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An ultra-weak pseudo-stress formulation of the Stokes problem and DPG approximation

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Chapter 1

Introduction

The basic equations in fluid dynamics are those of Navier-Stokes. In the case of a fluid at constant temperature, also called an isothermal fluid, these equations represent two physical laws: the conservation of mass and the conservation of linear momentum (see [18], Chapter 2). The variables that are involved in this model are the velocity $v(t,x)$, the pressure $p(t,x)$ and the density $\rho(t,x)$, where t represents the time and x the space location.
If the fluid is incompressible and homogeneous, then the variable $\rho(t,x)$ is simplified by a constant $\rho_0 > 0$. When velocity and pressure do not change over time, these fluids are modeled using the stationary Navier-Stokes equations

$$\begin{aligned} -\mu\Delta u + (u\cdot\nabla)u + \nabla p &= f, \\ \operatorname{div}(u) &= 0, \end{aligned} \tag{1.1}$$

in some domain Ω. If a fluid moves very slowly in a stationary flow the nonlinear term $(u\cdot\nabla)u$ of (1.1) can be neglected. This situation leads to a linear system of equations, called the Stokes equations

$$\begin{aligned} -\mu\Delta u + \nabla p &= f \text{ in } \Omega, \\ \operatorname{div}(u) &= 0 \text{ in } \Omega. \end{aligned} \tag{1.2}$$

The Stokes equations model the simplest incompressible flow problems.

The discontinuous Petrov Galerkin (DPG) method consists in applying Petrov-Galerkin approximations with optimal test functions to ultra-weak variational formulations. This method was introduced in its current form by Demkowicz and Gopalakrishnan in a series of papers [6], [7], [8], [19]. DPG methods for the Stokes problem can be found in [17].

In this work we present an ultra-weak formulation of the Stokes problem, using a pseudo-stress variable, and we discuss its well-posedness. After that we apply the DPG method to our problem to find a solution and analyze its convergence. The main difference is the pseudo-stress variable, which allows us to eliminate the pressure variable from the problem and also calculate the vorticity a posteriori. We note that [17] uses a similar formulation but does not eliminate the pressure from the system.

The remainder of this work is organized as follows:

- Chapter 2: We define the Sobolev spaces and their broken version. We recall some important results of the abstract setting of the DPG framework and for the broken variational forms. Finally we give a triangulation for the discrete space and some approximation properties.

- Chapter 3: Here we present the Stokes problem with the new pseudo-stress variable. We prove the existence of a unique solution of this new formulation and the $\inf - \sup$ conditions. In the last section of this chapter we present the Stokes problem in its ultra-weak formulation and prove its well-posedness.

- Chapter 4: We analyze the existence of a Fortin operator and we estimate the convergence rate of the error.

- Chapter 5: We present numerical experiments. We consider some smooth problems on a convex domain and one in an L-shaped domain. Also we consider a Lid-Driven cavity flow problem in a convex domain. Figure 1.1 shows the velocity field and pressure approximation of one of the smooth examples. In every example we show the predicted and observed convergence rates.

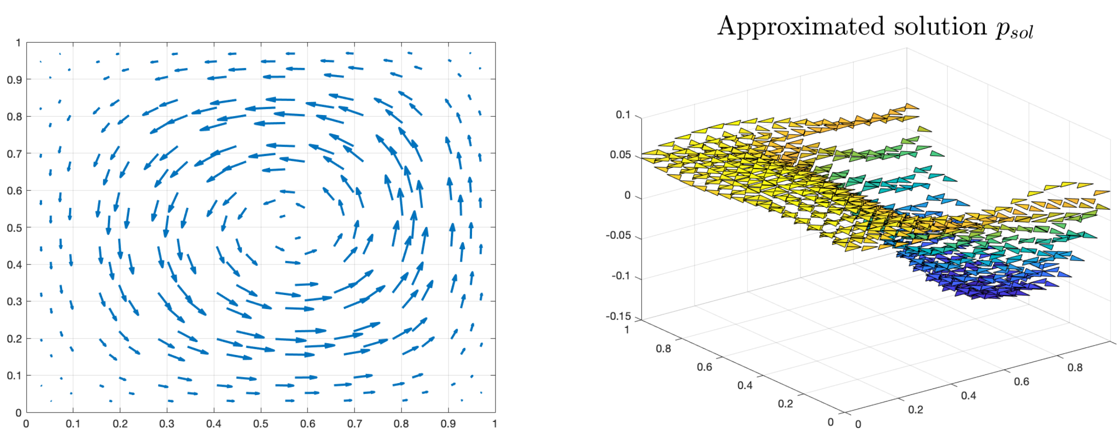

Figure 1.1: Velocity (left) and pressure (right) approximation of a smooth problem.

- In the final Chapter 6 we discuss our results and we propose some future lines of our investigation.

Chapter 2

Preliminaries

In this chapter we present the functional analytic framework of this work and a description of the DPG method. All these definitions and results can be found in [1], [2], [5], [11], [15], [9],[12] and [16].

2.1 Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^d$ ($d \geq 2$). The integration by parts formula yields that for every function with k continuous derivatives in Ω, $u \in C^k(\Omega)$, where k is a natural number, and for all infinitely differentiable functions with compact support $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega \phi D^\alpha u dx, \tag{2.1}$$

where α is a multi-index of order $|\alpha| = k$ and we are using the notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The left-hand side of equation (2.1) still makes sense if we only assume u to be locally integrable.

Definition 2.1. *Suppose u, v are locally integrable functions in Ω and α a multi-index. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx, \tag{2.2}$$

for all $\phi \in C_c^\infty(\Omega)$.

If there exists a weak α-th partial derivative of u, then it is uniquely defined almost everywhere, and thus is uniquely determined as an element of a Lebesgue space. On the other hand, if $u \in C^k(\Omega)$, the classical and the weak derivatives coincide.

Definition 2.2. *Let $1 \le p \le \infty$ and k a non-negative integer. The Sobolev space $W^{k,p}(\Omega)$ is defined as*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \forall |\alpha| \le k\},$$

where $D^\alpha u$ is the weak derivative.

Remark 1. *If $p = 2$ we write $H^k(\Omega) = W^{k,2}(\Omega)$ $(k = 0, 1, 2, \cdots)$. Note that $H^0(\Omega) = L^2(\Omega)$. We will focus on the $H^1(\Omega)$ space throughout this work.*

Definition 2.3. *If $u \in H^1(\Omega)$, we define its norm to be*

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)},$$

where $\|u\|_{L^2(\Omega)}$ is defined by

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2 dx \right)^{1/2}.$$

Definition 2.4. *We denote by $H^1_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ with respect to $\|\cdot\|_{H^1(\Omega)}$.*

We interpret $H^1_0(\Omega)$ as comprising those functions $u \in H^1(\Omega)$ such that "$u = 0$ on $\partial\Omega$". Now we discuss the possibility of assigning "boundary values" along $\partial\Omega$ to a function $u \in H^1(\Omega)$, assuming that Ω is Lipschitz. Now if $u \in C^1(\overline{\Omega})$, then clearly u has values on $\partial\Omega$ in the usual sense. The problem is that a typical function $u \in H^1(\Omega)$ is not in general continuous and is only defined almost everywhere in Ω. Since $\partial\Omega$ has d-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "u restricted to $\partial\Omega$". The notion of a trace operator resolves this problem.

Theorem 2.1 (Trace Theorem, [15] Theorems 3.37 and 3.38)**.** *Assume Ω is bounded and $\partial\Omega$ is Lipschitz. Define the trace operator $tr : D(\overline{\Omega}) \to D(\partial\Omega)$, where $D(\overline{\Omega}) = C^\infty_c(\overline{\Omega})$, by*

$$tr(u) = u|_{\partial\Omega}.$$

Then $tr(\cdot)$ has a unique extension to a bounded linear operator

$$tr : H^1(\Omega) \to tr(H^1(\Omega)),$$

and this extension has a continuous right inverse.

Definition 2.5. *We call $tr(u)$ the trace of u on $\partial\Omega$. Also, we define $H^{1/2}(\partial\Omega) = tr(H^1(\Omega))$ which is a Sobolev space with norm*

$$\|w\|_{H^{1/2}(\partial\Omega)} = \inf_{u \in H^1(\Omega),\, tr(u) = w} \|u\|_{H^1(\Omega)}, \text{ for all } w \in H^{1/2}(\partial\Omega).$$

Theorem 2.2 (Trace-zero functions in $H^1(\Omega)$, [15] Theorem 3.40)**.** *Assume Ω is bounded and $\partial\Omega$ is Lipschitz. Suppose furthermore that $u \in H^1(\Omega)$. Then*

$$u \in H^1_0(\Omega) \text{ if and only if } tr(u) = 0 \text{ on } \partial\Omega.$$

Definition 2.6. *We define*

$$H(\mathrm{div}, \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^d : \mathrm{div}(\boldsymbol{u}) \in L^2(\Omega)\}$$

which is a Hilbert space with the norm $\|\boldsymbol{u}\|^2_{H(\mathrm{div},\Omega)} = \|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\mathrm{div}(\boldsymbol{u})\|^2_{L^2(\Omega)}$.

Definition 2.7. *Define* $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))'$ *equipped with the dual norm*

$$\|u^*\|_{H^{-1/2}(\partial\Omega)} = \sup_{0\neq u\in H^{1/2}(\partial\Omega)} \frac{\langle u^*, u\rangle}{\|u\|_{H^{1/2}(\partial\Omega)}}.$$

Here, $\langle\cdot,\cdot\rangle$ *is an extension of the scalar product of* $L^2(\partial\Omega)$ *in the sense that when* $u^* \in L^2(\partial\Omega)$ *we can identify* $\langle u^*, u\rangle = \int_{\partial\Omega} u^* u$.

Theorem 2.3 (Trace theorem on $H(\mathrm{div}, \Omega)$, [11] Theorem 2.5)**.** *Assume* Ω *is bounded and* $\partial\Omega$ *is Lipschitz. Let* n *denote the outward unit normal vector on* $\partial\Omega$*. Then the mapping* $tr_n : \boldsymbol{u} \to \boldsymbol{u}\cdot n|_{\partial\Omega}$ *defined in* $D(\overline{\Omega})^d$ *can be extended by continuity to a linear and continuous mapping, from* $H(\mathrm{div}, \Omega)$ *into* $H^{-1/2}(\partial\Omega)$.

2.2 DPG Method

Let U, V be Hilbert spaces with norms $\|.\|_U, \|.\|_V$ and continuous dual spaces U', V'. We consider a bounded bilinear form $b : U \times V \to \mathbb{R}$ and given $\ell \in V'$, we seek a solution of the problem: find $u \in U$ such that

$$b(u, v) = \ell(v) \text{ for all } v \in V. \tag{2.3}$$

Here, the question of solvability of (2.3) arises and we will answer that question within the context of the inf-sup conditions.

2.2.1 Solvability

Theorem 2.4 ([9], [12] Theorem 2.5)**.** *Suppose that* U, V *are Hilbert spaces, then the following statements are equivalent:*

1. *For all* $\ell \in V'$*, there exists a unique* $u \in U$ *such that*

$$b(u, v) = \ell(v) \textit{ for all } v \in V. \tag{2.4}$$

2. $\{v \in V : b(u, v) = 0$ *for all* $u \in U\} = \{0\}$ *and there exists* $C > 0$ *such that*

$$\inf_{0\neq u\in U} \sup_{0\neq v\in V} \frac{b(u, v)}{\|u\|_U\|v\|_V} \geq C. \tag{2.5}$$

3. $\{u \in U : b(u, v) = 0$ *for all* $v \in V\} = \{0\}$ *and there exists* $C' > 0$ *such that*

$$\inf_{0\neq v\in V} \sup_{0\neq u\in U} \frac{b(u, v)}{\|u\|_U\|v\|_V} \geq C'. \tag{2.6}$$

Theorem 2.5 (Babuška's Theorem, [9] Theorem 2.6)**.** *Let U, V be Hilbert spaces and let $U_h \subseteq U, V_h \subseteq V$ be finite dimensional subspaces with $dim(U_h) = dim(V_h)$. Let $\ell \in V'$. If there exists a constant $C_h > 0$ such that*

$$\inf_{0 \neq u_h \in U_h} \sup_{0 \neq v_h \in V_h} \frac{b(u_h, v_h)}{\|u_h\|_U \|v_h\|_V} \geq C_h, \tag{2.7}$$

then, there exists a unique solution $u_h \in U_h$ with

$$b(u_h, v_h) = \ell(v_h) \text{ for all } v_h \in V_h. \tag{2.8}$$

Moreover, suppose that $u \in U$ satisfies (2.3)*. Then, there holds quasi-optimality*

$$\|u - u_h\| \leq C_{opt} \inf_{w_h \in U_h} \|u - w_h\|_U, \tag{2.9}$$

where $C_{opt} := C_b / C_h$, and C_b is such that $|b(u, v)| \leq C_b \|u\|_U \|v\|_V$ for all $u \in U, v \in V$.

2.2.2 Optimal test functions

When $U = V$, one usually chooses $U_h = V_h$, but still the $\inf - \sup$ condition has to be proved. For Petrov-Galerkin methods, the right choice of V_h for a given subspace U_h is non-trivial. Also, we want that the quasi-optimality (2.9) holds independent of the discretization parameter h.

Now, suppose that one of statements in Theorem 2.4 holds true. Given a finite dimensional space $U_h \subseteq U$, we want to find a subspace $V_h \subseteq V$ such that the discrete inf-sup condition is satisfied automatically and $C_h \geq C > 0$.

Let U, V be Hilbert spaces with scalar products $(\cdot, \cdot)_U, (\cdot, \cdot)_V$ and induced norms $\|\cdot\|_U^2 := (\cdot, \cdot)_U, \|\cdot\|_V^2 := (\cdot, \cdot)_V$. We define the *trial-to-test operator* $\Theta : U \to V$ by

$$(\Theta u, v)_V = b(u, v) \text{ for all } u \in U, v \in V. \tag{2.10}$$

Let $R_V : V \to V'$ be the Riesz isomorphism and the operator $B : U \to V'$ associated to $b(\cdot, \cdot)$ as follows:

$$\langle Bu, v \rangle_{V' \times V} := b(u, v).$$

Lemma 2.6 ([9], Lemma 2.8)**.** *The trial-to-test operator satisfies $\Theta = R_V^{-1} B$ and is hence a well-defined bounded linear operator. In particular, it holds that*

$$\|\Theta\| \leq C_b.$$

Moreover, if B is invertible, then Θ is invertible.

For some finite dimensional subspace $U_h \subseteq U$, we define the optimal test space V_h^Θ as the image of U_h under Θ,

$$V_h^\Theta := \Theta(U_h). \tag{2.11}$$

V_h^Θ is finite dimensional and the computation $v_h = \Theta u_h$ still involves the infinite dimensional test space V.
With the optimal test functions at hand, we are able to write down an idealized Petrov Galerkin method, the so-called *ideal PG method*: Given $\ell \in V'$, find $u_h \in U_h$ such that

$$b(u_h, v_h) = \ell(v_h) \text{ for all } v_h \in V_h^\Theta. \tag{2.12}$$

Theorem 2.7 ([9], Theorem 2.9)**.** *Let $U_h \subseteq U$ be a finite dimensional subspace and let $V_h^\Theta = \Theta(U_h)$ be the optimal space test space. If one of the statements in Theorem 2.4 holds, then, the discrete inf-sup condition* (2.7) *is satisfied for $V_h = V_h^\Theta$ and there holds $C_h = C$. Moreover, let $u \in U$, denote the solution of* (2.3) *and let $u_h \in U_h$ be the unique solution of the ideal PG method* (2.12)*. Then, quasi-optimality*

$$\|u - u_h\|_U \leq C_{opt} \inf_{w_h \in U_h} \|u - w_h\|_U, \tag{2.13}$$

holds, where $C_{opt} = C_b/C$.

