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Contents

LIST OF SYMBOLS

$\|G\|$; $\|g\|$	:	Size of the group G; order of the element $g \in G$.
$H \leq G$; $H < G$	:	H is subgroup of G; H is a proper subgroup of G.
$N \trianglelefteq G$; $N \triangleleft G$	:	N is a normal subgroup of G; N is a proper normal subgroup of G.
G/N	:	quotient group of G by the normal subgroup N.
$\|G : H\|$	:	index of H in G.
$G_1 \rtimes G_2$	:	semidirect product of G_1 by G_2.
$< g >$	:	subgroup generated by g.
$\mathbf{C}_G(H)$; $\mathbf{N}_G(H)$	:	centralizer of H in G; normalizer of H in G.
$\mathbf{Z}(G)$	:	center of G.
$[H, K]$	:	commutator subgroup of H and K, i.e, $[H,K] =< h^{-1}k^{-1}hk \ / \ h \in H, \ k \in K > .$
$\overline{H}$	:	the conjugacy class in G of the subgroup H.

$H \cong N$: H is isomorphic as groups to N.

$H \leq\sim G$: H embedded in G

$\mathbb{O}_p^H$: Orbit of p by the subgroup H

$\mathbb{O}_p$: Orbit of p by the whole group (G)

Ω_{G_p} : { left transversal of $N_G(G_p)$ in G} i.e.,
$\{l_1, ..., l_s \in G : G = \bigsqcup_{j=1}^{s} l_j N_G(G_p)\}$

$\mathbb{C}$: The Complex plane

$\mathbb{H}$: The upper half plane, i.e.,
$\{z \in \mathbb{C} : \Im z > 0\}$

Δ : The unit disc, i.e.,
$\{z \in \mathbb{C} : |z| < 1\}$

S, X, Y : use to be Riemann surfaces

S/G : The Orbit surface of S by the action of G

π_H : the intermediate Galois covering by the action of $H \leq G$, i.e.,
$\pi_H : S \to S/H$

π^H : the intermediate (possible non Galois) covering produced by the action of $H \leq G$, i.e. $\pi^H : S/H \to S/G$

ABSTRACT

We study group actions on Jacobian varieties, through the study of the action on the corresponding Riemann surfaces.

Some of our results are as follows, where S denotes a closed Riemann surface and G denotes a group acting conformally on S.

- We determine the ramification index for the Riemann-Hurwitz equations of the intermediate covers $S \to S/H$ and $S/H \to S/G$, for each subgroup H of G, in terms of (the number of fixed points of) the generators of the cyclic subgroups of the given group G.

- We also implement a computational algorithm (which runs on G.A.P.) giving all the information about the above intermediate coverings by subgroups of G (genus, cycle structure, ramification, etc.)

- We generalize the concept of signature of a cover of Riemann surfaces to the "geometric signature", which reflects the complete geometric structure of the lattice of intermediate covers in the situation given by a group G acting conformally on a closed Riemann surface S.

- With the aid of the geometric signature, we are able to describe the isotopical decomposition of the rational representation of the group G acting on JS, the Jacobian variety of S.

- We illustrate the theory developed by applying it to the case of the family of Weyl groups of type C_n, giving (in this case) new decompositions of JS as a product of generalized Prym varieties.

INTRODUCTION

For Jacobian varieties that admit action of group there are studies from different perspectives (c.f. [**6**], [**7**], [**13**], [**17**], [**18**], [**19**], [**20**]) what reflects their importance and interest; in this work, we analyze the action of groups on Jacobian varieties through the action on the corresponding Riemann surface, we give several results in this aspect to, finally, give the isotypical decomposition for the complexification of the Rational representation for the action of the group on the corresponding Jacobian variety.

The structure of this work is tacitly divided into three parts:

1. The first includes chapters 1 and 2, and it is a sort of summary of the basic definitions (chapter 1) and classical and/or modern but known results (chapter 2) that are used along this work.

2. The second part (chapters 3, 4, and 5) is the presentation of the results obtained in the course of this research.

   • In chapter 3, we introduce a new, and natural, concept which we call *the geometric signature* for the action of a finite group G on a Riemann surface; this sums up, among other things as will be seen all along this work, the information about the structure of the intermediate coverings for all the subgroups of G and, in fact, such structure determines, in turn, the geometric signature; at this point we also give a function that runs on the software G.A.P. that receives as data the geometric signature of the action and the subgroup of interest and gives as result the structure of the intermediate coverings associated to it.

   • In chapter 4, we show how a Riemann surface with action of an arbitrary group G with desired geometric signature is built; this opens a set of possibilities, among which we find the construction of surfaces in which the group G acts with certain desired behaviour for some intermediate coverings.

• In chapter 5, we present that the isotypical decomposition for the complexification of the Rational representation for the action of the group G on the Jacobian variety is determined by the geometric signature and viceversa, together with chapter 4 we can build surfaces in which the Rational representation for the action of G may contain a desired set of complex irreducible representations of G.

3. The third part, chapter 6, is an application of all presented above to concrete examples, first to the group of weyl of type $C_3 \cong \mathbb{Z}_2^3 \rtimes S_3$, that we have developed deeply enough, and then we give some results for all the $C_n \cong \mathbb{Z}_2^n \rtimes S_n$ family; these show, in particular, that the management of the geometric signature makes us control the existence of surfaces with action of groups such that the corresponding Jacobian variety be isogenous to products of desired Prym varieties associated to the intermediate coverings for the action of G on the Riemann surface.

It is worth observing that the concept of geometric signature does not correspond with the topological equivalence for group actions; it is different, perhaps related in some way. At this stage we show the information that summarizes: intermediate covering lattice, isotypical decomposition for the Rational representation; but we have still to explore, besides studying what is further needed to sum up, for instance, the isotypical decomposition for the Analytical representation.

CHAPTER 1

Definitions and Notation

In this chapter we introduce the basics, definitions and notations we will use in this work. We order them into sections, following the structure order of the following chapters.

1.1. Group actions

Throughout this work, a *Riemann surface* is a connected compact 1-complex manifold without boundary, i.e. all surfaces here are closed.

The number of elements on a group G is denoted by $|G|$.

An *action of a group G on a Riemann surface S* is given by a monomorphism of the group G onto a subgroup of $Aut(S)$ (analytical automorphism group of S). We will not differentiate between the abstract group G and the subgroup of the Automorphism Group, unless it will be strictly necessary; if $g \in G$, we will refer to the "automorphism" g.

Definition 1.1.1. (Covering) A *(smooth) covering* of S is a surjective continuous function $f : U \to S$ such that for each $v \in S$ there exists an open neighborhood W of v in S for which $f^{-1}(W)$ consists of a disjoint union of open sets U_i with $f|_{U_i}: U_i \to W$ a homeomorphism.

Definition 1.1.2. (Branched covering) A *branched covering* $P : U \to V$, between Riemann surfaces U and V, is by definition a surjective holomorphic map (in particular, non-constant). A point in U is a branch point for P if P fails to be locally one-to-one in there. The image of a branch point is a branch value of P. In this case, the set of branched points is a discrete set in U. If U and V are closed Riemann surfaces, then the set of branch points (values) is finite. We observe that $P : U - P^{-1}(B) \to V - B_P$ is a holomorphic covering, where $B_P \subset V$ is the set of branch values of P, of finite degree.

Definition 1.1.3. (Covering transformation) A *covering transformation* of a (smooth) covering $f : S \to M$ is a homeomorphism of S onto itself which interchanges points having the same projection on M. They form a group called *the Galois group of the covering*, $Gal(f : S \to M)$.

Definition 1.1.4. (Regular covering) The Galois group of a covering is called *fiber transitive* if there is a transformation of the group which carries any point P_1 over P into any other prescribed point P_2 over P. In this situation we called the covering $f : S \to M$ a *Galois or regular covering*.

Remark 1.1.1. In the situation of $f : S \to M$ being a covering, if M is a Riemann Surface there is a natural analytic structure on S that makes f be holomorphic and the covering transformations are conformal mappings from S onto itself.

For the following definitions, let us assume we have a Riemann surface S and G a group of conformal automorphisms of it.

Definition 1.1.5. (Stabilizer) The *stabilizer* of $p \in S$ is the (cyclic) subgroup of G,

$$G_p = \{g \in G : g(p) = p\}$$

Definition 1.1.6. (Orbit) The *orbit* of a point $p \in S$ is the set,

$$\mathbb{O}_p^G = \{y \in S : y = g(p) \text{ for some } g \in G\}$$

Remark 1.1.2. If the conformal group G is a finite group, the set of all orbits is also a Riemann surface.

Definition 1.1.7. (Orbit surface) We will denote by S/G the set of all orbits, and it is called the *Orbit or Quotient* surface.

DEFINITION 1.1.8. (Intermediate quotient) We will talk about *intermediate quotients*, meaning the orbit surface given by the action of a subgroup H of the full group G, which is acting on the surface. This situation produces two branched coverings, which we will denote by:

$$\pi_H : S \to S/H$$

$$\pi^H : S/H \to S/G$$

Obviously, the first one is always Galois and the second one will be Galois if and only if H is a normal subgroup of G.

DEFINITION 1.1.9. (fiber) Let $\pi : S \to S/G$ be the natural projection $\pi(s) = \mathbb{O}_s^G$, the set $\pi^{-1}(p)$, $p \in S/G$ is called *the fiber over* p.

If G is a finite group of conformal automorphisms of the Riemann surface S, then the natural projection map $\pi : S \to S/G$ is a regular branched covering.

DEFINITION 1.1.10. (Cycle structure of a covering) Let $f : X \to Y$ be a branched covering, B the set of branch values; for $b \in B$ consider its fiber $f^{-1}(b) = \{q_1, ..., q_s\} \subset X$, the *cycle structure* of f at b is an s-tuple $(n_1, ..., n_s)$ where n_i is the injectivity degree of f at q_i. This is, f is $n_i : 1$ at q_i.

REMARK 1.1.3. If the branched covering f is Galois, its cyclic structure at a branch value b will be a constant tuple $(n_b, ..., n_b)$, where the number n_b is the order of the stabilizer of the points in $f^{-1}(b)$ and the size of the tuple its index. The converse is not true. For this reason a Galois covering is described just by the numbers n_b for each branch value, this is called the signature of the group.

DEFINITION 1.1.11. (Signature of G on S) A branched covering $\pi_G : S \to S/G$ may be partially characterized by a vector of numbers $(\gamma; m_1, ..., m_r)$, called *the signature(or branching data) of G on S*, where γ is the genus of S/G, $r \leq 2\gamma + 2$ is the number of branch values of the covering and the m_i are positive integers associated to the branch values on S/G (they represent the degree of injectivity of π_G at that point).

Remark 1.1.4. The Riemann-Hurwitz equation, stated on corollary 2.1.2 next chapter, must be satisfied by the covering $S \to S/G$. This imposes restrictions on $|G|$ and the branching data that can occur.

Definition 1.1.12. (Type of a point) Consider G_i a (non-trivial) cyclic subgroup of G. A branch value $p \in S/G$ is named *"of type C_i"* (or a C_i-point), if the stabilizer of the points in its fiber are the elements of the complete conjugated class of G_i, C_i. Equivalently, if G_i is the stabilizer of at least one point in its fiber.

Remark 1.1.5.

1-. If there is a point $p \in S$ with non-trivial stabilizer, G_p, we find in its orbit points with stabilizer running on the complete conjugated class of G_p.

2-. We can take one representative of the C_i class, G_i, and talk about G_i-points. This way of talking could be a little bit ambiguous if we are on the situation we do not know all the conjugated of some cyclic subgroup G_i of G; for instance, we could have two cyclic subgroups G_1 and G_2 and on S/G points of type G_1 and G_2, if both of them belong to the same conjugated class of cyclic subgroups, those points will be points of the same type, we are just choosing different representatives of the same class, we are including the knowledge that they are actually conjugated, if we do not have that information, we will treat them as points of "different" type without ambiguity.

1.2. Fundamental Group and Monodromy representation

Definition 1.2.1. (Homotopy of loops) Let S be a Riemann surface, $q \in S$ a fixed base point.

1) A *loop based in* q is a continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = \gamma(1) = q$.

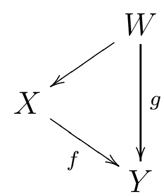
2) Two loops γ_1 and γ_2 are said *homotopics* if there exists a continuous function $G : [0,1] \times [0,1] \to S$ such that $G(0,t) = \gamma_1(t)$ and $G(1,t) = \gamma_2(t)$ for all t, and $G(s,0) = G(s,1) = q$ for all s.

Being homotopics is an equivalence relation in the set of all the loops based in q, this allows us to build what is called the Fundamental Group.

Definition 1.2.2. (Fundamental Group) The *Fundamental Group of S with base point q* is the set of all the homotopy classes of loops based in q, and is denoted by $\Pi_1(S,q)$.

DEFINITION 1.2.3. (Galois covering of a covering) Let X,Y be a Riemann Surfaces and $f : X \to Y$ an holomorphic function.

We define the *Galois covering of f* the minimum covering $g : W \to Y$ that is Galois, def. 1.1.3, and makes the following diagram to commute.



REMARK 1.2.1. It can also be defined as $g : W \to Y$ where W is a compact Riemann surface connected with the following universal property: If there exists a surface W' that accomplishes the property of the former definition, then there exists an isomorphism $h : W' \to W$ such that all the diagrams commute.

For a *(finite) presentation of a group* we will understand to give a finite set of generators and a finite set of relations which define the group.

DEFINITION 1.2.4. (Permutational representation on left cosets) Let G be a group, $N \leq G$. *The permutational representation associated to* N is given by the action of G on the right cosets of N, and is built as follows: consider $\Omega = \{g_1 N, ..., g_s N\}$ the set of left cosets of N, with $s =\mid K : N \mid$; we have that G acts in Ω by left multiplication, this produces a permutation of the elements in Ω, then we have a representation $\phi : G \to S_{|\Omega|}$.

Some basic fact about this homomorphism is that the kernel of this representation is

$$\ker \phi = \{k \in G : kg_i N = g_i N \ \ for\ all\ g_i N \in \Omega\}$$

$$\ker \phi = \{k \in G : g_i^{-1} k g_i \in N\ for\ all\ g_i \in G\}$$

$$\ker \phi = \cap_{g \in G} N^g$$

This motivates the next definition,

Definition 1.2.5. For N a subgroup of G, we define

$$CoreN = \bigcap_{g \in G} N^g,$$

it is a normal subgroup of G and maximal in N with the property of being normal in G.

Definition 1.2.6. (Monodromy representation)

We define the *Monodromy representation* of a covering $f : U \to V$, connected of degree d between Riemann surfaces, as the natural group homomorphism $\rho : \Pi_1(V, q) \longrightarrow S_d$ with transitive image in S_d.

It is built in the following way: Consider the fiber of q, $f^{-1}(q) = \{p_1, ..., p_n\}$; for $\sigma \in \Pi_1(V, q)$ we have

$$\rho(\sigma) = \begin{pmatrix} p_1 & \cdots & p_n \\ \tilde{\sigma}_1(1) & \cdots & \tilde{\sigma}_n(1) \end{pmatrix}$$

,
where $\tilde{\sigma}_i(1)$ is the final point of the lifting of σ starting from p_i.

Definition 1.2.7. (Push forward of curves) Let U and V be Riemann surfaces, and $F : U \to V$ a connected covering. There is a natural homomorphism $F_* : \Pi_1(U, p_1) \longrightarrow \Pi_1(V, q)$, where $p_1 \in F^{-1}(q)$. Given by,

$$\alpha \in \Pi_1(U, p_1) \mapsto F \circ \alpha \in \Pi_1(V, q)$$

It is well known F_* is injective, so $F_*(\Pi_1(U, p_1))$ is isomorphic to $\Pi_1(U, p_1)$

Definition 1.2.8. (Structure of cycles of a permutation) We will say that $\sigma \in S_n$ has a *structure of cycles* $(m_1, m_2, ..., m_k)$ if σ is written as a product of k disjoint cycles of orders m_i each one.

Remark 1.2.2. This is a good definition up to permutations of the factors, we will consider permuted tuples "as the same".

Definition 1.2.9. (Generating vector of type $(\gamma; m_1, ..., m_r)$, [**2**]) A $2\gamma+r$ tuple $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$ of elements of G is called a *generating vector of type* $(\gamma; m_1, ..., m_r)$ if the following are satisfied:

i) G is generated by the elements $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$;

ii) $order(c_i) = m_i$; and

iii) $\prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} c_j = 1$.

1.3. Group Representations

DEFINITION 1.3.1. (Linear representation) Let G be a (finite) group and V a vector space of finite dimension. A *linear representation* of G in V is a group homomorphism ρ from G to the group $GL(V)$; the space V is called the *representation space* or simply, by abuse of language, a *representation* of G.

DEFINITION 1.3.2. (Sub-representation) Let $\rho : G \to GL(V)$ be a linear representation and let W be a vector subspace of V. Suppose that W is *stable* under the action of G (we also say *invariant*), i.e. $x \in W$ implies $\rho(g)(x) \in W$ for all $g \in G$, then $\rho|_W : G \to GL(W)$ is a linear representation of G in W and W is said to be a *subrepresentation* of V.

DEFINITION 1.3.3. (Character of a representation) Let $\rho : G \to GL(V)$ be a linear representation of a finite group G in the vector space V. We can choose basis on V and for each $g \in G$, we define *the character of* ρ *on* g as

$$\chi_\rho(g) = Trace(\rho(g))$$

REMARK 1.3.1. It is a function on the field of definition of the vector space V, is the sum of the eigenvalues of $\rho(s)$ (counted with their multiplicities), and **does not depend on the choice of basis**.

DEFINITION 1.3.4. (Irreducible representation) Let $\rho : G \to GL(V)$ be a linear representation of G, we say it is *irreducible* or *simple* if V is not 0 and if no vector subspace of V is stable under G, except of course V and 0.

DEFINITION 1.3.5. (Scalar product) If ψ and χ are two characters of linear representations of a group G. We can define a scalar product between them as,

$$(\psi, \chi) = \frac{1}{|G|} \sum_{t \in G} \psi(t)\chi(t^{-1})$$

CHAPTER 2

Known Results

This chapter is devoted to expose some basic facts as prelims for the next chapters. It is divided on sections following the same structure as chapter 1.

2.1. Basic Facts about Group actions

First we include the well known *Riemann-Hurwitz equation*, it relates the genus of two surfaces,S and W, when there is a branch covering between them:

PROPOSITION 2.1.1. *Let $f : S \to W$ be a branch covering of degree d. Then,*

$$g_S = d(g_W - 1) \ + \ 1 \ + \ \frac{1}{2} \sum_{p \in S} (\beta_p)$$

where β_p is ramification index in $p \in S$, i.e. the injectivity degree of f at p minus 1*.*

In the particular case that the covering is given by a group action we have

COROLLARY 2.1.2. *Consider S a surface with G-action and S/G, the Orbit surface, of genus $g_{S/G}$. Then,*

$$g_S = \mid G \mid (g_{S/G} - 1) \ + \ 1 \ + \ \frac{\mid G \mid}{2} \sum_{p \in S} (1 - \frac{1}{\mid G_p \mid})$$

where G_p is the stabilizer of p.

This equation has been longer studied, and there are several presentations of it. In chapter 3 we give, among other results, another one, and we compare it with the one founded in [**9**].

In order to completeness we include some basic facts about double cosets from [**24**], we will use them on chapter 3

Proposition 2.1.3. *Let G be a group, H and K subgroups of G.*

1)A double coset is a set $HxK = \{hxk : h \in H, k \in K\}$

2) There is just one double coset containing $g \in G$; this is HgK.

3) The double cosets are disjoint. G is union (disjoint) of double cosets

4) A double coset is union of right cosets of H and of left cosets of K. The double coset HgK contains exactly $|H : K^{g^{-1}} \cap H|$ left cosets of K.

5) The cardinality of a double coset is given by:

$$|HgK| = |K||H : K^{g^{-1}} \cap H|$$

6) If n is the number of double cosets with respect to H and K in G. We have that:

$$|G| = |Hg_1K| + ... + |Hg_nK| = |K|(|H : K^{g_1^{-1}} \cap H| + ... + |H : K^{g_n^{-1}} \cap H|$$

Then

$$|G : K| = \sum_{i=1}^{n} |H : K^{g_i^{-1}} \cap H|$$

2.2. Fundamental Group and Presentation of a group

The construction of surfaces with a given action type for a given group G exposed on chapter 4 has being long studied, here we include some known results about that; some of the aspects shown in chapter 4 are unusual, however. As you will see there, we use the monodromy representation, Def. 1.2.6, of the fundamental group, to actually propose a construction of such a surface but controlling the intermediate quotients, by means of the concept of *geometric signature of G* which will be introduced on chapter 3.

The following propositions are actually results involving topological spaces, but we will state them using Riemann surfaces just for conserving the language.

Proposition 2.2.1. *All covering $F : U \to V$ has the property of* ***lifting curves****, this is for all curve γ and all $p \in F^{-1}(\gamma(0))$ in V there exists only one curve $\tilde{\gamma}$ such that $F \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = p$*

Proof. See [**11**], [**5**], [**23**]. □

Remark 2.2.1. A lifting of a curve $\gamma \subset V$ starting on p_i^*, will be closed if and only if $\gamma \in F_*(\Pi_1(U, p_i^*)$.

PROPOSITION 2.2.2. *Let V be a Riemann surface , $q \in V$ a base point. Then there exists a 1-1 correspondence between the set of Classes of isomorphisms of connected coverings $F : U \to V$ and the set of Conjugacy Classes of subgroups $H \leq \Pi_1(V, q)$.*

$$\left\{ \begin{array}{c} F : U \to V, \textit{ connected}, \\ \textit{up isomorphism} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Subgroups } H \leq \Pi_1(V, q) \\ \textit{up conjugation} \end{array} \right\}$$

PROOF. You can see a proof of it on [**11**], [**5**], [**23**] or [**12**], but we want to remark that for $p_1^* \in F^{-1}(q)$ the fundamental group based on it , $F_*(\Pi_1(U, p_1^*))$, corresponds to the subgroup H of $\Pi_1(V, q)$; i.e., $\Pi_1(U, p_1^*)$ is isomorphic to H. Moreover, you can see every point of U as the equivalence class consisting of pairs (p, C), where $p \in V$ and C is a curve that joins q to p; two pairs (p_1, C_1) and (p_2, C_2) will be identified as the same point on U if $p_1 = p_2$ and $C_1 C_2^{-1} \in H$ □

THEOREM 2.2.3. *Consider the covering $f : U \to V$, $\mathcal{K} := f_*(\Pi_1(U, p^*))$ and $\mathcal{F} := \Pi_1(V, q)$, with $p^* \in f^{-1}(q)$. Then, f is regular or Galois (i.e. its Galois group is transitive) if and only if $\mathcal{K}$ is a normal subgroup of $\mathcal{F}$; in this case, the group of covering transformations is isomorphic to $\mathcal{F}/\mathcal{K}$.*

PROOF. A complete one can be found in [**23**], here an sketch:

If $\mathcal{K}$ is not a normal subgroup of $\mathcal{F}$, there is a curve γ having a lift starting on p_i^* closed and another one starting on p_j^* opened, thus there is no covering transformation sending p_i^* to p_j^*.

