FACULTAD DE MATEMÁTICAS
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SOLUCIONES TIPO BURBUJA PARA ECUACIONES DIFERENCIALES PARCIALES NO LINEALES.

Dos ecuaciones locales y una ecuación no-local.

POR

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Contents

Introduction

There are many questions arising when studying a differential equation. As we know existence and uniqueness of solutions are two fundamental and higly nontrivial questions. In those cases in which we know solutions do exist, an interesting question is how such solutions would be; namely, if they are regular, if they develop singularities, where are they located and so on.

In this work, three nonlinear elliptic partial differential equations are studied; addressing both, the existence and uniqueness questions. Even more, using the reduction method of Lyapunov-Schmidt such solutions are constructed explicitly at main order, which provide a quite precise description of their shapes.

Problem 1.

In **Chapter** 1, we study the equation

$$\begin{aligned} &\Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \;\; u > 0, \qquad && \text{in} \quad \Omega \subset \mathbb{R}^2, \\ &u = 0, && \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain with smooth boundary, the exponent p satisfies $0 < p < 2$ and $a(x)$ is a nonnegative smooth function in Ω. This problem is the Euler-Lagrange equation for the functional

$$J^p_{a,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda}{p}\int_\Omega a(x)e^{u^p}dx, \quad u \in H^1_0(\Omega). \tag{2}$$

For later purpose we introduce the functional

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j) + \frac{2-p}{4p\pi}\sum_{j=1}^{K}\log a(\xi_j).$$

Here $G(x,y)$ denotes the Green's function for the negative Laplacian with Dirichlet boundary condition in Ω, namely

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x), & x \in \Omega, \\ G(x,y) = 0, & x \in \partial\Omega \end{cases}$$

and $H(x,y)$ its regular part, given by

$$H(x,y) = G(x,y) - \frac{1}{2\pi}\log\frac{1}{|x-y|}.$$

The main result in this chapter is theorem 1.3

Theorem 0.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, $0 < p < 2$, Define the set $Z \subset \Omega$*

$$Z := \{q \in \Omega : a(q) = 0\}.$$

We make the following assumption on $a(x)$.

(A_1)*: For any $q \in Z$, there exists $\alpha_q > 0$ such that*

$$a_q(x) = a(x)|x-q|^{-2\alpha_q}$$

is a strictly positive continuous function in a neighborhood of q.

(A_2) *Assume $Z \subset \Omega$ is a finite set. Let $K \geq 2$ be an integer, and $q_1, \cdots, q_m \in Z$ be* *distinct points so that*

$$\frac{2-p}{p}\alpha_{q_s} \neq 1, \cdots, K-1, \quad \textit{for any } s = 1, \cdots, m, \tag{3}$$

and

$$K = \sum_{s=1}^{m} K_s, \quad \textit{with } \; K_s = \max\left\{ k_s \in \mathbb{N} \;:\; 1 \leq k_s < \frac{2-p}{p}\alpha_{q_s} + 1 \right\}. \tag{4}$$

Then there is $\lambda_0 > 0$ small such that for any $0 < \lambda < \lambda_0$, Problem (1) has a family of solutions u_λ with the property:

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega a(x) e^{u_\lambda^p} dx = 8K\pi, \tag{5}$$

where ε satisfies

$$p\lambda \left(-\frac{4}{p} \log \varepsilon \right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{6}$$

Moreover, there exists an $K-$tuple $\tilde{\xi}^\lambda = (\tilde{\xi}_1^\lambda, \cdots, \tilde{\xi}_K^\lambda) \in (\Omega \backslash Z)^K$ such that

$$u_\lambda(x) = \left(-\frac{4}{p} \log \varepsilon \right)^{\frac{1-p}{p}} \left(8\pi \sum_{j=1}^{K} G(x, \tilde{\xi}_j^\lambda) + o(1) \right) \tag{7}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\overline{(\Omega \backslash Z)} \backslash \{\tilde{\xi}_1^\lambda, \cdots, \tilde{\xi}_K^\lambda\}$.

Furthermore, the functional defined in (2) satisfies the following expansion

$$J_{a,\lambda}^p(u_\lambda) = \frac{1}{p} \left(-\frac{4}{p} \log \varepsilon \right)^{\frac{2(1-p)}{p}} \left[\frac{8K\pi}{(2-p)p} \left[-2 + p \log 8 \right] - \frac{16K\pi}{p} \log \varepsilon - \frac{32\pi^2}{2-p} \Phi_{a,K}^p(\tilde{\xi}^\lambda) + O(|\log \varepsilon|^{-1}) \right] \tag{8}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$, and

$$\nabla \Phi^p_{a,K}(\tilde{\xi}^\lambda_1, \cdots, \tilde{\xi}^\lambda_K) \to 0,$$

as $\lambda \to 0$.

Problem 2

In **Chapter 2** we study the equation

$$\begin{aligned} \Delta u + a(x) u^p &= 0, \quad u > 0, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{9}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, p is a large exponent and $a(x)$ is a non negative, smooth function in Ω.

The functional of energy asociated to this equation is

$$J_p(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p+1} \int_\Omega a(\cdot) |u|^{p+1} dx, \; u \in H^1_0(\Omega). \tag{10}$$

The main theorem in Chapter 2 is

Theorem 0.2. *Let $\Omega \subset \mathbb{R}^2$ bounded and smooth domain, let $M \in \mathbb{N}$ and suppose that $a(x)$ satisfies*

Let us define the set $Z \subset \Omega$

$$Z := \{q \in \Omega : a(q) = 0\}$$

A_1*: For any $q \in Z$ there exists a non negative numbers $\alpha_q \notin \mathbb{N}$ such that*

$$a_q(x) = a(x)|x - q|^{-2\alpha_q}$$

is a strictly positive function in a neighborhood of q.

A_2: For any $q \in Z$, let M_q an integer with $0 \leq M_q < 1 + \alpha_q$ and define $M = \sum_{q \in Z} M_q$.

Then there exists p_M such that for any $p \geq p_M$, the problem (9) has a solution u_p which concentrates at M different point of Ω, i.e. as p goes to ∞

$$pa(x)u_p^{p+1} \rightharpoonup 8\pi e \sum_{j=1}^{M} \delta_{\xi_i}.$$

Furthermore, for any $\varepsilon > 0$,

$$u_p \to 0 \quad \textit{uniformly in } \Omega \setminus \left(\cup_{j=1}^{M} B(\xi_j, \varepsilon) \right)$$

and

$$\sup_{x \in B(\xi_j, \varepsilon)} u_p(x) \to \sqrt{e}.$$

In this case, we prescribe the number M of the point ξ of concentration.

Actually, the points of concentration of the solution are the nontrivial critical points of

$$\varphi_M(\xi_1, ..., \xi_M) = \sum_{j=1}^{M} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i \neq j}^{M} G(\xi_j, \xi_i) + \frac{1}{4\pi} \sum_{j=1}^{M} \log(a(\xi_j)).$$

Without loss of generality, the potential has the form

$$a(x) = \prod_{s=1}^{m} |x - q_s|^{2\alpha_s}.$$

Where, in this case $\alpha \notin \mathbb{N}$. That technical condition is imposed to apply the main theorem in [10].

Problem 3

In **Chapter** 3 we study the nonlocal problem

$$(-\Delta)^s u = \gamma \, |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^n, \tag{11}$$

where $n \geq 3$ and p is the fractional critical Sobolev exponent $p = \frac{n+2s}{n-2s}$ and $\gamma = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})}$.

For any $s \in (0,1)$, $(-\Delta)^s$ is the nonlocal operator defined as

$$(-\Delta)^s(x) = c(n,s)\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = c(n,s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

For $n \geq 3$, and for $s \in \left(\frac{1}{2}, 1\right)$, we construct a family of sing-changing solutions of the equation (11) which concentrates in k points of a regular polygon, extending a result from the local case, namely $s = 1$. More precisely, let $\xi_j^k = \sqrt{1-k^{-2}}(e^{\frac{2j\pi i}{k}}, \vec{0})$. For any integer k sufficiently large, there is a finite energy solution of problem (11)

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2s}{2}} U\left(\mu_k^{-1}(x-\xi_j)\right) + o(1),$$

where

$$\mu_k = \frac{[2^{\frac{n-2s}{2}} \sum_{j=1}^{\infty} j^{2s-n}]^{-1}}{k^2}(1+o(1)).$$

These solutions correspond to critical points of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 - \gamma \frac{n-2s}{2n} \int_{\mathbb{R}^n} |u|^{\frac{2n}{n+2s}}, \quad u \in \mathcal{D}^s(\mathbb{R}^n),$$

where

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \,:\, \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} < \infty\},$$

$$\|u\|_{\mathcal{D}^s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

Moreover,

$$J(u_k) = (k+1)J(U) + O(1).$$

Here $O(1)$ remains bounded and $o(1) \to 0$ uniformly as $k \to +\infty$.

One of the particular properties of this solutions is invariance under kelvin transform. These properties of symmetry are very useful in the linear problem; they allow us to study the solution inside the unit ball and compare this with the values outside the ball and obtain a priori bounds.

THE METHOD

We are going to explain, in general terms, how the reduction method works, and to highlight in each step, some particular details arising in each one of the three equations we consider (actually since problems 1 and 2 are similar, we just consider 1 and 3).

Let us consider the following equation

$$\Delta u + f(u) = 0, \quad \text{in } \Omega, \tag{12}$$

where $\Omega = \mathbb{R}^n$ or Ω is a bounded regular subset of $\mathbb{R}^n$ in which case the equation has a Dirichlet boundary condition, or Neumann boundary condition. The function f is regular, let us say $f \in C^1(\mathbb{R})$. Of course f may depend on $x \in \mathbb{R}^n$.

Step One:

Let us consider a function U which, eventually, may depend on additional parameters. This function U will be a candidate of a solution in the sense that $||\Delta U + f(U)||_* \to 0$ for some special norm $\|\cdot\|_*$.

In problem (1) our approximate function U is

$$U(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{K} PU_{\mu_j,\xi_j}(x),$$

where $PU_{\mu_j,\xi_j}(x)$ is the projection of $U_{\mu_j,\xi_j}(x) = \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2 a(\xi_j)}$ on $H_0^1(\Omega)$. This projection is necessary because we need to satisfy the boundary condition. Functions $U_{\mu_j,\xi_j}(x)$ are translations and dilatations of the solution of the Liouville equation

$$\Delta u + e^u = 0, \quad \text{on } \mathbb{R}^2.$$

To sketch why this function can approximate the solution of equation (1), consider the function

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p$$

with $\gamma^p = -\dfrac{4}{p}\log\varepsilon$. Thanks to (6), problem (1) reduces to

$$\begin{aligned} \Delta v + g(v) &= 0, \quad v > -p\gamma^p, \quad \text{in } \Omega_\varepsilon \\ v &= -p\gamma^p, \qquad\qquad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{13}$$

where

$$g(v) = a(\varepsilon y)\left(1 + \frac{v}{p\gamma^p}\right)^{p-1} e^{\gamma^p\left[\left(1+\frac{v}{p\gamma^p}\right)^p - 1\right]}.$$

Then, applying a change of variable to U, we get a candidate of the solution of equation (13)

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p.$$

When we are near to the points ξ we can use Taylor expansion to get, as $\varepsilon \to 0$

$$\Delta V_\lambda \sim -e^{w_i}$$

and

$$g(V_\lambda) \sim a(\xi_j)e^{\widetilde{w}_i} = e^{w_i}.$$

Then

$$\Delta V_\lambda + g(V_\lambda) \sim 0, \tag{14}$$

away from the points ξ_i the decaying of the function V_λ and the norm

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon}\left(\sum_{j=1}^{K}(1 + |y - \xi_j'|)^{-3} + \varepsilon^2\right)^{-1} |h(y)|$$

give us the nullity like

$$\|\Delta V_\lambda + g(V_\lambda)\|_* \sim 0 \tag{15}$$

as $\varepsilon \to 0$.

It is worth to mention the main role of the maximum principle applied to $PU_{\mu_j,\xi_j}(x)$, which allow us to write U as

$$U_\lambda(x) = \frac{8\pi}{p\gamma^{p-1}} \sum_{j=1}^{K} G(x,\xi_j)\left[1+O(\varepsilon^2)\right], \tag{16}$$

where the Green function $G(x,\xi)$ is singular at $\xi \in \Omega$.

A difficulty in this part was that this definition of U with this singular behavior (which allows us to think of solutions concentrated on points ξ_j), is not sufficient for the subsequent steps (more precisely in the nonlinear problem) so we need to improve the decaying in $\|\Delta V_\lambda + g(V_\lambda)\|_* \to 0$.

After a Taylor expansion when x is near to ξ_j we obtain

$$\begin{aligned} g(V_\lambda) &= [a(\xi_j)+O(\varepsilon)]e^{\widetilde{w}_j}\left[1+(\tfrac{p-1}{p})\tfrac{1}{\gamma^p}[\tfrac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j]+(\tfrac{p-1}{p})^2\tfrac{1}{\gamma^{2p}}[*]+O\left(\tfrac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\right] \\ &= e^{w_j}+\frac{p-1}{p}\frac{1}{\gamma^p}e^{w_j}[\tfrac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j]\cdots . \end{aligned}$$

Then we add a first correction term w_j^0 which satisfies

$$\Delta w_j^0 + e^{w_j}w_j^0 = -e^{w_j}\left[\frac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j\right] \leftrightarrow \Delta w_j^0 = -e^{w_j}\left[w_j^0+\frac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j\right],$$

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^{K}\left[PU_{\mu_j,\xi_j}(x)+\frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x)\right].$$

After that, there are new terms in the Taylor expansion

$$\begin{aligned} g(V_\lambda) &= [a(\xi_j)+O(\varepsilon)]e^{\widetilde{w}_j}\Big[1+(\tfrac{p-1}{p})\tfrac{1}{\gamma^p}[w_j^0+\tfrac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j]+(\tfrac{p-1}{p})^2\tfrac{1}{\gamma^{2p}}\Big[2\widetilde{w}_j w_j^0 \\ &\quad +\tfrac{1}{2}\left(w_j^0+\tfrac{(\widetilde{w}_j)^2}{2}\right)^2+w_j^0+\tfrac{p-2}{2(p-1)}(\widetilde{w}_j)^2+\tfrac{\widetilde{w}_j^3}{2}\Big]+O\left(\tfrac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\Big]. \end{aligned}$$

Again we add a correction term w_j^1 such that

$$\Delta w_j^1 + e^{w_j} w_j^1 = -e^{w_j}\left[2\widetilde{w}_j w_j^0 + \frac{1}{2}\left(w_j^0 + \frac{(\widetilde{w}_j)^2}{2}\right)^2 + w_j^0 + \frac{p-2}{2(p-1)}(\widetilde{w}_j)^2 + \frac{\widetilde{w}_j^3}{2}\right].$$

Finally the function U is

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^{K}\left[PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\right];$$

the above function is a good approximation in the context of our first step.