2.2.3 Ideal PG methods

We suppose that $B : U \to V'$ is invertible, so by Lemma 2.6 the trial-to-test operator Θ is invertible. Moreover, we suppose $U_h \subseteq U$ is a finite dimensional subspace.
We seek $u_h \in U_h$ such that

$$b(u_h, v_h) = \ell(v_h) \text{ for all } v_h \in V_h^\Theta.$$

Now, for $v_h \in V_h^\Theta$, there exists a unique $w_h \in U_h$ such that $v_h = \Theta(w_h)$, so that we can rewrite the equations above as

$$b(u_h, \Theta(w_h)) = \ell(\Theta(w_h)) \text{ for all } w_h \in U_h. \tag{2.14}$$

This is the *Galerkin formulation* of the ideal PG method. Define the bilinear form $a : U \times U \to \mathbb{R}$ by

$$a(u, w) := b(u, \Theta w) \text{ for all } u, w \in U.$$

Lemma 2.8 ([9], Lemma 2.10)**.** *The bilinear form $a : U \times U \to \mathbb{R}$ is bounded, symmetric and coercive, i.e.,*

$$\begin{aligned}
|a(u, w)| \leq& C_b^2 \|u\|_U \|w\|_U, \\
a(u, w) =& a(w, u), \\
C^2 \|u\|_U^2 \leq& a(u, u),
\end{aligned}$$

for all $u, w \in U$ where $C > 0$ denotes the $\inf - \sup$ *constant of $b(\cdot, \cdot)$.*

If we define $L := \ell(\Theta(\cdot)) \in U'$, then the solution $u \in U$ of (2.4) satisfies

$$a(u, w) = L(w) \text{ for all } w \in U, \tag{2.15}$$

and the ideal PG method (2.14) reads: Find $u_h \in U_h$ such that

$$a(u_h, w_h) = L(w_h) \text{ for all } w_h \in U_h. \tag{2.16}$$

Theorem 2.9 ([9], Theorem 2.11)**.** *Problems* (2.15), (2.16) *admit unique solutions* $u \in U$, $u_h \in U_h$, *and there holds optimality*

$$|||u - u_h||| = \inf_{w_h \in U_h} |||u - w_h|||, \tag{2.17}$$

and quasi-optimality (2.13)*, where* $|||\cdot||| := \|\Theta(\cdot)\|_V$ *is called the energy norm.*

Theorem 2.10 ([9], Theorem 2.13)**.** *The following statements are equivalent:*

1. $u_h \in U_h$ *is the unique solution of* (2.12)*.*
2. $u_h \in U_h$ *minimizes the residual in* V'*, i.e.,*

$$u_h = \arg \min_{w_h \in U_h} \|\ell - Bw_h\|_{V'}. \tag{2.18}$$

Lemma 2.11 ([9], Lemma 2.14)**.** *Let* $u_h \in U_h$ *be the solution of* (2.12)*. There holds*

$$\|\varepsilon_h\|_V = |||u - u_h||| \text{ and } \varepsilon_h \perp_V V_h^\Theta, \tag{2.19}$$

where $\varepsilon_h := \varepsilon(u_h) := R_V^{-1}(\ell - Bu_h) \in V$ *is the residual (representation) function.*

2.3 Trace operators and Breaking Sobolev spaces

Based on Carstensen et al. [5], we will work with infinite-dimensional (but mesh-dependent) spaces on an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. The mesh, denoted by Ω_h, is a disjoint partitioning of Ω into open elements K such that the union of their closures is the closure of Ω. The collection of element boundaries ∂K for all $K \in \Omega_h$, is denoted by $\mathcal{S}$. We assume that each element boundary ∂K is Lipschitz. For the most commonly occurring first order Sobolev spaces, namely $H^1(\Omega)$ and $H(\text{div}, \Omega)$, we define their broken versions as

$$\begin{aligned} H^1(\Omega_h) &= \{u \in L^2(\Omega) : u|_K \in H^1(K),\ K \in \Omega_h\}, \\ H(\text{div}, \Omega_h) &= \{\sigma \in (L^2(\Omega))^d : \sigma|_K \in H(\text{div}, K),\ K \in \Omega_h\}. \end{aligned}$$

These spaces are equipped with the norms

$$\|u\|^2_{H^1(\Omega_h)} := \sum_{K \in \Omega_h} \|u\|^2_{H^1(K)}, \quad \|\sigma\|^2_{H(\text{div},\Omega_h)} := \sum_{K \in \Omega_h} \|\sigma\|^2_{H(\text{div},K)}.$$

To recover the original Sobolev spaces from these broken spaces, we need traces. Let us consider these traces on each element K in Ω_h

$$
\begin{aligned}
tr^K_{\mathrm{grad}} u &= u|_{\partial K}, && u \in H^1(K),\\
tr^K_{\mathrm{div}} \sigma &= \sigma|_{\partial K} \cdot n_K, && \sigma \in H(\mathrm{div}, K).
\end{aligned}
$$

Here n_K denotes the unit outward normal on ∂K. These traces and n_K are well defined almost everywhere on ∂K, thanks to the assumption that ∂K is Lipschitz. The operators tr_{grad} and tr_{div} perform the above trace operation element by element on each of the broken spaces we defined previously, thus giving rise to linear maps

$$
tr_{\mathrm{grad}} : H^1(\Omega_h) \to \prod_{K \in \Omega_h} H^{1/2}(\partial K), \qquad tr_{\mathrm{div}} : H(\mathrm{div}, \Omega_h) \to \prod_{K \in \Omega_h} H^{-1/2}(\partial K).
$$

These maps are continuous and surjective. We define the spaces

$$
H^{1/2}(\mathcal{S}) = tr_{\mathrm{grad}} \left(H^1(\Omega) \right), \qquad H^{-1/2}(\mathcal{S}) = tr_{\mathrm{div}} \left(H(\mathrm{div}, \Omega) \right).
$$

If Ω_h consists of a single element, then $H^{1/2}(\mathcal{S})$ equals $H^{1/2}(\partial\Omega)$, but in general $tr_{\mathrm{grad}} \left(H^1(\Omega) \right) \subset tr_{\mathrm{grad}} \left(H^1(\Omega_h) \right)$ and analogously with $H^{-1/2}(\mathcal{S})$. We norm each of the above spaces by the quotient norms:

$$
\begin{aligned}
\|\widehat{u}\|_{H^{1/2}(\mathcal{S})} &= \inf_{u \in H^1(\Omega) \cap tr^{-1}_{\mathrm{grad}}\{\widehat{u}\}} \|u\|_{H^1(\Omega)},\\
\|\widehat{\sigma}\|_{H^{-1/2}(\mathcal{S})} &= \inf_{\sigma \in H(\mathrm{div}, \Omega) \cap tr^{-1}_{\mathrm{div}}\{\widehat{\sigma}\}} \|\sigma\|_{H(\mathrm{div}, \Omega)}.
\end{aligned}
$$

Also we define $H^{1/2}_0(\mathcal{S}) = tr_{\mathrm{grad}}(H^1_0(\Omega))$.
An important tool that we use is integration by parts, which can be extended to the Sobolev spaces as

$$
\langle tr^K_{\mathrm{div}} \tau, tr^K_{\mathrm{grad}} v \rangle_{\partial K} = (\mathrm{div}\, \tau, v)_K + (\tau, \nabla v)_K \text{ for all } \tau \in H(\mathrm{div}, K), v \in H^1(K). \tag{2.20}
$$

Then, for $u \in H^1(\Omega)$ with $tr_{\mathrm{grad}} u = \widehat{u}$ and $\sigma \in H(\mathrm{div}, \Omega)$ with $tr_{\mathrm{div}} \sigma = \widehat{\sigma}$, we observe that

$$
\langle \widehat{u}, \tau \cdot n \rangle_{\mathcal{S}} = \sum_{K \in \Omega_h} \langle tr^K_{\mathrm{div}} \tau, \widehat{u} \rangle_{\partial K} = \sum_{K \in \Omega_h} (\nabla u, \tau)_K + (u, \mathrm{div}\, \tau)_K, \text{ for all } \tau \in H(\mathrm{div}, \Omega_h), \tag{2.21}
$$

$$
\langle \widehat{\sigma}, v \rangle_{\mathcal{S}} = \sum_{K \in \Omega_h} \langle \widehat{\sigma}, tr^K_{\mathrm{grad}} v \rangle_{\partial K} = \sum_{K \in \Omega_h} (\sigma, \nabla v)_K + (\mathrm{div}\, \sigma, v)_K, \text{ for all } v \in H^1(\Omega_h). \tag{2.22}
$$

Lemma 2.12 ([5], Lemma 2.2). *The following identities hold for any $\widehat{\sigma}$ in $H^{-1/2}(\partial K)$ and any $\widehat{u}$ in $H^{1/2}(\partial K)$*

$$\|\widehat{\sigma}\|_{H^{-1/2}(\partial K)} = \sup_{0\neq u\in H^1(K)} \frac{|\langle \widehat{\sigma}, u\rangle_{\partial K}|}{\|u\|_{H^1(K)}} = \sup_{0\neq \widehat{u}\in H^{1/2}(\partial K)} \frac{|\langle \widehat{\sigma}, \widehat{u}\rangle_{\partial K}|}{\|\widehat{u}\|_{H^{1/2}(\partial K)}}, \tag{2.23a}$$

$$\|\widehat{u}\|_{H^{1/2}(\partial K)} = \sup_{0\neq \sigma\in H(\mathrm{div},K)} \frac{|\langle n\cdot\sigma, \widehat{u}\rangle_{\partial K}|}{\|\sigma\|_{H(\mathrm{div},K)}} = \sup_{0\neq \widehat{\sigma}\in H^{-1/2}(\partial K)} \frac{|\langle \widehat{\sigma}, \widehat{u}\rangle_{\partial K}|}{\|\widehat{\sigma}\|_{H^{-1/2}(\partial K)}}. \tag{2.23b}$$

Theorem 2.13 ([5], Theorem 2.3). *The following identities hold for any function $\widehat{\sigma}$ in $H^{-1/2}(\mathcal{S})$ and $\widehat{u}$ in $H^{1/2}(\mathcal{S})$,*

$$\|\widehat{\sigma}\|_{H^{-1/2}(\mathcal{S})} = \sup_{0\neq u\in H^1(\Omega_h)} \frac{|\langle \widehat{\sigma}, u\rangle_{\mathcal{S}}|}{\|u\|_{H^1(\Omega_h)}}, \tag{2.24a}$$

$$\|\widehat{u}\|_{H^{1/2}(\mathcal{S})} = \sup_{0\neq \sigma\in H(\mathrm{div},\Omega_h)} \frac{|\langle n\cdot\sigma, \widehat{u}\rangle_{\mathcal{S}}|}{\|\sigma\|_{H(\mathrm{div},\Omega_h)}}. \tag{2.24b}$$

For any $v \in H^1(\Omega_h)$ and $\tau \in H(\mathrm{div},\Omega_h)$,

$$v \in H^1_0(\Omega) \Leftrightarrow \langle \widehat{\sigma}, v\rangle_{\mathcal{S}} = 0 \qquad \forall \widehat{\sigma} \in H^{-1/2}(\mathcal{S}), \tag{2.25a}$$

$$\tau \in H_0(\mathrm{div},\Omega) \Leftrightarrow \langle \tau\cdot n, \widehat{u}\rangle_{\mathcal{S}} = 0 \qquad \forall \widehat{u} \in H^{1/2}(\mathcal{S}). \tag{2.25b}$$

2.4 Breaking variational forms

In this section we will recall some results from [5], which will be used in the analysis of the next chapter.
Let U_0 and V denote two Hilbert spaces and let V_0 be a closed subspace of V. We consider the bilinear form $b_0 : U_0 \times V \to \mathbb{R}$ satisfying the following assumption.

Assumption 2.1. *There is a positive constant c_0 such that*

$$\sup_{v\in V_0} \frac{|b_0(u,v)|}{\|v\|_V} \geq c_0\|u\|_{U_0}$$

It is a well-known result of Babuška and Nečas ([2], [16]) that Assumption 2.1 together with triviality of

$$Z_0 = \{v \in V_0 : b_0(u,v) = 0, \forall u \in U_0\} \tag{2.26}$$

guarantees wellposedness of the following variational problem: Given $\ell \in V_0'$, find $u \in U_0$ satisfying

$$b_0(u,v) = \ell(v) \quad \forall v \in V_0. \tag{2.27}$$

To describe a "broken" version of (2.27), we need another Hilbert space $\widehat{U}$, together with a continuous bilinear form $\widehat{b} : \widehat{U} \times V \to \mathbb{R}$. Define

$$b((u, \widehat{u}), v) = b_0(u, v) + \widehat{b}(\widehat{u}, v).$$

Clearly $b : U \times V \to \mathbb{R}$ is continuous, where $U = U_0 \times \widehat{U}$ is a Hilbert space under the Cartesian product norm. Now consider the following new broken variational formulation: Given $\ell \in V'$, find $u \in U_0$ and $\widehat{u} \in \widehat{U}$ satisfying

$$b((u, \widehat{u}), v) = \ell(v), \quad \forall v \in V. \tag{2.28}$$

Problems (2.28) and (2.26) are related under the following assumption.

Assumption 2.2. *The spaces V_0, V and $\widehat{U}$ satisfy*

$$V_0 = \{v \in V : \widehat{b}(\widehat{u}, v) = 0, \text{ for all } \widehat{u} \in \widehat{U}\} \tag{2.29}$$

and there is a positive constant $\widehat{c}$ such that

$$\widehat{c} \cdot \|\widehat{u}\|_{\widehat{U}} \leq \sup_{0 \neq v \in V} \frac{|\widehat{b}(\widehat{u}, v)|}{\|v\|_V} \quad \forall \widehat{u} \in \widehat{U} \tag{2.30}$$

Under this assumption, there is a simple result which shows that the broken form (2.28) inherits stability from the original unbroken form (2.26).

Theorem 2.14 ([5], Theorem 3.3)**.** *Assumptions (2.1) and (2.2) imply*

$$c_1 \|(u, \widehat{u})\|_U \leq \sup_{0 \neq v \in V} \frac{|b((u, \widehat{u}), v)|}{\|v\|_V}, \text{ for all } (u, \widehat{u}) \in U,$$

where c_1 is defined by

$$\frac{1}{c_1^2} = \frac{1}{c_0^2} + \frac{1}{\widehat{c}^2} \left(\frac{\|b_0\|}{c_0} + 1 \right)^2,$$

and $\|b_0\|$ denotes the smallest number with $|b_0(u, v)| \leq \|b_0\| \|u\|_{U_0} \|v\|_V$ holds for all $u \in U_0$ and all $v \in V$. Moreover, if

$$Z = \{v \in V : b((u, \widehat{u}), v) = 0 \text{ for all } u \in U_0 \text{ and } \widehat{u} \in \widehat{U}\},$$

then $Z = Z_0$. Consequently, if $Z_0 = \{0\}$, then (2.28) is uniquely solvable and the solution component u from (2.28) coincides with the solution of (2.27).

