FACULTAD DE MATEMÁTICAS
PONTIFICIA UNIVERSIDAD
CATÓLICA DE CHILE

PHD THESIS:

Topics in Ballistic and Transient Conditions for Random Walks in Random Environments

A Thesis submitted by Enrique Guerra Aguilar for the degree of Doctor en Matemáticas in the Pontificia Universidad Católica de Chile

Supervised by:

Alejandro F. Ramírez

July, 2016

Santiago, Chile

*Dedicado a mi esposa Stephanie Alfaro y a mis padres
Enrique Guerra Galaz y Patricia Aguilar Jara.*

Acknowledgements

I would here like to thank all people responsible for making this thesis more than a simple keyboard work. First of all, I express my sincere thanks to my thesis advisor Alejandro Ramírez for letting me to be his student. It was a real pleasure to work with him, I learnt much more than mathematics. He is undoubtedly my mentor and all my results were turned out by a deep exchange of probabilistic ideas. I also thank my thesis committee: Joaquin Fontbona, Gregorio Moreno, Rolando Rebolledo and Christophe Sabot. I offer my apologies for the delay in delivering the final thesis version. I have studied for more than a decade at the Pontificia Universidad Catolica de Chile. There I have met and interacted with many people. Among friends and staff whom I met, I would like to especially thank Dr. Moreno and Dr. Cabezas for sharing their knowledge through which often made me strengthen my own knowledge and like for the probability field. Also, I thank my office mate Alvaro Ferrada for his friendship, who is my friend from the very beginning of my graduate studies. I thank the partial support throughout my research to the Nucleo Milenio: Modelos Estocásticos de Sistemas Complejos y Desordenados, a mathematical community based at the Pontificia Universidad Católica and the Universidad de Chile. Regarding the Nucleo staff, for the constant help in non-mathematical matters I express my gratitude to: Consuelo Thiers, Maria Eugenia Heckman and Cecile Jourdan. Last but most important, I deeply thank with all of my heart my family: my wife Stephanie Alfaro who is the main source of inspiration in each mathematical idea that I have, and my parents: Enrique Guerra and Patricia Aguilar whom I owe all what I am.

Abstract

This thesis is devoted to the study of the stochastic process model called Random Walk in Random Environment (RWRE). To be precise, our research focuses on two kinds of random environments. The first one is the so called uniformly elliptic i.i.d. random environment. In this model it is conjectured that in dimensions $d \geqslant 2$ any random walk which is directionally transient is ballistic. The ballisticity conditions for RWRE somehow interpolate between directional transience and ballisticity and have served to quantify the gap which one needs to answer affirmatively this conjecture. Two important ballisticity conditions introduced by Sznitman [Sz02] in 2001 and 2002 are the so called conditions (T') and (T): given a slab of width L orthogonal to l, condition (T') in direction l is the requirement that the annealed exit probability of the walk through the side of the slab in the half-space $\{x : x \cdot l < 0\}$, decays faster than e^{-CL^γ} for all $\gamma \in (0,1)$ and some constant $C > 0$, while condition (T) in direction l is the requirement that the decay is exponential e^{-CL}. It is believed that (T') implies (T). We show that (T') implies at least an *almost* (in a sense to be made precise) exponential decay. The second class of random environment to be studied is a larger class which only requires a mixing condition on the environment law. As a matter of fact, the ballisticity conditions in this framework are not well-understood. Therefore our purpose is to find a connection between this strictly larger class of environments and the ballisticity conditions which have proved to be a powerful theoretical concept for random walks in an i.i.d. random environment. In that direction, we prove that every random walk in a uniformly elliptic random environment satisfying the cone mixing condition and a non-effective polynomial ballisticity condition with high enough degree has an asymptotic direction.

Contents

List of Figures

General Introduction

Random Walk in a Random Environment (RWRE) is a classical model of random motion in a random media. It was originally introduced as a toy model for replication of DNA chains and phase transition in alloys. We can describe a d- dimensional RWRE as the canonical Markov chain $(X_n)_{n \geqslant 0}$ with state space $\mathbb{Z}^d$, where its transition probabilities to nearest neighbor sites are random. In spite of its simplicity, when the dimension is larger than 1 its asymptotic laws are still not well-understood. This problem, is essentially due to the reversibility loss in the chain on averaging over the environment. Consequently this makes it hard to apply standard convergence methods in order to get asymptotic laws. Some progress has been done in that direction by means of the introduction of what are called ballisticity conditions. These conditions are essentially a functional control (for instance a polynomial control) of the probability that the walk exits from large slabs transversal to directions l' in a neighborhood of a given direction $l \in \mathbb{S}^{d-1}$ by the unlikely slab boundary side: the one which is in the direction $-l'$. The study of ballisticity conditions is the main focus of this thesis. To appropriately introduce them, we will now explain more precisely the model. Let $d \geqslant 1$ be a positive integer which will be thought as the underlying random walk dimension. We consider the $(2d-1)$- dimensional simplex $\mathcal{P}$ defined by:

$$\mathcal{P} := \{z \in \mathbb{R}^{2d} : \sum_{i=1}^{2d} z_i = 1, z_i \geqslant 0 \text{ for } i \in [1, 2d]\}.$$

Now, an environment $\omega := \omega(x, e)|_{x \in \mathbb{Z}^d, e \in \mathbb{Z}^d, |e|=1}$ is an element of the set $\Omega := (\mathcal{P})^{\mathbb{Z}^d}$ which specifies at each site $x \in \mathbb{Z}^d$ the transition probabilities of the walk. Throughout this chapter, by canonical σ- algebra on a product space we mean the σ-algebra generated by the cylinder measurable sets. For the time being, we assume that we have a given probability measure $\mathbb{P}$ on the canonical σ- algebra $\mathcal{W}$ in Ω.

For fixed $\omega \in \Omega$ and $x \in \mathbb{Z}^d$, one defines the *quenched law* $P_{x,\omega}$, as the law of the canonical Markov chain $(X_n)_{n\geqslant 1}$ starting from x, with state space $\mathbb{Z}^d$ and satisfying

$$P_{x,\omega}[X_0 = x] = 1$$

$$P_{x,\omega}[X_{n+1} = X_n + e \mid X_n, X_{n-1}, \dots X_0] = \omega(X_n, e) \text{ for } e \in \mathbb{Z}^d,\ |e| = 1.$$

We call $\mathcal{F}$ the canonical σ- algebra in $(\mathbb{Z}^d)^{\mathbb{N}}$, which is the σ- algebra in the walk path space. Furthermore, for a prescribed probability measure $\mathbb{P}$ one then defines the *annealed or averaged law* P_x as the semi-direct product $\mathbb{P} \otimes P_{x,\omega}$ on $\mathcal{W} \times \mathcal{F}$.

We consider two types of random environments: the first one will be the so-called i.i.d. random environment framework; the second one is a larger class satisfying a mixing condition. We start by defining the i.i.d. random environment. Let $\kappa \in (0, \frac{1}{2d}]$ and μ be a probability measure on $\mathcal{P}$ such that for each $x \in \mathbb{Z}^d$, $\omega(x, \cdot)$ distributes as μ, and μ is supported on the subset $\mathcal{P}_\kappa$ of $\mathcal{P}$ defined by:

$$\mathcal{P}_\kappa := \{z \in \mathbb{R}^{2d} : \sum_{i=1}^{2d} z_i = 1, z_i \geqslant \kappa \text{ for } i \in [1, 2d]\}.$$

This last restriction on the support of the law μ is called uniform ellipticity assumption. The random environment is now an element of the measurable probability space $\Omega_\kappa := (\mathcal{P}_\kappa)^{\mathbb{Z}^d}$ which is endowed with canonical $\sigma-$ algebra $\mathcal{W}_\kappa$ and the product measure $\mathbb{P} := \mu^{\otimes \mathbb{Z}^d}$. For the easy of notation, we shall drop κ when we talk about i.i.d random environments.

Before introducing the cone mixing condition we weaken the uniform ellipticity assumption. We say that $\mathbb{P}$ is *uniformly elliptic with respect to* l, denoted by $(UE)|l$, if the jump probabilities of the random walk are positive and larger than 2κ in those directions for which the projection on l is positive. In other words if $\mathbb{P}[\omega(0, e) > 0] = 1$ for $|e| = 1$ and if

$$\mathbb{P}\left[\min_{e\in\mathcal{E}} \ \omega(0, e) \geqslant 2\kappa\right] = 1,$$

where

$$\mathcal{E} := \bigcup_{i=1}^{d} \{sgn(l_i)e_i\} - \{0\} \tag{1}$$

and by convention $sgn(0) = 0$.

It will be convenient to define what is understood by a cone in this work. We let α be a small positive real number and R be a rotation such that

$$R(e_1) = l. \tag{2}$$

To define the cone, it will be useful to consider for each $i \in [2, d]$, the directions

$$l_{+i} = \frac{l + \alpha R(e_i)}{|l + \alpha R(e_i)|} \qquad \text{and} \qquad l_{-i} = \frac{l - \alpha R(e_i)}{|l - \alpha R(e_i)|}.$$

The cone $C(x, l, \alpha)$ centered in $x \in \mathbb{R}^d$ is defined as

$$C(x, l, \alpha)) := \bigcap_{i=2}^{d} \left\{ z \in \mathbb{R}^d : (z - x) \cdot l_{+i} \geqslant 0,\ (z - x) \cdot l_{-i} \geqslant 0 \right\}. \tag{3}$$

The following picture shows a cone centered at x in the lattice $\mathbb{Z}^2$

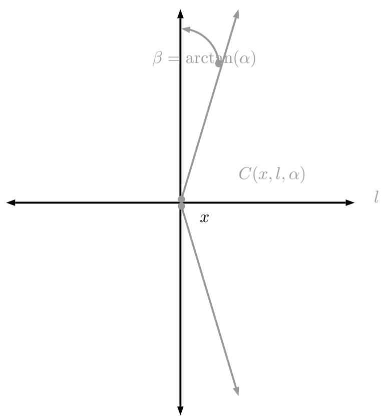

Figure 1: Cone

We are ready to state the cone mixing condition. Define the canonical shifts $\{\theta_x : x \in \mathbb{Z}^d\}$ by $\theta_x \omega(y) := \omega(x + y)$ for all $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$. Let us first recall the concept of ergodic measure. We say that a probability measure $\mathbb{P}$ is *stationary* if for all $x \in \mathbb{Z}^d$ and $A \in \mathcal{W}$ one has that $\mathbb{P}(\theta_x^{-1} A) = \mathbb{P}(A)$. We say that $\mathbb{P}$ is *ergodic*, if whenever $A \in \mathcal{W}$ is such that $A = \theta_x^{-1} A$ for all $x \in \mathbb{Z}^d$, one has that $\mathbb{P}(A) = 0$ or that $\mathbb{P}(A) = 1$. Now, let $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \phi(r) = 0$. We say that a stationary probability measure $\mathbb{P}$ satisfies the *cone mixing assumption* with respect to α, l and ϕ, denoted

$(CM)_{\alpha,\phi}|l$ if for every pair of events A, B, where $\mathbb{P}(A) > 0$, $A \in \sigma\{\omega(z,\cdot); z \cdot l \leqslant 0\}$, and $B \in \sigma\{\omega(z,\cdot); z \in C(rl, l, \alpha)\}$, it holds that

$$\left| \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} - \mathbb{P}[B] \right| \leqslant \phi(r|l|_1). \tag{4}$$

Thus, we can consider assumption $(CM)_{\alpha,\phi}|l$ as a restriction on the $\mathbb{P}$- dependence. As it was mentioned in [CZ01], it is important to allow strictly positive angles $\beta := \arctan(\alpha)$. Otherwise, when $\beta = 0$ and the cone mixing assumption is satisfied for each $l \in \mathbb{S}^{d-1}$, then the measure $\mathbb{P}$ is actually a finite range dependence law (see [Ze1] and [B1]). Furthermore, whenever a probability measure $\mathbb{P}$ satisfies the cone mixing assumption, it is ergodic (this will be proved in Chapter 2).

We will be dealing with three important asymptotic concepts:

- We say that the walk is *transient in the direction* l, if

$$P_0 \left[\lim_{n\to\infty} X_n \cdot l = \infty \right] = 1. \tag{5}$$

- We say that the walk is *ballistic in the direction* l if

$$P_0 \left[\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0 \right] = 1. \tag{6}$$

  Moreover, in this case we will also say that the walk has ballistic behavior.

- We say that a non-zero d- dimensional deterministic vector $\hat{v}$ is an *asymptotic direction* for the walk if

$$P_0 \left[\lim_{n\to\infty} \frac{X_n}{|X_n|} \to \hat{v} \right] \tag{7}$$

  holds.

It is straightforward to see that any RWRE which is ballistic in direction l is transient in the same direction. In the 1 dimensional case, the walk asymptotic behavior is well-understood, and the results come from Smith-Wilkinson [SW69], Solomon [So75] and Alili [Al99]. Define

$$\rho := \frac{\omega(0, -e_1)}{\omega(0, e_1)}.$$

We then have the following transience criteria:

Theorem 0.0.1 (Smith-Wilinson, Solomon, Alili)**.** *Suppose that* $\mathbb{P}$ *is ergodic and that* $\mathbb{E}[\ln(\rho)]$ *is defined (possibly* $\pm\infty$*), then*

(i) $\mathbb{E}[\ln \rho] < 0$ *implies* P_0*-a.s.* $\lim X_n = \infty$.

(ii) $\mathbb{E}[\ln \rho] > 0$ *implies* P_0*-a.s.* $\lim X_n = -\infty$.

(iii) $\mathbb{E}[\ln \rho] = 0$ *implies* P_0*-a.s.* $-\infty = \liminf X_n < \limsup X_n = \infty$.

As an example of the possible atypical behavior of RWRE, Sinai [Sin82] considered random walks in random environments satisfying the case (iii) of the previous theorem together with $0 < \mathbb{E}[(\ln \rho)^2] < \infty$, proving that: the position X_n of the random walk takes on values of order $\log^2(n)$. This is in contrast to the ordinary random walk typical asymptotic behavior of the random variable X_n which is of order $\sqrt{n}$. We also have a ballisticity criteria as follows:

Theorem 0.0.2 (Smith-Wilkinson, Solomon)**.** *Assume that* $\mathbb{P}$ *is i.i.d. and uniformly elliptic. One has that* P_0*-a.s.* $\frac{X_n}{n} \to v$, *where*

(i) *For* $\mathbb{E}[\rho] < 1$, $v = \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} > 0$.

(ii) *For* $\frac{1}{\mathbb{E}[\rho^{-1}]} \leqslant 1 \leqslant \mathbb{E}[\rho]$, $v = 0$.

(iii) *For* $1 < \frac{1}{\mathbb{E}[\rho^{-1}]}$, $v = \frac{1-\mathbb{E}[\rho^{-1}]}{1+\mathbb{E}[\rho^{-1}]} < 0$.

From Jensen inequality we can see that there exist random walks in i.i.d. random environments which are directionally transient with vanishing velocity. However in the higher dimensional case the last possibility is not expected as the following conjecture shows:

Conjecture 0.0.3 ($d \geqslant 2$.)**.** Any d- dimensional RWRE which is uniform elliptic, i.i.d. and transient in direction l, is ballistic in direction l.

As it was remarked above, this conjecture is not true when the dimension is 1. Informally, this conjecture says that traps are negligible when the dimension $d \geqslant 2$, and we mean by traps finite though arbitrary large regions in $\mathbb{Z}^d$ where the walk spends a long time with relatively high probability. In this direction, an intermediate problem has been solved by Simenhaus [Si07]

Theorem 0.0.4 (Simenhaus). *Assume that a d- dimensional random walk in a uniform elliptic i.i.d. random environment is transient in a neighborhood of the direction l. Then, there exists an asymptotic direction $\hat{v}$ for the random walk.*

The hypothesis of the previous theorem are actually equivalent. Indeed, the converse implication of Theorem 0.0.4 is a straightforward application of Kalikow's 0-1 law [K81]. The proof of this theorem strongly makes use of the independent structure of the environment. We will give some further comments about this result in Section 0.3. We want now to introduce the so-called ballisticity conditions and summarize what is known.

0.1 Some well-known results in uniformly elliptic i.i.d. random environments. (under $d \geqslant 2$.)

We present some important results for the uniformly elliptic i.i.d. random environment setting. The first result that we would like to mention is a relatively old one and comes from Kalikow in [K81] (we refer to this article for a further discussion).

0.1.1 On Kalikow's Condition.

In order to enlighten the nature of this condition, we will need some definitions. For a given set $U \in \mathbb{Z}^d$ we define its boundary ∂U by:

$$\partial U := \{y \in \mathbb{Z}^d - U : \ \exists z \in U, \ |y - z| = 1\},$$

and also define the first time of exit from the set U, which we denote T_U via:

$$T_U = \inf\{n \geqslant 0 : \ X_n \notin U\}.$$

Kalikow introduced a useful auxiliary Markov chain related to the original chain $(X_n)_{n \geqslant 0}$. More precisely, let U be a connected strict subset of $\mathbb{Z}^d$ with $0 \in U$, for $x \in U \cup \partial U$ we define the Kalikow's law $\widehat{P}_{x,U}$ as the law of the canonical Markov chain $(X_n)_{n \geqslant 0}$ (we keep the same notation because this makes sense in view of (0.1.1)) starting from x with state space in $U \cup \partial U$ and stationary transition probabilities given by:

$$\widehat{P}_U(x, x+e) = \begin{cases} \frac{E_0[\sum_{n=0}^{T_U} \mathbb{1}_{\{X_n = x\}} \omega(x,e)]}{E_0[\sum_{n=0}^{T_U} \mathbb{1}_{\{X_n = x\}}]}, & x \in U, |e|_1 = 1, \\ 1, & x \in \partial U,\ e = 0, \end{cases}$$

where the above expectations are finite thanks to the uniform ellipticity assumption. The previously mentioned main connection between this auxiliary chain and the original one is given by:

Theorem 0.1.1 (Kalikow)**.** *Assume $\widehat{P}_{0,U}[T_U < \infty]$. Then $P_0[T_U < \infty]$ and X_{T_U} has the same distribution under either $\widehat{P}_{0,U}$ or P_0.*

When $d \geqslant 2$, Kalikow's condition was the first condition used to prove asymptotic laws for RWRE. In the seminal result of [K81], Kalikow was able to prove directional transience under what is currently known as Kalikow's criteria. This is a priori a stronger requirement than Kalikow's condition. Before we define formally these concepts, we would like to heuristically explain what Kalikow's condition is and explain the general reasoning behind proofs of asymptotic laws for the walk under such an assumption. Kalikow's condition is essentially the existence of a positive local drift for the auxiliary Markov chains over all connected strictly subset $U \subset \mathbb{Z}^d$, with $0 \in U$. Standard arguments show that this implies ballistic behavior for the auxiliary Markov chains. We then transfer this ballistic behavior to the walk by means of (0.1.1) and some extra probabilistic arguments. Kalikow's condition with respect to some fixed direction $l \in \mathbb{S}^{d-1}$ is the following requirement:

Definition 0.1.2 (Kalikow's Condition)**.** *There exists a non-random real number $\delta > 0$, so that:*

$$\inf_{U,x} \sum_{e,\ |e|=1} \widehat{P}_U(x, x+e) e \cdot l > \delta$$

holds, where the infimum runs over all connected finite strict subsets $U \in \mathbb{Z}^d$ such that $0 \in U$.

As an example of what was mentioned in the previous paragraph, one can see that under this condition appealing to property (0.1.1) and Azuma's inequality, the following important result is satisfied:

Theorem 0.1.3 (Kalikow)**.** *Assume Kalikow's condition in direction l. Then*

$$P_0\left[\lim X_n \cdot l = \infty\right] = 1$$

.

We refer to [Ze1] for further details about the proof of this theorem using the ideas outlined here. This result was considerably improved by Sznitman and Zerner through the introduction of a renewal structure which is a higher dimensional analog of the one-dimensional theoretical construction introduced by Kesten in [Ke77], and which can be defined in directional transient case (see (5)). This renewal structure stems from a random time τ_1 which can be thought as the first time that the walk reaches a record level with respect to direction l and after this time the walk does never backtrack. One can use the renewal structure to prove the equivalence between the requirement of the ballisticity definition given in (6) with the following a priori stronger assumption (see [DR14]): P_0-a.s. one has that

$$\lim_{n\to\infty} \frac{X_n \cdot l}{n} \tag{8}$$

exists, is positive and constant. In this case, we can then define the *velocity* as

$$v := \lim_{n\to\infty} \frac{X_n}{n}.$$

Furthermore, from standard subsequence methods, it can be seen that the right candidate for the velocity v is

$$v := \frac{E_0[X_{\tau_1} \mid D = \infty]}{E_0[\tau_1 \mid D = \infty]}, \tag{9}$$

where D is the hitting time of the half space $\{z \in \mathbb{R}^d,\ z \cdot l < 0\}$ (c.f. (10)). Therefore a natural question is the following one: what kind of local condition on the environment does allow us to have a finite first moment for the random variable τ_1? In that direction, by means of a clever use of Kalikow's condition (see [SZ99]), Sznitman and Zerner proved that:

$$E_0[\tau_1] < \infty.$$

As a result, in view of (9) we obtain the following:

Theorem 0.1.4 (Sznitman and Zerner)**.** *Under Kalikow's condition with respect to direction l there exists a deterministic $v \in \mathbb{R}^d$, such that P_0- almost surely:*

$$\frac{X_n}{n} \to v$$

Moreover, one has that $v \cdot l > 0$.

In a subsequent article [Sz01], Sznitman was able to prove a Central Limit Theorem under Kalikow's condition. However we are mostly interested here in the ballistic conditions $(T^\gamma)|l$ for $\gamma \in (0, 1]$ which were defined in [Sz03].

0.1.2 Ballisticity Conditions: Stretched Exponential Decay, Effective Criterion and Polynomial Condition

As in the previous section, we begin with some definitions. We define for $a \in \mathbb{R}$, the stopping times T_a^l and $\widetilde{T}_a^l$ with respect to the canonical filtration of the walk by:

$$T_a^l := \inf\{n \geqslant 0 : \ X_n \cdot l \geqslant a\},$$

together with

$$\widetilde{T}_a^l := \inf\{n \geqslant 0 : \ X_n \cdot l \leqslant a\}.$$

It will also be convenient to define the stopping time D as the first time that the walk hits the random half-space $\{z \in \mathbb{Z}^d : (z - X_0) \cdot l < 0\}$:

$$D := \inf\{n \geqslant 0 : \ X_n \cdot l < X_0 \cdot l\}. \tag{10}$$

The underlying rough thought in the renewal structure is the following: under transience in direction l, P_0- a.s. there should exist a finite random time τ_1 such that X_{τ_1} is a record level in direction l, and after time τ_1 the walk never backtracks. In [SZ99] the authors prove that transience in direction l is equivalent to $P_0[\tau_1 < \infty] = 1$.

On the other hand, suppose that the walk is transient in direction l. For large L we consider the slab $A_{L,l}$ defined by

$$A_{L,l} := \{z : \ |z \cdot l| \leqslant L\}.$$

Elementary probabilistic arguments let us conclude that

$$P_0\left[X_{T_{A_{L,l}}}\cdot l<0\right]=P_0\left[\widetilde{T}^l_{-L}<\widetilde{T}^l_L\right]\to 0$$

as L goes to infinity. The ballisticity conditions introduced by Sznitman are *stretched exponential* controls for the above probabilities.

Definition 0.1.5 (Stretched Exponential Decay)**.** *Let $\gamma\in(0,1]$. We say that $(T^\gamma)|l$ holds, if*

$$\limsup_{L\to\infty}L^{-\gamma}\ln P_0\left[\widetilde{T}^{l'}_{-\tilde{b}L}<T^{l'}_L\right]<0, \tag{11}$$

for $\widetilde{b}>0$, and each l' in some neighborhood of l. We also say that $(T^{'})|l$ is fulfilled if $(T^\gamma)|l$ holds for each $\gamma\in(0,1)$, and we use for short the notation $(T)|l:=(T^1)|l$.

Let us remark that we can get rid in the previous definition the constant $\widetilde{b}$. This can be proved using the strategies developed in the proofs of Proposition 2.2.3 and Lemma 2.5.5.

From the definition, it is straightforward to see that for prescribed $\gamma_1,\gamma_2\in(0,1)$, with $\gamma_1<\gamma_2$ the following chain of implications holds:

$$(T)|l\ \to\ (T^{'})|l\ \to\ (T^{\gamma_2})|l\ \to\ (T^{\gamma_1})|l.$$

We actually expect even more: it is believed that all these conditions are equivalent. A non-negligible progress has been made regarding this conjecture. As a first step to address this question Sznitman proved in [Sz03], the implication:

$$(T^\gamma)|l\ \to\ (T^{'})|l, \tag{12}$$

for any $\gamma>1/2$. The tool used to prove this, is what is called the *effective criterion*, which is a higher dimensional version of standard ballisticity conditions for one-dimensional RWRE. In turn, it can be seen as the triggering condition in an induction probabilistic procedure. Its definition is a bit technical. Nevertheless, given its importance, we recall it here. Let $l\in\mathbb{S}^{d-1}$ be fixed and let R be a rotation on $\mathbb{R}^d$ so that $R(e_1)=l$. Let $L>2$ and $\tilde{L}>0$. Consider the box

$$B_{l,L,\tilde{L}}:=R((-(L-2),L+2)\times(-\tilde{L},\tilde{L})^{d-1})\cap\mathbb{Z}^d,$$

with the positive part of its boundary $\partial^+ B_{l,L,\tilde{L}}$ defined via

$$\partial^+ B_{l,L,\tilde{L}} := \partial B \cap \{x \in \mathbb{Z}^d,\ x \cdot l \geqslant L + 2,\ |R(e_i) \cdot x| < \tilde{L},\ i \geqslant 2\}.$$

We also attach three random variables p, q and ρ to this box , defined by the following relations

$$q(\omega) := P_{0,\omega}[X_{T_{B_{l,L,\tilde{L}}}} \notin \partial^+ B_{l,L,\tilde{L}}] = 1 - p(\omega)$$

along with

$$\rho = \frac{q(\omega)}{p(\omega)}$$

We are ready to define the effective criterion as follows:

Definition 0.1.6 (Effective Criterion). *Let $l \in \mathbb{S}^d$. Then the effective criterion with respect to l is satisfied if for some $L > c_2$, $\tilde{L} \in [3\sqrt{d}, L^3)$ and $a \in [0,1]$ the requirement*

$$\left\{ c_3 \left(\ln \left(\frac{1}{\kappa} \right) \right)^{3(d-1)} \tilde{L}^{d-1} L^{3d-2} \mathbb{E}[\rho^a] \right\} < 1,$$

Here, c_2 and c_3 are dimension dependent constants.

Notice that the effective criterion shares some similarities with the Solomon criterion $\mathbb{E}[\rho] < 1$ which ensures ballistic regime, as it can be seen from (0.0.2). We can be more precise yet with the statement of the equivalence in (12). Indeed the following theorem due to Sznitman uses the effective criterion as a pivotal condition.

Theorem 0.1.7 (Sznitman). *The following statements are equivalent:*

- *Effective Criterion with respect to l.*

- $(T')|l$.

- $(T^\gamma)|l$ *for* $1 > \gamma > 1/2$.

The proof of this theorem can be found in [Sz03]. Furthermore, in [BDR14] N. Berger, A. Drewitz and A. F. Ramírez proved the equivalence between this criterion and a polynomial decay of the probability entering in (11). Specifically, the polynomial condition in a non-effective form is the following requirement: let $M > 0$. The polynomial condition $(P^*)_M|l$ is satisfied if:

$$\lim_{L\to\infty} L^{-M} P_0 \left[\widetilde{T}^{l'}_{-\widetilde{b}L} < T^{l'}_{L}\right] = 0$$

for each l' in some neighborhood of l and $\widetilde{b} > 0$. One has the following:

Theorem 0.1.8 (Berger, Drewitz and Ramírez)**.** *Suppose that $(P^*)_M|l$ is satisfied for some $M > 15d + 5$. Then the Effective Criterion with respect to l is satisfied.*

Let us remark that actually in [BDR14], an effective version of the above polynomial condition on boxes was introduced. This means that it is a condition that in principle can be verified looking at the environment in large but finite boxes. The authors in this article also proved that the Effective Criterion of Sznitman is implied by their polynomial effective condition. Thus, using this *polynomial effective condition* one can avoid the use of the Effective Criterion to check ballisticity. On the other hand, using the equivalence between the Effective Criterion in direction l and $(T^{'})|l$ we conclude that

Theorem 0.1.9 (Berger, Drewitz, Ramírez and Sznitman)**.** *The following conditions are equivalent:*

- $(T^{'})|l$.
- $(T^{\gamma})|l$ *for* $1 > \gamma > 0$.
- $(P^*)_M|l$ *for* $M > 15d + 5$.
- *Effective Criterion with respect to* l.

As it was tacitly induced in the name given to $(T^{\gamma})|l$ for $\gamma \in (0, 1]$, under these conditions ballistic behavior is fulfilled. More precisely, combining (0.1.9) and Theorem 3.3 of [Sz03] we have that all the conditions in Theorem 0.1.9 satisfy :

$$P_0- \text{ a.s.}, \quad \frac{X_n}{n} \to v = \frac{E_0[X_{\tau_1} \mid D = \infty]}{E_0[\tau_1 \mid D = \infty]},$$

with $v \cdot l > 0$. Furthermore the random variable τ_1 has finite moments of any arbitrary order, and

$$B^n_\cdot := \frac{X_{[\cdot\ n]} - [\cdot\ n]v}{\sqrt{n}t}$$

converges in law on Skorohod space $D(R^+, R^d)$ under P_0 to the law of a non-degenerate Brownian Motion with matrix covariance given by

$$A = \frac{E_0[(X_{\tau_1} - \tau_1 v)^t (X_{\tau_1} - \tau_1 v) \mid D = \infty]}{E_0[\tau_1 \mid D = \infty]}.$$

It is then possible to show that the ballisticity conditions in direction l imply that

$$P_0\text{- a.s.}, \frac{T^l_u}{u} \to (v \cdot l)^{-1} \text{ as } u \to \infty.$$

Thus the walk escapes through direction l as if it had a local drift in direction l. Therefore the walk behavior is in concordance with the informal idea of what is meant by ballistic behavior. Besides seeing the effective criterion as a tool so as to get higher functional controls from lower ones on the walk exit probability by the unlikely side from slabs, we would like to mention that Sznitman in [Sz04] has found ballistic random walk examples satisfying $(T^{'})|l$ where the Kalikow's condition breaks down. As a result Kalikow's condition is not the weakest condition which ensures a ballistic behavior. Furthermore, it is conjectured that $(T^{'})|l$ is equivalent to ballisticity in direction l, which implies that Conjecture 0.0.3 can be rephrased as:

$$(T^{'})|l \;\leftrightarrow\; \text{the walk is transient in direction } l.$$

This ends our survey about ballisticity conditions in i.i.d. random environments.

0.2 Previous results for random walks in cone mixing random environments

In this section we would like to mention some results for random walks in random environments which are not i.i.d. The main result of Chapter 2 of this thesis is formulated in a framework of random walks in random environments which satisfy a mixing condition discussed in [CZ01], and called *cone mixing condition.* In [CZ01] it is proven that random walks in random environments satisfying a form of Kalikow's condition, cone mixing, and some important additional assumptions, are ballistic. A similar result was obtained by Rassoul-Agha in [RA03], where he assumes also Kalikow's condition and a mixing condition stronger that cone mixing called Dobrushin-Shlosman strong mixing assumption. Let us now describe these results.

