PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

FACULTY OF MATHEMATICS

# On Kurokawa's conjecture on the vanishing of Witten's zeta function at $s = -2$

by

Matías Bruna

Thesis submitted to the Faculty of Mathematics of Pontificia Universidad Católica de Chile, as one of the requirements to qualify for the academic Master's degree in Mathematics.

| | | |
|---|---|---|
| Advisors | : | Eduardo Friedman (UCh) |
| | | Ricardo Menares (PUC) |
| Committee | : | Sebastián Herrero (PUCV) |

August, 2022

Santiago, Chile

# Contents

# Introduction

The Riemann zeta function $\zeta(s)$, first studied in the 18th century by L. Euler as a real variable function and later by B. Riemann as a complex variable function, plays a crucial role in analytic number theory. Since then, many zeta functions have been defined, which have made important appearances in various areas of mathematics. It is therefore interesting to study their properties, for example their analytic continuation, special values at the integers, and decomposition as sum/product of simpler zeta functions.

Around 1990, related to his work on topological quantum field theory, E. Witten [Wit91] calculated the volumes of the moduli spaces of representations of the fundamental groups of two dimensional surfaces in terms of special values of a new zeta function attached to complex semi-simple Lie algebras $\mathfrak{g}$ at positive integers. Inspired by this result, in 1994 D. Zagier [Zag94] defined the Witten zeta function $\zeta_{\mathfrak{g}}^W(s)$ as

$$\zeta_{\mathfrak{g}}^W(s) := \sum_{\rho} \frac{1}{(\dim \rho)^s}, \tag{0.1}$$

where the sum runs over all isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{g}$. Using Weyl's dimension formula he computed

$$\zeta_{\mathfrak{sl}(2)}^W(s) = \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

and

$$\zeta_{\mathfrak{sl}(3)}^W(s) = \sum_{m,n=1}^{\infty} \frac{2^s}{m^s n^s (m+n)^s}. \tag{0.2}$$

With this in mind, for $G$ a compact topological group, N. Kurokawa & H. Ochiai [KO13] considered

$$\zeta_G^W(s) := \sum_{\rho \in \hat{G}} \frac{1}{(\dim \rho)^s}, \tag{0.3}$$

where the sum runs over the set of equivalence classes of irreducible unitary representations. If $G$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then from the correspondence between representations of $G$ and representations of $\mathfrak{g}$ it is clear that $\zeta_G^W(s) = \zeta_{\mathfrak{g}}^W(s)$. Thus (0.3) generalizes (0.1). They also noticed that if $G$ is a finite group, then

$$\zeta_G^W(-2) = \sum_{\rho \in \hat{G}} (\dim \rho)^2 = |G|$$

and

$$\zeta_{SU(2)}^W(-2) = \zeta_{\mathfrak{sl}(2)}^W(-2) = \zeta(-2) = 0.$$

Moreover, they proved that

$$\zeta_{SU(3)}^W(-2) = \zeta_{\mathfrak{sl}(3)}^W(-2) = 0 \quad \text{and} \quad \zeta_{SL_2(\mathbf{Z}_p)}^W(-2) = 0 \ \text{ for } p \neq 2,$$

where $\mathbf{Z}_p$ is the ring of $p-$adic integers. Finally, they conjectured that if $G$ is an infinite group then $\zeta_G^W(-2) = 0$.

Little is known so far about $\zeta_G^W(-2)$ for arbitrary compact topological groups. In fact, we have not been able to find a proof in the literature that $s = -2$ is a regular value of $\zeta_G^W(s)$ for such groups. Various authors (see for example [Ess97, Mat03, MT06, KMT10]) have defined a multi-variable versions of Witten's zeta function $\zeta_{\mathfrak{g}}^W(\mathbf{s})$, or zeta functions that include Witten's as a special case, and studied its meromorphic continuation. However they do not specialize their analysis to the case of $\zeta_{\mathfrak{g}}^W$, nor does it follow from their results that $s = -2$ is a regular value of $\zeta_{\mathfrak{g}}^W(s)$.

In 2004 E. Friedman & S. Ruijsenaars [FR04] studied the meromorphic continuation (in $s \in \mathbb{C}$) and special values of

$$\zeta_{N,n}(s, w \mid a_1, \ldots, a_N) := \sum_{k_1,\ldots,k_N=0}^{\infty} \prod_{j=1}^{n} (w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})^{-s},$$

where the $a_i$ and $w$ are elements of $\mathbb{C}^n$ whose coordinates $a_{ij}$ and $w_j$ have positive real parts. This was a slight generalization of Shintani's zeta function [Shi76] where $w$ was more restricted.

The main purpose of this thesis is to use the ideas developed in [FR04] to generalize some of their results by allowing $a_{ij} = 0$, but assuming that for each $i$ there is some $j$ such that $a_{ij} \neq 0$, and assuming further that there exists $\varepsilon > 0$ such that $-\pi/2 + \varepsilon < \arg(a_{ij}) < \pi/2 + \varepsilon$ whenever $a_{ij} \neq 0$. This generalization allows us to include Witten zeta function $\zeta_{\mathfrak{g}}^W$ as a special case of $\zeta_{N,n}$.

Specifically, in Chapter 1 we define the Shintani-Barnes zeta function $\zeta_{N,n}(s, w, \mathscr{M})$, prove some of its basic properties, and we also show several examples of zeta functions that are special cases of $\zeta_{N,n}$, including $\zeta_{\mathfrak{g}}^W$ (except for an exponential factor). We end Chapter 1 calculating explicitly $\zeta_{\mathfrak{g}}^W(s)$ for all classical Lie algebras and the exceptional type $G_2$. In Chapter 2 we prove that $\zeta_{N,n}$ and its multi-variable and integral versions $Z_{N,n}$ and $\mathcal{Z}_{N,n}$ extend to meromorphic functions. In particular, for $k \in \mathbb{N}_0$ we prove that $s = -k$ is always a regular value of $\zeta_{N,n}(s, w)$, it has at most poles of order $n$ which can only occur among the rational numbers of the form

$$s = \frac{N - l}{v}, \quad \text{with } v \in \{1, \ldots, n\} \quad \text{and} \quad l \in \mathbb{N}_0 \setminus \{N + kv : k \in \mathbb{N}_0\},$$

and we show that $\zeta_{N,n}(-k, w)$ is a polynomial in $w$ and find an upper bound for its degree. This result, together with the results from Section 1.3, imply that if $\mathfrak{g}$ is a complex semi-simple Lie algebra then $s = -2$ is always a regular value of $\zeta_{\mathfrak{g}}^W(s)$.

Chapter 1

# Preliminaries

## 1.1. Definitions and basic results

Let $\mathscr{M}$ be the $N \times n$ matrix defined by

$$\mathscr{M} := \{a_{ij}\},\ i \in \{1, \ldots, N\},\ j \in \{1, \ldots, n\},$$

where $a_{ij} \in \mathbb{C}$ are such that

$$\text{for each } i \in \{1, \ldots, N\} \text{ there exists } j \in \{1, \ldots, n\} \text{ such that } a_{ij} \neq 0, \tag{1.1}$$

i.e. every row has at least one non-zero value, and

$$\text{there exists } \varepsilon > 0 \text{ such that } -\pi/2 + \varepsilon < \arg(a_{ij}) \leq \pi/2 - \varepsilon \text{ whenever } a_{ij} \neq 0. \tag{1.2}$$

For $\mathrm{Re}(s) \gg 0$, we define the Shintani-Barnes zeta function by

$$\zeta_{N,n}(s, w, \mathscr{M}) = \zeta_{N,n}(s, w \mid a_1, \ldots, a_N) := \sum_{k_1, \ldots, k_N = 0}^{\infty} \prod_{j=1}^{n} (w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})^{-s}, \tag{1.3}$$

its multi-variable version

$$Z_{N,n}(\mathbf{s}, w, \mathscr{M}) := \sum_{k_1, \ldots, k_N = 0}^{\infty} \prod_{j=1}^{n} (w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})^{-s_j}, \tag{1.4}$$

and its integral version

$$\mathscr{Z}_{N,n}(s, w, \mathscr{M}) := \int_{x \in [0,\infty)^N} \prod_{j=1}^{n} (w_j + x_1 a_{1j} + \cdots + x_N a_{Nj})^{-s}\, dx, \tag{1.5}$$

where $a_i = (a_{i1}, \ldots, a_{in})$, $\mathbf{s} = (s_1, \ldots, s_n)$, $w \in \mathbb{C}^n$, $\mathrm{Re}(w_j) > 0$ for all $j \in \{1, \ldots, n\}$, and $dx$ is the Lebesgue measure on $\mathbb{R}^N$. Since $\mathrm{Re}(w_j) > 0$ and $\mathrm{Re}(a_{ij}) \geq 0$, in all these definitions we choose the principal branch of the logarithm to define the complex powers.

Note that if the $j-$th column of $\mathscr{M}$ has only zeros then the corresponding factor in (1.4) is $w_j^{-s_j}$, therefore to study the convergence of (1.4) we can safely assume that $\mathscr{M}$ has no zero columns.