2.5 Triangulation and discrete subspaces

Recall that we are working on an open bounded domain $\Omega \subset \mathbb{R}^d$ and a mesh Ω_h. We will work with meshes consisting of non-degenerated d-simplices, like triangles in the case of $d = 2$. The mesh $\Omega_h = \{K_1, K_2, \ldots, K_M\}$ is a regular triangulation of Ω, if

- Each element $K \in \Omega_h$ is a $d-$simplex, i.e., there exists $d+1$ different nodes $z_1, \ldots, z_{d+1} \in \overline{\Omega}$, which do not lay in a $(d-1)-$dimensional hypersurface with

  $$\overline{K} = \operatorname{conv}\{z_1, \ldots, z_{d+1}\}.$$

  The points in the set $\mathcal{N}(K) := \{z_1, \ldots, z_{d+1}\}$ are called nodes of the element K.

- For two different elements $K, K' \in \Omega_h$ it holds that

  $$\overline{K} \cap \overline{K'} = \operatorname{conv}\{\mathcal{N}(K) \cap \mathcal{N}(K')\}.$$

We define the nodes $\mathcal{N}$ of Ω_h by

$$\mathcal{N}(\Omega_h) := \bigcup_{K \in \Omega_h} \mathcal{N}(K)$$

Moreover, let $h_{\Omega_h} \in L^2(\Omega_h)$ denote the local mesh-size function

$$h_{\Omega_h}|_K := h_K := \operatorname{diam}(K) \text{ for } K \in \Omega_h,$$

and set $h := \max_{K \in \Omega_h} h_{\Omega_h}$. Shape-regularity of an element $K \in \Omega_h$ is defined by

$$\kappa(K) := \frac{\operatorname{diam}(K)^d}{|K|}$$

and $\kappa_{\Omega_h} := \max_{K \in \Omega_h} \kappa(K)$ is the shape-regularity of Ω_h. Also, let $\mathcal{E}$ denote the collection of all facets of the triangulation Ω_h, i.e., $E \in \mathcal{E}$ is given by

$$\overline{E} = \operatorname{conv}\{z_1, z_2, \ldots, z_d\}$$

for different nodes $z_1, \ldots, z_d \in \mathcal{N}(K)$ for some $K \in \Omega_h$.

For $q \in \mathbb{N}_0$ the space of elementwise polynomials is defined by

$$\mathcal{P}^q(\Omega_h) := \{v \in L^2(\Omega) : v|_K \text{ is polynomial of degree } \leq q, \text{ for all } K \in \Omega_h\}.$$

We approximate the solution component $u \in L^2(\Omega)$ by some $u_h \in \mathcal{P}^q(\Omega_h) \subseteq L^2(\Omega)$ and $\sigma \in L^2(\Omega)^d$ by some $\sigma_h \in (\mathcal{P}^q(K))^d \subseteq L^2(\Omega)^d$. To define approximation spaces for the numerical trace $\widehat{u} \in H_0^{1/2}(S)$ we proceed as follows: For $q \geq 1$, define $\mathcal{P}^q_{c,0}(\Omega_h) := \mathcal{P}^q(\Omega_h) \cap H_0^1(\Omega)$ and

$$\mathcal{P}^q_{c,0}(\mathcal{S}) := tr_{\text{grad}}(\mathcal{P}^q_{c,0}(\Omega_h)) \subseteq H_0^{1/2}(\mathcal{S}).$$

Finally, to define approximation of the numerical flux $\widehat{\sigma} \in H^{-1/2}(\mathcal{S})$ we use the Raviart-Thomas space: Let $\mathcal{P}^q_m(K)$ denote the space of homogeneous polynomials of degree q, and let $\boldsymbol{x} = (x, y)^T$ for $d = 2$ and let $\boldsymbol{x} = (x, y, z)^T$ for $d = 3$. The Raviart-Thomas element for $q \geq 0$ is defined by

$$\mathcal{RT}^q(K) := \mathcal{P}^q(K)^d + \boldsymbol{x}\mathcal{P}^q_m(K)$$

and the global Raviart-Thomas space is given by

$$\mathcal{RT}^q(\Omega_h) := \{\tau \in H(\mathrm{div}, \Omega) : \tau|_K \in \mathcal{RT}^q(K), \text{ for all } K \in \Omega_h\} \subseteq H(\mathrm{div}, \Omega).$$

Then,we define the approximation space

$$\mathcal{P}^q(\mathcal{S}) := tr_{\mathrm{div}}(\mathcal{RT}^q(\Omega_h)) \subseteq H^{-1/2}(\mathcal{S}).$$

For $q \in \mathbb{N}_0$, let $\Pi^q : L^2(\Omega) \to \mathcal{P}^q(\Omega_h)$ denote the L^2-orthogonal projection. It is known (see [10]) that

$$\min_{u_h \in \mathcal{P}^q(\Omega_h)} \|u - u_h\|_{L^2(\Omega)} = \|u - \Pi^q u\|_{L^2(\Omega)} \leq C_q h^s |u|_{H^s(\Omega)} \text{ for } s \in (0, q+1]. \tag{2.31}$$

Here $|\cdot|_{H^s(\Omega)}$ denotes the seminorm in $H^s(\Omega)$. Further, let $\Pi^{q+1}_{\mathrm{grad}} : H^{1+s}(\Omega) \to \mathcal{P}^{q+1}_c(\Omega_h) \cap H^1_0(\Omega)$, where $\mathcal{P}^{q+1}_c(\Omega_h) = \mathcal{P}^{q+1}(\Omega_h) \cap C(\overline{\Omega})$, denote an interpolation operator, such that

$$\|u - \Pi^{q+1}_{\mathrm{grad}} u\|_{H^1(\Omega)} \leq C_q h^s |u|_{H^{1+s}(\Omega)} \text{ for } s \in (1/2, q+1]. \tag{2.32}$$

Let $\Pi^q_{\mathcal{RT}} : H^1(\Omega)^2 \to \mathcal{RT}^q(\Omega_h)$ denote the Raviart-Thomas projection. It holds that

$$\|\sigma - \Pi^q_{\mathcal{RT}}\sigma\|_{L^2(\Omega)} \leq C_q h^s |\sigma|_{H^s(\Omega)} \text{ for } s \in (1/2, q+1]. \tag{2.33}$$

The constant $C_q = C_q(s) > 0$ in (2.31), (2.32) and (2.33) depends on shape-regularity of the mesh Ω_h and the fixed polynomial degree $q \in \mathbb{N}_0$ and s, but not on h.

Theorem 2.15 ([10], Theorem 5)**.** *Let $q \in \mathbb{N}_0$. Let $\sigma \in H^s(\Omega)^2 \cap H(\mathrm{div}, \Omega)$ for some $s \in [1, q+1]$. Then*

$$\|tr_{\mathrm{div}}(1 - \Pi^q_{\mathcal{RT}})\sigma\|_{H^{-1/2}(\mathcal{S})} \leq C \cdot h^s \|\sigma\|_{H^s(\Omega)} \tag{2.34}$$

The constant $C > 0$ only depends on the constant C_p.

2.6 Notational convention

We will write $A \lesssim B$ resp. $B \lesssim A$ if there exists a constant $C > 0$ that is independent of the maximal mesh-size h such that $A \lesssim CB$ resp. $B \lesssim CA$. Moreover, $A \cong B$ means that $A \lesssim B$ and $B \lesssim A$.

For $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^d$ recall the following operators:

$$\nabla f = \nabla \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_d \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial x_1} & \frac{\partial f_d}{\partial x_2} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix},$$

$$\Delta f = \begin{bmatrix} \displaystyle\sum_{k=1}^{d} \frac{\partial^2 f_1}{\partial x_k^2} \\ \vdots \\ \displaystyle\sum_{k=1}^{d} \frac{\partial^2 f_1}{\partial x_k^2} \end{bmatrix}.$$

Finally we will need some particular spaces

$$(L^2(\Omega))_*^{d\times d} = \{G \in (L^2(\Omega))^{d\times d} : \int_\Omega \mathrm{Tr}(G) = 0\}, \quad V = (H_0^1(\Omega))^d \times (H_\star(\mathrm{div};\Omega))^d,$$

with $(H_\star(\mathrm{div};\Omega))^d = \{M \in (H(\mathrm{div};\Omega))^d : \int_\Omega \mathrm{Tr}(M) = 0\}$. Here $\mathrm{Tr}(\cdot)$ means the trace of a matrix. This notation will be used in some proofs.

Chapter 3

The DPG method and the Stokes problem

3.1 The Stokes problem

Let Ω be a bounded and connected Lipschitz domain ($\Omega \subset \mathbb{R}^d$ with $d = 2, 3$) with boundary $\partial\Omega$. The Stokes flow problem consists of finding a vector valued function u (velocity), and a scalar function p (pressure) satisfying:

$$-\mu\Delta u + \nabla p = f \text{ in } \Omega, \tag{3.1a}$$

$$\operatorname{div}(u) = 0 \text{ in } \Omega, \tag{3.1b}$$

$$u|_{\partial\Omega} = u_0. \tag{3.1c}$$

Here, the momentum balance is expressed in (3.1a), the mass balance of the flow is in (3.1b), and (3.1c) are the Dirichlet boundary conditions. The force acting on the fluid is guided by f and the viscosity of the medium is $\mu > 0$, but for a simpler representation we set $\mu = 1$. The pressure solution of problem (3.1) is unique up to a constant. A canonical choice to fix the constant is to consider p with $\int_\Omega p = 0$.

Let I denote the identity matrix. We reformulate the Stokes problem by introducing the pseudostress M as new variable

$$M = \nabla u - p \cdot I.$$

Taking the trace of this matrix we get $\operatorname{Tr}(M) = \underbrace{\operatorname{div}(u)}_{=0} - d \cdot p$. Thus, we can eliminate the pressure in (3.1) and reformulate the Stokes problem as the following first order system:

$$-\operatorname{div}(M) = f, \tag{3.2a}$$

$$M - \frac{1}{d}\operatorname{Tr}(M) \cdot I - \nabla u = 0, \tag{3.2b}$$

$$u|_{\partial\Omega} = u_0. \tag{3.2c}$$

Note that equations (3.1b) and (3.1c) are now implicitly contained in (3.2b).

Theorem 3.1. *Let $f \in (L^2(\Omega))^d$, $F \in (L^2(\Omega))_*^{d\times d}$. Then, the problem*

$$-\operatorname{div}(M) = f, \tag{3.3a}$$
$$M - \frac{1}{d}\operatorname{Tr}(M)\cdot I - \nabla u = F, \tag{3.3b}$$
$$u|_{\partial\Omega} = 0, \tag{3.3c}$$

admits a unique solution $(u, M) \in V$ and the solution satisfies

$$\|u\|_{H^1(\Omega)} + \|M\|_{H(\mathrm{div};\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\right). \tag{3.4}$$

The constant $C > 0$ only depends on Ω.

For the demonstration of Theorem 3.1 we use three main steps. First we build an auxiliary mathematical system by introducing a scalar value $p = -\frac{1}{d}\operatorname{Tr}(M)$ which allows us to rewrite the original system in a variational form that has a unique solution for (u, p). Second, with this solution we can then find a solution for the initial variable M and prove that it satisfies the conditions of system (3.3). Third we prove that (3.3) admits a unique solution.

Proof. Note that if we take the trace in (3.3b) we obtain that

$$\operatorname{Tr}\left(M - \frac{1}{d}\operatorname{Tr}(M)\cdot I - \nabla u\right) = \operatorname{Tr}(F) \quad \Leftrightarrow \quad \operatorname{div}(u) = -\operatorname{Tr}(F).$$

Replacing (3.3b) in (3.3a) we obtain the following system

$$\begin{aligned} -\Delta u - \frac{1}{d}\nabla\operatorname{Tr}(M) &= f + \operatorname{div}(F), \\ \operatorname{div}(u) &= -\operatorname{Tr}(F). \end{aligned}$$

Naming $G = f + \operatorname{div}(F) \in (H^{-1}(\Omega))^d$ and $p = -\frac{1}{d}\operatorname{Tr}(M)$, this system written in its variational formulation reads as

$$(\nabla u, \nabla v) - (p, \operatorname{div}(v)) = (G, v), \tag{3.5a}$$
$$(\operatorname{div}(u), q) = -(\operatorname{Tr}(F), q) \tag{3.5b}$$

for all $v \in (H_0^1(\Omega))^d$ and $q \in L^2(\Omega)/\mathbb{R}$. Then, by mixed system theory for the Stokes problem (see [3], Theorem 8.2.1) there exists a unique solution $(u, p) \in (H_0^1(\Omega))^d \times L^2(\Omega)/\mathbb{R}$, and this solution satisfies

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C(\|G\|_{H^{-1}(\Omega)} + \|\operatorname{Tr}(F)\|_{L^2(\Omega)}).$$

Also, we have that

$$
\begin{aligned}
\|G\|_{H^{-1}(\Omega)} &= \sup_{0\neq v\in H_0^1(\Omega)} \frac{(f+\operatorname{div}(F),v)}{\|v\|_{H^1(\Omega)}} = \sup_{0\neq v\in H_0^1(\Omega)} \frac{(f,v)-(F,\nabla v)}{\|v\|_{H^1(\Omega)}} \\
&\leq \sup_{0\neq v\in H_0^1(\Omega)} \frac{\|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} \\
&\leq \|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)},
\end{aligned}
$$

For the trace of F note that

$$
\begin{aligned}
\|F\|^2_{L^2(\Omega)} &= \|F - \frac{1}{d}\operatorname{Tr}(F)\cdot I + \frac{1}{d}\operatorname{Tr}(F)\cdot I\|^2_{L^2(\Omega)} = \int_\Omega |(F - \frac{1}{d}\operatorname{Tr}(F)\cdot I) + \frac{1}{d}\operatorname{Tr}(F)\cdot I|^2 \\
&= \int_\Omega |(F - \frac{1}{d}\operatorname{Tr}(F)\cdot I)|^2 + 2\langle (F - \frac{1}{d}\operatorname{Tr}(F)\cdot I), \frac{1}{d}\operatorname{Tr}(F)\cdot I\rangle_{\boldsymbol{F}} + |\frac{1}{d}\operatorname{Tr}(F)\cdot I|^2 \\
&= \int_\Omega |(F - \frac{1}{d}\operatorname{Tr}(F)\cdot I)|^2 + |\frac{1}{d}\operatorname{Tr}(F)\cdot I|^2 \\
&= \|F - \frac{1}{d}\operatorname{Tr}(F)\cdot I\|^2_{L^2(\Omega)} + \|\frac{1}{d}\operatorname{Tr}(F)\cdot I\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Here $\langle\cdot,\cdot\rangle_{\boldsymbol{F}}$ is the Frobenius inner product, $|A|^2 = \langle A, A\rangle_{\boldsymbol{F}}$ and

$$
\langle (F - \frac{1}{d}\operatorname{Tr}(F)\cdot I), \frac{1}{d}\operatorname{Tr}(F)\cdot I\rangle_{\boldsymbol{F}} = \frac{1}{d}\operatorname{Tr}(F)^2 - \frac{1}{d^2} d\operatorname{Tr}(F)^2 = 0,
$$

and we get that $\|\operatorname{Tr}(F)\|_{L^2(\Omega)} \leq d\|F\|_{L^2(\Omega)}$. Consequently, there exists $C > 0$ such that

$$
\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}).
$$

Now, using the solution (u,p) of (3.5), we define $M = \nabla u - pI + F$ to find the solution for M. Taking the trace of M we see that $\operatorname{Tr}(M) = -d\cdot p$ (observe that $\operatorname{div}(u) = -\operatorname{Tr}(F)$). This means that M satisfies (3.3b). It still remains to show that M satisfies (3.3a), because we don't know if $\operatorname{div}(M) \in L^2(\Omega)$. For $\phi \in (C_0^\infty(\Omega))^d \subseteq (H_0^1(\Omega))^d$ we have that

$$
\begin{aligned}
\langle \overbrace{\operatorname{div}(M)}^{\text{Distribution}}, \phi\rangle = -\langle M, \nabla\phi\rangle &= -(M, \nabla\phi) \\
&= -(\nabla u - p\cdot I + F, \nabla\phi) \\
&= -(\nabla u, \nabla\phi) + (p, \operatorname{div}(\phi)) - (F, \nabla\phi) \\
&= -(f, \phi)
\end{aligned}
$$

which proves that $\operatorname{div}(M) = -f \in (L^2(\Omega))^d$. Thus, we have shown that $f \in (L^2(\Omega))^d$ and for $F \in (L^2(\Omega))^{d\times d}_*$ exists $(u, M) \in V$ that satisfies (3.3) and (3.4). Note that for $f = 0$ and $F = 0$ we get that

$$\begin{aligned} -\operatorname{div}(M) &= 0, \\ M - \frac{1}{d}\operatorname{Tr}(M)\cdot I - \nabla u &= 0, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