If $\mathcal{K}$ is a normal subgroup, we can find a covering transformation h taking p_1^* into any other point p_j^* lying over the same point $q_0 \in V$. We have seen every point $p^* \in U$ can be written in the form $p^* = (p, C)$, where p is the projection of p^* and C is the projection of the curve from p_1^* to p^*. Now let J^* be a path from $p_1^* \to p_j^*$, which projects into J. We define the mapping:

$$h_J : U \to U$$

$$(p, C) \mapsto (p, C \circ J)$$

The key thing is each covering transformation of f can be written as h_J for some closed curve J (because h is completely determined by $p_1^* \to p_j^*$, and take J as the projection of a curve from $p_1^* \to p_j^*$). Moreover, $h_{J_1} = h_{J_2}$ if and only if $J_1 J_2^{-1} \in \mathcal{K}$, so the group of covering transformations is isomorphic to the group of cosets of $\mathcal{K}$ and hence to $\mathcal{F}/\mathcal{K}$

□

LEMMA 2.2.4. *Let $F : U \to V$ be a connected covering and $\rho : \Pi_1(V,q) \to S_d$ the monodromy representation associated to this covering (def. 1.2.6). Then,*
1) U be connected implies $Im(\rho)$ is a transitive subgroup of S_d.
2) $F_*(\Pi_1(U,p_1)) = \rho^{-1}(S_{d-1}^{(1)})$.

PROOF. The first is classical.

The second is trivial but we will prove it anyway for showing some of the involved ideas. Consider $\tau \in \Pi_1(U,p_1)$ then $F(\tau) \in F_*(\Pi_1(U,p_1)) \leq \Pi_1(V,q)$ and $F(\tau)$ is such that its lifting starting from p_1 is closed, therefore

$$\rho(F(\tau)) = \begin{pmatrix} p_1 & p_2 & \cdots & p_d \\ p_1 & p_{i_2} & \cdots & p_{i_d} \end{pmatrix}$$

then $F_*(\Pi_1(U,p_1)) \subseteq \rho^{-1}(S_{d-1})$; on the other hand $\sigma \in \rho^{-1}(S_{d-1})$ is such that its lifting starting from p_1 is closed, therefore $\sigma \in F_*(\Pi_1(U,p_1))$. □

PROPOSITION 2.2.5. *Let V be a Riemann surface, $q \in V$ a base point. Then there exists a 1-1 correspondence between the set of Classes of isomorphisms of connected coverings $F : U \to V$ of degree d and the set of Homomorphisms $\rho : \Pi_1(V,q) \to S_d$ with transitive image up conjugation in S_d.*

$$\left\{ \begin{array}{c} F : U \to V, \text{ connected} \\ \text{covering of degree } d \\ \text{up isomorphism} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \rho : \Pi_1(V,q) \to S_d, \\ \text{Homomorphisms} \\ \text{with transitive image} \\ \text{up conjugation in } S_d \end{array} \right\}$$

PROOF. First, we associate to each finite connected covering $F : U \to V$ of degree d its Monodromy representation, following the Definition 1.2.6,

$$\rho : \Pi_1(V,q) \to S_d$$

The converse, given a homomorphism $\rho : \Pi_1(V,q) \to S_d$ with transitive image, consider the subgroup S_{d-1} that fixes the 1 and the set

$$H = \{\gamma \in \Pi_1(V,q) \text{ such that } \rho(\gamma) \in S_{d-1}\}$$

H is a subgroup of $\Pi_1(V,q)$ with $\mid \Pi_1(V,q) : H \mid= d$, according to proposition 2.2.2 there exists a finite covering $F_\rho : U_\rho \to V$ of degree d associated to H; by construction, its monodromy representation is ρ. □

We include a basic example here [**12**], it will be useful in the future.

EXAMPLE 2.2.1. Let D_N be the quotient of $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ by the translation $z \to z + N$; D_N is homeomorphic to a punctured disk and the quotient function is

$$\begin{aligned} \pi_N : \ & \mathbb{H} \to D_N \\ & z \mapsto \exp(2\pi i z/N) \end{aligned}$$

Consider the following covering of D_1

$F_N : D_N \to D_1$ given by $w_1 = w_N^N$ for $N \geq 2$,

where w_1 and w_N are coordinates in D_1 and D_N respectively; F_N is called the *N-power covering*. The Galois group of F_N is isomorphic to $\mathbb{Z}/N\mathbb{Z}$, we want to compute its monodromy representation: consider $q = 1/2^N \in D_1$ as a base point and $\gamma(t) = \exp(2\pi i t)/2^N$ for $t \in [0, 1]$ a generator of the $\Pi_1(D_1, q)$; using this, it is easy to demonstrate that the monodromy representation of this covering is $\rho : \Pi_1(D_1, q) \to S_N$ such that $\rho(\gamma) = (1\ 2\ ...\ N)$

The important conclusion that we get from this example is that the monodromy representation of the N-power covering between punctured disks is $\rho : \Pi_1(D_1, q) \to S_N$ such that $\rho(\gamma) = (1\ 2\ ...\ N)$ for γ generator of the $\Pi_1(D_1, q)$.

LEMMA 2.2.6. *Let $F : U \to V$ a connected covering, V a Riemann surface. Then, there is a complex structure on U such that F is a holomorphic map,* [**12**].

This implies the following corollary

COROLLARY 2.2.7. *Given V and q as in proposition 2.2.5, we have a 1-1 correspondence between the set of Classes of isomorphisms of holomorphic functions NON ramified $F : U \to V$ of finite degree d and the set of Homomorphisms with transitive image $\rho : \Pi_1(V, q) \to S_d$ up conjugation in S_d.*

LEMMA 2.2.8. *Let $F : X \to Y$ a connected branched covering, Y a Riemann surface. Then, there is a complex structure on X such that F is a holomorphic (branched) map.*

PROOF. A precise proof can be found in [**12**]. We include here just an sketch of one.

Let $B \subset Y$ be the set of branch values, consider de covering $F' : X \setminus F^{-1}(B) \to Y \setminus B$, applying Lemma 2.2.6, we have a complex structure on $X \setminus F^{-1}(B)$ which makes $F : X \setminus F^{-1}(B) \to Y \setminus B$ holomorphic; we can extend F to an holomorphic function over Y as follows: consider $b \in B$ and W a little open set such that $b \in W \subset Y$, then $W \setminus \{b\}$ is homeomorphic to a punctured disk; if W is small enough, $F^{-1}(W \setminus \{b\}) = \bigsqcup \tilde{U}_j$, without loosing generality we may assume that each $\tilde{U}_j$ is a punctured disk, $F\mid_{\tilde{U}_j}$ is the m_j-power function and W is contained in a chart of Y. So, we have in each $\tilde{U}_j$ a punctured chart of $X \setminus F^{-1}(B)$, consider U_j the open $\tilde{U}_j$ with the *corresponding branch point*, we have an only one extension of F to a holomorphic function $F : U_j \to W$.

We make this process for each $\tilde{U}_j$ in $F^{-1}(W)$ and for each $b \in B$; the result is a Compact Riemann Surface X, since X is the finite union of the closures of the U_j over the *branch values* and the preimage of W by a finite covering.

□

EXAMPLE 2.2.2. Using this on the Example 2.2.1, the covering F_N can be extended, in a natural way to, an analytical function between the non-punctured disks; having that the punctured point is a branch point of multiplicity N, this holomorphic function is called *N-power function.*

PROPOSITION 2.2.9. *Let Y be a compact Riemann surface, B a finite subset of Y and $q \in Y \setminus B$ a base point. Then we have a 1-1 correspondence between*

$$\left\{ \begin{array}{c} Isomorphism\ classes\ of \\ holomorphic\ maps \\ F : X \to Y \\ of\ degree\ d \\ whose\ branch\ points\ lie\ in\ B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \rho : \Pi_1(Y \setminus B, q) \to S_d, \\ Homomorphisms \\ with\ transitive\ image \\ up\ \ conjugation\ in\ S_d \end{array} \right\}$$

Moreover, for all $b \in B$ if γ is a small loop around b in $Y \setminus B$ based at q (small means that it is not around any other point of B), and if $\rho(\gamma)$ has cycle structure $(m_1, ..., m_k)$. Then we have k elements u_j in the fiber of b and the corresponding holomorphic function F_ρ has multiplicity m_j in u_j, for each j

PROOF. You could find a clear explanation on [**12**]; as before, we will include an sketch of proof in order to keep in mind the important things about this result.

1. First direction, consider X, Y compact Riemann surfaces and a non-constant holomorphic function of finite degree $F : X \to Y$, let B be the set of *branch values of* F. Set $V = Y \setminus B$ and $U = X \setminus F^{-1}(B)$, now $F\mid_U : U \to V$ is an smooth

covering of degree d which has a monodromy representation, as in corollary 2.2.7, we want to describe it for the loops that surround points of B: take $b \in B$, consider a small open set W such that $b \in W \subset Y$, then the punctured open set $W \setminus \{b\} \subset V$ is homeomorphic to a punctured disk; let $\{u_1, ..., u_k\}$ be the preimages of b in X (i.e. its fiber), choose W small enough such that $F^{-1}(W)$ splits in disjoint unions of open sets $U_1, ..., U_k$, with $u_i \in U_i$ for all i. Call $m_j = mult_{u_j} F$, therefore there are coordinates z_j in U_j and z in W such that F has the form $z = z_j^{m_j}$ in U_j. We have $U_j \setminus \{u_j\}$ homeomorphic to a punctured disk, the same as $W \setminus \{b\}$, and that F sends one into the other via the m_j-power function for $j = 1..k$, we can easily demonstrate by using the example 2.2.1 that the monodromy for a small loop γ around b has structure of cycles $(m_1, ..., m_k)$. This is: a product of disjoint cycles of length depending on the number of sheets that are glued in each point of its fiber and as many as the cardinality of this.

2. On the other direction, suppose we have a homomorphism $\rho : \Pi_1(Y \setminus B, q) \to S_d$, for some d with transitive image. To each homomorphism corresponds a covering $F_\rho : U_\rho \to Y \setminus B$ finite of degree d, built as it was explained in corollary 2.2.7. Now we apply lemma 2.2.8 to extend it to a holomorphic function over Y.

□

REMARK 2.2.2. We speak of branch values *contained* in B since the homomorphism associated to that analytical function may be such that $\rho(\gamma) = id$ (with γ small loop surrounding $b \in B$). In such a case the analytical function associated does not have a branch in b. Having always in mind this detail, we will keep on talking of branch values "in" B

As a corollary the following can be stated:

COROLLARY 2.2.10. *Consider $Y = \mathbb{P}^1$ and a finite subset $B = \{b_1, ..., b_n\}$ in Y. Then we have a 1-1 correspondence between*

$$\left\{ \begin{array}{c} Isomorphism\ classes \\ of\ holomorphic\ maps \\ F : X \to \mathbb{P}^1 \\ of\ degree\ d \\ whose\ branch\ points \\ lie\ in\ B \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} Conjugacy\ Classes \\ of\ n-tuples \\ \{\sigma_1, ..., \sigma_n\} \subset S_d \\ such\ that\ \sigma_1 \cdot ... \cdot \sigma_n = 1 \\ and\ < \sigma_1, ..., \sigma_n > \\ is\ a\ transitive\ subgroup\ of\ S_d \end{array} \right\}$$

Before going further, we will collect some of the former results in two useful summarizing theorems:

THEOREM 2.2.11. *Consider $f : X \to Y$ a branched covering of degree d, B the set of branch values, ρ the monodromy representation associated to it and*

$$\Gamma^{(i)} := \rho^{-1}(S_{d-1}^{(i)}) = \rho^{-1}(S_{d-1}^{(i)} \cap Im\rho) \quad i = 1..d$$

Then,

i)$\Gamma^{(i)}$ *is conjugated to* $\Gamma^{(j)}$ *for each* i, j.
ii) $\Pi_1(X \setminus f^{-1}(B), p_1) \cong f_*(\Pi_1(X \setminus f^{-1}(B), p_1)) = \Gamma^{(1)} \leq \Pi_1(Y \setminus B, q)$
iii) $|\Pi_1(Y \setminus B, q) : \Gamma^{(i)}| = d$
iv) $\ker \rho = \bigcap_{i=1}^{d} \Gamma^{(i)}$

THEOREM 2.2.12. *In the same situation as theorem 2.2.11, the following conditions are equivalent:*
i) f *is Galois (or regular)*
ii) The Galois group of f *acts transitively on* X
iii) $f_*(\Pi_1(X \setminus f^{-1}(B), p_1)) \trianglelefteq \Pi_1(Y \setminus B, q)$
iv) $\Gamma^{(i)} = \Gamma^{(j)}$ *for each* i, j
v) $ker\rho = \Gamma^{(i)}$
vi) $ker\rho = f_*(\Pi_1(X \setminus f^{-1}(B), p_1))$
vii) $Gal(f : X \to Y) \cong \operatorname{Im} \rho$

2.3. Galois covering of a covering

In this section we develop some results about the Galois covering of a covering, Def. 1.2.3, it is an interesting and useful concept. We were aware of this construction on a lecture of professor Recillas, from Mexico; as we do not have a reference for it, we have to develop the theory by our selves, using the known results about monodromy constructions and coverings.

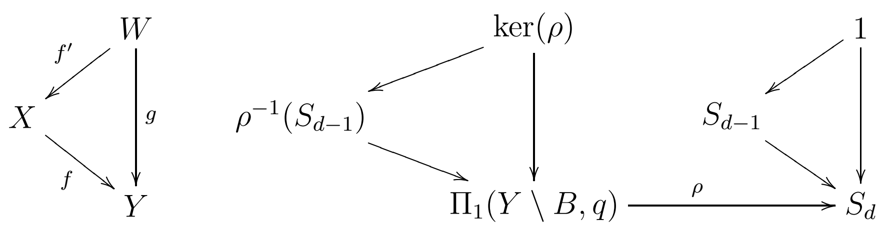
PROPOSITION 2.3.1. *Consider a (branched) covering $f : X \to Y$ of degree d between compact Riemann Surfaces X, Y. Let ρ be the monodromy representation associated to it; then, the Galois covering of f is the one associated to $ker\rho$.*

PROOF. According to what was presented in proposition 2.2.5, we have the covering $f : X \to Y$ corresponding to the subgroup $\rho^{-1}(S_{d-1})$. Following its definition, the Galois covering corresponds to the one associated to the maximal sub-group of the set

$$\mathcal{C} = \{N \leq \Pi_1(Y \setminus B, q) : N \trianglelefteq \Pi_1(Y \setminus B, q), N \leq \rho^{-1}(S_{d-1})\}$$

Consider $N \in \mathcal{C}$, so we have for all $\sigma \in N$ its lifting by f starting from p_1 is closed in $p_1 \in f^{-1}(q)$, since $\sigma \in \rho^{-1}(S_{d-1})$; as N is normal in $\Pi_1(Y \setminus B, q)$ its lifting starting from p_i is closed for all $p_i \in f^{-1}(q)$, then $\sigma \in \ker(\rho) \Rightarrow N \leq \ker(\rho)$. As $\ker(\rho) \in \mathcal{C}$, it must be the maximal element of $\mathcal{C}$. □

At this point from proposition 2.3.1, we have the following associated diagrams:



Proposition 2.3.2. *According to the former situation and diagrams, and setting $G := \mathrm{Im}(\rho) \leq S_d$. We can state:*
i)$f' : W \to X$ and $g : W \to Y$ are Galois.
ii) $Y = W/G$.
iii) $X = W/(S_{d-1} \cap G)$.

Proof. These are basic results from covering theory, we are using principally theorem 2.2.12:

i) $f'_*(\Pi_1(W \setminus g^{-1}(B), q_2)) \trianglelefteq \Pi_1(X \setminus f^{-1}(B), p_1)$

ii)
$$\begin{aligned} Gal(W \to Y) &= \frac{\Pi_1(Y \setminus B, q)}{g_*(\Pi_1(W \setminus g^{-1}(B), q_2))} \\ &= \frac{\Pi_1(Y \setminus B, q)}{ker(\rho)} \\ &\cong Im(\rho) = G \end{aligned}$$

$$\begin{aligned} \text{iii)} \quad Gal(W \to X) \quad &= \frac{\Pi_1(X \setminus f^{-1}(B), p_1)}{f'_*(\Pi_1(W \setminus g^{-1}(B), q_2))} \\ &= \frac{\rho^{-1}(S_{d-1})}{\rho^{-1}(id)} \\ &\cong S_{d-1} \cap G \end{aligned}$$

□

The next Lemma summarizes the key facts about all we have been saying about the relation among a covering f, its monodromy representation and its Galois covering.

Lemma 2.3.3. *Consider $f : X \to Y$ a (branched) covering of degree d between Riemann surfaces, let B be the set of branch values, $q \in Y \setminus B$ a base point, and $\rho : \Pi_1(Y \setminus B, q) \to S_d$ its monodromy representation; then, the Galois cover of f has Galois group isomorphic to* $\operatorname{Im} \rho$.

The next natural question is how does this Galois covering look like. To solve this, let us consider the following construction, where ϕ is the permutational representation of $G := \operatorname{Im} \rho$,

$$\begin{array}{ccc} \Pi_1(Y \setminus B, q) & \xrightarrow{\rho} & \operatorname{Im} \rho = G \\ & \searrow^{g_1 = \phi \circ \rho} & \downarrow \phi \\ & & S_{|G|} \end{array}$$

We have
i) $\ker(\rho) = \ker(g_1)$, and
ii) $g_1^{-1}(S_{|G|-1}) = \rho^{-1}(\{g \in G : g * 1 = 1\}) = \rho^{-1}(1) = \ker(\rho)$

Thus, the covering associated to $g_1^{-1}(S_{|G|-1})$, from proposition 2.2.9; which extends analytically to a function onto Y, with branch points in B and ramification index given by the structure of cycles of g_1, corresponds to the Galois covering of f because the subgroup of $\pi_1(Y \setminus B, q)$ written as $g_1^{-1}(S_{|K|-1})$ and the one written by $ker(\rho)$ are the same (no homomorphic but the same) subgroup of $\pi_1(Y \setminus B, q)$.

Then the Galois covering of f, $g : W \to Y$ has branch points in B and the cycle structure of it at every point of B is given by the structure of cycles of g_1.

It remains to describe how the ramification is over the points of B, which is equivalent to describe the structure of cycles of g_1. Consider $B = \{b_1, ..., b_r\}$, for every b_i there is a $\delta_i \in \Pi_1(Y \setminus B, q)$ that does not surround more points of B than b_i, we have for such δ_i its image by ρ, $c_i := \rho(\delta_i) \in G$; set,

$m_i =\mid c_i \mid$ and
$n_i =\mid K :< c_i >\mid$.
Then the structure of cycles of $g_1(\delta_i)$ is $\underbrace{(m_i, \cdots, m_i)}_{n_i}$, $i = 1..r$, since the structure is given by the action of G in the left cosets of 1; summarizing,

Proposition 2.3.4. *Given a branched covering $f : X \to Y$ between Riemann surfaces, with branch values in $B = \{b_1, ..., b_r\} \subset Y$ and associated monodromy representation ρ; its Galois covering, which is a branched analytical function $g : W \to Y$ between Riemann surfaces, has its branch values in the points $\{b_1, ..., b_r\}$ and each one has $|G :< \rho(\delta_i) > |$ preimages and it is $| < \rho(\delta_i) > | : 1$ at each one, where δ_i is a small loop surrounding b_i.*

2.4. Fuchsian Groups and Presentation of groups

Theorem 2.4.1. *Let Γ be a fuchsian group without elliptic elements, then*

$$\mathrm{Aut}(\Delta/\Gamma) \cong N(\Gamma)/\Gamma$$

where Δ is the unit disc and $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Aut}(\Delta)$.

Proof. See [**8**]. □

Theorem 2.4.2. *If Γ is a Fuchsian group with compact orbit space Δ/Γ of genus g. Then there are elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_r \in Aut(\Delta)$ such that the following hold.*

1) We have $\Gamma =< a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_r >$.

2)Defining relations for Γ are given by

$$c_1^{m_1}, c_2^{m_2}, \ldots, c_r^{m_r}, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} c_j$$

where the m_i are integers with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$.

3) Each non identity element of finite order in Γ lies in a unique conjugate of $< c_i >$ for suitable i. Furthermore, the cyclic groups $< c_i >$ are self-normalizing in Γ.

4) Each non identity element of finite order in Γ (a so-called elliptic element) has a unique fixed point in Δ. Each element of infinite order in Γ (a so-called hyperbolic element) acts fixed point freely on Δ.

Proof. See [**1**] □

THEOREM 2.4.3. *(Riemann's existence Theorem) A finite group G acts on a Riemann surface S of genus g with branching data $(\gamma; m_1, \ldots, m_r)$ if and only if the Riemann-Hurwitz equation, cor. 2.1.2, is satisfied and the group G has a generating vector of type $(\gamma; m_1, \ldots, m_r)$, Def. 1.2.9.*

PROOF. The theorem is cited on [**2**]. □

2.5. Group Representations

2.5.1. Basic facts about Linear representations. This section is devoted to cover some of the known facts about group representations, specially the ones used on chapter 5. Those were taken from [**22**].

THEOREM 2.5.1. *Every representation is a direct sum of irreducible representations*

THEOREM 2.5.2. *(Orthogonality relations for characters)*

1) If χ is a character of a complex irreducible representation, we have $(\chi, \chi) = 1$.

2) If χ and χ' are characters of two non isomorphic complex irreducible representations, we have $(\chi, \chi') = 0$.

THEOREM 2.5.3. *Let V be a linear representation of G with character ϕ, and suppose V decomposes into a direct sum of irreducible representations:*

$$V = W_1 \oplus \cdots \oplus W_s$$

then, if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product (ϕ, χ)

COROLLARY 2.5.4. *The number of W_i isomorphic to W does not depend on the chosen decomposition.*

This allows us to make the following definition,

DEFINITION 2.5.1. Let V be a linear representation of a finite group G, with character ψ. We call *the isotypical decomposition of V* to the decomposition of V on subrepresentations grouped by isomorphism. That is,

$$V = W_1^{(\psi,\chi_1)} \oplus \cdots \oplus W_s^{(\psi,\chi_s)}$$

where the scalar products (ψ, χ_j) could be 0.

THEOREM 2.5.5. *The number of complex irreducible representations of G (up isomorphism) is equal to the number of classes of G (conjugacy classes of elements).*

THEOREM 2.5.6. *The number of rational irreducible representations of G (up isomorphism) is equal to the number of cyclic subgroups of G up conjugation.*

We will assume more known facts like: the regular representation of a group, relations among the degrees of the representations, Schur index, etc.; almost everything can be found in [**22**] or [**4**].

