves as B_i in the order they were blown up, B_{11} being the last (-1)-curve. Then we have the chains

$$\mathcal{C}_1 : H, G_1, F_1, E_8, C, B_2, B_3, B_4, B_5, B_8, B_9, B_{10}$$

with self-intersections:

$$[5,2,6,2,6,2,2,2,4,2,2,2] = T(1,129,29)$$

and discrepancies:

$$d(\mathcal{C}_1) = \left(-\frac{100}{129}, -\frac{113}{129}, -\frac{126}{129}, -\frac{127}{129}, -\frac{128}{129}, -\frac{125}{129}, -\frac{122}{129}, -\frac{119}{129}, -\frac{116}{129}, -\frac{87}{129}, -\frac{58}{129}, -\frac{29}{129}\right)$$

and

$$\mathcal{C}_2 : B_7, B_6, B_1, G_2, L, E_2, E_1, Y, E_7, E_6, Z, E_3, E_4, X, A_1, A_2, A_3$$

with self-intersections:

$$[5,2,6,3,4,2,6,2,3,3,2,2,2,4,2,2,2] = T(1,1233,277)$$

and discrepancies:

$$d(\mathcal{C}_2) = \left(-\frac{956}{1233}, -\frac{1081}{1233}, -\frac{1206}{1233}, -\frac{1223}{1233}, -\frac{1230}{1233}, -\frac{1231}{1233}, -\frac{1232}{1233}, -\frac{1229}{1233}, -\frac{1226}{1233}, -\frac{1216}{1233}, -\frac{1189}{1233}, -\frac{1162}{1233}, -\frac{1135}{1233}, -\frac{1108}{1233}, -\frac{831}{1233}, -\frac{554}{1233}, -\frac{277}{1233}\right)$$

Since we chose three complete fibers, $\pi^* K_Z$ is $\mathbb{Q}$-effective. For nefness of K_Z we must only check at G_{10} since it is the only non-π-exceptional curve in $\pi^* K_Z$ that intersects a curve in some tail of (-2)-curves (excluding B_{11} which is positive for this coming from a P-resolution). For G_{10} we have

$$\pi^* K_Z \cdot G_{10} = \left(K_{S_3} + \frac{277}{1233} A_3 + \frac{1232}{1233} E_1\right) \cdot G_{10} = -1 + \frac{277}{1233} + \frac{1232}{1233} = \frac{92}{411}$$

Since there were 24 blow ups in total and the sum of lengths of the Wahl chains is 29, we obtain $K_Z^2 = 5$.

To see that K_Z is ample, by writing in S

$$K_S = -\frac{1}{2}(X + E_1 + Y + E_7 + E_6 + Z + E_3 + E_4) - \frac{1}{2}(C + L)$$

then

$$\pi^* K_Z = \frac{126}{129} F_1 + \left(1 + \frac{113}{129}\right) G_1 + G_3 + 2G_6 + \text{(effective divisor)}$$

so every curve coming from a n-multi-section intersects $\pi^* K_Z$ positively. Moreover clearly every component of fibers also intersects the divisor.

Lemma 7.1.2. The symplectic rational blow down of $(S, \mathcal{C}_1 + \mathcal{C}_2)$ is simply connected.

Proof. This is not as direct as other examples, as $\gcd(129, 1233) = 3$.

Let α_1 be a loop around H and α_2 be a loop around B_7. Recall that these loops generate fundamental groups of the lens spaces, which are isomorphic to $\mathbb{Z}/129^2\mathbb{Z}$ and $\mathbb{Z}/1233^2\mathbb{Z}$ respectively. By looking at 4.1 we see that α_1^{12899} is homotopic to a loop around B_{10}, which in turn is homotopic to $\alpha_2^{\pm 1}$ via B_{11}. This homotopy tells us that $(\alpha_1^{12899})^9$ contracts, which implies that α_1^9 contracts since $\gcd(12899, 129^2) = 1$. Note that the loop around F_1 is homotopic to α_1^9 so it contracts. This means that the loop around E_2, which is homotopic to α_1^9 via G_3 also contracts. This loop around E_2 is homotopic to α_2^{503}, and since $\gcd(1233^2, 503) = 1$, this means α_2 contracts. Finally looking at B_{11} again, we see that α_1^{12899} contracts, therefore α_1 contracts as well. $\square$

Since this example is obstructed, we do not know if there is a $\mathbb{Q}$-Gorenstein deformation of this singular surface. It would be surprising if it did, as the expected dimension of this component of the moduli space is 0.

Nonetheless we obtain the following.

