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A converse theorem for Jacobi–Maass forms and applications & Asymptotic distribution of Hecke points over $\mathbb{C}_p$ and applications

por

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A mis padres,
Enrique y Pilar.

Contents

About this thesis

This thesis is divided in two chapters, which are independent and can be read separately.

In the first chapter we provide a converse theorem for a certain class of non-holomorphic automorphic forms, called Jacobi–Maass forms. This new type of automorphic forms was defined by Pitale in [**27**] and it "contains" the space of holomorphic Jacobi forms and the space of skew-holomorphic Jacobi forms. More precisely, there exist explicit injective linear maps from the spaces of holomorphic or skew-holomorphic Jacobi forms, to spaces of Jacobi–Maass forms. As usual, our converse theorem gives a one-to-one correspondence between this new type of automorphic forms and certain family of Dirichlet series built from the Fourier coefficients (more precisely, Fourier–Whittaker coefficients in our case) of the corresponding forms. From this result we get three applications. The first one is a converse theorem for holomorphic Jacobi forms which generalizes a result of Martin [**24**]. The second application is a converse theorem for skew-holomorphic Jacobi forms. Finally, we apply our main theorem in order to give an alternative proof of a result of Pitale [**27**] on the existence of an explicit isomorphism between certain Jacobi–Maass forms (more precisely, Jacobi–Maass forms of index 1) and Maass forms of half-integral weight. This work was supervised by Prof. Yves Martin and all the results are contained in the paper

Herrero-Miranda, S.: *A converse theorem for Jacobi–Maass forms and applications.*
Journal of Number Theory **169** (2016) 41-61.

In the second chapter we study the asymptotic distribution of Hecke points in the moduli space of elliptic curves over $\mathbb{C}_p$ (a completion of an algebraic closure of $\mathbb{Q}_p$). Hecke points of degree n attached to an isomorphism class of elliptic curves $\overline{E}$ correspond to the collection of isomorphism classes $\overline{E'}$ which admit an isogeny $E \to E'$ of degree n. As application we describe the asymptotic distribution of singular moduli (j-invariants of CM elliptic curves) over $\mathbb{C}_p$ and give a finiteness result on certain collection of singular moduli over $\mathbb{C}$ generalizing a result of Habegger [**12**]. The main results in this chapter can be seen as p-adic analogues of classical results on the asympotic distribution of Hecke points over $\mathbb{C}$, going back to Selberg (see [**4**]), and on the asymptotic distribution of singular moduli over $\mathbb{C}$ by Duke [**9**] and Clozel-Ulmo [**4**]. This work was supervised by Prof. Juan Rivera-Letelier and is part of a project in collaboration with Ricardo Menares. All the results of this chapter are contained in the preprint

Herrero-Miranda, S., Menares, R., Rivera-Letelier, J.: *On the asymptotic distribution of Hecke points over $\mathbb{C}_p$ and applications.*

We warn the reader that this thesis is not intended to be used as a reference for the theory of automorphic forms or for the theory of elliptic curves. The general results needed to prove our main results are briefly recalled and kept to a minimum. In any case, general references are provided along the way.

The author naively hopes the reader will find this work useful or at least interesting.

CHAPTER 1

A converse theorem for Jacobi–Maass forms and applications

Introduction to Chapter 1

The main purpose of converse theorems is to establish a one-to-one correspondence between automorphic forms over some group, on the one hand, and Dirichlet series satisfying some nice analytic properties, on the other. One of the first such converse theorems was stated in [**13**] by E. Hecke, who showed an equivalence between the automorphy of a cusp form $f(\tau)$ over $\mathrm{SL}_2(\mathbb{Z})$ and the analytic continuation, functional equation and certain boundedness condition satisfied by a Dirichlet series $L(f, s)$ attached to $f(\tau)$. Many generalizations of Hecke's converse theorem have been established in other contexts. For example: for modular forms over congruence subgroups $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ by A. Weil [**34**], for Jacobi forms over $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ by Y. Martin [**24**], for certain non-holomorphic modular forms, later called Maass forms, by H. Maass [**20**], and for degree two Siegel modular forms by K. Imai [**14**] and T. Arakawa, I. Makino and F. Sato [**2**]. The converse theorem for GL_n automorphic representations is an outstanding achievement of several authors: for $n = 2$ by H. Jacquet and R. Langlands [**15**], for $n = 3$ by H. Jacquet, I. Piatetski-Shapiro and J. Shalika [**16**] and for general n by J. Cogdell and I. Piatetski-Shapiro [**6**].

In this chapter we provide a converse theorem for Jacobi–Maass forms, as introduced by A. Pitale [**27**]. A Jacobi–Maass form of weight k and index m is a smooth function $F(\tau, z)$ on $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ denotes the upper-half plane, which is an eigenfunction of a degree three differential operator $C^{k,m}$ and it satisfies a transformation property with respect to the action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{H} \times \mathbb{C}$, given by a non-holomorphic automorphy factor, analogous to the transformation property of holomorphic and skew-holomorphic Jacobi forms. In this thesis we also assume that F is holomorphic with respect to the z variable. We note that such condition is not a big restriction since any general Jacobi–Maass form can be obtained from finitely many such z-holomorphic Jacobi–Maass forms after applying certain differential operators (see remark in [**27**] p. 101).

This chapter is organized as follows: in Section 1 we fix notations and recall the definition and basic properties of holomorphic Jacobi forms, skew-holomorphic Jacobi forms and Maass forms of half-integral weight. Jacobi–Maass forms are introduced in Section 2. The main theorem of this chapter and some applications are stated in Sections 3 and 4, respectively. In Section 5 we adapt some results of Maass in order to get a certain converse theorem which will play a central role in the proof of our main theorem. Sections 6, 7 and 8 are devoted to proving our main theorem and our applications.

1. Preliminaries

1.1. The Modular group and the Jacobi group. Here and throughout this chapter $z = u+iv$ will denote a complex number and $\tau = x+iy$ a point on $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, thus u, v, x, y are real numbers with $y > 0$ and $i = \sqrt{-1}$.
For $r \in \mathbb{R}$ and $z \neq 0$ we put $z^r = e^{r(\log|z| + i\arg(z))}$ where $\arg(z)$ takes values on $(-\pi, \pi]$.
The modular group is defined as

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

This group acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The Jacobi group is defined as $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, where the group law is given explicitly by

$$(M_1, X_1)(M_2, X_2) = (M_1 M_2, X_1 M_2 + X_2).$$

The group Γ^J acts on $\mathbb{H} \times \mathbb{C}$ by

$$(M, X)(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \omega\tau + \xi}{c\tau + d} \right), \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X = (\omega, \xi).$$

1.2. Holomorphic Jacobi forms. We refer the reader to [**11**] for the general theory of holomorphic Jacobi forms.
For integers k, m with $m > 0$ consider the holomorphic automorphy factor

$$j_{k,m}((M, X), (\tau, z)) = e^{2\pi i m\left(-\frac{c(z+\omega\tau+\xi)^2}{c\tau+d} + \omega^2\tau + 2\omega z \right)} (c\tau + d)^{-k},$$

and the slash operator on smooth functions $F : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ defined by

$$(F|_{k,m}(M, X))(\tau, z) = j_{k,m}((M, X), (\tau, z)) F((M, X)(\tau, z)).$$

This gives an action of Γ^J on the space of holomorphic functions on $\mathbb{H} \times \mathbb{C}$.

Definition 1.1. *An holomorphic function $F : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of weight k ($k \in \mathbb{Z}$) and index m ($m \in \mathbb{Z}, m > 0$) with respect to Γ^J if*

(a) *$(F|_{k,m}(M, X))(\tau, z) = F(\tau, z)$ for all $(M, X) \in \Gamma^J$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$,*

(b) *F has a Fourier expansion of the form*

(1) $$F(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ 4mn - r^2 \geq 0}} a(n, r) e^{2\pi i n\tau} e^{2\pi i r z}.$$

If, in addition, F satisfies the condition $a(n, r) = 0$ for all $n, r \in \mathbb{Z}$ such that $4mn - r^2 = 0$, then we say that F is a Jacobi cusp form.

We denote the vector space of all Jacobi forms of weight k and index m with respect to Γ^J by $J_{k,m}$.

Theorem 1.2. *Let $F \in J_{k,m}$. The Fourier coefficients $a(n,r)$ in (1) depend only on the integer $4mn - r^2$, and on the class of r mod $2m$.*

For $\mu \in \{1\dots, 2m\}$ and integers $D \equiv -\mu^2 \pmod{4m}$ set

$$a_\mu(D) = a\left(\frac{D+\mu^2}{4m}, \mu\right) \tag{2}$$

and extend this definition to all D by setting $a_\mu(D) = 0$ if $D \not\equiv -\mu^2 \pmod{4m}$. The coefficients $a_\mu(D)$ have at most polynomial growth with respect to D.

For $\mu \in \{1, \dots, 2m\}$ define $f_\mu : \mathbb{H} \to \mathbb{C}$ by

$$f_\mu(\tau) = \sum_{D=0}^{\infty} a_\mu(D) e^{\pi i D\tau/2m}. \tag{3}$$

For later use, we rewrite this as

$$f_\mu(\tau) = c_\mu + \sum_{D=1}^{\infty} a_\mu(D) e^{\pi i D\tau/2m} \tag{4}$$

where $c_\mu = a_\mu(0)$.

A rearrangement of (1) gives the ϑ-decomposition

$$F(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau)\vartheta_{\mu,2m}(\tau, z), \tag{5}$$

where

$$\vartheta_{\mu,2m}(\tau, z) = \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} e^{\pi i r^2\tau/2m} e^{2\pi i r z}. \tag{6}$$

Theorem 1.3. *The map $F(\tau, z) \mapsto (f_1(\tau), \dots, f_{2m}(\tau))$ gives a $\mathbb{C}$-linear isomorphism between $J_{k,m}$ and the space of holomorphic vector-valued functions $(f_1(\tau), \dots, f_{2m}(\tau))$ satisfying*

$$f_\mu(\tau + 1) = e^{-2\pi i \frac{\mu^2}{4m}} f_\mu(\tau), \tag{7}$$

$$f_\mu\left(-\frac{1}{\tau}\right) = \frac{i^{1/2}}{\sqrt{2m}} \tau^{k-1/2} \sum_{j=1}^{2m} e^{\pi i \mu j/m} f_j(\tau). \tag{8}$$

1.3. Skew-holomorphic Jacobi forms. We refer the reader to [**32**] for the definition and basic properties of skew-holomorphic Jacobi forms.

For integers k, m with $m > 0$ consider the non-holomorphic automorphy factor

$$j^{sh}_{k,m}((M, X), (\tau, z)) = e^{2\pi i m\left(-\frac{c(z+\omega\tau+\xi)^2}{c\tau+d} + \omega^2\tau + 2\omega z\right)} \left(\frac{c\tau + d}{|c\tau + d|}\right)^{k-1} |c\tau + d|^{-k},$$

and the slash operator on smooth functions $\widetilde{F}: \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ defined by

$$(\widetilde{F}|^{sh}_{k,m}(M,X))(\tau,z) = j^{sh}_{k,m}((M,X),(\tau,z))\widetilde{F}((M,X)(\tau,z)).$$

This gives an action of Γ^J on the space of smooth functions on $\mathbb{H} \times \mathbb{C}$.

Definition 1.4. *A smooth function $\widetilde{F}: \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a skew-holomorphic Jacobi form of weight k ($k \in \mathbb{Z}$) and index m ($m \in \mathbb{Z}, m > 0$) with respect to Γ^J if*

(a) $(\widetilde{F}|^{sh}_{k,m}(M,X))(\tau,z) = \widetilde{F}(\tau,z)$ *for all* $(M,X) \in \Gamma^J$ *and* $(\tau,z) \in \mathbb{H} \times \mathbb{C}$,

(b) $\partial_{\overline{z}}\widetilde{F} = (8\pi i m\partial_\tau - \partial_z^2)\widetilde{F} = 0$,

(c) $\widetilde{F}$ *has a Fourier expansion of the form*

$$\widetilde{F}(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn-r^2\le 0}} a(n,r)e^{2\pi i\left(n\tau + \frac{iy(r^2-4mn)}{2m}\right)}e^{2\pi i r z}. \tag{9}$$

We denote the vector space of all skew-holomorphic Jacobi forms of weight k and index m with respect to Γ^J by $J^{sh}_{k,m}$.

An analogue of Theorem 1.2 holds for skew-holomorphic Jacobi forms. Thus we can define $a_\mu(D)$ by (2) and put

$$\widetilde{f}_\mu(\tau) = \sum_{D=0}^{\infty} a_\mu(-D)e^{\pi i\overline{\tau}(-D)/2m}, \tag{10}$$

which, for later use, we rewrite as

$$\widetilde{f}_\mu(\tau) = c_\mu + \sum_{D=1}^{\infty} a_\mu(-D)e^{\pi i\overline{\tau}(-D)/2m}, \tag{11}$$

where $c_\mu = a_\mu(0)$.

A rearrangement of (9) gives the ϑ-decomposition

$$\widetilde{F}(\tau,z) = \sum_{\mu=1}^{2m} \widetilde{f}_\mu(\tau)\vartheta_{\mu,2m}(\tau,z). \tag{12}$$

Theorem 1.5. *The map $\widetilde{F}(\tau,z) \mapsto (\widetilde{f}_1(\tau),\ldots,\widetilde{f}_{2m}(\tau))$ gives a $\mathbb{C}$-linear isomorphism between $J^{sh}_{k,m}$ and the space of antiholomorphic vector-valued functions $(\widetilde{f}_1(\tau),\ldots,\widetilde{f}_{2m}(\tau))$ satisfying*

$$\widetilde{f}_\mu(\tau+1) = e^{-2\pi i\frac{\mu^2}{4m}}\widetilde{f}_\mu(\tau), \tag{13}$$

$$\widetilde{f}_\mu\left(-\frac{1}{\tau}\right) = \frac{i^{1/2}}{\sqrt{2m}}\overline{\tau}^{k-1/2}\sum_{j=1}^{2m} e^{\pi i\mu j/m}\widetilde{f}_j(\tau). \tag{14}$$

1.4. Whittaker functions. We will make use of the Whittaker function $W_{\kappa,\mu} : \mathbb{R}^+ \to \mathbb{C}$ which can be defined as the unique solution of the Whittaker differential equation

$$\frac{\partial^2 w}{\partial y^2} + \left(\frac{1/4-\mu^2}{y^2} + \frac{\kappa}{y} - \frac{1}{4}\right) w = 0 \tag{15}$$

satisfying

$$W_{\kappa,\mu}(y) \sim e^{-y/2} y^{\kappa} \text{ for } y \to +\infty. \tag{16}$$

In particular, $W_{\kappa,-\mu} = W_{\kappa,\mu}$.

Any solution of (15) with at most polynomial growth is a constant multiple of $W_{\kappa,\mu}$. As a particular case, we have

$$W_{\kappa,\mu}(y) = e^{-y/2} y^{\kappa}, \quad \text{if} \quad \kappa(\kappa-1) = \mu^2 - \frac{1}{4}. \tag{17}$$

For general properties of Whittaker functions see [**23**].

1.5. Maass forms of half-integral weight. Let $\Gamma_0(4)$ denote the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ given by

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod 4 \right\}.$$

We will also consider the group $\mathfrak{G}$ of pairs $(M, \phi(\tau))$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\phi(\tau)$ is a function on $\mathbb{H}$ satisfying $\phi(\tau) = t\left(\frac{c\tau+d}{|c\tau+d|}\right)^{1/2}$ for some fixed $t \in \mathbb{C}$ depending on ϕ with $|t| = 1$. The group law of $\mathfrak{G}$ is given by

$$(M_1, \phi_1(\tau))(M_2, \phi_2(\tau)) = (M_1 M_2, \phi_1(M_2(\tau))\phi_2(\tau)).$$

Given $k \in \mathbb{Z}$ the group $\mathfrak{G}$ acts on smooth functions $f : \mathbb{H} \to \mathbb{C}$ by the slash operator $\|_{k-1/2}$ defined by

$$(f\|_{k-1/2}(M,\phi))(\tau) = f(M(\tau))\phi(\tau)^{-(2k-1)}.$$

There is a morphism $\Gamma_0(4) \to \mathfrak{G}$, given by $M \mapsto M^* = (M, j(M,\tau))$ where

$$j(M,\tau) = \left(\frac{c}{d}\right) \epsilon_d^{-1} \left(\frac{c\tau+d}{|c\tau+d|}\right)^{1/2}.$$

Here

$$\epsilon_d = \begin{cases} 1 & , \text{ if } d \equiv 1 \pmod 4, \\ i & , \text{ if } d \equiv 3 \pmod 4, \end{cases}$$

and $\left(\frac{c}{d}\right)$ is an extension of the usual Legendre symbol, when d is an odd prime, which is defined in [**30**] p. 442. Here we content ourselves with the fact that $\left(\frac{c}{d}\right) \in \{1, -1\}$.

Definition 1.6. *A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a Maass form of weight $k - \frac{1}{2}$ and eigenvalue λ ($\lambda \in \mathbb{C}$) with respect to $\Gamma_0(4)$ if*

(a) $(f\|_{k-1/2}M^*)(\tau) = f(\tau)$ *for all* $M \in \Gamma_0(4)$ *and* $\tau \in \mathbb{H}$,

(b) $\Delta_{k-1/2}f = \lambda f$,

(c) *for some* $\sigma > 0$, *we have* $f(\tau) = O(y^\sigma)$ *as* $y \to +\infty$.

Here Δ_ξ (for $\xi \in \mathbb{C}$) denotes the weight ξ Laplacian operator

$$\triangle_\xi = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \xi i y \frac{\partial}{\partial x} \tag{18}$$

acting on smooth functions on $\mathbb{H}$.

We will denote by $M_{k-1/2,\lambda}(4)$ the space of Maass forms of weight $k - \frac{1}{2}$ and eigenvalue λ with respect to $\Gamma_0(4)$.

As a first use of Whittaker functions, we recall that any $f(\tau)$ in $M_{k-1/2,\lambda}(4)$ has a Fourier–Whittaker expansion

$$f(\tau) = U_{2\lambda}(y)c + V_{2\lambda}(y)d + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\operatorname{sgn}(n)\frac{k-1/2}{2},\frac{i\varrho}{2}}\left(4\pi|n|y\right)e^{2\pi i n x}a(n) \tag{19}$$

where $\varrho^2 = -1 - 4\lambda$ and

$$(U_\lambda(y), V_\lambda(y)) = \begin{cases} \left(y^\alpha, y^\beta\right) & \text{, if } \lambda \neq -\frac{1}{2}, (\alpha,\beta) \text{ solutions of } X(X-1) = \frac{\lambda}{2}, \\ \\ \left(y^{1/2}, y^{1/2}\log(y)\right) & \text{, if } \lambda = -\frac{1}{2}. \end{cases} \tag{20}$$

Moreover, the coefficients $a(n)$ have at most polynomial growth with respect to $|n|$.

2. Jacobi–Maass forms

For integers k, m with $m > 0$ consider the non-holomorphic automorphy factor

$$j_{k,m}^{nh}((M,X),(\tau,z)) = e^{2\pi i m\left(-\frac{c(z+\omega\tau+\xi)^2}{c\tau+d}+\omega^2\tau+2\omega z\right)}\left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k},$$

and the slash operator on smooth functions $F : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ defined by

$$(F|_{k,m}^{nh}(M,X))(\tau,z) = j_{k,m}^{nh}((M,X),(\tau,z))F((M,X)(\tau,z)).$$

This gives an action of Γ^J on the space of smooth functions on $\mathbb{H}\times\mathbb{C}$.
Finally, define the differential operator

$$C^{k,m} = \frac{5}{8} - 2(\tau-\overline{\tau})^2\partial_\tau\partial_{\overline{\tau}} - (k-1)(\tau-\overline{\tau})\partial_{\overline{\tau}} - k(\tau-\overline{\tau})\partial_\tau + \frac{k(\tau-\overline{\tau})}{8\pi i m}\partial_z^2 + \frac{(\tau-\overline{\tau})^2}{4\pi i m}\partial_{\overline{\tau}}\partial_z^2.$$

Definition 2.1. *A smooth function $F : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is called a Jacobi–Maass form of weight k ($k\in\mathbb{Z}$), index m ($m\in\mathbb{Z}, m>0$) and eigenvalue λ ($\lambda\in\mathbb{C}$) with respect to Γ^J if*

(a) *$(F|_{k,m}^{nh}(M,X))(\tau,z) = F(\tau,z)$ for all $(M,X)\in\Gamma^J$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$,*
(b) *$F(\tau,z)$ is holomorphic in the z variable,*
(c) *$C^{k,m}F = \lambda F$,*
(d) *for some $\sigma>0$, we have $F(\tau,z) = O(y^\sigma e^{2\pi m v^2/y})$ as $y\to+\infty$.*

If, in addition, F satisfies the condition

$$\int_0^1\int_0^1 F(\tau+\omega, z+\xi)e^{-2\pi i(n\omega+r\xi)}d\omega d\xi = 0$$

for all $n, r\in\mathbb{Z}$ such that $4mn-r^2=0$, then we say that F is a Jacobi–Maass cusp form.

We denote the vector space of all Jacobi–Maass forms of weight k, index m and eigenvalue λ with respect to Γ^J by $\hat{J}_{k,m,\lambda}^{nh}$.

We mention here that the differential operator $C^{k,m}$ defined above is a simplified version of the one used by A. Pitale (see [**27**] p. 91). However, they coincide when applied to smooth functions that are holomorphic in the z variable, which is the case that we treat in this thesis. The original differential operator considered by Pitale is invariant under the slash operator $|_{k,m}^{nh}$ and this also holds for $C^{k,m}$.

Theorem 2.2. *Each $F\in\hat{J}_{k,m,\lambda}^{nh}$ has a Fourier–Whittaker expansion*

$$\begin{aligned} F(\tau,z) = &\sum_{\substack{n,r\in\mathbb{Z}\\ 4mn-r^2\neq 0}} a(n,r)y^{1/4}W_{\mathrm{sgn}(4mn-r^2)\frac{k-1/2}{2},\frac{i\ell}{2}}\left(\frac{\pi|4mn-r^2|y}{m}\right)e^{-\pi r^2y/2m}e^{2\pi i n x}e^{2\pi i r z} \\ &+ \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn-r^2=0}} \left(c(n,r)U_\lambda(y)+d(n,r)V_\lambda(y)\right)y^{1/4}e^{2\pi i n\tau}e^{2\pi i r z} \end{aligned} \tag{21}$$

where $\ell^2 = -1-2\lambda$ and the functions U_λ, V_λ are given by (20).