Since $\int_\Omega \operatorname{Tr}(M) = 0$ and since the Stokes problem admits a unique solution we conclude that $u = 0$ and $\operatorname{Tr}(M) = 0$. From (3.3b) we get that $M = 0$ and thus we conclude that $(u, M) = 0$. Overall, this shows that (3.3) admits a unique solution which satisfies (3.4). $\square$

We recast problem (3.3) into an ultra-weak variational formulation. Using test functions $(v, N) \in V$ and integrating by parts shows that

$$\begin{aligned} (M, \nabla v) &= (f, v) \\ (M, N) + (M, -\frac{1}{d}\operatorname{Tr}(N)\cdot I) + (u, \operatorname{div}(N)) &= (F, N), \text{ for all } (v, N) \in V. \end{aligned}$$

For $(u, M) \in W = (L^2(\Omega))^d \times (L^2(\Omega))^{d\times d}_*$, $(v, N) \in V$ we define the bilinear form $b : W \times V \to \mathbb{R}$ by

$$b((u, M); (v, N)) = (M, N - \frac{1}{d}\operatorname{Tr}(N)\cdot I + \nabla v) + (u, \operatorname{div}(N)). \tag{3.6}$$

Lemma 3.2. *$b : W \times V \to \mathbb{R}$ is bounded.*

Proof. Note that

$$\begin{aligned} (M, N)^2 &\le \|M\|^2_{L^2(\Omega)}\|N\|^2_{L^2(\Omega)} \\ &\le (\|M\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})(\|N\|^2_{L^2(\Omega)} + \|\operatorname{div}(N)\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}) \\ &= \|(u, M)\|^2_W(\|N\|^2_{H(\operatorname{div},\Omega)} + \|v\|^2_{H^1(\Omega)}) = \|(u, M)\|^2_W\|(v, N)\|^2_V, \\ (M, \operatorname{Tr}(N)\cdot I) &\le \|M\|_{L^2(\Omega)}\|\operatorname{Tr}(N)\cdot I\|_{L^2(\Omega)} \le d\|M\|_{L^2(\Omega)}\|N\|_{L^2(\Omega)} \le d\|(u, M)\|_W\|(v, N)\|_V, \\ (M, \nabla v) &\le \|M\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \le \|(u, M)\|_W\|(v, N)\|_V, \\ (u, \operatorname{div}(N)) &\le \|u\|_{L^2(\Omega)}\|\operatorname{div}(N)\|_{L^2(\Omega)} \le \|(u, M)\|_W\|(v, N)\|_V. \end{aligned}$$

Then, $b(\cdot, \cdot)$ is bounded.

$\square$

Lemma 3.3. *There are constants $c_1, c_2 > 0$ such that:*

$$\inf_{0\ne(u,M)\in W} \sup_{0\ne(v,N)\in V} \frac{b((u, M); (v, N))}{\|(v, N)\|_V\|(u, M)\|_W} \ge c_1, \tag{3.7a}$$

$$\inf_{0\ne(v,N)\in V} \sup_{0\ne(u,M)\in W} \frac{b((u, M); (v, N))}{\|(v, N)\|_V\|(u, M)\|_W} \ge c_2. \tag{3.7b}$$

Proof. For the first condition let us note that it suffices to demonstrate that for all $(u, M) \in W$

$$\sup_{0 \neq (v,N) \in V} \frac{b((u, M), (v, N))}{\|(v, N)\|_V} \geq \|(u, M)\|_W.$$

For this, let us consider the following system:

$$\begin{aligned} \operatorname{div}(N^*) &= u, \\ N^* - \frac{1}{d}\operatorname{Tr}(N^*) + \nabla v^* &= M, \\ v^*|_{\partial\Omega} &= 0. \end{aligned}$$

Note that this system is the adjoint problem of Theorem 3.1. By considering $N^* \to -N^*$ and $v^* \to -v^*$ Theorem 3.1 proves that there exists a unique solution which satisfies:

$$\|(v^*, N^*)\|_V \lesssim \|u\|_{L^2(\Omega)} + \|M\|_{L^2(\Omega)}.$$

Then, we have that:

$$\begin{aligned} \sup_{0 \neq (v,N) \in V} \frac{b((u, M); (v, N))}{\|(u, N)\|_V} &\geq \frac{b((u, M); (v^*, N^*))}{\|(v^*, N^*)\|_V} \\ &= \frac{\|u\|^2_{L^2(\Omega)} + \|M\|^2_{L^2(\Omega)}}{\|(v^*, N^*)\|_V} \\ &\gtrsim \frac{\|u\|^2_{L^2(\Omega)} + \|M\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)} + \|M\|_{L^2(\Omega)}} \\ &\cong \|(u, M)\|_W. \end{aligned}$$

For the second condition it suffices to see that (see Theorem 2.4)

$$\{b((u, M); (v, N)) = 0, \forall (u, M) \in W\} \Rightarrow (v, N) = 0.$$

Let us notice that if

$$b((u, M); (v, N)) = (u, \operatorname{div}(N)) + (M, N - \frac{1}{d}\operatorname{Tr}(N) + \nabla v) = 0,$$

for all $(u, M) \in W$ then, we can choose $u = \operatorname{div}(N) \in L^2(\Omega)^d$ y $M = N - \frac{1}{d}\operatorname{Tr}(N) \cdot I + \nabla v \in (L^2(\Omega))^{d \times d}_*$ and with this we have that

$$\|\operatorname{div}(N)\|^2_{L^2(\Omega)} + \|N - \frac{1}{d}\operatorname{Tr}(N) \cdot I + \nabla v\|^2_{L^2(\Omega)} = 0.$$

It follows that

$$
\begin{aligned}
\operatorname{div}(N) &= 0, \\
N - \frac{1}{d}\operatorname{Tr}(N) \cdot I + \nabla v &= 0,
\end{aligned}
$$

with $(v, N) \in V$. Again, Theorem 3.1 proves that this system has the unique solution $(v, N) = 0$, which finishes the proof.

□

3.2 Ultra weak formulation

We consider the variational formulation of (3.3), but using $v \in (H^1(\Omega_h))^d$ and $N \in (H(\operatorname{div}, \Omega_h))^d$ as test functions and integrating over $K \in \Omega_h$ gives

$$
\begin{aligned}
-\int_K \operatorname{div}(M) \cdot v &= \int_K f \cdot v, \\
\int_K M : N - \frac{1}{d}\int_K \operatorname{Tr}(M) I : N - \int_K \nabla u : N &= \int_K F : N.
\end{aligned}
$$

Integrating by parts that we get

$$
\begin{aligned}
\int_K M : \nabla v - \int_{\partial K} M \cdot n_K v &= \int_K f \cdot v, \\
\int_K M : N - \frac{1}{d}\int_K M : \operatorname{Tr}(N) I + \int_K u \cdot \operatorname{div}(N) - \int_{\partial K} u N \cdot n_K &= \int_K F : N.
\end{aligned}
$$

Summing over all $K \in \Omega_h$ we end up with

$$
\begin{aligned}
\sum_{K \in \Omega_h} \left(\int_K M : \nabla v - \int_{\partial K} M \cdot n_K v \right) &= \sum_{K \in \Omega_h} \int_K f \cdot v, \\
\sum_{K \in \Omega_h} \left(\int_K M : N - \frac{1}{d}\int_K M : \operatorname{Tr}(N) I + \int_K u \cdot \operatorname{div}(N) - \int_{\partial K} u N \cdot n_K \right) &= \sum_{K \in \Omega_h} \int_K F : N.
\end{aligned}
$$

Now, for $u \in (L^2(\Omega))^d$, $M \in (L^2(\Omega))^{d \times d}$, $v \in (H^1(\Omega_h))^d$ and $N \in (H(\operatorname{div}, \Omega_h))^d$, let us define

$$
\begin{aligned}
(u, \operatorname{div}_{\Omega_h}(N)) &= \sum_{K \in \Omega} (u, \operatorname{div}(N))_K, \\
(M, \nabla_{\Omega_h} v) &= \sum_{K \in \Omega} (M, \nabla v)_K.
\end{aligned}
$$

With the skeleton $\mathcal{S} = \{\partial K | K \in \Omega_h\}$ and traces $(\widehat{u}, \widehat{M}) \in (H^{1/2}(\mathcal{S}))^d \times (H^{-1/2}(\mathcal{S}))^d$ we recall that formally

$$\begin{aligned}
\langle \widehat{M}, v \rangle_{\mathcal{S}} &= \sum_{K \in \Omega} \int_{\partial K} M \cdot n_K v, \\
\langle \widehat{u}, N \cdot n \rangle_{\mathcal{S}} &= \sum_{K \in \Omega} \int_{\partial K} u N \cdot n_K.
\end{aligned}$$

We refer to Section 2.3 for more details. From this we get the following system: Find $(u, M, \widehat{u}, \widehat{M}) \in (L^2(\Omega))^d \times (L^2(\Omega))^{d\times d}_* \times (H^{1/2}_0(\mathcal{S}))^d \times (H^{-1/2}(\mathcal{S}))^d$ such that

$$(M, \nabla_{\Omega_h} v) - \langle \widehat{M}, v \rangle_S = (f, v), \tag{3.8a}$$

$$(M, N) - \frac{1}{d}(M, \mathrm{Tr}(N) I) + (u, \mathrm{div}_{\Omega_h}(N)) - \langle \widehat{u}, N \cdot n \rangle_S = (F, N), \tag{3.8b}$$

for all $(v, N) \in (H^1(\Omega_h))^d \times (H(\mathrm{div}, \Omega_h))^d$. Now, let us define $U_0 = (L^2(\Omega))^d \times (L^2(\Omega))^{d\times d}_*$, $\widehat{U} = (H^{1/2}_0(\mathcal{S}))^d \times (H^{-1/2}(\mathcal{S}))^d$, $V = (H^1(\Omega_h))^d \times (H(\mathrm{div}; \Omega_h))^d$ and $V_0 = (H^1_0(\Omega))^d \times (H_\star(\mathrm{div}; \Omega))^d \subseteq V$, and the bilinears forms $b_0 : U_0 \times V \to \mathbb{R}$ and $\widehat{b} : \widehat{U} \times V \to \mathbb{R}$ as follows

$$\begin{aligned}
b_0(u, M; v, N) &= (M, N - \frac{1}{d}\mathrm{Tr}(N) I + \nabla_{\Omega_h} v) + (u, \mathrm{div}_{\Omega_h}(N)), \\
\widehat{b}(\widehat{u}, \widehat{M}; v, N) &= \langle \widehat{M}, v \rangle_S + \langle \widehat{u}, N \cdot n \rangle_S,
\end{aligned}$$

and from here we define $b : U \times V \to \mathbb{R}$, where $U = U_0 \times \widehat{U}$ as

$$b(u, M, \widehat{u}, \widehat{M}; v, N) = b_0(u, M; v, N) - \widehat{b}(\widehat{u}, \widehat{M}; v, N).$$

For $(f, F) \in (L^2(\Omega))^d \times (L^2(\Omega))^{d\times d}_*$ we define $\ell : V \to \mathbb{R}$ by

$$\ell(v, N) = (f, v) + (F, N), \quad \forall (v, N) \in V.$$

Finally, the variational formulation (3.8) reads: Find $(u, M, \widehat{u}, \widehat{M}) \in U$ such that

$$b(u, M, \widehat{u}, \widehat{M}) = \ell(v, N), \quad \forall (v, N) \in V, \tag{3.9}$$

and our main result is

Theorem 3.4. *The bilinear form $b(\cdot, \cdot)$ is bounded and satisfies the inf-sup conditions. In particular, problem* (3.9) *admits a unique solution which satisfies*

$$\|(u, M, \widehat{u}, \widehat{M})\|_U \le C \|\ell\|_{V'}$$

Proof. Note that

$$b_0(u, M; v, N) = \sum_{K \in \Omega} (M, N - \frac{1}{d} \operatorname{Tr}(N) I + \nabla v)_K + (u, \operatorname{div}(N))_K,$$

and

$$\widehat{b}(\widehat{u}, \widehat{M}; v, N) = \sum_{K \in \Omega} \langle \widehat{M}, v \rangle_{\partial K} + \langle \widehat{u}, N \rangle_{\partial K}.$$

Then, using the same arguments from Lemma 3.2 we get that

$$\begin{aligned} |b_0(u, M; v, N)| &\leq \sum_{K \in \Omega} |(M, N)_K| + \frac{1}{d} |(M, \operatorname{Tr}(N) \cdot I)_K| + |(M, \nabla v)_K| + |(u, \operatorname{div}(N))_K| \\ &\leq \sum_{K \in \Omega} 4 \|(u, M)\|_{(L^2(K))^d \times (L^2(K))^{d \times d}} \|(v, N)\|_{H^1(K) \times H(\operatorname{div}; K)} \\ &= 4 \|(u, M)\|_{U_0} \|(v, N)\|_V. \end{aligned}$$

Boundedness of $\widehat{b}(\cdot, \cdot)$ follows from the boundednes of the trace operators defined in Section 2.3. With this we get that $b(\cdot, \cdot)$ is bounded. Now, note that by Lemma 3.3 we already get that

$$\sup_{(v,N) \in V_0} \frac{|b_0(u, M; v, N)|}{\|(v, N)\|_V} \geq \|(u, M)\|$$

and $\{(v, N) \in V : b(u, M, \widehat{u}, \widehat{M}; v, N) = 0 \text{ for all } (u, M, \widehat{u}, \widehat{M}) \in U\} = \{0\}$, see (3.7b). Also, by Theorem 2.13 we get that

$$\begin{aligned} V_0 &= \{(v, N) \in V : \widehat{b}(\widehat{u}, \widehat{M}; v, N) = 0 \text{ for all } (\widehat{u}, \widehat{M}) \in \widehat{U}\}, \\ \sup_{(v,N) \in V} &\frac{|\widehat{b}(\widehat{u}, \widehat{M}; v, N)|}{\|(v, N)\|_V} = \|(\widehat{u}, \widehat{M})\|_{\widehat{U}}. \end{aligned}$$