We first describe the main result in [CZ01], which ensures ballisticity under some conditions on the environment. Since mixing on cones is strictly weaker than the i.i.d. condition, it will not be surprising that we will have to strengthen the ballisticity conditions in order to ensure ballistic behavior. Even more, we will have to define approximate regeneration times, since the standard definition of them in the i.i.d. context does not work. For large fixed integer L we define $\tau_1(L)$ as the first time that the walk reaches a record level in direction l at time $\tau_1(L) - L$, and such in the following L steps after this time, the walk does successive steps in the direction l. Further, after time $\tau_1(L)$, the walk never exits the cone $C(X_n, l, \alpha)$ again. This random time is much larger than the standard regeneration time used in the i.i.d. case. In fact, it can be shown that both $\tau_1(L)$ and $X_{\tau_1(L)}$ are of order κ^{-L} as $L \to \infty$. We also need to switch the stopping time D defined in (10) by D', which is essentially defined as the first exit time of the set $C(0, l, \alpha)$. We now need a suitable extension of the Kalikow's condition. For V a finite, connected subset of $\mathbb{Z}^d$, with $0 \in V$, we let

$$\mathfrak{F}_{V^c} = \sigma\{\omega(z, \cdot) : z \notin V\}.$$

The *Kalikow's random walk* $\{X_n : n \geqslant 0\}$ with state space $V \cup \partial V$ and starting from $y \in V \cup \partial V$ is defined by the transition probabilities

$$\widehat{P}_V(x, x+e) := \begin{cases} \frac{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n = x\}} \omega(x,e) | \mathfrak{F}_{V^c}]}{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n = x\}} | \mathfrak{F}_{V^c}]}, & \text{for} \quad x \in V \text{ and } e \in U \\ 1 & \text{for} \quad x \in \partial V \text{ and } e = 0. \end{cases}$$

We denote by $\hat{P}_{y,V}$ the law of this random walk and by $\hat{E}_{y,V}$ the corresponding expectation. The following extension of the Kalikow's condition was introduced in [CZ01].

Definition 0.2.1 (Kalikow's conditional condition)**.** *Let* $\delta > 0$. *We say that Kalikow's* *conditional condition with respect to the direction l is satisfied if there exists a positive constant δ such that*

$$\inf_{V:x\in V} \widehat{d}_V(x)\cdot l \geqslant \delta,$$

where

$$\widehat{d}_V(x) := \widehat{E}_{x,V}[X_1 - X_0] = \sum_{|e|=1} e\widehat{P}_V(x, x+e)$$

denotes the drift of Kalikow's random walk at x, and the infimum runs over all finite connected subset V of $\mathbb{Z}^d$ such that $0 \in V$. We denote this condition by $(KC)_\delta$.

Finally, we set:

$$\mathfrak{F}_{0,L} := \sigma\left\{\omega(y,\cdot);\ y\cdot l \leqslant -\frac{L}{|u|_1}|u|_2\right\}.$$

The main result in [CZ01] is the following.

Theorem 0.2.2 (Comets and Zeitouni). *Consider a random walk in a random environment satisfying Kalikow's conditional condition $(KC)_\delta$, the cone mixing condition $(CM)_{\alpha,\phi}|l$ and the ellipticity assumption $(UE)|l$. Assume also that there exists a positive function $M(L)$ depending just on L such that for some $\vartheta > 1$ one has that*

$$\mathbb{P}[E_0[(\kappa^L \tau_1)^{\vartheta} \mid \mathfrak{F}_{0,L}] > M] = 0 \tag{13}$$

and satisfying $\lim_{L\to\infty} M(L)^{\frac{1}{\vartheta'}} \phi'(L)^{\frac{1}{\alpha}} = 0$, where $\vartheta' := \vartheta/(\vartheta-1)$ along with

$$\phi' := \frac{2\phi}{P_0[D' = \infty] - \phi}.$$

Then there exists a deterministic $v \in \mathbb{R}^d - \{0\}$ such that P_0- a.s.

$$\lim \frac{X_n}{n} \to v,$$

with $v\cdot l > 0$.

The integrability condition (13) is essentially required in order to establish a law of large numbers along a regeneration time sequence which is not i.i.d. In the i.i.d. case, and under the polynomial ballisticity condition $(P*)_M|l$ when $M \geqslant 15d+5$ (c.f. Theorem), it is satisfied (as a matter of fact any moment of τ_1 is finite). Nevertheless, the integrability condition (13) is quite unsatisfactory, since it is in general difficult to check wether a given random environments satisfies it or not. As a mater of fact, in [CZ01], a non-trivial example which satisfies (13) is given, but the argument presented there is not completely clear.

On the other hand, Rassoul-Agha in [RA03], under a mixing condition called Dobrushin-Shlosman strong mixing assumption (see [CZ01] or [RA03]) has proved ballistic behavior by means of a clever application of *the environment as seen from the random walk* technique. It is important to stand out that Rassoul-Agha has only assumed the usual Kalikow's condition. However it was mentioned above that Kalikow's condition is strictly stronger than condition (T') [Sz04].

On the other hand, further important results can be found for instance in [CZ02], [RA05] and [G14]. In [CZ02] the authors proved suitable versions of the central limit theorem for the random walk in two kinds of environments: cone mixing and Dobrushing strong mixing. In [RA05], the author has investigated conditional versions of the strong law of large numbers. There it is proved that under an elliptic assumption and Dobrushin-Shlosman strong mixing condition on the environment a weak version of the strong law of large number is satisfied. Finally, Guo in [G14] under similar assumptions gave an alternative proof of the result in [RA05] by means of regeneration arguments (instead of the theoretical tool used by Rassoul-Agha: the environment as seen from the random walk) and proved that there is at most one nonzero limit velocity when $d \geqslant 5$ (originally proved in the i.i.d. case by Berger in [Be08]).

In conclusion, in both of the articles [CZ01] and [RA03] some version of Kalikow's condition is assumed. Furthermore, neither these works nor the ones mentioned in the last paragraph, discuss possible adaptations of weaker ballisticity conditions like conditions (T), (T') or $(P)_M$, to environments which are not necessarily i.i.d., even less so asymptotic results under these kind of conditions. One of the objectives of this thesis, developed in Chapter 2, is to give a first indication about how should these ballisticity conditions be defined for cone mixing environments.

0.3 A brief explanation of the Thesis Results

In this section we will describe the results of each chapter.

0.3.1 Main Result for i.i.d. Random Environments

A problem left untouched in the quoted results of Section 0.1.2 is the following question:

Conjecture 0.3.1. *For a random walk in a uniformly elliptic i.i.d. environment, condition $(T^{'})|l$ is equivalent to $(T)|l$.*

Chapter 1 of this thesis addresses this question. The main result of Chapter 1 shows that condition $(T')|l$ implies an *almost* exponential decay for the exit probability of the random walk through the back side of slabs (which is very close to $(T)|l$). Specifically, for a given direction l and $L > 0$ we denote by $S_{l,L}$ the strip $\{x \in \mathbb{R}^d : \ |x \cdot l| \leqslant L\}$ and by $A_{l,L}$ the event that the walk starting from 0 exits $S_{l,L}$ through the side of $S_{l,L}$ where $x \cdot l < 0$. Now, for a given direction l and function $\gamma : [0, \infty) \to [0, 1]$ we say that the condition $(T)_{\gamma(L)}|l$ is satisfied if for all directions l' in a neighborhood of l there is a constant $c > 0$ such that asymptotically as $L \to \infty$ it is true that

$$P_0[A_{l',L}] = e^{-cL^{\gamma(L)}+o(L^{\gamma(L)})}.$$

It is straightforward to check that by definition, condition $(T)|l$ is equivalent to $(T)_{\gamma_1(L)}|l$ with

$$\gamma_1(L) = 1 - C\frac{1}{\log(L)},$$

for any $C > 0$. On the other hand, in [Sz03] Sznitman proved that $(T')|l$ implies $(T)_{\gamma_2(L)}|l$ with

$$\gamma_2(L) = 1 - C\frac{\log^{\frac{1}{2}}(L)}{\log(L)}.$$

In Chapter 1 of this thesis, we prove that $(T')|l$ implies $(T)_{\gamma_3(L)}|l$ with

$$\gamma_3(L) = 1 - \widetilde{C}\frac{\log_{n(L)}(L)}{\log(L)}, \tag{14}$$

where $\widetilde{C}$ is a positive constant and $n(L)$ a function with values in the positive integers that has limit infinity as $L \to \infty$. $\log_k$ denotes the function logarithm composed $k-1$ times with itself; i.e., $\log_k(x) = \overbrace{\log \circ \log \circ \log \circ \ldots \circ \log}^{k}(x)$. In spite that this result seems to be close to answering affirmatively Conjecture 0.3.1, it does not. Indeed, the function $n(L)$ of (14) is such that

$$\lim_{L \to \infty} \log_{n(L)}(L) = \infty.$$

The proof of this result relies on renormalization arguments which have the Effective Criterion as a seed condition.

0.3.2 Main Result for cone mixing random environments

Chapter 2 of this thesis is concerned with random walks in cone mixing random environments. The main result is the proof that under a non-effective polynomial ballisticity condition, these random walks have an asymptotic direction (see **??**). In what follows we will define this version of the polynomial ballisticity ccondition. Given $L, L' > 0$, $x \in \mathbb{Z}^d$, we define the boxes

$$B_{L,L',l}(0) := R\left((-L, L) \times (-L', L')^{d-1}\right) \cap \mathbb{Z}^d,$$

where R is a rotation on $\mathbb{R}^d$ such that $R(e_1) = l$. Define the *positive boundary* of $B_{L,L',l}(x)$, denoted by $\partial^+ B_{L,L',l}(0)$, as

$$\partial^+ B_{L,L',l}(0) := \partial B_{L,L',l}(0) \cap \{z : z \cdot l \geqslant L\},$$

Define also the half-space

$$H_{x,l} := \{y \in \mathbb{Z}^d : y \cdot l < x \cdot l\},$$

and the corresponding σ-algebra of the environment on that half-space

$$\mathcal{H}_{x,l} := \sigma(\omega(y) : y \in H_{x,l}).$$

Now, for $M \geqslant 1$, we say that the *non-effective polynomial* condition $(PC)_{M,c}|l$ is satisfied if there exists some $c > 0$ so that for $y \in H_{0,l}$ one has that

$$\lim_{L\to\infty} L^M \sup P_0 \left[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0), T_{B_{L,cL,l}(0)} < T_{H_{y,l}} | \mathcal{H}_{y,l} \right] = 0, \quad (15)$$

where the supremum is taken over all possible environments to the left of $y \cdot l$. We prove the existence of an asymptotic direction for random walks in random environment satisfying the condition $(CM)_{\alpha,\phi}|l$ under the assumptions $(PC)_{M,c}|l$ and $(UE)|l$, where the positive constants M, c and α satisfy the constraints:

$$M > 6d \text{ and } 0 < \alpha \leqslant \min\{\frac{1}{9}, \frac{1}{2c+1}\} \quad (16)$$

We will prove that the *non-effective polynomial* condition is weaker than the conditional version of Kalikow's condition introduced in [CZ01]. We would like to sketch the general strategy behind the proof of this result. As a first step we need to prove that:

$$P_0[D' = \infty] > 0. \quad (17)$$

Let us remark that we do not need a conditional version of the ballisticity assumption to prove this. To prove the claim (17), we have used renormalization type methods, so as to apply the polynomial condition. Specifically, using the assumption $(UE)|l$ we can and we do assume that the walk starting from 0 goes on a large distance through direction l up to a fixed point z with positive annealed probability, and starting from that point one can show that with a high probability the walk remains forever inside of each half-space: $H_{\pm i} := \{y : (y - z) \cdot l_{\pm i} \geqslant 0\}$, for $i \in [2, d]$. Finally the result follows from the definition of the cone. We refer to the proof of Proposition 2.4.1 in Chapter 2 for the precise argument. As a second step, we proved a strong integrability result of the regeneration position $X_{\tau_1(L)}$. Roughly speaking, we have proved that the *conditional* expectation of the second moment of the regeneration position is finite. These two steps are the core of the proof. Indeed using for instance similar arguments as the ones given in [CZ01] we can obtain the asymptotic direction $\hat{v}$. The main issue to integrate the second moment of the random variable $\kappa^L X_{\tau_1(L)}$ was to connect i.i.d. methods with the cone mixing model. We connect them by identifying how close (or far) the *old* τ_1 is from the *new* $\tau_1(L)$. The precise statement of the required integrability condition and its proof are given in Section 2.5 of Chapter 2.

On the other hand, the simpler Simenhaus's approach [Si07] does not work in cone mixing environment at least if we identify the random variable τ_1 with $\tau_1(L)$. The argument of [Si07] makes a strong use of i.i.d. assumption on the environment. The i.i.d. structure of the environment space is explicitly required in the renewal theorem to prove Zerner's formula (c.f. Lemma 2 in [Si07]) and in order to prove that the sequence $(Z_k) := \sup_{n\geqslant 0} |X_{n\wedge\tau_{k+1}} - X_{\tau_k}|$ is such that Z_n/n converges P_0-a.s. to 0 as $n \to \infty$. The first argument breaks down in the cone-mixing case, mainly because one cannot apply the renewal theorem without assuming some kind of strong integrability condition for the regeneration position. Furthermore, as an example to understand possible pathologies in the behavior of a random walk in a cone mixing environment, we provided an example of a random walk defined in a cone mixing environment which is directionally transient but not ballistic, showing that we cannot expect the ballisticity conjecture 0.0.3 to be valid outside of the i.i.d. setting. Consequently one could ask the following: what would be the kind of *natural* conditions which ensure that the random walk satisfies a strong law of large numbers with a non-vanishing limit velocity in this framework? We expect that the machinery developed in Chapter 2 could serve in a future work to prove ballistic behavior under a ballisticity condition similar to condition (T').

The two results are a joint work with Alejandro Ramírez.

Chapter 1

Almost exponential decay for the exit probability from slabs of ballistic RWRE

1.1 Introduction

The relationship between directional transience and ballisticity for random walks in random environment is one of the most challenging open questions within the field of random media. In the case of random walks in an i.i.d. random environment, several ballisticity conditions have been introduced which quantify the exit probability of the random walk through a given side of a slab as its width L grows, with the objective of understanding the above relation. Examples of these ballisticity conditions include Sznitman's (T') and (T) conditions [Sz02, Sz03]: given a slab of width L orthogonal to l, condition (T') in direction l is the requirement that the annealed exit probability of the walk through the side of the slab in the half-space $\{x : x \cdot l < 0\}$, decays faster than e^{-CL^γ} for all $\gamma \in (0,1)$ and some constant $C > 0$, while condition (T) in direction l is the requirement that the decay is exponential e^{-CL}. It is believed that condition (T'), is equivalent to condition (T). In this chapter we prove that condition (T') implies an *almost* exponential decay (see Theorem 1.1.2 for the precise meaning of this statement) of the corresponding exit probabilities. Our proof relies on a recursive renormalization scheme, where the a careful choice of fastly growing scales enables us to obtain the result. We use the equivalence between condition (T') [Sz03] and the $d \geqslant 2$ dimensional version of Solomon's criterion [So75], known as the effective criterion [Sz03].

Let us introduce the random walk in random environment model. For $x \in \mathbb{Z}^d$ denote its euclidean norm by $|x|_2$. Let $V := \{e \in \mathbb{Z}^d : |e|_2 = 1\}$ be the set of canonical vectors. Introduce the set $\mathcal{P}$ whose elements are $2d-$vectors $p(e)_{e \in \mathbb{Z}^d, |e|=1}$ such that

$$p(e) \geqslant 0, \text{ for all } e \in V, \sum_{e \in \mathbb{Z}^d, |e|=1} p(e) = 1.$$

We define an environment $\omega := \{\omega(x) : x \in \mathbb{Z}^d\}$ as an element of $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, where for each $x \in \mathbb{Z}^d$, $\omega(x) = \{\omega(x,e) : e \in V\} \in \mathcal{P}$. Consider a probability measure $\mathbb{P}$ on Ω endowed with its canonical product σ-algebra, so that an environment is now a random variable such that the coordinates $\omega(x)$ are i.i.d. under $\mathbb{P}$. The random walk in the random environment ω starting from $x \in \mathbb{Z}^d$ is the canonical Markov Chain $\{X_n : n \geqslant 0\}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ with *quenched law* $P_{x,\omega}$ starting from x, defined by the transition probabilities for each $e \in \mathbb{Z}^d$ with $|e| = 1$ by

$$P_{x,\omega}[X_{n+1} = X_n + e | X_0, \ldots, X_n] = \omega(X_n, e)$$

and

$$P_{x,\omega}[X_0 = x] = 1.$$

The *averaged* or *annealed law*, P_x, is defined as the semi-direct product measure

$$P_x = \mathbb{P} \times P_{x,\omega}$$

on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$. Whenever there is a $\kappa > 0$ such that

$$\inf_{e,x} \omega(x,e) \geqslant \kappa \qquad \mathbb{P} - a.s.$$

we will say that the law $\mathbb{P}$ of the environment is *uniformly elliptic*.

For the statement of the result, we need some further definitions. For each subset $A \subset \mathbb{Z}^d$ we define the first exit time of the random walk from A as

$$T_A := \inf\{n \geqslant 0 : X_n \notin A\}.$$

Fix a vector $l \in \mathbb{S}^{d-1}$ and $u \in \mathbb{R}$ then define the half-spaces $H^-_{u,l} := \{x \in \mathbb{Z}^d : x \cdot l < u\}$, $H^+_{u,l} := \{x \in \mathbb{Z}^d : x \cdot l > u\}$,

$$T^l_u := T_{H^-_{u,l}} = \inf\{n \geqslant 0,\ X_n \cdot l \geqslant u\}$$

and

$$\tilde{T}^l_u := T_{H^+_{u,l}} = \inf\{n \geqslant 0,\ X_n \cdot l \leqslant u\}.$$

For $\gamma \in (0, 1]$, we say that condition $(T)_\gamma|l$ holds with respect to direction $l \in \mathbb{S}^{d-1}$, if

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0[\tilde{T}^{l'}_{-L} < T^{l'}_L] < 0,$$

for all l' in some neighborhood of l. Furthermore, we define $(T')|l$ as the requirement that condition $(T)_\gamma|l$ is satisfied for all $\gamma \in (0, 1)$ and condition $(T)|_l$ as the requirement that $(T)_1|l$ is satisfied. In [Sz03], Sznitman proved that when $d \geqslant 2$ for every $\gamma \in (0.5, 1)$, $(T)_\gamma|l$ is equivalent to $(T')|l$. This equivalence was improved in [DR11] and [DR12] culminating with the work of Berger, Drewitz and Ramírez who in [BDR14] showed that for any $\gamma \in (0, 1)$, condition $(T)_\gamma|l$ implies $(T')|l$. As a matter of fact, in [BDR14], an effective ballisticity condition, which requires polynomial decay was introduced. To define this condition, consider $L, \tilde{L} > 0$ and $l \in \mathbb{S}^{d-1}$ and the box

$$B_{l,L,\tilde{L}} := R\left((-L, L) \times (-\tilde{L}, \tilde{L})^{d-1}\right) \cap \mathbb{Z}^d,$$

where R is a rotation defined by

$$R(e_1) = l. \tag{1.1}$$

Given $M \geqslant 1$ and $L \geqslant 2$, we say that the polynomial condition $(P)_M$ in direction l (also denoted by $(P)_M|l$) is satisfied on a box of size L if there exists and $\tilde{L} \leqslant 70L^3$ such that

$$P_0\left[X_{T_{B_{l,L,\tilde{L}}}} \cdot l < L\right] \leqslant \frac{1}{L^M}.$$

Berger, Drewitz and Ramírez proved in [BDR14] that there exists a constant c_0 such that whenever $M \geqslant 15d + 5$, the polynomial condition $(P)_M|l$ on a box of size $L \geqslant c_0$ is equivalent to condition $(T')|l$ (see also Lemma 3.1 of [CR14]). On the other hand, the following is still open.

Conjecture 1.1.1. *Consider a random walk in a uniformly elliptic random environment in dimension $d \geqslant 2$ and $l \in \mathbb{S}^{d-1}$. Then, condition $(T)|l$ is equivalent to $(T')|l$.*

To quantify how far are we presently from proving Conjecture 1.1.1, we will introduce now a family of intermediate conditions between conditions (T') and (T). Let $\gamma(L)$: $[0, \infty) \to [0, 1]$, with $\lim_{L\to\infty} \gamma(L) = 1$. Let $l \in \mathbb{S}^d$. We say that condition $(T)_{\gamma(L)}|_l$ is satisfied if

$$\limsup_{L\to\infty} L^{-\gamma(L)} \log P_0[\tilde{T}^{l'}_{-L} < T^{l'}_{L}] < 0, \tag{1.2}$$

for l' in a neighborhood of l. We will call $\gamma(L)$ the *effective parameter* of condition $(T)_{\gamma(L)}$. Note that condition (T) is actually equivalent to $(T)_{\gamma(L)}$ with an effective parameter given by

$$\gamma(L) = 1 - \frac{C}{\log L}, \tag{1.3}$$

for any constant $C \geqslant 0$.

In 2002 Sznitman [Sz03] was able to prove that (T') implies $(T)_{\gamma(L)}$ with effective parameter

$$\gamma(L) = 1 - \frac{C}{\log L}\sqrt{\log L}, \tag{1.4}$$

for some constant $C > 0$.

In this chapter, we are able to show that condition (T') implies condition $(T)_{\gamma(L)}$ with an effective parameter $\gamma(L)$ which is closer to the effective parameter for condition (T) given by (1.3). This is the first result since the introduction of condition (T') by Sznitman in 2002, which would give an indication that Conjecture 1.1.1 is true. To state it, let us introduce some notations. Throughout, for each $n \geqslant 1$, we will use the standard notation

$$\overbrace{\log \circ \cdots \circ \log}^{n} x,$$

for the composition of the logarithm function n times with itself, for all x in its domain, where the n superscript means that the composition is performed n times.

Theorem 1.1.2. *Let $d \geqslant 2$, $l \in S^{d-1}$ and $M \geqslant 15d + 5$. Assume that condition $(P)_M|l$ is satisfied on a box of size $L \geqslant c_0$. Then there exists a constant $C > 0$ and a function $n(L) : [0, \infty) \to \mathbb{N}$ satisfying $\lim_{L\to\infty} n(L) = \infty$, such that condition $(T)_{\gamma(L)}|_l$, c.f. (1.2), is satisfied with an effective parameter $\gamma(L)$ given by*

$$\gamma(L) = 1 - \frac{C}{\log L} \overbrace{\log \circ \cdots \circ \log}^{n(L)} L. \tag{1.5}$$

Remark 1.1.3. Note that the decay given by the effective parameter (1.5) of Theorem 1.1.2 is equivalent to the decay

$$\limsup_{L\to\infty} \frac{\overbrace{\log \circ \cdots \circ \log}^{n(L)-1} L}{L} \log P_0[\tilde{T}^{l'}_{-L} < T^{l'}_{L}] < 0,$$

for l' in a neighborhood of l.

Let us remark that a priori, even if $n(L) \to \infty$ as $L \to \infty$, it might happen that the composition of the logarithm $n(L)$ time is bounded. Nevertheless, in the case of Theorem 1.1.2, it turns out that

$$\lim_{L\to\infty} \overbrace{\log \circ \cdots \circ \log}^{n(L)} L = \infty.$$

Theorem 1.1.2 will be proven in the next section, but some remarks are in order. The strategy followed in the proof, roughly speaking, is to improve the iterative procedure used by Sznitman in [Sz02], to prove $(T)_{\gamma(L)}$ with $\gamma(L)$ given by (1.4), through the so called effective criterion introduced by Sznitman in [Sz03]. The iterative procedure used in [Sz03], in spirit is a renormalization argument, where the idea is to control the exit probability of the walk recursively from an initial scale L_0 to the final size of the slab

$L > L_0$ passing through a sequence of intermediate scales $L_0 < L_1 < \ldots < L_k = L$. To go from scale L_0 to scale L_1, a slab of width L_1 is subdivided into overlapping slabs of width L_0, and the walk is looked at its exit times from successive slabs of width L_0. Essentially, at these times the walk looks like a one dimensional random walk in random environment, for which one can control its exit probabilities through the expected value of ρ, where ρ is close to the quotient between the probability to exit a slab of width L_0 through its left side and the probability to exit it through its right side. Here, a triggering assumption is needed, which in our case is the effective criterion of Sznitman [Sz02] (the effective criterion is implied by the polynomial condition introduced by Berger, Drewitz and Ramírez in [BDR14]). This first step is the content of Proposition 2.1. A similar strategy is then used to pass from scale L_k to scale L_{k+1} for $k \geqslant 1$ (see Lemma 2.2). Nevertheless, reducing the movement of the random walk to a one dimensional walk, has a cost, which is a polynomial factor appearing in the recursion relations, and which somehow is the reason why one cannot go from the initial scale L_0 directly to L in one step. In this chapter, we modify Sznitman's argument, choosing a sequence of scales where L_{k+1} is much larger than L_k compared to Sznitman's approach, allowing us to work with a smaller number of intermediate steps in the recursion relation. The use of this new sequence of scales, produces at some points important difficulties in the proof which have to be properly handled.

1.2 Proof of Theorem 1.1.2

Throughout the rest of this section, we prove Theorem 1.1.2. Firstly, in subsection 1.2.1, we will introduce the basic notation which will be needed to implement the renormalization scheme, and we will recall a basic result of Sznitman which provides a bound for quantities involving the exit probability through the unlikely side of boxes which are inspired in techniques for used for one-dimensional random walks in random environment. In the second subsection, we will introduce a growth condition which will limit the maximal way in which the scales on the renormalization scheme can grow, while still giving a useful recurrence. In the third subsection we will choose an adequate sequence of scales satisfying the condition of subsection 1.2.2, and for which one can make computations. Finally, in subsection 1.2.4, Theorem 1.1.2 will be proven using the scales constructed in subsection 1.2.3 through the use of the effective criterion [Sz02].

1.2.1 Preliminaries and notation

The proof of Theorem 1.1.2 will follow the renormalization method used by Sznitman to prove Proposition 2.3 of [Sz02]. The idea is to use a renormalization procedure which somehow mimics a computation for a one-dimensional random walk in random environment, where one goes from one scale to the next (larger) one through formulas where the exit probabilities of the random walk through slabs at the smaller scales are involved.

Following Sznitman we introduce boxes transversal to direction l, which are specified in terms of $\mathcal{B} = (R, L, L', \tilde{L})$, where $L, L', \tilde{L}$ are positive numbers and R is the rotation defined in (2.3). The box attached to $\mathcal{B}$, is

$$B := R((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}) \cap \mathbb{Z}^d$$

and the positive part of its boundary is defined as

$$\partial_+ B := \partial B \cap \{x \in \mathbb{Z}^d,\, x \cdot l \geqslant L',\, |R(e_i) \cdot x| < \tilde{L},\, i \geqslant 2\}.$$

We can now define the following random variable depending on a given specification $\mathcal{B}$, analogous to the quotient in dimension $d = 1$ between the probability to jump to the left and the probability to jump to the right [SW69, So75], for $\omega \in \Omega$ as

$$\rho_{\mathcal{B}}(\omega) := \frac{q_{\mathcal{B}}(\omega)}{p_{\mathcal{B}}(\omega)},$$

where

$$q_{\mathcal{B}}(\omega) := P_{0,\omega}[X_{T_B} \notin \partial_+ B] =: 1 - p_{\mathcal{B}}(\omega).$$

The first step in the renormalization procedure will be to control the moments of $\rho_{\mathcal{B}}$ at the two first scales. To this end, consider positive numbers

$$3\sqrt{d} < L_0 < L_1,\;\; 3\sqrt{d} < \tilde{L}_0 < \tilde{L}_1$$

along with the box-specifications

$$\mathcal{B}_0 := (R, L_0 - 1, L_0 + 1, \tilde{L}_0)$$

and

$$\mathcal{B}_1 := (R, L_1 - 1, L_1 + 1, \tilde{L}_1).$$

It is convenient to introduce now the notation

$$q_0 := q_{\mathcal{B}_0},\ p_0 := p_{\mathcal{B}_0}, \qquad q_1 := q_{\mathcal{B}_1},\ p_1 := p_{\mathcal{B}_1},$$

and

$$\rho_0 := \rho_{\mathcal{B}_0},\ \ \rho_1 := \rho_{\mathcal{B}_1}. \tag{1.6}$$

Let also

$$N_0 := \frac{L_1}{L_0} \quad \text{and} \quad \tilde{N}_0 := \frac{\tilde{L}_1}{\tilde{L}_0}.$$

We will also need to introduce the constant

$$c_1(d) = c_1 := \sqrt{d}.$$

Note that for each pair of points $x, y \in \mathbb{Z}^d$, there exists a nearest neighbor path joining them which has less than $c_1|x-y|_2$ steps.

Let us now recall the following Proposition of Sznitman [Sz03].

Proposition 1.2.1. *There exist $c_2(d) > 3\sqrt{d}$, $c_3(d), c_4(d) > 1$, such that when $N_0 \geqslant 3, L_0 \geqslant c_2, \tilde{L}_1 \geqslant 48N_0\tilde{L}_0$, for each $a \in (0,1]$ one has that*

$$\mathbb{E}\left[\rho_1^{\frac{a}{2}}\right] \leqslant c_3 \left\{ \kappa^{-10c_1L_1} \left(c_4 \tilde{L}_1^{d-2} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E}[q_0] \right)^{\frac{\tilde{L}_1}{12N_0\tilde{L}_0}} \right.$$
$$\left. + \sum_{0 \leqslant m \leqslant N_0+1} \left(c_4 \tilde{L}_1^{d-1} \mathbb{E}[\rho_0^a] \right)^{\frac{[N_0]+m-1}{2}} \right\}. \tag{1.7}$$

1.2.2 The maximal growth condition on scales

We next recursively iterate inequality (1.7) at different scales which will increase as fast as possible, in the sense that a certain induction condition should enable us to push forward the recursion.

We next recursively iterate inequality (1.7) at different scales which will increase as fast as possible, in the sense that a certain induction hypothesis should enable us to push forward the recursion. Let

$$v := 8,\ \alpha := 240$$

and introduce two sequences of scales $L_k, \tilde{L}_k\ \ k \geqslant 0$, such that

$$L_0 \geqslant c_2\,, 3\sqrt{d} < \tilde{L}_0 \leqslant L_0^3 \tag{1.8}$$

and for $k \geqslant 0$

$$N_k \geqslant 7,\ L_{k+1} = N_k L_k,\ \tilde{L}_{k+1} = N_k^3 \tilde{L}_k, \tag{1.9}$$

as well as box-specifications

$$\mathcal{B}_k := (R, L_k - 1, L_k + 1, \tilde{L}_k).$$

Note that

$$\tilde{L}_{k+1} = \left(\frac{L_k}{L_0}\right)^3 \tilde{L}_0. \tag{1.10}$$

Introduce also the notation for the respective attached random variables

$$\rho_k := \rho_{\mathcal{B}_k}.$$

Throughout, we will adopt the notation

$$u_0 := \frac{3(d-1)}{L_0 \log \frac{1}{\kappa}}, \tag{1.11}$$

and for $k \geqslant 1$,

$$u_k := \frac{u_0}{v^k}. \tag{1.12}$$

We also let

$$c_5 := 2c_3c_4.$$

Condition (G)**.** *We say that the scales* $L_k, N_k, k \geqslant 0$ *satisfy condition* (G) *if*

$$u_k N_k \geqslant \alpha c_1 \text{ for } k \geqslant 0, \tag{1.13}$$

and if

$$c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1}L_{k+1}} \leqslant 1 \textit{ for } k \geqslant 0. \tag{1.14}$$

Let us now state the following lemma which generalizes Lemma 2.2 of Sznitman ([Sz03]), for scales satisfying condition (G). For completeness we include its proof.