Proof. From $F|_{k,m}^{nh}(M,Z) = F$ with $M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $X = (0,r)$, $n, r \in \mathbb{Z}$, we get $F(\tau + n, z + r) = F(\tau, z)$. This, and the fact that $F(\tau, z)$ is holomorphic in τ and smooth in z, gives

$$F(\tau, z) = \sum_{n,r\in\mathbb{Z}} a(n,r,y) e^{2\pi i n\tau} e^{2\pi i r z}.$$

Put $b(n,r,y) = a(n,r,y)e^{\pi r^2 y/2m} y^{-1/4}$. From the equality $C_{k,m}F = \lambda F$ we get

$$\begin{aligned}&(2m^2\lambda + \pi^2 r^4 y^2 + 4\pi m^2 ny - \pi m r^2 y + 16\pi^2 m^2 n^2 y^2 - 8\pi^2 mnr^2 y^2 - 8\pi k m^2 ny + 2\pi kmr^2 y)b(n,r,y)\\ &-4m^2y^2 b''(n,r,y)\\ =\ & 0.\end{aligned}$$

If $4mn - r^2 = 0$, then this simplifies to

$$2m^2\lambda b(n,r,y) - 4m^2 y^2 b''(n,r,y) = 0$$

or equivalently

$$\lambda b(n,r,y) - 2y^2 b''(n,r,y) = 0.$$

For $\lambda \neq -\frac{1}{2}$ we solve the differential equation and get

$$b(n,r,y) = c(n,r)y^{\alpha} + d(n,r)y^{\beta}$$

where α, β are the solutions of the quadratic equation $X(X-1) = \frac{\lambda}{2}$. Similarly, for $\lambda = -\frac{1}{2}$ we get

$$b(n,r,y) = d(n,r)y^{1/2} + c(n,r)y^{1/2}\log(y).$$

If $r^2 - 4mn \neq 0$, we put $b(n,r,y) = B\left(n, r, \frac{\pi|4mn-r^2|}{m} y\right)$ and get the differential equation

$$(2\lambda + y(1-2k)\mathrm{sgn}(4mn - r^2) + y^2)B(n,r,y) - 4y^2 B''(n,r,y) = 0.$$

This corresponds to the Whittaker differential equation (15) with parameters $\kappa = \mathrm{sgn}(4mn-r^2)\frac{k-1/2}{2}$ and μ satisfying $\frac{1}{4} + \frac{\lambda}{2} = \mu^2$. Putting $\mu = \frac{il}{2}$ we have $l^2 = -1 - 2\lambda$. Since

$$a(n,r,y)e^{2\pi i n x}e^{2\pi i r z_0} = \int_0^1 \int_0^1 F((x+iy)+p, z_0+q)e^{-2\pi i n p}e^{-2\pi i r q}dpdq$$

we get

$$|a(n,r,y)e^{-2\pi r \mathrm{Im}(z_0)}| \leq \int_0^1 \int_0^1 |F((x+iy)+p, z_0+q)|dpdq.$$

From the asymptotic growth of F we get, choosing $z_0 = -\frac{ryi}{2m}$, the asymptotic bound

$$|a(n,r,y)e^{\pi r^2 y/m}| = O(y^N e^{\pi r^2 y/2m}) \ , \quad \text{for } y \to \infty,$$

or equivalently

$$|a(n,r,y)e^{\pi r^2 y/2m}| = O(y^N) \ , \quad \text{for } y \to \infty.$$

From

$$B(n,r,y) = a\left(n, r, \frac{my}{\pi|4mn - r^2|}\right) e^{\pi r^2(my/\pi|4nm-r^2|)/2m} \left(\frac{my}{\pi|4nm - r^2|}\right)^{-1/4}$$

it follows that $B(n,r,y)$ has at most polynomial growth when $y \to \infty$. Hence

$$B(n,r,y) = a(n,r) W_{\operatorname{sgn}(4mn-r^2)\frac{k-1/2}{2}, \frac{il}{2}}(y).$$

This gives $a(n,r,y) = a(n,r) y^{1/4} W_{\operatorname{sgn}(4mn-r^2)\frac{k-1/2}{2}, \frac{i\ell}{2}}\left(\frac{\pi|4mn-r^2|y}{m}\right) e^{-\pi r^2 y/2m}$ and completes the proof of the theorem. ☐

We note that $F \in \hat{J}^{nh}_{k,m,\lambda}$ is a cusp form if and only if $c(n,r) = d(n,r) = 0$ for every n, r with $4mn - r^2 = 0$.

Theorem 2.3. *The Fourier–Whittaker coefficients $a(n,r)$ in (21) depend only on the integer $4mn - r^2$ and on the class of r mod $2m$. The coefficients $c(n,r), d(n,r)$ depend only on the class of r mod $2m$.*

Proof. From $F|^{nh}_{k,m}(M,X) = F$ with $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = (\lambda, \mu) \in \mathbb{Z}^2$ we get $F(\tau, z) = e^{2\pi i m(\lambda^2\tau + 2\lambda z)} F(\tau, z + \lambda\tau + \mu)$. Using the Fourier–Whittaker expansion of F we get

$$\begin{aligned}
&\sum_{4mn-r^2\neq 0} c(n,r) y^{1/4} W_{\operatorname{sgn}(4mn-r^2)\frac{k-1/2}{2}, \frac{il}{2}}\left(\frac{\pi|4mn-r^2|y}{m}\right) e^{-\pi r^2 y/2m} e^{2\pi i n x} e^{2\pi i r z} \\
&\quad + \sum_{4mn-r^2=0} \left(d(n,r,1)u_1(y) + d(n,r,2)u_2(y)\right) y^{1/4} e^{2\pi i n\tau} e^{2\pi i r z} \\
= &\sum_{4mn-r^2\neq 0} c(n,r) y^{1/4} W_{\operatorname{sgn}(4mn-r^2)\frac{k-1/2}{2}, \frac{il}{2}}\left(\frac{\pi|4mn-r^2|y}{m}\right) e^{-\pi(r+2\lambda m)^2 y/2m} e^{2\pi i(n+\lambda r+m\lambda^2)x} e^{2\pi i(r+2\lambda m)z} \\
&\quad + \sum_{4mn-r^2=0} \left(d(n,r,1)u_1(y) + d(n,r,2)u_2(y)\right) y^{1/4} e^{2\pi i(n+m\lambda^2+r\lambda)\tau} e^{2\pi i(r+2\lambda m)z}.
\end{aligned}$$

It follows that $a(n,r) = a(n + \lambda r + m\lambda^2, r + 2\lambda m)$ and the same holds for $c(n,r)$ and $d(n,r)$. This implies the result. ☐

As in the case of holomorphic and skew-holomorphic Jacobi forms, we put

$$a_\mu(D) = a\left(\frac{D+\mu^2}{4m}, \mu\right) \tag{22}$$

for $\mu \in \{1\ldots, 2m\}$ and integers $D \neq 0, D \equiv -\mu^2 \pmod{4m}$, and extend this definition to all $D \neq 0$ by setting $a_\mu(D) = 0$ if $D \not\equiv -\mu^2 \pmod{4m}$. Similarly, set

$$c_\mu = c\left(\frac{\mu^2}{4m}, \mu\right), \qquad d_\mu = d\left(\frac{\mu^2}{4m}, \mu\right) \tag{23}$$

for $\mu \in \{1, \ldots, 2m\}, \mu^2 \equiv 0 \pmod{4m}$ and extend this definition to all $\mu \in \{1, \ldots, 2m\}$ by setting $c_\mu = d_\mu = 0$ if $\mu^2 \not\equiv 0 \pmod{4m}$. The coefficients $a_\mu(D), c_\mu, d_\mu$ satisfy the following conditions:

$$a_\mu(D) = O(|D|^\sigma) \text{ for some } \sigma > 0, \tag{24}$$

$$a_\mu(D) = 0 \text{ if } D \not\equiv -\mu^2 \pmod{4m}, \tag{25}$$

$$c_\mu = d_\mu = 0 \text{ if } \mu^2 \not\equiv 0 \pmod{4m}. \tag{26}$$

We can now define $f_\mu : \mathbb{H} \to \mathbb{C}$ by

$$f_\mu(\tau) = U_\lambda(y)c_\mu + V_\lambda(y)d_\mu + \sum_{\substack{D\in\mathbb{Z}\\ D\neq 0}} W_{\mathrm{sgn}(D)\frac{k-1/2}{2},\frac{i\ell}{2}}\left(\frac{\pi|D|y}{m}\right) e^{\pi i Dx/2m} a_\mu(D). \tag{27}$$

A rearrangement of (21) gives the ϑ-decomposition

$$F(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau) y^{1/4} \vartheta_{\mu,2m}(\tau, z), \tag{28}$$

where the functions $\vartheta_{\mu,2m}$ are given by (6).

The relation between $F(\tau, z)$ and the vector $(f_1(\tau), \ldots, f_{2m}(\tau))$ is given explicitly in the following theorem of Pitale (Theorem 4.6 of [**27**]).

Theorem 2.4. *The map $F(\tau, z) \mapsto (f_1(\tau), \ldots, f_{2m}(\tau))$ gives an isomorphism between $\hat{J}^{nh}_{k,m,\lambda}$ and the space of vector-valued functions $(f_1(\tau), \ldots, f_{2m}(\tau))$ satisfying*

$$f_\mu(\tau + 1) = e^{-2\pi i \frac{\mu^2}{4m}} f_\mu(\tau), \tag{29}$$

$$f_\mu\left(-\frac{1}{\tau}\right) = \frac{i^{1/2}}{\sqrt{2m}} \left(\frac{\tau}{|\tau|}\right)^{k-1/2} \sum_{j=1}^{2m} e^{\pi i \mu j/m} f_j(\tau), \tag{30}$$

$$f_\mu(\tau) = O(y^\sigma) \text{ as } y \to +\infty, \text{ for some } \sigma > 0, \tag{31}$$

$$\triangle_{k-1/2} f_\mu = \frac{\lambda}{2} f_\mu. \tag{32}$$

We finish this section by stating another theorem of Pitale (Theorem 5.1 of [**27**]) which gives a nice relation between holomorphic or skewholomorphic Jacobi forms, and Jacobi–Maass forms.

Theorem 2.5. *Let $F(\tau, z)$ be a function defined by Fourier series as in (1) (resp. (9)). Then, $F(\tau, z)$ is an holomorphic (resp. skew-holomorphic) Jacobi form of weight k and index m if and only if $\widehat{F}(\tau, z) = y^{k/2} F(\tau, z)$ is a Jacobi–Maass form of weight k (resp. $1 - k$), index m and eigenvalue $\lambda = \frac{(k-1/2)(k-5/2)}{2}$.*

3. The main theorem

We associate to each $F(\tau, z)$ in $\hat{J}^{nh}_{k,m}$ a system of $4m$ Dirichlet series

$$L^{\pm}_{\mu}(F,s) = \sum_{D=1}^{\infty} \frac{a_\mu(\pm D)}{D^s}, \qquad 1 \le \mu \le 2m,$$

and put $L^{\pm}(F,s) = \begin{pmatrix} L^{\pm}_1(F,s) \\ \vdots \\ L^{\pm}_{2m}(F,s) \end{pmatrix}$. By (24) these Dirichlet series converge absolutely and uniformly on some half-plane $\mathrm{Re}(s) \gg 0$ and thus they define holomorphic functions on that region. We also put

$$\begin{aligned}
\Lambda(F,s) &= \left(\frac{\pi}{2m}\right)^{-s}\left(\mathfrak{A}_{k-1/2,\ell}(s)L^{+}(F,s) + \mathfrak{A}_{-k+1/2,\ell}(s)L^{-}(F,s)\right),\\
\Xi(F,s) &= \left(\frac{\pi}{2m}\right)^{-s}\left(\mathfrak{A}_{k-1/2,\ell}(s+1)L^{+}(F,s) - \mathfrak{A}_{-k+1/2,\ell}(s+1)L^{-}(F,s)\right),\\
\Omega(F,s) &= -2\Xi(F,s) + \left(k-\frac{1}{2}\right)\Lambda(F,s).
\end{aligned}$$

Here $\Gamma(s)$ denotes Euler's gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t}$ and

$$\mathfrak{A}_{r,\ell}(s) = 2^{r/2}\frac{\Gamma\left(\frac{1+i\ell}{2}+s\right)\Gamma\left(\frac{1-i\ell}{2}+s\right)}{\Gamma(s+1-\frac{r}{2})}\, {}_2F_1\left(\frac{1-i\ell-r}{2}, \frac{1+i\ell-r}{2}, s-\frac{r}{2}+1; \frac{1}{2}\right) \tag{33}$$

where ${}_2F_1(a,b,c;d)$ denotes the Gaussian hypergeometric function

$${}_2F_1\left(a_1,a_2,b_1;z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(b_1)_k}\frac{z^k}{k!}.$$

From [**22**] p. 217 we have

$$\mathfrak{A}_{r,\ell}(s) = \int_0^\infty W_{\frac{r}{2},\frac{i\ell}{2}}(2t)t^s\frac{dt}{t}, \quad \text{for } \mathrm{Re}(s) \gg 0. \tag{34}$$

Thus $\mathfrak{A}_{r,\ell}(s)$ is the Mellin transform of a Whittaker function.
It will be clear from the proofs in the next sections that $\Lambda(F,s)$ and $\Xi(F,s)$ correspond to the Mellin transforms of the vector-valued functions $(f_1(\tau),\dots,f_{2m}(\tau))$ and $\frac{\partial}{\partial x}(f_1(\tau),\dots,f_{2m}(\tau))$, respectively. Observe that the Ω-function is just a linear combination of $\Lambda(F,s)$ and $\Xi(F,s)$. It is introduced here to make the statement of our main theorem below simpler.

Theorem 3.1. *For $1 \le \mu \le 2m$ and $D \in \mathbb{Z}, D \neq 0$ let $c_\mu, d_\mu, a_\mu(D) \in \mathbb{C}$ satisfying conditions (24), (25) and (26). The following statements are equivalent:*

(A) *$F(\tau,z)$ defined by (28) and (27) is a Jacobi–Maass form of weight k, index m and eigenvalue λ.*

(B) *The functions* $\Lambda(F,s)$ *and* $\Omega(F,s)$ *defined for* $\mathrm{Re}(s) \gg 0$ *have meromorphic continuation to* $\mathbb{C}$ *satisfying the functional equations*

$$\Lambda(F,-s) = i^k \mathcal{M} \cdot \Lambda(F,s)$$
$$\Omega(F,-s) = -i^k \mathcal{M} \cdot \Omega(F,s)$$

with each

$$\Lambda^*(F,s) = \Lambda(F,s) + P_\lambda(s)C + Q_\lambda(s)D + i^k P_\lambda(-s)\mathcal{M}\cdot C + i^k Q_\lambda(-s)\mathcal{M}\cdot D,$$
$$\Omega^*(F,s) = \Omega(F,s) + \left(k - \frac{1}{2}\right)\left(P_\lambda(s)C + Q_\lambda(s)D - i^k P_\lambda(-s)\mathcal{M}\cdot C - i^k Q_\lambda(-s)\mathcal{M}\cdot D\right)$$

entire and BVS (bounded on vertical strips), where $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{2m} \end{pmatrix}, D = \begin{pmatrix} d_1 \\ \vdots \\ d_{2m} \end{pmatrix}$, $\mathcal{M}$ *denotes the* $2m$*-by-*$2m$ *matrix* $\frac{1}{\sqrt{2m}}\left(e^{\pi i uv/m}\right)_{u,v}$ *and* $P_\lambda(s), Q_\lambda(s)$ *are rational functions of* s *of degree at most* 2 *given explicitly by (50).*

We should clarify here that the implication (A)⇒(B) in our main theorem is probably well known to the experts, although no published proof has been found by the author. It is the proof of the implication (B)⇒(A) the more technical one and the main contribution of this thesis. It is also the reason why Theorem 3.1 is considered as a converse theorem; it is the converse of the easier implication (A)⇒(B). In this thesis we give nevertheless full proofs for each implication for sake of completeness.

4. Applications

In this section we state some applications of our main result. The first application is a generalization of a converse theorem for holomorphic Jacobi cusp forms due to Y. Martin [**24**] to non-cuspidal Jacobi forms.

Corolary 4.1. *For $1 \leq \mu \leq 2m$ and $D \in \mathbb{Z}, D > 0$ let $c_\mu, a_\mu(D) \in \mathbb{C}$ satisfying conditions (24), (25) and (26). The following statements are equivalent:*

(A1) *$F(\tau, z)$ defined by (5) and (4) is a Jacobi form of weight k and index m.*

(B1) *The function $\Lambda(F, s)$ defined by*

$$\Lambda(F,s) = \left(\frac{\pi}{2m}\right)^{-s+\frac{1}{4}-\frac{k}{2}} \Gamma\left(s - \frac{1}{4} + \frac{k}{2}\right) L\left(F, s - \frac{1}{4} + \frac{k}{2}\right),$$

$$L(F,s) = \begin{pmatrix} L_1(F,s) \\ \vdots \\ L_{2m}(F,s) \end{pmatrix},$$

$$L_\mu(F,s) = \sum_{D=1}^{\infty} \frac{a_\mu(D)}{D^s}, \qquad 1 \leq \mu \leq 2m,$$

for $\mathrm{Re}(s) \gg 0$, *has meromorphic continuation to $\mathbb{C}$ satisfying the functional equation*

$$\Lambda(F, -s) = i^k \mathcal{M} \cdot \Lambda(F, s) \tag{35}$$

with

$$\Lambda^*(F,s) = \Lambda(F,s) + \frac{1}{s - \frac{1}{4} + \frac{k}{2}} C + \frac{i^k}{\frac{k}{2} - \frac{1}{4} - s} \mathcal{M} \cdot C \tag{36}$$

entire and BVS, where $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{2m} \end{pmatrix}$.

In the case of Jacobi cusp forms we have $C = 0$ in the above corollary. In particular $\Lambda(F, s)$ is entire. That is the case treated in [**24**].

The second application of our main theorem is a converse theorem for skew-holomorphic Jacobi forms, analogous to Corollary 4.1.

Corolary 4.2. *For $1 \leq \mu \leq 2m$ and $D \in \mathbb{Z}, D > 0$ let $c_\mu, a_\mu(D) \in \mathbb{C}$ satisfying conditions (24), (25) and (26). The following statements are equivalent:*

(A2) *$\widetilde{F}(\tau, z)$ defined by (12) and (11) is a skew-holomorphic Jacobi form of weight k and index m.*

(B2) *The function* $\Lambda(\widetilde{F},s)$ *defined by*

$$\Lambda(\widetilde{F},s) = \left(\frac{\pi}{2m}\right)^{-s+\frac{1}{4}-\frac{k}{2}} \Gamma\left(s-\frac{1}{4}+\frac{k}{2}\right) L\left(\widetilde{F}, s-\frac{1}{4}+\frac{k}{2}\right),$$

$$L(\widetilde{F},s) = \begin{pmatrix} L_1(\widetilde{F},s) \\ \vdots \\ L_{2m}(\widetilde{F},s) \end{pmatrix},$$

$$L_\mu(\widetilde{F},s) = \sum_{D=1}^{\infty} \frac{a_\mu(-D)}{D^s}, \qquad 1 \le \mu \le 2m,$$

for $\mathrm{Re}(s) \gg 0$, *has meromorphic continuation to* $\mathbb{C}$ *satisfying the functional equation*

$$\Lambda(\widetilde{F},-s) \;=\; i^{1-k}\mathcal{M}\cdot\Lambda(\widetilde{F},s)$$

with

$$\Lambda^*(\widetilde{F},s) \;=\; \Lambda(\widetilde{F},s) + \frac{1}{s-\frac{1}{4}+\frac{k}{2}}C + \frac{i^{1-k}}{\frac{k}{2}-\frac{1}{4}-s}\mathcal{M}\cdot C$$

entire and BVS, where $C = \begin{pmatrix} c_1 \\ \vdots \\ c_{2m} \end{pmatrix}$.

Our last application is a new proof of an explicit isomorphism between a certain subspace of weight $k-1/2$ Maass forms, the so-called Kohnen's plus space, and Jacobi–Maass forms of weight k and index 1. This result was first proved by Pitale (Theorems 4.4 and 4.5 of [**27**]).

Let's define, for k even, the Kohnen's plus space by

$$M^+_{k-1/2,\lambda}(4) = \{f \in M_{k-1/2,\lambda}(4) : a(n) = 0 \text{ whenever } n \equiv 1,2 \pmod 4\}.$$

It is possible to define Kohnen's plus space for general k but this definition suffices for our purposes.

For $f(\tau)$ in $M^+_{k-1/2,\lambda}(4)$ with Fourier–Whittaker expansion (19) put

(37) $$f_1(\tau) = \sum_{n\in\mathbb{Z}} W_{\mathrm{sgn}(n+3/4)\frac{k-1/2}{2},\frac{i\varrho}{2}}\left(\pi\,|4n+3|\,y\right) e^{\pi i(4n+3)x/2} a(4n+3),$$

(38) $$f_2(\tau) = U_{2\lambda}(y)c + V_{2\lambda}(y)d + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{k-1/2}{2},\frac{i\varrho}{2}}\left(\pi|4n|y\right) e^{\pi i 4nx/2} a(4n),$$

(39) $$F(\tau,z) = f_1(\tau)y^{1/4}\vartheta_{1,2}(\tau,z) + f_2(\tau)y^{1/4}\vartheta_{2,2}(\tau,z).$$

Our third and last application is the following.

Corolary 4.3. *Suppose* k *is an even integer. Let* $c, d, a(n) \in \mathbb{C}$ *for* $n \in \mathbb{Z}, n \neq 0$ *with at most polynomial growth in* $|n|$ *and with* $a(n) = 0$ *whenever* $n \equiv 1, 2$ *(mod 4). The following conditions are equivalent:*

(A3) $f(\tau)$ *defined by (19) is in* $M^+_{k-1/2,\lambda}(4)$.