Then, by Theorem 2.14, (3.9) is uniquely solvable. □

Chapter 4

Practical DPG method

4.1 Fortin Operator

For the definition of the practical DPG method, we replace V by some finite dimensional subspace $V_k \subseteq V$ and define the discrete trial-to-test operator $\Theta_k : U \to V_k$ by

$$(\Theta_k \boldsymbol{u}, \boldsymbol{v}_k)_V = b(\boldsymbol{u}, \boldsymbol{v}_k) \text{ for all } \boldsymbol{v}_k \in V_k, \boldsymbol{u} \in U. \tag{4.1}$$

Let $U_h \subseteq U$ be some finite dimensional subspace and define the discrete optimal test space:

$$V_{h,k}^{\Theta} := \Theta_k(U_h)$$

Then, the practical DPG method reads: Find $\boldsymbol{u}_h \in U_h$ such that

$$b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h) \text{ for all } \boldsymbol{v}_h \in V_{h,k}^{\Theta}. \tag{4.2}$$

This problem is well-posed if we can prove the inf-sup conditions for $b : U_h \times V_{h,k}^{\Theta} \to \mathbb{R}$. We do this with the help of a Fortin operator.
We say $\Pi : V \to V_k$ is a Fortin operator if there exists $C_\Pi > 0$ such that

$$\|\Pi \boldsymbol{v}\|_V \leq C_\Pi \|\boldsymbol{v}\|_V \text{ and } b(\boldsymbol{u}_h, \boldsymbol{v}) = b(\boldsymbol{u}_h, \Pi \boldsymbol{v}) \text{ for all } \boldsymbol{u}_h \in U_h, \boldsymbol{v} \in V. \tag{4.3}$$

Theorem 4.1 ([9], Theorem 4.1)**.** *Let $U_h \subset U$ be a finite-dimensional subspace. Suppose that $\Pi : V \to V_k$ is a Fortin operator. Then, $b : U_h \times V_{h,k}^{\Theta} \to \mathbb{R}$ satisfies the inf-sup conditions. In particular* (4.2) *admits a unique solution $\boldsymbol{u}_h \in U_h$.*
Furthermore, let $\boldsymbol{u} \in U$ be the unique solution of (3.3)*. Then, there holds quasi-optimality*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_U \leq C_{opt} \min_{\boldsymbol{w}_h \in U_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_U \tag{4.4}$$

with $C_{opt} = C_b \cdot C_\Pi / C$ and C_b denotes the continuity constant of $b : U \times V \to \mathbb{R}$.

Recall that

$$U = (L^2(\Omega))^d \times (L^2(\Omega))_*^{d\times d} \times (H_0^{1/2}(\mathcal{S}))^d \times (H^{-1/2}(\mathcal{S}))^d,$$
$$V = (H^1(\Omega_h))^d \times (H(\mathrm{div}; \Omega_h))^d,$$

and define for $d = 2$

$$U_h = (\mathcal{P}^q(\Omega_h))^2 \times (\mathcal{P}^q(\Omega_h))_*^{2\times 2} \times (\mathcal{P}_{c,0}^{q+1}(\mathcal{S}))^2 \times (\mathcal{P}^q(\mathcal{S}))^2 \tag{4.5}$$

with $q \in \mathbb{N}_0$ and $(\mathcal{P}^q(\Omega_h))_*^{2\times 2} = (\mathcal{P}^q(\Omega_h))^{2\times 2} \cap (L^2(\Omega))_*^{d\times d}$. For the solvability of the resulting practical DPG method, we need at least $dim(V_k) \geq dim(U_h)$. We consider

$$V_k := (\mathcal{P}^{k_1}(\Omega_h))^2 \times (\mathcal{P}^{k_2}(\Omega_h))^{2\times 2} \tag{4.6}$$

with some $k_1, k_2 \in \mathbb{N}$. In order to construct operators that satisfy (4.6), one has to choose k_1, k_2 depending on the polynomial degree $q \in \mathbb{N}_0$ of the space U_h.

In the following we construct a Fortin operator $\Pi = (\Pi_\nabla^{k_1}, \Pi_{\mathrm{div}}^{k_2}) : V \to V_k$, where V_k is defined in 4.6. We follow [9] for the case $d = 2$. [13] also covers the general case $d \geq 3$.

Lemma 4.2 ([9], Lemma 4.3)**.** *Let $k = q+2$. There exists $\Pi_{\nabla,T}^k v$ such that for all $v \in H^1(K)$ there holds*

$$\left(\Pi_{\nabla,K}^k v - v, w\right)_K = 0 \quad \textit{for all } w \in \mathcal{P}^{q-1}(K), \tag{4.7a}$$

$$\langle tr_{\mathrm{grad}}^K(\Pi_{\nabla,K}^k v - v), \widehat{w}\rangle_{\partial K} = 0 \quad \textit{for all } \widehat{w} \in \mathcal{P}^q(\partial K). \tag{4.7b}$$

Moreover, there exists $C > 0$ depending only on shape-regularity and q such that

$$\|\Pi_{\nabla,K}^k v\|_{H^1(K)} \leq C\|v\|_{H^1(T)}, \tag{4.8a}$$

$$\|\Pi_{\nabla,K}^k v - v\|_K \leq Ch_K\|\nabla v\|_K, \tag{4.8b}$$

for all $v \in H^1(K)$.

Lemma 4.3 ([9], Lemma 4.5)**.** *Let $q \in \mathbb{N}_0$ and set $k := q + 2$. There exists $\Pi_{\mathrm{div},K}^k$: $H(\mathrm{div}, K) \to \mathcal{P}^k(K)^2$ such that for all $\tau \in H(\mathrm{div}, K)$ there holds*

$$\left(\Pi_{\mathrm{div},K}^k \tau - \tau, \sigma\right)_K = 0 \quad \textit{for all } \sigma \in \mathcal{P}^q(K)^2, \tag{4.9a}$$

$$\langle tr_{\mathrm{div}}^K(\Pi_{\mathrm{div},K}^k \tau - \tau), \widehat{\sigma}\rangle_{\partial K} = 0 \quad \textit{for all } \widehat{w} \in \mathcal{P}_c^{q+1}(\partial K). \tag{4.9b}$$

Moreover, there exists $C > 0$ depending only on q and shape-regularity such that

$$\|\Pi_{\mathrm{div},K}^k \tau\|_{H(\mathrm{div},K)} \leq C\|\tau\|_{H(\mathrm{div},K)} \quad \textit{for all } \tau \in H(\mathrm{div}, K). \tag{4.10}$$

The proof of Lemma 4.3 defines an operator over a reference element K_{ref} and then extend to the other elements with the Piola transformation. Details are in [9].

Theorem 4.4. *Let $q \in \mathbb{N}_0$, $d = 2$, and set $k := q + 2$. For $\boldsymbol{v} = (v, N) \in V$, where $v = (v_1, v_2)^T \in H^1(\Omega_h)^2$ and $N = (N_1, N_2)^T \in H(\mathrm{div}, \Omega_h)^2$, define $\Pi\boldsymbol{v} := (\Pi_\nabla^k v_1, \Pi_\nabla^k v_2, \Pi_{\mathrm{div}}^k N_1, \Pi_{\mathrm{div}}^k N_2)$ where*

$$\begin{aligned}\Pi^k_\nabla v_j &:= (\Pi^k_{\nabla,K_1} v_j|_{K_1}, \cdots, \Pi^k_{\nabla,K_M} v_j|_{K_M}),\\ \Pi^k_{\mathrm{div}} N_j &:= (\Pi^k_{\mathrm{div},K_1} N_j|_{K_1}, \cdots, \Pi^k_{\mathrm{div},K_M} N_j|_{K_M})\end{aligned}$$

with $M = \#\Omega_h$ and $j \in \{1,2\}$. Then, $\Pi : V \to V_k = (\mathcal{P}^k(\Omega_h))^2 \times (\mathcal{P}^k(\Omega_h))^{2\times 2}$ is a Fortin operator, i.e., satisfies (4.3) *with $C_\Pi > 0$ depending only on shape-regularity and the polynomial degree q.*
In particular, the practical DPG method (4.2) *is well-posed and there holds quasi-optimality*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_U \le C_{opt} \min_{\boldsymbol{w}_h \in U_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_U.$$

Proof. First, note that by Lemma 4.2-4.3 $\|\Pi \boldsymbol{v}\|_V \le C_\Pi \|\boldsymbol{v}\|_V$ for all $\boldsymbol{v} \in V$,where $C_\Pi > 0$ depends only on shape-regularity and the polynomial degree q.
Next, we prove that $b(\boldsymbol{u}_h, \boldsymbol{v}) = b(\boldsymbol{u}_h, \Pi\boldsymbol{v})$ for $\boldsymbol{v} \in V$, $\boldsymbol{u}_h \in U_h$. Recall that

$$b(\boldsymbol{u}_h, \boldsymbol{v}) = \left(M_h, N - \frac{1}{d}\operatorname{Tr}(N) \cdot I - \nabla_{\Omega_h} v\right) + (u_h, \operatorname{div}_{\Omega_h} N) - \langle \widehat{u}_h, N \cdot n\rangle_{\mathcal{S}} - \langle \widehat{M}_h, v\rangle_{\mathcal{S}},$$

where

$$\boldsymbol{u}_h = (u_h, M_h, \widehat{u}_h, \widehat{M}_h) \in U_h = (\mathcal{P}^q(\Omega_h))^2 \times (\mathcal{P}^q(\Omega_h))^{2\times 2} \times (\mathcal{P}^{q+1}_{c,0}(\mathcal{S}))^2 \times (\mathcal{P}^q(\mathcal{S}))^2.$$

Properties (4.7b),(4.9) directly show that

$$\begin{aligned}\langle \widehat{M}_h, v\rangle_{\mathcal{S}} &= \langle \widehat{M}_h, \Pi^{q+2}_\nabla v\rangle_{\mathcal{S}},\\ (M_h, N) &= (M_h, \Pi^{q+2}_{\mathrm{div}} N),\\ \langle \widehat{u}_h, N \cdot n\rangle_{\mathcal{S}} &= \langle \widehat{u}_h, (\Pi^{q+2}_{\mathrm{div}} N) \cdot n\rangle_{\mathcal{S}}.\end{aligned}$$

It can be proved (cf. [9], proof of Lemma 4.5) that $\operatorname{div} \Pi^{q+2}_{\mathrm{div}} = \Pi^{q+1} \operatorname{div}$, where $\Pi^q_K : L^2(K) \to \mathcal{P}^p(K)$ is the $L^2(K)-$orthogonal projection. Using this identity and $\Pi^{q+1} u_h = u_h$, since $u_h \in \mathcal{P}^q(\Omega_h)$, we infer

$$(u_h, \operatorname{div} N) = (u_h, \Pi^{q+1} \operatorname{div} N) = (u_h, \operatorname{div} \Pi^{q+2}_{\mathrm{div}} N).$$

For the term $(M_h, \nabla_{\Omega_h} v)$, note that $\operatorname{div} M_h \in (\mathcal{P}^{q-1}(\Omega_h))^2$ and $tr_{\mathrm{div}} M_h \in (\mathcal{P}^q(\mathcal{S}))^2$. Elementwise Integration by parts and (4.9) show

$$\begin{aligned}(M_h, \nabla_{\Omega_h} v) &= -(\operatorname{div} M_h, v) + \langle tr_{\mathrm{div}} M_h, v\rangle_S\\ &= -(\operatorname{div} M_h, \Pi^{q+2}_\nabla v) + \langle tr_{\mathrm{div}} M_h, \Pi^{q+2}_\nabla v\rangle_S = (M_h, \nabla_{\Omega_h} \Pi^{q+2}_\nabla v).\end{aligned}$$

Finally, for the term $(M_h, \operatorname{Tr}(N) \cdot I)$ we use (4.9) to get

$$(M_h, \operatorname{Tr}(N) \cdot I) = (\operatorname{Tr}(M_h) \cdot I, N) = (\operatorname{Tr}(M_h) \cdot I, \Pi^{q+2}_{\mathrm{div}} N) = (M_h, \operatorname{Tr}(\Pi^{q+2}_{\mathrm{div}} N)) \cdot I).$$

Putting all together shows that Π is indeed a Fortin operator. Theorem 4.1 concludes the proof. □

Recall that in equation (3.2) we introduce the pseudostress variable $M = \nabla u - p \cdot I$, where u is the velocity and p the pressure. We can recover p taking the trace of M. Note that if $u \in (H^{q+2}(\Omega))^2$ and $p \in H^{q+1}(\Omega)$, then $M \in (H^{q+1}(\Omega))^{2\times 2}$.

Theorem 4.5. *Let $\boldsymbol{u} \in U$ the unique solution of* (3.3) *and $\boldsymbol{u}_h \in U_h$ the solution of* (4.2)*. Under the assumptions of Theorem 4.4 and suppose that $u \in (H^{q+2}(\Omega))^2$ and $p \in H^{q+1}(\Omega)$, then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_U \lesssim h^{q+1}(\|u\|_{H^{q+2}(\Omega)} + \|p\|_{H^{q+1}(\Omega)})$$

Proof. Set $\boldsymbol{w}_h = (\Pi^q u, \Pi^q M, tr_{\text{grad}}\Pi^{q+1}_{\text{grad}} u, tr_{\text{div}}\Pi^q_{\text{div}} M) \in U_h$. Then, with the approximation properties (2.31), (2.32) and (2.33) we get

$$\|\boldsymbol{u} - \boldsymbol{w}_h\| \lesssim h^{q+1}\|u\|_{H^{q+1}(\Omega)} + h^{q+1}\|M\|_{H^{q+1}(\Omega)} + h^{q+1}\|u\|_{H^{q+2}(\Omega)} + \|tr_{\text{div}}(1 - \Pi^q_{\mathcal{RT}})M\|_{-1/2,\mathcal{S}}$$

The last term is tackled with Theorem 2.15. Using $M = \nabla u - p \cdot I$ together with the quasi-optimality from Theorem 4.4 finish the proof. □

4.2 Error estimator

We collect some results on the built-in error estimator, see [4], [9] for details. Let $\boldsymbol{u}_h \in U_h$ denote the solution of the practical DPG method (4.2). Let $\boldsymbol{\xi}_h \in U_h$ and $\ell \in V'$. Recall the error function $\varepsilon_h(\boldsymbol{\xi}_h) = R_V^{-1}(\ell - B\boldsymbol{\xi}_h)$ and its discrete form $\varepsilon_{h,k}(\boldsymbol{\xi}_h) = R_{V_k}^{-1}(\ell - B\boldsymbol{\xi}_h)$. We define the DPG error estimator by

$$\eta_{h,k}(\boldsymbol{\xi}_h) := \|\varepsilon_{h,k}(\boldsymbol{\xi}_h)\|_V = \|\ell - B\boldsymbol{\xi}_h\|_{V_k'}$$

An error estimator η is called reliable if there exists $C_{\text{rel}} > 0$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq C_{\text{rel}}\eta,$$

and efficient if there exists $C_{\text{eff}} > 0$ such that

$$C_{\text{eff}}\eta \leq \|\boldsymbol{u} - \boldsymbol{u}_h\|.$$

Now, given $\ell \in V'$, we define the data oscillation term by

$$\text{osc}(\ell) := \|\ell \circ (1 - \Pi)\|_{V'} = \sup_{0 \neq v \in V} \frac{\ell((1-\Pi)v)}{\|v\|_V}. \tag{4.11}$$

Then, define the overall estimator

$$\eta(\boldsymbol{\xi}_h)^2 := \eta_{h,k}(\boldsymbol{\xi}_h)^2 + \text{osc}(\ell)^2.$$

Theorem 4.6 ([9], Theorem 5.1)**.** *Let $\ell \in V'$ and $\boldsymbol{u} = B^{-1}F$ be the exact solution. Then, for $\boldsymbol{\xi}_h \in U_h$ there holds*

$$C^2\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_U^2 \leq (1 + 2C_\Pi^2)\eta_{h,k}(\boldsymbol{\xi}_h)^2 + 2\,\mathrm{osc}(\ell)^2, \tag{4.12a}$$

$$\eta_{h,k}(\boldsymbol{\xi}_h) \leq C_b\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_U, \tag{4.12b}$$

$$\mathrm{osc}(\ell) \leq C_b(1 + C_\Pi) \min_{\boldsymbol{w}_h \in U_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|. \tag{4.12c}$$

In particular, the overall estimator $\eta(\boldsymbol{\xi}_h)$ is efficient and reliable.