Another useful theorem which follows directly from the exposed on [**4**], relates the rational and complex irreducible representations of a group G.

THEOREM 2.5.7. *For each U, complex irreducible representation of G, we have:*

Let's consider $K_U = \mathbb{Q}(\chi_U(g) : g \in G)$ galoisian extension of $\mathbb{Q}$. The (unique) rational irreducible representation whose complexification "contains"U, let's call it θ, is like:

$$\theta \cong \bigoplus_{\sigma \in Gal(K_u:\mathbb{Q})} m_U U^\sigma$$

where m_U is the Schur index (over $\mathbb{Q}$) of U.

Now we include some known results, they were used on [**9**], for example; as we can not find a reference for it, we prove them here.

LEMMA 2.5.8. *Let $\rho : G \to GL(W)$ be a complex irreducible representation. Then*

$$dim_{\mathbb{C}}(W \otimes W^*)^G = 1$$

PROOF. Let's consider $\beta : G \to GL(W \otimes W^*)$ and the projection Ψ over $(W \otimes W^*)^G$:

$$\Psi = \frac{1}{\mid G \mid} \sum_{g \in G} \beta(g)$$

We know $Im(\Psi) = (W \otimes W^*)^G$ and $dim_{\mathbb{C}} Im(\Psi) = Trace(\Psi)$ So,

$$Tr(\Psi) = \frac{1}{\mid G \mid} \sum_{g \in G} Tr(\beta(g)) =$$

$$= \frac{1}{\mid G \mid} \sum_{g \in G} \chi_\rho(g) \bar{\chi_\rho}(g) = 1$$

□

COROLLARY 2.5.9. *Let V be a non irreducible complex representation, W an isotypical component (i.e. $V = W^n \oplus .... \oplus W_r^{n_r}$ with $\rho_i : G \to W_i$); then*

1. $\dim_{\mathbb{C}} \mathrm{Fix}_G(V \otimes W_i^*) =< \theta, \rho_i >$

2. G acts as $n\rho_0$ on the space $(V \otimes W^*)^G$, with ρ_0 the trivial representation of G.

3. The isotypical component $W_j^{n_j}$ is isomorphic, as G-modulo, to $(W_j \otimes (V \otimes W_j)^G)$

PROOF.

1. Analogous to demonstration for Lem. 2.5.8

2. $(V \otimes W^*)^G$ is a space fixed by G, by definition, so G acts there with the trivial representation ρ_0 but in dimension given in 1.

3. Just straightforward □

The previous results, specially Cor. 2.5.9, allow us to make the following and useful decomposition of a complex representation,

THEOREM 2.5.10. *Given a complex representation $\theta : G \to GL(V)$, we can write the isotypical decomposition for V as follows:*

$$V \cong \bigoplus_{\{W \in Irr_{\mathbb{C}} G\}} W \otimes (V \otimes W^*)^G$$

where $(V \otimes W^*)^G := Fix_G(V \otimes W^*)$

2.5.2. Basic facts about the rational and analytical Representations. According to [**16**], everytime that we have a Riemann Surface S with $G-$action, we have associated two representations for the action of G on the Jacobian variety of S, JS:

1) The Rational Representation, $\rho_r : G \to GL(H_1(S, \mathbb{Z}) \otimes \mathbb{Q})$

2) The Analytical Representation, $\rho_a : G \to GL(H^{1,0*}(S, \mathbb{C}))$

Both are related by,

$$\rho_r \otimes \mathbb{C} \cong \rho_a \oplus \overline{\rho_a}$$

We want to describe this representations, by describing the action induced on these spaces by the action of G on the surface S.

2.5.2.1. *Just thinking of vector spaces.* If we have a linear transformation between two vector spaces V and W, $T : V \to W$. This induces one between their duals: $T^* : W^* \to V^*$.

If we choose basis on V and W, and the canonical dual basis for V^* and W^*, we have that the matrix representation of T and T^* are related by $[T^*] = [T]^t$.

In the case that we have an action of a group G over a vector space V, this induces an action of G on V^* given by:

$G \times V^* \to V^*$
$(g, f) \mapsto f(g^{-1})$

And the matrices for each $g \in G$ on chosen basis are related by:

$$[g^*] = [g]^{-t}$$

2.5.2.2. *Action on the Homology groups.* We will follow the ideas given on [**5**]. Let's consider a Riemann surface S of genus g_S, then as a topological space it has a simplectic basis for the first homology group $H_1(S, \mathbb{Z})$:

$$\{\alpha_1, ..., \alpha_{g_S}, \beta_1, ..., \beta_{g_S}\}$$

and a dual one for the space of abelian differentials, $H^{1,0}(S, \mathbb{C})$:

$$\{w_1, ..., w_{g_S}\}$$

We are considering here the group G already given as automorphisms of S, i.e. $g \in G$ is an analytic function from S to S.

i) The action on $H_1(S, \mathbb{Z})$ is given by:

$$\begin{array}{lcl} G \times H_1(S, \mathbb{Z}) & \to & H_1(S, \mathbb{Z}) \\ (g, \gamma) & \mapsto & g_* := g \circ \gamma \end{array}$$

ii) The action on $H^{1,0}(S, \mathbb{C})$ is the pullback by g^{-1}, for each $g \in G$. We will explain this using local coordinates: consider $w \in H^{1,0}(S, \mathbb{C})$, $p \in S$ and a local coordinate z vanishing on p, so on this chart we have $w = \mu(z)dz$; ξ a local coordinate vanishing on $g(p)$, and $z = f(\xi)$ the local expression for g^{-1}. We define $g^*(w) := \mu(f(\xi))f\prime(\xi)d\xi$; we will think this last, abusing on the notation, as:

$$g^*(w) := \mu(g^{-1}(\xi))g^{-1}\prime(\xi)d\xi$$

Both i) and ii) actions are compatible in the sense:

$$\int_{g_*(\gamma)} g^*(w) = \int_{\gamma} w$$

EXAMPLE 2.5.1. At this time it is worthy to give an example which will show this actions. Let's consider the group $G =< \eta >\cong \mathbb{Z}/(7\mathbb{Z})$. Acting on a Riemann surface S with action of type $(0; 7, 7, 7)$. This means that the genus of S/G is 0, and we have 3 marked points with stabilizer the whole group G. At this point we know by Riemann-Hurwitz that S has genus 3. We will follow the ideas on [**15**] applied to our particular case of G, to find the analytical and rational representations:

1. There are just two "different" Riemann surfaces, i.e. they are not conformally equivalent, with this type of G-action:

$$k_1 : y^7 = x(x-1)$$

$$k_2 : y^7 = x^2(x-1)$$

2. Base of $H^{1,0}(k_i, \mathbb{C})$ and action:
i)Base of $H^{1,0}(k_1, \mathbb{C})$

$$\{w_1 = \frac{dx}{y^6}, w_2 = \frac{dx}{y^5}, w_3 = \frac{dx}{y^4}\}$$

And the action of η^* in this base is given by the matrix

$$[\eta^*] = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^3 \end{pmatrix}$$

ii)Base of $H^{1,0}(k_2, \mathbb{C})$

$$\{w_1 = \frac{xdx}{y^6}, w_2 = \frac{xdx}{y^5}, w_4 = \frac{dx}{y^4}\}$$

And the action of η^*

$$[\eta^*] = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^4 \end{pmatrix}$$

3. Fundamental group for k_1 and k_2. The fundamental group for k_i/G is

$$\Gamma^* =< \alpha, \beta, \gamma : \alpha^7 = \beta^7 = \gamma^7 = (\alpha\beta\gamma) = 1 >$$

And the fundamental group for each of the k_i are normal subgroups of this Γ^*.
i)For k_1:

$$\Gamma_1 =< \alpha^{-i}(\alpha\beta^{-1})\alpha^i, 0 \leq i \leq 6 >=<< \alpha\beta^{-1} >>$$

ii)For k_2:

$$\Gamma_2 =< \alpha^{-i}(\alpha^2\beta^{-1})\alpha^i, 0 \leq i \leq 6 >=<< \alpha^2\beta^{-1} >>$$

4. Presentation of the group G associated in each case (i.e. Γ^*/Γ_i)
i)$G =< x, y : x^7 = xy^{-1} = (xy)^7 = 1 >$
ii)$G =< x, y : x^7 = x^2y^{-1} = (xy)^7 = 1 >$

5. Action on $H_1(k_i, \mathbb{Z})$. We used the Polygon Method (and the previous presentations of G) to find the rational representation on both cases.

$$\text{i)}\quad [x] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \xi^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi^3 \end{pmatrix}$$

$$\text{ii)}\quad [x] = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} \xi^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}$$

Both representations are conjugated using a matrix in $SL(6, \mathbb{Z})$ which is not simplectic.

2.5.3. Little Groups method. We want to apply almost everything developed on this work to the family of Weyl groups of type $C_n = \mathbb{Z}_2^n \rtimes S_n$, chapter 6. In order to achieve that, we need to know first the irreducible representations of each group of this family. Fortunately, each of them are a semidirect product of an abelian normal subgroup with another, and there is a method known as *Little groups*, explained on [**22**]. We summarize here the steps involved on the method.

Let us consider $G = A \rtimes H$, with A abelian and a normal subgroup of G; the idea is to build all the irreducible representations of G starting from the irreducible representations of A and those from certain subgroups of H.

2.5.3.1. *Notation.* We will consider the irreducible (complex) representations of the group A and the following action of G on it.

$$\Omega_A = \{\text{irreducible representations for } A\} = \{\rho_i\}_{i\in I}$$

Consider the natural action of G in Ω_A given by:

$G \times \Omega_A \xrightarrow{\varphi} \Omega_A$
$(g, \rho_i) \to \rho_i^g : A \longrightarrow GL(V)$
(conjugated representation : $\rho_i^g(a) = \rho_i(g^{-1}ag)$)
We will write

$\Omega_A^H = \{\text{representatives of the orbits by } H \text{ in } \Omega_A\} = \{\rho_j\}_{j\in\Omega_{A/H}}$

For each $j \in \Omega_{A/H}$ consider:

$$H_j = \{h \in H : \varphi(h, \rho_j) = \rho_j)\} := Stb_H \rho_j$$

$$G_j = \{g \in G : \varphi(g, \rho_j) = \rho_j)\} := Stb_G \rho_j$$

It is easy to see that $G_j = A \rtimes H_j$

2.5.3.2. *Extension of ρ_j for G_j.* Let $\bar{\rho_j}$ be the natural extension for G_j defined by:

$$\bar{\rho_j}(g) = \begin{cases} \rho_j(1) & \text{for all } g \in H_j \\ \rho_j(g) & \text{for all } g \in A \end{cases}$$

which is an irreducible representation of G_j of degree 1. So, we obtain a set of irreducible representations of G_j which depends basically on the irreducible representations of A.

2.5.3.3. *Other representations of G_j.* In order to get the representations of G, following the method explained in [**22**], we need to consider more irreducible representations of G_j.

Consider $\{\ \sigma_{ij}\ \}_{i\in I}$ the irreducible representations of H_j. In the same form as before, we get a set of irreducible representations $\{\bar{\sigma_{ij}}\}_{i\in I}$ for G_j.

Besides, deg $\bar{\sigma_{ij}} =$ deg σ_{ij}.

2.5.3.4. *Tensor representation for G_j.* We will build a new irreducible representation of the group G_j. Taking the tensor product of the previous representation. It is got this way:

$$\theta_{ij} = \bar{\rho_j} \bigotimes \bar{\sigma_{ij}}$$

.

This is an irreducible representation of G_j.

2.5.3.5. *Induced representation for G.* Taking θ_{ij} of 2.5.3.4 we construct an irreducible representation of G, $\bar{\theta_{ij}}$, by inducing it to G in the classical way.

Therefore, deg $\bar{\theta_{ij}} =\mid G : G_j \mid$ deg θ_{ij}.

We will get as many representations as:

For each stabilizer G_j we will have as many representations as H_j has. And we have $\sharp\Omega_A^H$ different stabilizers.

The result of this method is that these are *all* the irreducible representations of G, [**22**].

2.6. Known Facts about decomposition of a Jacobian variety

Just like the isotypical decomposition of the complexification of the representation of a group G on the $H_1(S,\mathbb{Z})$ treated on chapter 5. We would like to give one for the representation on the $H^{1,0}(S,\mathbb{C})$ in order to give a decomposition of the Jacobian of a Riemann surface with G action. We include here some known results that will be useful finding the decomposition of a Jacobian with action of the group C_n that we give on chapter 6.

PROPOSITION 2.6.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two coverings of curves and let's take the composition $h = g \circ f$. Then there are two natural isogenies given as follows.*

$$ps : P(Y/Z) \times P(X/Y) \to P(X/Z), ps(y,x) = f^*(y) + x$$

and

$$\Psi : JZ \times P(Y/Z) \times P(X/Y) \to JX, \Psi(x,y,z) = h^*(z) + f^*(y) + x$$

PROOF. This is demonstrated and besides the kernel of these isogenies are characterized in [**16**]. □

PROPOSITION 2.6.2. *Let W be an irreducible representation of G such that there exists a subgroup $H < G$ with*

$$< \chi_W, \chi_{Ind_H^G(1)} >= 1$$

Then there exist $g_1, ..., g_s$ with $s = dim\, W$, elements of G such that

$$W = W^{g_1 H g_1^{-1}} + ... + W^{g_s H g_s^{-1}}$$

PROOF. See [**21**] □

PROPOSITION 2.6.3. *Let V be a representation of G and be W^n an isotypic component (a component of the isotypical decomposition). Suppose that there exist subgroups H and N of G, $H < N < G$, such that the following two conditions are accomplished:*

i) $< \chi_{Ind_H^G(1)}, \chi_W >= 1$, and

ii)$\chi_{Ind_H^G(1)} - \chi_{Ind_N^G(1)} = \chi_W$

then:

$V^H = V^N \oplus (W^n)^H$ *(orthogonal sum)*

PROOF. See [**21**] □

REMARK 2.6.1. Observe that conditions i) and ii) imply W is a rational representation.

If we apply the propositions 2.6.2 and 2.6.3 to coverings, we can deduce the following corollary.

COROLLARY 2.6.4. *Let X be a curve and $G \leq Aut(X)$. Since G acts on JX, in particular acts on T_eJC, so T_eJC is a representation of G. Let $H < N < G$ as before and denote by $X_H := X/H$, $X_N := X/N$ and the natural projections $\pi_H : X \to X_H$ and $\pi : X_H \to X_N$. Then:*

$$(d\pi_H^*)_e(T_eJX_H)) = (d\pi_N^*)_e(T_eJC_N) \oplus (W^n)^H$$

the sum being orthogonal, and by orthogonal cancellation we have that:

$$(W^n)^H = (d\pi_H^*)_e T_e P(C_H/C_N)$$

Moreover $n = \dim P(X_H/X_N)$, so $W^n = sd\pi_H^* T_e P(C_H/C_N)$ con $s =$ dim W, i.e. the isotypical component of JX corresponding to W is $P(X_H/X_N)^{\dim W}$.

REMARK 2.6.2. We read this as follows:

If $\theta : G \to GL(W)$ and fill the hypothesis of the corollary 2.6.4, then $n\theta$ acts in $s = dimW$ copies of $P(C_H/C_N)$ with $n = dim\ P(C_H/C_N)$.

CHAPTER 3

Characterization of intermediate quotients of a curve with G-action

3.1. Introduction

The goal proposed to this chapter, was to find a way of summarize the information of how the structure of intermediate covers for the action of G on a Riemann surface S is, this leads on the concept of *geometric signature of* G introduced here. It allows us to group the points of S, and then compute the genus of the intermediate quotients and the structure of them; this is, to describe for each $H \leq G$, the signature of the covering $\pi_H : S \to S/H$ and the cycle structure of $\pi^H : S/H \to S/G$, which is not in general a galois covering; in an algorithmic way.

A function in G.A.P. that gives this information: genus of S/H, signture for the action of H, cycle structure of the covering $\pi^H : S/H \to S/G$, etc., in a clear and organized manner was built.

Consider a closed Riemann surface S with action of a finite group G, and recall all de definitions given on chapter 1. We want to introduce now, new and useful definitions; the first one is a natural fusion between the definitions of *signature*, Def. 1.1.11, and *type of a branch value*, Def. 1.1.12.

Definition 3.1.1. (Geometric signature of G on S) Let S be a closed Riemann surface and G be a group of conformal automorphisms. Let $p_1, ..., p_r \in S$ be a maximal collection of non-equivalent branch points, respect to G. For each $j = 1, ..., r$, we consider its stabilizer G_j. We define *the geometric signature of* G *on* S as the tuple

$$(\gamma; [m_1, C_1], ..., [m_r, C_r])$$

where γ is the genus of S/G, m_j is the order of the stabilizer subgroup G_j and C_i is the conjugacy class represented by G_i.

Remark 3.1.1.

1.- We can see the *Geometric signature of G on S*, as given by a signature, $(\gamma; m_1, ..., m_r)$, of G on S, Def. 1.1.11, plus an specification of the type of the branch values, Def. 1.1.12, associated to the m_i's. It does not depend on the set $\{p_i\}$ of points we have taken, if we take a different one its stabilizers will be conjugated to the first ones, so the geometric signature will be the same. However, in order to make computations, we need to choose a set of representatives of those conjugated classes but it will be clear that the results do not depend on what set we have taken.

2.- It is clear that the m_i's are de order of the cyclic subgroups $G_i \in C_i$. So they are redundant information here, because we are given the subgroup so we know its order; we persist including it because, in that way, it looks more like a refined version of the usual signature.

3.- It could happen that two or more of the G_i will be different representatives of the same conjugated class, as we have discussed before, it will be no problem.

4.- If we know the conjugated classes of cyclic subgroups of G, we could group the branch values on the geometric signature and give the number of points of each type (allowing to be that number zero).

If we are asking for intermediate quotients by a subgroup H, we have two branched coverings, as we have seen:

$$\pi_H : S \to S/H$$

$$\pi^H : S/H \to S/G$$

Thus, to describe the first covering we need to give the signature of H on S. And to describe the second one we need to give the cycle structure of π^H at every branch value.

Definition 3.1.2. (Geometry structure of the action) We will understand by the *Geometry structure of the action*, to give the information about all the intermediate quotients concerning to genus and branch points, i.e. to give the *signature of the action of H on S* and the *cycle structure* for the covering π^H, for each subgroup $H \leq G$.

DEFINITION 3.1.3. (To pack) Let S be a surface with G-action, consider p a point in S/G and $\pi_G^{-1}(p)$ its fiber; we will say *"to pack (or to make packages) over p"* for grouping the points of $\pi_G^{-1}(p)$ in sets, each one containing points with the **same** stabilizer, we will call each one of these sets *"a package"* and we will refer as the *"the stabilizer associated to a package"* to the stabilizer of points of this package.

3.2. Intermediate coverings description

We know we can group the points of S in fibers over each point of S/G, definition 3.1.3 allows us to have the points of $\pi_G^{-1}(p)$ grouped according its stabilizer. This is, we *pack* the fiber: each package is formed by points of the same stabilizer and two different packages have different stabilizer; clearly, the stabilizers associated to a pack of one fiber form a complete conjugacy class of cyclic subgroups. This will became clear in the following, and it will allow us to make the computation of ramification index for the Riemann-Hurwitz formula in a very easy, clear and useful way.

REMARK 3.2.1.
1) $p \in S$ then $\sharp \mathbb{O}_p = \mid G : G_p \mid$
2) $q \in \mathbb{O}_p$ (i.e. there exists $h \in G$ such that $q = h(p)$) then $G_q = G_p^{h^{-1}} = \{hgh^{-1} : g \in G_q\}$

LEMMA 3.2.1. *Given S a Riemann surface with G action, consider $p \in S$ a branch point, $\mathbb{O}_p$ its orbit and G_p its stabilizer; then,*

1) *In $\mathbb{O}_p$ there are $\mid G : N_G(G_p) \mid$ packages with different stabilizer of G_p, conjugated to it by the elements of a left transversal of its normalizer on G ($N_G(G_p)$).*
2) *Every package has $\mid N_G(G_p) : G_p \mid$ points.*

PROOF.
1) Consider two different points $q_1, q_2 \in \mathbb{O}_p$, there exist $h_1, h_2 \in G$ such that $q_i = h_i(p)$, $i = 1, 2$; then,

$$\begin{aligned} G_{q_1} = G_{q_2} &\Leftrightarrow h_1 G_p h_1^{-1} = h_2 G_p h_2^{-1} \\ &\Leftrightarrow h_2^{-1} h_1 G_p (h_2^{-1} h_1)^{-1} = G_p \\ &\Leftrightarrow h_2^{-1} h_1 \in N_G(G_p) \end{aligned}$$

hence, they have the same stabilizer if and only if they are image of p by elements of G which define the same left coset of the normalizer of G_p.

2) As we know $\sharp\mathbb{O}_p =\mid G : G_p \mid$; from the first point of the proposition, we have $\mid G : N_G(G_p) \mid$ different packages (everyone with the same quantity of elements) in $\mathbb{O}_p$, so in each package there are:

$$\frac{\mid G : G_p \mid}{\mid G : N_G(G_p) \mid} =\mid N_G(G_p) : G_p \mid$$

points. □

DEFINITION 3.2.1. Let us define the following set:

$$\Omega_{G_p} = \{ \text{ left transversal of } N_G(G_p) \text{ in } G\}$$

REMARK 3.2.2.

1-. We have grouped the points of $\mathbb{O}_p$ in $\mid G : N_G(G_p) \mid$ packages considering the action of G on it, there are $\mid N_G(G_p) : G_p \mid$ points in each one having stabilizer $G_p^{l_j^{-1}}$, for all $l_j \in \Omega_{G_p}$; i.e. each element of Ω_{G_p} determines one package of $|N_G(G_p) : G_p|$ elements contained in $\mathbb{O}_p$.

2-. It is important to remark we have demonstrated that the existence of one point $p \in S$ with non trivial stabilizer, G_p, immediately implies the existence of $|N_G(G_p) : G_p|$ points with the same stabilizer G_p, besides the existence of the same amount of point with stabilizer running in the complete conjugate class of G_p; all of them produce just one branch value $b \in S/G$. That is the reason we prefer to describe the branch values, using the geometric signature for the action of G, instead the branch points.

3.2.1. Signature for the covering $\pi_H : S \to S/H$.

As the covering π_H is Galois, its structure is described by giving the signature for the action of H; i.e., we want to give the genus of S/G and the branch values of π_H, using the information of the geometric signature.

We will start finding the genus of S in function of the genus of S/H, because it will allow us to find the signature for the action of H.