In order to find a region of absolute convergence for each of the functions defined above, let

$$A_j := \{i \in \{1, \ldots, N\} : a_{ij} \neq 0\} \quad \text{and} \quad c := \min_{i,j}\{\mathrm{Re}(a_{ij}) : a_{ij} \neq 0\} > 0.$$

For $k_i \geq 0$, and letting $|A_j|$ denote the cardinality of $A_j$, we have

$$\begin{aligned}
|w_j + k_1 a_{1j} + \cdots + k_N a_{Nj}|^2 &= \left| w_j + \sum_{i \in A_j} k_i a_{ij} \right|^2 \\
&\geq \left( \mathrm{Re}(w_j) + \sum_{i \in A_j} k_i \cdot \mathrm{Re}(a_{ij}) \right)^2 \\
&\geq \frac{\mathrm{Re}(w_j)^2 + \sum_{i \in A_j} k_i^2 \mathrm{Re}(a_{ij})^2}{|A_j| + 1} \\
&\geq \left( \frac{\mathrm{Re}(w_j) + \sum_{i \in A_j} k_i \mathrm{Re}(a_{ij})}{|A_j| + 1} \right)^2 \\
&\geq \left( \frac{\mathrm{Re}(w_j) + c \sum_{i \in A_j} k_i}{|A_j| + 1} \right)^2 \\
&\geq \left( \frac{\mathrm{Re}(w_j) + c \sqrt{\sum_{i \in A_j} k_i^2}}{|A_j| + 1} \right)^2 .
\end{aligned}$$

From here we conclude that if $\mathrm{Re}(s_j) \geq 0 \ \forall j \in \{1, \ldots, n\}$ and $|\mathrm{Im}(s_j)|$ is contained in a compact set, then

$$\begin{aligned}
&\sum_{k_1, \ldots, k_N = 0}^{\infty} \prod_{j=1}^{n} |(w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})^{-s_j}| \\
&\quad = \sum_{k_1, \ldots, k_N = 0}^{\infty} \prod_{j=1}^{n} \left| w_j + \sum_{i \in A_j} k_i a_{ij} \right|^{-\mathrm{Re}(s_j)} \cdot \exp \left( \arg \left( w_j + \sum_{i \in A_j} k_i a_{ij} \right) \cdot \mathrm{Im}(s) \right) \\
&\quad \ll \prod_{j=1}^{n} \sum_{\substack{k_1, \ldots, k_N = 0 \\ k_i \geq 0 \text{ for } i \in A_j}} \left| w_j + \sum_{i \in A_j} k_i a_{ij} \right|^{-\mathrm{Re}(s_j)} \\
&\quad \leq \prod_{j=1}^{n} \sum_{\substack{k_1, \ldots, k_N = 0 \\ k_i \geq 0 \text{ for } i \in A_j}} \left( \frac{\mathrm{Re}(w_j) + c \sqrt{\sum_{i \in A_j} k_i^2}}{|A_j| + 1} \right)^{-\mathrm{Re}(s_j)} ,
\end{aligned}$$

and each series inside the product above clearly converges for $\mathbf{s} \in \mathbb{C}^n$ such that $\mathrm{Re}(s_j) > |A_j|$. We conclude that (1.4) converges absolutely, and therefore it defines an analytic function, for $(\mathbf{s}, w, \mathscr{M}) \in C_{N,n} \times D_n \times \mathscr{D}^*_{N,n}$, where

$$C_{N,n} := \{\mathbf{s} \in \mathbb{C}^n : \mathrm{Re}(s_j) > N \ \forall j \in \{1, \ldots, n\}\},$$

$$D_n := \{w \in \mathbb{C}^n : \mathrm{Re}(w_j) > 0 \ \forall j \in \{1, \ldots, n\}\},$$

and $\mathscr{D}^*_{N,n}$ is the set of $N \times n$ matrices $\mathscr{M} = \{a_{ij}\}$ that satisfy (1.1) and (1.2). In particular, (1.4) converges absolutely for $\mathrm{Re}(s_j) > N$. Taking all $s_j$ equal to $s$ in (1.4) we get (1.3). Therefore (1.3) converges absolutely and defines an analytic function for $(s, w, \mathscr{M}) \in \{s \in \mathbb{C} : \mathrm{Re}(s) > N\} \times D_n \times \mathscr{D}^*_{N,n}$. Replacing the sums by integrals, the same argument shows that (1.5) converges absolutely and defines an analytic function in the same region as $\zeta_{N,n}$.

A remarkable property of $\zeta_{N,n}$ is that they satisfy a recurrence relation: from (1.3) we have

$$\zeta_{N,n}(s, w + a_N \mid a_1, \dots, a_N) = \sum_{k_1,\dots,k_{N-1}=0}^{\infty} \sum_{k_N=1}^{\infty} \prod_{j=1}^{n} (w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})^{-s}.$$

Thus $\zeta_{N,n}$ satisfies

$$\zeta_{N,n}(s, w + a_N \,|\, a_1, \dots, a_N) - \zeta_{N,n}(s, w \,|\, a_1, \dots, a_N) = -\zeta_{N-1,n}(s, w \,|\, a_1, \dots, a_{N-1}), \quad (1.6)$$

where $\zeta_{0,n}(s, w) := \prod_{j=1}^{n} w_j^{-s}$. In Section 2.2 we will use this recurrence relation together with Theorem 2.1 to prove that $\zeta_{N,n}(-k, w, \mathscr{M})$ is a polynomial in $w$ when $k$ is a non-negative integer, and find an upper bound to its degree.

## 1.2. Examples of related zeta functions

In this section we show several examples of zeta functions which are actively studied and are special cases of $\zeta_{N,n}$ and $Z_{N,n}$. We begin with the origin of $\zeta_{N,n}$:

**Example 1.1** (Shintani, cf. [Shi76]). If $W(x) := \sum_{i=1}^{N} x_i a_i \in \mathbb{C}^n$ then

$$\zeta_{N,n}(s, W(x), \mathscr{M}) = \sum_{k_1,\dots,k_N=0}^{\infty} \prod_{j=1}^{n} \left( \sum_{i=1}^{N} a_{ij}(k_i + x_i) \right)^{-s}$$

gives the Shintani zeta function, originally of interest in Number Theory.

**Example 1.2** (Barnes, cf. [Bar04]). If $n = 1$, and letting superscript $^T$ stand for the transpose, then

$$\zeta_{N,1}(s, w, (a_1, \dots, a_N)^T) = \sum_{k_1,\dots,k_N=0}^{\infty} (w + k_1 a_1 + \cdots + k_N a_N)^{-s}$$

gives the Barnes zeta function.

**Example 1.3** (multiple zeta/Euler-Zagier zeta). If $N = n$, $\mathbf{s} = (s_1, \dots, s_N)$ and $\mathscr{M} = \{a_{ij}\}$ is the $N \times N$ matrix with $a_{ij} = 1$ if $i \geq j$ and $a_{ij} = 0$ if $i < j$, then

$$\begin{aligned} Z_{N,N}(\mathbf{s}, (1, \dots, N), \mathscr{M}) &= \sum_{k_1,\dots,k_n=0}^{\infty} (k_1 + 1)^{-s_1} (k_1 + k_2 + 2)^{-s_2} \cdots (k_1 + \cdots + k_N + N)^{-s_N} \\ &= \sum_{1 \leq k_1 < k_2 < \cdots < k_N} k_1^{-s_1} k_2^{-s_2} \cdots k_N^{-s_N} =: \zeta(s_1, \dots, s_N) \end{aligned}$$

gives the multiple zeta function, also called Euler-Zagier zeta function by some authors.

There is an extensive literature about this function, including its analytic continuation, special values, which are often called Multiple Zeta Values (MZV), and its decomposition into simpler zeta functions. For example, Euler proved the decomposition formula

$$2\zeta(1, n) = n\zeta(n + 1) - \sum_{i=1}^{n-2} \zeta(n - i)\zeta(i + 1) \quad \text{for } n \geq 2.$$

For further details see [BGF, Mat02, Mat06, Zha16].

**Example 1.4** (Mordell-Tornheim, cf. [Mor58, Tor50]). If $N = 2$, $n = 3$, $\mathbf{s} = (s_1, s_2, s_3)$ and $\mathscr{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ then

$$\begin{aligned} Z_{2,3}(\mathbf{s}, (1,1,2), \mathscr{M}) &= \sum_{k_1,k_2=0}^{\infty} (k_1+1)^{-s_1}(k_2+1)^{-s_2}(k_1+k_2+2)^{-s_3} \\ &= \sum_{l,k=1}^{\infty} l^{-s_1}k^{-s_2}(l+k)^{-s_3} =: \zeta_{MT}(s_1, s_2, s_3) \end{aligned}$$

gives the Mordell-Tornheim zeta function, and $\zeta_{MT}(s,s,s) = 2^{-s}\zeta_{\mathfrak{sl}(3)}(s)$ (see (0.2)). As we mentioned earlier, this function was studied in [KO13], but we would also like to mention the work of K. Onodera [Ono14] who calculated the values at all integers (along with the corresponding derivatives at non-positive integers), and D. Romik [Rom17] who calculated the exact pole locations and calculated their corresponding residues.

## 1.3. Witten zeta function

Given $\mathfrak{g}$ a complex semi-simple Lie algebra, we have

$$\zeta_{\mathfrak{g}}^{W}(s) = \sum_{\rho} \frac{1}{(\dim \rho)^s}$$

where the sum runs over all isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{g}$. In this section we will find a simpler formula for $\zeta_{\mathfrak{g}}^{W}(s)$, and calculate it in terms of explicit series in the case that $\mathfrak{g}$ is one of the classical Lie algebras or the exceptional type $G_2$. All the notation and results used in this section will be based in [Bou02, Chapter I, §1 and §4] unless otherwise stated.