(B3) $F(\tau, z)$ *defined by (39), (37) and (38) is a Jacobi–Maass form of weight* k*, index* 1 *and eigenvalue* 2λ.

In particular $f(\tau) \mapsto f_1(\tau)y^{1/4}\vartheta_{1,2}(\tau, z) + f_2(\tau)y^{1/4}\vartheta_{2,2}(\tau, z)$ *gives an isomorphism between* $M^+_{k-1/2,\lambda}(4)$ *and the space of Jacobi–Maass forms of weight* k*, index* 1 *and eigenvalue* 2λ.

5. An adaptation of Maass's converse theorem

In this section we prove a certain converse theorem which is based on the work of Maass [**20**] and [**22**]. We will see that the proof of Theorem 3.1 reduces to an application of this result.

Let's start with the statement. Fix real numbers r, N with $N > 0$. Let $p, q \in \mathbb{R}^+$, $c, d, \gamma, \eta, a(n), b(n) \in \mathbb{C}$ where $n \in \mathbb{Z}, n \neq 0$. Assume

$$a(n) = O(|n|^\sigma), b(n) = O(|n|^\sigma) \text{ when } |n| \to +\infty, \text{ for some } \sigma > 0. \tag{40}$$

Put

$$f(\tau) = U_\lambda(y)c + V_\lambda(y)d + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{r}{2},\frac{i\ell}{2}}\left(\frac{4\pi|n|y}{p}\right) e^{2\pi inx/p}a(n), \tag{41}$$

$$g(\tau) = U_\lambda(y)\gamma + V_\lambda(y)\eta + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{r}{2},\frac{i\ell}{2}}\left(\frac{4\pi|n|y}{q}\right) e^{2\pi inx/q}b(n). \tag{42}$$

Here $\lambda \in \mathbb{C}$ and $\ell^2 = -1 - 2\lambda$ (following the notation of previous sections). By straightforward computations we have $\triangle_r(f) = \frac{\lambda}{2}f$ and $\triangle_r(g) = \frac{\lambda}{2}g$. Define

$$L^\pm(f,s) = \sum_{n=1}^\infty \frac{a(\pm n)}{n^s}, \tag{43}$$

$$\Lambda(f,s) = \left(\frac{2\pi}{p\sqrt{N}}\right)^{-s}\left[\mathfrak{A}_{r,\ell}(s)L^+(f,s) + \mathfrak{A}_{-r,\ell}(s)L^-(f,s)\right], \tag{44}$$

$$\Xi(f,s) = \left(\frac{2\pi}{p\sqrt{N}}\right)^{-s}\left[\mathfrak{A}_{r,\ell}(s+1)L^+(f,s) - \mathfrak{A}_{-r,\ell}(s+1)L^-(f,s)\right], \tag{45}$$

$$\Omega(f,s) = -2\Xi(f,s) + r\Lambda(f,s), \tag{46}$$

and make similar definitions with g instead of f and q instead of p.

Our adaptation of Maass's converse theorem can be phrased as follows.

Theorem 5.1. *The following conditions are equivalent:*

(A4) $f\left(-\frac{1}{N\tau}\right) = \left(\frac{\tau}{|\tau|}\right)^r g(\tau)$.

(B4) *The functions* $\Lambda(f,s), \Omega(f,s), \Lambda(g,s), \Omega(g,s)$ *defined for* $\mathrm{Re}(s) \gg 0$ *have meromorphic continuation to* $\mathbb{C}$ *satisfying the functional equations*

$$\Lambda(f,-s) = i^r\Lambda(g,s), \tag{47}$$

$$\Omega(f,-s) = -i^r\Omega(g,s), \tag{48}$$

with each

$$\begin{aligned}\Lambda^*(f,s) &= \Lambda(f,s) + P_{\lambda,N}(s)c + Q_{\lambda,N}(s)d + i^rP_{\lambda,N}(-s)\gamma + i^rQ_{\lambda,N}(-s)\eta,\\ \Omega^*(f,s) &= \Omega(f,s) + r\Big(P_{\lambda,N}(s)c + Q_{\lambda,N}(s)d - i^rP_{\lambda,N}(-s)\gamma - i^rQ_{\lambda,N}(-s)\eta\Big),\end{aligned} \tag{49}$$

entire and BVS, where

$$(P_{\lambda,N}(s), Q_{\lambda,N}(s)) = \begin{cases} \left(\frac{N^{-\alpha/2}}{s+\alpha}, \frac{N^{-\beta/2}}{s+\beta}\right) & , \textit{ if } \lambda \neq -\frac{1}{2}, (\alpha,\beta) \textit{ solutions of } X(X-1) = \frac{\lambda}{2}, \\ \\ \left(\frac{N^{-1/4}}{s+1/2}, -\frac{N^{-1/4}}{(s+1/2)^2} - \frac{N^{-1/4}\log(\sqrt{N})}{s+1/2}\right) & , \textit{ if } \lambda = -\frac{1}{2}. \end{cases}$$

In the case $N = 1$ we simply write $(P_\lambda(s), Q_\lambda(s)) = (P_{\lambda,1}(s), Q_{\lambda,1}(s))$. In other words, we define

$$(P_\lambda(s), Q_\lambda(s)) = \begin{cases} \left(\frac{1}{s+\alpha}, \frac{1}{s+\beta}\right) & , \text{ if } \lambda \neq -\frac{1}{2}, (\alpha,\beta) \text{ as in (20)}, \\ \\ \left(\frac{1}{s+1/2}, -\frac{1}{(s+1/2)^2}\right) & , \text{ if } \lambda = -\frac{1}{2}. \end{cases} \tag{50}$$

Of course we could have stated Theorem 5.1 for functions with eigenvalue λ instead of $\lambda/2$. The reason why we have chosen the above formulation is that it is exactly the statement that we are going to use in the proof of Theorem 3.1. Maass's original work treats the case $\lambda = 0$.

Proof of Theorem 5.1. Suppose (A4) holds. By (40) and (16) the expression

$$\int_0^\infty \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{r}{2}, \frac{i\ell}{2}}\left(\frac{4\pi|n|y}{p\sqrt{N}}\right) a(n) y^s \frac{dy}{y}$$

is absolutely convergent for $\mathrm{Re}(s) \gg 0$. Interchanging summation and integration, and using (34), we obtain

$$\int_0^\infty \left[f\left(\frac{iy}{\sqrt{N}}\right) - U_\lambda\left(\frac{y}{\sqrt{N}}\right)c - V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right] y^s \frac{dy}{y} = \Lambda(f,s) \tag{51}$$

for $\mathrm{Re}(s) \gg 0$. Similarly we have

$$\int_0^\infty \left[g\left(\frac{iy}{\sqrt{N}}\right) - U_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma - V_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right] y^s \frac{dy}{y} = \Lambda(g,s),$$

$$\frac{-i}{\sqrt{N}}\int_0^\infty \frac{\partial f}{\partial x}\left(\frac{iy}{\sqrt{N}}\right) y^{s+1}\frac{dy}{y} = \Xi(f,s),$$

$$\frac{-i}{\sqrt{N}}\int_0^\infty \frac{\partial g}{\partial x}\left(\frac{iy}{\sqrt{N}}\right) y^{s+1}\frac{dy}{y} = \Xi(g,s),$$

for $\mathrm{Re}(s) \gg 0$. Making the change of variables $y \mapsto 1/y$ and using (A4), we conclude the equality

$$\int_0^1 f\left(\frac{iy}{\sqrt{N}}\right) y^s \frac{dy}{y} = i^r \int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right) y^{-s}\frac{dy}{y}.$$

Combining this with the identities

$$\int_0^1 U_\lambda\left(\frac{y}{\sqrt{N}}\right) y^s\frac{dy}{y} = P_{\lambda,N}(s), \qquad \int_0^1 V_\lambda\left(\frac{y}{\sqrt{N}}\right) y^s\frac{dy}{y} = Q_{\lambda,N}(s),$$

$$\int_1^\infty U_\lambda\left(\frac{y}{\sqrt{N}}\right) y^{-s}\frac{dy}{y} = -P_{\lambda,N}(-s), \qquad \int_1^\infty V_\lambda\left(\frac{y}{\sqrt{N}}\right) y^{-s}\frac{dy}{y} = -Q_{\lambda,N}(-s),$$

and (51), we get that $\Lambda(f,s)$ equals

$$\left(\int_0^1+\int_1^\infty\right)\left[f\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)c-V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^s\frac{dy}{y}$$
$$=i^r\int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right)y^{-s}\frac{dy}{y}-P_{\lambda,N}(s)c-Q_{\lambda,N}(s)d+\int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)c-V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^s\frac{dy}{y}$$
$$=i^r\int_1^\infty\left[g\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma-V_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right]y^{-s}\frac{dy}{y}-i^rP_{\lambda,N}(-s)\gamma-i^rQ_{\lambda,N}(-s)\eta$$
$$-P_{\lambda,N}(s)c-Q_{\lambda,N}(s)d+\int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)c-V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^s\frac{dy}{y}.$$

This gives us

$$\begin{aligned}\Lambda^*(f,s)=\;&\int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)c-V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^s\frac{dy}{y}\\ &+i^r\int_1^\infty\left[g\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma-V_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right]y^{-s}\frac{dy}{y}.\end{aligned}\tag{52}$$

From (40) and (16) is easy to see that both $f(iy)-U_\lambda(y)c-V_\lambda(y)d$ and $g(iy)-U_\lambda(y)\gamma-V_\lambda(y)\eta$ have exponential decay as $y\to+\infty$. We conclude that $\Lambda^*(f,s)$ has analytic continuation to an entire BVS function. Similarly, we obtain

$$\begin{aligned}\Lambda^*(g,s)=\;&\int_1^\infty\left[g\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma-V_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right]y^s\frac{dy}{y}\\ &+i^{-r}\int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-U_\lambda\left(\frac{y}{\sqrt{N}}\right)c-V_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^{-s}\frac{dy}{y}\end{aligned}\tag{53}$$

where $\Lambda^*(g,s)$ is defined as in (49) making the obvious substitution. Comparing (52) with (53) we deduce $\Lambda^*(F,-s)=i^r\Lambda^*(G,s)$ which implies (47).

For the Ω-functions we start by observing that

$$\begin{aligned}\Omega(f,s)&=\int_0^\infty\left[H\left(\frac{iy}{\sqrt{N}}\right)-rU_\lambda\left(\frac{y}{\sqrt{N}}\right)c-rV_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right]y^s\frac{dy}{y},\\ \Omega(g,s)&=\int_0^\infty\left[I\left(\frac{iy}{\sqrt{N}}\right)-rU_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma-rV_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right]y^s\frac{dy}{y},\end{aligned}$$

for $\mathrm{Re}(s)\gg0$, where

$$H(\tau)=2iy\frac{\partial f}{\partial x}(\tau)+rf(\tau),\qquad I(\tau)=2iy\frac{\partial g}{\partial x}(\tau)+rg(\tau).$$

By straightforward computations we have

$$H\left(\frac{-1}{Niy}\right)=-i^rI(iy).\tag{54}$$

Using (54) and following the arguments used to prove (52), we get

$$\begin{aligned}\Omega^*(f,s) &= \int_1^\infty \left[H\left(\frac{iy}{\sqrt{N}}\right) - rU_\lambda\left(\frac{y}{\sqrt{N}}\right)c - rV_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right] y^s\frac{dy}{y} \\ &\quad -i^r\int_1^\infty \left[I\left(\frac{iy}{\sqrt{N}}\right) - rU_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma - rV_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right] y^{-s}\frac{dy}{y}, \\ \Omega^*(g,s) &= \int_1^\infty \left[I\left(\frac{iy}{\sqrt{N}}\right) - rU_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma - rV_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta\right] y^s\frac{dy}{y} \\ &\quad -i^{-r}\int_1^\infty \left[H\left(\frac{iy}{\sqrt{N}}\right) - rU_\lambda\left(\frac{y}{\sqrt{N}}\right)c - rV_\lambda\left(\frac{y}{\sqrt{N}}\right)d\right] y^{-s}\frac{dy}{y}.\end{aligned}$$

As before we conclude that $\Omega^*(f,s)$ has analytic continuation to an entire BVS function satisfying $\Omega^*(f,-s) = -i^r\Omega^*(g,s)$, which in turns implies (48). This proves that (A4) implies (B4).

We now assume (B4). Choose $K \gg 0$ such that (44) holds for $\mathrm{Re}(s) = K$. By Stirling's asymptotic formula for $\Gamma(s)$ we deduce that $\mathfrak{A}_{\pm r,\ell}(s)$ has exponential decay as $|\mathrm{Im}(s)| \to +\infty$ when $\mathrm{Re}(s) = K$ (note that the extra factor coming from the Gaussian hypergeometric function is bounded when $|\mathrm{Im}(s)| \to +\infty$ and $\mathrm{Re}(s)$ is fixed). It follows that $\Lambda(f,s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $\mathrm{Re}(s) = K$. Applying the same argument for $\Lambda(g,s)$ and using the functional equation (47) we get $\Lambda(f,s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $\mathrm{Re}(s) = -K$. By comparison we have $\Lambda^*(f,s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $\mathrm{Re}(s) = \pm K$. We now make use of the classical Phragmén-Lindelöf Principle (Lemma 4.3.4. of [**25**]).

Lemma 5.2 (Phragmén-Lindelöf Principle)**.** *For two real numbers v_1, v_2 with $v_1 < v_2$, put*

$$\mathfrak{F} = \{s \in \mathbb{C} : v_1 \le \mathrm{Re}(s) \le v_2\}.$$

Let Φ be a holomorphic function on a domain containing $\mathfrak{F}$ satisfying

$$|\Phi(s)| = O(e^{|\mathrm{Im}(s)|^\delta}), \text{ for } |\mathrm{Im}(s)| \to +\infty$$

uniformly on $\mathfrak{F}$ with $\delta > 0$. For a real number b, if

$$|\Phi(s)| = O(|\mathrm{Im}(s)|^b), \text{ for } |\mathrm{Im}(s)| \to +\infty, \mathrm{Re}(s) \in \{v_1, v_2\}$$

then

$$|\Phi(s)| = O(|\mathrm{Im}(s)|^b), \text{ for } |\mathrm{Im}(s)| \to +\infty, \text{uniformly on } \mathfrak{F}.$$

A direct application of the Phragmén-Lindelöf Principle gives us $\Lambda^*(f,s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $-K \le \mathrm{Re}(s) \le K$, which implies the same bound for $\Lambda(f,s)$. From (47) we conclude that the same bound holds for $\Lambda(g,s)$ and the same argument applies for the functions $\Omega(f,s)$ and $\Omega(g,s)$.

Now we use (34) and Mellin inversion theorem to get

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=K} \mathfrak{A}_{r,\ell}(s)t^{-s}ds = W_{\frac{r}{2},\frac{i\ell}{2}}(2t)$$

as long as $K \gg 0$. We deduce

$$f\left(\frac{iy}{\sqrt{N}}\right) - U_\lambda\left(\frac{y}{\sqrt{N}}\right)c - V_\lambda\left(\frac{y}{\sqrt{N}}\right)d = \frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a(n)\int_{\mathrm{Re}(s)=K} \mathfrak{A}_{\mathrm{sgn}(n)r,\ell}(s)\left(\frac{2\pi|n|y}{p\sqrt{N}}\right)^{-s} ds$$

for $K \gg 0$. The expression above is absolutely convergent so we can interchange summation and integration and get

$$f\left(\frac{iy}{\sqrt{N}}\right) - U_\lambda\left(\frac{y}{\sqrt{N}}\right)c - V_\lambda\left(\frac{y}{\sqrt{N}}\right)d = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=K} \Lambda(f,s)y^{-s}ds. \tag{55}$$

Similarly

$$g\left(\frac{iy}{\sqrt{N}}\right) - U_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma - V_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=K} \Lambda(g,s)y^{-s}ds. \tag{56}$$

Since $\Lambda^*(f,s)$ is entire we have precise information about the poles of $\Lambda(f,s)$ and thus about the poles of $\Lambda(f,s)y^{-s}$. We can therefore apply the residue theorem to compute the line integral of $\Lambda(f,s)y^{-s}$ over the boundary of a big rectangle with vertical sides on $\mathrm{Re}(s) = -K$ and $\mathrm{Re}(s) = K$ respectively, use that $\Lambda(f,s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $|\mathrm{Re}(s)| \leq K$, and get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=K} \Lambda(f,s)y^{-s}ds - \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-K} \Lambda(f,s)y^{-s}ds\\ &= -U_\lambda\left(\frac{y}{\sqrt{N}}\right)c - V_\lambda\left(\frac{y}{\sqrt{N}}\right)d + i^rU_\lambda\left(\frac{1}{y\sqrt{N}}\right)\gamma + i^rV_\lambda\left(\frac{1}{y\sqrt{N}}\right)\eta\end{aligned} \tag{57}$$

for $K \gg 0$. Combining (55) and (57) we deduce

$$f\left(\frac{iy}{\sqrt{N}}\right) - i^rU_\lambda\left(\frac{1}{y\sqrt{N}}\right)\gamma - i^rU_\lambda\left(\frac{1}{y\sqrt{N}}\right)\eta = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-K} \Lambda(f,s)y^{-s}ds.$$

Making the change of variables $s \mapsto -s$, putting y^{-1} instead of y and using the functional equation (47) we get

$$f\left(\frac{i}{y\sqrt{N}}\right) - i^rU_\lambda\left(\frac{y}{\sqrt{N}}\right)\gamma - i^rU_\lambda\left(\frac{y}{\sqrt{N}}\right)\eta = \frac{i^r}{2\pi i}\int_{\mathrm{Re}(s)=K} \Lambda(g,s)y^{-s}ds.$$

By (56) we conclude

$$f\left(\frac{i}{y\sqrt{N}}\right) = i^r g\left(\frac{iy}{\sqrt{N}}\right). \tag{58}$$

Imitating all the previous computations, using H and $-I$ instead of f and g, we get

$$H\left(\frac{i}{y\sqrt{N}}\right) = -i^r I\left(\frac{iy}{\sqrt{N}}\right). \tag{59}$$

We claim that (58) together with (59) implies (A4). Indeed, define $A(\tau) = f\left(-\frac{1}{N\tau}\right)\left(\frac{\tau}{|\tau|}\right)^{-r} - g(\tau)$. It can be seen that

$$\triangle_r\left(f\left(-\frac{1}{N\tau}\right)\left(\frac{\tau}{|\tau|}\right)^{-r}\right) = \triangle_r(f)\left(-\frac{1}{N\tau}\right)\left(\frac{\tau}{|\tau|}\right)^{-r} = \frac{\lambda}{2}f\left(-\frac{1}{N\tau}\right)\left(\frac{\tau}{|\tau|}\right)^{-r}$$

thus $\triangle_r(A) = \frac{\lambda}{2}A$. It is a well known result from the theory of differential equations that solutions of elliptic equations with real-analytic coefficients, such as $(\triangle_r - \lambda/2)A = 0$, are themselves real-analytic in the interior of their domain (see [**26**]). It follows that $A(\tau)$ is real-analytic in $\mathbb{H}$ and we can write $A(\tau) = \sum_{n=0}^{\infty} x^n A_n(y)$ with

$$y^2((n+2)(n+1)A_{n+2}(y) + A_n''(y)) - riy(n+1)A_{n+1}(y) = \frac{\lambda}{2}A_n(y)\ , \text{ for } n \geq 0. \tag{60}$$

We conclude that

$$A_{n+2}(y) = \frac{\frac{\lambda}{2}A_n(y) - y^2 A_n''(y) + riy(n+1)A_{n+1}(y)}{y^2(n+2)(n+1)}\ , \text{ for } n \geq 0.$$

By (58) and (59) we have $A_0(y) = A_1(y) = 0$ thus $A_n(y) = 0$ for all $n \geq 0$ and we conclude $A(\tau) = 0$. This proves $(A4)$ and completes the proof of Theorem 5.1. □

6. Proof of Theorem 3.1

By Theorem 2.4 we know that $F(\tau,z)=\sum_{\mu=1}^{2m} f_\mu(\tau)\tilde{\vartheta}_{\mu,2m}(\tau,z)$ is a Jacobi–Maass form of the respective weight, index and eigenvalue, if and only if the system of functions $f_1(\tau),\dots,f_{2m}(\tau)$ satisfy conditions (29) to (32). Conditions (31) and (32) follow from (27). Condition (29) follows from (27) together with (25) and (26). We conclude that (A) is equivalent to the transformation formulas

$$f_\mu\left(-\frac{1}{\tau}\right)=\left(\frac{\tau}{|\tau|}\right)^{k-1/2} g_\mu(\tau)\ , \qquad 1\le\mu\le 2m$$

where

$$g_\mu\left(\tau\right)=\frac{i^{1/2}}{\sqrt{2m}}\sum_{j=1}^{2m} e^{\pi i\mu j} f_j(\tau). \tag{61}$$

Using (27) and following (41)-(46), with $N=1, r=k-\frac{1}{2}, p=4m$, we can define $L^{\pm}(f_\mu,s)$, $\Lambda(f_\mu,s)$, $\Xi(f_\mu,s)$, $\Omega(f_\mu,s)$. It is clear that $L^{\pm}(f_\mu,s)=L^{\pm}_\mu(F,s)$. We also have

$$\Lambda(F,s)=\begin{pmatrix}\Lambda(f_1,s)\\ \vdots\\ \Lambda(f_{2m},s)\end{pmatrix},\quad \Xi(F,s)=\begin{pmatrix}\Xi(f_1,s)\\ \vdots\\ \Xi(f_{2m},s)\end{pmatrix},\quad \Omega(F,s)=\begin{pmatrix}\Omega(f_1,s)\\ \vdots\\ \Omega(f_{2m},s)\end{pmatrix}.$$

Similarly, we can define $L^{\pm}(g_\mu,s),\Lambda(g_\mu,s),\Xi(g_\mu,s),\Omega(g_\mu,s)$ using (61) and $q=4m$. We have $L^{\pm}(g_\mu,s)=i^{1/2}\mathcal{M}\cdot L^{\pm}(F,s)$ and the same relation holds for the Λ and Ω-functions. In this context our main theorem is a direct application of Theorem 5.1 to the pairs of functions $f_\mu(\tau),g_\mu(\tau)$ with $1\le\mu\le 2m$. □