First Step In Problem 3

In equation (11), function U is

$$U_*(y) = U(y) - \sum_{j=1}^{k} U_j(y),$$

where

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2s}{2}}, \quad U_j(y) = \mu^{-\frac{n-2s}{2}}U(\mu^{-1}(y-\xi_j)).$$

with $\mu \sim k^{-2}$. The norm to study $E = (-\Delta)^s U_* - \gamma\,|U_*|^{p-1}U_*$ is

$$\|E\|_{**} := \|(1+|y|)^{n+2s-\frac{2n}{q}})E\|_{L^q(\mathbb{R}^n)}.$$

Here, to prove $\|E\|_{**} \to 0$ we divide the domain in balls with center in the boundary of $B(0,\sqrt{1-\mu^2}) \cap \mathbb{R}^2$. Each center, ξ_i, is a point of a regular polygon with k vertex. In this case since $\mu \sim k^{-2}$ (can't be prescribed); a necessary condition appear, $1 - \frac{n}{q} < 0$, namely, $q < n$. This is one of the technical reasons why $s \in (\frac{1}{2}, 1)$.

After constructing a good approximation of the solution of the equation, we are going to prove the existence of a function ϕ such that $u = U + \phi$ is a solution of problem (12). This function ϕ will be small compared with U and is called the error function.

Several ingredients will be used to prove the existence of the function ϕ. Notice that we can state our model equation (12) depending of variable ϕ:

$$\Delta\phi + f'(U)\phi + \Delta U + f(U) + f(U+\phi) - f(U) - f'(U)\phi = 0,$$

or equivalently

$$L(\phi) = -R - N(\phi),$$

where

$$L(\phi) = \Delta\phi + f'(U)\phi, \quad N(\phi) = f(U+\phi) - f(U) - f'(U)\phi, \quad R = \Delta U + f(U).$$

Step Two

In the second step we develop a linear theory for the linear equation

$$L(\phi) = h,$$

using well known theorems like Fredholm's alternative theorem and a priori estimates. To solve this equation, we need to study the kernel of operator L. Sadly not in all case we can obtain a full characterization of this set. In our case the asymptotic behavior of L, when we expand the domain

$$L = T + o(\epsilon),$$

where T is an operator with a better know Kernel, let us permit consider the kernel of T as an approximate kernel of L.

Having the elements of the approximate kernel of T, Z_i, we consider them on a new projected equation

$$L(\phi) = h + \sum_i c_i Z_i, \tag{17}$$

where the coefficients c_i depend on points ξ_i.

In Chapter 1 the linear operator is

$$L(\phi) := \Delta\phi + g'(V_\lambda)\phi,$$

which is approximated by

$$T(\phi) = \Delta\phi + \frac{8}{(1+|y|^2)^2}\phi \text{ As } \lambda \to 0.$$

In $\mathbb{R}^2$ this operator is non-degenerated in the class of bounded functions around a function

$$\varphi(y) = \log \frac{8}{(1+|y|^2)^2},$$

which is a bounded solution of the Liouville equation.

A main difficulty in this step is the comparison of the coefficients c_i with the solution ϕ of the equation (17), where one gets a priori bounds comparing some measures with a given function test; without this estimates it is no possible to apply the Fredholm alternative theorem. Among others we mention the technical results performed in (1.70):

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle h, Z_{ij}\chi_{2j}\rangle + c_{ij}\int_{\Omega_\varepsilon} \eta_j |Z_{ij}|^2.$$

where the function $a(x)$ must be controlled separating the integrability domains and its differentiability to find an a priori bound to c_{ij}

$$|c_{ij}| \leq C\|h\|_*.$$

The main objective of these estimates is to apply Fredholm Alternative theorem, so we use the Riez theorems to write the equation in the equivalent form $(Id - K)\phi = \tilde{h}$ where K is a compact operator, this part is difficult since we have to find the appropriate space to establish the compactness property of K. In this case we must consider a subspace of $H_0^1(\Omega)$

$$\mathbb{H} = \left\{\phi \in H_0^1(\Omega_\varepsilon) \; : \; \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 \quad \text{for } i = 1, 2, \; j = 1, 2, \cdots, K\right\},$$

to preserve the additional orthogonality condition coming from the projection over complement of kernel. Notice that the weight function η_j, a cut-off centered in ξ_j, guarantee integrability of $z_{ij}\phi$.

In **Chapter** 3 the linear problem associated with the s-Laplacian is studied in a different manner from the local case; in the literature one can find classical Sobolev injections and regularity theorems which can be located to a ball. In our case, the nonlocal nature prevents such a procedure without defining the Laplacian both inside and outside. Moreover, for the case $s \in (0, 1/2)$, laplacian has no trace, which led us to face the problem of a priori bounds from the definition, not making use of the maximum principle as in the local case. We have found in the literature an article [22] dealing with the same problem; the authors consider the problem in $\mathbb{R}^n$ and locate the operator to apply regularity theorems in a domain, wich generates a contradiction with the nature of the nonlocal problem on the whole space.

Some embedding theorems by holder inequality, see (3.32), gave us an extra condition $q < \frac{n}{s}$ and to find an admissible interval of q we have the condition $s \in \left(\frac{1}{2}, 1\right)$.

Step Three: Nonlinear Problem

The next aim is to prove the existence of a solution ϕ of an auxiliary projected problem

$$L(\phi) + R + N(\phi) = \sum_i c_i Z_i. \tag{18}$$

There are a least two ways to prove that.

One of this is to use the invertibility of the operator L. We can prove the existence of an operator A in a appropriate space such that equation (18) be equivalent to

$$\phi = A(\phi), \tag{19}$$

with A, a contracting mapping over an special region and, by the fixed point theorem, to solve equation (18). For example, in Chapter 2, we want to solve

$$\begin{cases} \Delta(V_\lambda+\phi)+g\left(V_\lambda+\phi\right)=\sum_{i=1}^{2}\sum_{j=1}^{K}c_{ij}Z_{ij}\eta_j, & \text{in } \Omega_\varepsilon, \\ \phi=0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon}\phi Z_{ij}\eta_j=0, \quad \text{for } i=1,2,\ j=1,\cdots,K. \end{cases} \tag{20}$$

Here, the special region under consideration is

$$\mathcal{F}_M:=\left\{\phi\in C(\bar{\Omega})\ :\ \|\phi\|_\infty\leq\frac{M}{|\log\varepsilon|^2}\right\}.$$

Estimates over ϕ, $N(\phi)$ in $\mathcal{F}_M$ together with the continuity of the operator A, (from the Riesz theorem), allow us to use a classical fixed point theorem over (19).

$$\|A(\phi_1)-A(\phi_2)\|_\infty\leq C|\log\varepsilon|^{-1}\|\phi_1-\phi_2\|_\infty.$$

When we consider the function V_λ without the corrected terms, $\|E_\lambda\|_*\leq C|\log(\varepsilon)|^{-1}$, one obtain for $A(\phi)$ (actually $A(\mathcal{F}_M)\not\subseteq\mathcal{F}_M$)

$$\|A(\phi_1)-A(\phi_2)\|_\infty\leq C\|\phi_1-\phi_2\|_\infty.$$

Which does not allow to apply the fixed point theorem.

Another way is to impose that $\phi=\sum_j\phi_j$ with $\phi_1,\phi_2,...\phi_{k+1}$ satisfying a nonlinear system, this system is not equivalent to our nonlinear problem, but implies the existence of the solution. This system is difficult to define. In Chapter 3, we consider equations which are ”localized” in each ball $B_i=B(\xi_i,\sqrt{1-\mu^2})$ (k of them) and $(\cup_i B_i)^c$. We try to prove the nonlinear problem using the dual norm, $L^{\frac{2n}{n-2s}}\to L^{\frac{2n}{n+2s}}$ (notice that the linear problem with a priori bound is natural, see (3.31)) but we were not able to control the therms (we think for the same reason that it doesn’t work in all interval (0,1) for s, with our norms) and to define a good region to apply the fixed point theorem (on each equation).

Final Step Problem 1

As a final step, we must find a configuration of parameters (center of bubble) in functions c_i such that $c_i \equiv 0$ for all i.

For this, we will see that there exist a non trivial relation between the existence of critical points of a functional of energy associated to the original equation and the configuration of the parameter which cancel the coefficient c_i; here is where the method get the name of reduction because we transform an infinite dimensional problem like: to find a critical point of an infinite dimensional function, to a finite problem (remember the parameter in c_i are finite). In Chapter 1, after getting the solution $U_\xi + \phi_\xi$ we consider the original problem in the weak sense. Associated to the original problem we have a functional of energy

$$J^p_{a,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda}{p}\int_\Omega a(x)e^{u^p}dx, \quad u \in H^1_0(\Omega),$$

$$\mathcal{J}_\lambda(\xi) = J^p_{a,\lambda}\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right), \tag{21}$$

where

$$\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right). \tag{22}$$

It is important to notice here that $D_\xi \mathcal{J}_\lambda(\xi) = 0$ implies $c_{ij}\left(\frac{\xi}{\varepsilon}\right) = 0$. Thus $U_\lambda + \tilde{\phi}$ is a critical point of $J^p_{a,\lambda}$, namely a solution of problem (1).

After that, to find the critical point of $\mathcal{J}_\lambda(\xi)$, we expand this function around the point $\xi = (\xi_1, \xi_2, ..., \xi_K)$

$$\mathcal{J}_\lambda(\xi) = \Theta(\xi) + O(\varepsilon).$$

We know that the function Θ has a special critical point "which remains with the error $O(\varepsilon)$", that kind of critical points are called nontrivial critical value. Of course that kind of expansion is difficult to find; one has to proceed carefully because everything count in this step and there is no standard method to proceed. Finally, the main problem is to find this special critical value.

Final Step Problem 3

Same ideas apply in problem 3. After a geometric argument, one needs to find $\delta > 0$ such that $c_{n+1}(\delta) = 0$ (just one of them) to prove the existence of the solution of problem (11). After a suitable reduction of the function $c(\delta)$, a correct application of intermediate value theorem gave us the result (we improve the original expansion thinking that it could help to expand the interval of definition of s but it did not give fruitful results).

CHAPTER 1

Liouville equation with potential in $\mathbb{R}^2$

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1.1 Introduction

We consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda a(x) u^{p-1} e^{u^p} = 0, \ \ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p < 2$. The function $a(x) \geq 0$ is smooth in Ω. This problem is the Euler-Lagrange equation for the functional

$$J^p_{a,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\lambda}{p}\int_\Omega a(x) e^{u^p} dx, \quad u \in H^1_0(\Omega). \tag{1.2}$$

If $a(x) \equiv 1$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This problem has been studied widely in the literature when $p = 1$. The asymptotic behavior of blowing up families of solutions can be referred to [1, 4, 27, 30, 31, 33]: in these works it has been established that if u_λ is an unbounded family of solutions to (1.3) for which $\lambda \int_\Omega e^{u_\lambda}$ remains uniformly bounded as $\lambda \to 0$, then there exists an integer K such that

$$\lambda \int_\Omega e^{u_\lambda}\, dx \to 8\pi K, \quad \text{as} \quad \lambda \to 0.$$

Moreover there are K points $\xi_1, \ldots, \xi_K$ in Ω, which are far away from the boundary of Ω and far away from each others, so that

$$\lambda e^{u_\lambda} \to \sum_{j=1}^{K} \delta_{\xi_j}$$

in the sense of measure. Furthermore, the location of the point $(\xi_1, \ldots, \xi_K)$ is known to be related to the critical points of the function

$$\Phi_K(\xi) = \sum_{j=1}^{K} H(\xi_j, \xi_j) + \sum_{i \neq j}^{K} G(\xi_i, \xi_j).$$

Here $G(x, y)$ denotes the Green's function for the negative Laplacian with Dirichlet boundary condition in Ω, namely

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x) & x \in \Omega; \\ G(x, y) = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

and $H(x,y)$ its regular part, given by

$$H(x,y) = G(x,y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}. \tag{1.5}$$

Concerning the reciprocal issue, several results are already known in the literature, we refer to [1, 19, 13]. In particulat, in [13] del Pino-Kowalczyk-Musso constructed bubbling solutions to problem (1.3) when $p = 1$. They showed that: *If the domain* Ω *is not simply connected*, and given any integer $K \geq 1$, there exist K points $\xi_1, \ldots, \xi_K$ in Ω and a family of solutions u_λ, for any λ sufficiently small, which blows-up at these K points in the sense that, as $\lambda \to 0$

$$\sup_{x \in \Omega \setminus \cup_{j=1}^K B(\xi_j, \delta)} u_\lambda(x) \to 0, \quad \text{and for any} \quad j = 1, \ldots, K, \quad \sup_{x \in B(\xi_j, \delta)} u_\lambda(x) \to \infty$$

for any positive fixed number δ. Furthermore,

$$\int_\Omega \lambda e^{u_\lambda} dx \to 8K\pi \quad \text{as } \lambda \to 0.$$

The location of these blow-up points $\xi_1, \ldots, \xi_K$ is not arbitrary: indeed they correspond to critical points of the function Φ_K defined above.

The results have been extended in [17] for the whole range of values of exponents p with $0 < p < 2$. This result was surprising, since the scenario changes completely when $p = 2$: this situation was previously treated in [16].

In this paper, we construct bubbling solutions to Problem (1.1), with a positive non trivial potential. When $p = 1$, this situation was already treated in [13], under the condition that the concentration points $(\xi_1, \ldots, \xi_K)$ belong to a region where the potential a is strictly positive. Our first result shows that this construction can be done for the whole range of exponent $0 < p < 2$.

Before stating our result, it is useful to introduce some notations. For an integer $K \geq 1$ and K distinct points ξ_j, $j = 1, \cdots, K$, in Ω, separated uniformly from each other and from the boundary $\partial\Omega$, write $\xi = (\xi_1, \cdots, \xi_K)$, let us define the following functional

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^{K} H(\xi_j, \xi_j) + \sum_{i\neq j}^{K} G(\xi_i, \xi_j) + \frac{2-p}{4p\pi} \sum_{j=1}^{K} \log a(\xi_j). \tag{1.6}$$

Definition 1.1. We say that ξ is a C^0-stable critical point of $\varphi : \mathcal{M} \to \mathbb{R}$ if for any sequence of function $\varphi_n : \mathcal{M} \to \mathbb{R}$ such that $\varphi_n \to \varphi$ uniformly on compact sets of $\mathcal{M}$, φ_n has a critical point ξ^n such that $\varphi_n(\xi^n) \to \varphi(\xi)$.