Let $N$ be the rank of $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{g})$ the set of all roots of $\mathfrak{g}$, $\Delta^+ = \Delta^+(\mathfrak{g})$ the set of positive roots, and $\Psi = \Psi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_N\}$ the fundamental system of $\Delta$. Let $\lambda_1, \dots, \lambda_N$ be the fundamental weights satisfying $(\lambda_i, \alpha_j) = \delta_{ij}$ (Kronecker's delta), where $(\cdot,\cdot)$ is the usual Euclidean inner product. Any dominant weight can be written as

$$\lambda = l_1\lambda_1 + \dots + l_N\lambda_N, \quad \text{with } l_1, \dots, l_N \in \mathbb{N}_0. \tag{1.7}$$

Let

$$\rho = \frac{1}{2}\sum_{\beta\in\Delta^+} \beta = \lambda_1 + \dots + \lambda_N,$$

and also let $V_\lambda$ be the representation space corresponding to the dominant weight $\lambda$. Using Weyl's dimension formula (see for example [Hal15, Theorem 10.18]), we get

$$\begin{aligned} \dim(V_\lambda) &= \prod_{\beta\in\Delta^+} \frac{(\beta, \lambda+\rho)}{(\beta,\rho)} \\ &= \prod_{\beta\in\Delta^+} \frac{(\beta, (l_1+1)\lambda_1 + \dots + (l_N+1)\lambda_N)}{(\beta, \lambda_1 + \dots + \lambda_N)}. \end{aligned}$$

Hence, writing $k_i = l_i + 1$ and summing over all dominant weights of the form (1.7) we have

$$\begin{aligned}\zeta_{\mathfrak{g}}^W(s) &= \sum_{\lambda} \prod_{\beta\in\Delta^+} \left(\frac{(\beta, k_1\lambda_1 + \cdots + k_N\lambda_N)}{(\beta, \lambda_1 + \cdots + \lambda_N)}\right)^{-s} \\ &= K(\mathfrak{g})^s \sum_{k_1,\ldots,k_N=1}^{\infty} \prod_{\beta\in\Delta^+} (\beta, k_1\lambda_1 + \cdots + k_N\lambda_N)^{-s},\end{aligned} \tag{1.8}$$

where

$$K(\mathfrak{g}) = \prod_{\beta\in\Delta^+} (\beta, \lambda_1 + \cdots + \lambda_N).$$

Now, it follows from the definitions of fundamental system and positive root that every $\beta \in \Delta^+$ can be written uniquely as

$$\beta = c(1, \beta)\alpha_1 + \cdots + c(N, \beta)\alpha_N,$$

where $c(i, \beta)$ are non-negative integers, not all zero, thus

$$\prod_{\beta\in\Delta^+} (\beta, k_1\lambda_1 + \cdots + k_N\lambda_N) = \prod_{\beta\in\Delta^+} (c(1, \beta)k_1 + \cdots + c(N, \beta)k_N)$$

and

$$K(\mathfrak{g}) = \prod_{\beta\in\Delta^+} (c(1, \beta) + \cdots + c(N, \beta)).$$

If we now define $n := |\Delta^+|$, we list $\Delta^+ = \{\beta_1, \ldots, \beta_n\}$, define $\mathscr{M} := \{a_{ij}\}$ where $a_{ij} := c(i, \beta_j)$, and $w = (w_1, \ldots, w_n)$ where $w_j := a_{1j} + \cdots + a_{Nj}$, then

$$\zeta_{\mathfrak{g}}^W(s) = K(\mathfrak{g})^s \cdot \zeta_{N,n}(s, w, \mathscr{M}).$$

Thus $\zeta_{\mathfrak{g}}^W$ is a special case of $\zeta_{N,n}$ except for an exponential factor, which is not relevant when studying the poles or zeros.

We will now use (1.8) and the results from [Bou02, Chapter VI, §4] to calculate explicitly $\zeta_{\mathfrak{g}}^W(s)$ for each of the classical Lie algebras, and the exceptional type $G_2$. For the rest of the section $e_i$ will always denote the $i-$th canonical vector ($e_i$ has $l + 1$ coordinates for $A_l$, 3 coordinates for $G_2$, and $l$ coordinates for the remainding cases treated here).

### 1.3.1. Type $A_l$: $\mathfrak{g} = \mathfrak{sl}(l + 1)$ $(l \geq 1)$

Here

$$\Delta = \{e_i - e_j : i \neq j, 1 \leq i \leq l + 1, 1 \leq j \leq l + 1\},$$
$$\Psi = \{\alpha_i := e_i - e_{i+1} : 1 \leq i \leq l\},$$
$$\Delta^+ = \{\alpha_i + \cdots + \alpha_{j-1} : 1 \leq i < j \leq l + 1\},$$

and the corresponding fundamental weights are

$$\lambda_i = e_1 + \cdots + e_i - \frac{i}{l+1}(e_1 + \cdots + e_{l+1}), \quad \text{for } 1 \leq i \leq l,$$

thus

$$\prod_{\beta\in\Delta^+}(\beta, k_1\lambda_1+\cdots+k_l\lambda_l) = \prod_{1\le i<j\le l+1}\sum_{r=1}^{l} k_r(\alpha_i+\cdots+\alpha_{j-1},\lambda_r)$$
$$= \prod_{1\le i<j\le l+1}(k_i+\cdots+k_{j-1}),$$

therefore

$$\zeta_{\mathfrak{g}}^W(s) = K(\mathfrak{g})^s \sum_{k_1,\ldots,k_l=1}^{\infty}\prod_{1\le i<j\le l+1}(k_i+\cdots+k_{j-1})^{-s}.$$

### 1.3.2. Type $B_l$: $\mathfrak{g}=\mathfrak{so}(2l+1)$ $(l\ge 2)$

Here

$$\Delta = \{\pm e_i : 1\le i\le l\}\cup\{\pm e_i\pm e_j : 1\le i<j\le l\},$$
$$\Psi = \{\alpha_i := e_i - e_{i+1} : 1\le i\le l-1\}\cup\{\alpha_l := e_l\},$$

$$\Delta^+ = \{\alpha_i+\cdots+\alpha_l : 1\le i\le l\}\cup\{\alpha_i+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_l : 1\le i<j\le l\}$$
$$\cup\{\alpha_i+\cdots+\alpha_{j-1} : 1\le i<j\le l\},$$

and the corresponding fundamental weights are

$$\lambda_i = e_1+\cdots+e_i,\quad \text{for } 1\le i\le l-1,\quad \text{and}\quad \lambda_l = \frac{1}{2}\left(e_1+\cdots+e_l\right).$$

Now we get

$$\zeta_{\mathfrak{g}}^W(s) = K(\mathfrak{g})^s\sum_{k_1,\ldots,k_l=1}^{\infty}\left(\prod_{1\le i<j\le l}(k_i+\cdots+k_{j-1}+2k_j+\cdots+2k_l)^{-s}\right.$$
$$\left.\cdot\prod_{i=1}^{l}(k_i+\cdots+k_l)^{-s}\cdot\prod_{1\le i<j\le l}(k_i+\cdots+k_{j-1})^{-s}\right).$$

### 1.3.3. Type $C_l$: $\mathfrak{g}=\mathfrak{sp}(2l)$ $(l\ge 2)$

Here

$$\Delta = \{\pm 2e_i : 1\le i\le l\}\cup\{\pm e_i\pm e_j : 1\le i<j\le l\},$$
$$\Psi = \{\alpha_i := e_i - e_{i+1} : 1\le i\le l-1\}\cup\{\alpha_l := 2e_l\},$$

$$\Delta^+ = \{\alpha_i+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_{l-1}+\alpha_l : 1\le i<j\le l\}$$
$$\cup\{\alpha_i+\cdots+\alpha_{j-1} : 1\le i<j\le l\}\cup\{2\alpha_i+\cdots+2\alpha_{l-1}+\alpha_l : 1\le i\le l\},$$

and the corresponding fundamental weights are

$$\lambda_i = e_1+\cdots+e_i,\quad \text{for } 1\le i\le l.$$

Now we get

$$\zeta_{\mathfrak{g}}^W(s) = K(\mathfrak{g})^s \sum_{k_1,\dots,k_l=1}^{\infty} \left( \prod_{1\le i<j\le l} (k_i+\cdots+k_{j-1}+2k_j+\cdots+2k_{l-1}+k_l)^{-s} \right.$$
$$\left. \cdot \prod_{1\le i<j\le l} (k_i+\cdots+k_{j-1})^{-s} \cdot \prod_{1\le i<j\le l} (2k_i+\cdots+2k_{l-1}+k_l)^{-s} \right).$$