7. Proof of Corollaries 4.1 and 4.2

From Theorem 2.5 we have that $F(\tau, z)$ is a Jacobi form of weight k and index m if and only if $\widehat{F}(\tau, z) = y^{k/2}F(\tau, z)$ is a Jacobi–Maass form of weight k index m and eigenvalue $\lambda = \frac{(k-1/2)(k-5/2)}{2}$ (observe that $\lambda \neq -\frac{1}{2}$). It is clear that $\widehat{F}(\tau, z)$ has a ϑ-decomposition

$$\widehat{F}(\tau, z) = \sum_{\mu=1}^{2m} \widehat{f}_\mu(\tau) y^{1/4} \vartheta_{\mu,2m}(\tau, z)$$

where $\widehat{f}_\mu(\tau) = y^{k/2-1/4} f_\mu(\tau)$. In this context we have $W_{\frac{k-1/2}{2}, \frac{i\ell}{2}}(t) = e^{-t/2}t^{k/2-1/4}$ (see (17)) so we can write

$$\widehat{f}(\tau) = y^{k/2-1/4}c_\mu + \left(\frac{\pi}{m}\right)^{-k/2+1/4} \sum_{D=1}^{\infty} W_{\frac{k-1/2}{2}, \frac{i\ell}{2}}\left(\frac{\pi D y}{m}\right) e^{\pi i D x/2m} a_\mu(D) D^{-k/2+1/4}.$$

Using the above Fourier–Whittaker expansion we get

$$L_\mu^+(\widehat{F}, s) = \left(\frac{\pi}{m}\right)^{-k/2+1/4} \sum_{D=1}^{\infty} \frac{a_\mu(D) D^{-k/2+1/4}}{D^s} = \left(\frac{\pi}{m}\right)^{-k/2+1/4} L_\mu(F, s - 1/4 + k/2),$$

$$L_\mu^-(\widehat{F}, s) = 0.$$

Similarly, using $\mathfrak{A}_{k-\frac{1}{2},\ell}(s) = 2^{k/2-1/4}\Gamma\left(s - 1/4 + k/2\right)$, we have

$$\Lambda(\widehat{F}, s) = \Lambda(F, s),$$
$$\Xi(\widehat{F}, s) = (s - 1/4 + k/2)\Lambda(F, s),$$
$$\Omega(\widehat{F}, s) = -2s\Lambda(F, s).$$

Choosing $\alpha = k/2 - 1/4$ in (50) gives $P_\lambda(s) = \frac{1}{s-1/4+k/2}$ thus

(62) $$\Lambda^*(\widehat{F}, s) = \Lambda^*(F, s),$$

(63) $$\Omega^*(\widehat{F}, s) = -2s\Lambda(F, s) + \left(k - \frac{1}{2}\right)\left(\frac{1}{s - \frac{1}{4} + \frac{k}{2}}C - \frac{i^k}{\frac{k}{2} - \frac{1}{4} - s}\mathcal{M} \cdot C\right).$$

If (A1) holds then $\widehat{F}$ is a Jacobi–Maass form and by Theorem 3.1 the function $\Lambda(\widehat{F}, s)$ has meromorphic continuation satisfying the functional equation $\Lambda(\widehat{F}, -s) = i^r\mathcal{M} \cdot \Lambda(\widehat{F}, s)$ with $\Lambda^*(\widehat{F}, s)$ entire and BVS. By the above equalities we conclude that (B1) holds.
Now suppose (B1) holds. Again, by the above computations, it follows directly that $\Lambda(\widehat{F}, s)$ has meromorphic continuation satisfying $\Lambda(\widehat{F}, -s) = i^r\mathcal{M} \cdot \Lambda(\widehat{F}, s)$ with $\Lambda^*(\widehat{F}, s)$ entire and BVS. It also follows that $\Omega(\widehat{F}, s)$ has meromorphic continuation satisfying $\Omega(\widehat{F}, -s) = -i^r\mathcal{M} \cdot \Omega(\widehat{F}, s)$ with $\Omega^*(\widehat{F}, s)$ entire. Using Stirling's asymptotic formula for $\Gamma(s)$ and the Phragmén-Lindelöf Principle is easy to see that $\Lambda(F, s) = O(|\mathrm{Im}(s)|^{-1})$ for $|\mathrm{Im}(s)| \to +\infty$ and $\mathrm{Re}(s)$ bounded, thus $\Omega^*(\widehat{F}, s)$ is BVS. By Theorem 3.1 we conclude that $\widehat{F}$ is a Jacobi–Maass form which implies that F is a Jacobi form. This completes the proof of Corollary 4.1.

It is not difficult to see that the above arguments also apply in the case of skew-holomorphic Jacobi forms. In this case we use that $\widetilde{F}(\tau, z)$ is a skew-holomorphic Jacobi form of weight k and index m if and only if $y^{k/2}\widetilde{F}(\tau, z)$ is a Jacobi–Maass form of weight $1-k$, index m and eigenvalue $\lambda = \frac{(k-1/2)(k-5/2)}{2}$. We will omit the details of the proof since it follows closely the arguments given above. □

8. Proof of Corollary 4.3

Since $\Gamma_0(4)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ we deduce that $f(\tau)$ is in $M^{+}_{k-1/2,\lambda}(4)$ if and only if it satisfies the following conditions:

(1) $f(\tau+1) = f(\tau)$,
(2) $f\left(\frac{\tau}{4\tau+1}\right) = \left(\frac{4\tau+1}{|4\tau+1|}\right)^{k-1/2} f(\tau)$,
(3) $\triangle_{k-1/2} f = \lambda f$,
(4) for some $\sigma > 0$, $f(\tau) = O(y^{\sigma})$ as $y \to +\infty$.

From (19) it follows that $f(\tau)$ satisfies conditions 1, 3 and 4. We conclude that (A4) is equivalent to condition 2. Making the substitution $\tau \mapsto \tau - \frac{1}{4}$, condition 2 becomes

$$f\left(-\frac{1}{16\tau} + \frac{1}{4}\right) = \left(\frac{\tau}{|\tau|}\right)^{k-1/2} f\left(\tau - \frac{1}{4}\right)$$

or equivalently

$$g\left(-\frac{1}{16\tau}\right) = \left(\frac{\tau}{|\tau|}\right)^{k-1/2} h\left(\tau\right)$$

where

$$g(\tau) = f\left(\tau + \frac{1}{4}\right) = U_{2\lambda}(y)c + V_{2\lambda}(y)d + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{k-1/2}{2}, \frac{i\varrho}{2}}\left(4\pi|n|y\right) e^{2\pi i n x} a(n) i^{n},$$

$$h(\tau) = f\left(\tau - \frac{1}{4}\right) = U_{2\lambda}(y)c + V_{2\lambda}(y)d + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} W_{\mathrm{sgn}(n)\frac{k-1/2}{2}, \frac{i\varrho}{2}}\left(4\pi|n|y\right) e^{2\pi i n x} a(n) i^{-n}.$$

Following (41)-(46) with $N = 16, r = k - 1/2, p = q = 1$ and $\ell = \varrho$, we can define the L, Λ, Ξ, Ω-functions associated to $g(\tau)$ and $h(\tau)$. We can also consider the corresponding functions associated to $F(\tau, z)$ in Theorem 3.1, choosing $m = 1, \ell = \varrho$. Simple computations yield

$$\begin{pmatrix} L^{\pm}(g,s) \\ L^{\pm}(h,s) \end{pmatrix} = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \cdot L^{\pm}(F,s) \tag{64}$$

and the same relation holds for the Λ, Ξ and Ω-functions. The functional equation $\Lambda(g,-s) = i^{k-1/2}\Lambda(h,s)$ is equivalent to

$$\begin{pmatrix} \Lambda(g,-s) \\ \Lambda(h,-s) \end{pmatrix} = \begin{pmatrix} 0 & i^{k-1/2} \\ i^{k+1/2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \Lambda(g,s) \\ \Lambda(h,s) \end{pmatrix}. \tag{65}$$

Using the corresponding relation (64) for the Λ-functions we deduce that the above functional equation is equivalent to

$$\Lambda(F,-s) = \frac{i^k}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \Lambda(F,s), \tag{66}$$

and a similar equivalence holds for the Ω-functions. Straightforward computations show that the analytic conditions in (B4) of Theorem 5.1 applied to $g(\tau)$ and $h(\tau)$ are equivalent to the corresponding analytic conditions in (B) of Theorem 3.1 applied to $F(\tau, z)$. The desired equivalence follows directly from these results. $\square$

CHAPTER 2

Asymptotic distribution of Hecke points over $\mathbb{C}_p$ and applications

Introduction to Chapter 2

Let K be an algebraically closed field and $\mathrm{Ell}(K)$ be the moduli space of elliptic curves over K. The usual j-invariant from the theory of elliptic curves gives us an identification $\mathrm{Ell}(K) \simeq K$. Given $\overline{E} \in \mathrm{Ell}(K)$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$ consider the divisor[1] over K given by

$$T_n(\overline{E}) = \sum_{C \leq E, |C| = n} [j(E/C)].$$

In this chapter we are interested in the asymptotic distribution of $T_n(\overline{E})$ when $K = \mathbb{C}_p$, as n goes to infinity.

In the classical case, i.e. when $K = \mathbb{C}$, the asymptotic distribution of $T_n(\overline{E})$ is well understood. In this case, the uniformization theory of elliptic curves over $\mathbb{C}$ gives us an identification $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} \simeq \mathrm{Ell}(\mathbb{C})$, where the group $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ as Möbius transformations. On $\mathbb{H}$ we have the hyperbolic measure $d\mu_{\mathbb{H}} = \frac{dxdy}{y^2}$, where $x = \mathrm{Re}(\tau)$ and $y = \mathrm{Im}(\tau)$. This measure is invariant under the action of $\mathrm{SL}(2, \mathbb{Z})$ and induces a measure $d\mu$ on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ with $d\mu(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = \frac{\pi}{3}$. The asymptotic distribution of $T_n(\overline{E})$ is then "governed" by the probability measure on $\mathbb{C}$ induced by $d\nu = \frac{3}{\pi} d\mu$ via the identifications $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} \simeq \mathrm{Ell}(\mathbb{C}) \simeq \mathbb{C}$ (see [**4**] for a precise statement of this result).

The purpose of this chapter is to study the p-adic analogue of the above result. There are some interesting differences between the classical and the p-adic case that we would like to point out here. Firstly, the asymptotic distribution of $T_n(\overline{E})$ depends on the type of reduction of $\overline{E}$ mod $\mathcal{M}_p$, where $\mathcal{M}_p$ denotes the unique maximal ideal of the ring of integers of $\mathbb{C}_p$. Secondly, in the case where $\overline{E}$ has supersingular reduction mod $\mathcal{M}_p$, the asymptotic distribution of $T_n(\overline{E})$ depends also on $\|n\|_p$, the p-adic norm of n. Furthermore, in the cases where we have a precise description of the asymptotic distribution of $T_n(\overline{E})$, the probability measure that describes this distribution is not supported on $\mathbb{C}_p$. Instead, this probability measure is supported at a single point of the Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}$ over $\mathbb{C}_p$.

Our work has application to the asymptotic distribution of singular moduli over $\mathbb{C}_p$ and to the finiteness of certain sets of singular moduli over $\mathbb{C}$, generalizing a result of Habegger [**12**].

This chapter is organized as follows: In Section 1 we fix notations and recall some basic facts about elliptic curves, isogenies, reduction of elliptic curves over $\mathbb{C}_p$ and Berkovich spaces, among other

[1] A divisor on a space X will be denoted by $\sum_{x \in X} n_x[x]$ where $n_x \in \mathbb{Z}$.

concepts needed to state our results. In Sections 2 and 3 our main theorem and some applications are stated. In sections 4, 5 and 6 we make a detailed study of the asymptotic distribution of $T_n(\overline{E})$ associated to an isomorphism class of elliptic curves $\overline{E}$ with bad, ordinary and supersingular reduction, respectively. The proof of the main theorem is given in Section 7 and the proofs of the applications are given in Sections 8 and 9.

1. Preliminaries

Here and throughout this section K will denote an algebraically closed field.

1.1. Elliptic curves. We refer the reader to [**31**] and [**19**] for the general theory of elliptic curves.

An elliptic curve E over K is a smooth projective curve of genus 1 defined over K together with a point $O \in E(K)$, where $E(K)$ denotes the set of K-rational points of E.

Every elliptic curve admits a Weierstrass model

$$E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3, \tag{67}$$

where $[X : Y : Z] \in \mathbb{P}^2$, with O corresponding to $[0 : 1 : 0]$. Moreover, there is a natural group law

$$\oplus : E \times E \to E$$

which makes E into a commutative algebraic group and where O corresponds to the neutral element.

We mention here that one can also define elliptic curves over commutative rings, but these will play a minor role in this thesis.

1.2. The j-invariant. To every elliptic curve E with Weierstrass model (67) one can attached the quantities

$$\begin{aligned}
b_2 &= a_1^2 + 4a_2, \\
b_4 &= 2a_4 + a_1a_3, \\
b_6 &= a_3^2 + 4a_6, \\
b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \\
c_4 &= b_2^2 - 24b_4, \\
c_6 &= b_2^3 + 36b_2b_4 - 216b_6, \\
\triangle &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \\
j &= c_4^3/\triangle.
\end{aligned}$$

It is known that $\triangle \neq 0$. This is called the discriminant of E and its non-vanishing is equivalent to E being smooth. From this it follows that $j \in K$. This is the so-called j-invariant of E. It is independent of the Weierstrass model chosen. Moreover, we have the following result.

Theorem 1.1. *Two elliptic curves over K are isomorphic (as algebraic groups) if and only if they have the same j-invariant. Moreover, every $j \in K$ is the j-invariant of some elliptic curve over K.*

1.3. The moduli space of elliptic curves. Let $\mathrm{Ell}(K)$ be the moduli space of elliptic curves over K. This is, $\mathrm{Ell}(K)$ is the space of isomorphism classes of elliptic curves over K. An element of $\mathrm{Ell}(K)$ will be denoted by $\overline{E}$ where E is any elliptic curve in the isomorphism class.

From Theorem 1.1 it follows that the j-invariant defines a bijection $j : \mathrm{Ell}(K) \to K$, which will be denoted by $\overline{E} \mapsto j(E)$.

1.4. Isogenies. An isogeny between two elliptic curves E, E' is a rational map $\phi : E \to E'$ satisfying $\phi(O) = O'$, where O, O' are the respective neutral elements. Every isogeny is a morphism of algebraic groups. If ϕ is not constant (equal to O') then ϕ is surjective.

Examples: (a) For every $n \in \mathbb{Z}$ the map $[n] : E \to E$ given by

$$[n](P) = P \oplus P \oplus \cdots \oplus P \ (n \text{ times})$$

is an isogeny. This is called the multiplication-by-n isogeny.

(b) Assume that $\mathrm{char}(K) = p > 0$ and that E is given by the Weierstrass model (67). Define $E^{(p)}$ as the elliptic curve with Weierstrass model

$$E^{(p)} : Y^2Z + a_1^pXYZ + a_3^pYZ^2 = X^3 + a_2^pX^2Z + a_4^pXZ^2 + a_6^pZ^3. \tag{68}$$

The map $E \to E^{(p)}$ given by $[X : Y : Z] \mapsto [X^p, Y^p, Z^p]$ is an isogeny, called the Frobenius map.

If E is an elliptic curve, we denote by $K(E)$ the function field of E. An isogeny $\phi : E \to E'$ induces, by composition with ϕ, a field morphism $\phi^* : K(E') \to K(E)$. If ϕ is constant, we have $\phi^*(K(E)) = \{0\}$. If ϕ is not constant, then $K(E)$ is a finite field extension of $\phi^*(K(E'))$. In this case one defines

$$\deg(\phi) = [K(E) : \phi^*(K(E'))],$$
$$\deg_s(\phi) = [K(E) : \phi^*(K(E'))]_s,$$
$$\deg_i(\phi) = [K(E) : \phi^*(K(E'))]_i,$$

and extend this definition by putting $\deg(0) = \deg_s(0) = \deg_i(0) = 0$. The functions $\deg, \deg_s, \deg_i$ are multiplicative and satisfy $\deg = \deg_s \cdot \deg_i$.

A non-constant isogeny is called separable, inseparable or purely inseparable if the field extension $K(E) \supset \phi^*(K(E'))$ is separable, inseparable or purely inseparable, respectively. Note that, when $\mathrm{char}(K) = 0$, all isogenies are separable.

Examples: (a) For every $n \in \mathbb{Z}$ coprime to $\mathrm{char}(K)$ the isogeny $[n] : E \to E$ is separable and $\deg([n]) = n^2$.

(b) Assume that $\mathrm{char}(K) = p$ and that $\phi : E \to E^{(p)}$ is the Frobenius map. Then ϕ is purely inseparable and $\deg(\phi) = p$.

For any non-constant isogeny $\phi : E \to E'$ the set

$$\mathrm{Ker}(\phi) = \{P \in E : \phi(P) = O'\}$$

is a finite subgroup of E of cardinality $\deg_s(\phi)$. In particular, if we define

$$E[n] = \mathrm{Ker}([n]), \ \text{for } n \in \mathbb{Z},$$

then $E[n]$ is a subgroup of cardinality n^2 when n is coprime to $\mathrm{char}(K)$.

Theorem 1.2. *Let C be a finite subgroup of an elliptic curve E. There exists an elliptic curve E' and a separable isogeny $\phi : E \to E'$ with $\mathrm{Ker}(\phi) = C$. Moreover, if E'' is another elliptic curve and $\psi : E \to E''$ is a separable isogeny with $\mathrm{Ker}(\psi) = C$, then there exists an isomorphism $\sigma : E' \to E''$ with $\sigma \circ \phi = \psi$.*

Since the curve E' in the above theorem is uniquely determined (up to isomorphism) by E and the subgroup C, we write $E' = E/C$. We also denote by $E \to E/C$ any separable isogeny with Kernel equal to C, as in the above theorem.

Given two isogenies $\phi : E \to E', \psi : E \to E''$ we say that they are equivalent if there exists an isomorphism $\sigma : E' \to E''$ satisfying $\sigma \circ \phi = \psi$. The above theorem implies that the isogeny $E \to E/C$ is unique up to equivalence.

1.5. The dual isogeny. If $\phi : E \to E'$ is a non-constant isogeny, then there exists a unique isogeny $\widehat{\phi} : E' \to E$ satisfying

$$\widehat{\phi} \circ \phi = [\deg(\phi)] \text{ on } E.$$

This isogeny is called the dual isogeny of ϕ. Its main properties are given in the following theorem.

Theorem 1.3. (a) $\phi \circ \widehat{\phi} = [\deg(\phi)]$ *on* E'.

(b) *If $\lambda : E' \to E''$ is another isogeny, then*

$$\widehat{\lambda \circ \phi} = \widehat{\phi} \circ \widehat{\lambda}.$$

(c) *If $\psi : E \to E'$ is another isogeny, then*

$$\widehat{\phi + \psi} = \widehat{\phi} + \widehat{\psi}.$$

(d) $\deg(\widehat{\phi}) = \deg(\phi)$.

(e) $\widehat{\widehat{\phi}} = \phi$.

1.6. The endomorphism ring of an elliptic curve. Given elliptic curves E, E' we put

$$\mathrm{Hom}(E, E') = \{\phi : E \to E' \text{ isogeny}\}.$$

This is a $\mathbb{Z}$-module of rank at most 4. In particular, we put

$$\mathrm{End}(E) = \mathrm{Hom}(E, E).$$

This is the endomorphism ring of the curve E where multiplication is given by composition. Note that one always has the injective group morphism $\mathbb{Z} \to \mathrm{End}(E)$ given by $n \mapsto [n]$.

Theorem 1.4. *If* $\mathrm{char}(K) = 0$*, then* $\mathrm{End}(E)$ *is isomorphic to $\mathbb{Z}$ or to an order in an imaginary quadratic field. If* $\mathrm{char}(K) = p > 0$*, then* $\mathrm{End}(E)$ *is isomorphic to an order in an imaginary quadratic field or to an order in the unique quaternion $\mathbb{Q}$-algebra ramified at p and ∞.*

By an order in a $\mathbb{Q}$-algebra we mean a subring which is finitely generated as $\mathbb{Z}$-module and which contains a $\mathbb{Q}$-basis of the $\mathbb{Q}$-algebra.

When $\mathrm{char}(K) = 0$ and $\mathrm{End}(E)$ is isomorphic to an order in an imaginary quadratic field, we say that E is a CM elliptic curve.

When $\mathrm{char}(K) = p > 0$ and $\mathrm{End}(E)$ is isomorphic to an order in an imaginary quadratic field (resp. in a quaternion algebra), we say that E is ordinary (resp. supersingular).

Example: Over $\overline{\mathbb{F}_p}$ an elliptic curve with j-invariant 0 is ordinary if and only if $p = 1 \pmod 3$.

There are several characterizations of ordinary (resp. supersingular) elliptic curves over fields of positive characteristic. We content ourselves with the following one.

Theorem 1.5. *Let E be an elliptic curve over a field of characteristic $p > 0$. Then:*

(a) *E is ordinary if and only if $E[p^i] \simeq \mathbb{Z}/p^i\mathbb{Z}$ for all $i \geq 1$.*

(b) *E is supersingular if and only if $E[p^i] = \{O\}$ for all $i \geq 1$.*

It is known that there are only finitely many isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_p}$. We will denote by

$$\{\overline{e_1}, \ldots, \overline{e_s}\} \subseteq \mathrm{Ell}(\overline{\mathbb{F}_p})$$

the set of such isomorphism classes.

1.7. The automorphism group of an elliptic curve. Given an elliptic curve E we put

$$\mathrm{Aut}(E) = \mathrm{End}(E)^{\times} = \{\phi : E \to E \text{ isomorphism}\}.$$

This is a group under composition. Note that we always have $\{[1], [-1]\}$ as a subgroup of $\mathrm{Aut}(E)$. The following theorem gives all the information that we need about this group.

Theorem 1.6. *Assume that* $\mathrm{char}(K) = 0$. *Then:*

(a) $\#\mathrm{Aut}(E) = 2$ *if and only if* $j(E) \neq 0, 1728$.

(b) $\#\mathrm{Aut}(E) = 4$ *if and only if* $j(E) = 1728$.