Theorem 7.1.3. There exists a singular surface Z with $K_Z^2 = 5$ and $\chi(Z) = 1$, which has only the singularities $\frac{1}{129^2}(1, 129 \cdot 29 - 1)$ and $\frac{1}{1233^2}(1, 1233 \cdot 277 - 1)$. Thus, it lies in $\overline{\mathcal{M}_{5,1}}$. Its minimal resolution can be symplectically blown down into a non-algebraic simply connected surface which is an exotic $\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}$.

7.1.3 An example of a wormhole

A wormhole happens when we find in a surface a chain with two extremal P-resolutions (for a precise definition, see [UV22, Definition 2.7]). It is called a wormhole because up to a small birrational surgery, the same surface appears in two places in the moduli space at once, assuming the wormhole conjecture holds.

Consider the same fibration as in Example 7.1.2. This time, select curves and blow up as in Figure 7.7, and call the surface S_1. At this point, H is a (-1)-curve, and since we are using two complete I_n fibers, by Remark 5.2.4 S_1 together with these curves is unobstructed, so again by Propositions 3.2.3 and 3.2.4, every example coming from a blow up of this configuration is unobstructed.

We blow up 8 more times as in Figure 7.8 and call the surface S_2. Call the exceptionals as follows

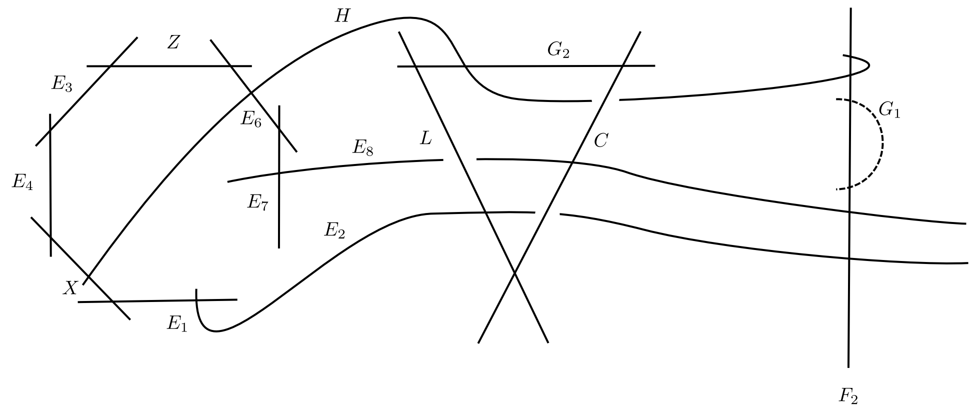


Figure 7.7: The surface S_1

$$G_3 \mapsto F_2 \cap E_2 \qquad G_4 \mapsto H \cap E_6 \qquad G_5 \mapsto L \cap G_2$$
$$G_6 \mapsto X \cap H \qquad G_7 \mapsto F_2 \cap H \qquad G_8 \mapsto E_7 \cap E_8$$
$$A_1, G_9 \mapsto C \cap E_8$$

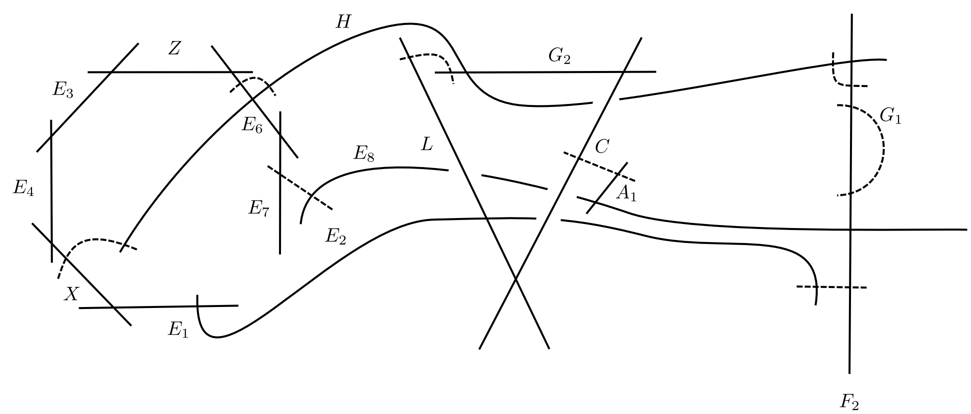


Figure 7.8: The surface S_2

Here we recognize the chain

$$\mathcal{C} : E_7, E_6, Z, E_3, E_4, X, E_1, E_2, L, C, G_2, H, F_2, E_8, A_1.$$

with self-intersections

$$[3, 3, 2, 2, 2, 3, 2, 2, 4, 5, 2, 4, 6, 3, 2] = \frac{267325}{104939}.$$

This chain has two extremal P-resolutions. The first one is obtained by blowing up 8 times at $X \cap E_1$. Call the resulting surface S_3. S_3 contains the chain