PROPOSITION 3.2.2. *Let S be a curve with action of G with geometric signature $\Gamma = (\gamma; [m_1, C_1], ..., [m_s, C_s])$; for each $H \leq G$, we will denote by S/H the quotient curve of S by the action of H; then Riemann-Hurwitz formula for the covering $S \xrightarrow{\pi_H} S/H$ can be expressed as follows*

$$g_S = \mid H \mid (g_{S/H} - 1) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \in \Omega_{G_i}} \mid N_G(G_i) : G_i \mid (\mid G_i^{l_j^{-1}} \cap H \mid - 1)$$

where G_i is a representative for the conjugated class C_i, and Ω_{G_i} is a left transversal of the normalizer in G of G_i, for all i.

PROOF. Consider Riemann-Hurwitz equation, corollary 2.1.2, now we associate to every point $p \in S$ its stabilizer in G, G_p. For the total covering each point p has ramification index $(\mid G_p \mid -1)$ and for the intermediate by H it will have $(\mid G_p \cap H \mid -1)$.

Therefore Riemann-Hurwitz formula for the intermediate quotient looks like:

$$g_S = \mid H \mid (g_{S/H} - 1) + 1 + \frac{1}{2} \sum_{p \in S} (\mid G_p \cap H \mid -1)$$

Now if we consider $q \in S/G$ we have that $\pi_G^{-1}(q) \subset S$, choose a point $p \in \pi_G^{-1}(q)$, therefore $\pi_G^{-1}(q) = \mathbb{O}_p$ (orbit by G); we have that q is then a point of G_p-type (with G_p the stabilizer of p in G); as we have $\mathbb{O}_p$ *packed*, we can index the sum on the points of S/H:

$$g_S = |H|(g_{S/H} - 1) + 1 + \frac{1}{2} \sum_{q \in S/G} \sum_{l \in \Omega_{G_q}} \mid N_G(G_q) : G_q \mid (\mid G_q^{l^{-1}} \cap H \mid -1)$$

Due to the only non tivial stabilizers are the cyclic subgroups of G including in the geometric signature Γ, the sum can be indexed upon them; in this way we get the equation of the proposition and the proof is complete. □

REMARK 3.2.3. If the previously proposed formula is taken and computed for $H = G$, we have the following:

$$g_S = \mid G \mid (g_{S/G} - 1) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \in \Omega_{G_i}} \mid N_G(G_i) : G_i \mid (\mid G_i^{l_j^{-1}} \cap G \mid - 1)$$

$$g_S = \mid G \mid (g_{S/G} - 1) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \in \Omega_{G_i}} \mid N_G(G_i) : G_i \mid (\mid G_i^{l_j^{-1}} \mid - 1)$$

$$g_S = \mid G \mid (g_{S/G} - 1) \ + \ 1 \ + \ \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \ \in \ \Omega_{G_i}} \mid N_G(G_i) : G_i \mid (\mid G_i \mid \ - 1)$$

therefore the second addition does not depend on l_j anylonger and it is only an addition $\sharp \ \Omega_{G_i}$ times the same term. As $\sharp \ \Omega_{G_i} = \mid G : N_G G_i \mid$ it follows:

$$g_S = \mid G \mid (g_{S/G} - 1) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} \mid G : N_G(G_i) \mid\mid N_G(G_i) : G_i \mid (\mid G_i \mid \ - 1)$$

therefore,

$$g_S = \mid G \mid (g_{S/G} - 1) \ + \ 1 \ + \ \frac{1}{2} \sum_{G_i \in \Gamma} \mid G : G_i \mid (\mid G_i \mid \ - 1)$$

Using proposition 3.2.2, we can describe, in the desired way, the intermediate coverings:

Genus for S/G:

PROPOSITION 3.2.3. *Let S be a curve with G action with geometric signature*

$$\Gamma := (\gamma; [m_1, C_1], ..., [m_r, C_r])$$

for each $H \leq G$, we will denote by S/H the quotient curve of S by the action of H; then, the genus of S/H is given by

$$\boxed{\begin{aligned} g_{S/H} \ &= \ \mid G : H \mid (g_{S/G} - 1) \ + \ 1 + \\ &+ \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \ \in \ \Omega_{G_i}} \frac{\mid N_G(G_i) : G_i \mid\mid G_i^{l_j^{-1}} \cap H \mid}{\mid H \mid} \left(\frac{\mid G_i^{l_j^{-1}} \mid}{\mid G_i^{l_j^{-1}} \cap H \mid} \ - \ 1 \right) \end{aligned}}$$

where G_i is a representative for the conjugated class C_i, and Ω_{G_i} is a left transversal of the normalizer in G of G_i, for all i.

PROOF. We have the following data for the points of S, for every $l_j \in \Omega_{G_i}$:
$\mid G_i^{l_j^{-1}} \cap H \mid$ which indicates how many sheets are glued at a branch value of G_i type (for the covering $\pi_G : S \to S/G$).
$\mid N_G(G_i) : G_i \mid$ which indicates how many times these sheets appear glued.
Then, to obtain the number of points in S/H that are in the fiber for $\pi^H : S/H \to S/G$ of a G_i-type branch value of π_G, it is enough to see how many images π_H has:

$$\frac{|N_G(G_i) : G_i|}{|\mathbb{O}_p^H|} = \frac{|N_G(G_i) : G_i||G_i^{l_j^{-1}} \cap H|}{|H|}$$

Each of these points will be marked with the number of sheets glued on it:

$$\frac{\mid G_i^{l_j^{-1}} \mid}{\mid G_i^{l_j^{-1}} \cap H \mid}$$

and we get the suggested Riemann-Hurwitz formula. □

REMARK 3.2.4. Up to here, not only a formula for intermediate genus has been given. If we follow carefully the demonstration of the proposition 3.2.3, we realize that we have actually found a way to describe the action in terms of signature of de action of H, even the geometric signature of the action of H if we want so, and the cyclic structure of the other covering, π^H.

Marked points for the action of $H \leq G$:

We have done all the work to find the signature for the action of H: we know that π_H is a galois covering, so the points of the fiber of every branch value of S/H have stabilizer of the same order, in fact they are conjugated in H.

Consider the following subset of Ω_{G_i}: pick one element $l_1 \in \Omega_{G_i}$, compute the value of $|G_i^{l_1^{-1}} \cap H|$ and build the set

$$L_1^i := \{l_j \in \Omega_{G_i} : |G_i^{l_j^{-1}} \cap H| = |G_i^{l_1^{-1}} \cap H|\}$$

Take now $l_2 \in \Omega_{G_i} \setminus L_1^i$,i.e. one of the resting elements of Ω_{G_i}, compute the value of $|G_i^{l_2^{-1}} \cap H|$ and form the corresponding set L_2^i as before; and so on.

What we are doing is to group the elements of Ω_{G_i} having the same value of $|G_i^{l_k^{-1}} \cap H|$. Obviously this is a finite algorithmic process. Let's call ν_i the number of sets L_k^i we have partitioning the set Ω_{G_i}.

Remark 3.2.5.

$$\sum_1^{\nu_i} |L_k^i| = |G : N_G(G_i)|$$

Using all the preceding, we have

Proposition 3.2.4. *Let S be a curve with action of G, with geometric signature*

$$\Gamma = (\gamma; [m_1, C_1], ..., [m_s, C_s])$$

then, for each $C_i \in \Gamma$ there are

$$|L_k^i| \left(\frac{|N_G(G_i) : G_i||G_i^{l_k^{-1}} \cap H|}{|H|} \right)$$

points marked with the number $|G_i^{l_k^{-1}} \cap H|$ on S/H for the action of $H \leq G$, $k = 1..\nu_i$.

Remark 3.2.6. We have for each $H \leq G$, the **signature for the action of H on S**, if we know the geometric signature for the G-action on S:

Elements of the signature tuple	Given by
Genus of S/H	Proposition 3.2.3
Marked points	Proposition 3.2.4

Remark 3.2.7. The problem with this, is we are *marking* also the points with trivial stabilizer for the action of H if they are points over a branch value for the action of G, technically they are not branch values for the action of H; we could solve this by saying: Do not consider the cases when $|G_i^{l_k^{-1}} \cap H|$ is trivial, but we will not do that, because they *are* special points, no branch values but they are pre-images of branch values for the covering $\pi^H : S/H \to S/G$.

3.2.2. Cycle structure for the covering $\pi^H : S/H \to S/G$. Doing as before, we know how the points over the branch values for the action of G are; all the former is happening over each point $b_i \in S/G$ branch value of C_i type, we know how the structure of cycles over this b_i is for the covering $\pi^H : S/H \to S/G$.

PROPOSITION 3.2.5. *Let S be a curve with G-action, with geometric signature*

$$\Gamma := (\gamma; [m_1, C_1], ..., [m_s, C_s])$$

over a branch value $b_i \in S/G$ of C_i type (for the total covering $\pi_G : S \to S/G$), $C_i \in \Gamma$, the cycle structure of π^H is given by a tuple of size

$$r_i := \sum_{k=1}^{\nu_i} |L_k^i| \left(\frac{|N_G(G_i) : G_i||G_i^{l_k^{-1}} \cap H|}{|H|} \right)$$

and form

$$(..., \underbrace{\frac{|G_i|}{|G_i^{l_k^{-1}} \cap H|}, \cdots, \frac{|G_i|}{|G_i^{l_k^{-1}} \cap H|}}_{|L_k^i| \left(\frac{|N_G(G_i):G_i||G_i^{l_k^{-1}} \cap H|}{|H|} \right)}, ...)$$

for all $C_i \in \Gamma$.

REMARK 3.2.8. The former propositions tell us that the geometric signature determines the geometric structure for the action of G, Def. 3.1.2, the converse is also true.

THEOREM 3.2.6. *Let S be a Riemann surface with G action, the Geometry structure of the action, Def. 3.1.2, is completely described by the geometric signature of the action of G, Def. 3.1.1.*

PROOF. As we have seen in propositions 3.2.3, 3.2.4 and 3.2.5, if we know the geometric signature, we now for all $H \leq G$ the signature for $\pi_H : S \to S/G$ and cycle structure for $\pi^H : S/H \to S/G$; i.e. the geometry structure of all the intermediate quotients.

Conversely, to have two different geometric signatures means that for at least one cyclic subgroup, the number of branch values of its type is different; if we take quotient by it, the quotient projection will have different branching data in both cases, actually different genus for this quotient. □

REMARK 3.2.9. We will think this theorem in terms of

Geometric signature for the G action $\Leftrightarrow$ Geometry structure of the action

COROLLARY 3.2.7. *Let S be a Riemann surface with G action, the geometric signature determines completely the geometry structure for the intermediate coverings by cyclic subgroups.*

REMARK 3.2.10. The former corollary means in particular: if we have different genus for the quotient by at least one cyclic subgroup, we have different geometric signature; we will use this on chapter 5.

3.2.3. Rewriting. Working a little with the expression presented in the proposition 3.2.3 we get the following relation between the genera of S/H and S/G:

$$g_{S/H} = |G:H|(g_{S/G}-1)+1+\frac{1}{2}\sum_{G_i\in\Gamma}\left(|G:H| - \sum_{l_j\in\Omega_{G_i}} \frac{|N_G(G_i):G_i||G_i^{l_j^{-1}}\cap H|}{|H|}\right)$$

Considering the following Lemma, we could write the equation on the proposition 3.2.3 in a different way.

LEMMA 3.2.8. *Let G be a finite group. Consider H and K subgroups and $H\backslash G/K$ a set of representatives of double cosets.*
Then:

$$|H\backslash G/K| = \sum_{l_j\,\in\,\Omega_K} \frac{|N_G(K):K||K^{l_j^{-1}}\cap H|}{|H|}$$

where Ω_K is left transversal of $N_G(K)$ (as before).

PROOF. We recall some of the facts of double cosets given on proposition 2.1.3.

Now consider the action of H in the left cosets of K (I_K):

$$H\times I_K \to I_K$$

given by multiplication for the left.

Be $g_iK\in I_K$. Then the stabilizer in H of g_iK is:

$$Stb_H(g_iK) = \{h\in H : hg_iK = g_iK\} = K^{g_i^{-1}}\cap H$$

Therefore the cardinality of the orbit of g_iK by H is:

$$|\mathbb{O}^H(g_iK)| = |H : K^{g_i^{-1}}\cap H|$$

For $i\in\{1..n\}$ where $n = |H\backslash G/K|$ is the number of double cosets. As the cardinality of $I_K = |G:K|$ and if we say that k is the number of different orbits (i.e. the cardinality of the set I_K/H). We have:

$$\sum_{i=1}^{k}|\mathbb{O}^H(g_iK)| = \sum_{i=1}^{k}|H : K^{g_i^{-1}}\cap H| = |G:K|$$

Comparing it with the point (6) of proposition 2.1.3, we have that $k = n$ this is,

$$|I_K/H| = |H\backslash G/K|$$

Now the idea is to get the cardinality of I_K/H through another way, to demonstrate our proposition.

Consider the action of G in the left cosets of K (I_K):

$$G \times I_K \to I_K$$

given by multiplication through the left.

Consider $g_iK \in I_K$. Then, the stabilizer in G of g_iK is:

$$Stb_G(g_iK) = \{g \in G : gg_iK = g_iK\} = K^{g_i^{-1}}$$

Two elements g_iK, g_jK in I_K have the same stabilizer in G if and only if g_i and g_j represent the same left coset of the N_GK in G. Because

$$K^{g_i^{-1}} = K^{g_j^{-1}} \Leftrightarrow g_i^{-1}g_j \in N_G(K)$$

Now, if to each point $g_iK \in I_K$ I associate its stabilizer in G (i.e. $Stb_G(g_iK)$) we will have the set I_K divided in $|G : N_G(K)|$ packages, in the same old sense of the word. Each package has $|N_G(K) : K|$ points (for I_K has $|G : K|$ elements) that have associated the **same** stabilizer.

Now consider the previous action but restricted to $H \leq G$ (i.e. $H \times I_K \to I_K$ multiplication by the left). The stabilizer in H of an element $g_iK \in I_K$ is:

$$Stb_H(g_iK) = Stb_G(g_iK) \cap H$$

Then the cardinality of the orbit by H of the element $g_iK \in I_K$ ($|\mathbb{O}^H(g_iK)|$) is:

$$|\mathbb{O}^H(g_iK)| = \frac{|H|}{Stb_H(g_iK)} = \frac{|H|}{Stb_G(g_iK) \cap H}$$

Previously we had assigned to every element of the set I_K its Stb_G, then considering the action of H we have that by each package of points with the **same** stabilizer in G we will have

$$\frac{\text{number of points in the package}}{\text{cardinality of the orbit}} = \frac{|N_G(K) : K||K^{g_i^{-1}} \cap H|}{|H|}$$

points in I_K/H.

The Stb_G associated to each package will be given by $K^{g_i^{-1}}$ if g_iK is in that package, with g_i representative of left coset of $N_G(K)$.

If we take all the representatives of left cosets of $N_G(K)$ we will have reviewed all the packages of points in I_K, then the whole of I_K.

Therefore, the cardinality of I_K/H is also given by:

$$\sum_{l_j \in \Omega_K} \frac{|N_G(K) : K||K^{l_j^{-1}} \cap H|}{|H|}$$

□

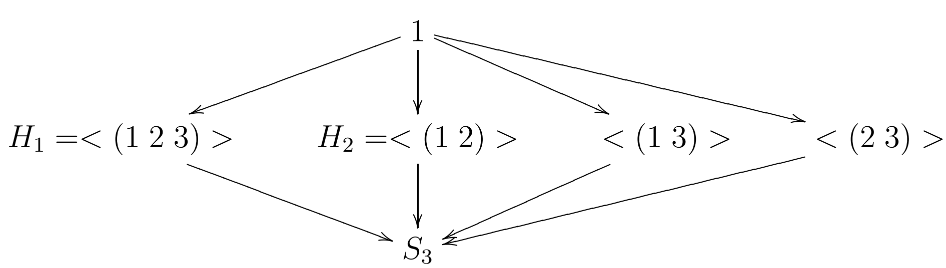
Remark 3.2.11. Using this proposition for $K = G_i$, the equation demonstrated in 3.2.3 is:

$$g_{S/H} = |G : H|(g_{S/G} - 1) \ + \ 1 \ + \ \frac{1}{2} \sum_{G_i \in \Gamma} (|G : H| - |H \backslash G/G_i|)$$

In particular this form of relating the genera of S/H and S/G is presented in [**9**] and it will be used later, the one presented on proposition 3.2.3 is very useful because, as we have seen, it allows us to determine the signature of the action of H and the cycle structure of the (non always Galois) covering $\pi^H : S/H \to S/G$.

Example 3.2.1. Here we include a little example of a covering analyzed with this machinery.

Consider S a Riemann surface with action of the symmetric group $G = S_3$.



G has two conjugacy classes of cyclic subgroups , then in S/G may there to be points of 2 types, (we only defined the type of the branch values, the regular points in S/G need not be counted); this is:

α_1 points of type $G_1 =< (1\ 2\ 3) >$ and

α_2 points of type $G_2 =< (1\ 2) >$.

In the following table the necessary data are presented:

Cyclic Subgroup G_i	Number of packages $\mid G : N_G(G_i) \mid$	Points per package $\mid N_G(G_i) : G_i \mid$	Right Cosets of $N_G G_i$ Γ_i	Stabilizer in each pack. $G_i^{l_j^{-1}}$
G_1	1	2	G	$<(1\ 2\ 3)>$
G_2	3	1	G_2 $G_2(1\ 2\ 3)$ $G_2(1\ 3\ 2)$	$<(1\ 2)>$ $<(1\ 3)>$ $<(2\ 3)>$

This permits to compute the intermediate genus of S/H_1 and S/H_2, the marked points for the H_i-action and to describe how the sheets are "glued" on the covering $S/H_i \to S/G$

$$g_S = 6(g_{S/G} - 1) + 1 + 2\alpha_1 + 3\alpha_2/2$$
$$g_{S/H_1} = 2(g_{S/G} - 1) + 1 + \alpha_2/2$$
$$g_{S/H_2} = 3(g_{S/G} - 1) + 1 + \alpha_1 + \alpha_2/2$$

Signature for π_{H_1}	$(g_{S/H_1};\ \underbrace{3,3}_{\alpha_1\text{-times}},\ \underbrace{1}_{\alpha_2\text{-times}})$
Signature for π_{H_2}	$(g_{S/H_2};\ \underbrace{1}_{\alpha_1\text{-times}},\ \underbrace{1,2}_{\alpha_2\text{-times}})$

We are able to give the structure of cycles of the covering, Def. 1.1.10:

Covering	degree	Type of the point	Structure
$S/H_1 \to S/G$	2	G_1 G_2	$(1,1)$ (2)
$S/H_2 \to S/G$	3	G_1 G_2	(3) $(1,2)$

The next one is an important example, it shows the fineness of the concept geometric signature, and its difference with the usual signature.

EXAMPLE 3.2.2. Considere as G the Weyl group of type C_3,

$$G := < x, y, z : ... > \rtimes < a, b : a^3, b^2, (ab)^2 > \cong \mathbb{Z}_2^3 \rtimes S_3$$

and two Riemann surfaces, S_1 and S_2 with G action, with the same signature, $(0; 6, 4, 2)$, but different geometric signature. We remark we do not know at this point if such surfaces actually exist, this problem is matter of the next chapter, but they do.

Consider the following representatives of different conjugacy classes of cyclic subgroups:

Representative	Order	Size of the class
$G_4 =< xyzb >$	2	6
$G_5 =< yzab >$	2	6
$G_7 =< xyab >$	4	3
$G_8 =< yab >$	4	3
$G_9 =< xa^2 >$	6	4

Consider the following non conjugated non cyclic subgroups of size 8:
$H_{23} =< y, z, xyzab >$
$H_{26} =< y, z, ab >$

S_1 Riemann surface with G action with geometric signature

$$\Gamma_1 = (0; [\overline{G_9}, 6], [\overline{G_7}, 4], [\overline{G_4}, 2])$$

S_2 Riemann surface with G action with geometric signature

$$\Gamma_2 = (0; [\overline{G_9}, 6], [\overline{G_8}, 4], [\overline{G_5}, 2])$$

If we take intermediate quotients by H_{23} and H_{26} we have:

Surface	Subgroup	genus of the quotient
S_1	H_{23}	0
	H_{26}	1
S_2	H_{23}	1
	H_{26}	0

So the lattice of the intermediate quotient, at least for these subgroups, is different, this opens new questions: do these surfaces exist?, are these actions topologicaly differents?, what is the importance of such a difference?; we will solve them step by step, the first one is the motivation for the next chapter.
You can see a *a picture* of this situation on Appendix B.

3.3. Function in G.A.P.

As we said, we create a function which runs over the software G.A.P. (*Group Algorithm Program*), let us consider a closed Riemann surface W with action of a group G with geometric signature $\Gamma := (\gamma; [m_1, C_1], ..., [m_r, C_r])$; the function considers as data G and the geometric signature Γ for the action of G, this is:

G : finite group that we want to study. It may be presented as a permutation group or a matrix group, and even as a quotient of a free group.

$l := [G_1, ..., G_r]$ a list that contains one representative for each $C_i \in \Gamma$.

H : the subgroup whose action is under study, i.e., the intermediate quotient we want to analyse.

$gwG : \gamma \in \Gamma$, i.e. the genus of W/G.

The file that contains the function, called "covering.g", must be recorded in the apropiated directory.

Once in G.A.P., we read the file :

gap>Read("covering.g");

and then we can execute the function with the given data:

gap>cove(l,G,H,gwG).

It gives the following results:

- Genus of W
- Genus of W/H
- Marked points in W/H, it says over which of the marked points of W/G given by the geometric signature, they are: this is given in pairs of numbers, [mark for the point, how many points they are].

  It is worth observing that it also specifies the points with trivial stabilizer, obviously only if these are over marked points in W/G, because it is relevant information to determine the structure of the covering $W/H \to W/G$.
- Cycle structure of the covering $W/H \to W/G$, this is given as a tuple that says which is the injectivity degree at the points in the fiber by the covering $W/H \to W/G$ for each marked point in W/G.
- Finally it gives the index of H in G and the $Core(H)$ in G. Because it is an important data which helps to solve a different question: the existence of a curve with this type of action, as we will show in the next chapter.

EXAMPLE 3.3.1. Continuing with the example 3.2.1, we will solve it with the G.A.P. function.

This text was extracted from an output file given by the function, we only show the part of the output file of interest at this stage.

Notation: "ls3"is a list that contain the representatives of the conjugate classes of cyclic subgroups of S_3.

```
gap> Read("covering.g");
gap> s3:=Group((1,2,3),(1,2));
gap> ls3:= [ Group([ (1,2,3) ]), Group([ (2,3) ]) ];
gap> h1s3:=Group((1,2,3));
gap> h2s3:=Group((1,2));
gap> cove(ls3,s3,h1s3,0);
```

For the covering $W_H \to W_G$ the structure of cycles is:

```
Group( [ (1,2,3) ] )→[ 1, 1 ]
Group( [ (2,3) ])→[ 2 ]
The index of H in G is:  2
```

```
gap> cove(ls3,s3,h2s3,0);
```

For the covering $W_H \to W_G$ the structure of cycles is:

```
Group( [ (1,2,3) ] )→[ 3 ]
Group( [ (2,3) ] )→[ 2, 1 ]
The index of H in G is:  3
```

It is clear this is enough information to compute.