(c) $\#\mathrm{Aut}(E) = 6$ *if and only if* $j(E) = 0$.

1.8. Hecke points. Given $\overline{E} \in \mathrm{Ell}(K)$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$ we define the divisor

$$T_n(\overline{E}) = \sum_{C \leq E, |C| = n} [j(E/C)],$$

where the sum runs over all the subgroups of order n of E. Thus $T_n(\overline{E})$ is an element of

$$\mathrm{Div}(K) = \bigoplus_{z \in K} \mathbb{Z}[z]$$

(i.e. the free abelian group spanned by the elements of K). Points in the support of $T_n(\overline{E})$ will be called Hecke points of order n associated to $\overline{E}$. Observe that, with our definition, Hecke points are elements of the field K. They are the j-invariants of the elliptic curves E' which admit an isogeny $E \to E'$ of degree n.

Recall that the degree of a divisor $D = \sum_{z\in K} n_z[z] \in \mathrm{Div}(K)$ is defined by

$$\deg(D) = \sum_{z\in K} n_z \in \mathbb{Z}.$$

In our case we have

$$\deg(T_n(\overline{E})) = \sigma_1(n) = \sum_{k>0, k/n} k, \text{ for all } \overline{E} \in \mathrm{Ell}(K), n \in \mathbb{N}.$$

1.9. Reduction of elliptic curves over $\mathbb{C}_p$. In what follows we fix a rational prime number $p > 0$. We will specialize to the cases $K = \mathbb{C}_p$ (a completion of an algebraic clousure of $\mathbb{Q}_p$) and $K = \overline{\mathbb{F}_p}$ (an algebraic closure of the finite field with p elements). The p-adic norm in $\mathbb{C}_p$ will be denoted by $\| \|_p$.

Let

$$\mathcal{O}_p = \{z \in \mathbb{C}_p : \|z\|_p \leq 1\}$$

be the ring of integers of $\mathbb{C}_p$ and

$$\mathcal{M}_p = \{z \in \mathbb{C}_p : \|z\|_p < 1\}$$

be the unique maximal ideal of $\mathcal{O}_p$. We denote by $\pi : \mathcal{O}_p \to \mathcal{O}_p/\mathcal{M}_p$ the reduction morphism. Here and throughout this chapter we fix an isomorphism $\mathcal{O}_p/\mathcal{M}_p \simeq \overline{\mathbb{F}_p}$ and identify this two spaces without further mention.

Note that for any disc of the form

$$D(a, 1) = \{z \in \mathbb{C}_p : \|z - a\| < 1\}, \text{ where } a \in \mathcal{O}_p,$$

we have $D(a, 1) = \pi^{-1}(\pi(a))$. Thus we will call $D(a, 1)$ a residue class.

We also denote by $\mathrm{Frob} : \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}$ the Frobenius morphism given by $\mathrm{Frob}(\zeta) = \zeta^p$.

Given $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ we may define its type of reduction (mod $\mathcal{M}_p$). We say that $\overline{E}$ has good reduction if $j(E) \in \mathcal{O}_p$. In this case there is a well defined reduced curve $\overline{\widetilde{E}} \in \mathrm{Ell}(\overline{\mathbb{F}_p})$ with $j(\widetilde{E}) = \pi(j(E))$. We say that $\overline{E}$ has ordinary reduction if $\overline{E}$ has good reduction and $\overline{\widetilde{E}}$ is ordinary. We say that $\overline{E}$ has supersingular reduction if $\overline{E}$ has good reduction and $\overline{\widetilde{E}}$ is supersingular. If $j(E) \notin \mathcal{O}_p$, we say that $\overline{E}$ has bad reduction.

We note that $\overline{E}$ has good reduction if and only if there exists a representative E in the isomorphism class with a Weierstrass model over $\mathcal{O}_p$ whose reduction $\widetilde{E}$ gives an elliptic curve over $\mathcal{O}_p/\mathcal{M}_p \simeq \overline{\mathbb{F}_p}$, or equivalently the discriminant of E is a unit in $\mathcal{O}_p$. Thus, when $\overline{E}$ has good reduction, we will assume that the representative E has this properties without further mention.

Let $\{\overline{e_1}, \ldots, \overline{e_s}\}$ be the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_p}$ as defined in Subsection 1.6. Define $D_i = \pi^{-1}(j(e_i))$ for $i \in \{1, \ldots, s\}$. These residue classes will be called supersingular. Their union is the set of j-invariants of those $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction.

It is known that Hecke points associated to $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ are of the same type of reduction than $\overline{E}$. The following lemma follows from this fact.

Lemma 1.7. (a) *If $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ has bad reduction, then $T_n(\overline{E})$ is supported on $\mathbb{C}_p \setminus \mathcal{O}_p$ for every $n \in \mathbb{N}$.*

(b) *If $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ has ordinary reduction, then $T_n(\overline{E})$ is supported on $\mathcal{O}_p \setminus \cup_{i=1}^s D_i$ for every $n \in \mathbb{N}$.*

(c) *If $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ has supersingular reduction, then $T_n(\overline{E})$ is supported on $\cup_{i=1}^s D_i$ for every $n \in \mathbb{N}$.*

1.10. Reduction of isogenies. It is known that if E, E' are elliptic curves over $\mathbb{C}_p$ with Weierstrass models over $\mathcal{O}_p$ with good reduction and $\phi : E \to E'$ is an isogeny, then there is an induced isogeny $\widetilde{\phi} : \widetilde{E} \to \widetilde{E'}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow & & \downarrow \\ \widetilde{E} & \xrightarrow[\widetilde{\phi}]{} & \widetilde{E'} \end{array}$$

is commutative (the vertical lines denote the reduction morphisms induced by $[X : Y : Z] \to [\pi(X) : \pi(Y) : \pi(Z)]$). Moreover, the map

$$\begin{array}{rcl} \mathrm{Hom}(E, E') & \to & \mathrm{Hom}(\widetilde{E}, \widetilde{E'}) \\ \phi & \mapsto & \widetilde{\phi} \end{array}$$

is injective.

1.11. The Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$. We refer the reader to [**3**] for details concerning Berkovich spaces.

By the Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ we mean the space of all the multiplicative seminorms on the polynomial ring $\mathbb{C}_p[X]$ which extend the p-adic norm on $\mathbb{C}_p$.

The Berkovich topology on $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is defined to be the weakest topology for which the function $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{R}$ given by $x \mapsto x(f)$ is continuous, for each $f \in \mathbb{C}_p[X]$. This space is locally compact, Hausdorff and path-connected. It contains $\mathbb{C}_p$ as a dense subspace via the map $\iota : \mathbb{C}_p \to \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ given by $\iota(z)(f) = \|f(z)\|_p$, for $z \in \mathbb{C}_p$ and $f \in \mathbb{C}_p[X]$. In $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ there is a canonical point, also called the Gauss point, corresponding to the Gauss norm $\|\sum a_n X^n\| = \max_n\{\|a_n\|_p\}$. We will denote by ζ_{can} this canonical point.

For $a \in \mathbb{C}_p$ and $r > 0$ define the open Berkovich discs

$$\begin{aligned} \mathcal{D}(a, r) &= \{x \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) : x(X - a) < r\}, \\ \mathcal{D}^\infty(a, r) &= \{x \in \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) : x(X - a) > r\}. \end{aligned}$$

Theorem 1.8. *A neighbourhood basis for ζ_{can} in the Berkovich topology is given by the collection of sets of the form*

$$\mathcal{A}(a_1, \ldots, a_t; R, r) = \mathcal{D}(0, R) \cap \left(\bigcap_{i=1}^{t} \mathcal{D}^{\infty}(a_i, r) \right), \tag{69}$$

where $a_1, \ldots, a_t \in \mathcal{O}_p$, $t \in \mathbb{N}$ and $0 < r < 1 < R$.

In the theory of Berkovich spaces, sets of the form (69) are called connected open affinoids.

2. The main theorem

Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$. The main theorem of this chapter gives a description of the asymptotic distribution of $T_n(\overline{E})$ as n goes to infinity. To that purpose, let's introduce some extra notations.

If X is a topological space and $D = \sum_{x\in X} n_x[x]$ is an effective divisor over X (meaning that $n_x \geq 0$ for every $x \in X$ and $\deg(D) > 0$) we define the probability measure

$$\overline{\delta}_D = \frac{1}{\deg(D)} \sum_{x\in X} n_x \delta_x$$

where δ_x denotes the atomic probability measure supported on $\{x\}$.

In our case we have the sequence of probability measures $(\overline{\delta}_{T_n(\overline{E})})_n$ on $\mathbb{C}_p$. One way of understanding the asymptotic distribution of $T_n(\overline{E})$ is by describing the (weak) limit of this sequence of probability measure.

We will use the Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ in order to formulate our result. Recall that there is an injective continuous map $\iota : \mathbb{C}_p \to \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ with dense image. The pushforward measure $\iota_*(\overline{\delta}_{T_n(\overline{E})})$ on $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is defined, as usual, by the formula

$$\iota_*(\overline{\delta}_{T_n(\overline{E})})(A) = \overline{\delta}_{T_n(\overline{E})}(\iota^{-1}(A)), \text{ for } A \subseteq \mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p).$$

Note that $\iota_*(\overline{\delta}_{T_n(\overline{E})})$ is simply $\overline{\delta}_{\iota_*(T_n(\overline{E}))}$ where

$$\iota_*(T_n(\overline{E})) = \sum_{C\leq E, |C|=n} [\iota(j(E/C))].$$

Our main theorem is the following.

Theorem 2.1. *If $\overline{E}$ has bad or ordinary reduction, then $\iota_*(\overline{\delta}_{T_n(\overline{E})}) \to \delta_{\zeta_{can}}$ weakly. If $\overline{E}$ has supersingular reduction, then $\iota_*(\overline{\delta}_{T_n(\overline{E})}) \to \delta_{\zeta_{can}}$ weakly if and only if $\|n\|_p \to 0$.*

Note that, given a sequence of effective divisors $(D_n)_n$ over $\mathbb{C}_p$, we have $\iota_*(\overline{\delta}_{D_n}) \to \delta_{\zeta_{can}}$ weakly if and only if $\iota_*(\overline{\delta}_{D_n})(\mathcal{A}) \to 1$ for any open neighbourhood $\mathcal{A}$ of ζ_{can}. By Theorem 1.8 we can assume that $\mathcal{A}$ is given by (69). In this case, we have

$$\iota^{-1}(\mathcal{A}) = D(0,R) \cap \left(\bigcap_{i=1}^{t} D^\infty(a_i, r) \right) \tag{70}$$

where

$$\begin{aligned} D(0,R) &= \{z \in \mathbb{C}_p : \|z\|_p < R\}, \\ D^\infty(a_i, r) &= \{z \in \mathbb{C}_p : \|z - a_i\| > r\}. \end{aligned}$$

Now, for a divisor $D = \sum_{z\in\mathbb{C}_p} n_z[z] \in \mathrm{Div}(\mathbb{C}_p)$ and $A \subseteq \mathbb{C}_p$, define the restriction of D to A as the divisor

$$D|_A = \sum_{z\in A} n_z[z].$$

We have

$$\iota_*(\overline{\delta}_D)(\mathcal{A}) = \frac{\deg(D|_{\iota^{-1}(\mathcal{A})})}{\deg(D)}.$$

It follows that $\iota_*(\overline{\delta}_{D_n}) \to \delta_{\zeta_{can}}$ weakly if and only

$$\overline{\delta}_{D_n}(A) = \frac{\deg(D_n|_A)}{\deg(D_n)} \to 1, \text{ as } n \to \infty$$

for every open set A of the form (70). This equivalence will be used in the proof of Theorem 2.1.

3. Applications

Our work has application to the asymptotic distribution of singular moduli over $\mathbb{C}_p$. A singular moduli is the j-invariant of a CM elliptic curve. Recall that CM elliptic curves E have $\mathrm{End}(E)$ isomorphic to an order in an imaginary quadratic field. It is known that for any such field $F = \mathbb{Q}(\sqrt{m})$, where m is a negative square-free integer, with ring of integer $\mathcal{O}_F$, the orders contained in F are precisely the subrings of the form

$$\mathcal{O}_{F,\ell} = \mathbb{Z} + \ell\mathcal{O}_F$$

where ℓ is a positive integer. One can define the discriminant of $\mathcal{O}_{F,\ell}$ as $\Delta_F \ell^2$ where

$$\Delta_F = \begin{cases} 4m & \text{if } m = 2,3 \pmod 4, \\ m & \text{if } m = 1 \pmod 4, \end{cases}$$

is the discriminant of F. We write $\mathrm{disc}(\mathcal{O}_{F,\ell}) = \Delta_F \ell^2$. Any negative integer of the form $d = \Delta_F \ell^2$ will be called a discriminant. It is known that every discriminant determines a unique order. The integer ℓ is called the conductor of the order. When $\ell = 1$ one calls d a fundamental discriminant. Note that we have $F = \mathbb{Q}(\sqrt{d})$.

For any such discriminant $d < 0$ let Λ_d be the divisor of j-invariants of CM elliptic curves over $\mathbb{C}_p$ with ring of endomorphism isomorphic to the unique imaginary quadratic order of discriminant d. More precisely, following the notation above, we put

$$\Lambda_d = \sum_{\substack{\overline{E} \in \mathrm{Ell}(\mathbb{C}_p) \\ \mathrm{End}(E) \simeq \mathbb{Z} + \ell\mathcal{O}_F}} [j(E)] \in \mathrm{Div}(\mathbb{C}_p).$$

Points in the support of Λ_d are called singular moduli of discriminant d over $\mathbb{C}_p$.

By the work of Deuring (see [**8**] or Theorem 12 in [**19**] Chapter 13 §4) we have the following dichotomy on the reduction type of CM elliptic curves over $\mathbb{C}_p$.

Lemma 3.1. *Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ be a CM elliptic curve with $\mathrm{End}(E) \simeq \mathbb{Z} + \ell\mathcal{O}_F$.*

(a) *If p splits completely in F, then $\overline{E}$ has ordinary reduction.*

(b) *If p ramifies completely or remains prime in F, then $\overline{E}$ has supersingular reduction.*

In particular, $\overline{E}$ has good reduction.

In the first case we say that d is p-ordinary, and in the second case we say that d is p-supersingular. As a first application of our work we prove the following result.

Theorem 3.2. *Suppose (d_n) is a sequence of discriminants with $d_n \to -\infty$. If each d_n is p-ordinay, then $\iota_*(\overline{\delta}_{\Lambda_{d_n}}) \to \delta_{\zeta_{can}}$ weakly. If each d_n is p-supersingular, then $\iota_*(\overline{\delta}_{\Lambda_{d_n}}) \to \delta_{\zeta_{can}}$ weakly if and only if $\|d_n\|_p \to 0$.*

We mention here that the asymptotic distribution of singular moduli over $\mathbb{C}$ was studied by Duke [**9**] and Clozel–Ullmo [**4**]. Our application is a p-adic analogue of their result.

As a second application of our work, we generalize a result of Habegger [**12**] on the finiteness of certain collection of singular moduli over $\mathbb{C}$. It is known that singular moduli are algebraic integers (see [**19**]). Habegger's work gives the finiteness of singular moduli that are algebraic units.

Our generalization gives a finiteness result for singular moduli that are S-units, where S is a finite set of rational primes satisfying an explicit congruence condition. More precisely, for a finite set of finite rational primes S we define

$$\mathrm{Sing}(S) = \{j \in \mathbb{C} : j \text{ singular moduli and } \mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j) \in \mathcal{O}_p^\times \text{ for all } p \notin S\}.$$

Here $\mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}} : \mathbb{Q}(j) \to \mathbb{Q}$ denotes the usual norm from Galois theory associated to the finite field extension $\mathbb{Q}(j) \supseteq \mathbb{Q}$.

Habegger's result states that $\mathrm{Sing}(\emptyset)$ is finite. Our generalization is the following.

Theorem 3.3. *If S is any finite set of rational primes congruent to* 1 *mod* 3*, then the set* $\mathrm{Sing}(S)$ *is finite.*

We mention that, as far as we know, no examples of singular moduli that are algebraic units have been found. This might suggest that $\mathrm{Sing}(\emptyset)$ is actually empty. Similarly, we haven't found any singular moduli in $\mathrm{Sing}(S)$ for any finite set S of rational primes as in the theorem above.

4. On Hecke points with bad reduction

Define $D(0,1)^* = \{q \in \mathbb{C}_p : 0 < \|q\|_p < 1\}$. For $q \in D(0,1)^*$ consider

$$\begin{aligned} b_2(q) &= 5\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 5q + 45q^2 + 140q^3 + \dots, \\ b_3(q) &= \sum_{n=1}^{\infty} \left(\frac{7n^2+5n^3}{12}\right) \frac{q^n}{1-q^n} = q + 23q^2 + 154q^3 + \dots, \end{aligned}$$

and define the Tate curve $\mathrm{Tate}(q)$ as the elliptic curve

$$\mathrm{Tate}(q) : Y^2Z + XYZ = X^3 - b_2(q)XZ^2 - b_3(q)Z^3.$$

This elliptic curve has j-invariant given by a convergent power series

$$j(q) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n \tag{71}$$

with $c_n \in \mathbb{Z}$. In particular

$$\|j(\mathrm{Tate}(q))\|_p = \|q\|_p^{-1} > 1 \tag{72}$$

By Tate uniformization theory for elliptic curves with bad reduction (see [**33**]) we know that the map

$$\begin{aligned} D(0,1)^* &\to \{\overline{E} \in \mathrm{Ell}(\mathbb{C}_p) : \|j(E)\|_p > 1\} \\ q &\mapsto \overline{\mathrm{Tate}(q)} \end{aligned}$$

is a bijection. Moreover, for each $q \in D(0;1)^*$ there exists an analytic isomorphism

$$\varphi_q : \mathbb{C}_p^*/q^{\mathbb{Z}} \to \mathrm{Tate}(q)$$

given explicitly by

$$z \mapsto \begin{cases} [x(z,q) : y(z,q) : 1] & \text{, if } z \notin q^{\mathbb{Z}}, \\ [0:1:0] & \text{, if } z \in q^{\mathbb{Z}}, \end{cases}$$

where

$$\begin{aligned} x(z,q) &= \sum_{n\in\mathbb{Z}} \frac{q^n z}{(1-q^n z)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}, \\ y(z,q) &= \sum_{n\in\mathbb{Z}} \frac{q^{2n} z^2}{(1-q^n z)^3} + \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}. \end{aligned}$$

Now, for each positive divisor k of n and each $\ell \in D(0,1)^*$ satisfying $\ell^k = q^{n/k}$, we have a subgroup of order n of $\mathbb{C}_p^*/q^{\mathbb{Z}}$ given by

$$C_{n,\ell} = \{z \in \mathbb{C}_p^* : z^{n/k} \in \ell^{\mathbb{Z}}\}/q^{\mathbb{Z}}.$$

These are all the subgroups of order n of $\mathbb{C}_p^*/q^{\mathbb{Z}}$. The map $z \mapsto z^{n/k}$ induces an analytic morphism $\mathbb{C}_p^*/q^{\mathbb{Z}} \to \mathbb{C}_p^*/\ell^{\mathbb{Z}}$ with Kernel $C_{n,\ell}$. It follows that

$$T_n(\overline{\mathrm{Tate}(q)}) = \sum_{\substack{\ell^k = q^{n/k} \\ k>0, k/n}} [j(\mathrm{Tate}(\ell))]. \tag{73}$$

The following result gives the asymptotic distribution of $T_n(\overline{\mathrm{Tate}(q)})$.

Theorem 4.1. *Let $q \in D(0,1)^*$. For any $R > 1$ and $\varepsilon > 0$ we have*

$$\deg(T_n(\overline{\mathrm{Tate}(q)})|_{\mathbb{C}_p \setminus D(0,R)}) = o(n^{\frac{1}{2}+\varepsilon}).$$

Proof. By (73) and (72) we have

$$\deg(T_n(\overline{\mathrm{Tate}(q)})|_{\mathbb{C}_p \setminus D(0,R)}) = \sum_{\substack{0<k\leq \sqrt{n}C \\ k/n}} k,$$

where $C = \sqrt{-\frac{\log(\|q\|_p)}{\log(R)}}$. If we denote by $d(n)$ the number of positive divisors of n, we get

$$\sum_{\substack{0<k\leq \sqrt{n}C \\ k/n}} k \leq \sqrt{n}Cd(n).$$

By [**1**] p. 296, we have $d(n) = o(n^{\varepsilon})$. This implies the desired result. $\square$

5. On Hecke points with ordinary reduction

Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with ordinary reduction and $D \subset \mathcal{O}_p$ a non-supersingular residue class, i.e. different from $D_1, \ldots, D_s$ (recall that $D_1 \cup D_2 \cup \cdots D_s$ equals the set of j-invariants of elliptic curves over $\mathbb{C}_p$ with supersingular reduction). Let e be an elliptic curve over $\overline{\mathbb{F}_p}$ with $D = \pi^{-1}(j(e))$. If n is a positive integer coprime to p then it is known that the reduction morphism $E \to \widetilde{E}$ induces a bijective map $E[n] \to \widetilde{E}[n]$. If for each subgroup C of E of order n we denote by $\widetilde{C}$ its reduction, then the map $C \mapsto \widetilde{C}$ is a bijection between the subgroups of order n of the respective elliptic curves. If $j(E/C) \in D$, then the isogeny $E \to E/C$ induces an isogeny $\phi_C : \widetilde{E} \to e$ such that $\mathrm{Ker}(\phi_C) = \widetilde{C}$ (simply because $D = \pi^{-1}(j(e))$). This defines an injective map

$$\begin{array}{rcl} \{C \leq E : \#C = n, j(E/C) \in D\} & \hookrightarrow & \mathrm{Hom}_n(\widetilde{E}, e) \\ C & \mapsto & \phi_C \end{array}$$

where $\mathrm{Hom}_n(\widetilde{E}, e)$ denotes de set of isogenies of degree n from $\widetilde{E}$ to e. This implies that

$$\deg(T_n(\overline{E})|_D) \leq \#\mathrm{Hom}_n(\widetilde{E}, e). \tag{74}$$

The following lemma gives an upper bound for $\#\mathrm{Hom}_n(\widetilde{E}, e)$.