In particular, if ξ is a strict local minimum or maximum point of φ, then ξ is C^0-stable critical point.

Let ε be a parameter, which depends on λ, defined as

$$p\lambda \left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{1.7}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$.

The result we have is the following.

Theorem 1.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, $0 < p < 2$ and K an integer with $K \geq 1$, assume that $a(x) \geq 0$ smooth in Ω, and $\xi^* = (\xi_1^*, \ldots, \xi_K^*)$ is a $\mathcal{C}^0-$stable critical point of $\Phi^p_{a,K}$. Then there exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$, Problem (1.1) has a solution u_λ, satisfies*

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega a(x) e^{u_\lambda^p} dx = 8K\pi, \tag{1.8}$$

where ε satisfies (1.7). Moreover, there exists an $K-$tuple $\xi^\lambda = (\xi_1^\lambda, \cdots, \xi_K^\lambda) \in \Omega^K$ such that $a(\xi_j^\lambda) > 0$, and as $\lambda \to 0$,

$$\Phi^p_{a,K}(\xi_1^\lambda, \cdots, \xi_K^\lambda) \to \Phi^p_{a,K}(\xi_1^*, \cdots, \xi_K^*),$$

and

$$u_\lambda(x) = \left(-\frac{4}{p}\log\epsilon\right)^{\frac{1-p}{p}} \left(8\pi\sum_{j=1}^{K} G(x,\xi_j^\lambda) + o(1)\right) \tag{1.9}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega}\backslash\{\xi_1^\lambda, \cdots, \xi_K^\lambda\}$. Furthermore

$$J_{a,\lambda}^p(u_\lambda) = \frac{1}{p}\left(-\frac{4}{p}\log\epsilon\right)^{\frac{2(1-p)}{p}} \left[\frac{8K\pi}{(2-p)p}[-2+p\log 8]\right.$$

$$\left. -\frac{16K\pi}{p}\log\varepsilon - \frac{32\pi^2}{2-p}\Phi_{a,K}^p(\xi^\lambda) + O(|\log\varepsilon|^{-1})\right] \tag{1.10}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

In [13], the authors consider also the case in which the potential $a(x)$ has a zero of type $|x-q|^\alpha$ for some point $q \in \Omega$. When $p = 1$ and $K < 1+\alpha$, they show the existence of a family of solutions u_λ to Problem (1.1) blowing up at K points of Ω, which remain far from q. This result was generalized by [10] in the case in which the potential a has several zeros $q_1, \ldots, q_m$, of type $|x-q_j|^{\alpha_j}$ respectively. She studies how the concentration phenomena is affected by the presence of several zeros for the potential. Our next result concerns a generalizations of these results when the exponent p belongs to the whole range $0 < p < 2$.

Define the set $Z \subset \Omega$ satisfies

$$Z := \{q \in \Omega : a(q) = 0\}.$$

We make the following assumption on $a(x)$.

(A_1)*: For any $q \in Z$, there exists $\alpha_q > 0$ such that*

$$a_q(x) = a(x)|x-q|^{-2\alpha_q}$$

is a strictly positive continuous function in a neighborhood of q.

(A_2) *Assume $Z \subset \Omega$ is a finite set. Let $K \geq 2$ be an integer, and $q_1, \cdots, q_m \in Z$ be distinct points so that*

$$\frac{2-p}{p}\alpha_{q_s} \neq 1, \cdots, K-1, \quad \textit{for any } s = 1, \cdots, m, \tag{1.11}$$

and

$$K = \sum_{s=1}^{m} K_s, \quad \textit{with } \; K_s = \max\left\{k_s \in \mathbb{N} \;:\; 1 \leq k_s < \frac{2-p}{p}\alpha_{q_s} + 1\right\}. \tag{1.12}$$

We have the following result.

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, $0 < p < 2$, and $a(x)$ satisfies $(A_1), (A_2)$. Then there is $\lambda_0 > 0$ small such that for any $0 < \lambda < \lambda_0$, Problem (1.1) has a family of solutions u_λ with the property:*

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega a(x) e^{u_\lambda^p} dx = 8K\pi, \tag{1.13}$$

where ε is defined in (1.7). Moreover, there exists an $K-$tuple $\tilde{\xi}^\lambda = (\tilde{\xi}_1^\lambda, \cdots, \tilde{\xi}_K^\lambda) \in (\Omega \backslash Z)^K$ such that as $\lambda \to 0$

$$\nabla \Phi_{a,K}^p(\tilde{\xi}_1^\lambda, \cdots, \tilde{\xi}_K^\lambda) \to 0,$$

and

$$u_\lambda(x) = \left(-\frac{4}{p}\log\varepsilon\right)^{\frac{1-p}{p}} \left(8\pi \sum_{j=1}^{K} G(x, \tilde{\xi}_j^\lambda) + o(1)\right) \tag{1.14}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\overline{(\Omega \backslash Z)} \backslash \{\tilde{\xi}_1^\lambda, \cdots, \tilde{\xi}_K^\lambda\}$. Furthermore

$$J_{a,\lambda}^p(u_\lambda) = \frac{1}{p}\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(1-p)}{p}} \left[\frac{8K\pi}{(2-p)p}\left[-2 + p\log 8\right]\right.$$

$$-\frac{16K\pi}{p}\log\varepsilon-\frac{32\pi^2}{2-p}\Phi^p_{a,K}(\tilde{\xi}^\lambda)+O(|\log\varepsilon|^{-1})\Big] \tag{1.15}$$

where $O(1)$ *uniformly bounded as* $\lambda\to 0$.

For the special case that Ω is the unit ball B in $\mathbb{R}^2$ and $a(x)=|x|^{2\alpha}$ with $\alpha>0$, that is, consider

$$\begin{cases}\Delta u+\lambda|x|^{2\alpha}u^{p-1}e^{u^p}=0,\ \ u>0 & \text{in } B;\\ u=0 & \text{on } \partial B,\end{cases} \tag{1.16}$$

where $\lambda>0$ is a small parameter. A direct consequence of Theorem 1.3 is that there exists a bubbling solution to (1.16) concentrating at points, which are outside the origin; furthermore the number of bubbling points depends on α. Set

$$K_\alpha=\max\left\{k\in\mathbb{N}\ :\ k<\frac{2-p}{p}\alpha+1\right\}.$$

The result we obtain for (1.16) can be stated as follows.

Theorem 1.4. *Let* $0<p<2$, *there exists* $\lambda_0>0$ *such that for any* $1\le K\le K_\alpha$, *for any* $0<\lambda<\lambda_0$, *the problem (1.16) has a solution* u_λ *which concentrates at* K *different points of* $B\backslash\{0\}$ *and*

$$\lim_{\lambda\to 0}\varepsilon^{\frac{2(2-p)}{p}}\int_B|x|^{2\alpha}e^{u_\lambda^p}dx=8K\pi, \tag{1.17}$$

where ε *satisfies (1.7). Moreover, (1.9) and (1.10) holds.*

Remark 1.5. To prove Theorem 1.3 we follow the approach developed in [10]: we apply a max-min argument to establish a topologically nontrivial critical value of $\Phi^p_{a,K}$ under the assumption (A_1) and (A_2) on $a(x)$ in any bounded smooth domain. Observe that we are not assuming the condition that domain is not simply connected. Observe that $Z=\emptyset$, the condition that Ω is not simply connected guarantees existence of a nontrivial critical value $\Phi^p_{a,K}$, see [13].

Remark 1.6. Theorem 1.4 is the special case of Theorem 1.3 for $a(x) = |x|^{2\alpha}$ and domain $\Omega = B$.

Remark 1.7. We construct bubbling solutions to (1.1), whose location of concentration occurs at points different from the zero set of the potential $a(x)$. The problem of finding solutions with additional concentration around at the zero points of $a(x)$ is of different type, indeed from the works [3, 2, 34] it follows that the contribution of each blow up point in the limit (1.13) is of $8\pi(1+\alpha)$. The asymptotic analysis in this situation is completely different.

In order to cover the case $p = 2$ in (1.1), we believe that a different approach is needed, given the known result for $a(x) \equiv 1$ contained in [16].

The paper is organized as follows: Section 1.2 is devoted to describing a first approximation solution to problem (1.1) and to estimate its error. Furthermore, we reduced problem into the finite-dimensional problem and solve it, we sketch it in section 1.4. In Section 1.5, we prove the main results.

1.2 The first approximation solution

In this Section, we build a good approximation solution and to estimate its error. Let us introduce the radially symmetric solutions of the following limit equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty,$$

which are given by the one parameter family of functions

$$w_\mu(z) = \log \frac{8\mu^2}{(\mu^2 + |z|^2)^2}. \tag{1.18}$$

Let K be an integer, set $\xi = (\xi_1, \cdots, \xi_K)$, let $\delta > 0$ small but fixed, define

$$\mathcal{O} := \left\{ \xi \in (\Omega \setminus Z)^K \ : \ \text{dist}(\xi_j, \partial(\Omega \setminus Z)) \geq \delta, |\xi_i - \xi_j| \geq \delta \text{ for } i \neq j \right\}. \tag{1.19}$$

Moreover, consider K positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all } j = 1, \ldots, K. \tag{1.20}$$

The parameters μ_j will be chosen properly later on. Define the function

$$\begin{aligned} U_{\mu_j,\xi_j}(x) &= \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2 a(\xi_j)} \\ &= w_{\mu_j}\left(\frac{x-\xi_j}{\varepsilon}\right) + 4\log\frac{1}{\varepsilon} - \log a(\xi_j). \end{aligned} \tag{1.21}$$

Let us denote $PU_{\mu_j,\xi_j}(x)$ the projection of U_{μ_j,ξ_j} into the space $H_0^1(\Omega)$, in other words, $PU_{\mu_j,\xi_j}(x)$ is the unique solution of

$$\begin{cases} \Delta PU_{\mu_j,\xi_j} = \Delta U_{\mu_j,\xi_j} & \text{in } \Omega; \\ PU_{\mu_j,\xi_j} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

By maximum principle, we have that for $\xi \in \mathcal{O}$ and μ_j satisfies (1.20), then

$$PU_{\mu_j,\xi_j}(x) = U_{\mu_j,\xi_j}(x) + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2) \tag{1.23}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$PU_{\mu_j,\xi_j}(x) = 8\pi G(x,\xi_j) + O(\mu_j^2\varepsilon^2) \tag{1.24}$$

in $C^1_{loc}(\overline{(\Omega\backslash Z)}\backslash\{\xi_j\})$ as $\varepsilon \to 0$, where $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are Green's function and its the regular part as defined in (1.4) and (1.5).

We now define the first ansatz is given by

$$U(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{K} PU_{\mu_j,\xi_j}(x),$$

with some number γ, to be fixed later on. We want to show that $U(x)$ is a good approximation for a solution to (1.1), and so that the solution to problem (1.1) like the formula $U(x)$ plus a small term. In order to perform the fixed point argument to find the lower order term, we need to improve our anstz, adding two other terms in the expansion of the solution. In order to do this, we set

$$w_j(y) = w_{\mu_j}(y - \xi_j') = \log \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2},$$

and

$$\widetilde{w}_j(y) = w_{\mu_j}(y) - \log a(\xi_j) = \log \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2 a(\xi_j)}. \tag{1.25}$$

Let w_j^i be the radial solution of

$$\Delta w_j^i + e^{w_j} w_j^i = e^{w_j} f^i \quad \text{in } \mathbb{R}^2, \quad \text{for} \quad i = 0, 1, \tag{1.26}$$

where

$$f^0 = -\left[\widetilde{w}_j + \frac{1}{2}(\widetilde{w}_j)^2\right],$$

and

$$f^1 = -\left[w_j^0 + \frac{p-2}{2(p-1)}(\widetilde{w}_j)^2 + \frac{1}{2}(w_j^0)^2 + \frac{1}{8}(\widetilde{w}_j)^4 + 2\widetilde{w}_j w_j^0 + \frac{1}{2}(\widetilde{w}_j)^3 + \frac{1}{2}w_j^0(\widetilde{w}_j)^2\right].$$

In fact, as shown in [20] (see also [7, 17]), there exists radially symmetric solutions with the properties that

$$w_j^i(y) = C_{ij} \log \frac{|y - \xi_j'|}{\mu_j} + O\left(\frac{1}{|y - \xi_j'|}\right) \quad \text{as } |y - \xi_j'| \to \infty, \tag{1.27}$$

for some explicit constants C_{ij}, which can be explicitly computed. In particular, when $i = 0$, the constant C_{0j} is given by

$$\begin{aligned} C_{0j} &= -8\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2a(\xi_j)}+\frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2a(\xi_j)}\right)^2\right]dt \\ &= 4\log 8-8-8\log\mu_j-4\log a(\xi_j). \end{aligned} \tag{1.28}$$

Let us define

$$w^0_{\mu_j,\xi_j}(x):=w^0_j\left(\frac{x}{\varepsilon}\right),\quad w^1_{\mu_j,\xi_j}(x):=w^1_j\left(\frac{x}{\varepsilon}\right)\quad \text{for } x\in\Omega.$$

Let $Pw^0_{\mu_j,\xi_j}$ and $Pw^1_{\mu_j,\xi_j}$ denote the projections into $H^1_0(\Omega)$ of $w^0_{\mu_j,\xi_j}$ and $w^1_{\mu_j,\xi_j}$, respectively. We write $y=\frac{x}{\varepsilon}$, $\xi_j'=\frac{\xi_j}{\varepsilon}$, by (1.27), we have that

$$Pw^i_{\mu_j,\xi_j}(x)=w^i_j\left(\frac{x}{\varepsilon}\right)-2\pi C_{ij}H(x,\xi_j)+C_{ij}\log(\mu_j\varepsilon)+O(\mu_j\varepsilon) \tag{1.29}$$

in $C^1(\bar{\Omega})$ as $\varepsilon\to 0$, and

$$Pw^i_{\mu_j,\xi_j}(x)=P\left(w^i_j\left(\frac{x}{\varepsilon}\right)\right)=-2\pi C_{ij}G(x,\xi_j)+O(\mu_j\varepsilon) \tag{1.30}$$

in $C^1_{loc}(\overline{(\Omega\backslash Z)}\backslash\{\xi_j\})$ as $\varepsilon\to 0$.