Lemma 1.2.2. *Consider scales* $L_k, N_k, k \geqslant 0$*, such that condition* (G) *is satisfied. Then, whenever* $L_0 \geqslant c_2$*,* $3\sqrt{d} \leqslant \tilde{L}_0 \leqslant L_0^3$*, and* $a_0 \in (0,1]$*, we have that*

$$\varphi_0 := c_4 \tilde{L}_1^{d-1} L_0 \mathbb{E}[\rho_0^{a_0}] \leqslant \kappa^{u_0 L_0}. \tag{1.15}$$

then for all $k \geqslant 0$*,*

$$\varphi_k := c_4 \tilde{L}_{k+1}^{d-1} L_k \mathbb{E}[\rho_k^{a_k}] \leqslant \kappa^{u_k L_k}. \tag{1.16}$$

with

$$a_k = a_0 2^{-k}, \ u_k = u_0 v^{-k}.$$

Proof. As in the proof of Lemma 2.2 of [Sz02], we can conclude by Proposition 1.2.1 that if $L_0 \geqslant c_2$ (note that by the choice of N_k in (1.9), the other conditions of Proposition 1.2.1 are satisfied) we have that for $k \geqslant 0$,

$$\varphi_{k+1} \leqslant c_3 c_4 \tilde{L}_{k+2}^{d-1} L_{k+1} \left\{ \kappa^{-10c_1 L_{k+1}} \varphi_k^{\frac{N_k^2}{12}} + \sum_{0 \leqslant m \leqslant N_k+1} \varphi_k^{\frac{[N_k]+m-1}{2}} \right\}. \tag{1.17}$$

We will now prove inequality (1.16) by induction on k using inequality (1.17). Since inequality (1.15) is identical to inequality (1.16) with $k = 0$, the induction hypothesis is satisfied for $k = 0$. We assume now that it is true for $k > 0$, along with inequality (1.13) of assumption (G) and conclude that

$$\kappa^{-10c_1 L_{k+1}} \varphi_k^{\frac{N_k^2}{24}} \leqslant \kappa^{-10c_1 L_{k+1}} \kappa^{N_k^2 \frac{L_k u_k}{24}} \leqslant 1. \tag{1.18}$$

Therefore, using (1.18) and the fact that $[N_k] - 1 \geqslant \frac{N_k}{2}$ because $N_k \geqslant 7$ we see that

$$\begin{aligned} \varphi_{k+1} &\leqslant c_3 c_4 \tilde{L}_{k+2}^{d-1} L_{k+1} \left\{ \varphi_k^{\frac{N_k^2}{24}} + L_{k+1} \varphi_k^{\frac{N_k}{4}} \right\} \\ &\leqslant c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \varphi_k^{\frac{N_k}{8}}, \end{aligned} \tag{1.19}$$

where we recall that $c_5 = 2c_3c_4$. Now, by the induction hypothesis (1.16) we see that

$$\varphi_k^{\frac{N_k}{8}} \leqslant \kappa^{u_{k+1} L_{k+1}}.$$

Substituting this into (1.19), we see that it is enough now to show that

$$c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \leqslant 1.$$

But this is true, by (1.14) of condition (G), the induction hypothesis and the inequality $\tilde{L}_{k+1} \leqslant L_{k+1}^3$ for $k \geqslant 0$ which follows by induction starting from (1.8). Indeed, using these facts,

$$c_5 \tilde{L}_{k+2}^{d-1} L_{k+1}^2 \varphi_k^{\frac{N_k}{8}} \leqslant c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1} L_{k+1}} \leqslant 1,$$

which ends the proof.

□

1.2.3 An adequate choice of fast-growing scales

We will now construct a sequence of scales $\{L_k : k \geqslant 0\}$ which satisfy condition (G), and for which Lemma 1.2.2 will eventually imply Theorem 1.1.2. This is not the fastest possible growing sequence of scales, but somehow it captures the best possible choice of $\gamma(L)$.

Let $\{f_k : k \geqslant 1\}$ be a sequence of functions from $[0,\infty)$ to $[0,\infty)$ defined recursively as

$$f_0(x) := 1,$$

$$f_1(x) := v^x$$

and for $k \geqslant 1$,

$$f_{k+1}(x) := f_k \circ f_1(x).$$

Let now, for $k \geqslant 0$,

$$N_k := \frac{\alpha c_1}{u_0} \frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right)}. \tag{1.20}$$

According to display (1.9), we have the following formula valid for $k \geqslant 0$,

$$L_{k+1} = f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{u_0}\right)^{k+1} L_0. \tag{1.21}$$

Lemma 1.2.3. *The condition*

$$u_k N_k \geqslant \alpha c_1 \quad \text{for} \quad k \geqslant 0$$

(c.f. (1.13) of condition (G)) is equivalent to

$$\frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right) v^k} \geqslant 1 \quad \text{for } k \geqslant 0. \tag{1.22}$$

Furthermore, the last relation is fulfilled.

Proof. Note that (1.22) can be easily verified for $k = 0, 1$ and 2. Therefore it is enough to prove inequality (1.22) for $k \geqslant 3$. For this purpose, we will first show that for all positive integers n, and a, $b \in [1, \infty)$, we have that

$$f_n(a+b) \geqslant f_n(a) f_n(b). \tag{1.23}$$

To prove (1.23), suppose that

$$A := \{n \in \mathbb{N} : f_n(a+b) < f_n(a) f_n(b) \text{ for some } a, b \geqslant 1\} \neq \varnothing.$$

Let m be the smallest element of A and remark that m is greater than 1. Also, note that

$$f_m(a+b) < f_m(a) f_m(b)$$

for some $a, b \geqslant 1$. However, note that for $a, b \geqslant 1$ one has that

$$v^{a+b} \geqslant v^a + v^b.$$

Furthermore, for each $k \geqslant 0$, the function $f_k(\cdot)$ is increasing. Therefore,

$$\begin{aligned} f_{m-1}(v^a) f_{m-1}(v^b) &= f_m(a) f_m(b) \\ > f_m(a+b) = f_{m-1}(v^{a+b}) &\geqslant f_{m-1}(v^a + v^b). \end{aligned}$$

This contradicts the minimality of m and hence $A = \varnothing$ which proves (1.23). Back to (1.22), note that

$$\frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right) v^k} \geqslant \frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]-1\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right)} \frac{f_{\left[\frac{k+2}{2}\right]}(1)}{v^k} \geqslant \frac{f_{\left[\frac{k+2}{2}\right]}(1)}{v^k} \geqslant 1,$$

where the first inequality was gotten using (1.23), the second one is a consequence of the inequality

$$\frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]-1\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right)} \geqslant 1,$$

valid for $k \geqslant 3$, and which can be proved in a straightforward fashion if we divide the argument according to whether k is even or odd, and the last inequality comes from the fact that

$$f_{[\frac{k+2}{2}]-1}(1) - k \geqslant 0 \qquad \text{for } k \geqslant 3. \tag{1.24}$$

Now, it is easy to verify inequality (1.24) when $k = 3$ and $k = 4$. Furthermore, the left hand of (1.24) is increasing as a function of $k \geqslant 2$ for k odd. Similarly, it is increasing for $k \geqslant 2$ for k even. We can therefore conclude, using induction that (1.24) is satisfied. This completes the proof of (1.22). □

Using Lemma 1.2.3 we can now obtain the following important lemma which gives conditions on the growth of a sequence of scale which ensure that (G) is satisfied.

Lemma 1.2.4. *There exists a constant $c_6(d)$ such that when $L_0 \geqslant c_6$, the scales $\{L_k : k \geqslant 0\}$ and $\{N_k : k \geqslant 0\}$ defined by (1.21) and (1.20) satisfy condition (G).*

Proof. By Lemma 1.2.3 we know that (1.13) of condition (G) is satisfied. We therefore just prove inequality (1.14) of condition (G). We need to show that there exists a constant $c(d, \kappa)$, such that whenever $L_0 \geqslant c(d, \kappa)$, for all $k \geqslant 0$ one has that

$$c_5 N_{k+1}^{3(d-1)} L_{k+1}^{3d-1} \kappa^{u_{k+1} L_{k+1}} \leqslant 1. \tag{1.25}$$

We will first show that there exists $c_7(d, \kappa) = c_7(d) > 0$, such that whenever $L_0 \geqslant c_7$, one has that for $k \geqslant 0$,

$$N_{k+1}^{3(d-1)} \kappa^{\frac{u_{k+1} L_{k+1}}{3}} \leqslant 1. \tag{1.26}$$

Now (1.26) is equivalent to

$$3(d-1) \log_v \left(\frac{\alpha c_1}{u_0} \frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \right) - \frac{L_0 u_0 f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right) \left(\frac{\alpha c_1}{v u_0}\right)^{k+1} \log_v\left(\frac{1}{\kappa}\right)}{3} \leqslant 0.$$

Therefore, (1.26) is equivalent to the bound for $k \geqslant 0$,

$$L_0 \geqslant \frac{\frac{9(d-1)}{u_0}\log_v\left(\frac{\alpha c_1}{u_0}\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{vu_0}\right)^{k+1}\log_v\left(\frac{1}{\kappa}\right)}. \tag{1.27}$$

Let us focus on right-hand side of inequality (1.27) . Note that it can be split as

$$\frac{\frac{9(d-1)}{u_0}\log_v\left(\frac{\alpha c_1}{u_0}\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{vu_0}\right)^{k+1}\log_v\left(\frac{1}{\kappa}\right)} + \frac{\frac{9(d-1)}{u_0}\log_v\left(\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{vu_0}\right)^{k+1}\log_v\left(\frac{1}{\kappa}\right)}. \tag{1.28}$$

Let us now try to find an upper bound for this expression independent on u_0 (or equivalently, on L_0). By the definition of u_0 (c.f. (1.11)) note that for $k \geqslant 0$ and $L_0 \geqslant \frac{3(d-1)}{\log\frac{1}{\kappa}}$ one has that,

$$\frac{1}{u_0}\frac{1}{\left(\frac{\alpha c_1}{vu_0}\right)^{k+1}} = \frac{1}{\left(\frac{\alpha c_1}{vu_0}\right)^{k}}\frac{1}{\left(\frac{\alpha c_1}{v}\right)} \leqslant \frac{1}{\left(\frac{\alpha c_1}{v}\right)^{k+1}}.$$

Substituting this into (1.28) we see that it is bounded from above by

$$\frac{9(d-1)\log_v\left(\frac{\alpha c_1}{u_0}\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{v}\right)^{k+1}\log_v\left(\frac{1}{\kappa}\right)} + \frac{9(d-1)\log_v\left(\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{v}\right)^{k+1}\log_v\left(\frac{1}{\kappa}\right)}. \tag{1.29}$$

Note that only the left-most term of (1.29) depends on L_0. Choose a constant $c_8(d,\kappa) = c_8(d) > 1$, such that if $L_0 \geqslant c_8$

$$\log_v\left(\frac{\alpha c_1}{u_0}\right) \leqslant L_0\frac{\log_v\left(\frac{1}{\kappa}\right)}{d-1}. \tag{1.30}$$

Then, when $L_0 \geqslant c_8$, we see using the fact that the left-most term of (1.29) is a decreasing function of $k \geqslant 0$ and from inequality (1.30), that it can be bounded from above by

$$L_0\frac{9v}{\alpha c_1} \leqslant L_0\frac{72}{240} \leqslant \frac{L_0}{3}. \tag{1.31}$$

Thus, whenever $L_0 \geqslant c_8$, from (1.28), (1.29) and (1.31), we see that (1.27) is satisfied if

$$L_0 \geqslant \frac{3}{2} \frac{9(d-1)\log_v \left(\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\left(\frac{\alpha c_1}{v}\right)^{k+1} \log_v\left(\frac{1}{\kappa}\right)}. \tag{1.32}$$

Therefore, in order to prove (1.26) it is enough to show that the right hand side of inequality (1.32) is bounded. To do this, it is enough to prove that the expression

$$\frac{\log_v \left(\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}$$

is bounded. Now,

$$\frac{\log_v \left(\frac{f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \leqslant \frac{\log_v \left(f_{\left[\frac{k+3}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right) \right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}. \tag{1.33}$$

Let us now remark that if k is even, then $\left[\frac{k+3}{2}\right] = \left[\frac{k+2}{2}\right]$ and $\left[\frac{k+1}{2}\right] = \left[\frac{k+2}{2}\right] - 1$. Therefore, in this case, the right-hand side of inequality (1.33) is smaller than

$$\frac{f_{\left[\frac{k+2}{2}\right]-1}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+2}{2}\right]-1\right)} = \frac{f_{\left[\frac{k+2}{2}\right]-1}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]-1}\left(v^{\left[\frac{k+2}{2}\right]-1}\right)}.$$

But, since for k fixed, the function $f_k(\cdot)$ is increasing, and since for $k \geqslant 0$ we have that

$$v^{\left[\frac{k+2}{2}\right]-1} \geqslant \left[\frac{k+2}{2}\right],$$

we see that the right-hand side of inequality (1.33) is bounded. Hence, for k even the right-most term of (1.33) is bounded by a constant $c_9(d,\kappa) = c_9(d) > 0$.

Suppose now that k is odd. Then $\left[\frac{k+3}{2}\right] = \left[\frac{k+2}{2}\right] + 1$ and $\left[\frac{k+1}{2}\right] = \left[\frac{k+2}{2}\right]$. Therefore, in this case, the right-hand side of inequality (1.33) is equal to

$$\frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)} = 1,$$

so that we can conclude that the right-hand side of inequality (1.33) is bounded, and hence that there is constant $c_{10}(d,\kappa) = c_{10}(d) > 0$ which is an upper bound for the right-hand side of inequality (1.27). We can hence conclude, taking $c_7(d) = \max\{c_9(d), c_{10}(d)\}$, that when $L_0 \geqslant c_7(d)$, then (1.26) holds.

As a second step to prove (1.25), we will show that it is possible to find a positive constant $c_{11}(d,\kappa) = c_{11}(d)$ such that when $L_0 \geqslant c_{11}$ one has that for all $k \geqslant 0$,

$$L_{k+1}^{3d-1}\,\kappa^{\frac{u_{k+1}L_{k+1}}{3}} \leqslant 1. \tag{1.34}$$

Inserting the definition (1.21) that defines L_k into this inequality, we see that it is enough to prove that

$$(3d-1)\log_v(L_{k+1}) - \frac{\log_v\left(\frac{1}{\kappa}\right) u_0 \left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right) L_0}{3} \leqslant 0. \tag{1.35}$$

If we show that for all $k \geqslant 0$, $L_0 \geqslant \frac{\log_v(L_{k+1})3(3d-1)}{\log_v\left(\frac{1}{\kappa}\right)u_0\left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}$, we have a proof of (1.35). But the right-hand side of this inequality can be written as

$$\frac{3(3d-1)\log_v\left[L_0\left(\frac{\alpha c_1}{u_0}\right)^{k+1}\right]}{\log_v\left(\frac{1}{\kappa}\right) u_0\left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} + \frac{3(3d-1)\log_v\left(f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}.$$

We need to establish a control with respect to L_0 in this expression. Only the first term depends on L_0 so we concentrate on the first term. Now, this term is decreasing with k. Therefore, it is smaller than

$$\frac{3(3d-1)\log_v\left[L_0\left(\frac{\alpha c_1}{u_0}\right)\right]}{\log_v\left(\frac{1}{\kappa}\right)\left(\frac{\alpha c_1}{v}\right)} = \frac{3(3d-1)\log_v\left(\frac{L_0^2\alpha c_1\log\left(\frac{1}{\kappa}\right)}{3(d-1)}\right)}{\log_v\left(\frac{1}{\kappa}\right)\left(\frac{\alpha c_1}{v}\right)}$$

From this last expression, it is clear that we can choose a constant $c_{12}(d,\kappa) = c_{12}(d) > 0$ such that whenever $L_0 \geqslant c_{12}(d)$ one has that

$$\frac{3(3d-1)\log_v\left[L_0\left(\frac{\alpha c_1}{u_0}\right)^{k+1}\right]}{\log_v\left(\frac{1}{\kappa}\right) u_0\left(\frac{\alpha c_1}{u_0 v}\right)^{k+1} f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)} \leqslant \frac{L_0}{3}. \tag{1.36}$$

Therefore, if $L_0 \geqslant c_{12}(d)$ and if

$$L_0 \geqslant \frac{3}{2}\frac{3(3d-1)\log_v\left(f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)\right)}{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)}, \tag{1.37}$$

we would have (1.34), whenever we could prove that the right hand side of (1.37) is bounded independently of $k \geqslant 0$. This can be proven in analogy to the previous computations made to show that the right-hand side of (1.32) is bounded. We have thus established the existence of a constant $c_{11}(d)$ such that (1.34) is satisfied whenever $L_0 \geqslant c_{11}(d)$.

On the other hand it is obvious that there is a constant $c_{13}(d)$, such that when $L_0 \geqslant c_{13}(d)$, for $k \geqslant 0$,

$$c_5 \kappa^{\frac{u_{k+1} L_{k+1}}{3}} \leqslant 1.$$

Finally, in order for inequality (1.14) of condition (G) to be fulfilled, it is enough to take $c_6(d) := \max\{c_7(d), c_{11}(d), c_{13}(d)\}$.

□

1.2.4 The effective criterion implies Theorem 1.1.2

We continue now showing how Lemma 1.2.2 with the appropriate choice of scales, enables us to use the effective criterion (see Theorem 2.4 of [Sz03] where it was introduced) to prove the decay of Theorem 1.1.2. Let us define for $x \in \mathbb{Z}^d$,

$$|x|_\perp := \max\{|x \cdot R(e_i)| : 2 \leqslant i \leqslant d\}.$$

Also, define for each $x \in \mathbb{Z}^d$, the canonical translation on the environments $t_x : \Omega \to \Omega$ as

$$t_x(\omega)(y) := \omega(x+y) \quad \text{for } y \in \mathbb{Z}^d.$$

For the statement of the following proposition and its proof, we will use the shorthand notation for each n,

$$\log_8^{(n)}(L) := \overbrace{\log_8 \circ \cdots \circ \log_8}^{n}(L).$$

Proposition 1.2.5. *There exist* $c_{15}(d) > 1$, $c_{14}(d) \geqslant 3\sqrt{d}$ *such that whenever* $L_0 \geqslant c_{14}$, $3\sqrt{d} \leqslant \tilde{L}_0 \leqslant L_0^3$, *and for the box specification* $\mathcal{B}_0 = (R, L_0 - 1, L_0 + 1, \tilde{L}_0)$, *the condition*

$$c_{15} \left(log \left(\frac{1}{\kappa} \right) \right)^{3(d-1)} \tilde{L}_0^{d-1} L_0^{3d-2} \inf_{a \in (0,1]} \mathbb{E}[\rho_0^a] < 1, \tag{1.38}$$

is satisfied (recall the definition of ρ_0 *in (1.6)), then there exist a constant* $c > 0$ *and a function* $n(L) : [0, \infty) \to \mathbb{N}$, *with* $n(L) \to \infty$ *as* $L \to \infty$, *such that*

$$\limsup_{L \to \infty} \; L^{-1} \exp\{c \log_8^{n(L)} L\} \log P_0(T_L^l \leqslant \tilde{T}_{-L}^l) < 0. \tag{1.39}$$

Remark 1.2.6. The assumption (1.38) of Proposition 1.2.5, is called the *effective criterion*, and was introduced by Sznitman in [Sz03].

Proof. Let us choose a sequence of scales $\{L_k : k \geqslant 0\}$ and $\{\tilde{L}_k : k \geqslant 0\}$ according to displays (1.21) and (1.10). With this choice of scales, as in the proof of Proposition 2.3 of Sznitman [Sz03], one can see that there are constants $c_{15}(d)$ and $c_{14} \geqslant \max\{c_6, c_2\}$ such that if $L_0 \geqslant c_{14}$ then condition (1.38) implies condition (1.15) of Lemma 1.2.2 with u_0 chosen according to (1.11). By Lemma (1.2.4), the chosen scales $\{L_k : k \geqslant 0\}$ and $\{\tilde{L}_k : k \geqslant 0\}$ satisfy condition (G). Therefore, since (1.15) of Lemma (1.2.2) is satisfied , we know that for all $k \geqslant 0$, inequality (1.16) is satisfied. The strategy to prove (1.39) will be similar to that employed in [Sz03] to prove Proposition 2.3: we will first choose an appropriate k so that L_k approximates a fixed scale L tending to ∞. Nevertheless, since here we are working with scales which are much larger than those used in [Sz03], we will have to be much more careful with this argument.

Let $L \geqslant L_0$. Then, there exists a unique integer $k = k(L)$ such that

$$L_k \leqslant L < L_{k+1}.$$

Note that to prove (1.39) it is enough to show that there exists a positive constant c_{16} such that for all $L \geqslant L_0$ one has that

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leqslant \frac{1}{c_{16}} \exp\left\{ -c_{16} L \exp\left\{ -\frac{1}{c_{16}} \log_8^{([\frac{k+1}{2}])}(L) \right\} \right\}. \tag{1.40}$$

In effect, since clearly $k \to \infty$ as $L \to \infty$, choosing $n(L) = \left[\frac{k+1}{2}\right]$ we have (1.39).

We will divide the proof of (1.40) into two cases.

Case 1. Assume that

$$L \leqslant \frac{2\alpha c_1}{u_0} v^k L_k. \tag{1.41}$$

Let

$$B := \left\{ x \in \mathbb{Z}^d : |x|_\perp \leqslant \left[\frac{L}{L_k} \right] \tilde{L}_k,\ x \cdot l \in (-L, L) \right\}.$$

From the inequality $\mathbb{E}[q_k] \leqslant \mathbb{E}[\rho_k^{a_k}]$, Lemma 1.2.2 and Chebyshev inequality, we see that if

$$\mathcal{H} := \{\omega \in \Omega : \exists x \in B \text{ such that } \ q_k \circ t_x(\omega) \geqslant \kappa^{\frac{1}{2} u_k L_k}\},$$

then

$$\mathbb{P}[\mathcal{H}] \leqslant \kappa^{\frac{1}{2} u_k L_k} \frac{|B|}{\tilde{L}_{k+1}^{d-1} L_k}.$$

Note that on $\mathcal{H}^c$, by the strong Markov property one has that

$$P_{0,\omega}[T_L^l \leqslant \tilde{T}_{-L}^l] \geqslant (1 - \kappa^{\frac{1}{2} u_k L_k})^{\left[\frac{L}{L_k}\right]+1}.$$

Therefore, since for $x \in [0, 1]$ and n natural one has that $(1 - x)^n \leqslant n(1 - x)$, for L large enough

$$\begin{aligned} P_0[\tilde{T}_{-L}^l < T_L^l] &\leqslant \left(\frac{|B|}{\tilde{L}_{k+1}^{d-1} L_k} + \frac{L}{L_k} + 1 \right) \kappa^{\frac{1}{2} u_k L_k} \\ &\leqslant 3 \times 2^d \left(\frac{L}{L_k} \right)^d \kappa^{\frac{1}{2} u_k L_k} \\ &\leqslant 3 \times 2^d \left(\frac{2\alpha c_1 v^k}{u_0} \right)^d \kappa^{\frac{1}{4} u_k L_k} \leqslant 1, \end{aligned} \tag{1.42}$$

where in the third inequality we have used our assumption on L (1.41). Hence, we can check that there is a constant c_{17}, such that for $k \geqslant 0$,

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leqslant \frac{1}{c_{17}} \exp\left\{ -c_{17} \frac{L_k}{v^k} \right\}. \tag{1.43}$$

Now, again by our assumption (1.41), observe that there is a constant c_{18} such that

$$\frac{L_k}{v^k} > c_{18}\frac{L}{v^{2k}}. \tag{1.44}$$

On the other hand, note that when $L_0 \geqslant \sqrt{\frac{3(d-1)}{\alpha c_1 \log\frac{1}{\kappa}}}$, we have by the choice of scales given in (1.21), that for $k \geqslant 1$

$$f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k}{2}\right]\right) \leqslant L_k \leqslant L. \tag{1.45}$$

Repeatedly taking logarithms in (1.45), we conclude that for $k \geqslant 1$

$$\frac{k}{4} \leqslant \left[\frac{k}{2}\right] \leqslant \log_8^{\left(\left[\frac{k+1}{2}\right]\right)}(L). \tag{1.46}$$

Then, substituting the inequalities (1.44) and (1.46) into (1.43), we see that there exists a positive constants c_{16} such that for $L \geqslant L_0$

$$P_0[\tilde{T}^l_{-L} < T^l_L] \leqslant \frac{1}{c_{16}}\exp\left\{-c_{16}L\exp\left\{-\frac{1}{c_{16}}\log_8^{\left(\left[\frac{k+1}{2}\right]\right)}(L)\right\}\right\}.$$

Now, (1.39) follows taking $n(L) = \left[\frac{k+1}{2}\right]$.

Case 2. Let us now assume that

$$L > \frac{2\alpha c_1}{u_0}v^k L_k.$$

Let m_k be the unique integer such that

$$m_k L_k \leqslant L < (m_k+1)L_k.$$

By the definition of m_k we have the inequality

$$m_k \geqslant \frac{\alpha c_1}{u_0}v^k. \tag{1.47}$$

We will now follow an approach similar to the one employed for *Case 1*, but using a sequence of scales which approximate L with a higher precision than the $\{L_k\}$ sequence. Let us define

$$\begin{aligned} S_1^k &:= m_k L_k, \qquad (1.48)\\ \widetilde{S}_1^k &:= m_k^3 \widetilde{L}_k,\\ S_2^k &:= m_k^2 L_k,\\ \widetilde{S}_2^k &:= m_k^6 \widetilde{L}_k, \end{aligned}$$

along with the box-specification $\widehat{\mathcal{B}} := (R, S_1^k - 1, S_1^k + 1, \widetilde{S}_1^k)$ and the random variable $\widehat{\rho}_k$ attached to this box-specification. In analogy with the proof of Lemma 1.2.2, we will prove that

$$(\widetilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_{k+1}}] \leqslant \kappa^{u_{k+1} S_1^k}. \tag{1.49}$$

For the time being, assume that this inequality is true. Let

$$\widehat{B} = \left\{ x \in \mathbb{Z}^d : |x|_\perp \leqslant \left[\frac{L}{S_1^k} \right] \tilde{S}_1^k,\ x \cdot l \in (-L, L) \right\}.$$

In analogy with the development of *Case 1*, using (1.49) we can arrive to the following inequality analogous to (1.42)

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leqslant \left(\frac{|\widehat{B}|}{(\widetilde{S}_2^k)^{d-1} S_1^k} + \frac{L}{S_1^k} + 1 \right) \kappa^{\frac{1}{2} u_{k+1} S_1^k}.$$

From here we conclude that there is a constant c_{19} such that for $k \geqslant 0$

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leqslant \frac{1}{c_{19}} \exp\left\{ -\frac{c_{19} S_1^k}{v^k} \right\} \tag{1.50}$$

Now, the computation $S_1^k = m_k L_k = (m_k + 1) L_k - L_k \geqslant L - \frac{u_0}{2\alpha c_1} v^{-k} L$, replaced at (1.50), gives us

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leqslant \frac{1}{c_{19}} \exp\left\{ -\frac{c_{19} L \left(1 - \frac{u_0}{2\alpha c_1} v^{-k}\right)}{v^k} \right\}$$

So that, there exists c_{20} such that

$$P_0(\tilde{T}_{-L}^l < T_L^l) \leqslant \frac{1}{c_{20}} \exp\left\{ -c_{20} \frac{L}{v^k} \right\}$$

Using now (1.46) we conclude that there is a constant c_{16} such that for $L \geqslant L_0$ one has that

$$P_0[\tilde{T}_{-L}^l < T_L^l] \leqslant \frac{1}{c_{16}} \exp\left\{ -c_{16} L \exp\left\{ -\frac{1}{c_{16}} \log_8^{([\frac{k+1}{2}])}(L) \right\} \right\}.$$

Choosing $n(L) = \left[\frac{k+1}{2}\right]$ we conclude the proof.

Now, we need to prove (1.49). Using Proposition 1.2.1, with $\widehat{\mathcal{B}}$ and $\mathcal{B}_k$ instead of $\mathcal{B}_1$ and $\mathcal{B}_0$, we have:

$$\mathbb{E}[\widehat{\rho}_k^{a_k+1}] \leqslant c_3 \left\{ \kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{12}} + \sum_{0 \leqslant j \leqslant m_k+1} \varphi_k^{\frac{m_k+j-1}{2}} \right\}$$

So that

$$(\widetilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_k+1}] \leqslant c_3 (S_2^k)^{d-1} S_1^k \left\{ \kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{12}} + \sum_{0 \leqslant j \leqslant m_k+1} \varphi_k^{\frac{m_k+j-1}{2}} \right\}$$

Now,

$$\kappa^{-10c_1 S_1^k} \varphi_k^{\frac{m_k^2}{24}} \leqslant \kappa^{-10c_1 S_1^k} \kappa^{\frac{m_k S_1^k u_k}{24}} \leqslant 1. \tag{1.51}$$

where the first inequality follows from inequality (1.47), the definition (1.48) of S_1^k and (1.12) of u_k, and from Lemma 1.2.4, which enables us to apply inequality (1.16) of Lemma 1.2.2, while the second inequality of (1.51) follows from the fact that $m_k u_k \geqslant 240 c_1$ for $k \geqslant 0$.

Then, inequality (1.51) and the fact that $m_k - 1 \geqslant \frac{m_k}{2}$, imply that

$$(\widetilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_k+1}] \leqslant c_3 (\widetilde{S}_2^k)^{d-1} S_1^k \left\{ \varphi_k^{\frac{m_k^2}{24}} + S_1^k \varphi_k^{\frac{m_k}{4}} \right\}.$$

So that

$$(\widetilde{S}_2^k)^{d-1} S_1^k \mathbb{E}[\widehat{\rho}_k^{a_k+1}] \leqslant 2c_3 (\widetilde{S}_2^k)^{d-1} (S_1^k)^2 \varphi_k^{\frac{m_k}{8}} \kappa^{u_{k+1} S_1^k}.$$

Where, it was used the result of Lemma 1.2.2. Finally, note that to finish the proof we have to show that for $k \geqslant 0$,

$$2c_3 (\tilde{S}_2^k)^{d-1} (S_1^k)^2 \varphi_k^{\frac{m_k}{8}} \leqslant 1. \tag{1.52}$$

By our definitions in (1.48),

$$(\widetilde{S}_2^k)^{d-1} (S_1^k)^2 = m_k^{6d-4} \widetilde{L}_k^{d-1} L_k^2.$$

Now, by Lemma 1.2.4 and its consequence Lemma 1.2.2, we have that $\varphi_k^{\frac{m_k}{8}} \leqslant \left(\kappa^{u_k L_k}\right)^{\frac{m_k}{8}} = \kappa^{u_{k+1} m_k L_k}$. Therefore, the left hand side of inequality (1.52) is smaller than

$$2c_k m_k^{6d-4} \tilde{L}_k^{d-1} L_k^2 \kappa^{u_{k+1} m_k L_k}.$$

However, as d is fixed, and k is large, it is clear that

$$\tilde{L}_k^{d-1} L_k^2 \kappa^{\frac{u_{k+1} m_k L_k}{2}} \leqslant 1$$

and

$$2c_3 m_k^{6d-4} \kappa^{\frac{u_{k+1} m_k L_k}{2}} \leqslant 1.$$

This completes the proof.

$\square$

It is now easy to check that Proposition 1.2.5 implies Theorem 1.1.2 with the function $\log x$ replaced by $\log_8 x$. Indeed, note that (1.38) is equivalent to the effective criterion. On the other hand, using the fact that for every $x > 0$, $\log x \geqslant \log_8 x$, we can then obtain Theorem 1.1.2.