$$[3, 3, 2, 2, 2, 8, 3, 2, 1, 3, 3, 2, 2, 2, 3, 2, 4, 5, 2, 4, 6, 3, 2].$$

Call the exceptional curves as B_i in the order they were blown up, B_8 being the exceptional at the middle. The first chain is

$$\mathcal{C}_1 : E_7, E_6, Z, E_3, E_4, X, B_5, B_7$$

with self-intersections

$$[3, 3, 2, 2, 2, 8, 3] = T(1, 28, 11)$$

and discrepancies

$$d(\mathcal{C}_1) = \left(-\frac{17}{28}, -\frac{23}{28}, -\frac{24}{28}, -\frac{25}{28}, -\frac{26}{28}, -\frac{27}{28}, -\frac{22}{28}, -\frac{11}{28}, \right)$$

The second chain is

$$\mathcal{C}_2 : B_6, B_4, B_3, B_2, B_1, E_1, E_2, L, C, G_2, H, F_2, E_8, A_1$$

with self-intersections

$$[3, 3, 2, 2, 2, 3, 2, 4, 5, 2, 4, 6, 3, 2] = T(1, 451, 117)$$

and discrepancies

$$d(\mathcal{C}_2) = \left(-\frac{274}{451}, -\frac{371}{451}, -\frac{388}{451}, -\frac{405}{451}, -\frac{422}{451}, -\frac{439}{451}, -\frac{444}{451}, -\frac{449}{451}, -\frac{450}{451}, -\frac{448}{451}, -\frac{446}{451}, -\frac{434}{451}, -\frac{354}{451}, -\frac{117}{451}\right)$$

By choosing $K_S = -\frac{1}{2}(C+L) - \frac{1}{2}F_2$, we see that π^*K_Z is $\mathbb{Q}$-effective, and furthermore K_Z is nef, since the only non π-exceptional curve in the divisor that intersects a (-2)-curve in a tail excluding B_8 is G_9, for which

$$\pi^*K_Z \cdot G_9 = \left(K_{S_3} + \frac{117}{451}A_1 + \frac{450}{451}C\right) \cdot G_9 = -1 + \frac{117}{451} + \frac{450}{451} = \frac{116}{451}$$

The total blow ups done were 18, and the sum of lengths of the Wahl chains is 22, so $K_Z^2 = 4$. We see that K_Z is ample since

$$\pi^*K_Z = \left(-\frac{1}{2} + \frac{449}{451}\right)L + \left(-\frac{1}{2} + \frac{450}{451}\right)C + \frac{448}{451}G_2 + \frac{1}{2}G_5 + \left(\frac{1}{2} + \frac{117}{451}\right)A_1 + G_9 + (\text{ effective divisor })$$

so every curve coming from n-multi-sections intersects π^*K_Z positively. Since $\gcd(28, 451) = 1$, the sphere B_8 immediately tells us that both loops around B_6 and B_7 contract, so the rational blow down of S_3 is simply connected. So this Z lives in the boundary of $\mathcal{M}_{4,1}$.

The second extremal P-resolution is given by blowing up 11 times at $L \cap C$. Call the resulting surface S_3'. It contains the chain

$$[3, 3, 2, 2, 2, 3, 2, 2, 7, 6, 3, 2, 1, 3, 3, 2, 2, 2, 3, 2, 6, 2, 4, 6, 3, 2].$$

Call the exceptional curves as B_i' in the order they were blown up, B_{11}' being the exceptional at the middle. The first chain is

$$\mathcal{C}_1 : E_7, E_6, Z, E_3, E_4, X, E_1, E_2, L, B_3', B_8', B_{10}'$$

with self-intersections

$$[3,3,2,2,2,3,2,2,7,6,3,2] = T(1,135,53)$$

and discrepancies

$$d(\mathcal{C}_1) = \left(-\frac{82}{135}, -\frac{111}{135}, -\frac{116}{135}, -\frac{121}{135}, -\frac{126}{135}, -\frac{131}{135}, -\frac{132}{135}, -\frac{133}{135}, -\frac{134}{135}, -\frac{130}{135}, -\frac{106}{135}, -\frac{53}{135}\right)$$