We define

$$U_\lambda(x)=\frac{1}{p\gamma^{p-1}}\sum_{j=1}^K\left[PU_{\mu_j,\xi_j}(x)+\frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\right]. \tag{1.31}$$

From (1.24) and (1.30), one has, away from the points ξ_j,

$$U_\lambda(x)=\frac{8\pi}{p\gamma^{p-1}}\sum_{j=1}^K G(x,\xi_j)\left[1-\frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4}-\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\frac{C_{1j}}{4}+O(\varepsilon^2)\right]. \tag{1.32}$$

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p, \quad \text{with } \gamma^p = -\frac{4}{p}\log\varepsilon.$$

Then problem (1.1) reduces to

$$\begin{cases} \Delta v + g(v) = 0, \ \ v > -p\gamma^p & \text{in } \Omega_\varepsilon; \\ v = -p\gamma^p & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.33}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$g(v) = a(\varepsilon y)(1 + \frac{v}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}. \tag{1.34}$$

Let us define the first approximation solution to (1.33) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{1.35}$$

with the numbers μ_j, $j = 1, \cdots, K$ defined by

$$\log\frac{8\mu_j^2}{a(\xi_j)} = \left[\frac{2(p-1)}{2-p}(1-\log 8) + \frac{8\pi}{2-p}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j)\right)\right]\left(1 + O\left(\frac{1}{\gamma^p}\right)\right). \tag{1.36}$$

Lemma 1.8. *We write $y = \varepsilon^{-1}x$, $\xi_j' = \varepsilon^{-1}\xi_j$. If μ_j, $j = 1, \cdots, K$, are given by (1.36), then for $|y - \xi_j'| < \delta/\varepsilon$ with δ sufficiently small but fixed, we have*

$$V_\lambda(y) = \widetilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y) + \theta(y), \tag{1.37}$$

with $\widetilde{w}_j$ defined by (1.25) and

$$w_j^0(y) := w_j^0\left(\frac{y-\xi_j'}{\mu_j}\right), \quad w_j^1(y) := w_j^1\left(\frac{y-\xi_j'}{\mu_j}\right).$$

and

$$\theta(y) = O(\varepsilon|y - \xi_j'|) + O(\varepsilon^2).$$

Proof. From (1.23), (1.24), (1.29), (1.30) and the fact that $U_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p = \widetilde{w}_j(y)$, we have

$$\begin{aligned}
V_\lambda(y) &= p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p \\
&= \sum_{j=1}^{K}\left[PU_{\mu_j,\xi_j}(x) + \tfrac{p-1}{p}\tfrac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x)\right] - p\gamma^p \\
&= PU_{\mu_j,\xi_j}(x) + \tfrac{p-1}{p}\tfrac{1}{\gamma^p}Pw^0_{\mu_j,\xi_j}(x) + \left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}Pw^1_{\mu_j,\xi_j}(x) - p\gamma^p \\
&\quad + \sum_{i\neq j}^{K}\left[PU_{\mu_i,\xi_i}(x) + \tfrac{p-1}{p}\tfrac{1}{\gamma^p}Pw^0_{\mu_i,\xi_i}(x) + \left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}Pw^1_{\mu_i,\xi_i}(x)\right] \\
&= U_{\mu_j,\xi_j}(x) - p\gamma^p + 8\pi H(x,\xi_j) - \log\tfrac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2) \\
&\quad + \tfrac{p-1}{p}\tfrac{1}{\gamma^p}\left[w^0_j(y) - 2\pi C_{0j}H(x,\xi_j) + C_{0j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + \left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\left[w^1_j(y) - 2\pi C_{1j}H(x,\xi_j) + C_{1j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon)\right] \\
&\quad + 8\pi\sum_{i\neq j}^{K}G(\xi_i,\xi_j)\left[1 - \tfrac{C_{0j}}{4}\tfrac{p-1}{p}\tfrac{1}{\gamma^p} - \tfrac{C_{1j}}{4}\left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\right] + O(\varepsilon^2) \\
&= \widetilde{w}_j(y) + \tfrac{p-1}{p}\tfrac{1}{\gamma^p}w^0_j(y) + \left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}w^1_j(y) + O(\varepsilon|y-\xi'|) + O(\varepsilon^2) \\
&\quad - \log\tfrac{8\mu_j^2}{a(\xi_j)} + \left[C_{0j}\tfrac{p-1}{p}\tfrac{1}{\gamma^p} + C_{1j}\left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\right](\log(\mu_j) + \log\varepsilon) \\
&\quad + 8\pi\left[1 - \tfrac{C_{0j}}{4}\tfrac{p-1}{p}\tfrac{1}{\gamma^p} - \tfrac{C_{1j}}{4}\left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\right]\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K}G(\xi_i,\xi_j)\right).
\end{aligned}$$

Since numbers μ_j satisfies (1.36), we note that $p\gamma^p = -4\log\varepsilon$, then find

$$\begin{aligned}
&- \log\tfrac{8\mu_j^2}{a(\xi_j)} + \left[C_{0j}\tfrac{p-1}{p}\tfrac{1}{\gamma^p} + C_{1j}\left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\right](\log(\mu_j) + \log\varepsilon) \\
&+ 8\pi\left[1 - \tfrac{C_{0j}}{4}\tfrac{p-1}{p}\tfrac{1}{\gamma^p} - \tfrac{C_{1j}}{4}\left(\tfrac{p-1}{p}\right)^2\tfrac{1}{\gamma^{2p}}\right]\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K}G(\xi_i,\xi_j)\right) \\
&= 0.
\end{aligned}$$

Thus (1.37) holds. □

We will look for solutions to (1.33) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (1.35), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (1.33) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} L(\phi) = -[E_\lambda + N(\phi)], & x \in \Omega_\varepsilon; \\ \phi = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{1.38}$$

where

$$L(\phi) := \Delta\phi + g'(V_\lambda)\phi, \tag{1.39}$$

$$E_\lambda := \Delta V_\lambda + g(V_\lambda), \tag{1.40}$$

$$N(\phi) := g(V_\lambda + \phi) - g(V_\lambda) - g'(V_\lambda)\phi. \tag{1.41}$$

We recall that $g(t) = a(\varepsilon y)(1 + \frac{t}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (1.38), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^{K} (1 + |y - \xi_j'|)^{-3} + \varepsilon^2 \right)^{-1} |h(y)| \tag{1.42}$$

for any $h \in L^\infty(\Omega_\varepsilon)$. With respect to this norm, the error term E_λ given in (1.40) can be estimated in the following way.

Lemma 1.9. *Let $\delta > 0$ be a small but fixed number and assume that the points $\xi \in \mathcal{O}$. There exists $C > 0$, such that we have*

$$\|E_\lambda\|_* \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{1.43}$$

for all λ small enough.

Proof. Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi'_j| > \frac{\delta}{\varepsilon}$, for all $j = 1, \cdots, K$, from (1.24) and (1.30) we have that

$$\Delta V_\lambda(y) = p\gamma^{p-1}\varepsilon^2 \Delta U(\varepsilon y) = O(\varepsilon^4).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{4\log\varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|} \tag{1.44}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$g(V_\lambda) = a(\varepsilon y)(1 + \frac{V_\lambda}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} O(1)$$

Thus if we are far away from the points ξ_j, or equivalently for $|y - \xi'_j| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\|\cdot\|_*$-norm, is relatively small. In other words, if we denote by 1_{outer} the characteristic function of the set $\{y \,:\, |y - \xi'_j| > \frac{\delta}{\varepsilon},\ j = 1, \ldots, K\}$, then in this region we have

$$\|E_\lambda 1_{\mathsf{outer}}\|_* \leq C \frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}}. \tag{1.45}$$

Let us now fix the index j in $\{1, \cdots, K\}$, for $|y - \xi'_j| < \frac{\delta}{\varepsilon}$, we have

$$\Delta V_\lambda(y) = -e^{w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\Delta w_j^0(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\Delta w_j^1(y) + O(\varepsilon^2). \tag{1.46}$$

On the other hand, for any $R > 0$ large but fixed, in the ball $|y - \xi'_j| < R_\varepsilon := R|\log\varepsilon|^\alpha$, with

$\alpha \geq 3$, we can use Taylor expansion to first get

$$(1+\frac{V_\lambda}{p\gamma^p})^{p-1} = 1+\frac{p-1}{p}\frac{1}{\gamma^p}\widetilde{w}_j+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}[w_j^0+\frac{p-2}{2(p-1)}(\widetilde{w}_j)^2]+(\frac{p-1}{p})^3\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|),$$

$$\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1] = \widetilde{w}_j+(\frac{p-1}{p})\frac{1}{\gamma^p}[w_j^0+\frac{(\widetilde{w}_j)^2}{2}]+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}(w_j^1+\widetilde{w}_jw_j^0)+\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|)$$

and

$$e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} = e^{\widetilde{w}_j}\left[1+(\frac{p-1}{p})\frac{1}{\gamma^p}[w_j^0+\frac{(\widetilde{w}_j)^2}{2}]\right.$$
$$\left.+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}[w_j^1+\widetilde{w}_jw_j^0+\frac{1}{2}(w_j^0+(\widetilde{w}_j)^2)^2]+\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|)\right]$$

Thus we obtain

$$\begin{aligned} g(V_\lambda) &:= a(\varepsilon y)(1+\tfrac{V_\lambda}{p\gamma^p})^{p-1}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &= [a(\xi_j)+O(\varepsilon)]e^{\widetilde{w}_j}\left[1+(\tfrac{p-1}{p})\tfrac{1}{\gamma^p}[w_j^0+\tfrac{(\widetilde{w}_j)^2}{2}+\widetilde{w}_j]\right. \\ &\qquad +(\tfrac{p-1}{p})^2\tfrac{1}{\gamma^{2p}}[w_j^1+2\widetilde{w}_jw_j^0+\tfrac{1}{2}(w_j^0+\tfrac{(\widetilde{w}_j)^2}{2})^2+w_j^0 \\ &\qquad \left.+\tfrac{p-2}{2(p-1)}(\widetilde{w}_j)^2+\tfrac{\widetilde{w}_j^3}{2}]+O\left(\tfrac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\right]. \end{aligned}$$

Thus, thanks to the fact that we have improved our original approximation with the terms w_j^0 and w_j^1, and the definition of $*$-norm, we get that

$$\|E_\lambda 1_{B(\xi_j',R_\varepsilon)}\|_* \leq \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3}, \quad \text{for any} \quad j=1,\dots,K. \tag{1.47}$$

Here $1_{B(\xi_j',R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j,R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon < |y-\xi_j'| < \frac{\delta}{\varepsilon}$, for any $j=1,\dots,K$, we have from one hand that $|\Delta V_\lambda(y)| \leq Ce^{w_j(y)}$, and also $|g(V_\lambda(y))| \leq Ce^{w_j(y)}$ as consequence of (1.37). This fact, together with (1.47) and (1.45) we obtain estimate (1.43). □

As the same proof of above Lemma, we have the following result.

Lemma 1.10. *For x very close to the point ξ_j in Ω, we have*

$$\|g'(V_\lambda) - e^{w_j}\|_* \to 0 \quad \text{as } \lambda \to 0, \tag{1.48}$$

and there exists some positive constant D_0 such that

$$g'(V_\lambda) \le D_0 \sum_{j=1}^{K} e^{w_j}. \tag{1.49}$$

Moreover,

$$\|g''(V_\lambda)\|_* \le C. \tag{1.50}$$

1.3 The Linear Problem

This section main objective is to study the invertibility of the linearized operator L.

Let us remember that for closed points to some ξ_j

$$\|g'(V_\lambda) - e^{w_j}\|_* \to 0 \quad \text{if } \lambda \to 0$$

Thus we can see that the operator L can be approximated by the family

$$L_j(\phi) = \Delta\phi + e^{w_j}\phi = \Delta\phi + \frac{8\mu_j^2}{\left(\mu_j^2 + |y - \xi_j'|^2\right)^2}\phi,$$

which we already know is non-degenerated , in the sense that the bounded solutions of $L_j(\phi) = 0$, excepting rescaling and traslation are (See [1]).

$$z_{0,j} = \partial_{\mu_j} w_{\mu_j} \quad z_{i,j}(y) = \partial_{y_i} w_{\mu_j}(y) \quad i = 1, 2.$$

Additionally we define

$$Z_{ij}(y) := z_{ij}(y - \xi'_j).$$

We considered $\xi = (\xi_1, ...\xi_K) \in \mathcal{O}$, additionally, let us consider a large but fixed number $R_0 > 0$ and a cut off function $\eta(\rho)$ with $\eta(\rho) = 1$ if $\rho < R_0$ and $\eta(\rho) = 0$ if $\rho > R_0 + 1$, and we denote

$$\eta_j(y) = \eta(|y - \xi'_j|).$$

Moreover, let $h \in C^{0,\alpha}(\Omega_\varepsilon)$, we consider the linear problem of finding a function ϕ and scalars $C_{ij} \quad i = 1, 2,\ j = 1, ..., K$ such that:

$$\begin{aligned} L(\phi) &= h + \sum_{j=1}^{2} \sum_{j=1}^{K} c_{ij} \eta_j Z_{ij} && \text{in } \Omega_\varepsilon \\ \phi &= 0 && \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi &= 0 && \text{for all } i = 1, 2,\ j = 1, .., K \end{aligned}$$

Lemma 1.11. *let $\tilde{\Omega}_\varepsilon := \Omega \setminus \cup_{j=1}^{K} B(\xi_j, R)$. There exists R large enough such that if $L(\phi) \leq 0$ in $\tilde{\Omega}_\varepsilon$ and $\phi \geq 0$ on $\partial\tilde{\Omega}_\varepsilon$ then $\phi \geq 0$ in $\tilde{\Omega}_\varepsilon$.*

Proof. We define the function

$$Z(y) = \sum_{j=1}^{K} z_0(a|y - \xi_j|), \quad y \in \Omega_\varepsilon,$$

where $z_0(r) = \frac{r^2 - 1}{r^2 + 1}$.