Since η is efficient and reliable this implies that

$$\eta(\boldsymbol{\xi}_h) \simeq \|\boldsymbol{u} - \boldsymbol{\xi}_h\|_U \text{ for all } \boldsymbol{\xi}_h \in U_h.$$

Since V_k is a broken space, the estimator can be localized, i.e.,

$$\eta_{h,k}^2 = \sum_{K \in \Omega} \|\ell - B\boldsymbol{u}_h\|_{V_k(K)}^2,$$

where $V_k(K)$ is the restricction of the space V_k to the element K. We use the local indicators to steer mesh-refinement in Chapter 5.

Chapter 5

Numerical Examples

We consider several examples to study our proposed method for the Stokes problem. We consider analytical solutions to measure the solution quality for different velocities u and pressures p under uniform mesh refinements. We then apply the adaptive refinement strategy to model a circulant segment with a re-entrant corner. Lastly, we analyze a lid-driven cavity problem with uniform and adaptive refinements. Most of the examples are from [14].
Recall that we introduced a variable $M = \nabla u - p \cdot I$. Taking the trace on both sides we recover the pressure p as follows

$$\mathrm{Tr}(M) = \underbrace{\mathrm{div}(u)}_{=0} - d \cdot p \Rightarrow p = -\frac{1}{d} \cdot \mathrm{Tr}(M).$$

For the approximate solutions we use U_h defined in (4.5) with $q \in \{0, 1\}$, and V_k defined in (4.6) for the test space, with $k_1 = k_2 = q + 2$. We expect that the total error behaves like $\|\boldsymbol{u} - \boldsymbol{u}_h\|_U = \mathcal{O}(h^{q+1}) = \mathcal{O}(N_{tot}^{-(q+1)/2})$, where $N_{tot} = dim(U_h)$ denotes the total degrees of freedom.

For the implementation of the condition $\int_\Omega \mathrm{Tr}(M) = 0$ we use Lagrange multipliers. Let $\{\psi_1, \ldots, \psi_t\}$ be a basis for $(\mathcal{P}^q(\Omega))^{2\times 2}$ and let $\mathcal{L} = (0, \ldots, 0, \int_\Omega \mathrm{Tr}(\psi_1), \ldots, \int_\Omega \mathrm{Tr}(\psi_t), 0, \ldots, 0)$. If the discrete problem can be represented as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{f},$$

where $\boldsymbol{x}$ are the coefficients of our approximated solution, we solve

$$\begin{bmatrix} \boldsymbol{A} & \mathcal{L}^T \\ \mathcal{L} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}.$$

From here we get that $\mathcal{L}\boldsymbol{x} = \int_\Omega \mathrm{Tr}(M_h) = 0$.

5.1 Smooth solution

First we study a smooth solution with different pressures to validate our method. The analytical expressions for the solution are:

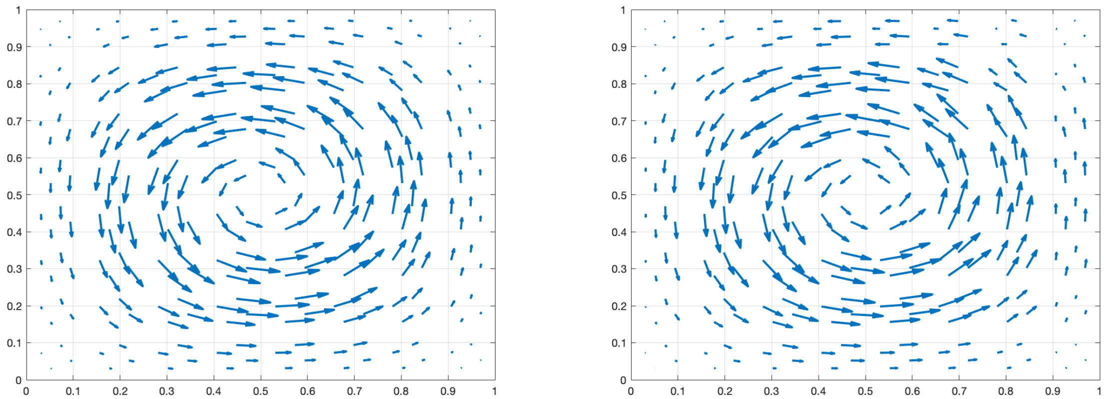

Figure 5.1: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem (5.1) with p_1.

$$u_{sol} = \begin{bmatrix} \pi \sin^2(\pi x) \cdot \sin(2\pi y) \\ -\pi \sin(2\pi x) \cdot \sin^2(\pi y) \end{bmatrix}, \tag{5.1a}$$

$$p_1 = x - \frac{1}{2}, \tag{5.1b}$$

$$p_2 = \left(x - \frac{1}{2}\right) \cdot \left(y - \frac{1}{2}\right), \tag{5.1c}$$

$$p_3 = \cos(2\pi x). \tag{5.1d}$$

Using the above pressures and the velocity, we construct the force term:

$$f_i = -\Delta u_{sol} + \nabla p_i, \; i \in \{1, 2, 3\} \tag{5.2}$$

Figures 5.1, 5.2 and 5.3 show the approximation of the velocity field of u_{sol} in (5.1) for the different pressures p_i and polynomial orders. Figures 5.4, 5.5 and 5.6 show the comparisons between the exact solution of the different pressures and its approximations for different polynomial orders. For the case of $q = 0$, the approximation of the pressures seems not to be good, however the convergence rates of the error look correct because they converge in the proper order as predicted by Theorem 4.5. In Figures 5.7, 5.8 and 5.9 we show the errors in L^2 of u, p_i and M_i, with $M_i = \nabla u - p_i \cdot I$ for $i \in \{1, 2, 3\}$. We plot these errors as a function of the mesh size h in logarithmic scale. Also we compare this curves with the convergence optimal order.

Also we study the manufactured solution from [14]. The analytical expressions of the solution are:

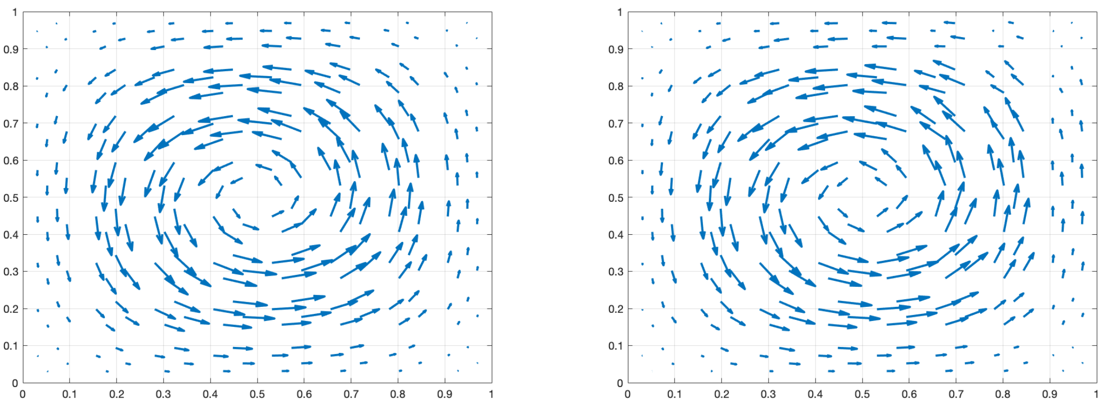

Figure 5.2: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem (5.1) with p_2.

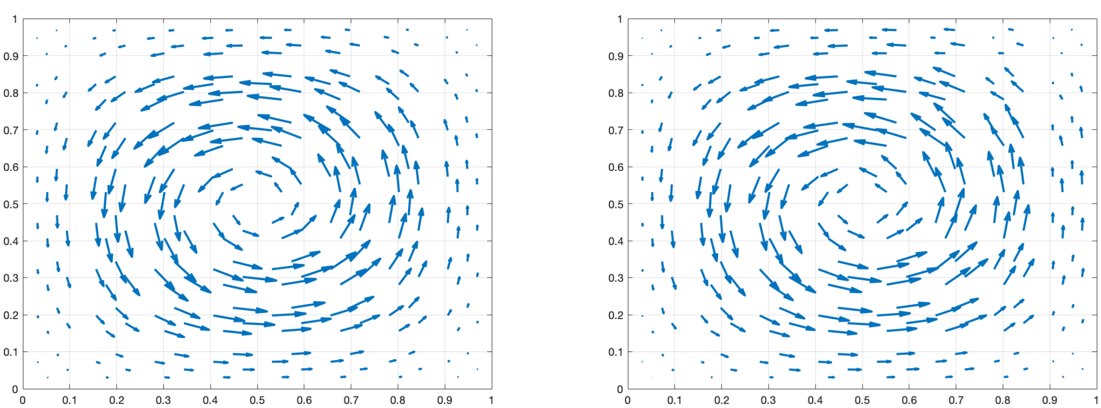

Figure 5.3: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem (5.1) with p_3.

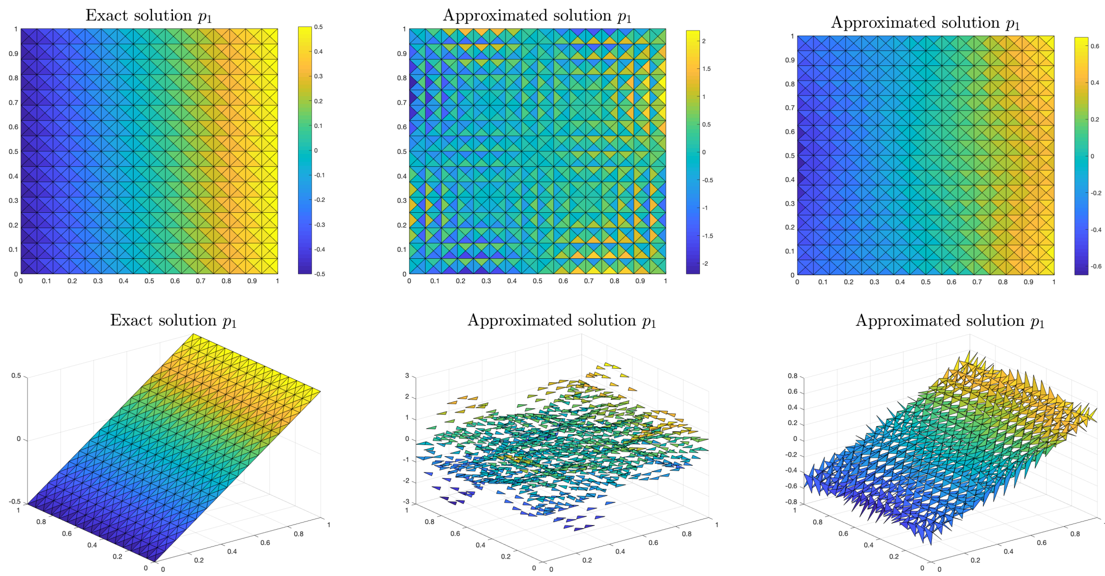

Figure 5.4: p_1 approximation compared to exact solution (left) for $q = 0$ (middle) and $q = 1$ (right) for problem (5.1).

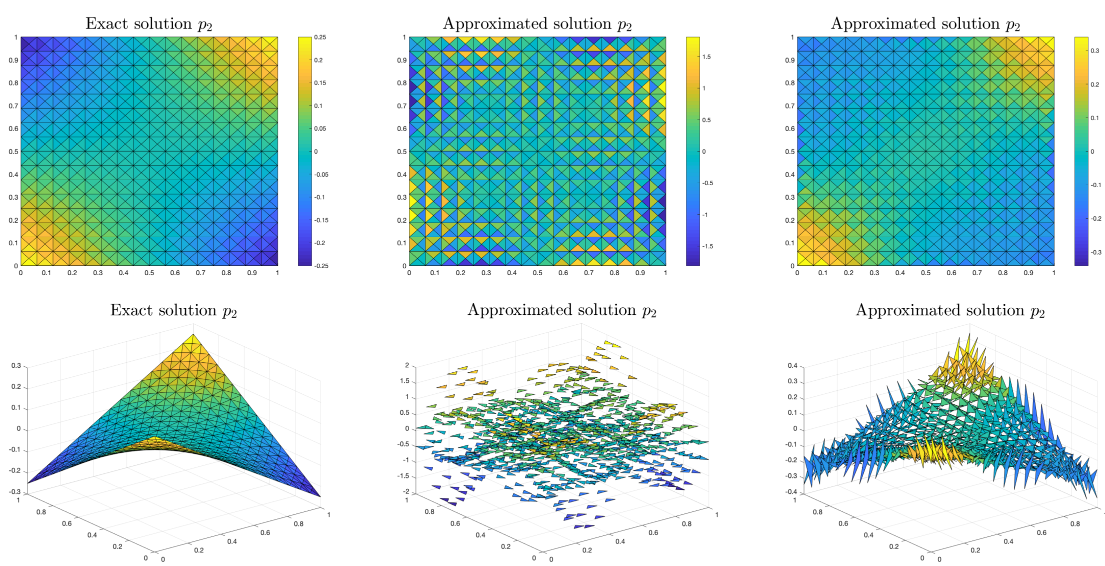

Figure 5.5: p_2 approximation compared to exact solution (left) for $q = 0$ (middle) and $q = 1$ (right) for problem (5.1).