Remark 1.2.7. Let us remark that somehow our choice of scales is optimal. More precisely, in this chapter we have tacitly assumed that the estimate in Proposition 1.2.1 cannot be improved in asymptotic terms. Once this is assumed, the requirements: (1.13) and (1.14) are sharp inequalities and one can verify that the inequality (1.33) is not satisfied for the choice of scales determined by

$$N_k' = \frac{\alpha c_1}{u_0} \frac{f_{\left[\frac{k+2}{2}\right]}\left(\left[\frac{k+2}{2}\right]\right)}{f_{\left[\frac{k+1}{2}\right]}\left(\left[\frac{k+1}{2}\right]\right)},$$

which implies: the scales $\{N_k'\}_{k \geqslant 0}$ do not satisfy condition (G). On the other hand, it is clear that the previous scale would give us the same result as in Theorem 1.1.2. Therefore a new idea should be introduced to prove the Conjecture 1.1.1.

Acknowledgments: We thank A.-S. Sznitman for suggesting to explore how close can one get to the exponential decay of condition (T') from the effective criterion via renormalization.

Chapter 2

Asymptotic Direction for Random Walk in Strong Mixing Environment

2.1 Introduction

Random walk in random environment is basic model of statistical mechanics while challenging questions about it remain open (see [Ze1] for a general overview). It is a simple but powerful model for a variety of phenomena including homogenization in disordered materials [M94], DNA chain replication [Ch62], crystal growth [T69] and turbulent behavior in fluids [?]. In the multidimensional setting a a widely open question is to establish relations between the environment at a local level and the long time behavior of the random walk. During last ten years, interesting progress has been achieved specially in the case in which the movement takes place on the hyper-cubic lattice $\mathbb{Z}^d$ and the environment is i.i.d., establishing relations between directional transience, ballisticity and the existence of an asymptotic direction and the law of the environment in finite regions. Nevertheless, to a great extent, these arguments are no longer valid when the i.i.d. assumption is dropped.

In this chapter we focus on the problem of finding local conditions on the environment which ensure the existence of an asymptotic direction for the random walk in contexts where the environment satisfies some mixing condition, but it is not necessarily i.i.d. To be more precise, we establish the existence of an asymptotic direction for random walks in random environments which are uniformly elliptic, are cone mixing [CZ01], and satisfy a non-effective version of the polynomial ballisticity condition introduced in [BDR14]. While they are directionally transient, these random walks may have a vanishing velocity even for dimensions $d > 1$.

For $x \in \mathbb{R}^d$, we denote by $|x|_1$, $|x|_2$ and $|x|_\infty$ its l_1, l_2 and l_∞ norms respectively. For each integer $d \geqslant 1$, we consider the $(2d-1)$-dimensional simplex $\mathcal{P}_d := \{z \in (\mathbb{R}^+)^{2d} : \sum_{i=1}^{2d} z_i = 1\}$ and $E := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$. We define the *environmental space* $\Omega := \mathcal{P}_d^{\mathbb{Z}^d}$ and endow it with its product σ-algebra. Now, for a fixed $\omega = \{\omega(y) : y \in \mathbb{Z}^d\} \in \Omega$, with $\omega(y) = \{\omega(y,e) : e \in U\} \in \mathcal{P}_d$, and a fixed $x \in \mathbb{Z}^d$, we consider the Markov chain $\{X_n : n \geqslant 0\}$ with state space $\mathbb{Z}^d$ starting from x defined by the transition probabilities

$$P_{x,\omega}[X_{n+1} = X_n + e \mid X_n] = \omega(X_n, e) \qquad \text{for} \quad e \in U. \tag{2.1}$$

We denote by $P_{x,\omega}$ the law of this Markov chain and call it a random walk in the environment ω. Consider a law $\mathbb{P}$ defined on Ω. We call $P_{x,\omega}$ the *quenched law* of the random walk starting from x. Furthermore, we define the semi-direct product probability measure on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ by

$$P_x(A \times B) := \int_A P_{x,\omega}(B) d\mathbb{P}$$

for each Borel-measurable set A in Ω and B in $(\mathbb{Z}^d)^{\mathbb{N}}$, and call it the *annealed or averaged law* of the random walk in random environment. The law $\mathbb{P}$ of the environment is said to be i.i.d. if the random variables $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under $\mathbb{P}$, *elliptic* if for every $x \in \mathbb{Z}^d$ and $e \in U$ one has that $\mathbb{P}[\omega(x,e) > 0] = 1$ while uniformly elliptic if there exists a $\kappa > 0$ such that $\mathbb{P}[\omega(x,e) \geqslant \kappa] = 1$ for every $x \in \mathbb{Z}^d$ and $e \in U$.

Let $l \in \mathbb{S}^{d-1}$. We say that a random walk is *transient in direction l* or just *directionally transient* if P_0-a.s. one has that

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

Furthermore, we say that it is *ballistic in direction l*

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

In the case in which the environment is elliptic and i.i.d., it is known that whenever a random walk is ballistic necessarily a law of large numbers is satisfied and in fact $\lim_{n\to\infty} \frac{X_n}{n} = v \neq 0$ is deterministic [DR14]. Furthermore, in the uniformly elliptic i.i.d. case, it is still an open question to establish wether or not in dimensions $d \geqslant 2$, every directionally transient random walk is ballistic (see [BDR14]).

On the other hand, we say that $\hat{v} \in \mathbb{S}^{d-1}$ is an *asymptotic direction* if P_0-a.s. one has that

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = \hat{v}.$$

For elliptic i.i.d. environments, Simenhaus established [Si07] the existence of an asymptotic direction whenever the random walk is directionally transient in an open set of $\mathbb{S}^{d-1}$. As it will be shown in Section 2.3, this statement is not true anymore when the environment is assumed to be ergodic instead of i.i.d., even if it is uniformly elliptic.

In this chapter we establish the existence of an asymptotic direction under three assumptions about the law $\mathbb{P}$ of the environment: a weak form of uniform ellipticity; cone mixing; a ballisticity condition demanding polynomial decay with high enough degree of the annealed exit probability of the random walk from the back and lateral side of boxes. All these assumption will be defined with respect to a fixed direction $l \in \mathbb{S}^{d-1}$. It will be shown in section 2.3, that there exist environments *almost* satisfying the above assumptions which are directionally transient but not ballistic. Here the term *almost* is used because in these examples the polynomial ballisticity condition is satisfied with a low degree. Let us describe these assumption with more precision.

Let $\kappa > 0$. We say that $\mathbb{P}$ is *uniformly elliptic with respect to* l, denoted by $(UE)|l$, if the jump probabilities of the random walk are positive and larger than 2κ in those directions which for which the projection on l is positive. In other words if $\mathbb{P}[\omega(0,e) > 0] = 1$ for $e \in E$ and if

$$\mathbb{P}\left[\min_{e\in\mathcal{E}} \omega(0,e) \geqslant 2\kappa\right] = 1,$$

where

$$\mathcal{E} := \cup_{i=1}^d \{sgn(l_i)e_i\} - \{0\} \tag{2.2}$$

and by convention $sgn(0) = 0$.

We will now introduce a certain mixing assumption for the environment $\mathbb{P}$. Let $\alpha > 0$ and R be a rotation such that

$$R(e_1) = l. \tag{2.3}$$

To define the cone, it will be useful to consider for each $i \in [2, d]$,

$$l_{+i} = \frac{l + \alpha R(e_i)}{|l + \alpha R(e_i)|} \qquad \text{and} \qquad l_{-i} = \frac{l - \alpha R(e_i)}{|l - \alpha R(e_i)|}.$$

The cone $C(x, l, \alpha)$ centered in $x \in \mathbb{R}^d$ is defined as

$$C(x, l, \alpha)) := \bigcap_{i=2}^{d} \left\{z \in \mathbb{R}^d : (z - x) \cdot l_{+i} \geqslant 0, \, (z - x) \cdot l_{-i} \geqslant 0\right\}. \tag{2.4}$$

Let $\phi : [0, \infty) \to [0, \infty)$ be such that $\lim_{r\to\infty} \phi(r) = 0$. We say that a stationary probability measure $\mathbb{P}$ satisfies the *cone mixing assumption* with respect to α, l and ϕ, denoted $(CM)_{\alpha,\phi}|l$, if for every pair of events A, B, where $\mathbb{P}(A) > 0$, $A \in \sigma\{\omega(z, \cdot); z \cdot l \leqslant 0\}$, and $B \in \sigma\{\omega(z, \cdot); z \in C(rl, l, \alpha)\}$, it holds that

$$\left|\frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} - \mathbb{P}[B]\right| \leqslant \phi(r|l|_1).$$

We will see that every stationary cone mixing measure $\mathbb{P}$ is necessarily ergodic. On the other hand, a cone-mixing environment can be such that the jump probabilities are highly dependent along certain directions.

We now introduce an assumption which is closely related to the effective polynomial ballistic condition introduced in [BDR14]. For each $A \subset \mathbb{Z}^d$ we define

$$\partial A := \{z \in \mathbb{Z}^d : z \notin A, \text{ there exists some } y \in A \text{ such that } |y - z| = 1\}.$$

Define also the stopping time

$$T_A := \inf\{n \geqslant 0 : X_n \notin A\}.$$

Given $L, L' > 0$, $x \in \mathbb{Z}^d$ and $l \in \mathbb{S}^{d-1}$ we define the boxes

$$B_{L,L',l}(x) := x + R\left((-L, L) \times (-L', L')^{d-1}\right) \cap \mathbb{Z}^d,$$

where R is defined in (2.3). The *positive boundary* of $B_{L,L',l}(x)$, denoted by $\partial^+ B_{L,L',l}(0)$, is

$$\partial^+ B_{L,L',l}(0) := \partial B_{L,L',l}(0) \cap \{z : z \cdot l \geqslant L\},$$

Define also the half-space

$$H_{x,l} := \{y \in \mathbb{Z}^d : y \cdot l < x \cdot l\},$$

and the corresponding σ-algebra of the environment on that half-space

$$\mathcal{H}_{x,l} := \sigma(\omega(y) : y \in H_{x,l}).$$

Now, for $M \geqslant 1$, we say that the *non-effective polynomial* condition $(PC)_{M,c}|l$ is satisfied if there exists some $c > 0$ so that for $y \in H_{0,l}$ one has that

$$\lim_{L\to\infty} L^M \sup P_0\left[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0), T_{B_{L,cL,l}(0)} < T_{H_{y,l}} | \mathcal{H}_{y,l}\right] = 0, \tag{2.5}$$

where the supremum is taken over all possible environments to the left of $y \cdot l$. It can be verified that for i.i.d. environments, this condition is implied by Sznitman's (T') condition [Sz03], and it is implied by the effective polynomial condition introduced in [BDR14]. Throughout this chapter, we will denote by $\mathbb{S}^{*d-1}$ the subset of $\mathbb{S}^{d-1}$ defined by

$$\mathbb{S}^{*d-1} := \{s \in \mathbb{S}^{d-1} : \text{ there exists } y \in \mathbb{R} - \{0\}, \text{ such that } ys \in \mathbb{Z}^d\}.$$

We can now state our main result.

Theorem 2.1.1. *Let $l \in \mathbb{S}^{*d-1} \cap$, $M > 6d$, $c > 0$ and $0 < \alpha \leqslant \min\{\frac{1}{9}, \frac{1}{2c+1}\}$. Consider a random walk in a random environment with stationary law satisfying the uniform ellipticity condition $(UE)|l$, the cone mixing condition $(CM)_{\alpha,\phi}|l$ and the non-effective polynomial condition $(PC)_{M,c}|l$. Then, there exists a deterministic $\hat{v} \in \mathbb{S}^{d-1}$ such that P_0-a.s. one has that*

$$\lim_{n\to\infty}\frac{X_n}{|X_n|}=\hat{v}.$$

As it will be explained in Section 2.3, Simenhaus's theorem which states that an asymptotic direction exists whenever the random walk is directionally transient in an open set of directions and the environment is i.i.d., is not true if the i.i.d. assumption is dropped. Somehow, Theorem 2.1.1 shows that if the i.i.d. assumption is weakened to cone mixing, while directional transience is strengthened to the non-effective polynomial condition, we still can guarantee the existence of an asymptotic direction.

In [CZ01], the existence of a strong law of large numbers is established for random walks in cone-mixing environments which also satisfy a version of Kalikow's condition, but under an additional assumption of existence of certain moments of approximate regeneration times. This assumption is unsatisfactory in the sense that it is in general difficult to verify if for a given random environment it is true or not. On the other hand, as it will be shown in Section 2.3, there exist examples of random walks in a random environment satisfying the cone-mixing assumption for which the law of large numbers is not satisfied, while an asymptotic direction exists. From this point of view, Theorem 1.1 is also a first step in the direction of obtaining scaling limit theorems for random walks in cone-mixing environments through ballisticity conditions weaker than Kalikow's condition, and without any kind of assumption on the moments of approximate regeneration times or of the position of the random walk at these times. On the other hand, in [RA03], a strong law of large numbers is proved for random walks which satisfy Kalikow's condition and Dobrushin-Shlosman's strong mixing assumption. The Dobrushin-Shlosman strong mixing assumption is stronger than cone-mixing, both because it implies cone-mixing in every direction and because it corresponds to a decay of correlations which is exponential.

A key step to prove Theorem 1.1 will be to establish that the probability that the random walk never exits a cone is positive through the use of renormalization type ideas, and only assuming the non-effective polynomial condition and uniform ellipticity. Using this fact, we will define approximate regeneration times as in [CZ01], showing that they have finite moments of order larger than one when we also assume cone-mixing. This part of the proof will require careful and tedious computations. Once this is done, the existence of an asymptotic direction can be deduced using for example the coupling approach of [CZ01].

We will now describe the general structure of the sections in this chapter. In Section 2.3, we will present two examples of random walks in random environments which exhibit a behavior which is not observed in the i.i.d. case, giving an idea of the kind of limitations given by the framework of Theorem 2.1.1. In Section 2.2, the meaning of the non-effective polynomial condition and its relation to other ballisticity conditions will be discussed. In Section 2.3, we will present two examples of random walks in random environments which exhibit a behavior which is not observed in the i.i.d. case, giving an idea of the kind of limitations given by the framework of Theorem 2.1.1. In Section 2.4, we will show that the non-effective polynomial condition implies that the probability that the random walk never exits a cone is positive. This will be used in Section 2.5 to prove that the approximate regeneration times have finite moments of order larger than one. Finally in Section 2.6, Theorem 2.1.1 will be proved using coupling with i.i.d. random variables.

2.2 Preliminary discussion

2.2.1 Non-effective polynomial condition and its relation with other directional transience conditions

Here we will discuss the relationship between the condition non-effective polynomial condition and other transience conditions. Furthermore we will show that the conditional non-effective polynomial condition is weaker than the conditional version of Kalikow's condition introduced by Comets and Zeitouni in [CZ01].

For reasons that will become clear in the next section, the following definition, which is actually weaker than the conditional non-effective polynomial condition, will be useful. Let $l \in \mathbb{S}^{d-1}$, $M \geqslant 1$ and $c > 0$. We say that condition $(P)_{M,c}|l$ is satisfied, and we call it the *non-effective polynomial condition* if there is a constant $c > 0$ such that

$$\overline{\lim}_{L\to\infty} L^M P_0[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0)] = 0.$$

It is straightforward to see that $(PC)_{M,c}|l$ implies $(P)_{M,c}|l$.

It should be pointed out, that for a fixed $\gamma \in (0,1)$, if both in the conditional and non-conditional non-effective polynomial conditions the polynomial decay is replaced by a stronger stretched exponential decay of the form e^{-L^γ}, one would obtain a condition defined on rectangles equivalent to condition $(T)_\gamma$ introduced by Sznitman in [Sz03], and also a conditional version of it. On the other hand, as we will see now, the conditional non-effective polynomial condition is implied by Kalikow's condition as defined in [CZ01] for environments which are not necessarily i.i.d. Let us recall this definition. For V a finite, connected subset of $\mathbb{Z}^d$, with $0 \in V$, we let

$$\mathfrak{F}_{V^c} = \sigma\{\omega(z,\cdot) : z \notin V\}.$$

The *Kalikow's random walk* $\{X_n : n \geqslant 0\}$ with state space in $V \cup \partial V$, starting from $y \in V \cup \partial V$ is defined by the transition probabilities

$$\widehat{P}_V(x,x+e) := \begin{cases} \frac{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n=x\}}\omega(x,e)|\mathfrak{F}_{V^c}]}{E_0[\sum_{n=0}^{T_{V^c}} \mathbb{1}_{\{X_n=x\}}|\mathfrak{F}_{V^c}]}, & \text{for} \quad x \in V \text{ and } e \in E \\ 1 & \text{for} \quad x \in \partial V \text{ and } e = 0. \end{cases}$$

We denote by $\hat{P}_{y,V}$ the law of this random walk and by $\hat{E}_{y,V}$ the corresponding expectation. The importance of Kalikow's random walk stems from the fact that

$$X_{T_{V^c}} \text{ has the same law under } \widehat{P}_{0,V} \text{ and under } P_0[\cdot|\mathfrak{F}_{V^c}] \tag{2.6}$$

(see ([K81])). Let $l \in \mathbb{S}^{d-1}$. We now define Kalikow's condition with respect to the direction l as the following requirement: there exits a positive constant δ such that

$$\inf_{V:x\in V} \widehat{d}_V(x) \cdot l \geqslant \delta,$$

where

$$\widehat{d}_V(x) := \widehat{E}_{x,V}[X_1 - X_0] = \sum_{e\in E} e\widehat{P}_V(x,x+e)$$

denotes the drift of Kalikow's random walk at x, and the infimum runs over all finite connected subset V of $\mathbb{Z}^d$ such that $0 \in V$. The following result shows that Kalikow's condition is indeed stronger that the conditional non-effective polynomial criteria.

Proposition 2.2.1. *Let $l \in \mathbb{S}^{d-1}$. Assume Kalikow's condition with respect to l. Then there exists an $r > 0$ such that for all $y \in H_{0,l}$ one has that*

$$\overline{\lim}_{L\to\infty} L^{-1} \sup \log P_0[X_{T_{B_{L,rL,l}(0)}} \notin \partial^+ B_{L,rL,l}(0), T_{B_{L,rL,l}(0)} < T_{H_{y,l}} | \mathcal{H}_{y,l}] < 0,$$

where the supremum is taken in the same sense as in (2.5). In particular, Kalikow's condition with respect to direction l implies $(PC)_{M,r}|l$ for all $M > 0$.

Proof. Suppose that Kalikow's condition is satisfied with constant $\delta > 0$. We will first assume that $y \cdot l \in (-L, 0)$. Let $c > 1$. For $y \in H_{0,l}$ and $L \geqslant 1$ consider the box

$$V := R\left([y \cdot l, L] \times \left(-\frac{c}{\delta}L, \frac{c}{\delta}L\right)^{d-1}\right).$$

Therefore, using (2.6) we find that

$$\begin{aligned} &P_0[X_{T_{B_{L,\frac{c}{\delta}L,l}(0)}} \notin \partial^+ B_{L,cL,l}(0), T_{B_{L,\frac{c}{\delta}L,l}(0)} < T_{H_{y,l}} | \mathfrak{F}_{V^c}] \\ &\leqslant P_0[X_{T_{\tilde{U}}} \cdot R(e_j) \geqslant \tfrac{c}{\delta}L \text{ for some } j \in [2,d], |X_{T_{\tilde{U}}} \cdot l| < L | \mathfrak{F}_{V^c}] \\ &= \widehat{P}_{0,V}[X_{T_{\tilde{U}}} \cdot R(e_j) \geqslant \tfrac{c}{\delta}L \text{ for some } j \in [2,d], |X_{T_{\tilde{U}}} \cdot l| < L]. \end{aligned} \tag{2.7}$$

Notice that on the set

$$\{X_{T_V} \cdot R(e_j) \geqslant \frac{c}{\delta}L \text{ for some } j, X_{T_V} \cdot l < L\},$$

one has $\widehat{P}_{0,V}$-a.s. that

$$T_V \geqslant \left[\frac{cL}{\delta}\right].$$

Thus, by means of the auxiliary martingale $\{M_n^V : n \geqslant 0\}$ defined by

$$M_n^V := X_n - X_0 - \sum_{j=0}^{n-1} \widehat{d}_V(X_j),$$

which has bounded increments (indeed bounded by 2) we can see that on $\{T_V > [\frac{cL}{\delta}]\}$, we have that for L large enough that

$$M^V_{[\frac{cL}{\delta}]} \cdot l < L - \left(\frac{cL}{\delta} - 1\right)\delta = (1-c)L + \delta < \frac{(1-c)L}{2} \tag{2.8}$$

$\widehat{P}_{0,V}$-a.s. Now, it will be convenient at this point to recall Azuma's inequality [Sz01]:

$$\widehat{P}_{0,V}[M_n^V \cdot w > A] \leqslant \exp\left\{-\frac{A^2}{8n}\right\} \quad \text{for} \quad A > 0, \ n \geqslant 0, \ |w| = 1,$$

for martingales with increments bounded by 2. Using this inequality and (2.8) we obtain that

$$\begin{aligned} &\widehat{P}_{0,V}[X_{T_{\tilde{U}}} \cdot R(e_j) > \tfrac{c}{\delta}L \text{ for some } j, X_{T_V} \cdot l \leqslant L] \\ &\leqslant \widehat{P}_{0,V}[T_V > \tfrac{cL}{\delta}] \\ &\leqslant \widehat{P}_{0,V}[M^V_{[\frac{cL}{\delta}]} \cdot (-l) > (c-1)L/2] \leqslant \exp\{-c_1 L\}, \end{aligned} \tag{2.9}$$

for a suitable positive constant c_1. Finally, coming back to (2.7), we can then conclude that

$$\overline{\lim}_{L\to\infty} L^{-1} \sup \log P_0[X_{T_{B_{L,rL,l}(0)}} \notin \partial^+ B_{L,rL,l}(0), T_{B_{L,rL,l}(0)} < T_{H_{y,l}} | \mathcal{H}_{y,l}] < 0,$$

where $r = \frac{c}{\delta}$. Let us now assume that $y \cdot l \leqslant -L$. By Lemma 1.1 in [Sz01] we know that there exists a positive constant ψ depending on δ such that for all V finite connected subsets of $\mathbb{Z}^d$ with $0 \in V$

$$e^{-\psi X_n \cdot l}$$

is a supermartingale with respect to the canonical filtration of the walk under Kalikow's law $\widehat{P}_{0,V}$. Thus, we have that

$$\widehat{P}_{0,V}[X_{T_V} \cdot l \leqslant -L] \leqslant \exp\{-\psi L\}$$

by means of stopping time theorem applied at time T_V. By an argument similar to the one developed for the case $y \cdot l \in (-L, 0)$, we can finish the estimate in the case $y \cdot l \leqslant L$. $\square$

2.2.2 Cone mixing and ergodicity

The main objective in this section is to establish the following: any stationary probability measure $\mathbb{P}$ defined on the canonical $\sigma-$ algebra $\mathfrak{F}$, which satisfies property $(CM)_{\phi,\alpha}|l$ is ergodic with respect to space-shifts. Before doing this, let us recall an ergodic notion. We say that $E \in \mathfrak{F}$ is an invariant set if :

$$\theta_x E := E$$

for all $x \in \mathbb{Z}^d$.

Theorem 2.2.2. *Assume that the probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *has the property* $(CM)_{\phi,\alpha}|l$ *and is stationary, then the probability measure* $\mathbb{P}$ *is* ergodic*, i.e. for any invariant set* $E \in \mathfrak{F}$ *we have:*

$$\mathbb{P}[E] \in \{0, 1\}.$$

Proof. Let $E \in \mathfrak{F}$ be an invariant set. From a theoretical measure fact, given $\epsilon > 0$ there exists a cylinder measurable set $A \in \mathfrak{F}$, so that:

$$\mathbb{P}[A \triangle E] < \epsilon.$$

Since A is a cylinder measurable set, it can be represented as:

$$A = \{\omega(x, \cdot) : \ x \in F, F \subset \mathbb{Z}^d, \ |F| < \infty,$$
$$\omega(x_i, \cdot) \in P_i, \text{ for } x_i \in F, \ P_i \in \mathcal{B}(\mathcal{P}_d)\},$$

where as a matter of definition $\mathcal{B}(\mathcal{P}_d)$ stands for the borelian $\sigma-$ algebra on the compact subset $\mathcal{P}_d$ of $\mathbb{R}^{2d}$. We choose L such that:

$$\phi(L) < \epsilon.$$

Plainly, for L we can find an $x \in \mathbb{Z}^d$ such that $\theta_x A$ and A are L separated on cones with respect to direction l , in other words:
There exists $y \in \mathbb{Z}^d$ such that:

$$A \in \sigma\{\omega(z, \cdot) : \ z \cdot l \leqslant y \cdot l - L\}$$

along with

$$\theta_x A \in \sigma\{\omega(z, \cdot) : \ z \in C(y, l, \alpha)\}.$$

We can suppose that $\mathbb{P}[E] > 0$, otherwise there is nothing to prove. So as to complete the proof we have to show that $\mathbb{P}[E] = 1$. Therefore taking ϵ small enough we can suppose further $\mathbb{P}[A] > 0$. Thus, using the cone mixing property, we get:

$$-\mathbb{P}[A]\phi(L) \leqslant \mathbb{P}[A \cap (\theta_x A)^c] - \mathbb{P}[A]\mathbb{P}[\Omega - A] \leqslant \mathbb{P}[A]\phi(L) \tag{2.10}$$

On the other hand, since E is an invariant set:

$$\mathbb{P}[\theta_x A \triangle E] = \mathbb{P}[\theta_x A \triangle \theta_x E] = \mathbb{P}[\theta_x(A \triangle E)] < \epsilon, \tag{2.11}$$

which implies:

$$\mathbb{P}[A \triangle \theta_x A] \leqslant \mathbb{P}[(A \triangle E) \cup (\theta_x A \triangle \theta_x E)] < 2\epsilon. \tag{2.12}$$

In turn, from inequality (2.12), it is clear that $\mathbb{P}[A \cap (\theta_x A)^c] < 2\epsilon$. Now, using the inequality (2.10) one has that

$$\mathbb{P}[A]\mathbb{P}[\Omega - A] \leqslant 2\epsilon + \mathbb{P}[A]\phi(L).$$

As a result, the inequalities

$$\begin{aligned} \mathbb{P}[E]\mathbb{P}[\Omega - E] &< (\mathbb{P}[A] + \epsilon)(\mathbb{P}[\Omega - A] + \epsilon) && (2.13) \\ &= \mathbb{P}[A]\mathbb{P}[\Omega - A] + \epsilon + \epsilon^2 && (2.14) \\ &< 4\epsilon + \phi(L) \leqslant 5\epsilon && (2.15) \end{aligned}$$

hold. Hence, from $\epsilon > 0$ was arbitrary this turns out that $\mathbb{P}[E]\mathbb{P}[\Omega - E] = 0$. Therefore if $\mathbb{P}[E] > 0$, this implies $\mathbb{P}[E] = 1$. □

2.2.3 Polynomial Decay implies Polynomial decay in a neighborhood

In this subsection we prove that whenever $(PC)_{M,c}|l$ holds, for prescribed positive constants M and c, then we can choose $2(d-1)$ directions where we still have polynomial decay although of less order. More precisely, we can prove the following:

Proposition 2.2.3. *Suppose that $(P)_{M,c}|l$ is satisfied with $c > 0$ for some $M > 6(d-1)$, then there exists an $\alpha > 0$ such that if we define for $i \in [2, d]$:*

$$l_{+i} := \frac{l \pm \alpha R(e_i)}{|l + \alpha R(e_i)|}$$

and

$$l_{-i} := \frac{l - \alpha R(e_i)}{|l - \alpha R(e_i)|},$$

then

$$(P)_{N,2c}|l_{\pm i}$$

is satisfied, where we can choose $N = \frac{M}{3} - 1$.

Therefore, if M fulfils the prescribed inequality in Theorem 2.2.3, then $(PC)_{M,c}|l$ implies for each $i \in [2, d]$ that $(P)_{N,2c}|l_{\pm i}$ is satisfied. The loss of degree in the polynomial condition is due to the requirement that the underlying boxes in the condition have the same dimensions in both l and $-l$ directions.

Proof of Proposition 2.2.3. We will just give the proof for direction l_{-2}, the other cases being analogous.
Throughout the proof we pick $\alpha \in (0,1)$ and we define the angle β by:

$$\beta := \arctan(\alpha). \tag{2.16}$$

Consider the specific rotation R'' on $\mathbb{R}^d$ defined by:

$$R'' := \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 & \dots & \dots & 0 \\ \sin(\beta) & \cos(\beta) & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

where this representation matrix is taken in the vector space base $\{R(e_1), R(e_2), \dots, R(e_d)\}$. It will be useful to define a new rotation

$$R' := R''R$$

together with the *rotated box* $\widetilde{B}_L(0)$ given by

$$\widetilde{B}_L(0) := R'\left([-L\lambda_1(\alpha), L\lambda_2(\alpha)] \times [-Lc\lambda_3(\alpha), Lc\lambda_3(\alpha)]^{d-1}\right) \cap \mathbb{Z}^d$$

where:

$$\lambda_1(\alpha) := \frac{1+\frac{1}{\alpha}}{\sqrt{\cot^2(\beta)+1}}$$
$$\lambda_2(\alpha) := \frac{\frac{1}{\alpha}-1}{\sqrt{\cot^2(\beta)+1}}$$
$$\lambda_3(\alpha) := \frac{\sqrt{(1-\cot(\beta))^2+(1-\tan(\beta))^2}}{|\tan(\beta)+\cot(\beta)|}$$

Notice that with these definitions, P_0- almost surely:

$$X_{T_{\tilde{B}_L(0)}} \notin \partial^+ \widetilde{B}_L(0) \Rightarrow X_{T_{B_{L,L,l}(0)}} \notin \partial^+ B_{L,L,l}(0). \tag{2.17}$$

The following figure shows the boxes involved in 2.17.

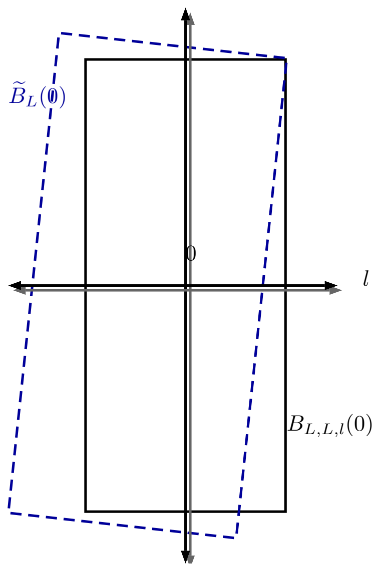

Figure 2.1: The choice of boxes.