REMARK 3.3.1. The function just presented does not determine if such covering does exist or not; it only says that if it exists, the structure of the covering for the action of $H \leq G$ will be the one that is given. Obviously it may be used as an instrument to verify the existence; for example if an intermediate surface appears with no an integer genus, it means that such an action of G, with that geometric signature, will not exist.

The source code of this function is found in Appendix A

CHAPTER 4

Construction of Riemann surfaces with G-action of a given type

4.1. Introduction

Here we use the facts given on chapter 2, specially the ones given on section 2.2 to solve the following problem: given a finite group G, when there is a Riemann surface S admitting the action of G with a given geometric signature; the fineness of this concept and its importance, will be clear by means of a very concrete example, $C_3 = \mathbb{Z}_2^3 \rtimes S_3$ acting on a Riemann surface of genus 3. We also study here the relation between the group structure and the galois covering of the intermediate coverings, Def. 1.2.3, we show that if we know the monodromy representation of one intermediate covering $\pi^H : S/H \to S/G$ to some $H \leq G$ with trivial Core, we know the geometric signature for the action of G, hence the complete lattice of intermediate quotient using chapter 3.

4.2. Construction Theorem

We have already enunciated the Riemann's existence theorem, theorem 2.4.3 it involves the usual signature for the action of G on S, we want to refine it and make another theorem involving the *geometric signature* of the action of G.

THEOREM 4.2.1. *Given a finite group G, there is a compact Riemann surface S of genus g where G acts with geometric signature $(\gamma; [m_1, C_1], \ldots, [m_r, C_r])$ if and only if the following two conditions hold*

i) The Riemann-Hurwitz equation, cor. 2.1.2,

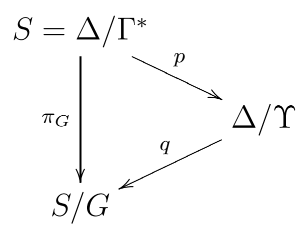
$$g = |G|(\gamma - 1) + 1 + \frac{|G|}{2} \sum_{i=1}^{r} (1 - \frac{1}{m_i})$$

is satisfied, and

ii) The group G has a tuple $(a_1, b_1, ..., a_\gamma, b_\gamma, c_1, ..., c_r)$ of generating vector of type $(\gamma; m_1, \ldots, m_r)$, Def. 1.2.9, with $< c_i > \in C_i$, for each i.

Proof. 1. Let's suppose first that there is a compact Riemann Surface, S, of genus g, where G acts with signature $(\gamma; m_1, \ldots, m_r)$:

- The uniformization theorem gives us the existence of $\Gamma^* \leq \mathrm{Aut}(\Delta)$, discrete, without elliptic elements, such that $S = \Delta/\Gamma^*$.
- By theorem 2.4.1, being Γ^* without elliptic elements, we have
$$\mathrm{Aut}(S) \cong N(\Gamma^*)/\Gamma^*$$
where $N(\Gamma^*)$ is the normalizer subgroup of Γ^* in $\mathrm{Aut}(\Delta)$.
- G acting on S means $G \leq\sim \mathrm{Aut}(S)$, using the former and one of the group isomorphism theorems, we have there is a Fuchsian group Υ such that $\Gamma^* \trianglelefteq \Upsilon \leq N(\Gamma^*)$ and $G \cong \Upsilon/\Gamma^*$.
- Consider the surfaces Δ/Υ, $S = \Delta/\Gamma^*$ and the coverings
$p : \Delta/\Gamma^* \to \Delta/\Upsilon$
$t : \Delta \to \Delta/\Upsilon$
$\nu : \Delta \to \Delta/\Gamma^*$
all of them are Galois, in particular
$$\mathrm{Gal}(p : \Delta/\Gamma^* \to \Delta/\Upsilon) \cong G$$
- There is a natural isomorphism $q : \Delta/\Upsilon \to S/G$ such that the following diagram commutes



define for $u \in \Delta/\Upsilon$, $q(u) := \pi_G(\tilde{u})$ where $\tilde{u} \in p^{-1}(u)$; we will identify S/G with Δ/Υ, so Υ uniformizes S/G.
- The geometric signature $\Gamma = (\gamma; [m_1, C_1], ..., [m_r, C_r])$ means that S/G has genus γ and there is a discrete set $\{x_1, ..., x_r\} \subset S/G$ of points on S/G marked with the number m_i meaning the size of the sabilizer of points in its fiber by π_G, the classes C_i mean such stabilizers are in those classes. With the first information we get, [**5**] theorem IV.9.12., a presentation for the group Υ, which uniformizes S/G:

$$\Gamma = < \alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma, \delta_1, ..., \delta_r : \delta_1^{m_1}, \delta_2^{m_2}, \ldots, \delta_r^{m_r}, \prod_{j=1}^{p} [\alpha_j, \beta_j] \prod_{i=1}^{r} \delta_i = 1 >$$

and the covering $t : \Delta \to \Delta/\Upsilon = S/G$ has as the set of branch values just $\{x_1, ..., x_r\}$ where is $m_i : 1$. The stabilizers of the preimages by t of a point x_i are the subgroup $< \delta_i >$ and all its conjugated, Theorem 2.4.2.

- Let us consider now the isomorphism $\theta : \Upsilon/\Gamma \to G$, then a generating vector of type given by Γ: $(\gamma; m_1, ..., m_r)$ for G is formed by the images by θ of the generating elements for Υ mod Γ^*:

$$(\theta([\alpha_1]_{\Gamma^*}), \theta([\beta_1]_{\Gamma^*}), \ldots, \theta([\alpha_g]_{\Gamma^*}), \theta([\beta_g]_{\Gamma^*}), \theta([\delta_1]_{\Gamma^*}), \ldots, \theta([\delta_r]_{\Gamma^*}))$$

- In order to show $\theta([\delta_i]_{\Gamma^*}) \in C_i \quad i = 1..r$ let's consider that the group acting on Δ/Υ is Υ/Γ^*, the a point $u \in \Delta$ having stabilizer $< \delta_i > \leq \Upsilon$, with image by t one $x_i \in S/G$, descends by ν to a point $\nu(u)$ with stabilizer

$$\frac{< \delta_i > \Gamma^*}{\Gamma^*} \leq \Upsilon/\Gamma^*$$

which corresponds to the subgroup $< \theta([\delta_i]_{\Gamma^*}) >$, hence the points $x_i \in S/G$ has type $< \theta([\delta_i]_{\Gamma^*}) >$ for the G action, then $< \theta([\delta_i]_{\Gamma^*}) > \in C_i$.

Conversely, suppose there are such a tuple of generating vectors $(a_1, b_1, ..., a_\gamma, b_\gamma, c_1, ..., c_r)$ for G. Consider Y a compact Riemann surface of genus γ, $B = \{q_1, ..., q_r\} \subset Y$ and $q \in Y \setminus B$ a base point; the Fundamental group for $Y \setminus B$ has presentation:

$$\Pi_1(Y \setminus B, q) =< \alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma, \delta_1, ..., \delta_r : \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] \prod_{i=1}^{r} \delta_i = 1 >$$

and the regular (permutational) representation of G, $\phi : G \to S_{|G|}$. Build the following group homomorphism:

$$\begin{array}{rcl}\rho:\Pi_1(Y\setminus B,q) & \rightarrow & S_{|G|}\\ \alpha_1 & \mapsto & \phi(a_1)\\ \vdots & & \vdots\\ \alpha_\gamma & \mapsto & \phi(a_\gamma)\\ \vdots & & \vdots\\ \beta_1 & \mapsto & \phi(b_1)\\ \vdots & & \vdots\\ \beta_\gamma & \mapsto & \phi(b_\gamma)\\ \delta_1 & \mapsto & \phi(c_1)\\ \vdots & & \vdots\\ \delta_r & \mapsto & \phi(c_r)\end{array}$$

The image of ρ is $\phi(G)$, a transitive subgroup of $S_{|G|}$, using Prop. 2.2.9 we build $f : S \to Y$ a branched covering of degree $|G|$ and branch values in B; this corresponds to the subgroup

$$H = \{[\gamma] \in \Pi_1(Y \setminus B, q) : \rho([\gamma]) \in S_{|G|-1} \cap \phi(G)\}$$

which in this case is $ker(\rho)$, hence $f : S \to Y$ is a regular covering with Galois group

$$\mathrm{Gal}(f : S \to Y) \cong \frac{\pi_1(Y \setminus B, q)}{f_*(\pi_1(S \setminus f^{-1}(B), p))} = \frac{\pi_1(Y \setminus B, q)}{ker(\rho)} \cong Im\rho \cong G$$

then G acts on S and $S/G \cong Y$, as Riemann-Hurwitz holds, the genus of S is g and the marked points are $\{q_1, ..., q_r\}$, using Prop. 2.2.9 we deduce that the cycle structure of $\phi(c_i)$ determines the size of the fiber of each q_i and the multiplicity of f there.
To show the type of each q_i, we have to consider the isomorphism presented on Theorem 2.2.3: it asociates to δ_i, loop surrounding q_i, the transformation corresponding presicely to the action of $\phi(c_i)$ on S.

The cyclic structure of F at a point q_i, will be given by the cyclic structure of $\rho(\delta_i) = \phi(c_i)$; it was described on proposition 2.3.4. □

COROLLARY 4.2.2. *Given a finite group G, for each set $\{g_i, i = 1..r\}$ of generators of G of orders n_i and product of order m, there is a Riemann surface S such that $G \leq Aut(S)$ and S/G having geometric signature* $(0; [n_1, < g_1 >], ..., [n_r, < g_r >], [m, < \prod_1^r g_i >])$.

REMARK 4.2.1.

1-. This is a bit more than an existence theorem, it actually gives a sort of characterization of the searched covering: the type of the branch values are given by the cyclic subgroups generated by the elements c_i in the generating vector; the cycle structure at every branch value is determined by the regular permutational representation, specifically by the cyclic structure of the c_i, it was characterized on proposition 2.3.4. Having a tuple of generating vectors for G, we know the type of the branch values of the constructed covering $f : S \to Y = S/G$, Thus we can apply chapter 3 and actually find the structure of every intermediate covering.

2-. For applying this theorem, you do not need a complete presentation of the group G (in terms of generators and relations) as is needed if you want to construct the fundamental polygon for the surface (Polygon Method); all you need here is a set of elements of G being *generating vectors* of the "right" type, you construct the surface as it was shown in the proof of theorem 4.2.1.

3-. You can use this theorem to construct surfaces with G action with some specific behavior on intermediate quotients: being of some fixed genus, its difference of genus (i.e. prym dimension) not being divisible by some number, etc. which can be useful, specially if you want to construct examples of Jacobian varieties isogenous to product of some fixed Prym varieties or anything else.

4.3. Intermediate Coverings and its Galois cover

We are going to analyze the structure of intermediate coverings, given by taking quotient by any subgroup H of G (Def.1.1.8); we show here there is a natural relation between $\pi^H : S/H \to S/G$ and the permutational representation associated to H (the representation in cosets of H of G).

PROPOSITION 4.3.1. *Let S be a Riemann surface with G action with geometric signature $\Gamma := (\gamma; [m_1, C_1], ..., [m_r, C_r])$, $B = \{b_1, ...b_r\} \subset S/G$ the set of branch values and $q \in S/G \setminus B$ a base point ; then for each $H \leq G$, the cycle structure for $\pi^H : S/H \to S/G$ over each point $b_i \in B$ is the cycle structure (Def. 1.2.8) of an element g_i such that $< g_i > \in C_i$ given by the permutational representation of G on the right cosets of H, $\phi_H : G \to S_{|G:H|}$.*

Proof. The covering π^H has its branch values, B_H, contained in B, suppose $\{b_h, ..., b_r\} = B \setminus B_H$ then

$$\Pi_1(S/G \setminus B_H, q) \cong \frac{\Pi_1(S/G \setminus B)}{<< \delta_j : j = h...r >>}$$

where δ_j is a small loop surrounding b_i, $i = h...r$; set

$\varsigma : \Pi_1(S/G \setminus B_H, q) \to \dfrac{\Pi_1(S/G \setminus B)}{<< \delta_j : j = h...r >>}$ such isomorphism, and

$pr : \Pi_1(S/G \setminus B) \to \dfrac{\Pi_1(S/G \setminus B)}{<< \delta_j : j = h...r >>}$ the natural projection.

Then, the monodromy representation for π^H, $\rho_H : \Pi_1(S/G \setminus B_H, q) \to S_{|G:H|}$ is given by

$$\delta \in \Pi_1(S/G \setminus B_H) \mapsto \phi_H \circ \rho(\widetilde{\delta})$$

where $\widetilde{\delta} \in pr^{-1}(\varsigma(\delta))$, $\rho : \Pi_1(S/G \setminus B) \to S_{|G|}$ is the monodromy representation for the total covering $\pi_G : S \to S/G$, and $\phi_H \circ \rho$ is

$$\begin{array}{ccc} \Pi_1(S/G \setminus B, q) & \xrightarrow{\rho} & G \leq S_{|G|} \\ & \searrow^{\phi_H \circ \rho} & \downarrow^{\phi_H} \\ & & S_{|G:H|} \end{array}$$

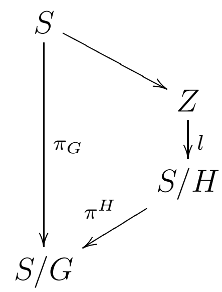
As the cyclic structure of a covering over a branch point is the ciclic structure of the image of a small loop surrounding it by the monodromy representation of the covering (Prop. 2.2.9), we have it is given by the permutational representation of G on the cosets of H.

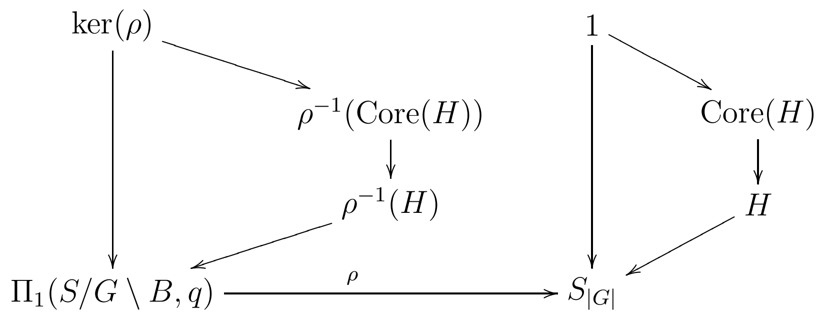
□

Now, we need to keep in mind the correspondence stated on proposition 2.2.2, U, V Riemann surfaces:

$$\left\{ \begin{array}{c} F : U \to V,\ connected, \\ up\ isomorphism \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} Subgroups\ H \leq \Pi_1(V, q) \\ up\ conjugation \end{array} \right\}$$

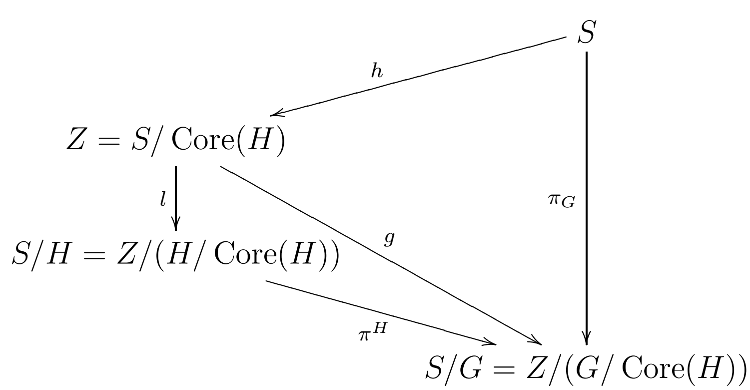
Proposition 4.3.2. *Let S be a Riemann surface with action of $G \leq S_{|G|}$, $\pi_G : S \to S/G$ the natural projection with branch values $B \subset S/G$, $q \in S/G \setminus b$ a base point and $\rho : \pi_1(S/G \setminus B, q) \to S_{|G|}$ the monodromy representation for π_G. Consider $H \leq G$ and $\pi^H : S/H \to S/G$, we have the following diagrams:*





then,
(i) the covering $\pi^H : S/H \to S/G$ corresponds to
$\rho^{-1}(H) \leq \pi_1(S/G \setminus B, q)$

(ii) the Galois covering for π^H, $g : Z \to S/G$, corresponds to the subgroup $\rho^{-1}(\mathrm{Core}(H)) \leq \pi_1(S/G \setminus B, q)$ and $\mathrm{Gal}(g : Z \to S/G) \cong G/\mathrm{Core}(H)$, then the groups acting on each surface are:



Proof. i) Follows directely from proposition 2.2.2.

ii) Is the same that demonstration of proposition 2.3.1, by definition, the Galois covering of π^H is the one corresponding to the maximal subgroup of the set

$$\mathcal{C} = \{\mathcal{N} \leq \Pi_1(S/G \setminus B, q) : \mathcal{N} \trianglelefteq \Pi_1(S/G \setminus B, q), \mathcal{N} \leq \rho^{-1}(H)\}$$

and that is exactly the main property of $\mathrm{Core}(H)$, hence of $\rho^{-1}(\mathrm{Core})$.

The Galois groups for each covering follow from clasical covering theory.

□

Corollary 4.3.3. *Let S be a compact Riemann surface with G action, given $H \leq G$, the Galois cover of $\pi^H : S/H \to S/G$ is $\pi_G : S \to S/G$ if and only if $Core(H) = \{1\}$*

Remark 4.3.1. To solve the question about the existence of a surface S with G action with a geometric signature Γ, if the group G has a subgroup H with trivial $\mathrm{Core}(H)$, then we can make the permutational representation of G on the cosets of such an H, and use it to find the generating vector of the desired type so we have the existence of S; the elements of the generating vector will tell us, by prop. **??**, the cycle structure of the covering π^H, and the Galois covering of it will be the surface with G action; but the generating vector tells us more: the type of the branch values for the total covering π_G, so we are able of finding the complete lattice of intermediate quotients.

Corollary 4.3.4. *Consider $f : X \to Y$ a branched covering of degree d between Riemann surfaces, B its set of branch values, $q \in Y \setminus B$ a base point, $\rho_f : \Pi_1(Y \setminus B, q) \to G \leq S_d$ and $g : W \to Y$ its Galois covering; then, for each $N \leq G$ we have*

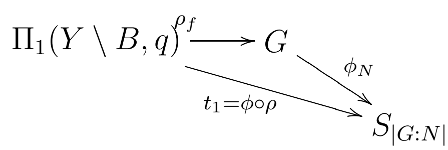
i) *The Galois cover of $\pi^N : W/N \to Y$ is g if and only if* $\mathrm{Core}(N) = \{1\}$.

ii) *The Galois cover of π^N is the corresponding to* $\rho_f^{-1}(\mathrm{Core}(N))$

iii) *Set $t : Z \to W/N$ the Galois covering of π^N, we have*

$$\begin{aligned} \mathrm{Gal}(t : Z \to W/N) &= N/\mathrm{Core}(N) \\ W/N &\cong Z/(N/\mathrm{Core}(N)) \\[1em] \mathrm{Gal}(W \to Z) &= \mathrm{Core}(N) \\ Z &\cong W/\mathrm{Core}(N) \\[1em] \mathrm{Gal}(Z \to Y) &= G/\mathrm{Core}(N) \\ Y &\cong Z/(G/\mathrm{Core}(N)) \end{aligned}$$

PROOF. i) It is just straight forward considering



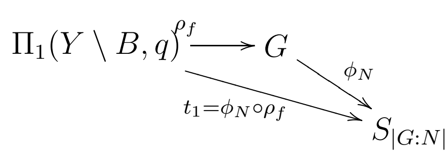
then t_1 is the monodromy representation for π^N because according with proposition 2.2.9, the covering corresponding to t_1 is the one associated to $t_1^{-1}(S_{|G:N|-1} \cap \phi_N(G))$, and

$$t_1^{-1}(S_{|G:N|-1} \cap \phi_N(G)) = \rho_f^{-1}(\{g \in G : gN = N\}) = \rho_f^{-1}(N)$$

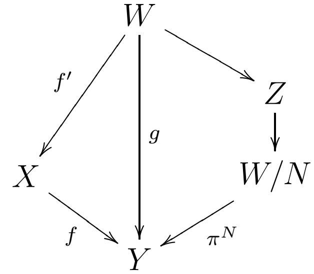
which corresponds to π^N. Now as ϕ_N is 1-1 if and only if $\mathrm{Core}(N) = \{1\}$ we have $\ker(t_1) = \ker(\rho_f)$ if and only if $\mathrm{Core}(N) = \{1\}$, hence the Galois covering of f and of π^N will be the same.

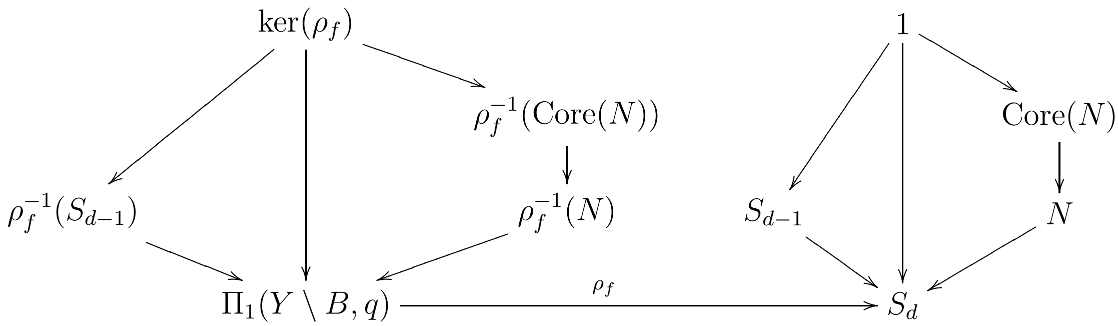
ii) Direct form propositions 4.3.2 and 2.3.2, using $X \cong W/(S_{d-1} \cap G)$.

iii) uses the same ideas involved on demonstration of proposition 2.3.2; considering



then $\ker(\rho_f) \trianglelefteq \ker(t_1)$, is a proper subgroup when $\mathrm{Core}(N)$ is not trivial, otherwise is equal; summarizing





$\square$

Example 4.3.1. Some times it could happen that there is a surface with G action for some signature but, if we specify the geometric signature there is no more; so we are being more specific with this concept.