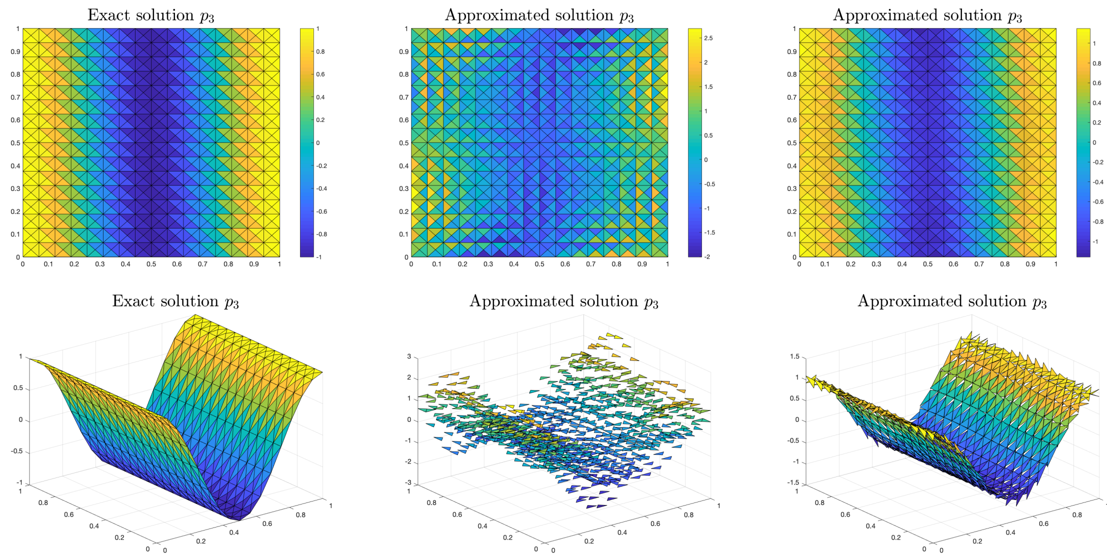

Figure 5.6: p_3 approximation compared to exact solution (left) for $q = 0$ (middle) and $q = 1$ (right) for problem (5.1).

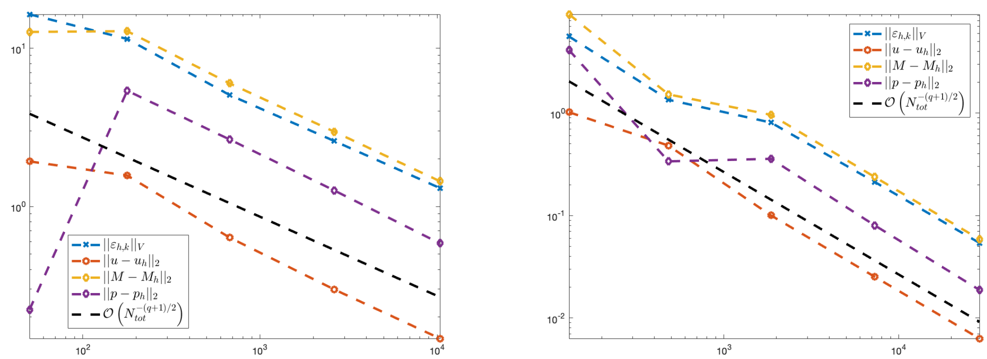

Figure 5.7: DPG error for problem (5.1) with p_1 for $q = 0$ (left) and $q = 1$ (right).

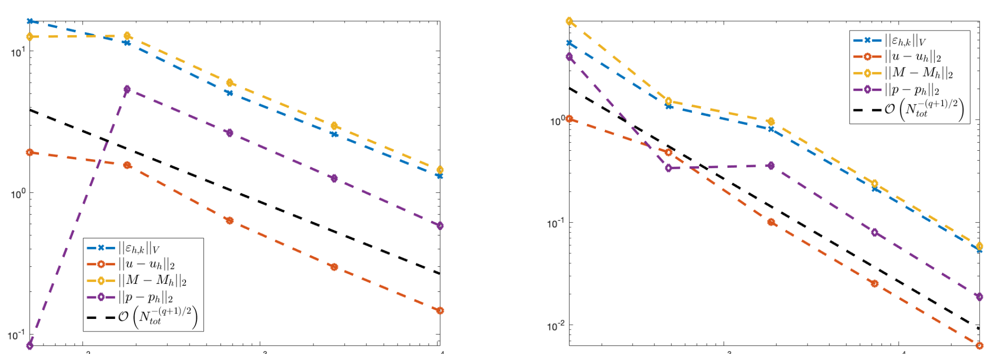

Figure 5.8: DPG error for the problem (5.1) with p_2 for $q = 0$ (left) and $q = 1$ (right).

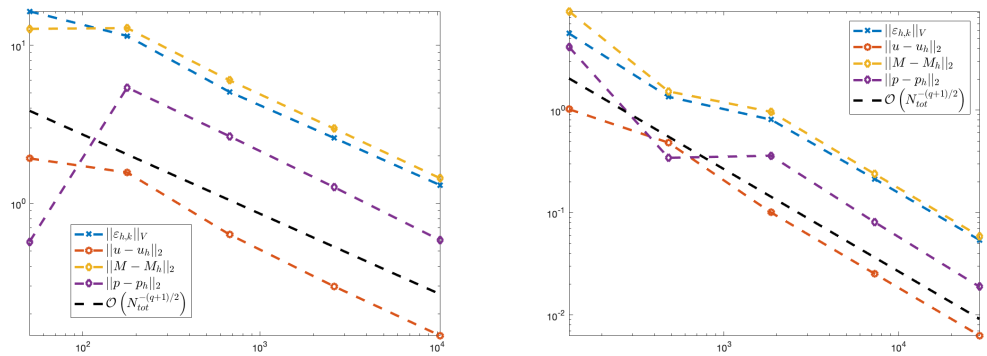

Figure 5.9: DPG error for problem (5.1) with p_3 for $q = 0$ (left) and $q = 1$ (right).

$$u_{sol} = \begin{bmatrix} (2e^x(-1+x)^2x^2(y^2-y)(-1+2y)) \\ (-e^x(-1+x)x(-2+x(3+x))(-1+y)^2y^2) \end{bmatrix}, \tag{5.3a}$$

$$\begin{aligned} p_{sol} = \quad & (-424 + 156e + (y^2-y)(-456 + e^x(456 + x^2(228 - 5(y^2-y)) \\ & +2x(-228 + (y^2-y)) + 2x^3(-36 + (y^2-y)) + x^4(12 + (y^2-y))))). \end{aligned} \tag{5.3b}$$

In Figure 5.10 we show the approximations of the velocity field with $q = 0, 1$. Figure 5.11 show the comparison between the exact solution of p, and its approximations for different polynomial orders and Figure 5.12 show the different errors.

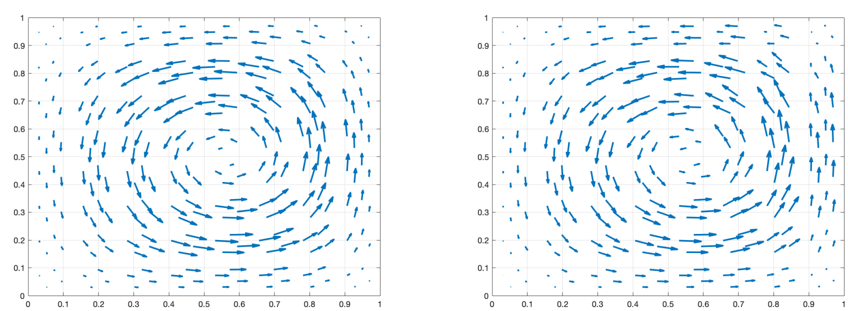

Figure 5.10: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem (5.3)

5.2 L-shaped domain and adaptive refinement

For the adaptive mesh refinement case, we consider the L-shaped $\Omega = (-1,1)^2 \setminus ((0,1) \times (-1,0))$ and the analytical solution for u_{sol} and p_{sol} from [14] given as

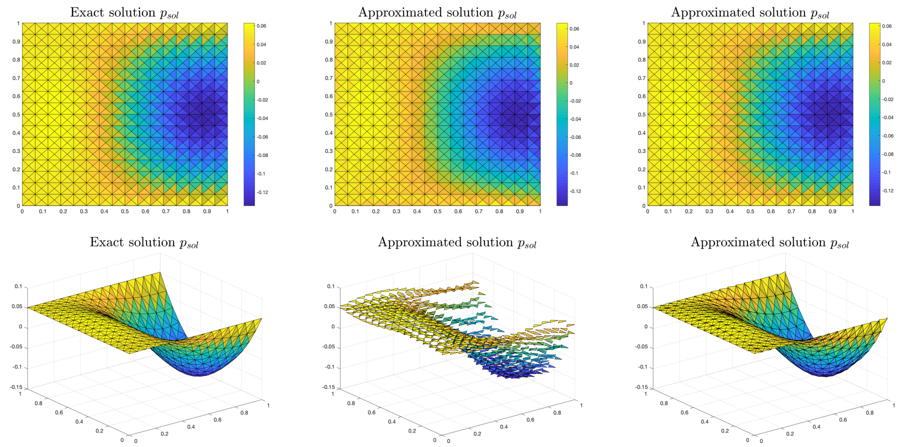

Figure 5.11: p_{sol} approximation compared to exact solution (left) for $q = 0$ (middle) and $q = 1$ (right) for problem (5.3).

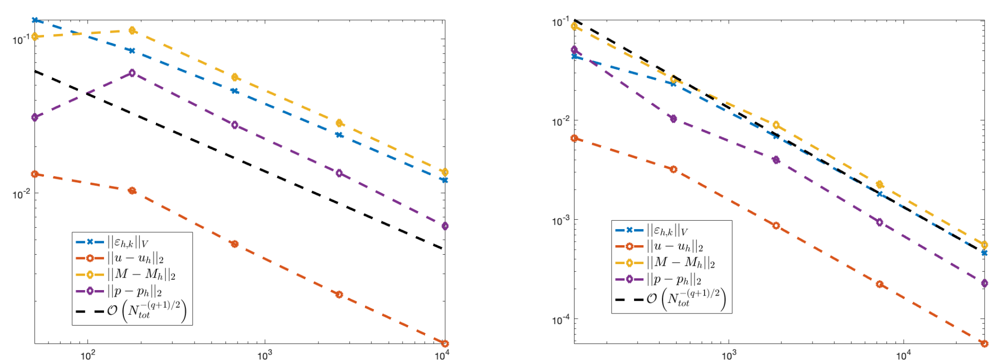

Figure 5.12: DPG error for problem (5.3) for $q = 0$ (left) and $q = 1$ (right).

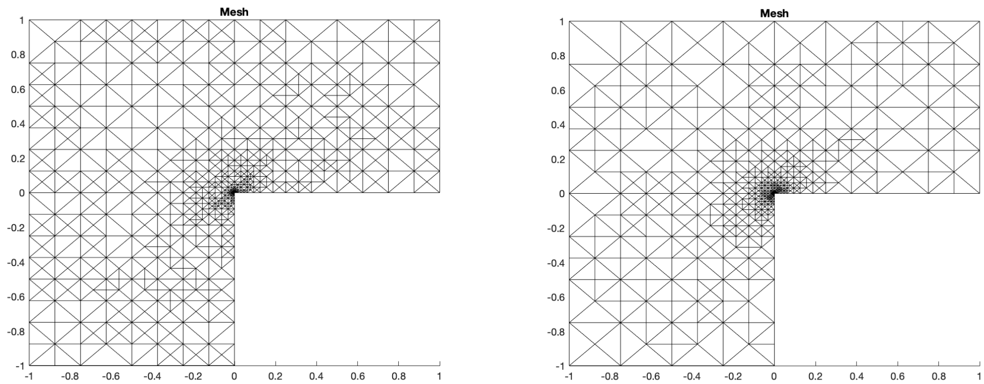

Figure 5.13: Adaptive mesh for $q = 0$ (left) and $q = 1$ (right) for problem (5.4).

$$u_{sol} = \begin{bmatrix} r^{\alpha}[(1+\alpha)\sin(\phi)\psi(\phi) + \cos(\phi)\partial_\phi\psi(\phi)] \\ r^{\alpha}[\sin(\phi)\partial_\phi\psi(\phi) - (1+\alpha)\cos(\phi)\psi(\phi)] \end{bmatrix}, \tag{5.4a}$$

$$p_{sol} = -\, r^{\alpha-1}[(1+\alpha)^2\partial_\phi\psi(\phi) + \partial^3_\phi\psi(\phi)]/(1-\alpha), \tag{5.4b}$$

with

$$\begin{aligned} \psi(\phi) = \quad & \frac{\sin((1+\alpha)\phi)\cos(\alpha\omega)}{(1+\alpha)} - \cos((1+\alpha)\phi) \\ & + \frac{\sin((\alpha-1)\phi)\cos(\alpha\omega)}{(1-\alpha)} + \cos((\alpha-1)\phi), \end{aligned} \tag{5.5}$$

$$\alpha = \frac{856399}{1572864}, \quad \omega = \frac{3\pi}{2}.$$

Here, (r, ϕ) denote polar coordinates centered in $(0, 0)$. We set the forcing term equal to zero and impose Dirichlet boundary conditions on the entire boundary, setting homogeneous values on the edges of the reentrant corner and nonhomogeneous ones on the complementary part. Figure 5.13 show meshes for different polynomials degrees q, Figure 5.14 show the approximation of the velocity field, Figure 5.15 show the pressure approximation and Figure 5.16 show the DPG errors. Also 5.17 show a comparison of the error using a uniform refinement of the mesh versus an adaptive refinement.

5.3 Lid driven cavity flow

As a last example, we consider the well-known lid-driven cavity flow problem (see [14]). We set the source term $f = 0$ and consider no-slip boundary conditions on the bottom, left,

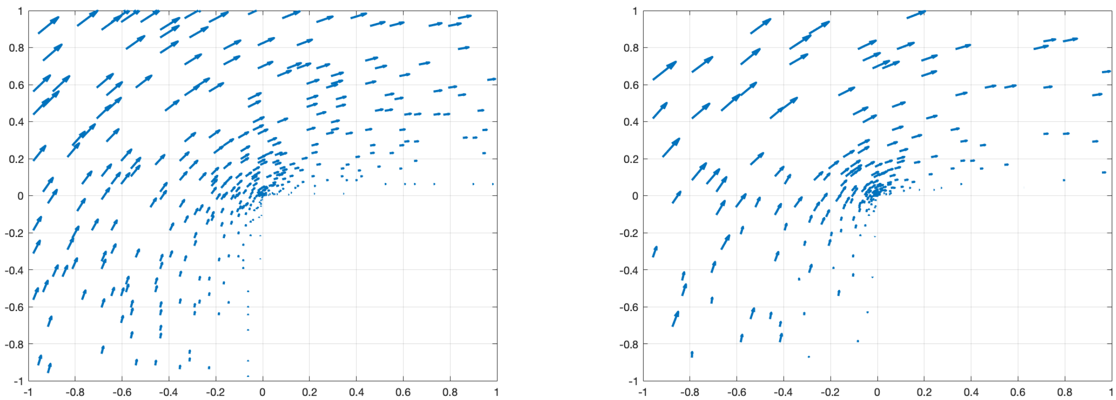

Figure 5.14: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem (5.4)

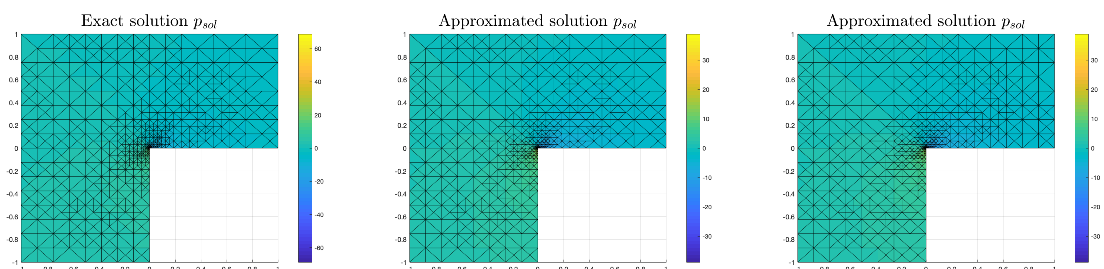

Figure 5.15: p_{sol} approximation compared to exact solution (left) for $q = 0$ (left) and $q = 1$ (right) for problem (5.4).