If a is taken small and fixed and R large enough such that $a|y - \xi'_j| > aR >> 1$ then $Z(y) > 0$; by (1.49) one has:

$$L(Z) \;= -\sum_{j=1}^{K} \frac{8a^2(a^2|y-\xi_j|^2-1)}{(1+a^2(|y-\xi_j'|^2))^3} + g'(V_\lambda)Z(y) \le -\sum_{j=1}^{K} \frac{c}{a^2|y-\xi_j|^4} + D_0 \sum_{j=1}^{K} e^{w_j} Z(y)$$

$$\le -\sum_{j=1}^{K} \frac{c}{a^2|y-\xi_j|^4} + \sum_{j=1}^{K} \frac{C}{|y-\xi_j|^4} \le 0$$

□

We consider for R as in lemma 1.11, the inner norm:

$$\|\phi\|_i := \sup_{y \in \cup_{j=1}^K B(\xi_j', R)} |\phi(y)|$$

Lemma 1.12. *Let $h \in L^\infty(\Omega_\varepsilon)$ if we consider the equation*

$$L(\phi) = h \quad \textit{in} \quad \Omega_\varepsilon \tag{1.51}$$

$$\phi = 0 \quad \textit{on} \quad \partial\Omega_\varepsilon, \tag{1.52}$$

then there exists $C > 0$ such that

$$\|\phi\|_\infty \le C[\|\phi\|_i + \|h\|_*]. \tag{1.53}$$

Proof. Let us take the following barrier

$$\tilde{\phi}(y) = 2\|\phi\|_i Z(y) + \|h\|_* \sum_{j=1}^{K} \psi_j(y)$$

where ψ_j is a solution of the equation:

$$-\Delta\psi_j = \frac{2}{|y-\xi_j'|^3} + 2\varepsilon^2, \qquad R < |y-\xi_j'| < \frac{M}{\varepsilon}, \tag{1.54}$$

$$\psi_j = 0 \qquad \text{if } |y-\xi_j'| = R, |y-\xi_j'| = \frac{M}{\varepsilon}, \tag{1.55}$$

where M is such that $\Omega_\varepsilon \subset B(\xi_j', \frac{M}{\varepsilon})$.

A direct computation shows that

$$\psi(r) = -\frac{2}{r} - \frac{\varepsilon^2 r^2}{2} + a\log(r) + b$$

where

$$a = \frac{\frac{2}{R} + \frac{\varepsilon^2 R^2}{2} - \frac{\varepsilon}{M} - \frac{M^2}{2}}{\log(\frac{\varepsilon R}{M})}$$

and

$$b = \frac{2}{R} + \frac{\varepsilon^2 R^2}{2} - a\log R$$

hence these functions have a uniform bound independent of ε as long as $1 < R < \frac{1}{2\varepsilon}$. By the maximum Principle one has $\psi_j \geq 0$; therefore, by the definition of Z(y) and for R large enough

$$\tilde{\phi}(y) \geq |\phi(y)| \text{ in } |y-\xi_j'| = R$$

$$\tilde{\phi} \geq 0 = \phi(y) \text{ on } \partial\Omega_\varepsilon$$

Moreover

$$\begin{aligned} L(\tilde{\phi}) &= 2\|\phi\|_i L(Z) + \|h\|_* L(\sum_{j=1}^{K} \psi_j) \leq \|h\|_* \sum_{j=1}^{K} (\Delta\psi_j + g'(V_\lambda)\psi_j) \\ &= \|h\|_* \sum_{j=1}^{K} \left(-\tfrac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + g'(V_\lambda)\psi_j \right) \\ &\leq \|h\|_* \sum_{j=1}^{K} \left(-\tfrac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + \tfrac{2KD_0}{R} e^{w_j} \right) \\ &\leq -\|h\|_* \left(\sum_{j=1}^{K} (1 + |y - \xi_j'|)^{-3} + \varepsilon^2 \right) \\ &\leq -|h(y)| \leq |L(\phi)(y)|. \end{aligned}$$

From Lemma 1.11 one has

$$|\phi| \leq |\tilde{\phi}(y)|; \quad y \in \Omega_\varepsilon$$

since ψ_j is uniformly bounded over ε, there exists $C > 0$ such that

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*].$$

□

Lemma 1.13. *We considered the equation*

$$L(\phi) = h \quad \textit{en } \Omega_\varepsilon \tag{1.56}$$

$$\phi = 0 \quad \textit{en } \partial\Omega_\varepsilon \tag{1.57}$$

$$\int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi = 0 \quad \textit{para } i = 0, 1, 2, \quad j = 1, 2..., K \tag{1.58}$$

then there exists positive numbers λ_0, C *such that, for all* $\xi \in \mathcal{O}$ *we have*

$$\|\phi\|_\infty \leq C\|h\|_*$$

for all $\lambda < \lambda_0$.

Proof. By contradiction:

We Suppose that there exits $\lambda_n \to 0$, $(\xi_1^n, \xi_2^n, ..., \xi_K^n) \in \mathcal{O}$, functions $\|h_n\|_* \to 0$, $\|\phi_n\|_\infty = 1$ and satisfies above equations. By Lemma 1.12, since $\|\phi_n\|_\infty = 1$ then $\|\phi_n\|_i > \kappa > 0$. Let $\hat{\phi}_n(z) = \phi_n((\xi_j^n)' + z)$, where the index j is such that $sup_{|y-(\xi_j^n)'|<R}|\phi_n| \geq \kappa$ we assume that this index j is the same for all n.

Local elliptic estimates imply that $\hat{\phi}_n$ converges uniformly over compacts to a bounded solutions $\hat{\phi} \neq 0$ de

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0 \quad \text{in } \mathbb{R}^2$$

the non degeneracy of the equation and the orthogonality condition give us the contradiction.

□

Lemma 1.14. *Let* $\delta > 0$ *be fixed and small. There exist positive numbers* λ_0 *and* C*, such that for any pioints* $\xi \in \mathcal{O}$*, and any solution* ϕ *to:*

$$\begin{cases} L(\phi) & = & h, \ \textit{in } \Omega_\varepsilon \\ \phi & = & 0 \ \textit{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi & = & 0 \ \textit{for } i = 1, 2, \quad j = 1, ...K \end{cases} \tag{1.59}$$

one has

$$\|\phi\|_\infty \leq C(-\log\varepsilon)\|h\|_*$$

For alla $\varepsilon < \varepsilon_0$.

Proof. Let $R > R_0 + 1$ be a large and fixed number, and $\hat{z}_0$ be the solution of the problem

$$\begin{cases} \Delta \hat{z}_{0j} + \frac{8\mu_j^2}{\left(\mu_j^2+|y-\xi_j'|^2\right)^2} \hat{z}_{0j} = 0, & \\ \hat{z}_{0j}(y) = z_{0j}(R) & \text{for } |y-\xi_j'| = R, \\ \hat{z}_{0j}(y) = 0 & \text{for } |y-\xi_j'| = \frac{\delta}{3\varepsilon}. \end{cases}$$

By computation, this function is explicitly given by

$$\hat{z}_{0j}(y) = z_{0j}(y) \left[1 - \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}} \right], \qquad r = |y - \xi_j'|.$$

Next we consider the radial smooth cut-off functions χ_1 and χ_2 withe the following properties:

$$0 \le \chi_1 \le 1, \;\; \chi_1 \equiv 1 \;\text{ in } B(0,R), \;\; \chi_1 \equiv 0 \;\text{ in }\; B(0,R+1)^c; \; and$$
$$0 \le \chi_2 \le 1, \;\; \chi_2 \equiv 1 \;\text{ in }\; B(0, \tfrac{\delta}{4\varepsilon}), \;\; \chi_1 \equiv 0 \;\text{ in }\; B\left(0, \tfrac{\delta}{3\varepsilon}\right)^c,$$

and $|\chi_2'(r)| \le C\varepsilon$, $|\chi_2''(r)| \le C\varepsilon^2$. Then we set

$$\chi_{1j}(y) = \chi_1(|y - \xi_j'|), \quad \chi_{2j}(y) = \chi_2(|y - \xi_j'|),$$

and define the test function

$$\tilde{z}_{0j} = \chi_{1j} Z_{0j} + (1 - \chi_{1j}) \chi_{2j} \hat{z}_{0j}.$$

Let ϕ be a solution to equation (1.59), we will modify ϕ so that the extra orthogonality conditions with respect to Z_{0j} is satisfies. We set

$$\tilde{\phi} = \phi + \sum_{j=1}^{K} d_j \tilde{z}_{0j}$$

with the number d_j is defined as

$$d_j = -\frac{\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} \eta_j |Z_{0j}|^2}.$$

Then

$$L(\tilde{\phi}) = h + \sum_{j=1}^{K} d_j L(\tilde{z}_{0j}), \tag{1.60}$$

and the orthogonality condition

$$\int_{\Omega_\varepsilon} \eta_j Z_{0i}\tilde{\phi} = 0, \quad \text{for all } i = 0, 1, 2,$$

hold. Then from the previous lemma we have the following estimate

$$\|\tilde{\phi}\|_\infty \le C[\|h\|_* + \sum_{j=1}^{K} |d_j| \|L(\tilde{z}_{0j})\|_*]. \tag{1.61}$$

Next, we show that

$$\|L(\tilde{z}_{0j})\|_* \le \frac{C}{\log \frac{1}{\varepsilon}}, \quad \text{and} \quad |d_j| \le C\left(\log \frac{1}{\varepsilon}\right)^2 \|h\|_*. \tag{1.62}$$

Indeed, we have

$$\begin{aligned} L(\tilde{z}_{0j}) &= 2\nabla\chi_{1j}\nabla(Z_{0j} - \hat{z}_{0j}) + \Delta\chi_{1j}(Z_{0j} - \hat{z}_{0j}) \\ &\quad +2\nabla\chi_{2j}\nabla\hat{z}_{0j} + \Delta\chi_{2j}\hat{z}_{0j} + O(\varepsilon^4). \end{aligned}$$

We consider the following four regions

$$\begin{aligned} \Omega_1 &= \{y \;:\; |y - \xi_j'| \le R\}, & \Omega_2 = \{y \;:\; R < |y - \xi_j'| < R + 1\}, \\ \Omega_3 &= \{y \;:\; R + 1 \le |y - \xi_j'| \le \frac{\delta}{4\varepsilon}\}, & \Omega_2 = \{y \;:\; \frac{\delta}{4\varepsilon} < |y - \xi_j'| < \frac{\delta}{3\varepsilon}\}. \end{aligned}$$

First, we note that $L(\tilde{z}_0) = O(\varepsilon^4)$ for $y \in \Omega_1 \cup \Omega_3$. For $y \in \Omega_2$, we have

$$\hat{z}_{0j} - Z_{0j} = -z_{0j}(r) \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}},$$

so that

$$|\hat{z}_{0j} - Z_{0j}| \leq \frac{C}{\log \frac{1}{\varepsilon}}.$$

Similarly, in this region, we have

$$|\hat{z}'_{0j} - Z'_{0j}| \leq \frac{C}{\log \frac{1}{\varepsilon}}.$$

On the other hand, for $y \in \Omega_4$, we have

$$\hat{z}_{0j}(r) \leq \frac{C}{\log \frac{1}{\varepsilon}}, \quad \text{and} \;\; \hat{z}'_{0j}(r) \leq \frac{C\varepsilon}{\log \frac{1}{\varepsilon}}.$$

Therefore, from the definition of the $*-$norm, we get

$$\|L(\tilde{z}_{0j})\|_* \leq \frac{C}{\log \frac{1}{\varepsilon}}, \tag{1.63}$$

where the number C depends in principle of the chosen large constant R.

Next we show the other inequality of (1.62) holds. Testing equation (1.60) against $\tilde{z}_{0l}$ we have

$$\langle \tilde{\phi}, L(\tilde{z}_{0l}) \rangle = \langle h, \tilde{z}_{0l} \rangle + d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle,$$

where $\langle f, g \rangle = \int_{\Omega_\varepsilon} fg$. This relation and (1.61) gives us that

$$d_l \langle L(\tilde{z}_{0l}), \tilde{z}_{0l} \rangle \leq C\|h\|_* [1 + \|L(\tilde{z}_{0l})\|_*] + C \sum_{j=1}^{K} |d_j| \|L(\tilde{z}_{0l})\|_*^2. \tag{1.64}$$

We want to measure the size of $\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle$. We decompose

$$\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle \;=\; \int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} + \int_{\Omega_4} L(\tilde z_{0l})\tilde z_{0l} + O(\varepsilon). \tag{1.65}$$

Since

$$\begin{aligned} \left|\int_{\Omega_4} L(\tilde z_{0l})\tilde z_{0l}\right| &\le C\int |\nabla\chi_{2l}||\nabla\hat z_{0l}||\hat z_{0l}| + C\int|\Delta\chi_{2l}||\hat z_{0l}|^2 + O(\varepsilon^2) \\ &\le \frac{C}{\left(\log\frac{1}{\varepsilon}\right)^2}. \end{aligned} \tag{1.66}$$

Moreover, for $y\in\Omega_2$, we have

$$\begin{aligned} \int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} &= 2\int\nabla\chi_{1l}\nabla(Z_{0l}-\hat z_{0l})\hat z_0 + \int\Delta\chi_{1l}(Z_{0l}-\hat z_{0l})\hat z_{0l} + O(\varepsilon) \\ &= \int\nabla\chi_{1l}\nabla(Z_{0l}-\hat z_{0l})\hat z_{0l} - \int\nabla\chi_{1l}(Z_{0l}-\hat z_{0l})\nabla\hat z_{0l} + O(\varepsilon), \end{aligned}$$

from the integration by parts. Now, we observe that in the considered region Ω_2, $|\hat z_{0l}-Z_{0l}|\le \frac{C}{\log\frac{1}{\varepsilon}}$, while $|\hat z_{0l}'|\sim\frac{1}{R^3}+\frac{1}{R}\frac{1}{\log\frac{1}{\varepsilon}}$. Then, for R is large but independent of ε we have

$$\left|\int\nabla\chi_{1l}(Z_{0l}-\hat z_{0l})\nabla\hat z_{0l}\right| \le \frac{C_1}{R^3}\frac{1}{\log\frac{1}{\varepsilon}},$$

with C_1 is a constant to be chosen independent R. Moreover

$$\begin{aligned} \int\nabla\chi_{1l}\nabla(Z_{0l}-\hat z_{0l})\hat z_{0l} &= 2\pi\int_R^{R+1}\chi_{1l}'(z_{0l}-\hat z_{0l})'\hat z_{0l} r\,dr \\ &= \frac{2\pi}{\int_R^{\frac{\delta}{3\varepsilon}}\frac{ds}{sz_{0l}^2}}\int_R^{R+1}\chi_{1l}'\left[1-\frac{4\mu_l^2r^2z_{0l}\int_R^r\frac{ds}{sz_{0l}^2}}{(\mu_l^2+r^2)^2}\right]dr \\ &= -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1+O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right], \end{aligned}$$

where C_2 is a positive constant independent on ε. Thus, choosing R large enough, we get

$$\int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} \sim -\frac{C_2}{\log\frac{1}{\varepsilon}}.$$

Combining this and (1.65), (1.66) we get

$$\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle \le -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1+O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right]. \tag{1.67}$$

From (1.63), (1.64) and (1.66) we have

$$|d_j| \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*.$$

We thus from estimate (1.61) that

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*.$$

□

We are ready to obtain the principal result of this section.