As a result we have got

$$P_0[X_{T_{\tilde{B}_L(0)}} \notin \partial^+ \widetilde{B}_L(0)] \leqslant L^{-M}.$$

Furthermore, a straightforward computation makes us see that the scale factor $\lambda_3(\alpha)$ is less than $\frac{4}{3}$ whenever $\alpha \leqslant \frac{1}{9}$. Therefore if we let the positive $\alpha \leqslant \frac{1}{9}$ one has that

$$\lambda_3(\alpha) \leqslant \frac{4}{3}. \tag{2.18}$$

For technical reasons, we need to introduce an auxiliary box. Specifically, we first set:

$$h := \frac{\frac{1}{\alpha}-1}{\sqrt{1+(\frac{1}{\alpha})^2}} = \frac{1-\alpha}{\sqrt{1+\alpha^2}}$$

and observe that $\frac{4}{5} < h < 1$. We then can introduce the new box $\overline{B}_{l-2,L}(0)$ defined by:

$$\overline{B}_{l-2,L}(0) := R'\left([-L(h+2), Lh] \times [-cL\lambda_3(\alpha), Lc\lambda_3(\alpha)]^{d-1}\right) \cap \mathbb{Z}^d.$$

From this definition, we obtain:

$$P_0[X_{T_{\overline{B}_{l-2,L}(0)}} \notin \partial^+ \overline{B}_{l-2,L}(0)] \leqslant L^{-M}.$$

In order to complete the proof, we claim that for large enough U the probability:

$$P_0[X_{T_{\overline{B}_{l-2,U}(0)}} \notin \partial^+ \overline{B}_{l-2,U}(0)].$$

has polynomial decay on U, where the box $\overline{B}_{l-2,U}(0)$ is defined by

$$\overline{B}_{l-2,U}(0) := R'([-U, U] \times [-2cU, 2cU]^{d-1}).$$

The general strategy to follow will be to stack smaller boxes up inside of $\overline{B}_{l-2,U}(0)$ and then using the Markov property along with *good environment sets* we will ensure that the walk exits from box $\overline{B}_{l-2,U}(0)$ by $\partial^+ \overline{B}_{l-2,U}(0)$ with probability bigger than $1 - 1/P(U)$, where P is a polynomial function. Specifically, we let

$$L := \frac{U}{h+2}. \tag{2.19}$$

We introduce a sequence of stopping times as follows:

$$T_1 = T_{\overline{B}_{l-2,L}(0)},$$

and for $i > 1$

$$T_i = T_{i-1} + T_1 \circ \theta_{T_{i-1}}.$$

For simplicity we write $\widehat{T}_1$ instead of $T_{\overline{B}_{l-2,U}(0)}$. In view of (2.18) and (2.19) it is clear that four *successful* exits of the walk from boxes of the $\overline{B}_{l-2,L}(0)$-type are sufficient to ensure that the walk exits from $\overline{B}_{l-2,U}(0)$ by its *positive* boundary. Therefore one sees that

$$\begin{aligned} P_0[X_{\widehat{T}_1} \in \partial^+ \overline{B}_{l-2,U}(0)] \geqslant P_0 \Big[& X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(0), \\ \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_1})\big) \circ \theta_{T_1}, & \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_2})\big) \circ \theta_{T_2}, \\ & \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_3})\big) \circ \theta_{T_3} \Big] \end{aligned} \tag{2.20}$$

In order to use (2.20), let i be a positive integer number and consider the lattice sets sequence $(F_i)_{i\geqslant 1}$ defined by:

$$F_1 = \partial^+ \overline{B}_{l-2,L}(0),$$

and for $i > 1$, we define by induction:

$$F_i = \bigcup_{y \in F_1} \partial^+ \overline{B}_{l-2,L}(y).$$

We now define for $i \geqslant 1$, the environment events G_i by:

$$G_i = \left\{ \omega \in \Omega : P_{y,\omega}\big[\big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_i})\big) \circ \theta_{T_i}\big] \geqslant 1 - L^{-\frac{M}{2}}, \text{ for each } y \in F_i \right\}$$

Plainly it is satisfied

$$\begin{gathered} P_0[X_{\widehat{T_1}} \in \partial^+ \overline{B}_{l-2,U}(0)] \geqslant \\ P_0\left[X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(0), \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_1})\big) \circ \theta_{T_1},\right. \\ \left.\big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_2})\big) \circ \theta_{T_2}, \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_3})\big) \circ \theta_{T_3}\right] \geqslant \\ P_0\left[X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(0), \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_1})\big) \circ \theta_{T_1},\right. \\ \left.\big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_2})\big) \circ \theta_{T_2}, \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_3})\big) \circ \theta_{T_3} \mathbb{1}_{G_3}\right] \end{gathered}$$

By the Markov property applied at time T_3 and the very meaning of G_3, we get that the last expression equals:

$$\begin{gathered} \textstyle\sum_{y \in F_3} \mathbb{E}\left[P_{0,\omega}\left[X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(0), \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_1})\big) \circ \theta_{T_1},\right.\right. \\ \left.\left.\big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_2})\big) \circ \theta_{T_2}\right] P_{y,\omega}[X_{T_{\overline{B}_{l-2,L}(y)}} \in \partial^+ \overline{B}_{l-2,L}(y)] \mathbb{1}_{G_3}\right] \geqslant \\ (1 - L^{-\frac{M}{2}}) \left(P_0\left[X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(0), \big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_1})\big) \circ \theta_{T_1},\right.\right. \\ \left.\left.\big(X_{T_1} \in \partial^+ \overline{B}_{l-2,L}(X_{T_2})\big) \circ \theta_{T_2}\right] - \mathbb{P}[(G_3)^c]\right) \end{gathered} \tag{2.21}$$

$$. \tag{2.22}$$

Repeating the above argument, one has the following upper bound for the right most expression of (2.21):

$$(1 - L^{-\frac{M}{2}})^4 - (1 - L^{-\frac{M}{2}})^3 \mathbb{P}[(G_1)^c] - (1 - L^{-\frac{M}{2}})^2 \mathbb{P}[(G_2)^c] - (1 - L^{-\frac{M}{2}}) \mathbb{P}[(G_3)^c]. \tag{2.23}$$

At this point, we would like to obtain for $i \in |[1,3]|$, an upper bound of the probabilities:

$$\mathbb{P}[(G_i)^c].$$

To this end, we first observe that Chevyshev's inequality and our hypothesis imply:

$$\mathbb{P}[(G_1)^c] \leqslant \sum_{y \in F_1} \mathbb{E}[\mathbb{1}_{\{P_{y,\omega}[X_{T_{\overline{B}_{l_{-2},L}(y)}} \in \partial^+ \overline{B}_{l_{-2},L}(y)] > L^{-\frac{M}{2}}\}}] \leqslant | F_1 | L^{-\frac{M}{2}}.$$

Clearly, we have the estimate $| F_1 | \leqslant \left(\frac{8}{3}L\right)^{d-1}$ (recall (2.18)). As a result, we have that:

$$\mathbb{P}[(G_1)^c] \leqslant \left(\frac{8}{3}L\right)^{d-1} L^{-\frac{M}{2}}. \tag{2.24}$$

By a similar procedure we can conclude that

$$\mathbb{P}[(G_2)^c] \leqslant \left(\frac{16}{3}L\right)^{d-1} L^{-\frac{M}{2}}. \tag{2.25}$$

and

$$\mathbb{P}[(G_3)^c] \leqslant \left(\frac{24}{3}L\right)^{d-1} L^{-\frac{M}{2}}. \tag{2.26}$$

Combining the estimates in (2.20)-(2.26) and the assumption $M \geqslant 6(d-1)$ we see that:

$$P_0[X_{\widehat{T_1}} \notin \partial^+ \overline{B}_{l_{-2},U}(0)] \leqslant 3\frac{6(8)^{d-1}}{2^{-\frac{M}{3}}} U^{-\frac{M}{3}}.$$

This ends the proof by choosing the required α as any number in the open interval $(0, \frac{1}{9})$. □

2.3 Examples of directionally transient random walks without an asymptotic direction and vanishing velocity

We will present two examples of random walks in random environment which exhibit the possible limitations of the hypothesis of a theorem stating the existence of an asymptotic direction and of a theorem stating the existence of a non-vanishing velocity for mixing environments.

Assumption **TNB**.

Let p be a random variable taking values in $(0, 1)$ such that there exists a unique $\kappa \in (1/2, 1)$ with the property that

$$E[\rho^\kappa] = 1 \quad \text{and} \quad E[\rho^\kappa \ln^+ \rho] < \infty,$$

where $\rho := (1 - p)/p$.

2.3.1 Random walk with a vanishing velocity but with an asymptotic direction

Let $\{p_i : i \in \mathbb{Z}\}$ be i.i.d. copies of p. Let e_1 and e_2 be the canonical vectors in $\mathbb{Z}^2$. Define an i.i.d. sequence of random variables $\{\omega_i : i \in \mathbb{Z}\}$ with $\omega_i = \{\omega_i(e_1), \omega_i(-e_1), \omega_i(e_2), \omega_i(-e_2)\}$, by

$$\omega_i(e_2) = \omega_i(-e_2) = \frac{1}{4},$$

$$\omega_i(e_1) = \frac{p_i}{2} \quad \text{and} \quad \omega_i(-e_1) = \frac{1}{2} - \frac{p_i}{2}.$$

Now consider the random environment $\omega = \{\omega((i, j)) : (i, j) \in \mathbb{Z}^2\}$ defined

$$\omega((i, j)) := \omega_i \quad \text{for all} \quad i, j \in \mathbb{Z}.$$

We will call $\mathbb{P}_1$ the law of the above environment and Q_1 the annealed law of the corresponding random walk starting from 0.

Theorem 2.3.1. *Consider a random walk in a random environment with law $\mathbb{P}_1$. Then, the following are satisfied:*

(i) Q_1-a.s.

$$\lim_{n \to \infty} X_n \cdot e_1 = \infty.$$

(*ii*) *Q_1-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

(*iii*) *In Q_1-probability*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = e_1.$$

(*iv*) *The law Q_1 satisfies the polynomial condition $(PC)_{M,c}$ is satisfied, with $M = \kappa - \frac{1}{2} - \varepsilon$ and $c = 1$, where ε is an arbitrary number in the interval $(0, \kappa - \frac{1}{2})$.*

Proof.

(*i*) We will describe a *one dimensional procedure* which will be used throughout the proofs of items (i) and (*ii*). Specifically, defining $(Y_n)_{n\geqslant 0} := (X_n \cdot e_i)_{n\geqslant 0}$ one has that it can be identified with the one dimensional RWRE which has quenched law $P_{0,\omega}$ starting from 0, defined by the transition probabilities:

$$P_{0,\omega}[Y_{n+1} = Y_n + e_1 \mid Y_n] = \widetilde{\omega}(Y_n, e_1) = p_{Y_n}/2,$$
$$P_{0,\omega}[Y_{n+1} = Y_n - e_1 \mid Y_n] = \widetilde{\omega}(Y_n, -e_1) = (1 - p_{Y_n})/2, \text{ and}$$
$$P_{0,\omega}[Y_{n+1} = Y_n \mid Y_n] = \widetilde{\omega}(Y_n, 0) = 1/2.$$

Since assumption **TNB** it follows that $\widetilde{E}_1[\ln[\widetilde{\rho}_0]] < 0$, where $\widetilde{\rho}_0 := \widetilde{\omega}(0, -e_1)/\widetilde{\omega}(0, e_1)$ and $\widetilde{E}_1$ denotes the corresponding expectation in this random environment. Now, from the transience criteria in [Ze1] Theorem 2.1.2 one has that Q_1- a.s.

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty.$$

(*ii*) Since $\kappa \leqslant 1$, using a *one dimensional procedure* for directions e_1 and e_2 and the strong law of large numbers for one dimensional RWRE ([Ze1], Theorem 2.1.9), we get Q_1-a.s.

$$\frac{X_n}{n} = \frac{(X_n \cdot e_1)e_1 + (X_n \cdot e_2)e_2}{n} \to 0.$$

(*iii*) We define the random variables N_1 and N_2 as horizontal and vertical steps performed by the walk X_n, respectively. By the very definition of this example, both of them distribute $B(n, \frac{1}{2})$ under the quenched law. Given $\varepsilon > 0$, we have to estimate the following probability:

$$Q_1\left[\left|\frac{(X_n \cdot e_1)e_1 + (X_n \cdot e_2)e_2}{\sqrt{(X_n \cdot e_1)^2 + (X_n \cdot e_2)^2}} - e_1\right| > \varepsilon\right] \;=\; Q_1\left[\left|\frac{\frac{(X_n \cdot e_1)}{n^\kappa}e_1 + \frac{(X_n \cdot e_2)}{n^\kappa}e_2}{\sqrt{\frac{(X_n \cdot e_1)^2}{n^{2\kappa}} + \frac{(X_n \cdot e_2)^2}{n^{2\kappa}}}} - e_1\right| > \varepsilon\right].$$

Clearly, $X_n \cdot e_2$ under the annealed law has the same law $\widetilde{P}$ of the one dimensional simple symmetric random walk $Z_{n'}$ at time $n' = N_2$, such that $\widetilde{P}$- a.s. $n'/n \to 1/2$ as $n \to \infty$. Therefore, since $\kappa > 1/2$ as a result one has

$$Q_1\left[\lim_{n\to\infty} \frac{X_n \cdot e_2}{n^\kappa} = 0\right] = \widetilde{P}\left[\lim_{N_2\to\infty} \frac{Z_{N_2}}{N_2^\kappa}\frac{1}{2^\kappa} = 0\right] = 1.$$

and also Q_1- a.s.

$$\lim_{n\to\infty} \frac{(X_n \cdot e_2)^2}{n^{2\kappa}} = 0.$$

On the other hand, using the convergence theorem of Kesten, Kozlov and Spitzer [KKS75], calling Y_{N_1} the one dimensional random walk in random environment corresponding to $X_n \cdot e_1$ and using a similar procedure as the one given above, we can see that

$$\frac{X_n \cdot e_1}{\sqrt{(X_n \cdot e_1)^2}} \to 1$$

in distribution, which turns out that this convergence is also in Q_1- probability and completes the proof.

(*iv*) For $j \in \{1, 2\}$ and a a positive real number, we define the stopping times $T_a^{e_j}$ and $\widetilde{T}_a^{e_j}$ by

$$T_a^{e_j} := \inf\{n \geqslant 0 : X_n \cdot e_j \geqslant a\} \tag{2.27}$$

along with

$$\widetilde{T}_a^{e_j} := \inf\{n \geqslant 0 : X_n \cdot e_j \leqslant a\} \tag{2.28}$$

Notice that for $c = 1$ and large L one has the following estimate

$$Q_1[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0)] \leqslant Q_1[\widetilde{T}_{-L}^{e_1} < T_L^{e_1}] + Q_1[T_L^{e_2} \wedge \widetilde{T}_{-L}^{e_2} < T_L^{e_1}]. \tag{2.29}$$

The first probability in the right most side of (2.29) has an exponential bound as it follows from the estimate in the proof of item *(iv)* in Theorem 2.3.2. Observe that the second probability in the right most side of (2.29) is less than or equal to

$$Q_1[T_L^{e_2} \wedge \widetilde{T}_{-L}^{e_2} \leqslant L^{2+\varepsilon}] + Q_1[L^{2+\varepsilon} < T_L^{e_1}].$$

Keeping the notations introduced in item *(iii)*. From the very definition of Z'_n, one sees that for large L, there exists a positive constant K_1 such that:

$$\begin{aligned} Q_1[T_L^{e_2} \wedge \widetilde{T}_{-L}^{e_2} \leqslant L^{2+\varepsilon}] \leqslant Q_1[|X_n \cdot e_2| \leqslant L, \text{ for all } n \in \mathbb{N}, 0 \leqslant n \leqslant L^{2+\varepsilon}] \leqslant \\ \widetilde{P}[Z_{N_2(n)} \leqslant L^{2+\varepsilon}, \text{ for all } n \in \mathbb{N}, 0 \leqslant n \leqslant L^{2+\varepsilon}] \leqslant \exp\{-K_1 L^{\varepsilon}\}. \end{aligned} \tag{2.30}$$

On the other hand, using the sharp estimate in Theorem 1.3 in [FGP10] and denoting $\hat{P}$ the law of underlying one dimensional random walk corresponding to the annealed law of $(X_n \cdot e_1)_{n \geqslant 0}$, we can see that for large L, there exists a positive constant K_2 such that:

$$\begin{aligned} Q_1[L^{2+\varepsilon} < T_L^{e_1}] \leqslant Q_1[X_{[L^{2+\varepsilon}]} \cdot e_1 < L] \leqslant \\ \hat{P}[Y_{N_1([L^{2+\varepsilon}])} < L] \leqslant K_2 L^{-(\kappa - 1/2 - \varepsilon)}. \end{aligned} \tag{2.31}$$

$$\tag{2.32}$$

Therefore, in view of the inequality (2.29) and the estimates (2.30)-(2.31), we complete the proof.

□

2.3.2 Directionally transient random walk without an asymptotic direction

Let $\{p_i : i \in \mathbb{Z}\}$ and $\{p'_j : j \in \mathbb{Z}\}$ be two independent i.i.d. copies of p. Following a similar procedure as in the previous example, we consider in the lattice $\mathbb{Z}^2$ the canonical vectors e_1 and e_2, and define the random environment $\omega = \{\omega((i,j)) : (i,j) \in \mathbb{Z}^2\}$ by,

$$\omega_{(i,j)}(e_1) = \frac{p_i}{2} \quad \text{and} \quad \omega_{(i,j)}(-e_1) = \frac{1}{2} - \frac{p_i}{2}.$$

together with

$$\omega_{(i,j)}(e_2) = \frac{p'_j}{2} \quad \text{and} \quad \omega_{(i,j)}(-e_2) = \frac{1}{2} - \frac{p'_j}{2}.$$

We call $\mathbb{P}_2$ the law of the above environment and Q_2 the annealed law of the corresponding random walk starting from 0.

Theorem 2.3.2. *Consider a random walk in a random environment with law $\mathbb{P}_2$. Then, the following are satisfied.*

(i) Let $l \in \mathbb{S}$. Then $l \cdot e_1 \geqslant 0$ and $l \cdot e_2 \geqslant 0$ if and only if Q_2-a.s.

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

(ii) Q_2-a.s.

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

(iii) There exists a non-deterministic $\hat{v}$ such that:

$$\frac{X_n}{|X_n|_2} \to \hat{v}.$$

in distribution.

(iv) There exists a $c > 1$ such that

$$\overline{\lim}_{L\to\infty} L^{-1} \log Q_2[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0)] < 0, \tag{2.33}$$

where $l = (1/\sqrt{2}, 1/\sqrt{2})$. Thus, condition $(T)|l$ [Sz02] is satisfied.

Proof.

(*i*) This amounts to prove that Q_2- a.s.

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty \text{ and } \lim_{n\to\infty} X_n \cdot e_2 = \infty.$$

Both assertions follow from the *one dimensional procedure*, Theorem 2.1.2 in [Ze1] and the assumption **TNB**.

(*ii*) This proof is similar to case (*ii*) of Theorem 2.3.1 .

(*iii*) Define a sequence $T_{i,j}$, for $i \geqslant 0,\ j \in \{1,2\}$ as follows: $T_{0,j} = 0$,

$$T_{1,j} = \inf\{n \geqslant 0 :\ (X_n - X_0)\cdot e_j > 0 \text{ or } (X_n - X_0)\cdot e_j < 0\}$$

and for $i \geqslant 2$

$$T_{i,j} = T_{1,j} \circ \theta_{T_{i-1,j}} + T_{i-1,j}.$$

Setting $Y_{n,j} := X_{T_{n,j}} \cdot e_j$, we see that for $j \in \{1,2\}$, the one dimensional random walks without transitions to itself at each site $(Y_{n,j})_{n\geqslant 0}$ are independent and their transitions at each site $i \in \mathbb{Z}^d$ are determined by p_i. Furthermore, for $j \in \{1,2\}$, the strong law of large numbers implies that Q_2- a.s.

$$\lim_{n\to\infty} \frac{T_{n,j}}{n} = 2. \tag{2.34}$$

We now apply the result of Kesten, Kozlov and Spitzer [KKS75] to see that there exist constants C_1 and C_2 such that

$$\left(\frac{Y_{n,1}}{n^\kappa}, \frac{Y_{n,2}}{n^\kappa}\right) \to \left(C_1\left(\frac{1}{S_{ca}^{1\ \kappa}}\right)^\kappa, C_2\left(\frac{1}{S_{ca}^{2\ \kappa}}\right)^\kappa\right)$$

in distribution, where for $j \in \{1,2\}$, $S_{ca}^{j\ \kappa}$ stand for two independent completely asymmetric stable laws of index κ, which are positive. Using (2.34) and properties of convergence in distribution we can see that

$$\frac{X_n}{|X_n|} = \frac{\frac{(X_n\cdot e_1)}{n^\kappa} e_1 + \frac{(X_n\cdot e_2)}{n^\kappa} e_2}{\sqrt{\frac{(X_n\cdot e_1)^2}{n^{2\kappa}} + \frac{(X_n\cdot e_2)^2}{n^{2\kappa}}}} \to \frac{\left(\frac{C_1}{S_{ca}^{1\ \kappa}}\right)^\kappa e_1 + \left(\frac{C_2}{S_{ca}^{2\ \kappa}}\right)^\kappa e_2}{\sqrt{\left(\frac{C_1}{S_{ca}^{1\ \kappa}}\right)^{2\kappa} + \left(\frac{C_2}{S_{ca}^{2\ \kappa}}\right)^{2\kappa}}}$$

in distribution. Therefore we have proved that the limit $\hat{v}$ is random.

(*iv*) A first step will be to prove the following decay

$$\limsup L^{-1} \log Q_2[\widetilde{T}^{e_j}_{-\widetilde{c}L} < T^{e_j}_{cL}] < 0$$

for arbitrary positive constants $\widetilde{c}$ and c (see (2.27) and (2.28) for the notations). We prove this in the case $j = 1$, the another case being similar. Following the notation introduced in Theorem 2.3.1 item (*i*) and denoting the greatest integer function by $[\cdot]$, we see that it is sufficient to prove that for large L there exists a positive constant $\widehat{C}$ such that:

$$\widetilde{E}_1[P_{0,\omega}[Y_n \text{ hits } [\hat{c}L] + 1 \text{ before } [cL] + 1]] \leqslant \exp\{-\widehat{C}L\}. \tag{2.35}$$

To this end, for a fixed random environment ω, if we denote $\mathfrak{V}^L_i$ to

$$P_{i,\omega}[Y_n \text{ hits } -[\hat{c}L] + 1 \text{ before } [cL] + 1],$$

the Markov property makes us see that $\mathfrak{V}^L_i$ satisfies the following difference equation for integer $i \in [\widetilde{c}L] + 2, [cL]]$

$$\mathfrak{V}^L_i = (1 - p_i)\mathfrak{V}^L_{i-1} + p_1\mathfrak{V}^L_{i+1}$$

with the constraints

$$\mathfrak{V}^L_{[\widetilde{c}L]+1} = 1 \text{ and } \mathfrak{V}^L_{[cL]+1} = 0.$$

This system can be solved by the method developed by Chung in [Ch67], Chapter 1, Section 12. Applying this one sees that

$$\mathfrak{V}^L_0 = \frac{\exp\{\sum_{-[\hat{c}L+1],0}\} + \ldots + \exp\{\sum_{-[\hat{c}L]+1,[cL]}\}}{1 + \exp\{\sum_{-[\hat{c}L]+1,-[\hat{c}L]+2}\} + \ldots + \exp\{\sum_{-[\hat{c}L]+1,[cL]}\}},$$

where we have adopted the notation introduced in [Sz02], $\sum_{z<m\leqslant z'} = \log \rho(m)$ and $\rho(m) = (1-p_m)/p_m$. A slight variation of the argument in [Sz02] page 744 completes the proof of claim (2.35). On the other hand, considering the probability

$$Q_2[X_{T_{B_{L,2L,l}(0)}} \notin \partial^+ B_{L,2L,l}(0)],$$

we observe that is clearly bounded from above by (see Figure 2.2)

$$Q_2[\widetilde{T}^{e_1}_{-\frac{\sqrt{2}}{2}L} < T^{e_1}_{\sqrt{2}L}] + Q_2[\widetilde{T}^{e_2}_{-\frac{\sqrt{2}}{2}L} < T^{e_2}_{\sqrt{2}L}]$$

In virtue of the claim (2.35) the last expression has an exponential bound and this finishes the proof.

□

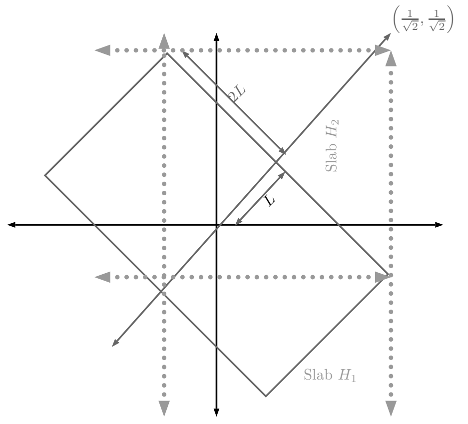

Figure 2.2: A geometric sketch to bound $Q_2[X_{T_{B_{L,2L,l}(0)}} \notin \partial^+ B_{L,2L,l}(0)]$.

2.4 Backtracking of the random walk out of a cone

Here we will provide a uniform control on the probability that a random walk starting form the vertex of a cone stays inside the cone forever. It will be useful to this end to define

$$D' := \inf\{n \in \mathbb{N} : X_n \notin \mathcal{C}(\alpha, l, X_0)\}, \tag{2.36}$$

where as before $l \in \mathbb{S}^{d-1}$.

Proposition 2.4.1. *Let $l \in \mathbb{S}^{d-1}$. Suppose that $(P)_{M,c}|l$ holds, for some $M > 6d - 3$. Then there exists a positive constant $c_2(d) > 0$ such that $P_0[D' = \infty] > c_2(d)$.*

In what follows we prove this proposition. With the purpose of making easier the reading, we introduce here some notations. Let $l' \in \mathbb{S}^{d-1}$ and choose a rotation R' on $\mathbb{R}^d$ with the property

$$R'(e_1) = l'$$

For each $x \in \mathbb{Z}^d$, real numbers $m > 0$, $c > 0$ and integer $i \geqslant 0$ we define the box

$$B_i(x) :=$$
$$x + R'\left((-2^{m+i}, 2^{m+i}) \times (-2c2^{m+i}, 2c2^{m+i})^{d-1}\right) \cap \mathbb{Z}^d$$

along with its "positive boundary"

$$\partial^+ B_i(x) := \partial B_i(x) \cap \{x + R'\left((2^{m+i}, \infty) \times \mathbb{R}^{d-1}\right)\}.$$

We also need slabs perpendicular to direction l'. Set

$$V_0(x) := x + R'\left([-2^m, 2^m] \times \mathbb{R}^{d-1}\right) \cap \mathbb{Z}^d$$

and for $i \geqslant 1$,

$$V_i(x) := x + R'\left(\left[-2^m, \sum_{j=0}^{i} 2^{m+j}\right] \times \mathbb{R}^{d-1}\right) \cap \mathbb{Z}^d.$$

The positive part of the boundary for this set is defined as

$$\partial^+ V_i(x) := \partial V_i(x) \cap \left\{x + R'\left(\left(\sum_{j=0}^{i} 2^{m+j}, \infty\right) \times \mathbb{R}^{d-1}\right)\right\}.$$

Furthermore, we will define recursively a sequence of stopping times as follows. First, let

$$T_0 := T_{B_0(X_0)}.$$

and for $i \geqslant 1$

$$T_i := T_{B_i(X_{T_{i-1}})} \circ \theta_{T_{i-1}} + T_{i-1}.$$

We now need to define the first time of entrance of the random walk to the hyperplane $R'\left((-\infty, 0) \times \mathbb{R}^{d-1}\right)$,

$$D_{l'} := \inf\{n \geqslant 0 : X_n \cdot l' < 0\}.$$

With these notations we can prove:

Lemma 2.4.2. *Assume $(P)_{N,2c}|l'$ where $c > 0$, for some $N > 2(d-1)$. Then, for all $m \in \mathbb{N}$ and $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geqslant 2^m\}$, we have that*

$$P_x[D_{l'} = \infty] \geqslant y(m)$$

where $y(m)$ does not depend on l' and satisfies $\lim_{m\to\infty} y(m) = 1$.