Lemma 5.1. *If e, e' are ordinary elliptic curves over $\overline{\mathbb{F}_p}$, then*

$$\#\mathrm{Hom}_n(e, e') = O_{\overline{e}, \overline{e'}}(n^{\frac{1}{2}}).$$

Proof. Choose a non constant isogeny $\phi_0 : e \to e'$ (if such an isogeny does not exist then $\mathrm{Hom}_n(e, e') = \{0\}$ and we are done) and put $N = \deg(\phi_0)$. The map $f \mapsto \widehat{\phi_0} \circ f$ defines an injective map $\mathrm{Hom}_n(e, e') \hookrightarrow \mathrm{End}_{nN}(e)$, where $\mathrm{End}_{nN}(e)$ denotes the set of endomorphisms of e of degree nN. Since e is ordinary, there exists a monomorphism $\mathrm{End}(e) \hookrightarrow \mathcal{O}_F$ where $\mathcal{O}_F$ is the ring of integers of some imaginary quadratic field $F = \mathbb{Q}(\sqrt{m})$, m a negative square-free integer. If $\mathrm{N}_{F/\mathbb{Q}}(x)$ denotes the usual norm of an element $x \in F$, then the image of $\mathrm{End}_{nN}(e)$ under the above monomorphism is contained in $\{x \in \mathcal{O}_F : \mathrm{N}_{F/\mathbb{Q}}(x) = nN\}$. We have

$$\mathcal{O}_F = \begin{cases} \mathbb{Z}[\sqrt{m}] & \text{if } m = 2, 3 \pmod 4, \\ \mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] & \text{if } m = 1 \pmod 4. \end{cases}$$

We conclude that

$$\#\mathrm{End}_{nN}(e) \leq \begin{cases} \#\{(x, y) \in \mathbb{Z}^2 : x^2 - y^2 m = nN\} & \text{if } m = 2, 3 \pmod 4, \\ \#\{(x, y) \in \mathbb{Z}^2 : x^2 + xy + y^2\left(\frac{1-m}{4}\right) = nN\} & \text{if } m = 1 \pmod 4. \end{cases}$$

In any case we have $\#\mathrm{End}_{nN}(e) = O_{\overline{e}, N}(n^{\frac{1}{2}})$. This proves the result since N depens on $\overline{e}$ and $\overline{e'}$. □

A direct application of the previous lemma together with (74) gives the following result.

Theorem 5.2. *For any $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with ordinary reduction and any non supersingular residue class $D \subset \mathcal{O}_p$ we have*

$$\deg(T_n(\overline{E})|_D) = O_{\widetilde{\overline{E}},D}(n^{1/2})$$

for n coprime to p.

We now proceed to study the Hecke points of order p^m associated to $\overline{E}$ when $\overline{E}$ has ordinary reduction. It is known that in this case $E[p]$ has p^2 elements, $\widetilde{E}[p]$ has p elements, and the reduction morphism $E[p] \to \widetilde{E}[p]$ is surjective. One defines $H(E)$, the canonical subgroup of E, as the unique subgroup of order p of E in the Kernel of the reduction morphism $E \to \widetilde{E}$. Let's also define $f : \mathcal{O}_p \setminus \cup_{i=1}^s D_i \to \mathcal{O}_p \setminus \cup_{i=1}^s D_i$ by the identity

$$f(j(E)) = j(E/H(E)).$$

A theorem of Deligne implies that f is an analytic function of degree p in $\mathcal{O}_p \setminus \cup_{i=1}^s D_i$. We refer the reader to [**10**] for a full treatment of Deligne's theorem. It is known that the reduction of the isogeny $E \to E/H(E) \bmod \mathcal{M}_p$ is equivalent to the Frobenius map $\widetilde{E} \to \widetilde{E}^{(p)}$ (see [**8**]). Since $j(\widetilde{E}^{(p)}) = j(\widetilde{E})^p$ it follows that

$$f(z) \equiv z^p \pmod{\mathcal{M}_p}. \tag{75}$$

We remark that (75) also follows from Deligne's theorem (see formulas (7.6)-(7.8) in [**10**]).

The next lemma gives a simple description of $T_{p^m}(\overline{E})$ which is crucial in our computations. For $z \in \mathbb{C}_p$ we consider $f^{-1}(z)$ as a divisor over $\mathbb{C}_p$, this is

$$f^{-1}(z) = \sum_{\substack{w \in \mathbb{C}_p \\ f(w)=z}} \mathrm{ord}_f(w)[w],$$

where $\mathrm{ord}_f(w)$ is defined in the usual way, as the smallest exponent in the power series expansion of f around w.

Lemma 5.3. *For any $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with ordinary reduction and any $m \in \mathbb{N}$ we have*

$$T_{p^m}(\overline{E}) = \sum_{i=0}^{m} f^{-m+i}(f^i(j(E))). \tag{76}$$

Proof. We start by observing that

$$T_p(\overline{E}) = [f(j(E))] + \sum_{\substack{C \le E, |C|=p \\ C \neq H(E)}} [j(E/C)].$$

Now, let C be an order p subgroup of E different form $H(E)$. The isogeny $\widehat{\phi}$ dual to $\phi : E \to E/C$ has Kernel $\phi(E[p])$. But $\phi(E[p])$ has order p and $\widetilde{\phi(E[p])} = \widetilde{\phi}(\widetilde{C}) = \{0\}$, thus $H(E/C) = \phi(E[p])$. This implies that the isogeny $E/C \to (E/C)/(H(E/C))$ is equivalent to $\widehat{\phi}$ and thus $f(j(E/C)) = j(E)$. This gives

$$T_p(\overline{E}) = [f(j(E))] + f^{-1}(j(E)) \tag{77}$$

and proves (76) when $m = 1$. Now, for $n \in \mathbb{N}$ define $\mathfrak{T}_n : \mathbb{C}_p \to \mathrm{Div}(\mathbb{C}_p)$ by the formula

$$\mathfrak{T}_n(j(E)) = T_n(\overline{E}) \tag{78}$$

Extending this definition by linearity we get a map $\mathfrak{T}_n : \mathrm{Div}(\mathbb{C}_p) \to \mathrm{Div}(\mathbb{C}_p)$. It is known (see Theorem 3.24 in [**30**] p. 63) that the recursive formula

$$\mathfrak{T}_{p^{m+2}} = \mathfrak{T}_p \circ \mathfrak{T}_{p^{m+1}} - p\mathfrak{T}_{p^m}, \quad \text{for } m \geq 0, \tag{79}$$

holds. Combining this with (77) we conclude (76) by induction on m. $\square$

For $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with ordinary reduction, let $\zeta = \pi(j(E))$ be the reduction of $j(E)$ and r be the exact period of ζ under the Frobenius morphism $\mathrm{Frob}(\zeta) = \zeta^p$. For $i \in \{0, 1, \ldots, r-1\}$ put $D^{(i)} = \pi^{-1}(\zeta^{p^i})$. From (75) it follows that the orbit of $j(E)$ under iteration by f is contained in the union $D^{(0)} \cup D^{(1)} \cup \cdots \cup D^{(r-1)}$. We have the following result.

Lemma 5.4. *There exists a unique point $z_0 \in D^{(0)}$ which is periodic under f with exact period r. Moreover, the following properties hold:*

(1) *For any $z \in D^{(0)}$ we have*

$$\|f^m(z_0) - f^m(z)\|_p \to 0, \ \textit{if } m \to \infty. \tag{80}$$

(2) *The points $f^k(z_0)$, for $k \in \{0, 1, \ldots, r-1\}$, are not ramification points for f.*

Proof. Let e be any elliptic curve over $\overline{\mathbb{F}_p}$ with $j(e) = \zeta$. A classical result of Deuring (see [**8**]) gives the existence of a unique (modulo isomorphism) elliptic curve $e^{\uparrow}$ over $\mathbb{C}_p$ reducing to e and satisfying $\mathrm{End}(e^{\uparrow}) \simeq \mathrm{End}(e)$. Moreover, the Frobenius morphism $e \to e^{(p)}$ lifts to an isogeny $e^{\uparrow} \to (e^{(p)})^{\uparrow}$ of degree p, thus $f(j(e)) = j((e^{(p)})^{\uparrow})$. Choosing $z_0 = j(e^{\uparrow})$ we get a point in $D^{(0)}$ of exact order r under f. The uniqueness of z_0 follows from (80) which in turn follows from the analytic properties of f. Indeed, we can write

$$f(z + z_0) - f(z_0) = u(z)z\prod_{i=2}^{p}(z - a_i), \text{ for } \ z \in \mathcal{M}_p,$$

where $a_2, \ldots, a_p \in \mathcal{M}_p$ and u is an analytic function satisfying $\|u(z)\|_p = 1$ for all $z \in \mathcal{M}_p$. This follows from classical results in p-adic analysis (see for example Theorem 1 in [**28**] Section 6.2.2.). We conclude that

$$\|f(z + z_0) - f(z_0)\|_p = \|z\|_p \prod_{i=2}^{p} \|z - a_i\|_p, \text{ for } \ z \in \mathcal{M}_p. \tag{81}$$

In particular, if $\|z\|_p \leq c < 1$ then

$$\|f(z + z_0) - f(z_0)\|_p \leq K_c \|z\|_p$$

where $K_c = \prod_{i=2}^{p} \max\{c, \|a_i\|_p\}$. Applying the same argument in each $D^{(i)}$ we get

$$\|f^m(z+z_0) - f^m(z_0)\|_p \leq (K'_c)^m \|z\|_p, \text{ for } m \geq 1, \|z\|_p \leq c,$$

for some positive constant $K'_c < 1$. This implies (80). Finally, let's prove that the points $f^k(z_0)$, for $k \in \{0, 1, \ldots, r-1\}$, are not ramification points for f. We can assume $k = 0$, i.e. $f^k(z_0) = z_0$. If z_0 is a ramification point for f then $f(z_0)$ is a ramification value. By Lemma 5.3, the corresponding elliptic curve $(e^{(p)})^{\uparrow}$ has two subgroups $C_1, C_2 \leq (e^{(p)})^{\uparrow}$ of order p, both different from the canonical subgroup $H((e^{(p)})^{\uparrow})$, with $(e^{(p)})^{\uparrow}/C_1 \simeq (e^{(p)})^{\uparrow}/C_2$. We will make use of the following.

Claim. Assume that $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ has ordinary reduction. If C_1, C_2 are two subgroups of order p of E, different from the canonical subgroup $H(E)$, with $E/C_1 \simeq E/C_2$, then $\sigma(C_1) = C_2$ for some $\sigma \in \mathrm{Aut}(E)$. In particular, the order of f at $[E/C]$ equals $\#\{\sigma(C) : \sigma \in \mathrm{Aut}(E)\}$.

Proof of the Claim. If we put $E' = E/C_1$, then there exist two isogenies $\psi_1, \psi_2 \in \mathrm{Hom}(E, E')$ with $\mathrm{Ker}(\psi_1) = C_1$ and $\mathrm{Ker}(\psi_2) = C_2$. For $i = 1, 2$ the dual isogeny $\widehat{\psi_i} \in \mathrm{Hom}(E', E)$ has Kernel $\psi_i(E[p])$, which is equal to $H(E')$. It follows that $\sigma \circ \widehat{\psi_1} = \widehat{\psi_2}$ for some $\sigma \in \mathrm{Aut}(E)$. We get $\sigma(C_1) = \sigma(\widehat{\psi_1}(E[p])) = \widehat{\psi_2}(E[p]) = C_2$. This proves the first part of the claim. The last statement of the claim follows from this by using a technical result of algebraic geometry (see [**5**] pp. 29-30).

A direct application of the previous claim gives $\sigma(C_1) = C_2$ for some $\sigma \in \mathrm{Aut}((e^{(p)})^{\uparrow})$. In particular $\sigma \neq [\pm 1]$ thus, by Theorem 1.6, we must have $j((e^{(p)})^{\uparrow}) \in \{0, 1728\}$. This implies that $r = 1$ and $f(z_0) = z_0$. For the corresponding elliptic curve $e^{\uparrow}$ this gives $e^{\uparrow} \simeq e^{\uparrow}/C$ for some subgroup $C \leq e^{\uparrow}$ of order p different from $H(e^{\uparrow})$. Now, for any $\tau \in \mathrm{Aut}(e^{\uparrow})$ we have $\tau(C) = C$. Indeed, fix isogenies $\phi, \psi : e^{\uparrow} \to e^{\uparrow}$ with $\mathrm{Ker}(\phi) = C$, $\mathrm{Ker}(\psi) = \tau(C)$. The reduced isogenies $\widetilde{\phi}, \widetilde{\psi} \in \mathrm{End}(e)$ have Kernel $e[p]$, thus $\widetilde{\phi} = \widetilde{\alpha} \circ \widetilde{\psi}$ for some $\widetilde{\alpha} \in \mathrm{Aut}(e)$. Since the reduction of isogenies map $\mathrm{End}(e^{\uparrow}) \to \mathrm{End}(e)$ is an isomorphism, we get an automorphism $\alpha \in \mathrm{Aut}(e^{\uparrow})$ satisfying $\phi = \alpha \circ \psi$. This implies $C = \tau(C)$. By the above claim we deduce that z_0 is not a ramification point for f. □

We will give another characterization of the periodic points of f. For this, we need the following result (see Theorem 12 in [**19**] Chapter 13 §4).

Lemma 5.5. *Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ be a CM elliptic curve with ordinary reduction. If $\mathrm{End}(E) \simeq \mathbb{Z} + \ell\mathcal{O}_F$ and $\ell = p^s m$, $s \geq 0, (m, p) = 1$ then $\mathrm{End}(\widetilde{E}) \simeq \mathbb{Z} + m\mathcal{O}_F$. In particular, if $(\ell, p) = 1$ then the reduction map $\mathrm{End}(E) \to \mathrm{End}(\widetilde{E})$ is an isomorphism.*

Our second characterization of the periodic points of f is the following.

Theorem 5.6. *The unique periodic point $z_0 \in D^{(0)}$ of f is the unique j-invariant in $D^{(0)}$ corresponding to an elliptic curve $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ whose endomorphism ring is an order of conductor coprime to p in an imaginary quadratic field.*

Proof. From the proof of Theorem 5.4 we know that $z_0 = j(e^{\uparrow})$ where $\overline{e} \in \mathrm{Ell}(\overline{\mathbb{F}_p})$ satisfies $D^{(0)} = \pi^{-1}(j(e))$, and $e^{\uparrow}$ is defined as the unique elliptic curve (modulo isomorphism) over $\mathbb{C}_p$ reducing to e and satisfying $\mathrm{End}(e^{\uparrow}) \simeq \mathrm{End}(e)$. By Lemma 5.5 we conclude that $e^{\uparrow}$ is the unique elliptic curve (modulo isomorphism) reducing to e and whose endomorphism ring is an order of conductor coprime to p. $\square$

We conclude this section with the following theorem regarding the asymptotic distribution of $T_{p^m}(\overline{E})$ for $m \to \infty$, when $\overline{E}$ has ordinary reduction.

Theorem 5.7. *Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with ordinary reduction, $\zeta = \pi(j(E))$ the reduction of $j(E)$ and $D^{(i)} = \pi^{-1}(\zeta^{p^i})$ where $i \in \{0, 1, \ldots, r-1\}$ and r is the exact period of ζ under the Frobenius morphism $\mathrm{Frob}(\zeta) = \zeta^p$. Then*

$$\mathrm{supp}(T_{p^m}(\overline{E})) \subset \bigcup_{i=0}^{r-1} D^{(i)}, \text{ for all } m \in \mathbb{N}. \tag{82}$$

Moreover, for any closed disc B properly contained in some $D^{(i)}$ we have

$$\deg(T_{p^m}(\overline{E})|_B) = O_{\overline{E},B}(p^{m/2}). \tag{83}$$

Proof. A direct application of (75) and Lemma 5.3 gives (82). We now proceed to prove (83). For simplicity we assume $r = 1$, the general case following the same reasoning. By Lemma 5.4 and (81) we have

$$\|f(z+z_0) - z_0\|_p = \|z\|_p \prod_{i=2}^{p} \|z - a_i\|_p, \text{ for } z \in \mathcal{M}_p,$$

where $z_0 \in D^{(0)}$ and $a_2, \ldots, a_p \in \mathcal{M}_p \setminus \{0\}$. Putting $\rho = \prod_{i=2}^p \|a_i\|_p$ and $\eta = \min\{\|a_2\|_p, \ldots, \|a_p\|_p\}$, we have

$$\|f(z+z_0) - z_0\|_p = \|z\|_p \rho, \text{ if } \|z\|_p < \eta. \tag{84}$$

We also have

$$\|f^m(z+z_0) - z_0\|_p \le K_c^m c, \text{ if } m \ge 1, \|z\|_p \le c < 1, \tag{85}$$

where $K_c = \prod_{i=2}^p \max\{c, \|a_i\|_p\}$. Now, choose $k_1 \in \mathbb{N}$ such that $\|f^{k_1}(j(E)) - z_0\|_p < \eta$. We proceed by considering two cases.

Case 1: Assume that $f^{k_1}(j(E)) \neq z_0$. Choose $k_2 \in \mathbb{N}$ such that $K_c^{k_2} c < \|f^{k_1}(j(E)) - z_0\|_p$. From (84) and (85) we conclude

$$f^{-m+i}(f^i(j(E)))|_{D(z_0,c)} = 0, \text{ for } m \ge k_2 + k_1, 0 \le i \le \left[\frac{m + k_1 - k_2}{2}\right]. \tag{86}$$

Indeed, if $\|z\|_p \le c$ and $0 \le i \le k_1$ then

$$\|f^{m-i}(z) - z_0\|_p \le K_c^{m-i} c \le K_c^{k_2} c < \|f^{k_1}(j(E)) - z_0\|_p \le \|f^i(j(E)) - z_0\|_p,$$

thus $f^{-m+i}(f^i(j(E)))|_{D(z_0,c)} = 0$. As for $k_1 < i \leq \left[\frac{m+k_1-k_2}{2}\right]$, we have $m - i - k_2 \geq i - k_1$ thus

$$\|f^{m-i}(z) - z_0\|_p = \rho^{m-i-k_2}\|f^{k_2}(z) - z_0\|_p < \rho^{i-k_1}\|f^{k_1}(j(E)) - z_0\|_p = \|f^i(j(E)) - z_0\|_p,$$

hence $f^{-m+i}(f^i(j(E)))|_{D(z_0,c)} = 0$. This proves (86). Such result, combined with Lemma 5.3, gives

$$\deg(T_{p^m}(\overline{E})|_{D(z_0,c)}) \leq 1 + p + \ldots + p^{\left[\frac{m+k_1-k_2}{2}\right]-1} = O_{\overline{E},c}(p^{m/2}).$$

Choosing $c < 1$ such that $B \subseteq D(z_0, c)$ we get (83).

CASE 2: Assume that $f^{k_1}(j(E)) = z_0$. We can also assume that k_1 is the smallest such positive integer. Choose $k_3 \in \mathbb{N}$ with $K_c^{k_3} c < \eta$. If $\|z\|_p \leq c$, $m \geq k_1 + k_3$ and $0 \leq i < k_1$ then

$$\|f^{m-i}(z + z_0) - z_0\|_p \leq K_c^{k_3} c < \eta \leq \|f^i(j(E)) - z_0\|_p,$$

hence $f^{-m+i}(f^i(j(E)))|_{D(z_0,c)} = 0$. As for $k_1 \leq i \leq m$ we have

$$f^{-m+i}(f^i(j(E)))|_{D(z_0,c)} = f^{-m+i}(z_0)|_{D(z_0,c)}.$$

But $f^{k_3}(D(z_0, c)) \cap f^{-m+i}(z_0) = \{z_0\}$, thus

$$z \in D(z_0, c), f^{m-i}(z) = z_0 \Rightarrow f^{k_3}(z) = z_0.$$

It follows that $\deg(f^{-m+i}(z_0)|_{D(z_0,c)}) \leq \deg(f^{-k_3}(z_0)|_{D(z_0,c)})$ for $k_1 \leq i \leq m$, which implies

$$\deg(T_{p^m}(\overline{E})|_{D(z_0,c)}) \leq (m - k_1 + 1)p^{k_3} = O_{\overline{E},c}(1).$$

This implies (83) if we assume that $B \subseteq D(z_0, c)$. This completes our proof.

□

6. On Hecke points with supersingular reduction

In this section we study the distribution of $T_n(\overline{E})$ when $\overline{E}$ has supersingular reduction. We will make use of the theory of classical modular forms, modular forms in the sense of Katz [**17**] and formal groups. For a brief account of this theories we refer the reader to Appendix A and B.

For an even integer $k \geq 4$, let E_k be the Eisenstein series of weight k (and level one). This is the unique modular form over $\mathbb{C}$ whose Fourier expansion is

$$\mathrm{E}_k(q) = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where b_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d/n, d>0} d^{k-1}$. For $k = p-1, p \geq 5$ prime, the p-adic norm of $\frac{2k}{b_k}$ is $\frac{1}{p}$, hence E_{p-1} is defined over $\mathbb{Q} \cap \mathcal{O}_p$. By Theorem 9.7 we can consider E_{p-1} as a modular form in the sense of Katz (see [**17**] or Appendix A) over $\mathbb{Q} \cap \mathcal{O}_p$. In particular, for any elliptic curve E over $\mathcal{O}_p$ and a non-vanishing holomorphic differential form ω on E, we get an element $\mathrm{E}_{p-1}(E, \omega) \in \mathcal{O}_p$. The rule $(E, \omega) \mapsto \mathrm{E}_k(E, \omega)$ satisfies the following properties.