Consider $D_4 = < x, y : x^4, y^2, (xy)^2 >$ and make the following questions:

Question 1: Is there any surface with D_4 action with signature $(0; 4, 2, 2)$?
Answer: Yes, the Riemann sphere; there D_4 acts with geometric signature $(0; [4, \overline{x}], [2, \overline{y}], [2, \overline{xy}])$

Question 2: Is there any surface with D_4 action with geometric signature $(0; [4, \overline{x}], [2, \overline{x^2}], [2, \overline{x^2}])$
Answer: No, because we can not find a generating vector of D_4 with the elements belonging to such conjugacy classes

Example 4.3.2. We continue with the example 3.2.2, the Weyl group $C_3 \cong \mathbb{Z}_2^3 \rtimes S_3$, we want to show that these, *a priori*, two surfaces really exist; we are considering S_1 with G action with geometric signature

$$\Gamma_1 = (0; [\overline{G_9}, 6], [\overline{G_7}, 4], [\overline{G_4}, 2])$$

and S_2 Riemann surface with G action with geometric signature

$$\Gamma_2 = (0; [\overline{G_9}, 6], [\overline{G_8}, 4], [\overline{G_5}, 2])$$

where the subgroups involved are

Representative	Order	Size of the class
$G_4 =< xyzb >$	2	6
$G_5 =< yzab >$	2	6
$G_7 =< xyab >$	4	3
$G_8 =< yab >$	4	3
$G_9 =< xa^2 >$	6	4

Applying Theorem 4.2.1, we check first that Riemann- Hurwitz holds, and then we need to find a generating vector for each case:

a) For Γ_1 consider $c_1 = xa^2$, $c_2 = xyab$ y $c_3 = xyzb$, we can prove by hand, or G.A.P., that:

i) $G =< c_1, c_2, c_3 >$

ii) order$(c_1) = 6$, order$(c_2) = 4$ and order$(c_3) = 2$

iii) $c_1 * c_2 * c_3 = 1$

iv) $< c_1 >\in \overline{G_9}$, $< c_2 >\in \overline{G_7}$ and $< c_3 >\in \overline{G_4}$.

b) For Γ_2 consider $c_1 = xa^2$ $c_2 = zab$ and $c_3 = b$, with:

i) $G =< c_1, c_2, c_3 >$

ii) order$(c_1) = 6$, order$(c_2) = 4$ and order$(c_3) = 2$

iii) $c_1 * c_2 * c_3 = 1$

iv) $< c_1 >\in \overline{G_9}$, $< c_2 >\in \overline{G_8}$ and $< c_3 >\in \overline{G_5}$.

Conclusion: The two situation actually exist, we can see a picture of them on appendix B.

Remark 4.3.2. In this case we can be more precise: if we saw the list of topological classification for G actions on small genus [**2**], 3 in this case, we can see there is only one topological action of a group of size 48 with quotient $(0; 6, 4, 2)$; so this concept is different from topological equivalence, and two equivalent situations topologically speaking can produce different intermediate lattices, equivalents by an external automorphism of the group. However, the situation is even more fine, as we will show on Chapter 5, this two situations have another difference: the decomposition of the rational representation for the action of G on the Jacobian of the surface.

We can feel then, that the concept geometric signature is gathering an information which were lost when we just look the topological equivalence of coverings, it is capturing more information about the embedding of the group G on the automorphisms of the surface.

Considering the following facts, we are able to find the equation of this surface S:

1) $C_3 \cong S_4 \times \mathbb{Z}_2$
2) S_4 is the group of automorphisms of the cube and of the regular octahedron.
3) S is hyperelliptic, it has 8 points of Weierstrass, these should be the vertexes of a cube inscribed in the Riemann sphere (or centers of faces of a regular octahedron).
4) The curve of Bolza: $B: \ y^2 = x(x^4 - 1)$ has its points of Wierstrass in such a way that they are vertexes of a regular octahedron. In fact $\text{Wier}(B) = \{0, \infty, 1, -1, i, -i\}$

   The element $g \in \text{Aut}(B)$ that fixes the vertexes $i, -i$ has order 4 is of the form:

$$g(x) = \frac{x-1}{x+1}$$

   it permutes the vertexes of the cube as shown in the figure 1. We solve for R_1 and R_2, using the action of g

$$\begin{array}{lcl} R_1 - iR_1 & \mapsto & -R_1 - iR_1 \\ -R_2 - iR_2 & \mapsto & R_2 - iR_2 \end{array}$$

   and we find the Weierstrass points for S:

$$\text{Wier}(S) = \{\frac{(1+i)(\sqrt{3}-1)}{2}, \frac{-(1+i)(\sqrt{3}-1)}{2}, \frac{(-1+i)(\sqrt{3}-1)}{2}, \frac{-(-1+i)(\sqrt{3}-1)}{2}, \frac{(1+i)(\sqrt{3}+1)}{2}, \frac{-(1+i)(\sqrt{3}+1)}{2}, \frac{(-1+i)(\sqrt{3}+1)}{2}, \frac{-(-1+i)(\sqrt{3}+1)}{2}\}$$

   Then

$$S : y^2 = \prod_{v \in \text{Wier}(S)} (x - v)$$

   Developing this, we get an algebraic expression for the curve S:

$$S : y^2 = x^8 + 14x^4 + 1$$

Having the algebraic expresion for S, we are able to find its full group of automorphisms, we did it: it is a group of size 48, and is isomorphic to C_3, it has one conjugacy class of function of order 4, one of order 2 and one of order 6 fixing points on S, and one class of order 4 and one of order 2 which do not fix point and G is generated by the two set of functions, so we can make two different concretizations of G as $\mathrm{Aut}(S)$ for each geometric signature Γ_i:

Γ_1	Γ_2
$g9 \mapsto$ $x \longrightarrow \frac{x+i}{x-i}$, $y \longrightarrow \frac{4y}{(x-i)^4}$	$g9 \mapsto$ $x \longrightarrow \frac{x+i}{x-i}$, $y \longrightarrow \frac{4y}{(x-i)^4}$
$g7 \mapsto$ $x \longrightarrow \frac{x-1}{x+1}$, $y \longrightarrow \frac{-4y}{(x+1)^4}$	$g7 \mapsto$ $x \longrightarrow \frac{x-1}{x+1}$, $y \longrightarrow \frac{4y}{(x+1)^4}$
$g4 \mapsto$ $x \longrightarrow \frac{i}{x}$, $y \longrightarrow \frac{-y}{x^4}$	$g4 \mapsto$ $x \longrightarrow \frac{i}{x}$, $y \longrightarrow \frac{y}{x^4}$
$g8 \mapsto$ $x \longrightarrow \frac{x-1}{x+1}$, $y \longrightarrow \frac{4y}{(x+1)^4}$	$g8 \mapsto$ $x \longrightarrow \frac{x-1}{x+1}$, $y \longrightarrow \frac{-4y}{(x+1)^4}$
$g5 \mapsto$ $x \longrightarrow \frac{i}{x}$, $y \longrightarrow \frac{y}{x^4}$	$g5 \mapsto$ $x \longrightarrow \frac{i}{x}$, $y \longrightarrow \frac{-y}{x^4}$

where $G_j =< g_j >$ for $i \in \{4, 5, 7, 8, 9\}$.

FIGURE 1. Action of g

CHAPTER 5

Isotypical decomposition of the Rational Representation

5.1. Introduction

According to [**16**], everytime that we have a Riemann Surface S with $G-$action we have associated two representations for the cation of G on the Jacobian variety corresponding to S:

1) The Rational Representation, $\rho_{RAC} : G \to GL(H_1(S,\mathbb{Z}) \otimes \mathbb{Q})$

2) The Analytical Representation, $\rho_a : G \to GL(H^{1,0*}(S,\mathbb{C}))$

Both are related by,

$$\rho_{RAC} \otimes \mathbb{C} \cong \rho_a \oplus \overline{\rho_a}$$

This chapter is devoted to find the isotypical decomposition for the complexification of the Rational Representation. In order to achieve this, we find the multiplicity of each complex irreducible representation in terms of the geometric signature of the G action.

5.2. Isotypical decomposition for the complexification of the rational representation

Let's consider the following definition and theorems 2.5.10 and 2.5.7,

Definition 5.2.1. Let U be a complex irreducible representation of G, Consider de galois group associated to the field extension $K_U = \mathbb{Q}(\chi_U(g) : g \in G)$. We will call the set $\{U^\sigma : \sigma \in \mathrm{Gal}(K_U : \mathbb{Q})\}$ a Complete Galois Class of U

Using the former definition, we can write the isotypical decomposition for any rational representation of G in the following way, PROPOSITION 5.2.1. *Given a set $\{U_1, ..., U_r\}$ constructed by taking from the set of all the complex irreducible representations of a given group G, one representative of each Complete Galois Class; every rational representation γ of G can be written as,*

$$\gamma \otimes \mathbb{C} \cong \bigoplus_{i=1}^{r} (\bigoplus_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} U_i^{\sigma}) \otimes V_i$$

where,

$r=$ number of rational irreducible representations up equivalence (i.e. number of cyclic subgroups of G up conjugation)

$K_i = \mathbb{Q}(\chi_{U_i}(g) : g \in G)$

$V_i = (\gamma \otimes U_i^*)^G$

So, in the specific case of the group acts on Riemann Surface, we have an immediate corollary for the above proposition

COROLLARY 5.2.2. *Let S be a Riemann surface with G-action. Consider ρ_a and ρ_{RAC} the analytical and rational representations for the action of G on the Jacobian variety corresponding to S, then*

$$\rho_{RAC} \otimes \mathbb{C} \cong \bigoplus_{i=1}^{r} (\bigoplus_{\sigma \in Gal(K_i:\mathbb{Q})} U_i^{\sigma}) \otimes V_i$$

where $r=$ number of rational irreducible representations up equivalence, $K_i = \mathbb{Q}(\chi_{U_i}(g) : g \in G)$ and $V_i = (\gamma \otimes U_i^*)^G$

REMARK 5.2.1. Let W be a vector space, $G \leq GL(W)$, and suppose we have a G-equivariant decomposition of W $W = U \oplus V$, then $W^H = U^H \oplus V^H$

Using the former remark and corollary 5.2.2, we have

PROPOSITION 5.2.3. *Let G be a group acting on a Riemann surface and ρ_{RAC} the Rational representation for the action of G on the Jacobian variety of S, for each $H \leq G$ we have*

$$(\rho_{RAC} \otimes \mathbb{C})^H \cong \bigoplus_{i=1}^{r} (\bigoplus_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} U_i^{\sigma})^H \otimes V_i$$

where $r=$ number of rational irreducible representations up equivalence, $K_i = \mathbb{Q}(\chi_{U_i}(g) : g \in G)$ and $V_i = (\gamma \otimes U_i^*)^G$

Comparing dimensions:

$$\dim((\rho_{RAC}\otimes\mathbb{C})^H) = \sum_{i=1}^{r}(\sum_{\sigma\in\mathrm{Gal}(K_i:\mathbb{Q})}\dim(U_i^\sigma)^H)\cdot\dim(V_i)$$

We know $\dim((\rho_{RAC}\otimes\mathbb{C})^H) = 2g_{S/H}$, where S is the surface where G is acting, and $g_{S/H}$ is the genus of the orbit (quotient) surface by the H-action, we will consider the set $\{H_1, ...H_r\}$ of all the cyclic subgroups of G up isomorphism, to get a system of r equations and r incognitos which are $\dim(V_i)$ representing the multiplicities of each complex irreducible representation on the isotypical decomposition of the Rational Representation ρ_{RAC}.

The system looks like:

$$2g_{S/H_j} = \sum_{i=1}^{r}\left(\sum_{\sigma\in\mathrm{Gal}(K_i:\mathbb{Q})}\dim(U_i^\sigma)^{H_j}\right)\cdot\dim(V_i) \text{ for } i,j=1..r \qquad \text{(E)}$$

The next steps are:

1. To show that the system (E) admits a well defined solution, equivalently to show that the matrix:

$$\Omega = \left(a_{ij} := \sum_{\sigma\in\mathrm{Gal}(K_i:\mathbb{Q})}\dim(U_i^\sigma)^{H_j}\right)_{i,j}$$

formed by the coefficients of the system is invertible.

2. To find a solution.

Proposition 5.2.4. *The matrix Ω defined above is invertible.*

Proof. First consider the Complex Character table of G: the rows are indexed by representatives of the conjugacy classes of elements of G, c_i, sorted by order (from less to more, $|c_i| \leq |c_{i+1}|$); the columns are indexed by the complex irreducible characters,χ_i, sorted by packages of Complete Galois Classes and the coefficients of the table are, as always, the character evaluated on the representative.

A picture to make it clearer: (Let's consider σ as being in the appropriated Galois group, to simplify the notation)

	χ_1	χ_1^σ	...	χ_j	χ_j^σ	...
c_1	$\chi_1(c_1)$	$\chi_1^\sigma(c_1)$	...	$\chi_j(c_1)$	$\chi_j^\sigma(c_1)$	...
...	...	...	...	...	...	...
c_s	$\chi_1(c_s)$	$\chi_1^\sigma(c_s)$	...	$\chi_j(c_s)$	$\chi_j^\sigma(c_s)$	...

So the Character table defines a square matrix, A, of size corresponding to the number of complex irreducible representations up equivalence, s; due to the orthogonality relations of characters, this matrix is invertible.

Let's take A and produce another matrix, B, by adding the columns associated to representations of the same Complete Galois Class; this will be no longer a square matrix, but it will be of maximal rank, which is $r =$ number of cyclic subgroups up conjugation.

Let's call θ_j to the following class function

$$\theta_j := \sum_{\sigma \in \mathrm{Gal}(K_j:\mathbb{Q})} \chi_j^\sigma$$

A picture of B at this point:

	θ_1	...	θ_j	...	θ_r
c_1	$\theta_1(c_1)$	...	$\theta_j(c_1)$	...	$\theta_r(c_1)$
...	...	...	...	...	...
c_s	$\theta_1(c_s)$	...	$\theta_j(c_s)$	...	$\theta_r(c_s)$

REMARK 5.2.2. If two elements, c_1 and c_2, generate conjugated cyclic subgroups of G then $\theta_j(c_1) = \theta_j(c_2)$ for $j = 1..r$.

Using remark 5.2.2, we can erase the rows corresponding to elements which generate conjugated subgroups from the B matrix. Now we have a new square matrix, of size r, which is invertible; a picture of our new B,

	θ_1	...	θ_j	...	θ_r
c_1	$\theta_1(c_1)$	...	$\theta_j(c_1)$	...	$\theta_r(c_1)$
...	...	...	...	...	...
c_r	$\theta_1(c_r)$	...	$\theta_j(c_r)$	...	$\theta_r(c_r)$

Now we are abusing on the notation, because c_i represents now not only an element conjugacy class but a lot of them. We will denote $\bar{c_i}$ when we mean all the elements of the conjugacy class of c_i joined with all the elements of the conjugacy class of c_j where c_i and c_j generate conjugated cyclic subgroups of G.

Let's analyze Ω now, it is a square matrix of size r and its coefficients are given by:

$$a_{ij} := \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} = \frac{1}{|H_i|} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \sum_{h \in H_i} \chi_j(h)$$

rearranging the sums and using our definition of θ_j and $\bar{c_k}$, we have

$$a_{ij} = \frac{1}{|H_i|} \sum_{h \in H_i} \theta_j(h) = \frac{1}{|H_i|} \sum_{k=1}^{r} \theta_j(c_k)|\bar{c_k} \cap H_i|$$

Now we sort Ω just like B, this is: the rows indexed by cyclic subgroups of G up conjugation, sorted by size ($|H_i| \leq |H_{i+1}|$) and the columns indexed by θ_j sorted in the same order as they are in B.

Using the fact that the order of the elements of a subgroup are always less or equal than the size of the subgroup, we are allowed to state that every row i of Ω is made by elementary operations involving the rows $k = 1..i$ of B. Finally, as B is invertible, this leads on Ω invertible.

□

At this moment we know that the system (E) has a unique solution. We will offer it and test it, but firs a useful Lemma,

LEMMA 5.2.5. *Let G be a finite group. Consider H_j and G_k subgroups and $H_j \backslash G / G_k$ a set of representatives of double cosets.*
Then:

$$\frac{1}{|H_j|} \sum_{a \in H_j} \frac{|G||G_k \cap \bar{a}|}{|G_k||\bar{a}|} = \sharp(H_j \backslash G / G_k)$$

where $\bar{a}$ means the conjugacy class of a.

PROOF. We know from chapter 3, specifically demonstration of Lemma 3.2.8, that $\sharp(H_j \backslash G/G_k)$ represents the number of orbits by the action of H_j on the set I_{G_k} of right cosets of G_k on G. We will show that the left term represents the same number.

i) The cardinality of the orbit by $a \in H_j$ is

$$\frac{|\bar{a}|}{|\bar{a} \cap G_k|}$$

ii) Number of orbits produced by $a \in H_j$

$$\frac{|G : G_k||\bar{a} \cap G_k|}{|\bar{a}|}$$

So, the number of orbits by the whole H_j is

$$\sum_{a \in H_j} \frac{|G||G_k \cap \bar{a}|}{|G_k||\bar{a}|}$$

□

THEOREM 5.2.6. *For each non trivial complex irreducible representation $\theta_i : G \to GL(U_i)$ of a group G acting on a Riemann surface S, with geometric signature $\Gamma = (\gamma; [C_1, m_1], ..., [C_t, m_t])$; the multiplicity, n_i, of θ_i on the isotypical decomposition of $\rho_{RAC} \otimes \mathbb{C}$ is given by*

$$\boxed{n_i = 2\dim(U_i)(\gamma - 1) + \sum_{G_k \in \Gamma} (\dim(U_i) - \dim(U_i^{G_k})}$$

where the G_k is a representative of the conjugacy class $C_k \in \Gamma$, $j = 1, ..., t$.

PROOF. We are considering: $\{\theta_1, ..., \theta_i, ..., \theta_s\}$ all the complex irreducible representations of G up isomorphism, $\{H_1, ..., H_j, ..., H_r\}$ all the cyclic subgroups of G up conjugation, $s \geq r$, and the system (E). The idea for proving this theorem, is to replace the expression proposed for n_i in it, which corresponds to $\dim(V_i)$ in the system (E), and the expression for g_{S/H_j}, given in remark 3.2.11:

$$g_{S/H_j} = |G : H_j|(\gamma - 1) + 1 + \frac{1}{2} \sum_{G_k \in \Gamma} (|G : H_j| - |H_j \backslash G/G_k|) \quad (3.2.11)$$

in the system (E); and see that they work. As a system has a unique solution, there we go.

For the trivial representation U_1, we want to write its multiplicity in the same fashion, we have:

- $n_1 = \dim(V_1) = 2\gamma$
- $\dim(U_1) = 1$
- $\dim(U_1) - \dim(U_1)^H = 0$ for every $H \leq G$.

then we can write,

$$n_1 = \dim(V_1) = 2 + 2\dim(U_1)(\gamma - 1) + \sum_{G_k \in \Gamma} (\dim(U_i) - \dim(U_i^{G_k}))$$

Besides, we can decompose each equation on system (E):

$$2g_{S/H_j} = \sum_{\sigma\in \mathrm{Gal}(K_1:\mathbb{Q})} \dim(U_1^\sigma)^{H_j} \dim(V_1) + \sum_{i=2}^{r} \left(\sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \right) \dim(V_i)$$

equivalently

$$2g_{S/H_j} = \sum_{\sigma\in \mathrm{Gal}(K_1:\mathbb{Q})} \dim(U_1^\sigma)^{H_j} n_1 + \sum_{i=2}^{r} \left(\sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \right) n_i \qquad (EE)$$

for $j = 1, ..r$. Replacing the expressions for n_1 and n_i for $i = 2..r$ (the non trivial ones) on the system (EE), we have:

$$2g_{S/H_j} = \sum_{\sigma\in \mathrm{Gal}(K_1:\mathbb{Q})} \dim(U_1^\sigma)^{H_j} \left(2 + 2\dim(U_1)(\gamma-1) + \sum_{G_k\in\Gamma} (\dim(U_i) - \dim(U_i^{G_k})) \right) + \sum_{i=2}^{r} \left(\sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \right) \left(2\dim(U_i)(\gamma-1) + \sum_{G_k\in\Gamma} (\dim(U_i) - \dim(U_i^{G_k})) \right)$$

Grouping the term for U_1 with the sum and simplifying, the system looks like:

$$g_{S/H_j} = 1 + \sum_{i=1}^{r} \sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \left(\dim(U_i)(\gamma-1) + \frac{1}{2} \sum_{G_k\in\Gamma} (\dim(U_i) - \dim(U_i^{G_k})) \right) \qquad \text{(EEE)}$$

Now we compare (EEE) term by term with the expression for g_{S/H_j} from 3.2.11:

(i) Term corresponding to the factor $(\gamma - 1)$:

$$\sum_{i=1}^{r} \sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \dim(U_i) \ \ vs \ \ \frac{|G|}{|H_j|}$$

For the left term we have:

$$\sum_{i=1}^{r} \sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \dim(U_i) = \frac{1}{|H_j|} \sum_{h\in H_j} \left(\sum_{i=1}^{r} \sum_{\sigma\in \mathrm{Gal}(K_i:\mathbb{Q})} \chi_j^\sigma(h) \chi_j(id) \right)$$

but $\chi_j(id) = \chi_j^\sigma(id)$ for all σ in the right Galois group and all j so, the former is equal to:

$$= \frac{1}{|H_j|} \sum_{h \in H_j} \chi_{REG}(h) = \frac{|G|}{|H_j|}$$

(ii)Term associated to Γ

$$\sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \left(\frac{1}{2} \sum_{G_k \in \Gamma} (\dim(U_i) - \dim(U_i^{G_k})) \right)$$

$$vs \;\; \frac{1}{2} \sum_{G_k \in \Gamma} \left(\frac{|G|}{|H_j|} - \sharp(H_j \backslash G / G_k) \right)$$

We rearrange the sums on the first term :

$$\frac{1}{2} \sum_{G_k \in \Gamma} \left(\sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} (\dim(U_i) - \dim(U_i^{G_k})) \right)$$

and using (i) we just need to prove

$$\sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \dim(U_i^\sigma)^{H_j} \dim(U_i^{G_k}) = \sharp(H_j \backslash G / G_k)$$

The left term is given by,

$$\frac{1}{|H_j||G_k|} \sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \left(\sum_{a \in H_j} \sum_{b \in G_k} \bar{\chi_i^\sigma}(a) \chi_i^\sigma(b) \right) \quad \text{(l.t.)}$$

At this point is useful to remember:

a.

$$\sum_{\chi \in \mathrm{Irr}_{\mathbb{C}} G} \chi(\bar{g}) \chi(g) = \frac{|G|}{\sharp \bar{g}}$$

Equivalently,

$$\sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \bar{\chi_i^\sigma}(g) \chi_i^\sigma(g) = \frac{|G|}{\sharp \bar{g}}$$

b.

$$\sum_{\chi \in \mathrm{Irr}_{\mathbb{C}} G} \chi(\bar{g_1}) \chi(g_2) = 0$$

Equivalently,

$$\sum_{i=1}^{r} \sum_{\sigma \in \mathrm{Gal}(K_i:\mathbb{Q})} \bar{\chi_i^{\sigma}}(g_1)\chi_i^{\sigma}(g_2) = 0$$

if g_1 is not conjugated to g_2.