Proof. From the fact that $(P)_{N,2}|l'$ holds, we can (and we do) assume that there exists a $m > 0$ large enough, such that for any positive integer i one has that

$$P_0[X_{T_{B_i(0)}} \in \partial^+ B_i(0)] \geqslant 1 - 2^{-N(m+i)} \tag{2.37}$$

holds. By stationarity, we have for $x \in \mathbb{Z}^d$:

$$P_x[X_{T_{B_i(x)}} \in \partial^+ B_i(x)] \geqslant 1 - 2^{-N(m+i)}. \tag{2.38}$$

Throughout this proof, let us choose $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geqslant 2^m\}$. For reasons that will be clear through the proof, we need to estimate for $i \geqslant 1$ the following probability

$$I_i := P_x[X_{T_{V_i(x)}} \in \partial^+ V_i(x)], \tag{2.39}$$

and with this aim, in view of (2.38), we have

$$I_0 \geqslant P_x[X_{T_{B_0(x)}} \in \partial^+ B_0(x)] \geqslant 1 - 2^{-Nm} \geqslant 1 - 2^{-N\frac{m}{2}}.$$

Now, as a preliminary computation for the recursion, we begin to estimate I_1. Note that

$$I_1 \geqslant P_x[X_{T_0} \in \partial^+ B_0(X_0), (X_{T_{B_1(X_0)}} \in \partial^+ B_1(X_0)) \circ \theta_{T_0}]. \tag{2.40}$$

Using the strong Markov property at time T_0 we then see that

$$\begin{aligned} I_1 \geqslant \textstyle\sum_{y \in \partial^+ B_0(x)} \mathbb{E}\Big[& P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), X_{T_0} = y] \\ & \times P_{y,\omega}[X_{T_{B_1(y)}} \in \partial^+ B_1(y)]\Big] \\ \geqslant \textstyle\sum_{y \in \partial^+ B_0(x)} \mathbb{E}\Big[& P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), X_{T_0} = y] \\ & \times P_{y,\omega}[X_{T_{B_1(y)}} \in \partial^+ B_1(y)] \mathbb{1}_{G_0}\Big], \end{aligned} \tag{2.41}$$

where

$$G_0 := \{w \in \Omega : P_{y,\omega}[X_{T_{B_1(y)}} \in \partial^+ B_1(y)] > 1 - 2^{-N\frac{m}{2}}, \text{ for all } y \in \partial^+ B_0(x)\}.$$

Thus, it is clear that

$$I_1 \geqslant \left(1 - 2^{-N\frac{m}{2}}\right)\left(P_x[X_{T_0} \in \partial^+ B_0(X_0)] - \mathbb{P}[(G_0)^c]\right). \tag{2.42}$$

Notice that by (2.38) and Chebyshev's inequality

$$\begin{aligned}\mathbb{P}[(G_0)^c] &\leqslant \textstyle\sum_{y\in\partial^+ B_0(x)} \mathbb{P}[P_{y,\omega}[X_{T_{B_1(y)}} \notin \partial^+ B_1(y)] \geqslant 2^{-N\frac{m}{2}}] \\ &\leqslant \textstyle\sum_{y\in\partial^+ B_0(x)} P_y[X_{T_{B_1(y)}} \notin \partial^+ B_1(y)] 2^{N\frac{m}{2}} \\ &= |\partial^+ B_0(x)| 2^{N\frac{m}{2}}] P_0[X_{T_{B_1(0)}} \notin \partial^+ B_1(0)] \\ &\leqslant (2c2^{m+1})^{d-1} 2^{N\left(\frac{m}{2}-(m+1)\right)} \leqslant (2c2^{m+1})^{d-1} 2^{-N\frac{m}{2}}. \end{aligned} \tag{2.43}$$

Plugging (2.43) into (2.42) we see that

$$I_1 \geqslant (1 - 2^{-N\frac{m}{2}})(1 - 2^{-N\frac{m}{2}} - (2c2^{m+1})^{d-1} 2^{-N\frac{m}{2}}). \tag{2.44}$$

Hereafter we can do the general recursive procedure. For this end, we define for $i \geqslant 1$

$$\begin{aligned} J_i := P_0[X_{T_0} \in \partial^+ B_0(X_0), (X_{T_{B_1(X_0)}} \in \partial^+ B_1(X_0)) \circ \theta_{T_0}, \ldots \\ \ldots, (X_{T_{B_i(X_0)}} \in \partial^+ B_i(X_0)) \circ \theta_{T_{i-1}}]. \end{aligned} \tag{2.45}$$

It is straightforward that $I_i \geqslant J_i$. Furthermore, through induction on $i \geqslant 1$, we will establish the following claim

$$J_i \geqslant (1 - 2^{-N\frac{(m+i-1)}{2}}) \left[J_{i-1} - 2^{-N\frac{(m+i-1)}{2}} \left(\sum_{j=0}^{i-1} 2c2^{(m+j)+1} \right)^{d-1} \right]. \tag{2.46}$$

To prove this, we first define the *extended boundary* of the pile of boxes at a given step as

$$F_0 := \partial B_0(x) \cap \{x + R'((2^m, \infty) \times \mathbb{R}^{d-1}))\},$$

and for $i \geqslant 2$

$$F_{i-1} := \partial \left\{\cup_{y\in F_{i-2}} B_{i-1}(y)\right\} \cap \{x + R'((2^{m+i-1}, \infty) \times \mathbb{R}^{d-1}))\}.$$

Using these notations, we can apply the strong Markov property to (2.45) at time T_{i-1}, to get that

$$\begin{aligned} J_i = \textstyle\sum_{y\in F_{i-1}} \mathbb{E}\,[P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), \ldots \\ \ldots, (X_{T_{B_{i-1}(X_0)}} \in \partial^+ B_{i-1}(X_0)) \circ \theta_{T_{i-2}}, X_{T_{i-1}} = y] P_{y,\omega}[X_{T_{B_i(X_0)}} \in \partial^+ B_i(X_0)]\Big]. \end{aligned}$$

Following the same strategy used to deduce (2.44), it will be convenient to introduce for each $i \geqslant 2$ the event

$$G_{i-1} := \{\omega \in \Omega : P_{y,\omega}[X_{T_{B_i(y)}} \in \partial^+ B_i(y)] > 1 - 2^{-N^{\frac{(m+i-1)}{2}}}, \text{ for all } y \in F_{i-1}\}.$$

Inserting the indicator function of the event G_{i-1} into (2.45) we get that

$$J_i \geqslant \sum_{y \in F_{i-1}} \mathbb{E}\Big[P_{x,\omega}[X_{T_0} \in \partial^+ B_0(X_0), \ldots, (X_{T_{B_{i-1}(X_0)}} \in \partial^+ B_{i-1}(X_0)) \circ \theta_{T_{i-2}}, X_{T_{i-1}} = y] \times P_{y,\omega}[X_{T_{B_i(X_0)}} \in \partial^+ B_i(X_0)] \mathbb{1}_{G_{i-1}}\Big].$$

By the same kind of estimation as in (2.42), we have

$$J_i \geqslant (1 - 2^{-N^{\frac{(m+i-1)}{2}}})\,(J_{i-1} - \mathbb{P}[(G_{i-1})^c])\,. \tag{2.47}$$

We need to get an estimate for $\mathbb{P}[(G_{i-1})^c]$. We do it repeating the argument given in (2.43). Let us first remark that

$$|F_{i-1}| \leqslant \left(\sum_{j=0}^{i-1} 2c2^{(m+j)+1}\right)^{d-1}, \tag{2.48}$$

holds. Indeed, the case in which $l' = e_1$ gives the maximum number for $|F_{i-1}|$. Keeping (2.48) in mind we get that

$$\begin{aligned} P_x[(G_{i-1})^c] &\leqslant \textstyle\sum_{y \in F_{i-1}} \mathbb{P}\left[P_{y,\omega}\left[X_{T_{B_i(y)}} \notin \partial^+ B_i(y)\right] \geqslant 2^{-N^{\frac{(m+i-1)}{2}}}\right] \\ &\leqslant \textstyle\sum_{y \in F_{i-1}} P_y[X_{T_{B_i(y)}} \notin \partial^+ B_i(y)] 2^{N^{\frac{(m+i-1)}{2}}} \\ &\leqslant \left(\textstyle\sum_{j=0}^{i-1} 2c2^{(m+j)+1}\right)^{d-1} 2^{-N^{\frac{(m+i-1)}{2}}}. \end{aligned} \tag{2.49}$$

Therefore, combining (2.49) and (2.47) we prove claim (2.46). Iterating (2.46) backward, from a given integer i, we have got

$$J_i \geqslant J_1 \left[\prod_{h=1}^{i-1} (1 - 2^{-N^{\frac{(m+h)}{2}}})\right] - \sum_{j=1}^{i-1} a_j 2^{-N^{\frac{m+j}{2}}} \prod_{k=j}^{i-1} (1 - 2^{-N^{\frac{(m+k)}{2}}}), \tag{2.50}$$

where we have used for short

$$a_j := \left(\sum_{i=0}^{j} c2^{(m+i)+1}\right)^{d-1} \leqslant (2c)^{d-1} 2^{(m+j+2)(d-1)}.$$

The same argument used to derive (2.44) can be repeated to conclude that

$$J_1 \geqslant (1 - 2^{-N\frac{m}{2}})(1 - 2^{-N\frac{m}{2}} - (2c2^{m+1})^{d-1}2^{-N\frac{m}{2}}). \tag{2.51}$$

Replacing the right hand side of (2.51) into (2.50), and together to the fact $I_i \geqslant J_i$, we see that

$$I_i \geqslant \left[\prod_{h=0}^{i-1}(1 - 2^{-N\frac{m+h}{2}})\right](1 - 2^{-N\frac{m}{2}}) - \sum_{j=0}^{i-1} a_j 2^{-N\frac{(m+j)}{2}} \prod_{k=j}^{i-1}(1 - 2^{-N\frac{(m+k)}{2}}). \tag{2.52}$$

Now we can finish the proof. First, observe that

$$P_x[D_{l'} = \infty] \geqslant I_\infty,$$

where as a matter of definition

$$I_\infty := \lim_{i\to\infty} I_i$$

(this limit exists, because it is the limit of a decreasing sequence of real numbers bounded from below). By the condition $N > 2(d-1)$, we get that for each $m \geqslant 1$ one has that for all $j \geqslant 1$,

$$a_j\ 2^{-\frac{M(m+j)}{2}} \leqslant (8c)^{d-1}2^{-\vartheta\frac{(m+j)}{2}},$$

where ϑ stands for the positive number so that $N = 2(d-1) + \vartheta$. Thus all the products and series in (2.52) converge and we have that for all $m \geqslant 1$ and $x \in \{z \in \mathbb{Z}^d : z \cdot l' \geqslant 2^m\}$

$$P_x[D_{l'} = \infty] \geqslant y(m),$$

where

$$y(m) := \left[\prod_{h=0}^{\infty}(1 - 2^{-N\frac{(m+h)}{2}})\right](1 - 2^{-N\frac{m}{2}}) - \sum_{j=0}^{\infty} a_j 2^{-N\frac{(m+j)}{2}} \prod_{k=j}^{\infty}(1 - 2^{-N\frac{(m+k)}{2}}).$$

Clearly for each $m \geqslant 1$, $y(m)$ does not depend on the direction l' and $\lim_{m\to\infty} y(m) = 1$, which completes the proof. □

With the previous Lemma, we now have enough tools to prove Proposition 2.4.1. Before this, we need a definition of geometrical nature.

We will say that a sequence $(x_0, \ldots, x_n)$ of lattice points is a *path* if for every $1 \leqslant i \leqslant n-1$, one has that x_i and x_{i-1} are nearest neighbors. Furthermore, we say that this path is *admissible* if for every $1 \leqslant i \leqslant n-1$ one has that

$$(x_i - x_{i-1}) \cdot l \neq 0.$$

Proof of Proposition 2.4.1. Assume $(P)_{M,c}|l$, where $M > 6(d-1)+3$ which is the condition of the statement of the Proposition 2.4.1. We appeal to Proposition (2.2.3) and assumption $(P)_{M,c}|l$ to choose an $\alpha > 0$ such that for all $i \in [2, d]$

$$(P)_{N,2c}|l_{\pm i}$$

is satisfied with

$$N := \frac{M}{3} - 1 > 2(d-1). \tag{2.53}$$

From now on, let m be any natural number satisfying

$$y(m) > 1 - \frac{1}{2(d-1)}, \tag{2.54}$$

where $y(m)$ is the function given in Lemma 2.4.2. Note that there exists a constant $c_3(d)$ such that for all $x \in \mathbb{Z}^d$ contained in $C(\alpha, l, R(2^m e_1))$ and such that $|R(2^m e_1) - x|_1 \leqslant 1$ one has that there exists an admissible path with at most $c_3 2^m$ lattice points joining 0 and x. We denote this path by

$$(0, y_1, \ldots, y_n = x)$$

noting that $n \leqslant c_3 2^m$.

The general idea to finish the proof is to push forward the walk up to site x with the help of uniform ellipticity in direction l and then we make use of Lemma (2.4.2) to ensure that the walk remains inside the cone.

Therefore, by (2.53) and Lemma (2.4.2) we can conclude that for all $2 \leqslant i \leqslant d$ one has that

$$P_x[D_{l_{i+}} = \infty] \geqslant y(m), \tag{2.55}$$

along with

$$P_x[D_{l_{i-}} = \infty] \geqslant y(m). \tag{2.56}$$

Define the event that the random walk starting from 0 following that path $(0, y_1, \ldots, y_n)$ as

$$A_n := \{(X_0, \ldots, X_n) = (0, y_1, \ldots, y_n)\}.$$

Now notice that

$$P_0[D' = \infty] \geqslant$$

$$P_0\left[A_n, (D_{l_{i-}} = \infty) \circ \theta_n, (\ D_{l_{i+}} = \infty) \circ \theta_n \text{ for } 2 \leqslant i \leqslant d\right]. \tag{2.57}$$

On the other hand, by definition of the annealed law, together with the strong Markov property we have that

$$\begin{aligned} P_0[A_n, (D_{l_{i-}} = \infty) \circ \theta_n, (\ D_{l_{i+}} = \infty) \circ \theta_n \text{ for } 2 \leqslant i \leqslant d] = \\ \mathbb{E}\left[P_{0,\omega}[A_n], P_{x,\omega}[D_{l_{i-}} = \infty, D_{l_{i+}} = \infty \text{ for } 2 \leqslant i \leqslant d]\right]. \end{aligned} \tag{2.58}$$

Using the uniform ellipticity assumption $(UE)|l$, along with (2.55) and (2.56), we can see that (2.58) is bounded from below by

$$(2\kappa)^{c_3 2^m} \left(1 - 2(d-1)(1 - y(m))\right). \tag{2.59}$$

By virtue of our choice of m in (2.54), we see that there exists a constant c_2 just depending on the dimension (we recall that m is fixed at this point of the proof), such that

$$c_4 := (2\kappa)^{c_3 2^m} \left(1 - 2(d-1)(1 - y(m))\right) > 0 \tag{2.60}$$

Finally, in view of the inequalities (2.57) and (2.58) it follows that

$$P_0[D' = \infty] \geqslant c_4.$$

□

2.5 Polynomial control of regeneration positions

In this section, we define an approximate regeneration times as done in [CZ01], which will depend on a distance parameter $L > 0$. We will then show that these times, assuming $(PC)_{M,c}|l$ for M large enough, and cone-mixing, when scaled by κ^L, define approximate regeneration positions with a finite second moment.

2.5.1 Preliminaries

We recall the definition of approximate renewal time given in [CZ01]. Let $W := \mathcal{E} \cup \{0\}$ [c.f. (2.2)] and endow the space $W^{\mathbb{N}}$ with the canonical $\sigma-$algebra $\mathcal{W}$ generated by the cylinder sets. For fixed $\omega \in \Omega$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots) \in W^{\mathbb{N}}$, we denote by $P_{\omega,\varepsilon}$ the law of the Markov chain $\{X_n\}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$, so that $X_0 = 0$ and with transition probabilities defined for $z \in \mathbb{Z}^d$, $e, |e| = 1$ as

$$P_{\omega,\varepsilon}[X_{n+1} = z + e | X_n = z] = \mathbb{1}_{\{\varepsilon_n = e\}} + \frac{\mathbb{1}_{\{\varepsilon_n = 0\}}}{1 - \kappa|\mathcal{E}|}[\omega(z, e) - \kappa \mathbb{1}_{\{e \in \mathcal{E}\}}].$$

Call $E_{\omega,\epsilon}$ the corresponding expectation. Define also the product measure Q, which to each sequence of the form $\varepsilon \in W^{\mathbb{N}}$ assigns the probability $Q(\varepsilon_1 = e) := \kappa$, if $e \in \mathcal{E}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}|$, and denote by E_Q the corresponding expectation.

Now let $\mathfrak{G}$ be the σ-algebra on $(\mathbb{Z}^d)^{\mathbb{N}}$ generated by cylinder sets, while $\mathfrak{F}$ be the σ-algebra on Ω generated by cylinder sets. Then, we can define for fixed ω the measure

$$\overline{P}_{0,\omega} := Q \otimes P_{\omega,\varepsilon}$$

on the space $(W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{W} \times \mathfrak{G})$, and also

$$\overline{P}_0 := \mathbb{P} \otimes Q \otimes P_{\omega,\varepsilon}$$

on $(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathfrak{F} \times \mathcal{W} \times \mathfrak{G})$, denoting by $\bar{E}_{0,\omega}$ and $\bar{E}_0$ the corresponding expectations. A straightforward computation makes us conclude that the law of $\{X_n\}$ under $\bar{P}_{0,\omega}$ coincides with its law under $P_{0,\omega}$ and that its law under $\overline{P}_0$ coincides with its law under P_0.

Let q be a positive real number such that for all $1 \leqslant i \leqslant d$,

$$u_i := l_i q$$

is an integer. Define now the vector $u := (u_1, \dots, u_d)$. From now on, we fix a particular sequence $\overline{\varepsilon}$ in $\mathcal{E}$ of length $p := |u|_1$ whose components sum up to u:

$$\overline{\varepsilon} := (\overline{\varepsilon}_1, \dots, \overline{\varepsilon}_p),$$

together with

$$\overline{\varepsilon}_1 = \overline{\varepsilon}_2 = \dots = \overline{\varepsilon}_{|u_1|} := \operatorname{sgn}(u_1) e_1,$$
$$\overline{\varepsilon}_{|u_1|+1} = \overline{\varepsilon}_{|u_1|+2} = \dots = \overline{\varepsilon}_{|u_1|+|u_2|} := \operatorname{sgn}(u_2) e_2$$
$$\vdots$$
$$\overline{\varepsilon}_{p-|u_d|+1} = \dots = \overline{\varepsilon}_p := \operatorname{sgn}(u_d) e_d.$$

Without loss of generality we can assume that $l_1 \neq 0$. And by taking α small enough that

$$\overline{\varepsilon}_1, \overline{\varepsilon}_1 + \overline{\varepsilon}_2, \dots \overline{\varepsilon}_p$$

are inside of $\mathcal{C}(0, l, \alpha)$. For $L \in p\mathbb{N}$ consider the sequence $\overline{\varepsilon}^{(L)}$ of length L, defined as the concatenation L/p times with itself of the sequence $\bar{\varepsilon}$, so that

$$\overline{\varepsilon}^{(L)} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_p, \dots, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_p).$$

Consider the filtration $\mathcal{G} := \{\mathcal{G}_n : n \geqslant 0\}$ where

$$\mathcal{G}_n := \sigma((\varepsilon_i, X_i), i \leqslant n).$$

Define $S_0 := 0$,

$$S_1 :=$$
$$\inf\{n \geqslant L : \ X_{n-L} \cdot l > \max\{X_m \cdot l : m < n - L\}, \ (\varepsilon_{n-L}, \dots, \varepsilon_{n-1}) = \overline{\varepsilon}^{(L)}\}$$

together with

$$R_1 := D' \circ \theta_{S_1} + S_1.$$

We can now recursively define for $k \geqslant 1$,

$$S_{k+1} :=$$
$$\inf\{n \geqslant R_k : \ X_{n-L} \cdot l > \max\{X_m \cdot l : m < n-L\}, \ (\varepsilon_{n-L}, \ldots, \varepsilon_{n-1} = \overline{\varepsilon}^{(L)}))\}$$

and

$$R_{k+1} := D' \circ \theta_{S_{k+1}} + S_{k+1}.$$

Clearly,

$$0 = S_0 \leqslant S_1 \leqslant R_1 \leqslant \cdots \infty,$$

the inequalities are strict if the left member of the corresponding inequality is finite, and the sequences $\{S_k : k \geqslant 0\}$ and $\{R_k : k \geqslant 0\}$ are $\mathcal{G}$-stopping times. On the other hand, we can check that $\bar{P}_0-$a.s. one has that $S_1 < \infty$ along with the fact $\bar{P}_0-$a.s. on the set

$$\{\lim X_n \cdot l = \infty\} \cap \{R_k < \infty\} \text{ one has too that } \qquad (2.61)$$
$$S_{k+1} < \infty.$$

Put

$$K := \inf\{k \geqslant 1 : S_k < \infty, R_k = \infty\}$$

and define the *approximate regeneration time*

$$\tau^{(L)} := S_K. \qquad (2.62)$$

We see that the random variable $\tau^{(L)}$ is the first time n in which the walk has reached a record at time $n-L$ in direction l, and then the walk goes on L steps in the direction l by means of the action of $\overline{\varepsilon}^{(L)}$ to finally after this time n never exits the cone $C(X_n, l, \alpha)$.

The following lemma is required to show that the approximate renewal times are $\bar{P}_0$-a.s. finite. Its can be proved using a slight variation of the argument given in page 517 of Sznitman [Sz03].

Lemma 2.5.1. *Consider a random walk in a random environment. Let $l \in \mathbb{S}^{*d-1}$, $M \geqslant d+1$ and $c > 0$ and assume that $(PC)_{M,c}|l$ is satisfied. Then the random walk is transient in direction l.*

Proof. For the sake of completeness, we are going to sketch the steps so as to obtain the claim of the theorem.

Step 1. Notice that any $M > 0$ gives

$$P_0[\limsup_{n\to\infty} X_n \cdot l = \infty] = 1.$$

From one can easily show that

$$P_0[\limsup_{n\to\infty} X_n \cdot l = \infty] = 1$$

if and only if

$$\lim_{L\to\infty} P_0[\inf\{n \geqslant 0 : \ X_n \cdot l \geqslant L\} \ = \ \infty] = 0.$$

Step 2. Following the argument on page 517 of [Sz03], we have got to get rid the order of the *positive boundary of a box* plus some order which makes possible to apply Borel-Cantelli Lemma. It can be seen that a term in M of $d-1$ suffices to get rid the order of the *positive boundary* and therefore $M \geqslant (d-1)+2 = d+1$ is enough to get:

$$P_0[\lim_{n\to\infty} X_n \cdot l = \infty] = 1.$$

□

We make note a trivial remark that the random walk is transient in direction u also.

We can now prove the following stronger version of Lemma 2.2 of [CZ01].

Lemma 2.5.2. *Assume $(CM)_{\alpha,\phi}|l$, $(UE)|l$ and $(PC)_{M,c}$ for $M > 6d-3, c > 0$. Then there exists a positive $L_0 \in |u|_1\mathbb{N}$, such that*

$$\phi(L_0) + P_0[D' < \infty] < 1,$$

and $\tau^{(L)} < \infty$, P_0-a.s. are fulfilled for each $L \geqslant L_0$, $L \in |u|_1\mathbb{N}$.

Proof. Following the arguments in the proof of Lemma 2.2. of [CZ01] (using u instead of l), one has that:

$$\bar{P}_0[R_k < \infty] \leqslant (\phi(L_0) + P_0[D' < \infty])^k \tag{2.63}$$

From the assumption $(CM)_{\alpha,\phi}|l$, we have $\phi(L) \to 0$ as $L \to \infty$. On the other hand, by Lemma 2.4.1,

$$P_0[D' < \infty] < 1.$$

Therefore, we can find a L_0 with the property:

$$\phi(L) + P_0[D' < \infty] < 1,$$

for all $L \geqslant L_0,\ L \in \mathbb{N}|l|_1$.

Then, via Borel-Cantelli Lemma, one has that $\bar{P}_0-$ almost surely

$$\inf\{n \geqslant 1 :\ R_n = \infty\} < \infty, \tag{2.64}$$

holds. Now, observe that $\bar{P}_0-$ almost surely:

$$\inf\{n \geqslant 1 :\ R_n = \infty\} = \inf\{n \geqslant 1 : R_{n-1} < \infty\ R_n = \infty\} \tag{2.65}$$

In turn, using (2.61) which is satisfied in view of Lemma 2.5.1, turns out that

$$\inf\{n \geqslant 1 : S_n < \infty\ R_n = \infty\} = K < \infty$$

$\bar{P}_0-$ almost surely. □

Finally, we can state the following proposition, which gives a control on the second moment of the position of the random walk at the first regeneration position. Define for $x \in \mathbb{Z}^d$ and $L > 0$ the $\sigma-$algebra

$$\mathfrak{F}_{x,L} := \sigma\left\{\omega(y,\cdot);\ y\cdot u \leqslant x\cdot u - \frac{L}{|u|_1}|u|_2\right\}.$$

Proposition 2.5.3. *Fix $l \in \mathbb{S}^{*d-1}$, $\alpha > 0$, $M > 0$ and $\phi : [0,\infty) \to [0,\infty)$ such that $\lim_{r\to\infty}\phi(r) = 0$. Assume that $0 < \alpha < \min\{\frac{1}{9}, \frac{1}{2c+1}\}$ and that $(CM)_{\alpha,\phi}|l$, $(UE)|l$ and $(PC)_{M,c}|_l$ hold. Then, there exists a constant c_5, such that*

$$\bar{E}_0[(\kappa^L X_{\tau(L)}\cdot l)^2|\mathfrak{F}_{0,L}] \leqslant c_5. \tag{2.66}$$

2.5.2 Preparatory results

Now we are in position to prove the main proposition of this section. Before we do this, we will prove a couple of lemmas.

Lemma 2.5.4. *Assume that $(CM)_{\alpha,\phi}|l$ holds. Then, for each $x \in \mathbb{Z}^d$ one has that*

$$|\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}] - P_0[D' = \infty]| \leqslant \phi(L)$$

holds a.s.

Proof. For each $A \in \mathfrak{F}_{x,L}$, we define

$$\nu[A] := \mathbb{E}[P_{x,\omega}[D' = \infty]\mathbb{1}_A] \tag{2.67}$$

and

$$\mu[A] := (P_0[D' = \infty] + \phi(L))\,\mathbb{P}[A] - \nu[A]. \tag{2.68}$$

Clearly (2.67) defines a measure on $(\Omega, \mathfrak{F}_{x,L})$. We will show that (2.68) also. Indeed, take an $A \in \mathfrak{F}_{x,L}$ and note that $P_{x,\omega}[D = \infty]$ is $\sigma\{\omega(y,\cdot),\, y \in C(x,l,\alpha)\}$-measurable. Therefore, by assumption $(CM)_{\alpha,\phi}|l$ one has that

$$\nu[A] \leqslant P_0[D' = \infty]\mathbb{P}[A] + \phi(L)\mathbb{P}[A].$$

Consequently, (2.68) defines a measure μ on $(\Omega, \mathfrak{F}_{x,L})$. Consider the increasing sequence $\{A_n : n \geqslant 1\}$ of $\mathfrak{F}_{x,L}$-measurable sets defined by

$$A_n := \left\{\omega \in \Omega : \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}] > P_0[D' = \infty] + \phi(L) + \frac{1}{n}\right\}$$

and define

$$A := \bigcup_{n \geqslant 1} A_n.$$

Observe that for each $n \geqslant 1$ we have that

$$\begin{aligned} 0 \leqslant \mu(A_n) &= (P_0[D = \infty] + \phi(L))\mathbb{P}[A_n] - \mathbb{E}[\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}]\mathbb{1}_{A_n}] \\ &\leqslant -\tfrac{1}{n}\mathbb{P}[A_n]. \end{aligned}$$

Therefore, one has that for each $n \geqslant 1$, $\mathbb{P}[A_n] = 0$ and consequently $\mathbb{P}[A] = 0$. Observing that

$$A = \{\omega \in \Omega : \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}] > P_0[D' = \infty] + \phi(L)\},$$

we see that

$$\mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}] - P_0[D' = \infty] \leqslant \phi(L). \tag{2.69}$$

One can prove that

$$-\phi(L) \leqslant \mathbb{E}[P_{x,\omega}[D' = \infty]|\mathfrak{F}_{x,L}] - P_0[D' = \infty]$$

following the same argument used to show (2.69), but changing the event $\{D' = \infty\}$ by $\{D' < \infty\}$.

□

The second lemma that will be needed to prove Proposition 2.5.3 is the following one. To state it define

$$\mathfrak{M} := \sup_{0 \leqslant n \leqslant D'} (X_n - X_0) \cdot u,$$
$$D'(0) := \inf\{n \geqslant 0 : \; X_n \notin C(0, l, \alpha)\},$$

and for $a \in \mathbb{R}$

$$T^l_a := \inf\{n \geqslant 0 : \; X_n \cdot l \geqslant a\} \qquad \text{and}$$
$$\bar{T}^l_a := \inf\{n \geqslant 0 : \; X_n \cdot l > a\}. \tag{2.70}$$

Lemma 2.5.5. *Let $M > 4d + 1$ and*

$$2c + 1 \leqslant \frac{1}{\alpha}. \tag{2.71}$$

Assume that $(PC)_{M,c}|l$ is satisfied. Then, there exists $c_6 = c_6(d) > 0$ such that a.s. one has that

$$E_0[\mathfrak{M}^2, D' < \infty|\mathfrak{F}_{0,L}] \leqslant c_6.$$

$\mathbb{P}-$ *almost surely.*

Proof. To simplify the proof, we will show that the second moment of

$$\mathfrak{M}' := \sup_{0\leqslant n\leqslant D'} (X_n - X_0)\cdot l$$

is bounded from above. Note that

$$E_0[\mathfrak{M}'^2, D' < \infty | \mathfrak{F}_{0,L}] \leqslant P_0[D' < \infty \mid \mathfrak{F}_{0,L}]$$
$$+ \textstyle\sum_{m\geqslant 0} 2^{2(m+1)} P_0[2^m \leqslant \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,L}]. \tag{2.72}$$

Therefore, it is enough to obtain an appropriate upper bound of the probability when m is large

$$P_0[2^m \leqslant \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,L}].$$

Note that,

$$P_0[2^m \leqslant \mathfrak{M}' < 2^{m+1}, D' < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant P_0[T^l_{2^m} < D' < \infty, T^l_{2^{m+1}} \circ \theta_{T_{2^m}} > D'(0) \circ \theta_{T_{2^m}} \mid \mathfrak{F}_{0,L}]$$
$$\leqslant P_0[X_{T^l_{2^m}} \notin \partial^+ B_{2^m, c2^m, l}(0), T^l_{2^m} < D' < \infty \mid \mathfrak{F}_{0,L}]$$
$$+ P_0[X_{T^l_{2^m}} \in \partial^+ B_{2^m, c2^m, l}(0), T^l_{2^{m+1}} \circ \theta_{T_{2^m}} > D'(0) \circ \theta_{T_{2^m}} \mid \mathfrak{F}_{0,L}]. \tag{2.73}$$

Using $(PC)_{M,c}|l$, we get the following upper bound for the first term of the rightmost expression in (2.73),

$$P_0[X_{T_{B_{2^m,c2^m,l}(0)}} \notin \partial^+ B_{2^m,2^m,l}(0), (X_n)_{0\leqslant n\leqslant T_{B_i(0)}} \subset (H_{0,l})^c | \mathcal{H}_{0,l}]$$
$$\leqslant 2^{-Mm}. \tag{2.74}$$

As for the second term in the rightmost expression in (2.73), it will be useful to introduce the set

$$F_m := \partial^+ B_{2^m, c2^m, l}(0).$$

Now, by the strong Markov property we have the bound

$$P_0[X_{T_{B_{2^m,c2^m,l}(0)}} \in \partial^+ B_{2^m,2^m,l}(0), T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,L}]$$
$$\leqslant \textstyle\sum_{y \in F_m} P_y[T^l_{2^{m+1}} > D'(0) \mid \mathfrak{F}_{0,L}]. \quad (2.75)$$

In order to estimate this last conditional probability, we obtain a lower bound for its complement as follows. To simplify the computations which follow, for each $x \in \mathbb{Z}^d$ we introduce the notation

$$B_x := B_{2^{m-1},c2^{m-1},l}(x).$$

Now, note that under the assumption (2.71) we have that

$$c\left(2^m + 2^{m-1}\right) \leqslant \cot(\beta)2^{m-1},$$

which implies that the boxes B_y and B_z, for all $y \in F_m$ and $z \in \partial^+ B_y$, are inside the cone $C(0, l, \alpha)$ (see Figure 2.3).

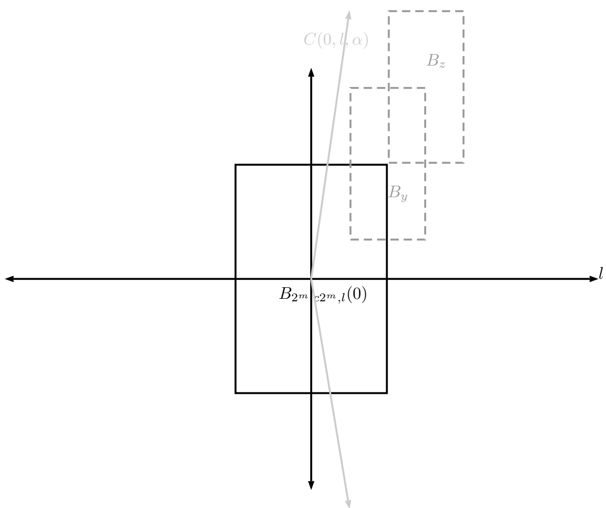

Figure 2.3: The boxes B_y and B_z are inside of $C(0, l, \alpha)$.

Therefore, fixing $y \in F_m$, it follows that

$$P_y[T^l_{2^{m+1}} < D'(0) \mid \mathfrak{F}_{0,L}] \geqslant$$
$$\textstyle\sum_{z \in \partial^+ B_y} \mathbb{E}[P_{y,\omega}[X_{T_{B_y}} \in \partial^+ B_y,$$
$$X_{T_{B_y}} = z, (X_{T_{B_z}} \in \partial^+ B_z) \circ \theta_{T_{B_y}}]|\mathfrak{F}_{0,L}]. \quad (2.76)$$

To estimate the right-hand side of the above inequality, it will be convenient to introduce the set

$$\bar{F}_m := \partial[\cup_{y \in F_m} B_y] \cap \{R([2^{m-1} + 2^m, \infty) \times \mathbb{R}^{d-1})\},$$

and the event

$$G_{\bar{F}_m} := \{\omega \in \Omega : \; P_{z,\omega}[X_{T_{B_z}} \in \partial^+ B_z] > \\ 1 - 2^{-\frac{M(m-1)}{2}}, \text{ for all } z \in \bar{F}_m\}.$$

Using the strong Markov property, we can now bound from below the right-hand side of inequality (2.76) by

$$(1 - 2^{-\frac{M(m-1)}{2}}) \left(P_y[X_{T_{B_y}} \in \partial^+ B_y | \mathfrak{F}_{0,L}] - P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,L}] \right). \tag{2.77}$$

In turn, by means of the polynomial condition and the fact that the boxes B_y and B_z are inside the cone $C(0, l, \alpha)$ we see that (2.77) is greater than or equal to

$$(1 - 2^{-\frac{M(m-1)}{2}}) \left(1 - 2^{-M(m-1)} - P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,L}]\right). \tag{2.78}$$

Now, note that

$$P_y[(G_{\bar{F}_m})^c | \mathfrak{F}_{0,L}] \leqslant \sum_{x \in \bar{F}_m} 2^{\frac{M(m-1)}{2}} P_x[X_{T_{B_x}} \notin \partial^+ B_x | \mathfrak{F}_{0,L}] \\ \leqslant |\bar{F}_m| 2^{-\frac{M(m-1)}{2}} \leqslant (4c)^{d-1} 2^{m(d-1)} 2^{-\frac{M(m-1)}{2}}. \tag{2.79}$$

where in the first inequality we have used Chebyshev inequality, in the second one the assumption that $(PC)_{M,c}|_l$ is satisfied and in the third one the bound $|\bar{F}_{2m}| \leqslant (4c)^{d-1} 2^{m(d-1)}$.