Lemma 6.1. (a) $\mathrm{E}_{p-1}(E, \omega)$ *depens only on the $\mathcal{O}_p$-isomorphism class of the pair (E, ω).*

(b) $\mathrm{E}_{p-1}(E, \lambda\omega) = \lambda^{1-p}\mathrm{E}_{p-1}(E, \omega)$ *for any $\lambda \in \mathcal{O}_p^{\times}$.*

(c) *The formation of* $\mathrm{E}_{p-1}(E, \omega)$ *commutes with arbitrary extension of scalars.*

Another important feature of E_{p-1} is that its reduction mod p is the Hasse invariant A, which is a modular form in the sense of Katz over $\mathbb{F}_p$ which vanishes only over supersingular elliptic curves. For $p = 2$ and 3 it is not possible to lift A to a modular form of level one over a characteristic zero ring. However, for $p = 2$ (resp. 3) we can take E_4 (resp. E_6) as a lifting of A^4 (resp. A^3). Now, for a general rational prime p and $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction, define

$$v_p(\overline{E}) = \begin{cases} -\log_p(\|\mathrm{E}_{p-1}(E, \omega)\|_p) & \text{if } p \geq 5, \\ \\ -\frac{1}{3}\log_3(\|\mathrm{E}_6(E, \omega)\|_3) & \text{if } p = 3, \\ \\ -\frac{1}{4}\log_2(\|\mathrm{E}_4(E, \omega)\|_2) & \text{if } p = 2, \end{cases}$$

where E is any representative defined over $\mathcal{O}_p$ in the isomorphism class $\overline{E}$, and ω is any non-vanishing holomorphic differential form on E. By Lemma 6.1 we see that this definition does not depend on the particular choice of E and ω. Also, since the Hasse invariant vanishes over supersingular elliptic curves, we have $0 < v_p(\overline{E}) \leq \infty$ for any $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction.

Following Katz's denomination, we say that $\overline{E}$ is not too supersingular if $v_p(\overline{E}) < \frac{p}{p+1}$. By a theorem of Lubin (see Theorems 3.1 and 3.10.7 in [**17**]) there is a nice theory of canonical subgroups for not too supersingular elliptic curves. More specifically, if $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ is not too supersingular, then there exists an order p subgroup $H(E)$ of E, called the canonical subgroup of E, such that the rule $\overline{E} \mapsto \mathfrak{f}(\overline{E}) = \overline{E/H(E)}$ satisfies the following properties.

Lemma 6.2. (a) *If* $v_p(\overline{E}) < \frac{1}{p+1}$, *then* $v_p(\mathfrak{f}(\overline{E})) = pv_p(\overline{E})$.

(b) *If* $\frac{1}{p+1} < v_p(\overline{E}) < \frac{p}{p+1}$, *then* $v_p(\mathfrak{f}(\overline{E})) = 1 - v_p(\overline{E})$ *and* $\mathfrak{f}^2(\overline{E}) = \overline{E}$.

(c) *If* $v_p(\overline{E}) \geq \frac{p}{p+1}$, *then there exist precisely* $p+1$ *supersingular elliptic curves* $\overline{E'} \in \mathrm{Ell}(\mathbb{C}_p)$ *with* $v_p(\overline{E'}) = \frac{1}{p+1}$ *and* $\mathfrak{f}(\overline{E'}) = \overline{E}$.

(d) *If* $0 < v_p(\overline{E}) < \frac{p}{p+1}$, *then there exist* p *supersingular elliptic curves* $\overline{E'} \in \mathrm{Ell}(\mathbb{C}_p)$ *with* $v_p(\overline{E'}) = \frac{1}{p}v_p(\overline{E})$ *and* $\mathfrak{f}(\overline{E'}) = \overline{E}$.

Note that $\mathfrak{f}$ induces a map

$$f : \left\{ j(E) \in \bigcup_{i=1}^{s} D_i : v(\overline{E}) < \frac{p}{p+1} \right\} \to \bigcup_{i=1}^{s} D_i$$

given by

$$f(j(E)) = j(\mathfrak{f}(\overline{E})).$$

This is an extension of the map $f : \mathcal{O}_p \setminus \cup_{i=1}^{s} D_i \to \mathcal{O}_p \setminus \cup_{i=1}^{s} D_i$ introduced in Section 5.

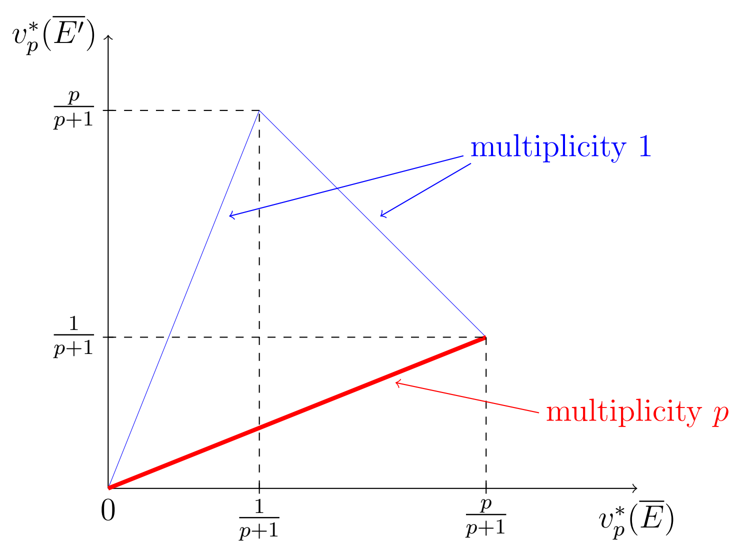
As a consequence of Lemma 6.2 we have the following result which should be compared with Lemma 5.3.

Lemma 6.3. *For any* $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ *with supersingular reduction we have*

$$T_p(\overline{E}) = \begin{cases} f^{-1}(j(E)) + [f(j(E))] & \text{if } v_p(E) < \frac{p}{p+1}, \\ \\ f^{-1}(j(E)) & \text{if } v_p(E) \geq \frac{p}{p+1}. \end{cases} \tag{87}$$

The following picture gives a simple representation of T_p in terms of the "parameter"

$$v_p^*(\overline{E}) = \min\left\{ v_p(\overline{E}), \frac{p}{p+1} \right\}.$$



The multiplicity one part of T_p corresponds to f.

The following result gives a useful relation between $v_p(\overline{E})$ and $j(E)$.

Lemma 6.4. *For each $i \in \{1, \ldots, s\}$ there exists a unique point $\mathfrak{j}_i \in D_i \cap \overline{\mathbb{Q}}$ such that*

$$v_p(\overline{E}) = -\sum_{i=1}^{s} \alpha_i \log_p(\|j(E) - \mathfrak{j}_i\|_p)$$

for all $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction, where

$$\alpha_i = \begin{cases} 1 & \text{if } p \geq 5, \mathfrak{j}_i \not\equiv 0, 1728, \\ \frac{1}{3} & \text{if } p \geq 5, \mathfrak{j}_i \equiv 0, \\ \frac{1}{2} & \text{if } p \geq 5, \mathfrak{j}_i \equiv 1728, \\ \frac{1}{6} & \text{if } p = 3, \\ \frac{1}{12} & \text{if } p = 2. \end{cases}$$

Proof. Let's first assume $p \geq 5$. We can write $p-1$ uniquely in the form $p-1 = 12m + 4\delta + 6\varepsilon$ with $m \in \mathbb{Z}, m \geq 0$ and $\delta, \varepsilon \in \{0, 1\}$. It is known that we can write

$$\mathrm{E}_{p-1} = \Delta^m \mathrm{E}_4^\delta \mathrm{E}_6^\varepsilon P(j)$$

where Δ is the unique modular form of weight 12 with Fourier expansion

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q + \ldots,$$

and $P(X)$ is a monic polynomial over $\mathbb{Q} \cap \mathcal{O}_p$ of degree $\left[\frac{p}{12}\right]$. Using the well known identities ${\mathrm{E}_4}^3 = \Delta j$ and $\mathrm{E}_6^2 = \Delta(j - 1728)$ we get

$$\mathrm{E}_{p-1}^{12} = \Delta^{p-1} j^{4\delta} (j - 1728)^{6\varepsilon} P(j)^{12}.$$

By a result of Deligne (Théorème 3 in [**29**]) the reduction of $Q(X) = X^\delta (X - 1728)^\varepsilon P(X)$ is the supersingular polynomial, i.e. the polynomial over $\mathbb{F}_p$ whose roots are the j-invariants of the supersingular elliptic curves over $\overline{\mathbb{F}_p}$. This gives

$$Q(X) = \prod_{i=1}^{s} (X - \mathfrak{f}_i)$$

where $\mathfrak{j}_i \in D_i$ for $i \in \{1, \ldots, s\}$. Now, for any pair (E, ω) having good reduction we have $\Delta(E, \omega) \in \mathcal{O}_p^\times$, thus

$$\|\mathrm{E}_{p-1}(E, \omega)\|_p^{12} = \|j(E)\|_p^{4\delta} \|j(E) - 1728\|_p^{6\varepsilon} \prod_{\substack{i=1 \\ \mathfrak{j}_i \neq 0, 1728}}^{s} \|j(E) - \mathfrak{j}_i\|_p^{12}. \tag{88}$$

This implies the result when $p \geq 5$. For $p = 2, 3$ we use the identities $\mathrm{E}_4^3 = \Delta j, \mathrm{E}_6^2 = \Delta(j - 1728)$ and the fact that 0 is the only supersingular j-invariant. This finishes the proof of the lemma. $\square$

Remark. It follows from the previous proof that if $p \geq 5$ and $\mathfrak{j}_i \equiv 0$ (resp. $\mathfrak{j}_i \equiv 1728$), then $\mathfrak{j}_i = 0$ (resp. $\mathfrak{j}_1 = 1728$). For $p = 2$ (resp. $p = 3$) we have $s = 1$ and $\mathfrak{j}_1 = 0$ (resp. $\mathfrak{j}_1 = 0$). On the other hand, for $\mathfrak{j}_i \not\equiv 0, 1728$ we note that $\mathfrak{j}_i$ is not a singular moduli. Indeed, the point $\mathfrak{j}_i$ corresponds, under a fixed isomorphism $\mathbb{C}_p \simeq \mathbb{C}$, to $j(\tau_i)$ where $\tau_i \in \mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ and $j : \mathbb{H} \to \mathbb{C}$ is the unique holomorphic modular function with Fourier expansion (71). Here τ_i is a zero of E_{p-1} not equivalent to ρ or i under the action of $\mathrm{SL}_2(\mathbb{Z})$. In [**18**] Kohnen proves that such τ_i is transcendental. This implies the result since singular moduli are j-values of quadratic irrationals in $\mathbb{H}$.

We now proceed to describe the asymptotic distribution of $T_{p^m}(\overline{E})$ in the supersingular reduction case. Let's define

$$I_k = \left\{ j(E') \in \bigcup_{i=1}^{s} D_i : \overline{E'} \in \mathrm{Ell}(\mathbb{C}_p), v_p(\overline{E'}) \in \left] \frac{p^k}{p+1}, \frac{p^{k+1}}{p+1} \right[\right\}$$

for $k \in \{0, -1, -2, \ldots\}$, and

$$J_0 = \left\{ j(E') \in \bigcup_{i=1}^{s} D_i : \overline{E'} \in \mathrm{Ell}(\mathbb{C}_p), v_p(\overline{E'}) \geq \frac{p}{p+1} \right\},$$

$$J_k = \left\{ j(E') \in \bigcup_{i=1}^{s} D_i : \overline{E'} \in \mathrm{Ell}(\mathbb{C}_p), v_p(\overline{E'}) = \frac{p^{k+1}}{p+1} \right\},$$

for $k \in \{-1, -2, \ldots\}$. By Lemma 6.2 we have

$$f^{-1}(I_k) = I_{k-1} \text{ for } k \leq -1, \text{ and } f^{-1}(I_0) = I_1 \cup I_0, \tag{89}$$

and also

$$f^{-1}(J_k) = J_{k-1} \text{ for } k \leq 0. \tag{90}$$

The asymptotic distribution of $T_{p^m}(\overline{E})$ is given by the following theorem.

Theorem 6.5. *Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction. For any $k \in \{0, -1, -2, \ldots\}$, we have*

$$\deg(T_{p^m}(\overline{E})|_{I_k \cup J_k}) = O_{\overline{E},k}(1).$$

Proof. Define $\mathcal{C} : \mathrm{Div}(\mathrm{Ell}(\mathbb{C}_p)) \to \bigoplus_{k=-\infty}^{0} (\mathbb{Z}I_k \oplus \mathbb{Z}J_k)$ by

$$\mathcal{C}(D) = \sum_{k=-\infty}^{0} \deg(D|_{I_k}) I_k + \deg(D|_{J_k}) J_k.$$

Assume that $j(E) \in I_K$ for some $K \in \{0, -1, -2, \ldots\}$. By straightforward computations, using Lemma 6.3, the recursive formula (79) and (89), we get

$$\mathcal{C}(T_{p^{K+m}}(\overline{E})) = \sum_{i=0}^{-K} p^{i+m} I_{-2i-m} + \sum_{i=0}^{m-1} p^i I_{-i}, \text{ for all } m \geq 1.$$

Now, assume that $j(E) \in I_K$ for some $K \in \{0, -1, -2, \ldots\}$. As before, using Lemma 6.3, the recursive formula (79) and (90), we get

$$\mathcal{C}(T_{p^{K+m}}(\overline{E})) = \sum_{i=1}^{-K} p^{i+m} J_{-2i-m} + (p+1) \sum_{i=1}^{m/2} p^{2i-1} J_{-2i-1} + J_0$$

for $m \geq 1$ even, and

$$\mathcal{C}(T_{p^{K+m}}(\overline{E})) = \sum_{i=1}^{-K} p^{i+m} J_{-2i-m} + (p+1) \sum_{i=0}^{(m-1)/2} p^{2i} J_{-2i}$$

for $m \geq 1$ odd. In all possible cases we conclude

$$\deg(T_{p^m}(\overline{E})|_{I_k \cup J_k}) = O_{\overline{E},k}(1).$$

This completes the proof. □

We conclude this section with the following lemma which implies that the Hecke points of order n, associated to an elliptic curve with supersingular reduction, remain away from ζ_{can}, as long as n is coprime to p.

Lemma 6.6. *Suppose that $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ has supersingular reduction and C is an order n subgroup of E with $(n,p) = 1$. If $v_p(\overline{E}) < 1$ then $v_p(\overline{E/C}) = v_p(\overline{E})$.*

Proof. Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction and $\widehat{E}$ its formal group. By [**17**] section (3.10.5), there exists a parameter X for $\widehat{E}$ such that the multiplication-my-p map is given by the power series

$$[p](X) = pX + aX^p + \sum_{m \geq 2} c_m X^{m(p-1)+1}$$

with $a \equiv \mathrm{E}_{p-1}(E, \omega) \mod p\mathcal{O}_p$, $\|c_m\|_p \leq \frac{1}{p}$ for $m \not\equiv 1 \mod p$ and $\|c_{p+1}\|_p = 1$. From $\|a - \mathrm{E}_{p-1}(E,\omega)\|_p \leq \frac{1}{p}$ we deduce

$$\|a\|_p > \frac{1}{p} \Rightarrow v_p(\overline{E}) = -\log_p(\|a\|_p) < 1, \tag{91}$$

and similarly

$$v_p(\overline{E}) < 1 \Rightarrow v_p(\overline{E}) = -\log_p(\|a\|_p) < 1. \tag{92}$$

Suppose now that $v_p(\overline{E}) < 1$ and let $\phi : E \to E'$ be an isogeny with $\mathrm{Ker}(\phi) = C, \#C = n, (n,p) = 1$. Let $\widehat{E}$ and $\widehat{E'}$ be the formal groups of E and E' with parameters X, Y respectively, such that the corresponding multiplication-by-p maps are given by power series

$$[p](X) = pX + aX^p + \sum_{m \geq 2} c_m X^{m(p-1)+1},$$

$$[p](Y) = pY + bY^p + \sum_{m \geq 2} d_m Y^{m(p-1)+1},$$

as above. Since $v_p(\overline{E}) < 1$ we have $\|a\|_p > \frac{1}{p}$ by (92). The isogeny ϕ induces a morphism of formal groups

$$\phi(X) = t_1 X + t_2 X^2 + t_3 X^3 + \ldots$$

between $\widehat{E}$ and $\widehat{E'}$ with $t_i \in \mathcal{O}_p$ for all i. Since $\deg(\phi) = n$ is coprime to p, we have that ϕ is an isomorphism, or equivalently $t_1 \in \mathcal{O}_p^\times$ (see [**31**] Chapter IV). By the identity $[p] \circ \phi = \phi \circ [p]$ we get

$$\begin{aligned} & p(t_1 X + t_2 X^2 + t_3 X^3 + \ldots) + b(t_1 X + t_2 X^2 + t_3 X^3 + \ldots)^p + \ldots \\ = \; & t_1(pX + aX^p + \ldots) + t_2(pX + aX^p + \ldots)^2 + \ldots \end{aligned}$$

Comparing the X^p-coefficients we have

$$pt_p + bt_1^p = t_1 a + t_p p^p.$$

Since $\|a\|_p > \frac{1}{p}$ and $\|t_1\| = 1$ it follows that $\|t_1 a + t_p(p^p - p)\|_p = \|a\|_p$ and $\|bt_1^p\|_p = \|b\|_p$, thus $\|b\|_p = \|a\|_p > \frac{1}{p}$. By (91) we conclude

$$v_p(\overline{E'}) = -\log_p(\|b\|_p) = -\log_p(\|a\|_p) = v_p(\overline{E}).$$

This completes the proof of the lemma.

□

7. Proof of Theorem 2.1

Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$, A an open set as in (70) and $\varepsilon > 0$. If $\overline{E}$ has bad reduction, then $\overline{E} = \overline{\mathrm{Tate}(q)}$ for some $q \in D(0,1)^*$. By Lemma 1.7 and Theorem 4.1 we have

$$1 - \overline{\delta}_{T_n(\overline{E})}(A) = \frac{\deg(T_n(\overline{\mathrm{Tate}(q)})|_{\mathbb{C}_p \setminus D(0,R)})}{\sigma_1(n)} = O_{\overline{E},A,\varepsilon}(n^{-\frac{1}{2}+\varepsilon})$$

since $\sigma_1(n) \geq n$. This proves the result. Now, assume that $\overline{E}$ has ordinary reduction. Let's write $n = g_n p^{r_n}$ where $g_n \in \mathbb{Z}_{\geq 1}, r_n \in \mathbb{Z}_{\geq 0}$ and $(g_n, p) = 1$. Choose $K \in \mathbb{N}$ such that $\pi(a_1), \ldots, \pi(a_t) \in \mathbb{F}_{p^K}$ and let $\mathfrak{D}$ be the union of the residue classes $\pi^{-1}(\zeta)$ with $\zeta \in \mathbb{F}_{p^K}$. By Theorem 5.2 we have

$$\deg(T_{g_n}(\overline{E})|_{\mathfrak{D}}) = O_{\overline{E},A}(g_n^{\frac{1}{2}}).$$

It is well known (see Theorem 3.24 in [**30**] p. 63) that we have

$$\mathfrak{T}_{g_n p^{r_n}} = \mathfrak{T}_{g_n} \circ \mathfrak{T}_{p^{r_n}} = \mathfrak{T}_{p^{r_n}} \circ \mathfrak{T}_{g_n} \tag{93}$$

(recall that $\mathfrak{T}_n(j(E)) = T_n(\overline{E})$). This, together with (82) gives

$$\deg(T_n(\overline{E})|_{\mathfrak{D}}) = \sigma_1(p^{r_n}) O_{\overline{E},A}(g_n^{\frac{1}{2}})$$

hence

$$1 - \overline{\delta}_{T_n(\overline{E})}(A) \leq \frac{\deg(T_n(\overline{E})|_{\mathfrak{D}})}{\sigma_1(n)} = O_{\overline{E},A}(g_n^{-\frac{1}{2}}).$$

Here we have used that $\sigma_1(n) = \sigma_1(p^{r_n})\sigma_1(g_n) \geq \sigma_1(p^{r_n}) g_n$. This proves the result for $g_n \gg 0$. On the other hand, if we suppose that (g_n) is bounded above by some positive constant C, then by (93) and Theorem 5.7 we have

$$1 - \overline{\delta}_{T_n(\overline{E})}(A) \leq \sum_{i=1}^{t} \frac{\deg(T_n(\overline{E})|_{D^-(a_i,r)})}{\sigma_1(n)} = O_{\overline{E},A,C}(n^{-\frac{1}{2}})$$

where $D^-(a_i, r) = \{z \in \mathbb{C}_p : \|z - a_i\| \leq r\}$. This completes the proof of the result when $\overline{E}$ has ordinary reduction. Finally, assume $\overline{E}$ has supersingular reduction. By Lemma 1.7 we can assume that each $D^-(a_i, r)$ is contained in some supersingular residue class. From Theorem 6.4 we can choose $K \in \mathbb{N}$ such that

$$\bigcup_{i=1}^{t} D^-(a_i, r) \subseteq \bigcup_{i=-K}^{0} (I_k \cup J_k).$$

From Lemma 6.6 and (93) we get

$$1 - \overline{\delta}_{T_n(\overline{E})}(A) \leq \sum_{i=-K}^{0} \frac{\deg(T_n(\overline{E})|_{I_k \cup J_k})}{\sigma_1(n)} = \sum_{i=-K}^{0} \frac{\deg(T_{p^r}(\overline{E})|_{I_k \cup J_k})}{\sigma_1(p^r)}$$

if $n = gp^r$ with $(g, p) = 1$. From this and Theorem 6.3 we get

$$1 - \delta_{T_n(\overline{E})}(A) = O_{\overline{E},A}(\|n\|_p).$$

Thus, if $\|n\|_p \to 0$ we have $\delta_{T_n(\overline{E})}(A) \to 1$. Lemma 6.6 implies that the condition $\|n\|_p \to 0$ is also necessary in order to have $\delta_{T_n(\overline{E})}(A) \to 1$ for every A as in (70). This completes the proof of our main theorem.