Proposition 1.15. *There exist positive numbers λ_0 and C, such that for $\xi \in \mathcal{O}$, there is unique solution $\phi = T_{\lambda_0}(h)$ to:*

$$\begin{aligned} L(\phi) &= h + \sum_{j=1}^{2}\sum_{j=1}^{K} c_{ij}\eta_j Z_{ij} && \text{in } \Omega_\varepsilon \\ \phi &= 0 && \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi &= 0 && \text{for all } i=1,2,\ j=1,..,K, \end{aligned} \tag{1.68}$$

for all $\lambda < \lambda_0$. Moreover

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_* \tag{1.69}$$

We just considered the orthogonality conditions whit respect to the elements of the approximate kernel due to translation.

Proof. Let us consider the cut-off function χ_{2j} introduced before. Testing equation (1.68) against $Z_{ij}\chi_{2j}$ to get

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle h, Z_{ij}\chi_{2j}\rangle + c_{ij}\int_{\Omega_\varepsilon} \eta_j |Z_{ij}|^2. \tag{1.70}$$

Moreover

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle \phi, L(Z_{ij}\chi_{2j})\rangle$$

We have

$$L(Z_{ij}\chi_{2j}) = \Delta\chi_{2j}Z_{ij} + 2\nabla Z_{ij}\nabla\chi_{2j} + \varepsilon O((1+r)^{-3}),$$

with $r = |y - \xi_j'|$. Since $\Delta\chi_{2j} = O(\varepsilon^2)$, $\nabla\chi_{2j} = O(\varepsilon)$, and $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we get

$$L(Z_{ij}\chi_{2j}) = O(\varepsilon^3)\varepsilon O((1+r)^{-3}).$$

Then we have

$$|\langle L(\phi), Z_{ij}\chi_{2j}\rangle| = |\langle \phi, L(Z_{ij}\chi_{2j})\rangle| \le C\varepsilon\|\phi\|_\infty.$$

From the previous Lemma we find

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\right] \tag{1.71}$$

Combining this with (1.70) and (1.71)

$$|c_{ij}| \le C\left[\|h\|_* + \varepsilon\log\frac{1}{\varepsilon}\sum_{l,m}|c_{lm}|\right]. \tag{1.72}$$

Then,

$$|c_{ij}| \le C\|h\|_*.$$

Combining this with (1.71)) we obtain the estimate

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*$$

.

Next prove the solvability assertion. We consider the Hilbert space

$$\mathbb{H} = \left\{\phi \in H_0^1(\Omega_\varepsilon) \;:\; \int_{\Omega_\varepsilon}\phi Z_{ij}\eta_j = 0 \quad \text{for } i = 1,2,\; j = 1,2,\cdots,K\right\},$$

endowed with the usual inner product $\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}\nabla\phi\nabla\psi$. Problem 1.68, expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}(W\phi - h)\psi\,dx, \qquad \text{for all } \psi \in \mathbb{H},$$

where $W = g'(V_\lambda)$. With the aid of Riesz's representation theorem, this equation gets rewritten in $\mathbb{H}$ in the operator form

$$(Id - R)\phi = \tilde{h}, \tag{1.73}$$

for certain $\tilde{h} \in \mathbb{H}$, where R is a compact operator in $\mathbb{H}$. The homogeneous equation $\phi = R\phi$ in $\mathbb{H}$, which is equivalent to (1.68) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (1.69). Now, Fredholm's alternative guarantees unique solvability of (1.73) for any $\tilde{h} \in \mathbb{H}$. This finishes the proof.

□

Lemma 1.16. *The operator T is differentiable whit respect to the variable $(\xi_1,...\xi_K) \in \mathcal{O}$. Moreover one has the estimate*

$$\|\partial_{(\xi'_m)_l}T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*$$

Proof. Let $\phi = T(h)$ where ϕ satisfies the equation

$$L(\phi) = h + \sum_{ij} c_{ij} Z_{ij} \eta_j$$

whit additional conditions, for some unique constants c_{ij}. Formally $Z = \partial_{(\xi'_m)_l}\phi$ should satisfies

$$L(Z) = -\partial_{(\xi'_m)_l}(g'(V_\lambda))\phi + \sum_{i=1}^{2} c_{im}\partial_{(\xi'_m)_l}(\eta_m Z_{im}) + \sum_{i=1}^{2}\sum_{j=1}^{K} d_{ij} Z_{ij}\eta_j \tag{1.74}$$

whit $d_{ij} = \partial_{(\xi'_m)_l} C_{ij}$ and the orthogonality conditions become

$$\int_{\Omega_\varepsilon} Z_{im}\eta_m Z = -\int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{im}\eta_m)\phi$$

We consider the projected function

$$\tilde{Z} = Z + \sum b_{im}\eta_m Z_{im}$$

such that

$$\int_{\Omega_\varepsilon} \eta_j Z_{ij}\tilde{Z} = 0$$

then

$$b_{im}\int_{\Omega_\varepsilon} \eta_m Z_{im}^2 = \int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{im}\eta_m)\phi$$

We write the equation (1.74) in the way that (1.68)

$$\begin{cases} L(\tilde{Z}) = f + \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{K} b_{im}\eta_m Z_{im} & \text{in } \Omega_\varepsilon \\ \tilde{Z} = 0 & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \eta_j Z_{im}\tilde{Z} = 0 & \text{for } i = 0,1,2 \end{cases} \tag{1.75}$$

and applied proposition (1.68) to get

$$\tilde{Z} = T_{\lambda_0}(f) \tag{1.76}$$

where f satisfies

$$\|f\|_* \leq C\|\phi\|_\infty$$

using (1.69) we find

$$\|\partial_{(\xi'_m)_l} T_\lambda(h)\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|f\|_* \leq C\left(\log\frac{1}{\varepsilon}\right)\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_*$$

arguing by definition it shows that indeed $\partial_{(\xi'_m)_l}\phi = Z$ □

1.4 The Finite Dimensional Reduction

Following the approach in [17] for $a(x) = 1$, we have the following result.

Lemma 1.17. *There exists $\lambda_0 > 0$ and a constant $C > 0$ such that for any $\lambda \in (0, \lambda_0)$ and each $\xi \in \mathcal{O}$, then there exists a unique ϕ satisfying*

$$\begin{cases} \Delta(V_\lambda + \phi) + g\left(V_\lambda + \phi\right) = \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij} Z_{ij}\eta_j & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 \quad \text{for } i = 1, 2,\ j = 1, \cdots, K. & \end{cases} \tag{1.77}$$

for some $c_{ij} \in \mathbb{R}$. Moreover,

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|^2},$$

Furthmore, the map $\xi' \mapsto \phi \in H_0^1(\Omega_\varepsilon)$ is $\mathcal{C}^1$, and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|}.$$

We included the proof just for completeness.

Proof. From Proposition 1.15 equation (1.77) is equivalent to find ϕ such that

$$\phi = T_\lambda(-(N(\phi) + E_\lambda)) := A(\phi) \tag{1.78}$$

where

$$\|A(\phi)\|_\infty \leq C \left(\log \frac{1}{\varepsilon}\right) [\|N(\phi)\|_* + \|E_\lambda\|_*] \,. \tag{1.79}$$

To $N(\phi)$ we have there exist $s \in (0,1)$ such that

$$|N(\phi)| \leq C|g''(V_\lambda + s\phi)||\phi|^2 \leq C|g''(V_\lambda + s\phi)|\|\phi\|_\infty^2$$

From the previous step, we know that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$ and from (1.50)

$$\|g''(V_\lambda)\|_* \leq C$$

then we get

$$\|N(\phi)\|_* \leq C\|\phi\|_*$$

we combine this with 1.79

$$\|A(\phi)\|_\infty \leq C|\log\varepsilon| \left(C\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|^3}\right).$$

For a given number $M > 0$, let us consider the region

$$\mathcal{F}_M := \left\{\phi \in C(\bar{\Omega}) \ : \ \|\phi\|_\infty \leq \frac{M}{|\log\varepsilon|^2}\right\}.$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ.

Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, using standard argument on mean value integral, one has

$$\|N(\phi_1) - N(\phi_2)\|_* \leq C\left(\max_{i=1,2} \|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

Thanks to (1.50) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\|N(\phi_1)-N(\phi_2)\|_* \leq C\|g''(V_\lambda)\|_*(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty \leq C(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty.$$

Then we have

$$\|A(\phi_1) - A(\phi_2)\|_\infty \leq C|\log\varepsilon|\|N(\phi_1) - N(\phi_2)\|_* \leq C|\log\varepsilon|\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi'_1, \cdots, \xi'_K) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{(\xi'_j)_i}\phi$ exists for $i = 1, 2$. Since $\phi = T_\lambda\left(-(N(\phi) + E_\lambda)\right)$, formally that

$$\partial_{(\xi'_j)_i}\phi = (\partial_{(\xi'_j)_i}T_\lambda)\left(-(N(\phi) + E_\lambda)\right) + T_\lambda\left(-(\partial_{(\xi'_j)_i}N(\phi) + \partial_{(\xi'_j)_i}E_\lambda)\right).$$

From Lemma 1.16, we have

$$\|\partial_{(\xi'_j)_i}T_\lambda\left(-(N(\phi) + E_\lambda)\right)\|_\infty \leq C|\log\varepsilon|^2\|N(\phi) + E_\lambda\|_* \leq C\tfrac{1}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}\partial_{(\xi'_j)_i}N(\phi) &= [g'(V_\lambda + \phi) - g'(V_\lambda) - g''(V_\lambda)\phi]\partial_{(\xi'_j)_i}V_\lambda + \partial_{(\xi'_j)_i}[g'(V_\lambda) - e^{w_j}]\phi \\ &\quad +[g'(V_\lambda + \phi) - g'(V_\lambda)]\partial_{(\xi'_j)_i}\phi + [g'(V_\lambda) - e^{w_j}]\partial_{(\xi'_j)_i}\phi.\end{aligned}$$

Then,

$$\|\partial_{(\xi'_j)_i} N(\phi)\|_* \le C\left\{\|\phi\|_\infty^2 + \tfrac{1}{|\log\varepsilon|}\|\phi\|_\infty + \|\partial_{(\xi'_j)_i}\phi\|_\infty\|\phi\|_\infty + \tfrac{1}{|\log\varepsilon|}\|\partial_{(\xi'_j)_i}\phi\|_\infty\right\}.$$

Since $\|\partial_{(\xi_j)_i} E_\lambda\|_* \le \frac{C}{|\log\varepsilon|^3}$, and by Proposition 1.15 we then have

$$\|\partial_{(\xi'_j)_i}\phi\|_\infty \le \frac{C}{|\log\varepsilon|},$$

for all $i = 1, 2$, $j = 1, \cdots, K$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (1.78). This concludes proof of the Lemma. □

After problem (1.77) has been solved, we find a solution to problem (1.38), if we can find a point $\xi' = \frac{\xi}{\varepsilon} = (\xi'_1, \ldots, \xi'_K)$ such that coefficients $c_{ij}(\xi')$ in (1.77) satisfy

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, 2, \;\; j = 1, \cdots, K.$$

We now introduce the finite dimensional restriction $\mathcal{J}_\lambda(\xi) : \mathcal{O} \to \mathbb{R}$, given by

$$\mathcal{J}_\lambda(\xi) = J^p_{a,\lambda}\left(\left(U_\lambda + \tilde{\phi}\right)(x, \xi)\right)$$

where

$$\left(U_\lambda + \tilde{\phi}\right)(x, \xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left(\left(V_\lambda + \phi\right)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$$

with V_λ defined in (1.35), ϕ is the unique solution to problem (1.77) given by Lemma 1.17.

The following result can be proved by using standard arguments, see Lemma 5.1 in [17].

Lemma 1.18. *For all $\lambda > 0$ sufficiently small, the functional $\mathcal{J}_\lambda(\xi)$ is of class C^1. Moreover, if $\xi \in \mathcal{O}$ is a critical point of $\mathcal{J}$, then $U_\lambda + \tilde{\phi}$ is a critical point of $J^p_{a,\lambda}$, namely a solution to the problem (1.1).*

Next we need to write the expansion of $\mathcal{J}_\lambda(\xi)$ as λ goes to 0.

Lemma 1.19. *Let $\delta > 0$ be fixed. There exist positive number λ_0, such that μ_j are given by (1.36), for any $0 < \lambda < \lambda_0$, the following expansion holds*

$$p\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \mathcal{J}_\lambda(\xi) = \frac{8K\pi}{(2-p)p}\left[-2+p\log 8\right] - \frac{16K\pi}{p}\log\varepsilon - \frac{32\pi^2}{2-p}\Phi^p_{a,K}(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi) \tag{1.80}$$

uniformly for any points $(\xi_1, \ldots, \xi_K) \in \mathcal{O}$, where

$$\Phi^p_{a,K}(\xi) = \sum_{j=1}^{K} H(\xi_j, \xi_j) + \sum_{i\neq j}^{K} G(\xi_i, \xi_j) + \frac{2-p}{4p\pi}\sum_{j=1}^{K}\log a(\xi_j).$$

Furthermore

$$p\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \nabla_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \frac{32\pi^2}{2-p}\nabla_{(\xi_m)_l}\Phi^p_{a,K}(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi) \tag{1.81}$$

In (1.80) and (1.81), the function θ_λ denotes a smooth function of the points ξ, which is uniformly bounded, as $\lambda \to 0$, for points $\xi \in \mathcal{O}$.