Consequently inserting the estimates (2.79) into (2.78) and combining this with inequality (2.76) we conclude that

$$P_y[T^l_{2^{m+1}} \leqslant D'(0) \mid \mathfrak{F}_{0,L}] \geqslant \; (1 - 2^{-\frac{M(m-1)}{2}}) \times \\ (1 - 2^{-\frac{M(m-1)}{2}} - (4c)^{d-1} 2^{m(d-1)} 2^{-\frac{M(m-1)}{2}}) \\ \geqslant 1 - 3(4c)^{d-1} 2^{m(d-1)} 2^{-\frac{M(m-1)}{2}}. \tag{2.80}$$

Using the bound (2.80) in (2.75), together with the estimate $|F_m| \leqslant (2c)^{d-1}2^{m(d-1)}$, we see that

$$P_0[X_{T_{B_{2^m,c2^m,l}}(0)} \in \partial^+ B_{2^m,2^m,l}(0), T^l_{2^{m+1}} \circ \theta_{T^l_{2^m}} > D'(0) \circ \theta_{T^l_{2^m}} \mid \mathfrak{F}_{0,L}] \leqslant 3(4c)^{2(d-1)}2^{2m(d-1)}2^{-\frac{M(m-1)}{2}}. \tag{2.81}$$

Combining the estimates (2.81), (2.74), (2.73) with (2.72) we conclude that

$$\begin{aligned} &E_0[\mathfrak{M}'^2, D' < \infty|\mathfrak{F}_{0,L}] \\ &\leqslant 1 + 4(4c)^{2(d-1)} \textstyle\sum_{m\geqslant 0} 2^{2(m+1)}2^{2m(d-1)}2^{-\frac{M(m-1)}{2}} \\ &\leqslant 1 + 4(4c)^{2(d-1)} \textstyle\sum_{m\geqslant 0} 2^{-m} \leqslant c_6, \end{aligned}$$

where in the second to last inequality we have used the fact that $M > 4d + 1$ and c_6 is a constant that does not depend on L. This completes the proof of the Lemma. □

2.5.3 Proof of Proposition 2.5.3

To simplify the computations, we introduce the notation

$$b = b(L) := P_0(D' < \infty) + \phi(L),$$
$$b' = b'(L) := P_0(D' = \infty) + \phi(L)$$

and $E_{\mathbb{P}\otimes Q} := \mathbb{E}E_Q$. Furthermore, it will be necessary to define for each $j \geqslant 0$ and $n \geqslant L+j$ the events

$$D_{j,n} := \{\varepsilon \in W^{\mathbb{N}} : (\varepsilon_m, \ldots, \varepsilon_{m+L-1}) \neq \bar{\varepsilon}^{(L)} \text{ for all } j \leqslant m \leqslant j+n-L+1\}.$$

The following lemma, whose proof is presented in Appendix 2.7, will be useful in the proof of Proposition 2.5.3.

Lemma 2.5.6. *There exists a constant c_7 such that for all $n \geqslant L^2$ one has that*

$$Q[D_{0,n}] \leqslant (1 - c_7 L^2 \kappa^L)^{[\frac{n}{L^2}]}.$$

We now present the proof of Proposition 2.5.3, divided in several steps. For the sake of simplicity, we will write τ instead of $\tau^{(L)}$.

Step 0. We first note that

$$\bar{E}_0[(X_\tau \cdot u)^2 \mid \mathfrak{F}_{0,L}] = \\ \sum_{k=1}^{\infty}\sum_{k'=0}^{k-1} \bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,L}]. \tag{2.82}$$

Throughout the subsequent steps of the proof we will estimate the right-hand side of (2.82).

Step 1. Here we will prove the following estimate valid for all $k \geqslant 1$ and $0 \leqslant k' < k$.

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,L}] \\ &\leqslant b' b^{k-k'-1} \bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k'+1} < \infty \mid \mathfrak{F}_{0,L}].\end{aligned} \tag{2.83}$$

Furthermore, define the set

$$H^L := \left\{ y \in \mathbb{Z}^d : y \cdot u \geqslant L \frac{|u|_2}{|u|_1} \right\}.$$

Then, for each $0 \leqslant k' < k$, one has that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,L}] \\ &= \sum_{n \geqslant 1, x \in H^L} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k = n, \\ &\qquad X_{S_k} = x, D' \circ \theta_n = \infty \mid \mathfrak{F}_{0,L}] \\ &= \sum_{n \geqslant 1, x \in H^L} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k = n, X_n = x] \\ &\qquad P_{\vartheta_x \omega,\ \theta_n \varepsilon}[D' = \infty] \mid \mathfrak{F}_{0,L}] \\ &= \sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, X_{S_k} = x] \\ &\qquad P_{x,\omega}[D' = \infty] \mid \mathfrak{F}_{0,L}],\end{aligned} \tag{2.84}$$

where here for each $x \in \mathbb{Z}^d$, ϑ_x denotes the canonical space shift in Ω so that $\vartheta_x \omega(y) = \omega(x+y)$, while for each $n \geqslant 0$, θ_n denotes the canonical time shift in the space W so that $(\theta_n \epsilon)_m = \epsilon_{n+m}$, in the first equality we have used the fact that the value of $X_{S_k} \cdot u \geqslant X_{S_1} \cdot u$, in the second equality the Markov property and in the last equality we have used the independence of the coordinates of ϵ and the fact that the law of the random walk is the same under $P_{x,\omega}$ and under $E_Q P_{\vartheta_x \omega, \theta_n \epsilon}$.

Moreover, by the fact that the first factor inside the expectation of the right-most expression of (2.84) is $\mathfrak{F}_{x,L}$-measurable, the right-most expression in (2.84) is equal to

$$\sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, X_{S_k} = x] \\ \mathbb{E}[P_{x,\omega}[D' = \infty] \mid \mathfrak{F}_{x,L}] \mid \mathfrak{F}_{0,L}]. \tag{2.85}$$

Applying next Lemma 2.5.4 to (2.85), we see that

$$\begin{aligned} \textstyle\sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, X_{S_k} = x] \\ \times \mathbb{E}[P_{x,\omega}[D' = \infty] \mid \mathfrak{F}_{x,L}] \mid \mathfrak{F}_{0,L}] \\ \leqslant b' \bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty \mid \mathfrak{F}_{0,L}]. \end{aligned} \tag{2.86}$$

Next, observe that for $k' < k$ one has that

$$\begin{aligned} &\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty \mid \mathfrak{F}_{0,L}] \\ &= \bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, R_{k-1} < \infty \mid \mathfrak{F}_{0,L}] \\ &= \textstyle\sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k-1} < \infty, X_{S_{k-1}} = x, \\ &\qquad D' \circ \theta_{S_{k-1}} < \infty] \mid \mathfrak{F}_{0,L}] \\ &= \textstyle\sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k-1} < \infty, X_{S_{k-1}} = x] \\ &\qquad P_{x,\omega}[D' < \infty] \mid \mathfrak{F}_{0,L}] \\ &= \textstyle\sum_{x \in H^L} \mathbb{E}[\bar{E}_{0,\omega}[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k-1} < \infty, X_{S_{k-1}} = x] \\ &\qquad \mathbb{E}[P_{x,\omega}[D' < \infty] \mid \mathfrak{F}_{x,L}] \mid \mathfrak{F}_{0,L}]. \end{aligned} \tag{2.87}$$

By Lemma 2.5.4, we have that $\mathbb{E}[P_{x,\omega}[D' < \infty] \mid \mathfrak{F}_{x,L}] \leqslant b = P_0[D' < \infty] + \phi(L)$. Using this inequality to estimate the last term in (2.87), we see that

$$\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant b\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k-1} < \infty \mid \mathfrak{F}_{0,L}].$$

By induction on k we get that

$$\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant b^{k-k'-1}\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_{k'+1} < \infty \mid \mathfrak{F}_{0,L}]. \tag{2.88}$$

Combining (2.88) with (2.86) we obtain (2.83).

Step 2. For $k \geqslant 1$ we define

$$M_k := \sup_{0 \leqslant n \leqslant R_k} X_n \cdot u. \tag{2.89}$$

Define also the sets parametrized by k and $n \geqslant 0$

$$A_{n,k} := \left\{\varepsilon \in W^{\mathbb{N}} : \left(\varepsilon_{t_k^{(n)}}, \varepsilon_{t_k^{(n)}+1}, \ldots, \varepsilon_{t_k^{(n)}+L-1}\right) = \overline{\varepsilon}^{(L)}\right\} \tag{2.90}$$

and

$$B_{n,k} := \left\{\varepsilon \in W^{\mathbb{N}} : \left(\varepsilon_{t_k^{(j)}}, \varepsilon_{t_k^{(j)}+1}, \ldots, \varepsilon_{t_k^{(j)}+L-1}\right) \neq \overline{\varepsilon}^{(L)} \text{ for all } 0 \leqslant j \leqslant n-1\right\}, \tag{2.91}$$

where we define the sequence of stopping times [c.f. (2.70)] parameterized by k and recursively on $n \geqslant 0$ by

$$t_k^{(0)} := \bar{T}^l_{M_k}$$

and the successive times where a record value of the projection of the random walk on l is achieved by

$$t_k^{(n+1)} := \bar{T}^l_{X_{t_k^{(n)}} \cdot u}.$$

In this step we will show that for all $k \geqslant 0$ one has that

$$\begin{aligned}&\bar{E}_0[(X_{S_{k+1}}\cdot u)^2-(X_{S_k}\cdot u)^2, S_{k+1}<\infty|\mathfrak{F}_{0,L}]\\ &\leqslant \textstyle\sum_{n=0}^{L^2-1}\bar{E}_0[(X_{S_{k+1}}\cdot u)^2-(X_{S_k}\cdot u)^2, t_k^{(n)}<\infty, A_{n,k}\mid \mathfrak{F}_{0,L}]\\ &+\textstyle\sum_{n=L^2}^{\infty}\bar{E}_0[(X_{S_{k+1}}\cdot u)^2-(X_{S_k}\cdot u)^2, t_k^{(n)}<\infty, B_{n,k}, A_{n,k}\mid \mathfrak{F}_{0,L}],\end{aligned} \tag{2.92}$$

To prove (2.92), we have to introduce some further notations. Now, note that on the event $A_{n,k}\cap B_{n,k}$ one has that

$$S_{k+1}=t_k^{(n)}+L.$$

Thus, as a consequence of the definition of S_{k+1}, one has that $\bar{P}_0$-a.s.

$$\{S_{k+1}<\infty\}\subset\bigcup_{n\geqslant 0}\{t_k^{(n)}<\infty, B_{n,k}, A_{n,k}\}. \tag{2.93}$$

Display (2.92) now follows directly from (2.93).

Step 3. Here we will derive an upper bound for the two sums appearing in the right-hand side in (2.92). In fact, we will prove that there is a constant c_8 such that for all $k\geqslant 1$ one has that

$$\begin{aligned}&\textstyle\sum_{n=0}^{L^2-1}\bar{E}_0[(X_{S_{k+1}}\cdot u)^2-(X_{S_k}\cdot u)^2, t_k^{(n)}<\infty, A_{n,k}\mid \mathfrak{F}_{0,L}]\\ &\leqslant c_8\kappa^L\left(L^4 b^{k-1}+L^2\bar{E}_0[X_{S_k}\cdot u, S_k<\infty|\mathfrak{F}_{0,L}]\right)\end{aligned} \tag{2.94}$$

and

$$\begin{aligned}&\textstyle\sum_{n=L^2}^{\infty}\bar{E}_0[(X_{S_{k+1}}\cdot u)^2-(X_{S_k}\cdot u)^2, t_k^{(n)}<\infty, B_{n,k}, A_{n.k}\mid \mathfrak{F}_{0,L}]\\ &\leqslant c_8\textstyle\sum_{n=L^2}^{\infty}\kappa^L(1-c_7\kappa^L)^{\left[\frac{n}{L^2}\right]}\left((n+L)^2 b^{k-1}\right.\\ &\left.+(n+L)\bar{E}_0[X_{S_k}\cdot u, S_k<\infty|\mathfrak{F}_{0,L}]\right).\end{aligned} \tag{2.95}$$

Note that for all $n\geqslant 0$ one has that

$$X_{t_k^{(n+1)}}\cdot u\leqslant X_{t_k^{(n)}}\cdot u+|u|_\infty,$$

and hence by induction on n we get that

$$X_{t_k^{(n)}} \cdot u \leqslant M_k + (n+1)|u|_\infty.$$

Therefore, if we set

$$L' := \frac{L|u|}{|u|_1} + |u|_\infty \leqslant c_9 L, \tag{2.96}$$

where c_9 is a constant depending on l and d, we can see that P_0-a.s on the event $\{t_k^{(n)} < \infty,\ A_{n,k}\}$ one has that

$$X_{S_{k+1}} \cdot u \leqslant N_{k,n} := M_k + n|u|_\infty + L'. \tag{2.97}$$

Therefore, for all $0 \leqslant n \leqslant L^2 - 1$ one has that

$$\begin{aligned}
&\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,L}] \\
&\leqslant \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,L}] \\
&= \textstyle\sum_{j=0}^{\infty} \sum_{x \in \mathbb{Z}^d} E_{\mathbb{P}\otimes Q}[E_{\omega,\varepsilon}[N_{k,n}^2 - (X_{S_k} \cdot u)^2, \\
&\quad t_k^{(n)} = j, X_j = x] \mathbb{1}_{\{(\varepsilon_j, \ldots, \varepsilon_{j+L-1}) = \bar{\varepsilon}^{(L)}\}} \mid \mathfrak{F}_{0,L}] \\
&\leqslant \kappa^L \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,L}],
\end{aligned} \tag{2.98}$$

where in the equality we have applied the Markov property and in the second inequality the fact that Q is a product measure and that $R_k \leqslant t_k^{(n)}$. Similarly for all $n \geqslant L^2$ one has that

$$\begin{aligned}
&\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,L}] \\
&\leqslant \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,L}] \\
&\leqslant \textstyle\sum_{j=0}^{\infty} \sum_{j'=j+n}^{\infty} \sum_{y \in \mathbb{Z}^d} E_{\mathbb{P}\otimes Q}\ [E_{\omega,\varepsilon}[N_{k,n}^2 - (X_{S_k} \cdot u)^2, \\
&X_{t_k^{(0)}} = y, t_k^{(0)} = j] P_{\theta_y \omega, \theta_j \varepsilon}[D_{j,n}, t_k^{(n)} = j'] \mathbb{1}_{\{(\varepsilon_{j'}, \ldots, \varepsilon_{j'+L-1}) = \bar{\varepsilon}^{(L)}\}}] \mid \mathfrak{F}_{0,L}] \\
&\leqslant \kappa^L Q[D_{0,n}] \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,L}] \\
&\leqslant \kappa^L (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L}\right]} \bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,L}],
\end{aligned} \tag{2.99}$$

where in the second inequality we have used the Markov property, in the third one the fact that $R_k \leqslant t_k^{(0)}$ and in the last one Lemma 2.5.6.

Now, by displays (2.98) and (2.99), to finish the proof of inequalities (2.94) and (2.95) it is enough to prove that there is a constant c_{10} such that

$$\begin{aligned} &\bar{E}_0[N_{k,n}^2 - (X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,L}] \\ &\leqslant c_{10} \left((n+L)^2 b^{k-1} + (n+L)\bar{E}_0[X_{S_k} \cdot u, S_k < \infty | \mathfrak{F}_{0,L}] \right), \end{aligned} \tag{2.100}$$

using the fact that $n \leqslant L^2 - 1$ in the left-hand side of inequality (2.94). To prove (2.100), the following identity will be useful

$$\begin{aligned} N_{k,n}^2 - (X_{S_k} \cdot u)^2 &= (M_k - X_{S_k} \cdot u)^2 \\ +2(n|u|_\infty + L')(M_k - X_{S_k} \cdot u) &+ 2(n|u|_\infty + L')X_{S_k} \cdot u \\ +2(M_k - X_{S_k} \cdot u)X_{S_k} \cdot u &+ (n|u|_\infty + L')^2. \end{aligned} \tag{2.101}$$

We will now insert this decomposition in the left-hand side of (2.100) and bound the corresponding expectations of each term. Let us begin with the expectation of the last term. Note that by an argument similar to the one developed in *Step 1* we have that

$$\bar{E}_0[(n|u|_\infty + L')^2, R_k < \infty | \mathfrak{F}_{0,L}] \leqslant c_{11}(n+L)^2 b^k, \tag{2.102}$$

for some constant c_{11}. Similarly, the expectation of the first term of the right-hand side of display (2.101) can be bounded using Lemma 2.5.5, so that

$$\begin{aligned} &\bar{E}_0[(M_k - X_{S_k} \cdot u)^2, R_k < \infty \mid \mathfrak{F}_{0,L}] \\ &= \sum_{x \in H^L} \mathbb{E}[\bar{P}_{0,\omega}[S_k < \infty, X_{S_k} = x] E_x[\mathfrak{M}^2, D' < \infty \mid \mathfrak{F}_{x,L}] \mid \mathfrak{F}_{0,L}] \\ &\leqslant c_6 b^{k-1}. \end{aligned} \tag{2.103}$$

Again, for the expectation of the second term of the right-hand side of display (2.101), we have that

$$\begin{aligned} &\bar{E}_0[2(n|u|_\infty + L')(M_k - X_{S_k} \cdot u), R_k < \infty \mid \mathfrak{F}_{0,L}] \\ &\leqslant c_{12} b^{k-1}(n+L), \end{aligned} \tag{2.104}$$

for some suitable positive constant c_{12}. For the expectation of fourth term of the right-hand side of (2.101), we see by Lemma 2.5.5 that

$$\bar{E}_0[2(M_k - X_{S_k} \cdot u)X_{S_k} \cdot u, R_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant 2\sqrt{c_6}\bar{E}_0[X_{S_k} \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}]. \quad (2.105)$$

Finally, for the expectation of third term of the right-hand side of (2.101) we have that

$$\bar{E}_0[2(n|u|_\infty + L')X_{S_k} \cdot u, R_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant c_{12}b(n + L)\bar{E}_0[X_{S_k} \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}]. \quad (2.106)$$

Using the bounds (2.106), (2.105), (2.104), (2.103) and (2.102) we obtain inequality (2.100).

Step 4. Here we will derive for all $k \geqslant 1$ the inequality

$$\bar{E}_0[X_{S_k} \cdot u, S_k < \infty | \mathfrak{F}_{0,L}]$$
$$\leqslant \sum_{k'=0}^{k-1} b^{k-k'-1} \Big(\sum_{n=0}^{L^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,L}] +$$
$$\sum_{n=L^2}^{\infty} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,L}] \Big). \quad (2.107)$$

Note that

$$\bar{E}_0[X_{S_k} \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$= \sum_{k'=0}^{k-1} \bar{E}_0[(X_{S_{k'+1}} - X_{S_{k'}}) \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}]. \quad (2.108)$$

By an argument similar to the one used in *Step 1* we see that for $k' < k$ one has that

$$\bar{E}_0[(X_{S_{k'+1}} - X_{S'_k}) \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant b^{k-k'-1}\bar{E}_0[(X_{S_{k'+1}} - X_{S'_k}) \cdot u, S_{k'+1} < \infty \mid \mathfrak{F}_{0,L}]. \quad (2.109)$$

Now, we can use inclusion (2.93) in order to get that

$$\bar{E}_0[(X_{S_{k'+1}} - X_{S'_k}) \cdot u, S_{k'+1} < \infty \mid \mathfrak{F}_{0,L}]$$
$$\leqslant \sum_{n=0}^{L^2-1} \bar{E}_0[(X_{S_{k'+1}} - X_{S'_k}) \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,L}]$$
$$+ \sum_{n=L^2}^{\infty} \bar{E}_0[(X_{S_{k'+1}} - X_{S'_k}) \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,L}], \quad (2.110)$$

where the events $A_{n,k'}$ and $B_{n,k'}$ are defined in (2.90) and (2.91). Using the fact that on the event $\{t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'}\}$ one has that P_0-a.s.

$$(X_{S_{k'+1}} - X_{S_{k'}}) \cdot u \leqslant N_{k',n} - X_{S'_k} \cdot u,$$

we see that the right-hand side of (2.110) is bounded by the right-hand side of (2.107), which is what we want to prove.

Step 5. Here we will obtain an upper bound for the terms in the first summation in (2.110). Indeed, note that on $R_{k'} \leqslant t_{k'}^{(n)}$, by an argument similar to the one used to derive inequality (2.98), we have that for all $0 \leqslant n \leqslant L^2$ and $0 \leqslant k' \leqslant k-1$

$$\begin{aligned} &\bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,L}] \\ &\leqslant \kappa^L \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,L}]. \end{aligned} \tag{2.111}$$

Step 6. Here we will obtain an upper bound for the terms in the second summation in (2.110), showing that for all $n \geqslant L^2$ and $0 \leqslant k' \leqslant k-1$,

$$\begin{aligned} &E_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,L}] \\ &\leqslant \kappa^L \left(1 - c_7 L^2 \kappa^L\right)^{\left[\frac{n}{L}\right]} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,L}]. \end{aligned} \tag{2.112}$$

Now note that

$$\begin{aligned} &E_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, B_{n,k'}, A_{n,k'} \mid \mathfrak{F}_{0,L}] \\ &\leqslant \textstyle\sum_{j=0}^{\infty} \sum_{j' \geqslant j+n} \sum_{y \in \mathbb{Z}^d} E_{\mathbb{P} \otimes Q} \, [E_{\omega,\varepsilon}[N_{k',n} - X_{S_{k'}} \cdot u, \\ &X_{t_{k'}^{(0)}} = y, t_{k'}^{(0)} = j] P_{\theta_y \omega, \theta_j \varepsilon}[D_{j,n}, t_{k'}^{(n)} = j'] \mathbb{1}_{\{(\varepsilon_{j'}, \ldots, \varepsilon_{j'+L-1}) = \bar{\varepsilon}^{(L)}\}}] \mid \mathfrak{F}_{0,L}] \\ &= \kappa^L Q[D_{0,n}] \mathbb{E}[\bar{E}_{0,\omega}[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(0)} < \infty] \mid \mathfrak{F}_{0,L}]. \end{aligned} \tag{2.113}$$

Using Lemma 2.5.6 to estimate $Q[D_{0,n}]$ we conclude the proof of inequality (2.112).

Step 7. Here we will show that there exist constant c_{13} and c_{14} such that

$$\sum_{n=0}^{L^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,L}] \leqslant c_{13} \kappa^L L^4 b^{k'-1} \tag{2.114}$$

and

$$\sum_{n=L^2}^{\infty} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'}, B_{n,k'} \mid \mathfrak{F}_{0,L}] \leqslant 4c_{14}\kappa^{-L} b^{k'-1}. \tag{2.115}$$

Let us first note that by an argument similar to the one used to derive the bound in Step 1 (through Lemmas 2.5.4 and 2.5.5), we have that

$$\bar{E}_0[N_{k',n} - X_{S'_k} \cdot u, R_{k'} < \infty] \leqslant (n|u|_\infty + L' + c_{15}) b^{k'-1}, \tag{2.116}$$

where $c_{15} := \sqrt{c_6}$. Let us now prove (2.114). Indeed, note that by Step 5 and (2.116) we then have that

$$\begin{aligned} &\textstyle\sum_{n=0}^{L^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'} \mid \mathfrak{F}_{0,L}] \\ &\leqslant \kappa^L \textstyle\sum_{n=0}^{L^2-1} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,L}] \\ &\leqslant c_{13}\ L^4\ \kappa^L b^{k'-1}, \end{aligned} \tag{2.117}$$

for some suitable constant c_{13}. Let us now prove (2.115). First note that

$$\begin{aligned} &\textstyle\sum_{n=L^2}^{\infty} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, t_{k'}^{(n)} < \infty, A_{n,k'}, B_{n,k'} \mid \mathfrak{F}_{0,L}] \\ &\leqslant \textstyle\sum_{n=L^2}^{\infty} \kappa^L (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L}\right]} \bar{E}_0[N_{k',n} - X_{S_{k'}} \cdot u, R_{k'} < \infty \mid \mathfrak{F}_{0,L}] \\ &\leqslant b^{k'-1} \textstyle\sum_{n=L^2}^{\infty} \kappa^L (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L}\right]} (n|u|_\infty + L' + c_{15}) \\ &\leqslant c_{16} b^{k'-1} \textstyle\sum_{n=L^2}^{\infty} n\kappa^L (1 - c_{33} L^2 \kappa^L)^{\left[\frac{n}{L^2}\right]}. \end{aligned} \tag{2.118}$$

for some constant c_{16}, where in the first inequality we have used *Step 6* and in the second we have used inequality (2.116). Finally notice that using the fact that for $n \geqslant L^2$ one has that $n \leqslant 2L^2 \left[\frac{n}{L^2}\right]$, we get that

$$\begin{aligned} &\textstyle\sum_{n=L^2}^{\infty} n\kappa^L (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L^2}\right]} \leqslant 2\kappa^L L^2 \textstyle\sum_{n=L^2}^{\infty} \left[\frac{n}{L^2}\right] (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L^2}\right]} \\ &= 2L^4 \kappa^L \textstyle\sum_{m=1}^{\infty} m(1 - c_7 L^2 \kappa^L)^m \leqslant \frac{2}{(c_7)^2} \kappa^{-L}. \end{aligned}$$

Using this estimate in (2.118) we obtain (2.115).

Step 8. Here we finish the proof of Proposition 2.5.3 combining the previous steps we have already developed. Combining inequality (2.107) proved in *Step 4* with inequalities (2.114) and (2.115) proved in *Step 7*, we see that there is a constant c_{17} such that

$$\bar{E}_0[X_{S_k} \cdot u, S_k < \infty \mid \mathfrak{F}_{0,L}] \leqslant c_{17} k b^{k-2} \kappa^{-L}. \tag{2.119}$$

Thus, by inequality (2.94) proved in *Step 3*, we have that

$$\sum_{n=0}^{L^2-1} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, A_{n,k} \mid \mathfrak{F}_{0,L}] \leqslant c_{18} L^4 k b^{k-2}. \tag{2.120}$$

for certain positive constant c_{18} On the other hand, combining inequality (2.95) proved in Step 3 with (2.119), we see that there exists a constant c_{19} such that

$$\begin{aligned}\textstyle\sum_{n=L^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,L}]\\ \leqslant c_{19} \textstyle\sum_{n=L^2}^{\infty} \kappa^L (1 - c_7 L^2 \kappa^L)^{\left[\frac{n}{L^2}\right]} \big((n+L)^2 b^{k-1}\\ +(n+L) k b^{k-2} \kappa^{-L}\big)\,.\end{aligned} \tag{2.121}$$

Now, note that for some constant c_{20} one has that

$$\textstyle\sum_{n=L^2}^{\infty} (n+L)^2 (1 - c_7 L^2 \kappa^L)^{[\frac{n}{L^2}]} \leqslant c_{20}\ \kappa^{-3L} \quad \text{and} \tag{2.122}$$

$$\textstyle\sum_{n=L^2}^{\infty} (n+L) (1 - c_7 L^2 \kappa^L)^{[\frac{n}{L^2}]} \leqslant c_{20}\ \kappa^{-2L}. \tag{2.123}$$

Substituting (2.122) and (2.123) into (2.121) we see that

$$\begin{aligned}\textstyle\sum_{n=L^2}^{\infty} \bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, t_k^{(n)} < \infty, B_{n,k}, A_{n,k} \mid \mathfrak{F}_{0,L}] \leqslant\\ c_{21} \kappa^{-2L} b^{k-2} k,\end{aligned} \tag{2.124}$$

for some suitable positive constant c_{21}. Substituting (2.121) and (2.124) into inequality (2.92) of Step 2, we then conclude that there is a constant c_{22} such that

$$\bar{E}_0[(X_{S_{k+1}} \cdot u)^2 - (X_{S_k} \cdot u)^2, S_{k+1} < \infty | \mathfrak{F}_{0,L}] \leqslant c_{22} \kappa^{-2L} b^{k-2} k. \tag{2.125}$$

Substituting (2.125) into (2.83) of Step 1, we get that

$$\begin{aligned}\bar{E}_0[(X_{S_{k'+1}} \cdot u)^2 - (X_{S_{k'}} \cdot u)^2, S_k < \infty, D' \circ \theta_{S_k} = \infty \mid \mathfrak{F}_{0,L}]\\ \leqslant b' b^{k+1} k'.\end{aligned} \tag{2.126}$$

From the fact that $\sum_{k=1}^{\infty}\sum_{k'=0}^{k-1} b^{k+1}k' < \infty$ together with (2.126 and (2.82) of *Step 0*, we conclude that

$$\bar{E}_0[(X_\tau \cdot u)^2 | \mathfrak{F}_{0,L}] \leqslant c_{23}\kappa^{-2L},$$

for some constant $c_{23} > 0$, which proves the proposition.

2.6 Proof of Theorem 2.1.1

In this section we will prove Theorem 2.1.1 using Proposition 2.5.3 proved in Section 2.5. First in Subsection 2.6.1, we will define an approximate sequence of regeneration times. In Subsection 2.6.2, we will show through this approximate regeneration time sequence, that there exists an approximate asymptotic direction. In Subsection 2.6.3, we will use the approximate asymptotic direction to prove Theorem 2.1.1.

2.6.1 Approximate regeneration time sequence

As in [CZ01], we define approximate regeneration by the recursively by $\tau_1^{(L)} := \tau$ [c.f.(2.62)] and for $i \geqslant 2$

$$\tau_i^{(L)} := \tau_1^{(L)} \circ \theta_{\tau_{i-1}^{(L)}} + \tau_{i-1}^{(L)}.$$

We will drop the dependence in L on $\tau_1^{(L)}$ when it is convenient for us, using the notation τ_i instead $\tau_i^{(L)}$. Let us define σ-algebras corresponding to the information of the random walk and the ε process up to the first regeneration time and of the environment ω at a distance of order L to the left of the position of the random walk at this regeneration time as

$$\begin{aligned}\mathcal{H}_1 := \sigma(&\tau_1^{(L)}, X_0, \varepsilon_0, \ldots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}},\\ &\{\omega(y,\cdot) :\ y \cdot u < u \cdot X_{\tau_1^{(L)}} - L|u|/|u|_1\}).\end{aligned}$$

Similarly define for $k \geqslant 2$

$$\mathcal{H}_k := \sigma(\tau_1^{(L)}, \ldots, \tau_k^{(L)}, X_0, \varepsilon_0, \ldots, \varepsilon_{\tau_k^{(L)}-1}, X_{\tau_k^{(L)}},$$
$$\{\omega(y,\cdot) : \ y \cdot u < u \cdot X_{\tau_k^{(L)}} - L|u|/|u|_1\}). \qquad (2.127)$$

Let us now recall Lemma 2.3 of [CZ01], stated here under the condition $P_0[D' = \infty] > 0$ [c.f. (2.36)] instead of Kalikow's condition.