### 1.3.4. Type $D_l$: $\mathfrak{g}=\mathfrak{so}(2l)$ $(l\ge 3)$

Here

$$\Delta = \{\pm e_i \pm e_j : 1\le i<j\le l\},$$
$$\Psi = \{\alpha_i := e_i - e_{i+1} : 1 \le i \le l-1\} \cup \{\alpha_l := e_{l-1}+e_l\},$$

$$\begin{aligned}\Delta^+ = &\{\alpha_i+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l : 1\le i<j\le l-2\}\\ &\cup\{\alpha_i+\cdots+\alpha_{j-1} : 1\le i<j\le l\}\cup\{\alpha_i+\cdots+\alpha_l : 1\le i\le l-2\}\\ &\cup\{\alpha_i+\cdots+\alpha_{l-2}+\alpha_l : 1\le i\le l-2\}\cup\{\alpha_l\},\end{aligned}$$

and the corresponding fundamental weights are

$$\lambda_i = e_1+\cdots+e_i, \quad \text{for } 1\le i\le l-2,$$
$$\lambda_{l-1} = \frac{1}{2}(e_1+\cdots+e_{l-1}-e_l) \quad \text{and} \quad \lambda_l = \frac{1}{2}(e_1+\cdots+e_l).$$

Now we get

$$\zeta_{\mathfrak{g}}^W(s) = K(\mathfrak{g})^s \sum_{k_1,\dots,k_l=1}^{\infty} \left( \prod_{1\le i<j\le l-2} (k_i+\cdots+k_{j-1}+2k_j+\cdots+2k_{l-2}+k_{l-1}+k_l)^{-s} \right.$$
$$\cdot k_l^{-s} \cdot \prod_{1\le i\le l-2} (k_i+\cdots+k_{l-2}+k_l)^{-s} \cdot \prod_{1\le i<j\le l} (k_i+\cdots+k_{j-1})^{-s}$$
$$\left. \cdot \prod_{1\le i\le l-2} (k_i+\cdots+k_l)^{-s} \right).$$

### 1.3.5. Type $G_2$

Here

$$\Delta = \{\pm(e_i-e_j) : 1\le i<j\le 3\}\cup\{\pm(2e_1-e_2-e_3), \pm(2e_2-e_1-e_3), \pm(2e_3-e_1-e_2)\},$$
$$\Psi = \{\alpha_1 := e_1-e_2, \alpha_2 := -2e_1+e_2+e_3\},$$
$$\Delta^+ = \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\},$$

and the corresponding fundamental weights are

$$\lambda_1 = -e_2+e_3, \quad \lambda_2 = -e_1-e_2+2e_3.$$

Now we get

$$K(\mathfrak{g}) = 1(1+1)(2+1)(3+1)(3+2) = 120$$

and

$$\zeta_{\mathfrak{g}}^{W}(s) = 120^s \sum_{k_1,k_2=1}^{\infty} k_1^{-s}(k_1+k_2)^{-s}(2k_1+k_2)^{-s}(3k_1+k_2)^{-s}(3k_1+2k_2)^{-s}.$$

Similarly, $\zeta_{\mathfrak{g}}^{W}(s)$ can be calculated for the remaining exceptional types ($F_4$, $E_6$, $E_7$ and $E_8$), but we do not include the calculations since the resulting expressions are much larger.

Chapter 2

# Main result and consequences

## 2.1. Notation

We begin introducing the notation that will be used for the rest of the Chapter.

Given a matrix $\mathscr{M}$ as in Chapter 1, let $z = z(\mathscr{M})$ be the maximum number of zeros in any one row of $\mathscr{M}$, and let $\mathrm{Inj}(z+1, n)$ be the set of injective functions from $\{1, \ldots, z+1\}$ to $\{1, \ldots, n\}$. For non-zero $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$ we define the order of $v$ by

$$j = \mathrm{ord}(v) \iff v_i = 0 \quad \forall i < j \text{ and } v_j \neq 0,$$

and for each $j \in \{1, \ldots, z+1\}$ let

$$F_j(\mathscr{M}) := \{i \in \{1, \ldots, N\} : \mathrm{ord}(a_i) = j\}.$$

In summary, for each $\tau \in \mathrm{Inj}(z+1, n)$ we have $z+1$ numbers from 1 to $n$, and we think of $\tau$ as being an ordering of these numbers: $\tau(1)$ being the first, $\tau(2)$ second, etc. With this in mind, $F_j(\mathscr{M})$ tells which rows of $\mathscr{M}$ are such that the first $j-1$ entries are zero and the $j-$th entry is non-zero.

The following definitions will be useful to simplify the notation of Theorem 2.1, since they are what is obtained naturally from the study of the analytic continuation. For each $\tau \in \mathrm{Inj}(z+1, n)$ and $v \in \mathbb{C}^n$ we define $v^\tau := (v_{\tau(1)}, \ldots, v_{\tau(z+1)}, v_{q^\tau_{z+2}}, \ldots, v_{q^\tau_n})$, where $q^\tau_{z+2}, \ldots, q^\tau_n$ are the remaining indices in increasing order. We use this to define, for $k \in \{n-z, \ldots, n\}$, the set

$$\begin{aligned}\mathrm{Ap}(k) := \{\{p_1, \ldots, p_k\} \subseteq \{1, \ldots, n\} : \exists \tau \in \mathrm{Inj}(z+1, n) \text{ s.t.}\\ \{q^\tau_{z+2}, \ldots, q^\tau_n, \tau(n-k+1), \tau(n-k+2), \ldots, \tau(z+1)\} = \{p_1, \ldots, p_k\}\}.\end{aligned}$$

Given $\mathscr{M}$ a matrix with rows $a_i$ and $\tau \in \mathrm{Inj}(z+1, n)$, let $\mathscr{M}^\tau$ denote the matrix with rows $a_i^\tau$, and use this to define, for $r \in \{n-z, \ldots, n\}$ and $\{p_1, \ldots, p_r\} \in \mathrm{Ap}(r)$, the numbers

$$m(p_1, \ldots, p_r) := \max_{\substack{\tau \in \mathrm{Inj}(z+1,n) \text{ s.t.} \\ \{q^\tau_{z+2}, \ldots, q^\tau_n, \tau(n-r+1), \tau(n-r+2), \ldots, \tau(z+1)\} = \{p_1, \ldots, p_r\}}} \left\{ \sum_{j=n-r+1}^{z+1} |F_j(\mathscr{M}^\tau)| \right\},$$

and

$$m_r := \max_{\tau \in \mathrm{Inj}(z+1,n)} \left\{ \sum_{j=n-r+1}^{z+1} |F_j(\mathscr{M}^\tau)| \right\}.$$

Lastly, from now on empty sums (products) are always taken to be 0 (1, respectively).

## 2.2. Analytic continuation of $\zeta_{N,n}$, $Z_{N,n}$ and $\mathcal{Z}_{N,n}$

Our main result is the following Theorem, which generalizes [FR04, Proposition 2.1], [Mat03, Theorem 3] and is the analog of [FP12, Theorem 3] but for a different class of functions:

**Theorem 2.1.** The functions

$$\frac{\zeta_{N,n}(s,w,\mathscr{M})\cdot\Gamma(s)^{z+1}}{\Gamma\left(ns-N\right)\cdot\prod_{k=n-z}^{n-1}\Gamma(ks-m_k)}\quad\text{and}\quad\frac{\mathcal{Z}_{N,n}(s,w,\mathscr{M})\cdot\Gamma(s)^{z+1}}{\Gamma\left(ns-N\right)\cdot\prod_{k=n-z}^{n-1}\Gamma(ks-m_k)}$$

extend to analytic functions on $\mathbb{C}\times D_n\times\mathscr{D}_{N,n}^*$. In particular, for fixed $(w,\mathscr{M})\in D_n\times\mathscr{D}_{N,n}^*$, the functions $s\mapsto\zeta_{N,n}(s,w,\mathscr{M})$ and $s\mapsto\mathcal{Z}_{N,n}(s,w,\mathscr{M})$ are meromorphic and have at most poles of order $z+1$, which can only occur among the rational numbers of the form

$$s=\frac{m_v-l}{v},\quad\text{with } v\in\{n-z,\ldots,n\}\quad\text{and}\quad l\in\mathbb{N}_0\setminus\{m_v+kv:k\in\mathbb{N}_0\},$$

and have no poles for $s=-k$ with $k\in\mathbb{N}_0$. Similarly, the function

$$\frac{Z_{N,n}(\mathbf{s},w,\mathscr{M})\cdot H(\mathbf{s},\mathscr{M})}{\Gamma\left(-N+\sum_{k=1}^n s_k\right)\cdot\prod_{k=n-z}^{n-1}\prod_{\{p_1,\ldots,p_k\}\in\mathrm{Ap}(k)}\Gamma\left(-m(p_1,\ldots,p_k)+\sum_{j=1}^k s_{p_j}\right)}$$

extends to an analytic function on $\mathbb{C}^n\times D_n\times\mathscr{D}_{N,n}^*$, where

$$H(\mathbf{s},\mathscr{M}):=\begin{cases}\prod_{j=1}^n\Gamma(s_j) & \text{if } z=n-1\\ 1 & \text{if } z\neq n-1\end{cases}.$$

The case $z=0$ (i.e. $\mathrm{Re}(a_{ij})>0$ for all $i,j$) is precisely [FR04, Proposition 2.1], and the case $w_j=\sum_{i=1}^N a_{ij}$ is precisely [Mat03, Theorem 3]. Also note that [FP12, Theorem 3] does not apply here because $\prod_{j=1}^n(w_j+x_1a_{1j}+\cdots+x_Na_{Nj})$ does not necessarily satisfy Mahler's Hypothesis (see [FP12, p. 5]), but we manage to get the same conclusion. Finally, we would like to mention that the existence of meromorphic continuation of $\zeta_{N,n}(s,w,\mathscr{M})$ can be deduced from D. Essouabri's work [Ess97, Théorème 2], but our results are simpler to prove and give more detailed information about the poles that cannot be deduced from his work. For example, the fact that $s=-k$ is always a regular value of $\zeta_{N,n}(s,w,\mathscr{M})$ for all $k\in\mathbb{N}_0$ does not seem to follow from [Ess97].