Proof. Define

$$I^p_{a,\lambda}(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2\, dy - \int_{\Omega_\varepsilon} a(\varepsilon y) e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}\, dy.$$

By direct calculation,

$$J^p_{a,\lambda}\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I^p_{a,\lambda}\left(\left(V_\lambda + \phi\right)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right), \tag{1.82}$$

Using the fact $\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left(\left(V_\lambda + \phi\right)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$, we then have

$$\mathcal{J}_\lambda(\xi) - J^p_{a,\lambda}\left(U_\lambda(\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}}\left[I^p_{a,\lambda}(V_\lambda + \phi) - I^p_{a,\lambda}(V_\lambda)\right]$$

Since by construction $DI^p_{a,\lambda}(V_\lambda+\phi)[\phi]=0$, we get

$$\mathcal{J}_\lambda(\xi)-J^p_{a,\lambda}\left(U_\lambda(\xi)\right)=\frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2I^p_{a,\lambda}(V_\lambda+t\phi)\phi^2(1-t)\,dt$$

$$=\frac{1}{p^2\gamma^{2(p-1)}}\int_0^1\left[\int_{\Omega_\varepsilon}(E_\lambda+N(\phi))\phi+\int_{\Omega_\varepsilon}[g'(V_\lambda)-g'(V_\lambda+t\phi)]\phi^2\right](1-t)\,dt$$

Since $\|E_\lambda\|_*\le\frac{c}{|\log\varepsilon|^3}$, $\|\phi\|_{L^\infty(\Omega_\varepsilon)}\le\frac{c}{|\log\varepsilon|^2}$, $\|N(\phi)\|_*\le\frac{c}{|\log\varepsilon|^4}$ and (1.50), we get that

$$\left|\mathcal{J}_\lambda(\xi)-J^p_{a,\lambda}\left(U_\lambda(\xi)\right)\right|\le\frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3}. \tag{1.83}$$

Next we expand $J^p_{a,\lambda}\left(U_\lambda(\xi)\right)$. We first have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\nabla\left(U_\lambda(\xi)\right)|^2\\
=\;&\frac{1}{2}\frac{1}{p^2\gamma^{2(p-1)}}\Bigg\{\sum_{j=1}^K\int_\Omega|\nabla PU_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\\
&\quad+\frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^K\int_\Omega\nabla PU_{\mu_j,\xi_j}(x)\nabla Pw^0_{\mu_j,\xi_j}(x)\\
&\quad+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^K\int_\Omega\nabla PU_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\
&\quad+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\sum_{j=1}^K\int_\Omega|\nabla Pw^0_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla Pw^0_{\mu_l,\xi_l}\nabla Pw^0_{\mu_j,\xi_j}\right]\\
&\quad+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^K\int_\Omega\nabla Pw^0_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\
&\quad+\left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\sum_{j=1}^K\int_\Omega|\nabla w^1_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla Pw^1_{\mu_l,\xi_l}\nabla Pw^1_{\mu_j,\xi_j}\right]\Bigg\}
\end{aligned}\tag{1.84}$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. We note that PU_{μ_j,ξ_j} satisfies

$$-\Delta PU_{\mu_j,\xi_j} = \varepsilon^2 a(\xi_j) e^{U_{\mu_j,\xi_j}}, \quad \text{in } \Omega, \quad PU_{\mu_j,\xi_j} = 0 \quad \text{on } \partial\Omega.$$

Then we have

$$\begin{aligned}
&\int_\Omega |\nabla PU_{\mu_j,\xi_j}(x)|^2\,dx = \varepsilon^2 \int_\Omega a(\xi_j) e^{U_{\mu_j,\xi_j}} PU_{\mu_j,\xi_j}(x) \\
= &\ \varepsilon^2 \int_\Omega a(\xi_j) e^{U_{\mu_j,\xi_j}} \left(U_{\mu_j,\xi_j}(x) + 8\pi H(x,\xi_j) - \log \frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2 \varepsilon^2) \right) \\
= &\int_\Omega \frac{8\varepsilon^2 \mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} \left(\log \frac{1}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} + 8\pi H(x,\xi_j) + O(\mu_j^2\varepsilon^2) \right) \\
= &\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \left(\log \frac{1}{(1+|z|^2)^2} + 8\pi H(\xi_j + \varepsilon\mu_j z, \xi_j) - 4\log(\varepsilon\mu_j) \right) + O(\mu_j^2\varepsilon^2) \\
= &\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \log \frac{1}{(1+|z|^2)^2} + 8\pi \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \left(H(\xi_j + \varepsilon\mu_j z, \xi_j) - H(\xi_j,\xi_j) \right) \\
&+8\pi \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} H(\xi_j,\xi_j) - 4\log(\varepsilon\mu_j) \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} + O(\mu_j^2\varepsilon^2).
\end{aligned} \tag{1.85}$$

But

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} = 8\pi + O(\varepsilon), \tag{1.86}$$

and

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \log \frac{1}{(1+|y|^2)^2} = -16\pi + O(\varepsilon). \tag{1.87}$$

Moreover,

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \left(H(\xi_j + \varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j) \right) = O(\varepsilon). \tag{1.88}$$

Therefore from (1.85)-(1.88) and (1.36), we have

$$\begin{aligned}
&\int_\Omega |\nabla PU_{\mu_j,\xi_j}(x)|^2\,dx \\
=\;& -16\pi + 64\pi^2 H(\xi_j,\xi_j) - 32\pi\log\varepsilon - 16\pi\log(8\mu_j^2) + 16\pi\log(8) + O\left(\frac{1}{\gamma^p}\right) \\
=\;& -16\pi + 64\pi^2 H(\xi_j,\xi_j) - 32\pi\log\varepsilon + 16\pi\log(8) - 16\pi\log a(\xi_j) + O\left(\frac{1}{\gamma^p}\right) \\
& -16\pi\left[\frac{2(p-1)}{2-p}(1-\log 8) + \frac{8\pi}{2-p}\left(H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j)\right)\right]. \qquad (1.89)
\end{aligned}$$

Now, we calculate that

$$\begin{aligned}
&\sum_{l\neq j}\int_\Omega \nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\,dx = \sum_{l\neq j}\int_\Omega \varepsilon^2 a(\xi_l)e^{U_{\mu_l,\xi_l}}PU_{\mu_j,\xi_j} \\
=\;& \sum_{l\neq j}\int_\Omega \frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\left(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2 a(\xi_j)} + 8\pi H(x,\xi_j) - \log\frac{8\mu_j^2}{a(\xi_j)} + O(\mu_j^2\varepsilon^2)\right) \\
=\;& \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_l z+\xi_l-\xi_j|^2)^2} + 8\pi H(\xi_l+\varepsilon\mu_l z,\xi_j)\right) + O(\mu_j^2\varepsilon^2) \\
=\;& \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2} 8\pi G(\xi_l,\xi_j) + O(\mu_j^2\varepsilon^2) \\
=\;& 64\pi^2\sum_{l\neq j} G(\xi_l,\xi_j) + O(\mu_j^2\varepsilon^2).
\end{aligned} \qquad (1.90)$$

Thus, from (1.84), (1.89) and (1.90) we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla U_\lambda(x)|^2\,dx \\
=\;& \frac{1}{p^2\gamma^{2(p-1)}}\Bigg\{-8K\pi - 16K\pi\log\varepsilon + 8K\pi\log(8) - 8K\pi\frac{2(p-1)}{2-p}(1-\log 8) \\
& -8\pi\sum_{j=1}^{K}\log a(\xi_j) - \frac{32\pi^2 p}{2-p}\left(\sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j)\right) + O\left(\frac{1}{|\log\varepsilon|}\right)\Bigg\}.
\end{aligned} \qquad (1.91)$$

Finally, let us evaluate the second term in the energy

$$
\begin{aligned}
\frac{\lambda}{p}\int_\Omega a(x)e^{(U_\lambda)^p}\,dx &= \frac{\lambda}{p}\int_\Omega a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\
&= \frac{\lambda}{p}\sum_{j=1}^K\int_{B(\xi_j,\tilde\delta)} a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\
&\quad +\frac{\lambda}{p}\int_{\Omega\setminus\bigcup_{j=1}^K B(\xi_j,\tilde\delta)} a(x)e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\
&:= I+II.
\end{aligned}
\tag{1.92}
$$

First we observe that

$$
II=\lambda\Theta_\lambda(\xi) \tag{1.93}
$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. On the other hand,

$$
\begin{aligned}
I &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^K\int_{B(\xi_j',\tilde\delta/\varepsilon)} a(\varepsilon y)e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}(V_\lambda)(y)\right)^p-1\right]}\,dy\\
&= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^K\int_{B(\xi_j',\tilde\delta/\varepsilon)} a(\varepsilon y)e^{\left\{\widetilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right\}}(1+O(\frac{1}{\gamma^p}))dy\\
&= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^K\int_{B(0,\frac{\tilde\delta}{\mu_j\varepsilon})}\frac{8}{(1+|y|^2)^2}\left(1+O(\frac{1}{\gamma^p})\right)\,dy\\
&= \frac{1}{p^2\gamma^{2(p-1)}}8K\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right),
\end{aligned}
\tag{1.94}
$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. From (1.92)-(1.94) we get

$$
\frac{\lambda}{p}\int_\Omega a(x)e^{(U_\lambda)^p}\,dx=\frac{1}{p^2\gamma^{2(p-1)}}8K\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \tag{1.95}
$$

Therefore, from (1.83), (1.91), (1.95) and (1.7) we get that (1.80) hold.

Let us now prove the validity of (1.81). Fix $m \in \{1,\ldots,K\}$ and $l \in \{1,2\}$. We have

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda\right](1+O(\frac{1}{\gamma^p})). \qquad (1.96)$$

On the one hand, if we multiply equation in (1.77) against $\partial_{(\xi'_m)_l}V_\lambda$, we get

$$\int_{\Omega_\varepsilon}(\Delta\upsilon_\xi + g(\upsilon_\xi))\partial_{(\xi'_m)_l}V_\lambda = \sum_{i=1}^{2}\sum_{j=1}^{K} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda$$

where $\upsilon_\xi = (V_\lambda+\phi)(y,\xi') = (V_\lambda+\phi)(\frac{x}{\varepsilon},\frac{\xi}{\varepsilon})$. On the other hand, we have that

$$\partial_{(\xi_m)_l}U_\lambda(x) = \frac{\varepsilon^{-1}}{p\gamma^{p-1}}\partial_{(\xi'_m)_l}V_\lambda(\frac{x}{\varepsilon}).$$

Putting together these information, we have that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega\left[\Delta(U_\lambda+\tilde{\phi}) + \lambda a(x)(U_\lambda+\tilde{\phi})^{p-1}e^{(U_\lambda+\tilde{\phi})^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)(1+o(1)).$$

Furthermore, since $\|\tilde{\phi}\|_{L^\infty(\Omega)} \le \frac{C}{\gamma^{p-1}}\|\phi\|_{L^\infty(\Omega_\varepsilon)}$, by definition of U_λ we have that

$$(U_\lambda+\tilde{\phi})(x) = U_\lambda(x)\left(1+O(\frac{1}{\gamma^p})\right) \quad \text{in} \quad \Omega.$$

Hence, by means of integrations by parts, and the boundary conditions satisfied by U_λ, we get that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega\left[\Delta U_\lambda + \lambda a(x)U_\lambda^{p-1}e^{U_\lambda^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)(1+O(\frac{1}{\gamma^p})),$$

where $O(1)$ here denotes a smooth function of the points ξ, which is uniformly bounded as

$\lambda \to 0$. We thus conclude that

$$\partial_{(\xi_m)_l}\mathcal{J}_\lambda(\xi) = \left(\int_\Omega \left[-\nabla U_\lambda \nabla \partial_{(\xi_m)_l} U_\lambda + \lambda a(x) U_\lambda^{p-1} e^{U_\lambda^p} \partial_{(\xi_m)_l} U_\lambda\right]\right)(1 + O(\frac{1}{\gamma^p})).$$

Computations analogous to the ones we performed to get expansion (1.80) give us the validity of (1.81). This concludes the proof of the Lemma. □

1.5 Proof of the main results

1.5.1 Proof of Theorem 1.2

Proof of Theorem 1.2. According to Lemma 1.18, we have a solution to (1.1) if we find a critical point ξ^λ of $\mathcal{J}_\lambda(\xi)$, it is equivalent to finding a critical point of the function $\widetilde{\mathcal{I}}(\xi) : \mathcal{O} \to \mathbb{R}$ defined by

$$\widetilde{\mathcal{I}}(\xi) = \frac{2-p}{32\pi^2}\left[-\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}\mathcal{J}_\lambda(\xi) + \frac{8K\pi}{2-p}\left[-2+\log 8\right] - \frac{16K\pi}{p}\log\varepsilon\right].$$

From Lemma 1.19, we have

$$\widetilde{\mathcal{I}}(\xi) = \Phi^p_{a,K}(\xi) + o(1), \tag{1.97}$$

where $o(1) \to 0$ uniformly for any points $(\xi_1, \dots, \xi_K) \in \mathcal{O}$, and $\Phi^p_{a,K}(\xi)$ defined by (1.6). By assumption that $\xi^* = (\xi_1^*, \dots, \xi_K^*)$ is a $\mathcal{C}^0$ stable critical point of $\Phi^p_{a,K}$, by Definition 1.1, there exists a critical point $\xi_\lambda^* \in \mathcal{O}$ of $\widetilde{\mathcal{I}}$ such that $\widetilde{\mathcal{I}}(\xi_\lambda^*) \to \widetilde{\mathcal{I}}(\xi^*)$. Moreover, up to a subsequence, $\xi_\lambda^* \to \xi^\lambda$ as $\lambda \to 0$, with $\Phi^p_{a,K}(\xi^\lambda) = \Phi^p_{a,K}(\xi^*)$.

Futhermore, expansion (1.8) follows from (1.7) and (1.95), while (1.9) holds as a direct consequence of the construction of U_λ. Expansion (1.10) is consequence of (1.80). □

1.5.2 Proof of Theorem 1.3

Proof of Theorem 1.3. According to Theorem 1.2, the proof of Theorem 1.4 reduces to show that function $\Phi^p_{a,K}$has a $\mathcal{C}^0-$critical point. For $a(x)=|x|^{2\alpha}$ and $\Omega=B$ is the unit ball in $\mathbb{R}^2$. Follow the approach in [21], we obtain that this holds.