Lemma 2.6.1. *Let $l \in \mathbb{S}^{*d-1}$, $\alpha > 0$ and ϕ be such that $\lim_{r\to\infty} \phi(r) = 0$. Consider a random walk in a random environment satisfying the cone-mixing assumption with respect to α, l and ϕ and uniformly elliptic with respect to l. Assume that L is such that*

$$\phi(L) < P_0[D' = \infty].$$

Then, $\mathbb{P}-$a.s. one has that

$$\left|\bar{P}_0[\{X_{\tau_k+\cdot} - X_{\tau_k}\} \in A \mid \mathcal{H}_k] - \bar{P}_0[\{X_\cdot\} \in A | D' = \infty]\right| \leqslant \phi'(L),$$

for all measurable sets $A \subset (\mathbb{Z}^d)^{\mathbb{N}}$, where

$$\phi'(L) := \frac{2\phi(L)}{(P_0[D' = \infty] - \phi(L))}.$$

Proof. For $k = 1$, the argument given in page 890 of ([CZ01]) still works without any change. With the purpose of showing that the result continues being true under the *weaker* assumptions here, we complete the induction argument in the case $k = 2$. To this end, we consider a positive $\mathcal{H}_2-$ measurable function h of the form $h = h_1 \cdot (h_2) \circ \theta_{\tau_1}$ ($\cdot$ denotes usual function multiplication), such that h_1, is $\mathcal{H}_1-$ measurable and h_2 is $\mathcal{H}'_1$ measurable, where the $\sigma-$ algebra $\mathcal{H}'_1$ is defined as :

$$\mathcal{H}'_1 : \ = \ \sigma(\tau_1^{(L)}, X_0, \varepsilon_0, \ldots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}},$$
$$\{\omega(y,\cdot) : \ u \cdot y \leqslant u \cdot X_{\tau_1^{(L)}} - L\frac{|u|}{|u|_1}, \ y \in C(X_0, l, \alpha)\}).$$

We let A be a measurable set of the path space, for short we will write $\mathbb{1}_A := \mathbb{1}_{\{(X_n - X_0)_{n\geqslant 0} \in A\}}$. By the strong Markov property and using that $\tau_1 < \infty$ within an event of full P_0 probability, we get:

$$\bar{E}_0[h \mathbb{1}_A \circ \theta_{\tau_2}] \leqslant \sum_{n\geqslant 1} E_0[h \mathbb{1}_A \circ \theta_{\tau_2}]$$
$$\bar{E}_0[h \mathbb{1}_A \circ \theta_{\tau_2}\ \mathbb{1}_{K=n} \circ \theta_{\tau_1}, \tau_1 < \infty]$$
$$\sum_{t\geqslant 1} \bar{E}_0[h \mathbb{1}_A \circ \theta_{\tau_2}\ , \tau_1 = t]$$
$$\sum_{t\geqslant 1} \bar{E}_0[h_1 \cdot (h_2 \circ \theta_{\tau_1}) \mathbb{1}_A \circ \theta_{\tau_2}\ , S_t < \infty, D' \circ \theta_{S_t} = \infty]. \tag{2.128}$$

Now, notice that for given $t \in \mathbb{N}, m \in \mathbb{N}, x \in \mathbb{Z}^d$, we can find a random variable $h_{1,t,m,x}$ measurable with respect to $\sigma(\{\omega(y,\cdot) : y \cdot u < x \cdot u - L\frac{|u|}{|u|_1}\}, \{X_i\}_{i<m})$ such that it coincides with h_1 on the event $\{\tau_1 = S_t = m, X_{S_t} = x\}$, therefore (2.128) equals

$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \bar{E}_0[h_{1,t,m,x}(h_2 \circ \theta_{\tau_1}) \mathbb{1}_A \circ \theta_{\tau_2}\ \mathbb{1}_{S_t = m, D'\circ\theta_m = \infty, X_m = x}]$$
$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \bar{E}_0[h_1 \mathbb{1}_{S_t = m, X_m = x D'\circ\theta_m = \infty} \mathbb{1}_A \circ \theta_{\tau_2} h_2 \circ \theta_{\tau_1}]$$
$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} E_{\mathbb{P}\otimes Q}[E_{0,\omega,\varepsilon}[h_{1,t,m,x} \mathbb{1}_{S_t = m, X_m = x} \mathbb{1}_{D'\circ\theta_m = \infty} \mathbb{1}_A \circ \theta_{\tau_2} h_2 \circ \theta_{\tau_1}]]$$
$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} E_{\mathbb{P}\otimes Q}[E_{0,\omega,\varepsilon}[h_{1,t,m,x} \mathbb{1}_{S_t = m, X_m = x}] E_{\theta_x\omega, \theta_m\varepsilon}[\mathbb{1}_{D' = \infty} \mathbb{1}_A \circ \theta_{\tau_1} h_2]]. \tag{2.129}$$

We now work out the following expression

$$E_{\theta_x\omega, \theta_m\varepsilon}[\mathbb{1}_{D' = \infty} \mathbb{1}_A \circ \theta_{\tau_1} h_2]$$
$$\sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} E_{\theta_x\omega, \theta_m\varepsilon}[\mathbb{1}_{D' = \infty} \mathbb{1}_A \circ \theta_{\tau_1} h_2, S_n = j, X_{S_n} = z, D' \circ \theta_j = \infty]. \tag{2.130}$$

Observe that, as in the case of h_1, for fixed x and m, we consider the probability measure $P_{\theta_x\omega,\theta_m\varepsilon}$. Then we can find a measurable function $h_{2,j,n,z}$ with respect to $\sigma(\{\omega(y,\cdot) : y \cdot u \leqslant z \cdot u - L\frac{|u|}{|u|_1},\ y \in C(x,l,\alpha)\}, \{X_i\}_{i<j})$, which coincides with h_2 on the event $\{\tau_1 = S_n = j, X_{S_n} = z, D' = \infty\}$, furthermore note that $D' = \infty$ depends up to $(j-1)$ coordinate in ε (recall that $\{D' = \infty\} \in \mathcal{H}_1$), hence we can apply the Markov property to get that the last expression in (2.130) is equal to:

$$\sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant 1}} E_{\theta_x\omega, \theta_m\varepsilon}[h_{2,j,n,z}, \mathbb{1}_{S_n = j, X_{S_n} = z, D' = \infty}] P_{\theta_z\omega, \theta_j\varepsilon}[A \cap \{D' = \infty\}]. \tag{2.131}$$

Using (2.131), it follows that (2.128) is equal to:

$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} E_{\mathbb{P}\otimes Q}[E_{0,\omega,\varepsilon}[h_{1,t,m,x}\mathbb{1}_{S_t=m,X_m=x}]\cdot$$
$$E_{\theta_x\omega,\theta_m\varepsilon}[h_{2,j,n,z},\mathbb{1}_{S_n=j,X_{S_n}=z,D'=\infty}]P_{\theta_z\omega,\theta_j\varepsilon}[A\cap\{D'=\infty\}]]$$

Following [CZ01], we can write down the expression above as

$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} E_{\mathbb{P}\otimes Q}[[E_{0,\omega,\varepsilon}[h_{1,t,m,x}\mathbb{1}_{S_t=m,X_m=x}]]\cdot$$
$$E_{\theta_x\omega,\theta_m\varepsilon}[h_{2,j,n,z}\mathbb{1}_{S_n=j,X_{S_n}=z,D'=\infty}]]\bar{P}_0[A\cap\{D'=\infty\}]+\rho(A),$$

where

$$\rho(A):=\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} Cov_{\mathbb{P}\otimes Q}[f_{t,m,x,j,n,z},g_{j,z}],$$

with:

$$f_{t,m,x,j,n,z}:=E_{0,\omega,\varepsilon}[h_{1,t,m,x}\mathbb{1}_{S_t=m,X_m=x}]E_{\theta_x\omega,\theta_m\varepsilon}[h_{2,j,n,z},\mathbb{1}_{S_n=j,X_{S_n}=z,D'=\infty}]$$

and

$$g_{j,z}:=P_{\theta_z\omega,\theta_j\varepsilon}[A\cap\{D'=\infty\}].$$

On the other hand, since assumption $(CM)_{\phi,\alpha}|l$, the estimate

$$\rho(A)\leqslant\phi(L)\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} E_{\mathbb{P}\otimes Q}[E_{0,\omega,\varepsilon}[h_{1,t,m,x}\mathbb{1}_{S_t=m,X_m=x}]\cdot$$
$$E_{\theta_x\omega,\theta_m\varepsilon}[h_{2,j,n,z}\mathbb{1}_{S_n=j,X_{S_n}=z,D'=\infty}]]$$

holds for all measurable set A in the path space, in particular applying this for $A=\mathbb{Z}^d$ turns out the estimate:

$$\sum_{t\geqslant 1, m\geqslant 1, x\in\mathbb{Z}^d} \sum_{\substack{z\in C(x,l,\alpha)\\ n\geqslant 1, j\geqslant m+1}} E_{\mathbb{P}\otimes Q}[E_{0,\omega,\varepsilon}[h_{1,t,m,x}\mathbb{1}_{S_t=m,X_m=x}]\cdot$$
$$E_{\theta_x\omega,\theta_m\varepsilon}[h_{2,j,n,z}\mathbb{1}_{S_n=j,X_{S_n}=z,D'=\infty}]]\leqslant$$
$$(P_0[D'=\infty]-\phi(L))^{-1}\bar{E}_0[h].$$

From now on, we can follow the same sort of argument as in ([CZ01]), in order to conclude that

$$\|\,\bar{P}_0[\{X_{\tau_2+n}-X_{\tau_2}\}\in\cdot\mid\mathcal{H}_2]-\bar{P}_0[\{X_n\}\in\cdot\mid D'=\infty]\,\|_{var}\leqslant\phi'(l).$$

Therefore the second step induction is complete. □

2.6.2 Approximate asymptotic direction

We will show that a random satisfying the cone mixing, uniform ellipticity assumption and the non-effective polynomial condition with high enough degree has an approximate asymptotic direction. The exact statement is given below. It will also be shown that the right order in which the random variable X_{τ_1} grows as a function of L is κ^{-L}.

Proposition 2.6.2. *Let $l \in \mathbb{S}^{*d-1}$, ϕ be such that $\lim_{r\to\infty} \phi(r) =$, $c > 0$, $M > 6d$ and $0 < \alpha < \min\{\frac{1}{9}, \frac{1}{2c+1}\}$. Consider a random walk in a random environment satisfying the cone mixing condition with respect to α, l and ϕ and the uniform ellipticity condition with respect to l. Assume that $(PC)_{M,c}|l$ is satisfied. Then, there exists a sequence η_L such that $\lim_{L\to\infty} \eta_L = 0$ and $\bar{P}_0$-a.s.*

$$\limsup_{n\to\infty} \left| \frac{\kappa^L X_{\tau_n}}{n} - \lambda_L \right| < \eta_L, \tag{2.132}$$

where for all $L \geqslant 1$,

$$\lambda_L := \bar{E}_0[\kappa^L X_{\tau_1} \mid D' = \infty]. \tag{2.133}$$

Furthermore,

$$|\lambda_L|_2 \geqslant c_{270}\kappa^{-L}, \tag{2.134}$$

for some constant c_{270}.

We first prove inequality (2.132) of Proposition 2.6.2. We will follow the argument presented for the proof of Lemma 3.3 of [CZ01]. For each integer $i \geqslant 1$ define the sequence

$$\overline{X}_i := \kappa^L (X_{\tau_i} - X_{\tau_{i-1}}),$$

with the convention $\tau_0 = 0$. Using Lemma 2.6.1 and Lemma 3.2 of [CZ01], we can enlarge the probability space where the sequence $\{X_i : i \geqslant 1\}$ so that there we have the following properties:

(1) There exist an i.i.d. sequence $\{(\widetilde{X}_i, \Delta_i) : i \geqslant 2\}$ of random vectors with values in $(\kappa^L \mathbb{Z}^d, \{0,1\})$, such that $\widetilde{X}_2$ has the same distribution as $\overline{X}_1$ under the measure $\bar{P}_0[\cdot | D' = \infty]$ while Δ_2 has a Bernoulli distribution on $\{0,1\}$ with $\bar{P}_0[\Delta_i = 1] = \phi'(L)$.

(2) There exists a sequence $\{Z_i : i \geqslant 2\}$ of random variables such that for all $i \geqslant 2$ one has that

$$\overline{X}_i = (1 - \Delta_i)\widetilde{X}_i + \Delta_i Z_i. \tag{2.135}$$

Furthermore, for each $i \geqslant 2$, Δ_i is independent of Z_i and of

$$\mathcal{G}_i := \sigma\{\overline{X}_j : j \leqslant i - 1\}.$$

We will call P the common probability distribution of the sequences $\{\overline{X}_i : i \geqslant 2\}$, $\{\widetilde{X}_i : i \geqslant 2\}$, $\{Z_i : i \geqslant 2\}$ and $\{\Delta_i : i \geqslant 2\}$, and E the corresponding expectation. From (2.135) note that

$$\frac{1}{n}\sum_{i=1}^{n} \overline{X}_i = \frac{\overline{X}_1}{n} + \frac{1}{n}\sum_{i=2}^{n} \widetilde{X}_i - \frac{1}{n}\sum_{i=2}^{n} \Delta_i \widetilde{X}_i + \frac{1}{n}\sum_{i=1}^{n} \Delta_i Z_i. \tag{2.136}$$

Let us now examine the behavior as $n \to \infty$ of each of the four terms in the left-hand side of (2.136). Clearly, the first term tends to 0 as $n \to \infty$. For the second term, note that on the event $\{D' = \infty\}$, one has that $| \overline{X}_1 |_2^2 \leqslant c_{24}(\overline{X}_1 \cdot l)^2$ for some constant c_{24}. Therefore, by Proposition 2.5.3, and the fact that $\widetilde{X}_2$ has the same distribution as $\overline{X}_1$ under $\bar{P}_0[\cdot | D' = \infty]$, we see that

$$E[|\widetilde{X}_2|_2^2] = \bar{E}_0[|\overline{X}_1|_2^2 | D' = \infty] \leqslant c_{24} \bar{E}_0[(\overline{X}_1 \cdot l)^2 | D' = \infty] < c_{25}, \tag{2.137}$$

for a suitable constant c_{25}. Hence, by the strong law of large numbers, we actually have that P-a.s.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n} \widetilde{X}_i = \lambda_L. \tag{2.138}$$

For the third term in the left-hand side of (2.136) we have by Cauchy-Schwartz inequality that

$$\left|\frac{1}{n}\sum_{i=2}^{n}\Delta_i\widetilde{X}_i\right|_2 \leqslant \left(\frac{1}{n}\sum_{i=2}^{n}|\widetilde{X}_i|^2\right)^{\frac{1}{2}}\left(\frac{1}{n}\sum_{i=2}^{n}\Delta_i\right)^{\frac{1}{2}}. \tag{2.139}$$

Again by (2.137) and Proposition 2.5.3, we know that there is a constant c_{26} [c.f. (2.66)] such that P-a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}|\widetilde{X}_i|_2^2 = \bar{E}_0[|\overline{X}_1|_2^2|D' = \infty] \leqslant c_{26}.$$

As a result, from (2.139) we see that

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{i=2}^{n}\Delta_i\widetilde{X}_i\right|_2 \leqslant \sqrt{c_{26}\phi'(L)}. \tag{2.140}$$

For the fourth term of the left-hand side of (2.136), we note setting $\overline{Z}_i^{(L)} := E[Z_i \mid \mathcal{G}_i]$ that

$$M_n^j := \sum_{i=2}^{n}\frac{\Delta_i(Z_i - \overline{Z}_i)\cdot e_j}{i} \quad \text{for } n \geqslant 2,\ j \in \{1, 2, \ldots, n\}$$

is a martingale with mean zero with respect to the filtration $\{\mathcal{G}_i : i \geqslant 1\}$. Thus, from the Burkholder-Gundy inequality [W91], we know that there is a constant c_{27} such that for all $j \in \{1, 2, \ldots, d\}$

$$E\left[\left(\sup_n M_n^j\right)^2\right] \leqslant c_{27}E\left[\sum_{i=2}^{\infty}\frac{|\Delta_i(Z_i - \overline{Z}_i)|_2^2}{i^2}\right]. \tag{2.141}$$

Now, since (2.135), note that for all $i \geqslant 2$, $|\Delta_i Z_i| \leqslant |\bar{X}_i|$. It follows that there exists a constant c_{28} such that

$$E[|Z_i|_2^2|\mathcal{G}_i] \leqslant \frac{1}{\phi'(L)}E_0[|\overline{X}_1|_2^2, D' = \infty|\mathfrak{F}_{0,L}] \leqslant \frac{1}{\phi'(L)}c_{28}, \tag{2.142}$$

where we have used Proposition 2.5.3 and Lemma 2.5.4 in the second inequality. So that by (2.141) we see that the martingale $\{M_n^j : n \geqslant 1\}$ converges $P-$a.s. to a random variable for any $j \in \{1, 2, \ldots, d\}$. Thus, by Kronecker's lemma applied to each component $j \in \{1, 2, \ldots, d\}$, we conclude that P-a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\Delta_i(Z_i-\overline{Z}_i)=0. \tag{2.143}$$

Now, note from (2.142) that there is a constant c_{29} such that

$$|\overline{Z}_i|_2 \leqslant E[|Z_i|_2^2 \mid \mathcal{G}_i]^{\frac{1}{2}} \leqslant c_{29}\phi'(L)^{-\frac{1}{2}}. \tag{2.144}$$

Therefore, P-a.s. we have that

$$\begin{aligned}\limsup_{n\to\infty}\left|\tfrac{1}{n}\textstyle\sum_{i=2}^{n}\Delta_i\overline{Z}_i\right|_2 &\leqslant c_{29}\phi'(L)^{-\frac{1}{2}}\limsup_{n\to\infty}\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\Delta_i\\ &\leqslant c_{29}\phi'(L)^{\frac{1}{2}}.\end{aligned} \tag{2.145}$$

Substituting (2.145), (2.140) and (2.138) into (2.136), we conclude the proof of inequality (2.132) provided we set $\eta_L = c_{30}\phi'(L)^{\frac{1}{2}}$ for some constant c_{30}.

Let us now prove the inequality (2.134). By an argument similar to the one presented in [CZ01] to show that the random variable τ_1 has a lower bound of order κ^{-L}, we can show that $X_{\tau_1}\cdot l$ is bounded from below by the sum $S := \sum_{i=1}^{N} U_i$, where $\{U_i : i \geqslant 1\}$ are i.i.d. random variables taking values on $\{1,2,\ldots,L\}$ with law $P[U_i = n] = \kappa^n$ for $1 \leqslant n \leqslant L$, while $N := \min\{i \geqslant 1 : U_i = L\}$. It is clear then that

$$E[X_{\tau_1}\cdot l] \geqslant E[N] = c_{31}\kappa^{-L},$$

for some constant c_{31}.

2.6.3 Proof of Theorem 2.1.1

It will be enough to prove that there is a constant c_{32} such that for all $L \geqslant 1$ one has that

$$\limsup_{n\to\infty}\left|\frac{X_n}{|X_n|_2}-\frac{\lambda_L}{|\lambda_L|_2}\right|_2 < c_{260}\frac{\eta_L}{\lambda_L}. \tag{2.146}$$

Indeed, by compactness, we know that we can choose a sequence $\{L_m, m \geqslant 1\}$ such that

$$\lim_{m\to\infty}\frac{\lambda_{L_m}}{|\lambda_{L_m}|_2} = \hat{v}, \tag{2.147}$$

exists. On the other hand, by the inequality (2.134) of Proposition 2.5.3, we know that $\lim_{m\to\infty} \frac{\eta_{L_m}}{\lambda_{L_m}} = 0$. Now note that by the triangle inequality and (2.146), for every $m \geqslant 1$ one has that

$$\limsup_{n\to\infty} \left| \frac{X_n}{|X_n|_2} - \hat{v} \right|_2 \leqslant c_{32} \frac{\eta_{L_m}}{\lambda_{L_m}} + \left| \frac{\lambda_{L_m}}{|\lambda_{L_m}|_2} - \hat{v} \right|_2 . \tag{2.148}$$

Taking the limit $m \to \infty$ in (2.148) using (2.147) we prove Theorem 2.1.1.

Let us hence prove inequality (2.146). Choose a nondecreasing sequence $\{k_n : n \geqslant 1\}$, P- a.s. tending to $+\infty$ so that for all $n \geqslant 1$ one has that

$$\tau_{k_n} \leqslant n < \tau_{k_n+1}.$$

Notice that

$$\frac{X_n}{|X_n|_2} = \left(\frac{X_n - X_{\tau_{k_n}}}{|X_n|_2} \right) + \left(\frac{X_{\tau_{k_n}}}{k_n} \frac{k_n}{|X_n|_2} \right). \tag{2.149}$$

On the other hand, we assume for the time being, that for large enough L we have proved that

$$\limsup_{n\to\infty} \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} = 0. \tag{2.150}$$

Note first that (2.150) implies that

$$\limsup_{n\to\infty} \frac{|X_n - X_{\tau_{k_n}}|_2}{|X_n|_2} = 0. \tag{2.151}$$

Indeed, note that $|X_n|_2 \geqslant X_n \cdot l \geqslant X_{\tau_{k_n}} \cdot l \geqslant k_n L \frac{|l|_2}{|l|_1}$, which in combination with (2.150) implies (2.151). Also, from (2.150) and the fact that

$$\frac{|X_{\tau_{k_n}}|_2}{k_n} - \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} \leqslant \frac{|X_n|_2}{k_n} \leqslant \frac{|X_{\tau_{k_n}}|_2}{k_n} + \frac{|X_n - X_{\tau_{k_n}}|_2}{k_n}, \tag{2.152}$$

we see that

$$\limsup_{n\to\infty} \left| \frac{\kappa^L |X_n|_2}{k_n} - |\lambda_L|_2 \right|_2 \leqslant \eta_L. \tag{2.153}$$

Combining (2.151) and (2.153) with (2.149) we get (2.146). Thus, it is enough to prove the claim in (2.150). To this end, note that

$$\frac{|X_n - X_{\tau_{k_n}}|_2}{k_n} \leqslant \sup_{j \geqslant 0} \frac{|X_{(\tau_{k_n}+j)\wedge \tau_{k_n+1}} - X_{\tau_{k_n}}|_2}{k_n} \tag{2.154}$$

We now consider the sequence $\widehat{X}_{k \geqslant 1} := \left(\kappa^L \ \sup_{j \geqslant 0} |X_{(\tau_k+j)\wedge(\tau_{k+1})} - X_{\tau_k}|\right)_{k \geqslant 1}$, a coupling decomposition as in the proof of Proposition 2.6.2 turns out; in a enlarged probability space P if necessary, the existence of two i.i.d. sequences $(\mathrm{X}_k)_{k \geqslant 1}$, $(\Delta_k)_{k \geqslant 1}$ and a sequence $(Y_k)_{k \geqslant 1}$, such that P supports the following:

- For $k \geqslant 1$, the common law of X_k is the same as $\widehat{X}_1$ under $\bar{P}[\cdot \mid D' = \infty]$, and one has that Δ_k is Bernoulli with values in the set $\{0, 1\}$ independent of $\mathcal{G}_k$ and $\mathrm{P}[\Delta_k = 1] = \phi'(L)$.
- P- almost surely for $k \geqslant 1$, we have the decomposition:

$$\widehat{X}_k = (1 - \Delta_k)\mathrm{X}_k + \Delta_k Y_k$$

Furthermore, quite similar arguments as the ones given in the proof of Proposition 2.6.2 allow us to conclude that:

$$\sum_{j=1}^{n} \frac{|\mathrm{X}_j|}{n} \to \mathrm{E}[|\widehat{X_1}| \mid D' = \infty] < \infty,$$

$$\sum_{j=1}^{n} \frac{\Delta_j(Y_j - \widetilde{Y}_j)}{n} \to 0 \text{ and}$$

$$\sum_{j=1}^{n} \frac{|\Delta_j \widetilde{Y}_j|}{n} \leqslant c_{240} \phi'(L)^{\frac{1}{2}}. \tag{2.155}$$

(2.156)

where $\widetilde{Y}_j := \mathrm{E}[Y_j \mid \mathcal{G}_j]$. Therefore, using the following inequality

$$\frac{\widehat{X}_k}{k} = \frac{\mathrm{X}_k}{k} + \frac{\Delta_k(Y_k - \widetilde{Y}_k)}{k} + \frac{\Delta_k \widetilde{Y}_k}{k}, \tag{2.157}$$

implies that

$$\frac{\mathrm{X}_k}{k} \to_{k \to \infty} 0 \tag{2.158}$$

The proof is finished.

2.7 Appendix

Proof of Lemma 2.5.6

Here we will prove Lemma 2.5.6. Let us first remark that it will be enough to show that there exists a constant $c_{33} > 0$ such that for all $L \in |u|_1\mathbb{N}$

$$Q[D_{0,L^2}] \leqslant 1 - c_{33}L^2\kappa^L. \tag{2.159}$$

Indeed, using this inequality and the product structure of Q, for all $n \geqslant L^2$ one has that

$$Q[D_{0,n}] \leqslant (1 - c_7L^2\kappa^L)^{\left[\frac{n}{L^2}\right]}.$$

In order to prove (2.159), for $j = L^2 - L$ and $i = 0, 1, \ldots, j$ consider the events

$$A_i = \{\varepsilon : \ (\varepsilon_i, \ldots, \varepsilon_{i+L-1}) = \overline{\varepsilon}^{(L)}\}.$$

Then, by the inclusion-exclusion principle we have that

$$Q[(D_{0,L^2})^c] \geqslant \sum_{0\leqslant j_1 \leqslant j} Q[A_{j_1}] - \sum_{0\leqslant j_1<j_2\leqslant j} Q[A_{j_1} \cap A_{j_2}]. \tag{2.160}$$

Now, note that

$$\begin{aligned}
\sum_{0\leqslant j_1<j_2\leqslant j} Q[A_{j_1} \cap A_{j_2}] &\leqslant j\kappa^{L+1} + (j-1)\kappa^{L+2} + \ldots \\
\ldots + (j-L+1)\kappa^{2L} &+ (j-L)\kappa^{2L} + \ldots + (j-(j-1))\kappa^{2L} \\
\leqslant j\kappa^L \sum_{n=1}^{L} \kappa^n &+ \kappa^{2L}(j-L)^2 \leqslant L^2\kappa^L \frac{1-\kappa^{L+1}}{1-\kappa} + L^4\kappa^{2L} \\
&\leqslant c_{34}L^2\kappa^L,
\end{aligned} \tag{2.161}$$

for some constant c_{34}. Since $Q[A_i] = \kappa^L$ for all $1 \leqslant i \leqslant j$, we conclude from (2.160) and (2.161) that there is a constant c_{33} such that

$$Q[D_{0,L^2}] = 1 - Q[(D_{0,L^2})^c] \leqslant 1 - c_{33}L^2\kappa^L.$$

This finishes the proof.

Acknowledgments: I wish to thank for a private communication about Sznitman's ballisticity conditions to Alexander Drewitz.

Bibliography

[Al99] S. Alili. *Asymptotic behaviour for random walks in random environments.* J. Appl. Probab. 36, no. 2, 334-349, (1999).

[BDR14] N. Berger, A. Drewitz and A.F. Ramírez. *Effective Polynomial Ballisticity Conditions for Random Walk in Random Environment.* Comm. Pure. Appl. Math. (2014).

[Be08] N. Berger. *Limiting velocity of high-dimensional random walk in random environment.* Ann. Probab. 36, no. 2, 728-738, (2008).

[B1] R. Bradley. *A caution on mixing conditions for random fields.* Statist. Probab. Lett. 8, 489–491, (1989).

[CR14] D. Campos and A.F. Ramírez. *Ellipticity criteria for ballistic behavior of random walk in random environment.* Probab. Theory Related Fields 160, no. 1-2, 189-251 (2014).

[Ch62] A.A. Chernov. *Replication of a multicomponent chain by the lightning mechanism.* Biophysics, 12, 336-341 (1962).

[Ch67] K. L. Chung. *Markov Chains: With Stationary Transition Probabilities.* Grundlehren der mathematischen Wissenschaften, (1967).

[CZ01] F. Comets and O. Zeitouni. *A law of large numbers for random walks in random mixing environments.* Ann. Probab. 32, no. 1B, 880–914, (2004).

[CZ02] F. Comets and O. Zeitouni. *Gaussian fluctuations for random walks in random mixing environments.* Probability in mathematics. Israel J. Math. 148, 87–113, (2005).

[DR11] A. Drewitz and A.F. Ramírez. *Ballisticity conditions for random walk in random environment.* Probab. Theory Related Fields 150, no. 1-2, 61-75 (2011).

[DR12] A. Drewitz and A.F. Ramírez. *Quenched exit estimates and ballisticity conditions for higher-dimensional random walk in random environment.* Ann. Probab. 40, no. 2, 459-534 (2012).

[DR14] A. Drewitz and A.F. Ramírez. *Selected topics in random walks in random environment.* Topics in percolative and disordered systems, 23-83, Springer Proc. Math. Stat., 69, Springer, New York, (2014).

[FGP10] A. Fribergh, N. Gantert and S. Popov. *On slowdown and speedup of transient random walks in random environment.* Probab. Theory Relat. Fields 147, 43, (2010).

[GR15] E. Guerra and A.F. Ramírez. *Almost exponential decay for the exit proobability from slabs of ballistic random walks in random environments*

[G14] X. Guo. *On the limiting velocity of random walks in mixing random environment.* Ann. Inst. H. Poincarè Probab. Statist. 50, no. 2, 375-402, (2014).

[K81] S. Kalikow. *Generalized random walks in random environment.* Ann. Probab. 9, 753–768, (1981).

[Ke77] H. Kesten. *A renewal theorem for random walk in a random environment.* Proc. Symposia Pure Math. 31, 67–77, (1977).

[KKS75] H. Kesten, M. V. Kozlov and F. Spitzer. *A limit law for random walk in a random environment.* Compositio Mathematica 30, 2, 145-168, (19 75).

[M94] S. Molchanov. *Lectures on random media.* Lectures on probability theory (Saint-Flour, 1992), 242-411, Lecture Notes in Math., 1581, Springer, Berlin, (1994).

[RA03] F. Rassoul-Agha. *The point of view of the particle on the law of large numbers for random walks in a mixing random environment.* Ann. Probab. 31 , no. 3, 1441–1463, (2003).

[RA05] F. Rasoul-Agha. *On the Zero-One Law and the Law of Large Numbers for Random Walk in Mixing Random Environment.* Electron. Commun. Probab. 10 , no. 5, 36-44, (2005).

[Sh03] L. Shen. *On ballistic diffusions in random environment.* Ann. Inst. H. Poincarè Probab. Statist. 39 , no. 5, 839-876, (2003).

[Si07] F. Simenhaus. *Asymptotic direction for random walks in random environments.* Ann. Inst. H. Poincarè Probab. Statist. 43 , no. 6, 751–761, (2007).

[Sin82] Ya. G. Sinai Theory Probab. Appl. 27, no.2, 256–268. (1982). *The Limiting Behavior of a One-Dimensional Random Walk in a Random Medium.*

[SW69] W. Smith and W. Winkinson. *On branching processes in random environment.* Ann. Math. Statist. 40, 814-827, (1969).

[So75] F. Solomon. *Random walks in a random environment.* Ann. Probability 3, 1-31, (1975).

[SZ99] A.S. Sznitman and M. Zerner. *A law of large numbers for random walks in random environment.* Ann. Probab. 27 , no. 4, 1851–1869, (1999).

[Sz01] A.S. Sznitman. *Slowdown estimates and central limit theorem for random walks in random environment.*J. Eur. Math. Soc. 2, no. 2, 93–143 (2000).

[Sz02] A.S. Sznitman. *On a class of transient random walks in random environment.* Ann. Probab. 29 (2), 724-765 (2001).

[Sz03] A.S. Sznitman. *An effective criterion for ballistic behavior of random walks in random environment.* Probab. Theory Related Fields 122, no. 4, 509-544 (2002).

[Sz04] A.S. Sznitman. *On new examples of ballistic random walks in random environment* Ann. Probab. 31, (1), 285-322 (2003).

[T69] D.E. Temkin. *The theory of diffusionless crystal growth.* Journal of Crystal Growth, 5, no. 3, 193-202 (1969).

[W91] D. Williams. *Probability with martingales.* Cambridge Univ. Press (1991).

[Ze1] O. Zeitouni . *Lecture notes on random walks in random environment.* Lecture Notes in Math. Springer, New York (2004).