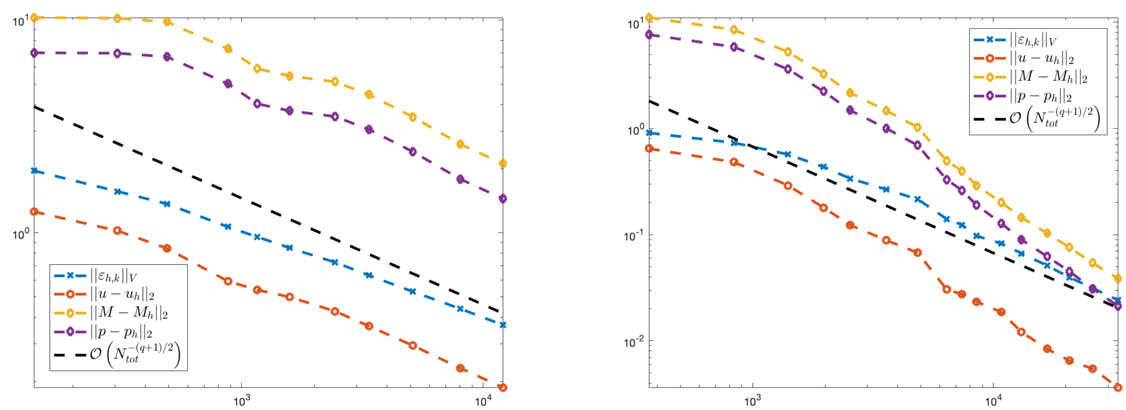

Figure 5.16: DPG error for problem (5.4) for $q = 0$ (left) and $q = 1$ (right).

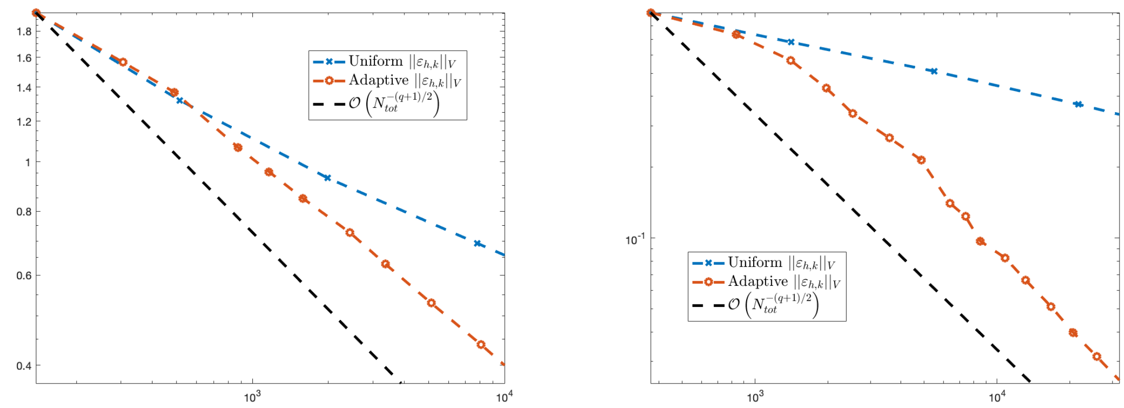

Figure 5.17: Comparison of the DPG error between an adaptive refinement vs. a uniform refinement of the mesh, for $q = 0$ (left) and $q = 1$ (right).

and right boundaries ($u = (0, 0)$). At the top, as Figure 5.18 shows, we impose the velocity profile $\boldsymbol{u}_{top} = (u_1(x), 0)$ (see [18], Example D.4)

$$u_1(x) = \begin{cases} 1 - \frac{1}{4}\left(1 - \cos\left(\frac{x_1 - x}{x_1} \cdot \pi\right)\right)^2 & \text{for } x \in [0, x_1] \\ 1 & \text{for } x \in (x_1, 1 - x_1) \\ 1 - \frac{1}{4}\left(1 - \cos\left(\frac{x - (1 - x_1)}{x_1} \cdot \pi\right)\right)^2 & \text{for } x \in [1 - x_1, 1] \end{cases} \tag{5.6}$$

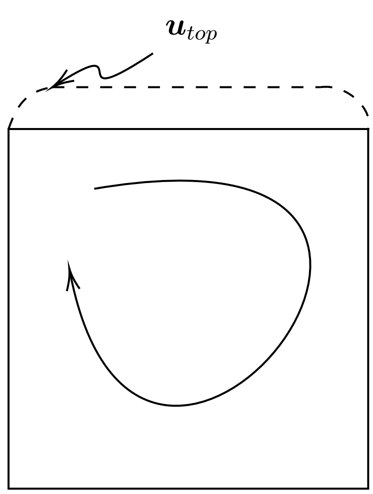

Figure 5.18: Boundary conditions for the lid-driven cavity flow.

The simulation of this example were performed with $x_1 = 0.2$. Figure 5.19 show the adaptive

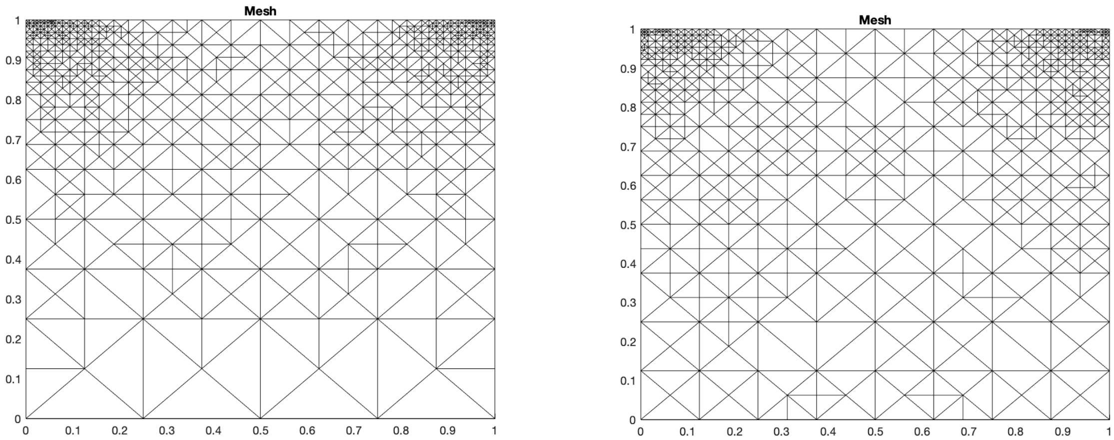

Figure 5.19: Adaptive mesh for $q = 0$ (left) and $q = 1$ (right) for problem from Section 5.3

meshes, Figures 5.20 and 5.21 show the velocity field and pressure approximation respectively and Figure 5.22 shows the DPG error. All figures show the cases $q = 0, 1$.

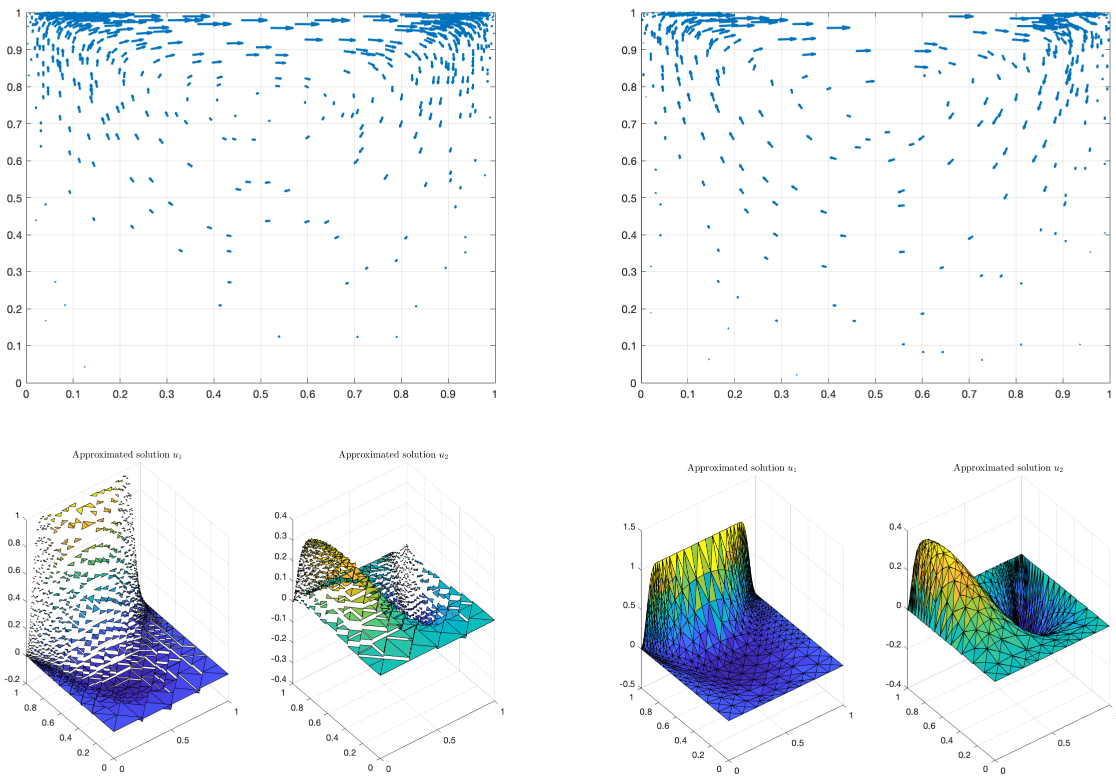

Figure 5.20: Approximation of the velocity for $q = 0$ (left) and $q = 1$ (right) for problem from Section 5.3

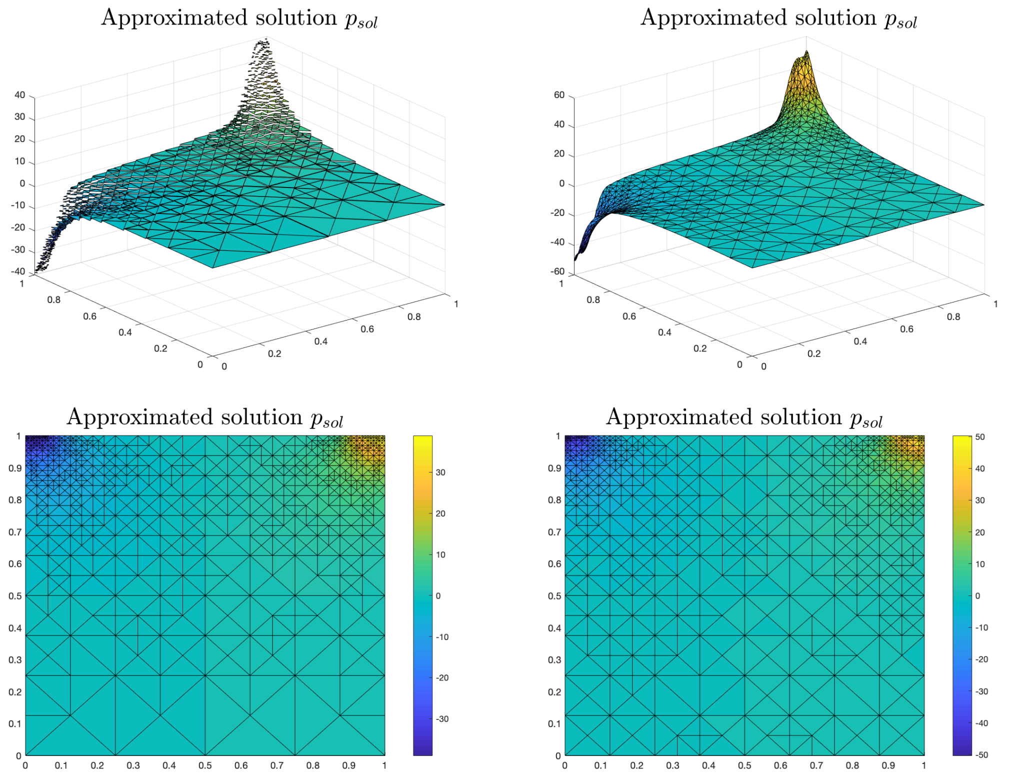

Figure 5.21: Approximation of the pressure for $q = 0$ (left) and $q = 1$ (right) for problem from Section (5.3).

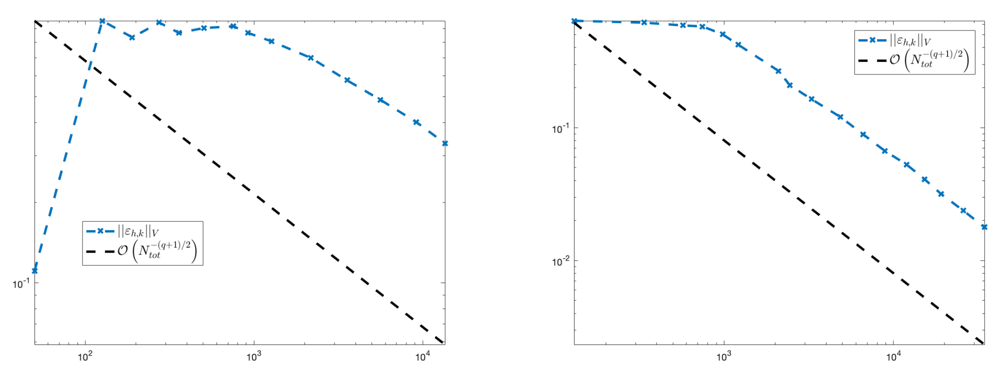

Figure 5.22: DPG error for problem from Section (5.3) for $q = 0$ (left) and $q = 1$ (right).

Chapter 6

Conclusions

In this work we have presented an ultra-weak formulation of the Stokes problem with a pseudo-stress variable and the approximation of its solution with the DPG method. Our formulation is similar to the one in [17] but we eliminate the pressure variable from the system. In Chapter 2 we saw the tools needed for the ultra-weak formulation and also the presentation of the DPG method. Theorem 3.1 shows that our reformulation has a unique solution and in the rest of Chapter 3 we see that its ultra-weak formulation also has a unique solution, if the inf-sup conditions are met (see Theorem 3.4). Chapter 4 shows us the existence of a Fortin operator and thus ensures the well-posedness of the discrete problem as well as the quasi-optimality of the DPG method.
In Chapter 5 we discuss numerical examples that empirically show our theoretical development and the different rates of convergence for different degrees of polynomial approximation. The poor approximation of the pressures that we observe (at least visually) in Problem (5.1) is striking, even though the convergence rate is correct. A possible solution to this problem could be to increase the polynomial degree for the trace of the pseudostress variable in order to improve the approximation for the pressure, which in our scheme is defined using the trace of the pseudostress.
In [10] superconvergence in the primal variable is achieved by doing element-by-element post-processing for the Poisson problem. We believe that the same results can be proved for the Stokes problem, using the formulation proposed in this work.

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