We will first prove the following Lemma:

**Lemma 2.2.** Let $\mathrm{Re}(s_j)>N$, $\mathrm{Re}(w_j)>0$ for $j\in\{1,\ldots,n\}$, and

$$I(\mathbf{s},w,\mathscr{M}):=\int_E\frac{\prod_{j=1}^n e^{-w_jT_j}}{\prod_{i=1}^N\left(\sum_{j=1}^n a_{ij}T_j\right)}f(T)\prod_{j=1}^n T_j^{s_j-1}\,dT,\tag{2.1}$$

where

$$E:=\{T\in(0,\infty)^n:T_1\geq T_2\geq\ldots\geq T_{z+1}\ \text{ and }\ T_{z+1}\geq T_l\ \forall l\geq z+2\},$$

and

$$f(T):=\prod_{i=1}^N\varphi\left(\sum_{j=1}^n a_{ij}T_j\right)\quad,\quad\varphi(r):=\frac{r}{1-e^{-r}}.\tag{2.2}$$

Then the function

$$\frac{I(\mathbf{s},w,\mathscr{M})}{\Gamma\left(-N+\sum_{j=1}^{n}s_j\right)\cdot\prod_{l=z+2}^{n}\Gamma(s_l)\cdot\prod_{k=2}^{z+1}\Gamma\left(\sum_{j=k}^{n}s_j-\sum_{j=k}^{z+1}|F_j(\mathscr{M})|\right)}$$

extends to an analytic function on $\mathbb{C}^n\times D_n\times\mathscr{D}^*_{N,n}$.

**Proof.** For $T\in E$ we set $\sigma_1:=T_1$, $\sigma_k:=T_k/T_{k-1}$ for $k\in\{2,\dots,z+1\}$, and $\mu_l:=T_l/T_{z+1}$ for $l\in\{z+2,\dots,n\}$. These range over $\sigma_1\in(0,\infty)$, $\sigma_j\in(0,1)$ for $j\in\{2,\dots,z+1\}$ and $\mu_l\in(0,1)$ for $l\in\{z+2,\dots,n\}$. The coordinates $T_k$ can be written as

$$T_k=\prod_{j=1}^{k}\sigma_j\quad\text{for } k\in\{1,\dots,z+1\},$$

and

$$T_l=\mu_l\cdot\prod_{j=1}^{z+1}\sigma_j\quad\text{for } l\in\{z+2,\dots,n\}.$$

Using the new coordinates we have

$$\prod_{j=1}^{n}T_j^{s_j-1}=\prod_{j=1}^{z+1}\sigma_j^{\sum_{k=j}^{n}(s_k-1)}\cdot\prod_{l=z+2}^{n}\mu_l^{s_l-1},$$

and $\frac{\partial T_i}{\partial\sigma_j}=\frac{\partial T_i}{\partial\mu_j}=0$ for $i<j$. Therefore the corresponding Jacobian matrix is lower triangular, and the Jacobian determinant equals

$$\prod_{j=1}^{z+1}\frac{\partial T_j}{\partial\sigma_j}\cdot\prod_{l=z+2}^{n}\frac{\partial T_l}{\partial\mu_l}=\prod_{j=1}^{z+1}\left(\prod_{k=1}^{j-1}\sigma_k\right)\cdot\prod_{l=z+2}^{n}\left(\prod_{j=1}^{z+1}\sigma_j\right)=\prod_{j=1}^{z+1}\sigma_j^{n-j}.$$

Lastly, denoting $F_j:=F_j(\mathscr{M})$, note that for fixed $j\in\{1,\dots,z+1\}$ we have

$$\prod_{i\in F_j}\left(\sum_{j=1}^{n}a_{ij}T_j\right)=\prod_{i\in F_j}\left(a_{ij}T_j+\sum_{k=j+1}^{n}a_{ik}T_k\right)=T_j^{|F_j|}\cdot\prod_{i\in F_j}\left(a_{ij}+\sum_{k=j+1}^{n}a_{ik}\frac{T_k}{T_j}\right),$$

thus, considering $\sigma':=(\sigma_2,\dots,\sigma_{z+1})$ and $\mu:=(\mu_{z+2},\dots,\mu_n)$, we get

$$\begin{aligned}\prod_{i=1}^{N}\left(\sum_{j=1}^{n}a_{ij}T_j\right)&=\prod_{j=1}^{z+1}\left(\prod_{i\in F_j}\left(\sum_{k=1}^{n}a_{ik}T_k\right)\right)\\&=\prod_{j=1}^{z+1}T_j^{|F_j|}\cdot\prod_{j=1}^{z+1}\left(\prod_{i\in F_j}\left(a_{ij}+\sum_{k=j+1}^{n}a_{ik}\frac{T_k}{T_j}\right)\right)\\&=\prod_{j=1}^{z+1}\sigma_j^{\sum_{k=j}^{z+1}|F_k|}\cdot\underbrace{\prod_{j=1}^{z+1}\left(\prod_{i\in F_j}\left(a_{ij}+\sum_{k=j+1}^{z+1}a_{ik}\prod_{r=j+1}^{k}\sigma_r+\prod_{r=j+1}^{z+1}\sigma_r\sum_{l=z+2}^{n}a_{il}\mu_l\right)\right)}_{y(\sigma',\mu)},\end{aligned}$$

where we have used that $\{1, \dots, N\}$ is the disjoint union of the $F_j$'s. From the previous computations, after simplifying, (2.1) becomes

$$I(\mathbf{s}, w, \mathscr{M}) = \int_{\sigma_1=0}^{\infty} \sigma_1^{-N-1+\sum_{j=1}^{n} s_j} \cdot e^{-\sigma_1 w_1} \int_{\sigma',\mu} g(\sigma_1, \sigma', \mu) \cdot p(\mathbf{s}, \sigma', \mu)\, d\sigma'\, d\mu\, d\sigma_1, \quad (2.3)$$

where

$$g(\sigma_1, \sigma', \mu) := \frac{\tilde{f}(\sigma_1, \sigma, \mu) \cdot \prod_{j=2}^{z+1} e^{-w_k \prod_{k=1}^{j} \sigma_k} \cdot \prod_{l=z+2}^{n} e^{-w_l \mu_l \prod_{j=1}^{z+1} \sigma_j}}{y(\sigma', \mu)},$$

with

$$\tilde{f}(\sigma_1, \sigma', \mu) := \prod_{i=1}^{N} \varphi \left( \sum_{j=1}^{z+1} a_{ij} \prod_{k=1}^{j} \sigma_k + \prod_{j=1}^{z+1} \sigma_j \cdot \sum_{l=z+2}^{n} a_{il} \mu_l \right),$$

$$\int_{\sigma',\mu} := \int_{\sigma_2=0}^{1} \cdots \int_{\sigma_{z+1}=0}^{1} \int_{\mu_{z+2}=0}^{1} \cdots \int_{\mu_n=0}^{1}, \quad d\sigma' := d\sigma_2 \cdots d\sigma_{z+1}, \quad d\mu := d\mu_{z+2} \cdots d\mu_n,$$

and

$$p(\mathbf{s}, \sigma', \mu) := \prod_{j=2}^{z+1} \sigma_j^{-1+\sum_{k=j}^{n} s_k - \sum_{k=j}^{z+1} |F_k|} \cdot \prod_{l=z+2}^{n} \mu_l^{s_l - 1}. \quad (2.4)$$

Note that $g$ is infinitely differentiable in $(\sigma_1, \sigma', \mu)$ on an open neighborhood of $[0, \infty) \times [0,1]^z \times [0,1]^{n-z-1}$ because the numerator is, and the denominator does not vanish because of condition (1.2) and the fact that $a_{ij} \neq 0$ for $j \in \{1, \dots, n\}$, $i \in F_j$. Now, using integration by parts over $\mu_n$ in (2.3), for $\mathrm{Re}(s_j) > N$ we get

$$\begin{aligned}
&\int_{\mu_n=0}^{1} g(\sigma_1, \sigma', \mu) \cdot \mu_n^{s_n - 1}\, d\mu_n \\
&= \frac{g(\sigma_1, \sigma', \mu_{z+2}, \dots, \mu_{n-1}, 1)}{s_n} - \frac{1}{s_n} \int_{\mu_n=0}^{1} \frac{\partial g}{\partial \mu_n}(\sigma_1, \sigma', \mu) \cdot \mu_n^{s_n}\, d\mu_n \\
&= \frac{1}{s_n} \int_{\mu_n=0}^{1} \left( (s_n + 1) g(\sigma_1, \sigma', \mu_{z+2}, \dots, \mu_{n-1}, 1) - \frac{\partial g}{\partial \mu_n}(\sigma_1, \sigma', \mu) \right) \cdot \mu_n^{s_n}\, d\mu_n \\
&= \frac{1}{s_n} \int_{\mu_n=0}^{1} g_0(\mathbf{s}, \sigma_1, \sigma', \mu) \cdot \mu_n^{s_n}\, d\mu_n,
\end{aligned}$$