Applying both we know that the left term, expression (l.t.), is different from 0 only when b be conjugated to a, in this case it is equal to $\frac{|G|}{|\bar{a}|}$ and it happens on $|G_k \cap \bar{a}|$ elements of G_k. Now we use Lemma 5.2.5 and the theorem is proved. □

COROLLARY 5.2.7. *Given a finite group G such that all its rational irreducible representations are absolutely irreducibles, acting on a Riemann surface with geometric signature $\Gamma = (\gamma; [m_1, C_1], ..., [m_t, C_t])$. For each complex (rational) irreducible representation $\theta_i : G \to GL(U_i)$ of G its multiplicity, n_i, on the isotypical decomposition of the Analytical Representation,ρ_a, associated to the action of G on the corresponding Jacobian variety is:*

$$\boxed{n_i = \dim(U_i)(\gamma - 1) + \frac{1}{2} \sum_{G_k \in \Gamma} (\dim(U_i) - \dim(U_i^{G_k}))}$$

where the G_k is a representative of the conjugacy class $C_k \in \Gamma$, $k = 1, ..., t$.

Which is the result in [**9**]

If we write the isotypical decomposition for the representation $\rho_{RAC} \otimes \mathbb{C}$ using the already computed multiplicities of each complex irreducible representation, and we compute its character, we get the following

COROLLARY 5.2.8. *The character of the rational representation is given by*

$$\chi_{RAC} = 2\chi_{trivial} + (2g_{S/G} - 2 + r)\chi_{REG} - \sum_{G_i \in \Gamma} \chi_{Ind_{G_i}^{G} 1}$$

As it was given on [**3**]. This shows in particular that the Rational Character is an integer combination of Induced characters from the trivial character of the cyclic subgroups which are stabilizers for the G-action, and the trivial and regular ones.

REMARK 5.2.3. We have seen on Theorem 5.2.6, the geometric signature for the action of a group G on a Riemann surface S, determines the isotypical decomposition for the complexification of the Rational representation for the action of G on the corresponding Jacobian variety, the converse is also true.

THEOREM 5.2.9. *Let S be a Riemann surface with G action, the isotypical decomposition for the complexification of the Rational representation for the action of G on the corresponding Jacobian variety, is completely determined by the geometric signature of the action of G, Def. 3.1.1.*

PROOF. Forward is a direct application of Theorem 5.2.6.

Conversely, if we have two different geometric signatures by Corollary 3.2.7 we know the genus of the intermediate quotients by all cyclic subgroups are different in at least one, so the system $(E):\Omega(X)=B$ have the same matrix Ω, but different vector B, then both solutions must be different. □

REMARK 5.2.4. We will think this theorem in terms of

$$\text{Geometric signature for the } G \text{ action} \quad \Leftrightarrow \quad \text{Isotypical decomposition for } \rho_R AC \otimes \mathbb{C}$$

EXAMPLE 5.2.1. We continue with the example 3.2.2, the Weyl group $C_3 \cong \mathbb{Z}_2^3 \rtimes S_3$, we want to show that this two different concretization of G as the group of automorphisms of $S: y^2 = x^8 + 14x^4 + 1$ give two different decomposition for the Rational representation associated to the action of G on the Jacobian variety of S. We are considering one G action with geometric signature

$$\Gamma_1 = (0; [\overline{G_9}, 6], [\overline{G_7}, 4], [\overline{G_4}, 2])$$

and other with G action with geometric signature

$$\Gamma_2 = (0; [\overline{G_9}, 6], [\overline{G_8}, 4], [\overline{G_5}, 2])$$

where the subgroups involved are

Representative	Order	Size of the class
$G_4 =< xyzb >$	2	6
$G_5 =< yzab >$	2	6
$G_7 =< xyab >$	4	3
$G_8 =< yab >$	4	3
$G_9 =< xa^2 >$	6	4

Consider the character table of C_3,

$$\begin{bmatrix} & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 & \theta_9 & \theta_{10} \\ Id & 1 & 1 & 2 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\ xyz & 1 & 1 & 2 & -1 & -1 & -2 & -3 & -3 & 3 & 3 \\ xy & 1 & 1 & 2 & 1 & 1 & 2 & -1 & -1 & -1 & -1 \\ xyzb & 1 & -1 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 \\ zyb & 1 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & 1 \\ x & 1 & 1 & 2 & -1 & -1 & -2 & 1 & 1 & -1 & -1 \\ zyab & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ xyza^2 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ yza^2 & 1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ zab & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

If we compute the multiplicity of each $\theta_i, \quad i = 2, ..., 10$ for each geometric signature, Γ_1 and Γ_2 using theorem 5.2.6, we obtain only one different form 0 for each case, but a different one:

Geometric signature	Irreducible representation	Multiplicity in $\rho_{RAC} \otimes \mathbb{C}$
Γ_1	θ_7	2
Γ_2	θ_8	2

Moreover, we know C_3 has all its Rational irreducible representations absolutely irreducible, then we have constructed two examples of actions of G where the corresponding Analytical representation (associated to the action on the Jacobian variety) is isomorphic to just one irreducible representation, different in each case.

CHAPTER 6

Example $C_n = \mathbb{Z}_2^n \rtimes S_n$

The goal of this chapter is to show how all the already exposed can be applied in a more complex example: the family of Weyl groups of C_n type.

Finally, this tool will be used to C_3 to find surfaces and coverings with the appropriate intermediate structure in order to get the wished Pryms in the decomposition of the Jacobian of S, with WS a Riemann surface with C_3 action.

6.1. Existence of coverings $S \to \mathbb{P}_1$ with Galois group C_n

THEOREM 6.1.1. *C_n can be represented as a transitive subgroup of S_{2n}, for $n \geq 3$*

PROOF. From [**24**] consider $s_i = (i\ i+1)(n+i\ n+i+1)$ for $i < n$, $s_n = (n\ 2n)$. Then $C_n =< s_i : i = 1..n >$,transitive subgroup of S_{2n}. Besides we can observe that:
1) $s_n * s_{n-1} * s_{n-2} * ... * s_1 = (1\ 2\ 3\ ...\ 2n) = c$
2) $c * s_1 = (2\ 3\ ..n-1\ n\ n+2\ n+3\ ..\ 2n)$
Therefore we have

$$C_n =< c, s_1 : c^{2n} = s_1^2 = (c * s_1)^{2n-2} = ... >$$

$$C_n \cong< (1\ 2\ ...\ 2n), (1\ 2)(n+1\ n+2), (2\ 3\ ..n-1\ n\ n+2\ n+3\ ..\ 2n) >$$

□

PROPOSITION 6.1.2. $S_{2n-1} \cap C_n \cong C_{n-1}$

PROOF. It is obvious from the representation given in [**24**]. □

REMARK 6.1.1. Now, consider $C_n =< (1\ 2\ ...\ 2n), (1\ 2)(n+1\ n+2), (2\ 3\ ..n-1\ n\ n+2\ n+3\ ..\ 2n) >$, just to avoid a difficult notation. We can state the following theorem which is an extension of example 4.3.2,

THEOREM 6.1.3. *For each $n = 3... \in \mathbb{Z}$, there exists a Riemann surface, S, admitting the action of $C_n = \mathbb{Z}_2^n \rtimes S_n$, with geometric signature*
$(0; [2n, < (1\ 2\ ...\ 2n) >], [2n-2, < (2\ 3\ ..n-1\ n\ n+2\ n+3\ ..\ 2n) >],$
$[2, < (1\ 2)(n+1\ n+2) >)$

PROOF. Actually it is a corollary of Theorems 4.2.1 and 6.1.1, and remark 6.1.1

□

REMARK 6.1.2. We can compute certain intermediate genus:

- The genus of S is given by Riemann-Hurwitz:

$$g_S = 2^{n-2}(n-2)!(n^2 - 3n + 1) + 1$$

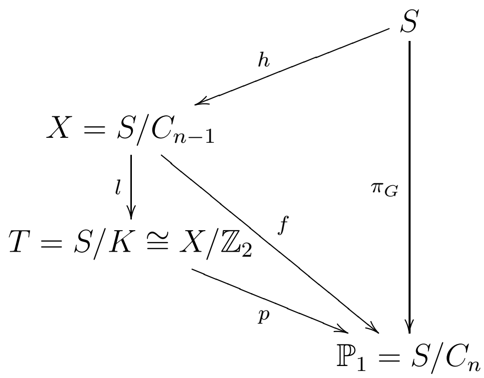
- The genus of the curve associated to $C_{n-1} < C_n$, X, is given by Riemann-Hurwitz, considering that its the ramification is given by the cyclic structure of the representation of C_n given in the theorem 6.1.1. So,

$$g_X = 2n(0-1) + 1 + \frac{1}{2}((2n-1) + (2n-2-1)(2-1+2-1)) = 0$$

- the Galois covering of $f : X \to \mathbb{P}_1$ is $\pi_G : S \to \mathbb{P}_1$, as a conclusion we can say that f is a rational function of degree $2n$ with 3 branch values with Galois group C_n
- Up to now we have the following surfaces:
  1) $\mathbb{P}_1 = S/C_n$
  2) X, the surface associated the subgroup $S_{2n-1} \cap C_n = C_{n-1}$
  3) S the Surface associated to the kernel of the monodromy representation corresponding to f.
  4) Considering that: the covering $f : X \to \mathbb{P}_1$ is of degree $2n$, the center of C_n is $Z(C_n) =< (1\ n+1)(2\ n+2)...(n\ 2n) >$ of size 2, and $Z(C_n) \cap C_{n-1} = \emptyset$. We have that the subgroup $K = C_{n-1} \times Z(C_n)$ has associated a surface:

$$T = S/K \cong X/Z(C_n)$$

The following diagram sums up the surfaces what have been identified up to now.



Where π_G,h,l and $h \circ l$ are Galois.

6.2. Irreducible representations of C_n

The idea is to use the method of the "Little groups ", explained in section 2.5.3.

PROPOSITION 6.2.1. *The Group of Weyl $C_n = \mathbb{Z}_2^n \rtimes \mathbf{S}_n$ has the following amount of (complex or rational) irreducible representations:*

$$2\sum_{i=0}^{k+1} P(n-i)P(i) \quad para\ n = 2k+1 \quad y$$

$$2\sum_{i=0}^{k-1} P(n-i)P(i) \ + \ P^2(k) \quad para\ n = 2k$$

Where $P(j)$ is the number of possible forms to obtain j as the addition of positive numbers (up permutation, and $P(0) := 1$).

We will show more than that: we will give a form to obtain all the irreducible representations of C_n, we will see that they are in direct relationship with the irreducible representations of S_i for $i = 2..n$. Let's consider the following concretization of C_n:

$$\mathbb{Z}_2^n = < x_j = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad / \quad j = 1, ..., n \ >$$

$$H = \mathbf{S}_n = \left\langle a = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\rangle$$

6.2.1. Set Ω_A. We will follow the steps explained in the former section.

So, the set of the irreducible representations of $\mathbb{Z}_2^n$, Ω_A will be,

$$\Omega_A = \{\rho_0, ..., \rho_{2^n-1}\}$$

6.2.1.1. Ω_A^H. In order to get the representatives of the action of G on Ω_A the following fact is taken into account.

PROPOSITION 6.2.2. *There is a group isomorphism between Ω_A and $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ (n-times). Such that the action φ of H in Ω_A is equivalent to the standard action (of permutation of coordinates)of H in $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$.*

PROOF. Consider
$\Phi : \Omega_A \longrightarrow \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ given by
$\rho_i \xrightarrow{\Phi} (\bar{\chi_{\rho_i}}(x_1), ..., \bar{\chi_{\rho_i}}(x_n))$
with

$$\bar{\chi_{\rho_i}}(x_j) = \begin{cases} 0 & \text{if } \chi_{\rho_i}(x_j) = 1 \\ 1 & \text{if } \chi_{\rho_i}(x_j) = -1 \end{cases}$$

Such an isomorphism accomplishes the proposition. □

Given the former,to find the representatives of the orbits of H in Ω_A, it is equivalent to find them in $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ looking at the standard action. We will refer indistinctly to the representation ρ or to the corresponding tuple $(0, 0, ..1, 0)$, ($\Phi(\rho)$).
As the standard action is to permute coordinates, we have that the orbits are conformed of n-tuples that have the same amount of "zeros"(or of "ones"). We will name O_i the orbit of the elements with "i ones "(therefore $(n - i)$ zeros).

The cardinality of O_i is $\binom{n}{i}$.
As representative we will take the tuple:

$$\begin{cases}(0...01...1) & \text{if } i < n-i \\ (1...10...0) & \text{if } i > n-i\end{cases}$$

There are n+1 different orbits (with 0 ones to n ones). Therefore n+1 different representatives.

$$\Omega_A^H = \{(0...0), (0..01), (0..011), ..., (11..1)\} := \{\rho_0, ..., \rho_n\}$$

So the stabilizers in $H = S_n$ are:

$H_0 = Stb_H\rho_0 = H$

$H_1 = Stb_H\rho_1 \simeq S_{n-1}$ (It is the S_{n-1} that fixes x_n, i.e. $S_{n-1} =< (1\ 2...(n-1)),\ (1\ 2) >$

$H_2 = Stb_H\rho_2 \simeq S_{n-2} \times S_2$ (with $S_{n-2} =< (1\ 2...(n-2)), (1\ 2) >$ and $S_2 =< (n-1\ n) >$

...

We must to make a difference between the cases even and odd:

1) $n = 2k$:

$H_k = Stb_H\rho_k \simeq S_k \times S_k$ (With the first generated by $(1\ 2...k),\ (1\ 2)$, and the second by $(k+1...n),\ (k+1\ k+2)$).

$H_{k+1} = H_{k-1}$

$H_{k+2} = H_{k-2}$

...

$H_n = H_0$

2) $n = 2k+1$:

$H_{k+1} \simeq S_{k+1} \times S_k$ (With $S_{k+1} =< (1\ 2...k+1), (1\ 2) >$ and $S_k =< (k+2...n), (k+2\ k+3) >$)

$H_{k+2} = H_{k+1}$

$H_{k+3} = H_k$

...

$H_n = H_0$

And $G_j = Stb_G\rho_j = A \rtimes H_j = \mathbb{Z}_2^n \rtimes H_j$

6.2.2. Representations for G_j. As in the procedure already explained,
1) Let $\bar{\rho_j}$ be the extension of ρ_j (considering H_j act trivially).
2) Sean $\{\sigma_{ij}\}_{i\in I}$ the irreducible representations of H_j and doing the part of A act trivially, we get a set $\{\bar{\sigma_{ij}}\}_{i\in I}$, of irreducible representations for G_j.

6.2.3. Tensor representation for G_j. Taking the representations for G_j obtained in the former section, the representation $\theta_{ij} = \bar{\rho_j} \bigotimes \bar{\sigma_{ij}}$ is produced.

REMARK 6.2.1. If $G_j = \mathbb{Z}_2^n \rtimes (S_{n-i} \times S_i)$ Then there are $P(n-i)P(i)$ representations θ_{ij} got in this way.

6.2.4. Representations of $G = C_n$. Finally we build the representation $\psi_{ij} = \operatorname{Ind}_{G_j}^G \theta_{ij}$ For all the representations θ_{ij} got in the previous point, for $i = 0..n$
Some conclusions:
1) Due to $G_0 = G$, we have that $\theta_{i0} = \overline{\rho_0} \bigotimes \bar{\sigma_{i0}}$ are already representations of G. Of the type σ_{i0} for S_n and trivial for $\mathbb{Z}_2^n$.

2) $G_n = G$, therefore $\theta_{in} = \bar{\rho_n} \bigotimes \bar{\sigma_{in}}$ is representation of G, in the way σ_{in} for S_n and the generators of $\mathbb{Z}_2^n$ act as $-Id$.

3) From all the former we conclude the proposition with which we begin the chapter, in fact:
For $n = 2k$, we have

$$G_i = \mathbb{Z}_2^n \rtimes (S_{n-i} \times S_i) \text{ for } i = 0..k-1$$

then there are $P(n-i)P(i)$ representations θij that will be induced as irreducible for G.

Besides $G_i = G_{n-i}$ for $i = 0..k-1$.

For $G_k = \mathbb{Z}_2^n \rtimes (S_k \times S_k)$ we will have $P^2(k)$ representations θ_{ij} that will be induced for G.

So it is clear from when the proposed formula for the amount of irreducible representations of $G = C_n$ for the case n even was deduced
.

For $n = 2k+1$, we make an analogous analysis that permits to get the proposed formula.

6.2.5. Example $n = 3$**.** $C_3 = \mathbb{Z}_2^3 \rtimes S_3$:

In this case $n = 2 * 1 + 1$ then $k = 1$. Therefore it must have $2\sum_{i=0}^{1} P(3-i)P(i) = 2(3+2) = 10$ irreducible representations.

Consider the set of irreducible representations of $\mathbb{Z}_2^n$ $\Omega = \{\rho_0, ..., \rho_7\}$. As it was shown we have $n+1 = 4$ representatives of the orbits in Ω by the action of C_3:

ρ_0 associated to the tuple (000)
ρ_1 associated to (001)
ρ_2 associated to (110)
ρ_3 associated to (111)

Consider $\mathbb{Z}_2^n$ generated by x, y, z, then the previous information is translated in:
1) ρ_0 the trivial representation of $\mathbb{Z}_2^n$
2) $\rho_1(x) = \rho_1(y) = 1$ and $\rho_1(z) = -1$
3) $\rho_2(x) = \rho_2(y) = -1$ and $\rho_2(z) = 1$
4) $\rho_3(x) = \rho_3(y) = \rho_3(z) = -1$

Therefore the stabilizers are:
i)$G_0 = G_3 = C_3$, then here we already have the following representations of $G = C_3$:
1) θ_1 the trivial representation.
2) θ_2 Given by,

$$\theta_2(x) = \theta_2(y) = \theta_2(z) = \theta_2(a) = 1 \text{ and } \theta_2(b) = -1$$

3) θ_3 Given by,

$$\theta_3(x) = \theta_3(y) = \theta_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\theta_3(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\theta_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

4) θ_4 Given by,

$$\theta_4(x) = \theta_4(y) = \theta_4(z) = -1, \theta_4(a) = \theta_4(b) = 1$$

5) θ_5 Given by,

$$\theta_5(x) = \theta_5(y) = \theta_5(z) = \theta_2(b) = -1, \theta_5(a) = 1$$

6) θ_6 Given by,

$$\theta_6(x) = \theta_6(y) = \theta_6(z) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\theta_6(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\theta_6(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

ii) $G_1 = \mathbb{Z}_2^3 \rtimes < b >$, for which we have the following representations we will use to induce irreducible of G.

1) θ_{10} such that
$\theta_{10}(x) = \theta_{10}(y) = \theta_{10}(b) = 1$, $\theta_{10}(z) = -1$

2) θ_{11} such that $\theta_{11}(x) = \theta_{11}(y) = 1$, $\theta_{11}(z) = \theta_{11}(b) = -1$

iii) $G_2 = \mathbb{Z}_2^3 \rtimes < b >$, for which we have the following irreducible representations.

1)θ_{20} such that $\theta_{20}(x) = \theta_{20}(y) = -1$,$\theta_{20}(z) = \theta_{20}(b) = 1$

2) θ_{21} such that $\theta_{21}(x) = \theta_{21}(y) = \theta_{21}(b) = -1$, $\theta_{21}(z) = 1$

When inducing these representations for G, we find the following (complex or rational) irreducible representations for G:

1) $\theta_7 = Ind_{G_1}^{G} \theta_{10}$ such that,

$$\theta_7(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \theta_7(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\theta_7(z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \theta_7(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\theta_7(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2) $\theta_8 = Ind_{G_1}^{G}\theta_{11}$ such that,

$$\theta_8(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \theta_8(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\theta_8(z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \theta_8(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\theta_8(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

3) $\theta_9 = Ind_{G_2}^{G}\theta_{20}$ such that,

$$\theta_9(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \theta_9(y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\theta_9(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \theta_9(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\theta_9(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4) $\Theta_{10} = Ind_{G_2}^{G}\theta_{21}$ such that,

$$\Theta_{10}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Theta_{10}(y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Theta_{10}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \Theta_{10}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Theta_{10}(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Thus, θ_1 ... Θ_{10} are all the irreducible representations of C_3.

The following is the table of characters for C_3:

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	Θ_{10}
Id	1	1	2	1	1	2	3	3	3	3
xyz	1	1	2	−1	−1	−2	−3	−3	3	3
xy	1	1	2	1	1	2	−1	−1	−1	−1
$xyzb$	1	−1	0	−1	1	0	−1	1	1	−1
zyb	1	−1	0	1	−1	0	−1	1	−1	1
x	1	1	2	−1	−1	−2	1	1	−1	−1
$zyab$	1	−1	0	1	−1	0	1	−1	1	−1
$xyza^2$	1	1	−1	−1	−1	1	0	0	0	0
yza^2	1	1	−1	1	1	−1	0	0	0	0
zab	1	−1	0	−1	1	0	1	−1	−1	1

6.3. Decomposition of the Jacobian of a curve with action of C_n

We study the decomposition of the Jacobian variety of a Curve that admits the action of C_n as subgroup of its group of Automorphisms.

The idea is to say as much as possible about the factors that appear in such decomposition. This is, to solve some questions such as:

What representations will always act in product of Pryms of intermediate quotients.

Where will they act.

Which ones will not act "alone" in subvarieties geometrically described as Pryms.

How do the latter ones group themselves to do it.

Unfortunately not all have been solved, some partially. I hope in a near future to shed more light on this respect.

Finally we apply all what has been presented, in this chapter the same as in the formers, to study the particular case of C_3. In this case, the decomposition will be in 9 factors (i.e. one less than the number of irreducible representations C_3 has, it happens here that one of those never "act alone "on a Prym variety) and in each factor C_3 acts with one of its representations, except for one where we have the action of the sum of 2 representations.

We recall here all the known results presented on section 2.6.

As it has already been shown in former chapters C_n is a group of the type "little group ", i.e. $C_n = A \rtimes H$ with A normal subgroup and abelian. Its irreducible representations were got in section **??** (they are all its complex irreducible representations but they all are also rational).

Let's remember some items:

$G = A \rtimes H$, $\Omega = \{\rho_1, ..., \rho_l\}$ set of irreducible representations of A. Due to A is abelian, they are all of degree 1. In the case of C_n we have $l = 2^n$.

H acts in Ω by conjugacy (i.e., each representation is sent to its conjugacy through elements of H). In our example Ω/H has cardinality $n+1$. Each element of Ω/H was represented by an n-tuple of $0'$s and $1'$s one per each generator of $A =< x_i >$ obeying to the rule 1 if $\rho_j(x_i) = -1$ and 0 if $\rho_j(x_i) = 1$ in this way we form the $j-$ tuple (of $n+1$).