8. Proof of Theorem 3.2

Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ be a CM elliptic curve whose endomorphism ring has discriminant d_n. Put $d_n = f_n(g_n p^{r_n})^2$ where $f_n < 0$ is a fundamental discriminant, $g_n \in \mathbb{Z}_{\geq 1}, r_n \in \mathbb{Z}_{\geq 0}$ and $(g_n, p) = 1$. Assume first that each d_n is p-ordinary. We will also assume that $f_n g_n \to -\infty$. Let $D \subset \mathcal{O}_p$ be any ordinary residue class and let $\overline{e} \in \mathrm{Ell}(\overline{\mathbb{F}_p})$ be the unique elliptic curve with $D = \pi^{-1}(j(e))$. By Lemma 5.5, the reduced curve $\widetilde{E}$ has $\mathrm{disc}(\mathrm{End}(\widetilde{E})) = f_n g_n^2$. If $\mathrm{disc}(\mathrm{End}(e)) = \mathfrak{d}\mathfrak{g}^2$, with $\mathfrak{d}$ fundamental, then

$$\Lambda_{d_n}|_D \neq 0 \Rightarrow f_n = \mathfrak{d}, g_n = \mathfrak{g}.$$

Since $f_n g_n \to -\infty$ we conclude that

$$\Lambda_{d_n}|_D = 0 \text{ for } n \gg 0.$$

This proves the result. We now assume that the sequence $(f_n g_n)$ is bounded. Without lost of generality we can assume $f_n = f_0$ and $g_n = g_0$ are both fixed. By a formula of Zhang (see Proposition 4.2.1 in [**35**]) we have

$$\mathfrak{T}_\ell\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right) = \sum_{c/\ell, c>0} R_{f_0}\left(\frac{\ell}{c}\right)\frac{\Lambda_{f_0 c^2}}{w_{f_0 c^2}}, \text{ for any } \ell \in \mathbb{Z}_{\geq 0}. \tag{94}$$

Here $R_{f_0}(n)$ is the number of integral ideals of norm n in $\mathcal{O}_{\mathbb{Q}(\sqrt{f_0})}$ and $w_{f_0 c^2} = \#(\mathbb{Z} + c\mathcal{O}_{\mathbb{Q}(\sqrt{f_0})})^\times/\mathbb{Z}^\times$. By the usual Moebius inversion formula we get

$$\frac{\Lambda_{f_0 \ell^2}}{w_{f_0 \ell^2}} = \sum_{c/\ell, c>0} R_{f_0}^{-1}\left(\frac{\ell}{c}\right)\mathfrak{T}_c\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right).$$

Here $R_{f_0}^{-1}$ is the inverse of R_{f_0} with respect to convolution of arithmetic functions. Replacing ℓ by $\ell_n = g_0 p^{r_n}$ we get

$$\Lambda_{d_n} = w_{d_n} \sum_{c/\ell_n, c>0} R_{f_0}^{-1}\left(\frac{\ell_n}{c}\right)\mathfrak{T}_c\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right).$$

For any discriminant $d < 0$ put $h_d = \#\Lambda_d$. As pointed out in [**4**] pp. 203-204, we have $h_{d_n} \geq h_{f_0}\ell_n$ and also $|R_{f_0}^{-1}(n)| = O_{\varepsilon, f_0}(n^\varepsilon)$ for any $\varepsilon > 0$.
If B is any close disc properly contained in $\mathcal{O}_p$ then we have

$$\deg\left(\mathfrak{T}_c\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right)\Bigg|_B\right) = O_{f_0, g_0, B}(c^{\frac{1}{2}}), \quad \text{for any } c/\ell_n, n \in \mathbb{N}, \tag{95}$$

by Theorem 5.7. We conclude

$$\deg(\Lambda_{d_n}|_B) = O_{f_0, g_0, B, \varepsilon}\left(\sum_{c/\ell_n, c>0}\left(\frac{\ell_n}{c}\right)^\varepsilon c^{\frac{1}{2}}\right) = O_{f_0, g_0, B, \varepsilon}(\ell_n^{\frac{1}{2}+\varepsilon}).$$

Using $h_{d_n} \geq h_{f_0}\ell_n$ we get

$$\frac{\deg(\Lambda_{d_n}|_B)}{h_{d_n}} = O_{f_0, g_0, B, \varepsilon}(\ell_n^{-\frac{1}{2}+\varepsilon}).$$

This proves the first part of the result. Now, assume that each d_n is p-supersingular. Recall that for $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with supersingular reduction we define $v_p^*(\overline{E}) = \min\left\{v_p(\overline{E}), \frac{p}{p+1}\right\}$. We will need the following result which is contained in [**7**]. For sake of completeness we give a proof of it.

Lemma 8.1. *Let $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ be a CM elliptic curve with supersingular reduction. If p^r exactly divides the conductor of $\mathrm{End}(E) \subseteq F$ (F an imaginary quadratic field), then*

$$v_p^*(\overline{E}) = \begin{cases} \frac{p^{1-r}}{p+1} & \text{if } p \text{ is inert in } F, \\ \\ \frac{p^{-r}}{2} & \text{if } p \text{ ramifies in } F. \end{cases}$$

Proof of Lemma 8.1. Let $\overline{E_1}, \overline{E_2}$ be two CM elliptic curves with supersingular reduction and isomorphic endomorphism rings. By Theorem 5 in [**19**] Chapter 8 §1, there exists an isogeny between E_1 and E_2 of degree coprime to p. By Lemma 6.6 we have $v_p^*(\overline{E_1}) = v_p^*(\overline{E_2})$. This proves that $v_p^*(E)$ depends only on $\mathrm{End}(E)$, when E has CM. Now, if $\mathrm{End}(E)$ has discriminant $d = f_0(gp^r)^2$, where f_0 is fundamental and $(g,p) = 1$, then by (94) there exists an isogeny $E_0 \to E$ of degree gm where $(m,p) = 1$ and E_0 is a CM elliptic curve with $\mathrm{disc}(\mathrm{End}(E_0)) = f_0p^{2r}$. It follows that $v_p^*(E_0) = v_p^*(E)$. Thus, replacing E by E_0 if necessary, we can assume that $\mathrm{End}(E_0)$ has conductor p^r. If p is inert in F then, by (94), we have

$$\mathfrak{T}_p\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right) = \frac{\Lambda_{f_0p^2}}{w_{f_0p^2}}. \tag{96}$$

From Lemma 6.3 we deduce that $v_p^*(E) = \frac{p}{p+1}$ if $r = 0$. If follows immediately from this and (96) that $v_p^*(E) = \frac{1}{p+1}$ if $r = 1$. The general case follows by induction on r by using (94). Now, assume that p ramifies in F. In this case there exists an element in $\mathcal{O}_F$ with norm pm, where $(m,p) = 1$. If $r = 0$ this gives an endomorphism of E of degree pm. By Lemmas 6.3 and 6.6 we must have $v_p^*(E) = \frac{1}{2}$. Now, by (94) we have

$$\mathfrak{T}_p\left(\frac{\Lambda_{f_0}}{w_{f_0}}\right) = \frac{\Lambda_{f_0}}{w_{f_0}} + \frac{\Lambda_{f_0p^2}}{w_{f_0p^2}}. \tag{97}$$

It follows that $v_p^*(E) = \frac{1}{2p}$ if $r = 1$. Again, the general case follows by induction on r by using (94). □

From Lemmas 8.1 and 6.4 we conclude that $\iota_*(\overline{\delta}_{\Lambda_{d_n}}) \to \delta_{\zeta_{can}}$ if and only if $r_n \to \infty$. This is equivalent to $\|d_n\|_p \to 0$ since fundamental discriminants are square-free or four times a square-free. This completes the proof of Theorem 3.2.

9. Proof of Theorem 3.3

We will need the following result.

Lemma 9.1. *Let p be a rational prime number congruent to* 1 *mod* 3*. There exists a constant $C_p > 0$ such that, for every CM elliptic curve $\overline{E} \in \mathrm{Ell}(\mathbb{C}_p)$ with $j(E) \neq 0$ we have $\|j(E)\|_p \geq C_p$.*

Proof of Lemma 9.1. Let $\overline{E}$ be a CM elliptic curve over $\mathbb{C}_p$ with $0 < \|j(E)\|_p < 1$. Since $p \equiv 1 \pmod 3$ the curve $\overline{E}$ has ordinary reduction. It is known that the elliptic curve $\overline{E'}$ with j-invariant 0 over $\mathbb{C}_p$ has endomorphism ring with discriminant -3 (which is fundamental). It follows, by Lemma 5.5, that the ring of endomorphisms of $\overline{E}$ must have discriminant $D = -3p^{2m}$, for some integer $m \geq 0$. By (94) we deduce that $j(E)$ is in the support of $T_{p^m}(\overline{E'})$. But Lemma 5.4 implies that $f(0) = 0$, thus we have $f^m(j(E)) = 0$ by (76). The result follows from (84). $\square$

We now proceed with the proof of Theorem 3.3, which is based on ideas of Habegger. Let S be a finite set of primes congruent to 1 mod 3. For each $p \in S$ we fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, thus any singular modular j will be considered as both element of $\mathbb{C}$ and of $\mathbb{C}_p$ indistinctly. We want to prove the finiteness of

$$\mathrm{Sing}(S) = \{j \in \mathbb{C} : j \text{ singular moduli and } \|\mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j)\|_p = 1 \text{ for all } p \notin S\}.$$

For any singular moduli j we denote by $h(j)$ the absolute logarithmic Weil height of j. Since every singular moduli is an algebraic integer, $h(j)$ is given by

$$h(j) = \frac{1}{[\mathbb{Q}(j) : \mathbb{Q}]} \sum_{\sigma} \log \max\{1, |\sigma(j)|\}$$

where the sum runs over the field embeddings $\sigma : \mathbb{Q}(j) \hookrightarrow \mathbb{C}$. We have

$$h(j) = \frac{1}{[\mathbb{Q}(j) : \mathbb{Q}]} \sum_{|\sigma(j)|>1} \log|\sigma(j)| = \frac{1}{[\mathbb{Q}(j) : \mathbb{Q}]} \left(\log|\mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j)| - \sum_{|\sigma(j)|<1} \log|\sigma(j)| \right).$$

For $j \in \mathrm{Sing}(S)$ we use the classical product formula $\prod_{p\leq\infty} \|\alpha\|_p = 1$ for $\alpha \in \mathbb{Q}$ and get

$$h(j) = -\frac{1}{[\mathbb{Q}(j) : \mathbb{Q}]} \left(\sum_{p\in S} \log \|\mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j)\|_p + \sum_{|\sigma(j)|<1} \log|\sigma(j)| \right)$$

By Lemma 9.1 there exists a constant $C_S > 0$ such that

$$\log \|\mathbf{N}_{\mathbb{Q}(j)/\mathbb{Q}}(j)\|_p \geq [\mathbb{Q}(j) : \mathbb{Q}] \log(C_S)$$

for any $p \in S$ and $j \in \mathrm{Sing}(S)$, thus

$$h(j) \leq -\log(C_S) - \frac{1}{[\mathbb{Q}(j) : \mathbb{Q}]} \sum_{|\sigma(j)|<1} \log|\sigma(j)|. \tag{98}$$

Now, by [**12**] Lemma 1 there exist absolute constants $c_1, c_2 > 0$ such that

$$h(j) \geq c_1 \log|D| - c_2 \tag{99}$$

for any singular moduli j of discriminant D. On the other hand, by [**12**] Lemma 4 there exists an absolute constant $c_3 > 0$ such that

$$\log|j| \geq -c_3 \log|D| \tag{100}$$

for any singular moduli $j \neq 0$. This, together with (98) gives

$$\begin{aligned} h(j) &\leq -\log(C_S) - \frac{1}{[\mathbb{Q}(j):\mathbb{Q}]}\left(\sum_{|\sigma(j)|<\varepsilon} \log|\sigma(j)| + \sum_{\varepsilon\leq|\sigma(j)|<1} \log|\sigma(j)|\right) \\ &\leq -\log(C_S) + \frac{n(j,\varepsilon)}{[\mathbb{Q}(j):\mathbb{Q}]} c_3 \log|D| - \log\varepsilon, \end{aligned}$$

for any $\varepsilon \in (0,1]$, where $n(j,\varepsilon)$ is the number of field embeddings $\sigma : \mathbb{Q}(j) \hookrightarrow \mathbb{C}$ with $|\sigma(j)| < \varepsilon$. By [**12**] Lemma 2 (which is proved using equidistribution of singular moduli over $\mathbb{C}$!) there exists an absolute constant $c_4 > 0$ independent of ε such that $n(j,\varepsilon) \leq c_4\varepsilon^{2/3}[\mathbb{Q}(j):\mathbb{Q}]$ if $[\mathbb{Q}(j):\mathbb{Q}]$ is sufficiently large depending on ε. This gives the upper bound

$$h(j) \leq -\log(C_S) + c_3c_4\varepsilon^{2/3}\log|D| - \log(\varepsilon).$$

Since $|D|$ grows with $[\mathbb{Q}(j):\mathbb{Q}]$, we get a contradiction with (99) if $c_3c_4\varepsilon^{2/3} \leq \frac{c_1}{2}$ and $[\mathbb{Q}(j):\mathbb{Q}]$ is large. We conclude that $[\mathbb{Q}(j):\mathbb{Q}]$ is bounded for $j \in \mathrm{Sing}(S)$. A classical result of Heilbronn and Hecke states that there are only finitely many singular moduli j with $[\mathbb{Q}(j):\mathbb{Q}]$ bounded by a given constant. This implies the result.

Appendix A: Classical modular forms and K-modular forms

In this appendix we define classical modular forms and modular forms in the sense of Katz, both of level one. One can also define modular forms of arbitrary level, but we don't need them in this thesis.

Definition 9.2. *A classical modular form of weight $k \in \mathbb{Z}$ (and level one) is an holomorphic function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, satisfying the following properties:*

(a) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

(b) *f has a Fourier expansion of the form*

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i\tau}$ and $a_n \in \mathbb{C}$.

Example: (a) For every even integer $k \geq 4$ the Eisenstein series defined by

$$\mathrm{E}_k(q) = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where b_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d>0,d/n} d^{k-1}$, is a modular form of weight k.

(b) There exists a unique modular form Δ of weight 12 with Fourier expansion

$$\Delta(q) = q \prod_{i=1}^{\infty} (1 - q^n)^{24} = q - 24q + \ldots.$$

We will see that classical modular forms can be considered as modular forms in the sense of Katz [**17**], which we call K-modular forms for simplicity. Before that, we need to recall some basic facts on the uniformization theory of elliptic curves over $\mathbb{C}$.

Let $\mathcal{L}$ be the set of all lattices in $\mathbb{C}$. A complex torus is a Riemann surface of the form $\mathbb{C}/L$ where $L \in \mathcal{L}$. Consider

$$\mathbf{T} = \{(T, \omega) : T \text{ complex torus}, \omega \text{ non-zero holomorphic differential on T}\}/\sim$$

where $(T_1, \omega_1) \sim (T_2, \omega_2)$ if there exists an analytic isomorphism $\phi : T_1 \to T_2$ with $\phi_*(\omega_1) = \omega_2$. In particular $(\mathbb{C}/L, \lambda dz) = (\mathbb{C}/\lambda L, dz)$ in $\mathbf{T}$.

Theorem 9.3. *The map $L \mapsto (\mathbb{C}/L, 2\pi i dz)$ gives a bijection between $\mathcal{L}$ and $\mathbf{T}$.*

Now consider

$$\mathbf{E} = \{(E, \omega) : E \text{ complex elliptic curve}, \omega \text{ non-zero holomorphic differential}\}/\sim$$

where $(E_1, \omega_1) \sim (E_2, \omega_2)$ if there exists an isomorphism $\phi : T_1 \to T_2$ with $\phi_*(\omega_1) = \omega_2$.

Recall that for any lattice L we have the Weierstrass function

$$\wp(z, L) := \frac{1}{z^2} + \sum_{\lambda \in L \setminus \{0\}} \frac{1}{z - \lambda^2} - \frac{1}{\lambda^2}.$$

The map $z \mapsto [\wp(z, L) : \wp'(z, L) : 1]$, for $z \notin L$, induces an isomorphism $\mathbb{C}/L \to E_L$ where E_L is the elliptic curve

$$E_L : Y^2 Z = 4X^3 - g_4(L) X Z^2 - g_6(L) Z^3$$

with

$$\begin{aligned} g_4(L) &= 60 \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^4}, \\ g_6(L) &= 140 \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^6}. \end{aligned}$$

Under this isomorphism the differential dz corresponds to dX/Y.

Theorem 9.4. *The map* $(\mathbb{C}/L, \lambda dz) \mapsto (E_L, \lambda dX/Y)$ *gives a bijection between* $\mathbf{T}$ *and* $\mathbf{E}$.

In particular, for $\tau \in \mathbb{H}$, the pair $(\mathbb{C}/2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}, dz) = (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, 2\pi i dz)$ corresponds, via the previous bijections, to the pair

$$\left(Y^2 Z = 4X^3 - \frac{E_4(q)}{12} X Z^2 + \frac{E_6(q)}{216} Z^3, dX/Y \right)$$

where $q = e^{2\pi i \tau}$ and

$$\begin{aligned} E_4(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\ E_6(q) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \end{aligned}$$

are the Eisenstein series of weight 4 and 6, respectively (recall that $q = e^{2\pi i \tau}$).

Making the substitutions $X = x + \frac{1}{12}$, $Y = x + 2y$ this pair correspond to

$$\left(\text{Tate}(q) : y^2 z + xyz = x^3 - b_2(q) x z^2 - b_3(q) z^3, \omega_{can} = \frac{dx}{2y + x} \right)$$

where

$$\begin{aligned} b_2(q) &= 5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 5q + 45q^2 + 140q^3 + \dots, \\ b_3(q) &= \sum_{n=1}^{\infty} \left(\frac{7n^2 + 5n^3}{12} \right) \frac{q^n}{1 - q^n} = q + 23q^2 + 154q^3 + \dots. \end{aligned}$$

Now, if $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k \in \mathbb{Z}$ we can define a function $F : \mathcal{L} \to \mathbb{C}$ by

$$F(L) = w_2^{-k} f\left(\frac{w_1}{w_2}\right), \text{ if } L = \mathbb{Z}w_1 + \mathbb{Z}w_2, \frac{w_1}{w_2} \in \mathbb{H}.$$

We can recover f from F by using the identity $f(\tau) = F(\mathbb{Z} + \tau\mathbb{Z})$. Using the bijections $\mathcal{L} \simeq \mathbf{T} \simeq \mathbf{E}$ we can consider F as a function on $\mathbf{E}$. We have

$$f(q) = \sum_{n=0}^{\infty} a_n q^n = F\Big(\mathrm{Tate}(q), \omega_{can}\Big).$$

Theorem 9.5. *The map $f \mapsto F$ stablish a bijection between the space of modular forms of weight k and functions $F : \mathbf{E} \to \mathbb{C}$ satisfying:*

(a) *$F(E, \lambda\omega) = \lambda^{-k} F(E, \omega)$ for all $\lambda \in \mathbb{C}^*$ and $(E, \omega) \in \mathbf{E}$.*
(b) *$F(\mathrm{Tate}(q), \omega_{can}) \in \mathbb{C}[[q]]$.*

Note that we can consider $\mathrm{Tate}(q)$ as an elliptic curve over $\mathbb{Z}[[q]]$. We now define K-modular forms.

Definition 9.6. *A K-modular form of weight $k \in \mathbb{Z}$ (and level one) over a ring R is a rule f which assigns to every pair $(E/A, \omega)$ consisting of an elliptic curve E over an R-algebra A together with an non-zero holomorphic differential ω on E, an element $f(E/A, \omega) \in A$, satisfying the following properties:*

(a) *$f(E/A, \omega)$ depends only on the A-isomorphism class of the pair $(E/A, \omega)$.*
(b) *$f(E/A, \lambda\omega) = \lambda^{-k} f(E/A, \omega)$ for every $\lambda \in A^\times$.*
(c) *The formation of $f(E/A, \omega)$ commutes with extension of scalar. More precisely, if $g : A \to A'$ is a ring morphism, then $f(E_{A'}/A', \omega) = g(f(E/A, \omega))$.*

Note that, with this definition, we have

$$f(\mathrm{Tate}(q)/(\mathbb{Z}[[q]] \otimes R), \omega_{can}) \in \mathbb{Z}[[q]] \otimes R.$$

This means that f has naturally a Fourier expansion with coefficients in R. In the context of K-modular forms one calls $f(\mathrm{Tate}(q)/(\mathbb{Z}[[q]] \otimes R), \omega_{can})$ the q-expansion of f.

From Theorem 9.5 it follows that classical modular forms define K-modular forms over $\mathbb{C}$.

We finish this section with the following result.

Theorem 9.7 (The q-expansion principle). *Let $S \to R$ be an inclusion of rings. Suppose that f is a K-modular form over R whose q-expansion has coefficients in S. Then f is a K-modular form over S.*

Appendix B: Formal groups

A (one-parameter commutative) formal group $\mathcal{F}$ defined over a ring R is a power series $F \in R[[X, Y]]$ satisfying

(a) $F(X, 0) = X$ and $F(0, Y) = Y$.
(b) $F(X, Y) = F(Y, X)$.
(c) $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

One calls $F(X, Y)$ the formal group law of $\mathcal{F}$ and writes $\mathcal{F} = (\mathcal{F}, F)$.

It can be proved that if $(\mathcal{F}, F)$ is a formal group over a ring R, then there exists a unique power series $i(X) \in R[[X]]$ such that $F(X, i(X)) = 0$.

Let $(\mathcal{F}, F), (\mathcal{G}, G)$ be two formal groups defined over R. A morphism $\mathcal{F} \to \mathcal{G}$ defined over R is a power series with no constant term $f(T) \in R[[T]]$ satisfying

$$f(F(X, Y)) = G(f(X), f(Y)).$$

Example: One can define morphisms $[n] : \mathcal{F} \to \mathcal{F}$ inductively for $n \in \mathbb{Z}$ by

$$\begin{aligned} [0](T) &= 0, \\ [m+1](T) &= F([m](T), T), \\ [m-1](T) &= F([m](T), i(T)). \end{aligned}$$

One calls $[n]$ the multiplication-by-n map.

Given an elliptic curve E over a ring R, there is a simple way of construction a formal group $\widehat{E}$ over R. This construction can be explicitly described using a Weierstrass model for E (see [**31**] Chapter IV). We don't need this explicit construction to the purposes of this thesis. We content ourselves with the fact that every isogeny between elliptic curves induces a morphism between the corresponding formal groups. More precisely, if E, E' are elliptic curves, then there is an injective morphism of commutative groups

$$\mathrm{Hom}(E, E') \to \mathrm{Hom}(\widehat{E}, \widehat{E'})$$

where $\mathrm{Hom}(\widehat{E}, \widehat{E'})$ denotes the set of all morphisms of formal groups $E \to \widehat{E}$. When $E = E'$, this morphism is also multiplicative and sends the multiplication-by-n isogeny in E to the multiplication-by-n map in $\widehat{E}$.

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