with the obvious definition of $g_0$. Repeating the integration by parts $M$ more times we get

$$\int_{\mu_n=0}^{1} g(\sigma_1, \sigma', \mu) \cdot \mu_n^{s_n - 1}\, d\mu_n = \left( \prod_{p=0}^{M} \frac{1}{s_n + p} \right) \cdot \int_{\mu_n=0}^{1} g_M(\mathbf{s}, \sigma_1, \sigma', \mu) \cdot \mu_n^{s_n + M}\, d\mu_n,$$

where $g_M$ is a finite sum of $\mu_n-$derivatives of $g$ and some specializations of them at $\mu_n = 1$, with coefficients which are monomials in $s_n$. The same procedure applied to the remaining $\mu$'s and $\sigma$'s replaces each $\mu_l^{s_l - 1}$ in (2.4) by $\mu_l^{s_l + M}$. Repeating the integration by parts over $\sigma_j$ $(2 \leq j \leq z+1)$ enough times it also replaces each

$$\sigma_j^{-1+\sum_{k=j}^{n} s_k - \sum_{k=j}^{z+1} |F_k|}$$

in (2.4) by

$$\sigma_j^{M+\sum_{k=j}^{n} s_k}.$$

We conclude that

$$I = T_M(\mathbf{s}) \int_{\sigma_1=0}^{\infty} \sigma_1^{-N-1+\sum_{j=1}^n s_j} \cdot e^{-\sigma_1 w_1} \int_{\sigma',\mu} g_*(\mathbf{s},\sigma_1,\sigma',\mu)\cdot p_M(\mathbf{s},\sigma',\mu)\, d\sigma'\, d\mu\, d\sigma_1, \tag{2.5}$$

where

$$T_M(\mathbf{s}) := \prod_{l=z+2}^{n} \prod_{p=0}^{M} \frac{1}{s_l+p} \cdot \prod_{j=2}^{z+1} \prod_{p=-\sum_{k=j}^{z+1}|F_k|}^{M} \frac{1}{p+\sum_{k=j}^n s_k}, \tag{2.6}$$

$$p_M(\mathbf{s},\sigma',\mu) := \prod_{j=2}^{z+1} \sigma_j^{M+\sum_{k=j}^n s_k} \cdot \prod_{l=z+2}^{n} \mu_l^{s_l+M}, \tag{2.7}$$

and

$$g_*(\mathbf{s},\sigma_1,\sigma',\mu) = \sum_u c_u(\mathbf{s}) d_u(\sigma_1,\sigma',\mu) \tag{2.8}$$

is again a finite sum, $c_u$ are monomials in $\mathbf{s}$ and $d_u$ are partial derivatives of $g$ with respect to some $\sigma_j$ and $\mu_l$, and some specializations of these at 1.

Now, we use the finite Taylor expansion of $d_u$ centered at $\sigma_1 = 0$ with the integral form of the remainder, to obtain

$$d_u(\sigma_1,\sigma',\mu) = \sum_{l=0}^{M-1} \underbrace{\frac{1}{l!}\frac{\partial^l d_u}{\partial \sigma_1^l}(0,\sigma',\mu)}_{b_{u,l}(\sigma',\mu)} \cdot \sigma_1^l + \sigma_1^M \cdot \underbrace{\frac{1}{M!}\int_0^1 \frac{\partial^M d_u}{\partial \sigma_1^M}(\sigma_1 y,\sigma',\mu)\cdot(1-y)^{M-1}\, dy}_{r_{u,M}(\sigma_1,\sigma',\mu)}, \tag{2.9}$$

and is clear from their definition that $b_{u,l}$ is bounded and $|r_{u,M}|$ is bounded above for $(\sigma_1,\sigma',\mu) \in [0,\infty) \times [0,1]^z \times [0,1]^{n-z-1}$ by a polynomial in $\sigma_1$.

Replacing (2.9) in (2.8), and then all this into (2.5), we get

$$\begin{aligned} I = T_M(\mathbf{s}) \sum_u c_u(\mathbf{s}) \Bigg( & \sum_{l=0}^{M-1} \frac{\Gamma(-N+l+\sum_{j=1}^n s_j)}{w_1^{-N+l+\sum_{j=1}^n s_j}} \int_{\sigma',\mu} b_{u,l}(\sigma',\mu)\cdot p_M(\mathbf{s},\sigma',\mu)\, d\sigma'\, d\mu \\ & + \int_{\sigma_1=0}^{\infty} \int_{\sigma',\mu} e^{-\sigma_1 w_1} \cdot \sigma_1^{M-N-1+\sum_{j=1}^n s_j} \cdot r_{u,M}(\sigma_1,\sigma',\mu) \cdot p_M(\mathbf{s},\sigma',\mu)\, d\sigma'\, d\mu\, d\sigma_1 \Bigg). \end{aligned} \tag{2.10}$$

From the definition (2.6) of $T_M(\mathbf{s})$ we have that

$$\frac{T_M(\mathbf{s})}{\prod_{l=z+2}^n \Gamma(s_l) \cdot \prod_{k=2}^{z+1} \Gamma\left(\sum_{j=k}^n s_j - \sum_{j=k}^{z+1} |F_j|\right)}$$

is analytic, and together with (2.7) and (2.10) we conclude that

$$\frac{I(\mathbf{s},w,\mathscr{M})}{\Gamma\left(-N+\sum_{j=1}^n s_j\right) \cdot \prod_{l=z+2}^n \Gamma(s_l) \cdot \prod_{k=2}^{z+1} \Gamma\left(\sum_{j=k}^n s_j - \sum_{j=k}^{z+1} |F_j|\right)}$$

extends analytically to the domain given by

$$\sum_{k=j}^n \mathrm{Re}(s_k) > -M-1 \text{ for } j \in \{2,\dots,z+1\},\ \sum_{j=1}^n \mathrm{Re}(s_j) > N-M,$$

$$\mathrm{Re}(s_l) > -M-1 \text{ for } l \in \{z+2,\dots,n\} \quad \text{and} \quad (w,\mathscr{M}) \in D_n \times \mathscr{D}_{N,n}^*.$$

As $M$ is arbitrary we get the desired result. □

**Proof of Theorem 2.1.** We follow [FR04] closely. Using the integral definition of the $\Gamma-$function and the absolute convergence of $Z_{N,n}(\mathbf{s}, w, \mathscr{M})$ for $\mathbf{s} \in C_{N,n}$ and $\mathrm{Re}(s_j) > 0$, we have

$$Z_{N,n}(\mathbf{s}, w, \mathscr{M}) \prod_{j=1}^{n} \Gamma(s_j) = \int_{[0,\infty)^n} h(t) \prod_{j=1}^{n} t_j^{s_j - 1} \, dt, \tag{2.11}$$

where

$$\begin{aligned} h(t) :=& \sum_{k_1,\ldots,k_N=0}^{\infty} \prod_{j=1}^{n} \exp(-t_j(w_j + k_1 a_{1j} + \cdots + k_N a_{Nj})) \\ =& \prod_{j=1}^{n} e^{-w_j t_j} \cdot \prod_{i=1}^{N} \left( \sum_{k_i=0}^{\infty} \exp(-k_i(a_{i1}t_1 + \cdots + a_{in}t_n) \right) \\ =& \frac{\prod_{j=1}^{n} e^{-w_j t_j}}{\prod_{i=1}^{N} \left( \sum_{j=1}^{n} a_{ij} t_j \right)} f(t), \end{aligned}$$

since $\mathrm{Re}(\sum_{j=1}^{n} a_{ij} t_j) > 0$ except on a subset of measure 0 of $[0, \infty)^n$, and $f(t)$ is given by (2.2). For each $\tau \in \mathrm{Inj}(z+1, n)$ let

$$E_\tau := \{t \in [0,\infty)^n : t_{\tau(1)} \geq t_{\tau(2)} \geq \ldots \geq t_{\tau(z+1)} \text{ and } t_{\tau(z+1)} \geq t_{q_j^\tau} \ \forall j \in \{z+2, \ldots, n\}\},$$

where $q_j^\tau$ is defined as in Section 2.1. Write

$$[0, \infty)^n = \bigcup_{\tau \in \mathrm{Inj}(z+1,n)} E_\tau,$$

where the union is disjoint up to sets of measure 0. With this, (2.11) becomes

$$Z_{N,n}(\mathbf{s}, w, \mathscr{M}) \prod_{j=1}^{n} \Gamma(s_j) = \sum_{\tau \in \mathrm{Inj}(z+1,n)} \int_{E_\tau} h(t) \prod_{k=1}^{n} t_k^{s_k - 1} \, dt =: \sum_{\tau \in \mathrm{Inj}(z+1,n)} I_\tau(\mathbf{s}, w, \mathscr{M}). \tag{2.12}$$

For $t \in E_\tau$ set $T_k = t_{\tau(k)}$ for $k \in \{1, \ldots, z+1\}$ and $T_l = t_{q_l^\tau}$ for $l \in \{z+2, \ldots, n\}$, so that the corresponding Jacobian determinant is clearly 1. For each $\tau \in \mathrm{Inj}(z+1, n)$ we have

$$I_\tau(\mathbf{s}, w, \mathscr{M}) = I(\mathbf{s}^\tau, w^\tau, \mathscr{M}^\tau),$$

where $I$ is as in Lemma 2.2 and $\mathscr{M}^\tau$ is defined as in Section 2.1. Thus

$$\frac{I_\tau(\mathbf{s}, w, \mathscr{M})}{\Gamma\left(-N + \sum_{j=1}^{n} s_j\right) \cdot \prod_{l=z+2}^{n} \Gamma(s_{q_l^\tau}) \cdot \prod_{k=2}^{z+1} \Gamma\left( \sum_{j=k}^{z+1} (s_{\tau(j)} - |F_j^\tau|) + \sum_{l=z+2}^{n} s_{q_l^\tau} \right)},$$

extends to an analytic function on $\mathbb{C}^n \times D_n \times \mathscr{D}_{N,n}^*$.