Remember that here it is necessary to deal separately the cases n even and odd, but the stabilizer in G of each one of the $\rho_j \in \Omega/H$ (called $G_i = A \rtimes H_i$) is completely characterized. See chapter **??**, in this chapter we will refer to them as G_j and by this will be understood the subgroup that stabilizes $\rho_j \in \Omega/H$ which, just as a reminder, has the form $G_j = A \rtimes (S_k \times S_{n-k})$, for some value of k in direct relationship with the tuple that represents ρ_j.

Remark 6.3.1. 1) It is relatively clear that such G_j are maximal subgroups.

2) First, if we concentrate ourselves to the representations of degree one of these G_j, we can say the following,

a) $G_1 = G_{n+1} = G$ from which 4 representations of degree 1 of G are produced. They are the only ones of this degree.

This leads in other interesting fact: The conmutator subgroup of C_n, G' must have index 4 in C_n, and it is normal. Actually, we can describe it as

$$C'_n = \mathbb{Z}_2^{n-2} \rtimes A_n$$

b) $G_2 = A \rtimes S_{n-1}$ (and G_n that is the same) produce 2 representations of degree 1 which must be induced (or lifted) to G, then, in this way, we produce two representations of degree n (and two others for G_n).

c) The rest of the G_j produce 4 representations of degree 1 each one, which must be induced for G, getting so the description of a great number of irreducibles of G as induced by representations of degree 1 of subgroups clearly determined.

3) All the representations here considered are those got by means of "little groups ". Those of G as those of G_j. We are not interested, for example, in ALL the irreducibles of degree 1 for the G_j, but only in those which appear for the method (chapter **??**).

The following is a series of propositions that, in combination with the corollary 2.6.4, permit us to describe the abelian subvariety where each one of these acts.

PROPOSITION 6.3.1. *Let σ_{ij} be irreducible representation of degree 1 of $G_j = A \rtimes H_j$ for ρ_j given by the tuple*

$$(\overbrace{0, ..., 0}^{n-j}, \underbrace{1, ..., 1}_{j})$$

Then its kernel is the subgroup.

$$K_{ij} =< x_1, ..., x_{n-j}, x_{n-j+i} * x_{n-j+(i+1)} \text{ for } i \in \{1..(j-1)\} > \rtimes H_j$$

REMARK 6.3.2. K_{ij} has index 2 in G_j.

PROPOSITION 6.3.2. *$< Ind_{G_j}^{G} 1, \theta >= 0$ for all $\theta \in Irr(G)$ such that θ is induced by irreducible representation of G_j*

That is to say, if speak in the language of the corollary 2.6.4, the subgroup G_j serve as N.

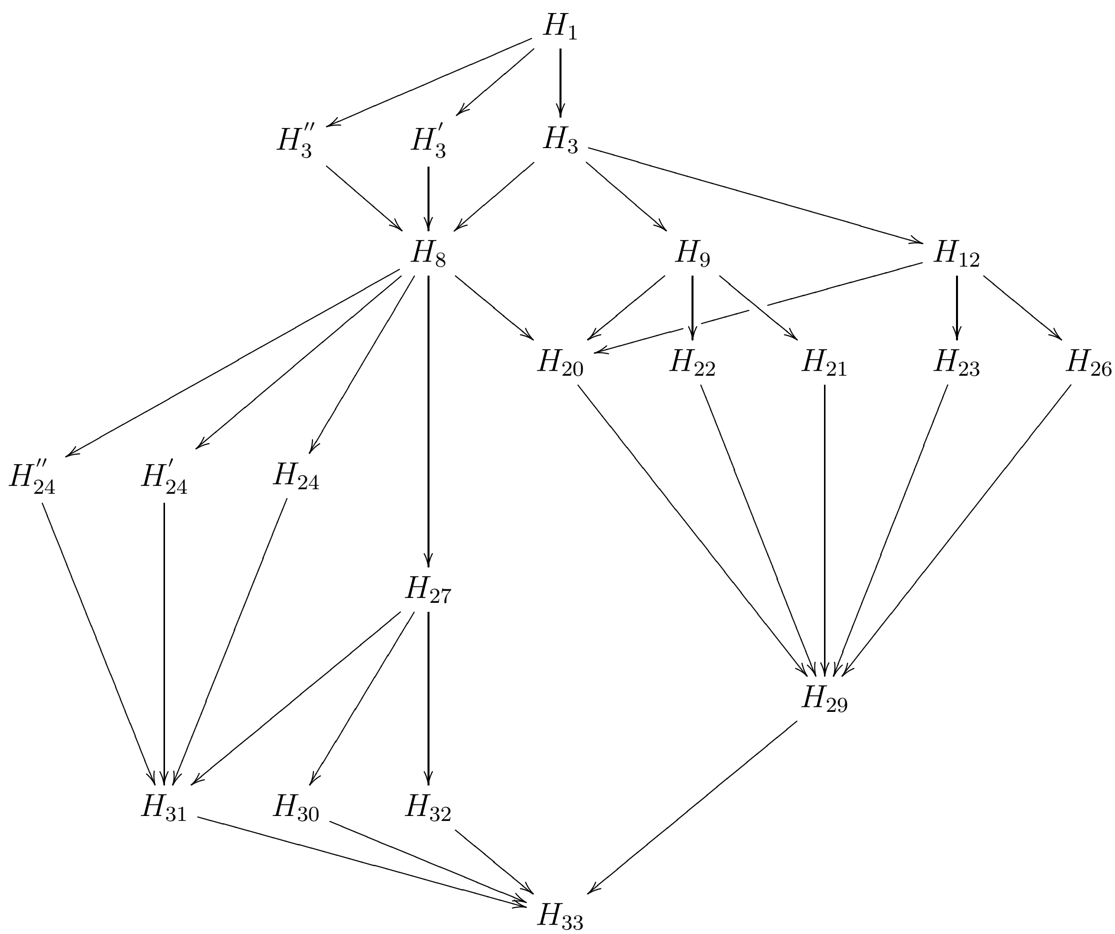
PROPOSITION 6.3.3. *Let σ_{ij} be irreducible representation of degree 1 of G_j and such that $\mid G_j : K_{ij} \mid= 2$. Then*

$$Ind_{K_{ij}}^{G} 1 - Ind_{G_j}^{G} 1 = Ind_{G_j}^{G} \sigma_{ij}$$

This proposition in combination with the corollary 2.6.4 tells us that $nInd_{G_j}^{G}\sigma_{ij}$, acts in s copies of $P(C_{K_{ij}}/C_{G_j})$ with $n = dim\ P(C_{K_{ij}}/C_{G_j})$ and $s =\mid G : G_j \mid$

6.3.1. Decomposition for C_3. Consider now $C_3 = \mathbb{Z}_2^3 \rtimes S_3 =< x, y, z > \rtimes < a, b >$, with x, y, z, a, b of the form presented in the section 6.2
Consider the following subgroups:
$H_{33} = C_3$
$H_{32} =< yz, xz, yza^2, xzb > \simeq \mathbb{Z}_2^2 \rtimes S_3$
$H_{31} =< xz, yz, yza^2, xyzb > \simeq \mathbb{Z}_2^2 \rtimes \phi(S_3)$ where $\phi(S_3) =< a, xyzb >$
$H_{30} =< xyz, xz, yz, yza^2 > \simeq \mathbb{Z}_2^3 \rtimes A_3$
$H_{29} =< xyz, yz, xz, xyzab > \simeq \mathbb{Z}_2^3 \rtimes S_2$
$H_{27} =< yz, xz, yza^2 >$
$H_{26} =< yz, z, yzab > \simeq \mathbb{Z}_2^2 \rtimes S_2$
$H_{24} =< yz, xz, xyzb > \simeq \mathbb{Z}_2^2 \rtimes \phi(S_2)$
$H_{24}^{'} =< xz, yz, xyzab >$
$H_{24}^{''} =< xz, yz, xyza^2b >$
$H_{23} =< y, z, xyzab >$
$H_{22} =< xyz, yz, yxab >$
$H_{21} =< xyz, yz, xyzab >$
$H_{20} =< x, y, z >$
$H_{12} =< z, yz >$
$H_9 =< x, yz >$
$H_8 =< yz, xz >$
$H_3 =< zy >$
$H_3^{'} =< yz >$
$H_3^{''} =< xy >$
$H_1 = Id$

In order to see the relations of inclusion see the following reticulate.



Consider a curve W that admits the action of C_3. We have a diagram of coverings associated to the reticulate of the subgroups previously mentioned, we also have action in the Jacobians of these curves. We will call

$W_j := W/H_j$

$JW_j := JW/H_j$

$P(j/k)$ to the Prym variety of corresponding covering $JW_j \longrightarrow JW_k$.

With these notations we can enunciate the following theorem:

THEOREM 6.3.4. *Let W be a curve such that $C_3 \ < \ Aut(W)$, consider the subgroups and intermediate coverings formerly explained. Then we have an isogenea*

$$\Phi : JW_G \times P(31/33) \times P(30/33) \times P(32/33) \times 2P(24/31) \times 3P(22/29) \times 3P(21/29) \times 3P(23/29) \times 3P(26/29) \to JW$$

The demonstration of this theorem is followed by series of propositions that use results presented in [**14**].

6.3.2. Demonstration of the theorem 6.3.4. Consider the subgroups described in the section 6.3.1, W curve such that $C_3 < Aut(W)$ and intermediate coverings obtained and denoted as in the section 6.3.1.

We have the following:

PROPOSITION 6.3.5. *There is an isogeny* $\phi_1 : JW_{33} \times P(31/33) \times P(27/31) \times P(8/27) \times P(3/8) \times P(1/3) \to JW$

PROOF. It is the recursive application of the proposition 2.6.1 □

PROPOSITION 6.3.6. *There is an isogeny*

$$\phi_2 : P(30/33) \times P(32/33) \to P(27/31)$$

PROOF. It follows from $H_{33}/H_{27} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we apply what has been stated in [**14**] for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

PROPOSITION 6.3.7. *There is an isogeny*

$$\phi_3 : P(24/31) \times P(24/31) \to P(8/27)$$

PROOF. Hence from the fact that $H_{31}/H_8 \simeq S_3$ and we apply what has been stated in [**14**] for the group S_3. □

PROPOSITION 6.3.8. *There is an isogeny*

$$\phi_4 : P(3/8) \times P(3/8) \to P(1/3)$$

PROOF. It follows from $H_8/H_1 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we apply what has been stated in [**14**] for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

PROPOSITION 6.3.9. *There is an isogeny*

$$\phi_5 : P(9/20) \times P(12/20) \to P(3/8)$$

PROOF. Hence from $H_{20}/H_3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we apply what has been stated in [**14**] for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

PROPOSITION 6.3.10. *There is an isogeny*

$$\phi_6 : P(22/29) \times P(21/29) \to P(9/20)$$

PROOF. Due to $H_{29}/H_9 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we apply has been stated in [**14**] for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

PROPOSITION 6.3.11. *There is an isogeny*

$$\phi_7 : P(23/29) \times P(26/29) \to P(12/20)$$

PROOF. Due to $H_{29}/H_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we apply what has been stated in [**14**] for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

In order to demonstrate the theorem 6.3.4 we "substitute" the factors of the isogeny of the proposition 6.3.5 by its corresponding isogeny stated in the propositions from the 6.3.6 to the 6.3.11, and we build the isogenea Φ of the theorem 6.3.4.

6.3.3. Genera of the intermediate coverings that appear in the Theorem 6.3.4. C_3 has 10 conjugate classes of cyclic subgroups, a set of representatives is:
$g_0 = id$
Of order 2:
$g_1 =< xyz >$
$g_2 =< zy >$
$g_3 =< z >$
$g_4 =< xyzb >$
$g_5 =< yzab >$
Of order 3:
$g_6 =< yza^2 >$
Of order 4:
$g_7 =< yxab >$
$g_8 =< yab >$
Of order 6:
$g_9 =< xa^2 >$

Suppose that for each one there are α_i points marked in the quotient W/C_3 and that this has genus γ. Then we can compute the genera of the intermediate surfaces, this is useful because it allows us to compute the dimensions of the subvarieties that appear in the decomposition of the Jacobian variety under study. This is an application of chapter 3

List of genera:

$$g_W = 48\gamma-47+12(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+16\alpha_6+18(\alpha_7+\alpha_8)+20\alpha_9$$
$$g_{W_{29}} = 3(\gamma-1)+1+(\alpha_4+\alpha_5+\alpha_7+\alpha_8)/2+\alpha_6+\alpha_9)$$
$$g_{W_{21}} = 6(\gamma-1)+1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+2(\alpha_6+\alpha_7+\alpha_8+\alpha_9)$$
$$g_{W_{22}} = 6(\gamma-1)+1+(\alpha_2+\alpha_3)+3(\alpha_4+\alpha_5+\alpha_7+\alpha_8)/2+2(\alpha_6+\alpha_9)$$
$$g_{W_{23}} = 6(\gamma-1)+1+3(\alpha_1+\alpha_5+\alpha_7)/2+\alpha_2+\alpha_4+\alpha_3/2+2(\alpha_6+\alpha_8)+5\alpha_9/2$$
$$g_{W_{26}} = 6(\gamma-1)+1+3(\alpha_1+\alpha_4+\alpha_8)/2+\alpha_2+\alpha_5+2(\alpha_6+\alpha_7)+\alpha_3/2+5\alpha_9/2$$
$$g_{W_{30}} = 2(\gamma-1)+1+(\alpha_4+\alpha_5+\alpha_7+\alpha_8)/2$$
$$g_{W_{31}} = 2(\gamma-1)+1+(\alpha_1+\alpha_3+\alpha_5+\alpha_7+\alpha_9)/2$$
$$g_{W_{32}} = 2(\gamma-1)+1+(\alpha_1+\alpha_3+\alpha_4+\alpha_8+\alpha_9)/2$$
$$g_{W_{28}} = 4(\gamma-1)+1+\alpha_2+\alpha_3+\alpha_6+\alpha_9+(\alpha_4+\alpha_5)/2+3(\alpha_7+\alpha_8)/2$$
$$g_{W_8} = 12(\gamma-1)+1+3(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7+\alpha_8)+4\alpha_6+5\alpha_9$$
$$g_{W_{27}} = 4(\gamma-1)+1+\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7+\alpha_8+\alpha_9$$
$$g_{W_{24}} = 6(\gamma-1)+1+3(\alpha_1+\alpha_3+\alpha_5+\alpha_7)/2+\alpha_4+\alpha_8+2\alpha_6+5\alpha_9/2$$

6.3.4. Representations that act in the factors of the decomposition given in the Theorem 6.3.4. Here we use a lot of the results exposed in the former chapters and sections.

Applying the Corollary 2.6.4 and due to we have the dimensions of the Prym that appear in the decomposition of the Theorem 6.3.4. We determine that in each factor ($n-times$) one complex (rational) irreducible representation acts, except for one factor, in which an addition of two of them acts. As always $n = dimP(C_H/C_N)$.

The representations are those found in section 6.2.5.

This is summed up in the following table:

Factor	Dimensión	Representación
P(31/33)	$\gamma-1+(\alpha_1+\alpha_3+\alpha_5+\alpha_7+\alpha_9)/2$	θ_5
P(30/33)	$\gamma-1+(\alpha_4+\alpha_5+\alpha_7+\alpha_8)/2$	θ_2
P(32/33)	$\gamma-1+(\alpha_1+\alpha_3+\alpha_4+\alpha_8+\alpha_9)/2$	θ_4
2P(24/31)	$4\gamma-4+\alpha_1+\alpha_3+\alpha_5+\alpha_7+\alpha_4+\alpha_8+2*\alpha_6+2*\alpha_9$	$\theta_3+\theta_6$
3P(22/29)	$3\gamma-3+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_7+a\alpha_8+\alpha_6+\alpha_9$	Θ_{10}
3P(21/29)	$3\gamma-3+\alpha_2+\alpha_3+\alpha_6+\alpha_9+\alpha_4/2+\alpha_5/2+3\alpha_7/2+3\alpha_8/2$	θ_9
3P(23/29)	$3\gamma-3+3\alpha_1/2+\alpha_5+\alpha_7+\alpha_2+\alpha_4/2+\alpha_3/2+\alpha_6+3\alpha_8/2+3\alpha_9/2$	θ_8
3P(26/29)	$3\gamma-3+3\alpha_1/2+\alpha_4+\alpha_8+\alpha_2+\alpha_5/2+\alpha_6+3\alpha_7/2+\alpha_3/2+3\alpha_9/2$	θ_7
JW_G	γ	θ_1

6.3.5. Construction of Surfaces with action of C_3 given by a determined representation. The idea here is to use the exposed on chapter 4 to construct surfaces with the property that its Jacobian is isogeneous to a power of Pryms therefore, G acts over it with a power of just one irreducible representation.

Observing in the previous table the dimension of each factor in the decomposition of the Jacobian of a curve with action of C_3. We search values for γ y α_i $i = 1..9$ such that all the dimensions be zero except one. We will have, then, if C_3 acts with these stabilizers in any surface W, the decomposition of the Jacobian of this one will be constituted by only one factor in which C_3 acts with a determined representation.

To build such W we use Theorem 4.2.1, we consider for all the cases the subgroup H_{26} presented in section 6.3.1.

We will only give the set of generating vectors using a permutational representation of C_3 associated to the subgroup H_{26}. The rest of the construction is the one presented one Theorem 4.2.1. The representations are found in section 6.2.5.

6.3.5.1. *Representation θ_8.* The values $\gamma = 0$, $\alpha_5 = \alpha_8 = \alpha_9 = 1$ and $\alpha_i = 0$ in the other cases, are those such that $\dim P(23/29) = 1$ and all the others zero. Consider the function $\phi : C_3 \to S_6$ given by:

$$\phi(b) = (1\ 2)(4\ 5)$$

$$\phi(yab) = (2\ 3\ 5\ 6)$$

$$\phi(xa^2) = (1\ 2\ 3\ 4\ 5\ 6)^{-1}$$

The conclusion here is: There is a curve W which Jacobian variety is isogeniuos to $3P(23/29)$, as the dimension of $P(23/29)$ is 1, we have that this jacobian is isognious to a product of the same eliptic curve (3-times). And in this product C_3 acts with its irreducible representation θ_8.

6.3.5.2. *Representation θ_7.* In order that $\dim P(26/29) = 1$ and the others zero. Consider $\gamma = 0$, $\alpha_4 = \alpha_7 = \alpha_9 = 1$, $\alpha_i = 0$ in another case. Consider the function $\phi : C_3 \to S_6$ given by:

$$\phi(xyzb) = (1\ 5)(2\ 4)(3\ 6)$$

$$\phi(yxab) = (1\ 4)(2\ 6\ 5\ 3)$$

$$\phi(xa^2) = (1\ 3\ 5\ 4\ 6\ 2)^{-1}$$

Conclusion: There is a curve W which jacobian variety is isognious to a product of the same eliptic curve (3-times).

$$JW \approx 3P(26/29)$$

And in this product C_3 acts with its irreducible representation θ_7.

6.3.5.3. *Representation* Θ_{10}. Consider the values $\gamma = 0$, $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 1$, $\alpha_i = 0$ in another case. Those that produce $\dim P(22/29) = 1$, other dimensions equal zero.

The following permutational representation is the one that demonstrates the existence of W where C_3 acts with those stabilizers: $\pi : C_3 \to S_6$ given by:

$$\phi(z) = (3\ 6)$$
$$\phi(xyzb) = (1\ 5)(2\ 4)(3\ 6)$$
$$\phi(yzab) = (2\ 6)(3\ 5)$$
$$\phi(yza^2) = (1\ 5\ 3)(2\ 6\ 4)$$

Conclusion: There is a curve W with jacobian variety isogenious to a product of the same elliptic curve (3-times).

$$JW \approx 3P(22/29)$$

And in this product C_3 acts with its irreducible representation Θ_{10}.

Remark 6.3.3. From the analysis of the equations of the dimensions presented in the former table, we understand that there are no values of γ and α_i that permit us to make an analogous process for the other representations.

APPENDIX

A. Code

```
cubr:=function(l,G,H,gwG)
local a,ng,rtng,s,i,j,t,index,B,b,bi,gw,gwH,u,u2,sig,m,k,c;
a:=Size(l);
ng:=[];
for i in [1..a] do
ng[i]:=Normalizer(G,l[i]);
od;
rtng:=[];
for i in [1..a] do
rtng[i]:=RightTransversal(G,ng[i]);
od;
s:=[];
for i in [1..a] do
s[i]:=[];
od;
for i in [1..a] do
for j in [1..Size(rtng[i])] do
s[i][j]:=ConjugateGroup(l[i],rtng[i][j]);
od;
od;
t:=[];
for i in [1..a] do
t[i]:=[];
od;
for i in [1..a] do
for j in [1..Size(rtng[i])] do
t[i][j]:=Intersection(s[i][j],H);
od;
od;
index:=[];
for i in [1..a] do
index[i]:=Index(ng[i],l[i]);
```

```
od;
B:=[];
for i in [1..a] do
B[i]:=0;
od;
for i in [1..a] do
for j in [1..Size(rtng[i])] do
B[i]:=B[i]+index[i]*(Size(t[i][j])-1);
od;
od;
b:=0;
for i in [1..a] do
b:=b+B[i];
od;
bi:=0;
gw:=0;
gwH:=0;
for i in [1..a] do
bi:=bi+1-1/Size(l[i]);
od;
gw:=Size(G)*(gwG-1)+1+(Size(G)*bi)/2;
gwH:=(gw-1-b/2)/Size(H) +1;
u:=[];
for i in [1..a] do
u[i]:=[];
od;
for i in [1..a] do
for j in [1..Size(rtng[i])] do
u[i][j]:=index[i]*Size(t[i][j]);
od;
od;
u2:=[];
for i in [1..a] do
u2[i]:=Collected(u[i]);
od;
sig:=[];
for i in [1..a] do
sig[i]:=[];
od;
for i in [1..a] do
for j in [1..Size(u2[i])] do
sig[i][j]:=[u2[i][j][1]/index[i],(u2[i][j][1]*u2[i][j][2])/Size(H)];
```

```
od;
od;
m:=[];
for i in [1..a] do
m[i]:=[];
od;
for i in [1..a] do
c:=1;
for j in [1..Size(sig[i])] do
for k in [1..sig[i][j][2]] do
m[i][c+k-1]:=(Size(l[i]))/sig[i][j][1];
od;
c:=c+sig[i][j][2];
od;
od;
Print("\n");
Print("genus of W=",gw,"\n");
Print("The signature of W/H is: \n");
Print("genus of W/H:",gwH,"\n");
for i in [1..a] do
Print(sig[i],"that they are on the point marked by " ,l[i],"\n");
od;
Print("\n");
Print("For the covering W/H–¿W/G the structure of cycles is: \n");
for i in [1..a] do
Print(l[i],"→" ,m[i],"\n");
od;
Print("The index of H in G is:",Index(G,H),"\n");
Print("Core(H)=",Core(G,H),"\n");
end;
```

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