The second chain is

$$\mathcal{C}_2 : B_9', B_7', B_6', B_5', B_4', B_2', B_1', C, G_2, H, F_2, E_8, A_1$$

with self-intersections

$$[3,3,2,2,2,3,2,6,2,4,6,3,2] = T(1,265,104)$$

and discrepancies

$$d(\mathcal{C}_2) = \left(-\frac{161}{265}, -\frac{218}{265}, -\frac{228}{265}, -\frac{238}{265}, -\frac{248}{265}, -\frac{258}{265}, -\frac{261}{265}, -\frac{264}{265}, -\frac{263}{265}, -\frac{262}{265}, -\frac{255}{265}, -\frac{208}{265}, -\frac{104}{265}\right)$$

Again, by choosing $K_S = -\frac{1}{2}(C+L) - \frac{1}{2}F_2$ we see that $\pi^* K_{Z'}$ is $\mathbb{Q}$-effective. K_Z is nef, because excluding B_{11}', the only non π-exceptional curve that intersects a (-2)-curve in the tail of a Wahl chain is G_9, for which

$$\pi^* K_{Z'} \cdot G_9 = \left(K_{S_3} + \frac{104}{265}A_1 + \frac{264}{265}C\right) \cdot G_9 = -1 + \frac{104}{265} + \frac{264}{265} = \frac{103}{265}$$

The total blow ups done were 21, and the sum of lengths of the Wahl chains is 25, so $K_{Z'}^2 = 4$. We see that $K_{Z'}$ is ample the same way we saw K_Z is ample. Note that

$$\begin{aligned}\pi^* K_Z = &\left(-\frac{1}{2} + \frac{264}{265}\right)C + \left(-\frac{1}{2} + \frac{164}{165}\right)L + \frac{263}{265}G_2 + \frac{1}{2}G_5 + \left(\frac{1}{2} + \frac{104}{265}\right)A_1 + G_9 + \frac{261}{265}B_1' \\ &+ \left(\frac{1}{2} + \frac{258}{265}\right)B_2' + \left(1 + \frac{130}{135}\right)B_3' + \left(\frac{5}{2} + \frac{248}{265}\right)B_4' + \left(\frac{9}{2} + \frac{238}{265}\right)B_5' + \left(\frac{13}{2} + \frac{228}{265}\right)B_6' \\ &+ \left(\frac{17}{2} + \frac{218}{265}\right)B_7' + \left(\frac{21}{2} + \frac{106}{135}\right)B_8' + \left(20 + \frac{161}{265}\right)B_9' + \left(\frac{63}{2} + \frac{53}{135}\right)B_{10}' + \frac{105}{2}B_{11}' \\ &+ (\text{ effective divisor })\end{aligned}$$

so every curve coming from an n-multi-section intersects $\pi^* K_{Z'}$ positively.

Lemma 7.1.4. The symplectic rational blow down of $(S_3', \mathcal{C}_1 + \mathcal{C}_2)$ is simply connected.

Proof. Let α_1 be a loop around E_7 and α_2 be a loop around B_9'. We have $\gcd(135,265) = 5$, so through the sphere B_{11}' we can determine that $\alpha_1^{25} = \alpha_2^{25} = e$. By 4.1, the loop around F_2 is homotopic to $\alpha_2^{2645} = \alpha_2^{20}$ and the loop around H is homotopic to $\alpha_2^{767} = \alpha_2^{17}$. Since these two curves are connected by the sphere G_7, it implies that $\left(\alpha_2^{17}\right)^5$ is homotopic to $\left(\alpha_2^{\pm 20}\right)^5 = e$, and since $\gcd(17,25) = 1$, it also implies $\alpha_2^5 = e$. In particular, the loop around F_2 contracts, and since it is homotopic to $\alpha_2^{\pm 17}$, it also contracts, which in turn gives us that α_2 contracts. We finally use B_{11}' again to see that α_1 also contracts, concluding that the symplectic rational blow down of S_3' is simply connected. $\square$

We can deduce the following.

Theorem 7.1.5. There exist two different singular surfaces Z and Z' with $K_Z^2 = K_{Z'}^2 = 4$ and $\chi(Z) = \chi(Z') = 1$ where Z has only the singularities $\frac{1}{28^2}(1, 28 \cdot 11 - 1)$ and $\frac{1}{451^2}(1, 451 \cdot 117 - 1)$, and Z' has only the singularities $\frac{1}{135^2}(1, 135 \cdot 53 - 1)$ and $\frac{1}{265^2}(1, 265 \cdot 104 - 1)$, both with no obstructions to deformations, and thus they lie in the border of $\mathcal{M}_{4,1}$ at two distinct points. They both are extremal P-resolutions of a surface Y with a single singularity $\frac{1}{267325}(1, 104939)$. A general member of the component of either Z or Z' is smooth, simply connected and an exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$.

7.2 Source Code

The source code of the program, instructions on how it works and thousands of examples can be found at `https://github.com/jereyes4/Wahl_Chains`

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