We now examine

$$\prod_{k=2}^{z+1} \Gamma \left( \sum_{j=k}^{z+1} (s_{\tau(j)} - |F_j^\tau|) + \sum_{l=z+2}^{n} s_{q_l^\tau} \right) \tag{2.13}$$

as $\tau$ varies. First note that for $k = 2$ the number of $s_j$'s that appear inside (2.13) is $n-1$, and as $k$ varies we get that all the number of appearances from $n - z$ to $n - 1$ occur. With this in mind, for fixed $r \in \{n - z, \ldots, n - 1\}$ we choose $\{p_1, \ldots, p_r\} \subseteq \{1, \ldots, n\}$ a subset of indices. Now, note that the previous indices appear inside some factor of (2.13) if and only if there exists some $\tau \in \mathrm{Inj}(z+1,n)$ such that $\{q^\tau_{z+2},\dots,q^\tau_n\} \subseteq \{p_1,\dots,p_r\}$, and the remaining $p_j$'s are exactly $\tau(n-r+1)$, $\tau(n-r+2)$, ..., $\tau(z+1)$. For such $\tau$ we conclude that the corresponding number inside (2.13) is the negative of

$$\sum_{j=n-r+1}^{z+1} |F^\tau_j|. \tag{2.14}$$

If we take the maximum of (2.14) over all such $\tau \in \mathrm{Inj}(z+1,n)$ we get $m(p_1,\dots,p_r)$, which means that

$$\prod_{k=n-z}^{n-1} \prod_{\{p_1,\dots,p_k\}\in \mathrm{Ap}(k)} \Gamma\left(-m(p_1,\dots,p_k) + \sum_{j=1}^{k} s_{p_j}\right)$$

contains all the poles that appear in (2.13) as $\tau$ varies. Similarly, note that

$$\prod_{\substack{j\in\{1,\dots,n\} \text{ s.t. } \exists \tau\in \mathrm{Inj}(z+1,n)\\ \text{with } j\notin\{\tau(1),\dots,\tau(z+1)\}}} \Gamma(s_j)$$

contains all the poles that appear in $\prod_{l=z+2}^{n}\Gamma(s_{q^\tau_l})$ as $\tau$ varies. From this discussion and (2.12) we conclude the result for $Z_{N,n}$.

If we take $s_j = s$ for all $j$ then now we only care how many $s_j$'s appear inside (2.13), not which ones, and by a similar argument to the one above we see that $ks$ appears inside (2.13) for all $k \in \{n-z,\dots,n-1\}$ and $\tau \in \mathrm{Inj}(z+1,n)$. Again, for fixed $r \in \{n-z,\dots,n-1\}$ the corresponding number inside (2.12) is the negative of (2.14), and if we take the maximum of (2.14) over all $\tau \in \mathrm{Inj}(z+1,n)$ we get $m_r$. Putting all of the above together, we conclude the result for $\zeta_{N,n}$. Lastly, for $\mathscr{Z}_{N,n}$ we can use the same trick as in (2.11) to obtain

$$\mathscr{Z}_{N,n}(s,w,\mathscr{M})\Gamma(s)^n = \int_{[0,\infty)^n} \tilde{h}(t) \prod_{j=1}^{n} t_j^{s-1}\, dt,$$

where

$$\begin{aligned}\tilde{h}(t) := &\int_{[0,\infty)^N} \prod_{j=1}^{n} \exp(-t_j(w_j + x_1a_{1j} + \dots + x_Na_{Nj}))\, dx\\ =&\frac{\prod_{j=1}^{n} e^{-w_jt_j}}{\prod_{i=1}^{N}\left(\sum_{j=1}^{n} a_{ij}t_j\right)}\end{aligned}$$

and from here on the argument is exactly the same as before, replacing $f(t)$ with 1. □

## 2.3. Consequences of Theorem 2.1

**Remark 2.3.** Note that, except for a factor of $w_j^{-s_j}$, (1.4) remains the same if $\mathscr{M}$ has a column of zeros, in which case we can remove that column and obtain a new matrix with a lower $n$. Similarly, (1.4) does not change if the rows of $\mathscr{M}$ are permuted, and the same happens if we permute the columns of $\mathscr{M}$ and the corresponding $w_j$'s accordingly. If after applying some of these operations to $\mathscr{M}$ we obtain a matrix $\mathscr{M}'$ that can be written as diagonal blocks $\mathscr{M}_1 \oplus \ldots \oplus \mathscr{M}_p$, where $\mathscr{M}_j$ has $N_j$ rows and $n_j$ columns, then we get a product decomposition

$$\zeta_{N,n}(s, w, \mathscr{M}) = \zeta_{N_1,n_1}(s, w_1', \mathscr{M}_1) \cdot \ldots \cdot \zeta_{N_p,n_p}(s, w_p', \mathscr{M}_p),$$

where $w_j'$ is obtained from $w$. If we now apply Theorem 2.1 to each $\zeta_{N_p,n_p}$ we get better information about the poles of $\zeta_{N,n}$, as we illustrate in the next example:

**Example 2.4.** If $N = 5$, $n = 7$ and

$$\mathscr{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

then $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$, where

$$\mathscr{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{M}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Applying Theorem 2.1 directly to $\zeta_{5,7}(s, w, \mathscr{M})$ we get that $z(\mathscr{M}) = 5$, $m_2 = 1$, $m_3 = m_4 = 3$ and $m_5 = m_6 = 4$, therefore we conclude that $\zeta_{5,7}(s, w, \mathscr{M})$ has at most poles of order up to 6 at

$$\begin{aligned} &\left\{\frac{1-k}{2} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{3-k}{3} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{3-k}{4} : k \in \mathbb{N}_0\right\} \\ &\cup \left\{\frac{4-k}{5} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{4-k}{6} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{5-k}{7} : k \in \mathbb{N}_0\right\}. \end{aligned} \tag{2.15}$$

On the other hand, applying Theorem 2.1 to $\zeta_{2,4}(s, (w_1, w_2, w_3, w_4), \mathscr{M}_1)$ we get $z(\mathscr{M}_1) = 2$ and $m_2 = m_3 = 1$, so it has at most poles of order 3 at

$$\left\{\frac{1-k}{2} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{1-k}{3} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{2-k}{4} : k \in \mathbb{N}_0\right\},$$

and with $\zeta_{3,3}(s, (w_5, w_6, w_7), \mathscr{M}_2)$ we obtain $z(\mathscr{M}_2) = 1$ and $m_2 = 1$, therefore it has at most simple poles at

$$\left\{\frac{1-k}{2} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{3-k}{3} : k \in \mathbb{N}_0\right\}.$$

Finally, we conclude that

$$\zeta_{5,7}(s, w, \mathscr{M}) = \zeta_{2,4}(s, (w_1, w_2, w_3, w_4), \mathscr{M}_1) \cdot \zeta_{3,3}(s, (w_5, w_6, w_7), \mathscr{M}_2)$$

has at most simple poles at $s = 1$ and $s = 2/3$, at most poles of order 3 at

$$\left\{\frac{2-k}{4} : k \in \mathbb{N}_0\right\},$$

and at most poles of order 4 at

$$\left\{\frac{1-k}{2} : k \in \mathbb{N}_0\right\} \cup \left\{\frac{1-k}{3} : k \in \mathbb{N}_0\right\},$$

which is a considerable improvement with respect to what was obtained in (2.15).

**Example 2.5** (Example 1.4 revisited)**.** If $N = 2$, $n = 3$, $w = (1, 1, 2)$, and

$$\mathscr{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

then

$$\zeta_{2,3}(s, w, \mathscr{M}) = 2^{-s} \cdot \zeta^{W}_{\mathrm{SU}(3)}(s) = \sum_{k,l=1}^{\infty} \frac{1}{(kl(k+l))^s}. \tag{2.16}$$

In this case is clear that $z = 1$ and $m_2 = 1$, therefore from Theorem 2.1 we conclude that (2.16) extends to a meromorphic function with at most simple poles at

$$\left\{ \frac{1}{2} - k : k \in \mathbb{N}_0 \right\} \cup \left\{ \frac{1}{3} - k : k \in \mathbb{N}_0 \right\} \cup \left\{ \frac{2}{3} - k : k \in \mathbb{N}_0 \right\}.$$

As we mentioned before, this example was previously studied in [KO13, Ono14, Rom17] and they proved, among other things, that in fact the only poles are

$$\left\{ \frac{2}{3} \right\} \cup \left\{ \frac{1}{2} - k : k \in \mathbb{N}_0 \right\},$$

and $\zeta_{2,3}(s, w, \mathscr{M}) = 0$ for $s = -1, -2, -3, \ldots$.

We now list some Theorems which are consequences of Theorem 2.1, which apply with exactly the same proof as in their corresponding articles thanks to the analytical continuation that we have already developed. We include the proofs and the authors of each Theorem for the sake of completeness.

**Theorem 2.6** (cf. Theorem 4 (1), [FP12])**.** For $s$ outside the possible set of poles of $\mathscr{Z}_{N,n}(s, w, \mathscr{M})$ given by Theorem 2.1 and $(w, \mathscr{M}) \in D_n \times \mathscr{D}^*_{N,n}$, we have

$$\mathscr{Z}_{N,n}(s, w, \mathscr{M}) = \int_{t \in [0,1]^N} \zeta_{N,n}(s, w + W(t), \mathscr{M})\, dt,$$

where $W(x) := \sum_{i=1}^{N} x_i a_i$.

**Proof.** For $\mathrm{Re}(s) > N$ the integral and series defining $\zeta_{N,n}$ and $\mathscr{Z}_{N,n}$ are absolutely convergent. Thus we have

$$\begin{aligned}
&\int_{t \in [0,1]^N} \zeta_{N,n}(s, w + W(t), \mathscr{M})\, dt \\
&\qquad = \int_{t \in [0,1]^N} \sum_{k_1, \ldots, k_N = 0}^{\infty} \prod_{j=1}^{n} (w_j + (k_1 + t_1)a_{1j} + \cdots + (k_N + t_N)a_{Nj})^{-s}\, dt \\
&\qquad = \sum_{k_1, \ldots, k_N = 0}^{\infty} \int_{x \in k + [0,1]^N} \prod_{j=1}^{n} (w_j + x_1 a_{1j} + \cdots + x_N a_{Nj})^{-s}\, dx \\
&\qquad = \int_{x \in [0,\infty)^N} \prod_{j=1}^{n} (w_j + x_1 a_{1j} + \cdots + x_N a_{Nj})^{-s} \\
&\qquad = \mathscr{Z}_{N,n}(s, w, \mathscr{M}).
\end{aligned}$$

By analytic continuation, the Raabe formula

$$\mathscr{Z}_{N,n}(s,w,\mathscr{M}) = \int_{t\in[0,1]^N} \zeta_{N,n}(s,w+W(t),\mathscr{M})\,dt$$

holds for all $s$ outside the possible set of poles of $\mathscr{Z}_{N,n}(s,w,\mathscr{M})$ given by Theorem 2.1. $\square$

**Theorem 2.7** (cf. Proposition 3.1, [FR04]). The functions of $w=(w_1,\ldots,w_n)$, defined for $\mathrm{Re}(w_j)>0$ $(1\le j\le n)$, given by

$$P_{kn+N,N,n}(w) := \zeta_{N,n}(-k,w,\mathscr{M}), \quad k\in\mathbb{N}_0$$

are polynomials of degree at most $n(k+1)-1+\sum_{j=1}^n |A_j|$, with $A_j$ as defined in Section 1.1.

**Proof.** Given a multi-index $I=(I_1,\ldots,I_n)$ of weight $|I| := \sum_{j=1}^n I_j$, let $\partial_w^J$ be the differential operator $\frac{\partial^{|I|}}{\partial w_1^{I_1}\cdots\partial w_n^{I_n}}$. For $\mathrm{Re}(s)>N$ the series defining $\zeta_{N,n}$ converges absolutely, thus direct differentiation yields

$$\begin{aligned}
&\partial_w^I\,\zeta_{N,n}(s,w,\mathscr{M}) \\
&\quad = (-1)^{|I|}\left(\prod_{j=1}^n\prod_{p=0}^{I_j-1}(s+p)\right)\sum_{k_1,\ldots,k_N=0}^{\infty}\prod_{j=1}^n(w_j+k_1a_{1j}+\cdots+k_Na_{Nj})^{-s-I_j} \\
&\quad = (-1)^{|I|}Z_{N,n}((s+I_1,\ldots,s+I_n),w,\mathscr{M})\prod_{j=1}^n\prod_{p=0}^{I_j-1}(s+p).
\end{aligned} \tag{2.17}$$

The above series converges absolutely for $s+I_1>|A_1|,\ldots,s+I_n>|A_n|$ by the results in Section 1.1, thus, by analytic continuation, it represents $\partial_w^I\zeta_{N,n}(-k,w,\mathscr{M})$ for $I_1\ge k+|A_1|+1,\ldots,I_n\ge k+|A_n|+1$. Since the product term in (2.17) vanishes for $s=-k$ and $I=(k+|A_1|+1,\ldots,k+|A_n|+1)$, we conclude that $\zeta_{N,n}(-k,w,\mathscr{M})$ is a polynomial in $w$ of degree at most $n(k+1)-1+\sum_{j=1}^n|A_j|$. $\square$

Finally, we state the following Theorem which is an immediate consequence of Theorem 2.1 and the results of Section 1.3:

**Theorem 2.8.** Given $\mathfrak{g}$ a complex semi-simple Lie algebra, let $N$ denote its rank, and $n=|\Delta^+(\mathfrak{g})|$ denote the number of positive roots associated to $\mathfrak{g}$. Then $\zeta_{\mathfrak{g}}^W(s)$ extends to an analytic function on $\mathbb{C}$, it has at most poles of order $n$ which can only occur among the rational numbers of the form

$$s=\frac{N-l}{v}, \quad \text{with } v\in\{1,\ldots,n\} \quad \text{and} \quad l\in\mathbb{N}_0\setminus\{N+kv : k\in\mathbb{N}_0\},$$

and has no poles for $s=-k$ with $k\in\mathbb{N}_0$.

# References


[Bar04] E. W. Barnes. *On the theory of the multiple gamma function.* Trans. Camb. Philos. Soc., **19** (1904), 374-425.

[Bou02] N. Bourbaki. *Lie Groups and Lie Algebras. Chapters 4-6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[BGF] J. I. Burgos Gil, J. Fresán. *Multiple zeta values: from numbers to motives*. Clay Mathematics Proceedings (to appear).

[Ess97] D. Essouabri. *Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications à la théorie analytique des nombres*. Ann. Inst. Fourier (Grenoble) **47** (1997)

[FP12] E. Friedman, A. Pereira. *Special values of Dirichlet series and zeta integrals*. Int. J. Number Theory **8** (2012), no. 3, 697-714.

[FR04] E. Friedman, S. Ruijsenaars. *Shintani-Barnes zeta and gamma functions*. Adv. Math. **187** (2004), no. 2, 362-395.

[Hal15] B. C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics, 222. Springer, Cham, 2015.

[KMT10] Y. Komori, K. Matsumoto, H. Tsumura. *On Witten multiple zeta-functions associated with semisimple Lie algebras II*. J. Math. Soc. Japan **62** (2010), no. 2, 355-394.

[KO13] N. Kurokawa, H. Ochiai. *Zeros of Witten zeta functions and absolute limit*. Kodai Math. J. **36** (2013), no. 3, 440-454.

[Mat02] K. Matsumoto. *On analytic continuation of various multiple zeta-functions*. In *Number theory for the millennium II (Urbana, IL, 2000)*, 417–440, A K Peters, Natick, MA, 2002.

[Mat03] K. Matsumoto. *On Mordell-Tornheim and other multiple zeta-functions*. In *Proceedings of the Session in Analytic Number Theory and Diophantine Equations*, 17 pp., Bonner Math. Schriften, **360**, Univ. Bonn, Bonn, 2003.

[Mat06] K. Matsumoto. *Analytic Properties of Multiple Zeta-Functions in Several Variables*. In *Number Theory: Tradition and Modernization*, 153-173, Dev. Math., **15**, Springer, New York, 2006.

[Mor58] L. J. Mordell. *On the Evaluation of Some Multiple Series*. J. London Math. Soc. **33** (1958), 368-371.

[MT06] K. Matsumoto, H. Tsumura. *On Witten multiple zeta-functions associated with semisimple Lie algebras I*. Ann. Inst. Fourier (Grenoble) **56** (2006), no. 5, 1457-1504.

[Ono14] K. Onodera. *A functional relation for Tornheim's double zeta functions*. Acta Arith. **162** (2014), no. 4, 337-354.

[Rom17] D. Romik. *On the number of $n$-dimensional representations of $SU(3)$, the Bernoulli numbers, and the Witten zeta function*. Acta Arith. **180** (2017), no. 2, 111-159.

[Shi76] T. Shintani. *On evaluation of zeta functions of totally real algebraic number fields at non-positive integers*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 2, 393-417.

[Tor50] L. Tornheim. *Harmonic Double Series*. Amer. J. Math. **72** (1950), 303-314.

[Wit91] E. Witten. *On quantum gauge theories in two dimensions*. Comm. Math. Phys. **141** (1991), no. 1, 153-209.

[Zag94] D. Zagier. *Values of Zeta Functions and Their Applications*. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, 497-512. Progr. Math., **120**, Birkhäuser, Basel, 1994.

[Zha16] J. Zhao. *Multiple zeta functions, multiple polylogarithms and their special values*. Series on Number Theory and its Applications **12**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.