Indeed, for $\rho\in(0,1)$, we set

$$\xi_{j,\rho}=\left(\rho\cos\frac{2\pi(j-1)\pi}{K},\rho\sin\frac{2\pi(j-1)\pi}{K}\right)\quad\text{for any } j=1,\dots,K.$$

Then by symmetric, we have

$$\Phi^p_{a,K}(\xi_\rho)=K\left[H(\xi_{1,\rho},\xi_{1,\rho})+\sum_{i=2}^{K}G(\xi_{1,\rho},\xi_{i,\rho})+\frac{(2-p)\alpha}{2p\pi}\log\rho\right].$$

Thus it is equivalent to find a $\mathcal{C}^0-$critical point of

$$F(\rho)=H(\xi_{1,\rho},\xi_{1,\rho})+\sum_{i=2}^{K}G(\xi_{1,\rho},\xi_{i,\rho})+\frac{(2-p)\alpha}{2p\pi}\log\rho.$$

In the unit ball of $\mathbb{R}^2$ we have

$$\begin{aligned}G(x,y)&=\frac{1}{2\pi}\log\frac{1}{|x-y|}-\frac{1}{2\pi}\log\frac{1}{\sqrt{|x|^2|y|^2+1-2(x,y)}},\\ H(x,x)&=-\frac{1}{2\pi}\log\frac{1}{1-|x|^2}.\end{aligned}$$

Hence

$$F(\rho)=\frac{1}{2}\log(1-\rho^2)+\frac{1}{2\pi}\left(\frac{2-p}{p}\alpha-(K-1)\right)\log\rho+\frac{1}{2\pi}\sum_{i=2}^{K}\log\frac{\sqrt{\rho^4+1-2\rho^2(\xi_1^*,\xi_i^*)}}{|\xi_1^*-\xi_i^*|}$$

Here

$$\xi_j^*=\left(\cos\frac{2\pi(j-1)\pi}{K},\sin\frac{2\pi(j-1)\pi}{K}\right)\quad\text{for any } j=1,\dots,K.$$

If $\frac{2-p}{p}\alpha - (K-1) > 0$, that is $K < \frac{2-p}{p}\alpha + 1$, we find that

$$\lim_{\rho\to 1^-} F(\rho) = \lim_{\rho\to 0^+} F(\rho) = -\infty.$$

Then there exists $\rho_0 \in (0,1)$ such that

$$F(\rho_0) = \max_{\rho\in(0,1)} F(\rho),$$

and ρ_0 is a $\mathcal{C}^0-$critical point of $F(\rho)$. This completes the proof of Theorem 1.4. □

1.5.3 Proof of Theorem 1.4

Proof of Theorem 1.4. According to the result of Theorem 1.2, the proof of Theorem 1.3 reduces to show that, for any $K \geq 1$ the function $\Phi^p_{a,K}$ has a non trivial critical values in some open set $\mathcal{M}$, compactly contained in $(\Omega\backslash Z)^K$. This fact has already been established in [10] under some minor modifications. For completeness, we recall here the principal ingredients employed to characterize a topological nontrivial critical value of $\Phi^p_{a,K}$ in some set $\mathcal{M}$, compactly contained in $(\Omega\backslash Z)^K$. We refer the reader to [10] for a complete proof of each step.

From the assumptions (A_1) and (A_2), without loss of generality we write

$$a(x) = \prod_{s=1}^{m} |x - q_s|^{2\alpha_s}.$$

Then we have

$$\begin{aligned}\Phi^p_{a,K}(\xi) &= \sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j) - \frac{(2-p)}{p}\sum_{j=1}^{K}\sum_{s=1}^{m}\frac{\alpha_s}{2\pi}\log\frac{1}{|\xi_j - q_s|} \\ &= \sum_{j=1}^{K} H(\xi_j,\xi_j) + \sum_{i\neq j}^{K} G(\xi_i,\xi_j) - \frac{(2-p)}{p}\sum_{j=1}^{K}\sum_{s=1}^{m}\alpha_s G(\xi_j,q_s) \\ &\quad \underbrace{+\frac{(2-p)}{p}\sum_{j=1}^{K}\sum_{s=1}^{m}\alpha_s H(\xi_j,q_s)}_{O(1)}. \end{aligned} \tag{1.98}$$

Define the set

$$\mathcal{M} := \left\{\xi = (\xi_1,\cdots,\xi_K) \in (\Omega\backslash Z)^K \ : \ \xi_i \neq \xi_j \text{ if } i \neq j\right\}.$$

Define the set

$$\begin{aligned}\mathcal{D} &= \Bigg\{\xi \in \mathcal{M} \,\Big|\, \Psi(\xi) := \sum_{j=1}^{K} H(\xi_j,\xi_j) - \frac{(2-p)}{p}\sum_{j=1}^{K}\sum_{s=1}^{m}\alpha_s G(\xi_j,q_s) \\ &\qquad\qquad - \sum_{i\neq j}^{K} G(\xi_i,\xi_j) + O(1) > -M\Bigg\}\end{aligned} \tag{1.99}$$

where $M > 0$ is a sufficiently large number to be chosen. We have that $\mathcal{D}$ is compactly contained in $\mathcal{M}$.

From A_2, we write set $\{1, 2, \dots, K\} = I_1 \cup I_2 \cup \dots \cup I_m$ where

$$
\begin{aligned}
I_1 &= \{1, \dots, K_1\}, \\
I_2 &= \{K_1 + 1, \dots, K_1 + K_2\}, \\
&\dots \\
I_s &= \{K_1 + \dots + K_{s-1} + 1, \dots, K_1 + \dots + K_{s-1} + K_s\}, \\
&\dots \\
I_m &= \{K_1 + \dots + K_{m-1} + 1, \dots, K\}.
\end{aligned}
$$

Let us fix angles θ_q $(q \in Z)$ and a number $\delta \in (0, \frac{\pi}{2})$ sufficiently small such that the cones

$$\left\{ q + \rho e^{i(\theta_q + \theta)} \ : \ \rho \geq 0, \ \theta \in [-\delta, \delta] \right\}, \quad q \in Z \tag{1.100}$$

are disjoint from one another. Moreover, we assume

$$dist(q, \partial\Omega) > 2\delta \ \ \forall \, q \in Z, \qquad |q_i - q_j| > 4\delta \ \ \forall \, q_i, q_j \in Z, i \neq j. \tag{1.101}$$

Now we define $K-$tuple

$$\xi_0 = (\xi_1^0, \dots, \xi_K^0)$$

by

$$\xi_j^0 = q_s + \frac{3}{2}\delta e^{i(\theta_{q_s} + j\frac{\delta}{K})} \quad \forall j \in I_s, \ \ s = 1, \dots, m.$$

Let us set an annulus with radii δ and 2δ centered in q_s, that is

$$U_s := \left\{ \xi \in \mathbb{R}^2 \ : \ \delta < |\xi - q_s| < 2\delta \right\},$$

and consider the $K-$tuple $\xi = (\xi_1, \dots, \xi_K)$ belongs to the open set

$$\left\{ \xi \in U_1^{K_1} \times \dots \times U_m^{K_m} \ : \ |\xi_i - \xi_j| > M^{-1} \ \ \forall \, i \neq j \right\}. \tag{1.102}$$

The choice of δ in (1.100) and (1.101) implies that $\xi_i^0 \neq \xi_j^0$ for $i \neq j$, then we have that ξ_0 belongs to (1.102) provided that M sufficiently large. Then we define

$$W := \text{ the connected component (1.102) } containg \ \xi_0$$

$$\mathcal{K} := \overline{W}, \qquad \mathcal{K}_0 = \left\{ \xi \in \mathcal{K} : \min_{i \neq j} |\xi_i - \xi_j| = M^{-1} \right\}.$$

From these facts, we get that

(P1) $\mathcal{D}$ is an open set, $\mathcal{K}$ and $\mathcal{K}_0$ are compact sets, $\mathcal{K}$ is connected and

$$\mathcal{K}_0 \subset \mathcal{K} \subset \mathcal{D} \subset \bar{\mathcal{D}} \subset \mathcal{M}.$$

Let us $\mathcal{F}$ to be the class of all continuous maps $\eta : \mathcal{K} \to \mathcal{D}$ with the property that there exists a continuous homotopy $\Gamma : [0,1] \times \mathcal{K} \to \mathcal{D}$ such that

$$\Gamma(0,\cdot) = id, \ \ \Gamma(1,\cdot) = \eta, \ \ \Gamma(t,\xi) = \xi \ \ \forall t \in [0,1], \ \forall \xi \in \mathcal{K}_0.$$

In [10], the following facts are proven:

(P2):

$$\Phi^* := \sup_{\eta \in \mathcal{F}} \min_{\xi \in \mathcal{K}} \Phi^p_{a,K}(\eta(\xi)) < \min_{\xi \in \mathcal{K}_0} \Phi^p_{a,K}(\xi)$$

(P3): for every $\xi \in \partial\mathcal{D}$ such that $\Phi^p_{a,K}(\xi) = \Phi^*$, $\partial\mathcal{D}$ is smooth at ξ and there exists a vector τ_ξ tangent to $\partial\mathcal{D}$ at ξ so that $\tau_\xi \cdot \nabla\Phi^p_{a,K}(\xi) \neq 0$.

Under (P1), (P2) and (P3), a critical point $\xi \in \mathcal{D}$ of $\Phi^p_{a,K}$ with $\Phi^p_{a,K}(\xi) = \Phi^*$ exists, as a standard deformation argument involving the gradient flow of $\Phi^p_{a,K}$ shows. This finishes the proof of Theorem 1.3. □

CHAPTER 2

Concentrating Solutions to Potential Supercritical Equation

Danilo Garrido

2.1 Introduction

In this paper we are concerned with the study of the following boundary value problem

$$\Delta u + a(x)u^p = 0 \text{ in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, p is a large exponent and $a(x)$ is a positive and smooth function in Ω.

If $a(x) \equiv 1$, problem (2.1) becomes

$$\Delta u + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

This problem has been study in [20], where the authors established the existence of a family of solutions which concentrate at different points of Ω, as p goes to $+\infty$; the existence of this solution is related with a topological condition on the domain Ω. More precisely, when Ω is not simply connected, given any integer $m \geq 1$ there exists $p_m > 0$ such that for all $p \geq p_m$ problem (2.2) has a solution u_p which concentrate at m different points in Ω. The location of such points is related to critical points of the function

$$\phi_m(\xi_1, ..., \xi_m) = \sum_{j=1}^{m} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i\neq j}^{m} G(\xi_i, \xi_j).$$

Here $G(x, y)$ is the Green function in Ω, solution of the equation

$$\begin{aligned} -\Delta_x G(x, y) &= \delta_y, \qquad && x \in \Omega \\ G(x, y) &= 0, && x \in \partial\Omega \end{aligned} \tag{2.3}$$

and $H(x, y)$ its regular part defined as $H(x, y) = G(x, y) - \dfrac{1}{2\pi} \log \dfrac{1}{|x - y|}$. If the domain Ω is not simply connected condition then a non trivial critical point for ϕ_m indeed does exist.

This construction was motivated by a previous result on the singularly perturbed Liouville equation

$$\Delta u + \varepsilon^2 e^u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We refer the reader to [1], [13] and [19]. In particular, in [13] it is proven the existence of solutions to the above equations with any number of concentration points, as $\varepsilon \to 0$ under the assumption that Ω is not simply connected. The main fact is the existence of a critical value of ϕ_m when Ω is not simply connected. This condition ensures the existence of an open set $\mathcal{D}$ in Ω^m with smooth boundary where ϕ_m links at a non-trivial critical level $\mathcal{C}$.

If $a(x) = |x|^{2\alpha}$, $\alpha > 0$, with $\alpha \notin \mathbb{N}$, Problem (2.1) was considered in [21]. The authors consider the case in which Ω is the unit ball and construct a positive solution u_p of Problem (2.1) concentrating outside the origin at k symmetric points as p goes to $+\infty$ and $k \leq K = \max\{n \in \mathbb{N}; n < \alpha + 1\}$. Notice that in this case, the topological condition over

Ω is not necessary. The construction of a first aproximation of the solution is based on the one-parameter family solution of the equation

$$\begin{cases} \Delta u + |u|^{2\alpha} e^u = 0 \quad \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |x|^{2\alpha} e^u < +\infty \end{cases} \tag{2.4}$$

for $\alpha \notin \mathbb{N}$, given by

$$U_\delta(x) = \log \frac{8(\alpha+1)\delta^2}{(\delta^2 + |x|^{2(\alpha+1)})^2}, \quad \delta > 0. \tag{2.5}$$

This approximation, up to a suitable normalization, looks like a sum of functions like (2.5) centered at several points $\xi_1, ..., \xi_k$. In [21] the authors consider also the case in which the continuous function $a(x) \geq 0$ has several zeros q_j of order $\alpha_j \notin \mathbb{N}$. In this case, there exists a solution to (2.1) centered at m zeros of the function a and at points $\xi_1, \ldots, \xi_k$ different from the zeros of $a(x)$. The existence of such solutions depends on the existence of a C^0-stable critical point of the function

$$\Phi(\xi) = \sum_{i=1}^{k} \left[H(\xi_i, \xi_i) + \sum_{j=1, j\neq i}^{k} G(\xi_i, \xi_j) + \frac{1}{4\pi} \log a(\xi_i) + 2 \sum_{j=1}^{m} (1+\alpha_j) G(\xi_i, q_j) \right].$$

This paper is devoted to construct solutions to problem (2.1) in the same context considered in [21], namely in the case the function $a(x)$ has several zeros. We obtain a type of solutions different from the ones found in [21].

Let us define the set $Z \subset \Omega$

$$Z := \{q \in \Omega : a(q) = 0\} \tag{2.6}$$

A$_1$: For any $q \in Z$ there exists a non negative numbers $\alpha_q \notin \mathbb{N}$ such that

$$a_q(x) = a(x)|x-q|^{-2\alpha_q}$$

is a strictly positive function in a neighborhood of q.

A_2: For any $q \in Z$, let M_q an integer with $0 \leq M_q < 1 + \alpha_q$ and define $M = \sum_{q \in Z} M_q$.

We will construct solutions to Problem (2.1) concentrating at $M = \sum_{q \in Z} M_q$ points in $\Omega \setminus Z$. The location of these M concentrating points is related to the critical points of the function

$$\varphi_M(\xi_1, ..., \xi_M) = \sum_{j=1}^{M} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i \neq j}^{M} G(\xi_j, \xi_i) + \frac{1}{4\pi} \sum_{j=1}^{M} \log(a(\xi_j)).$$

To construct the solution u_p we are going to consider an approximation for the solution which is different from the one consideres in [21], which produces a different type of solutions. This approximation looks like a sum of solutions of Liouville equation in $\mathbb{R}^2$ with an additionally term

$$U_{\delta,\xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2 a(\xi)}. \tag{2.7}$$

Indeed, for any $\mu > 0$ and any point $\xi \in \mathbb{R}^2$, these functions satisfy the limit equation

$$\Delta u + a(\xi) e^u = 0, \quad \text{in} \quad \mathbb{R}^2.$$

This paper generalizes the result in [21], with an algebraically different solution, and without the condition of the existence of a C^0-stable critical point for φ_M. Instead of that, we follow the ideas in [20] to prove a C^1 expansion of the functional of energy and we used the result in [10] to prove the existence of the nontrivial critical value of φ_M.

The result we obtain is the following:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ bounded and smooth, let $M \in \mathbb{N}$ and suppose that $a(x)$ satisfies A_1-A_2 condition. Then there exists p_M such that for any $p \geq p_M$, the problem (2.1) has a solution u_p which concentrates at M different point of Ω, i.e. as p goes to ∞*

$$p a(x) u_p^{p+1} \rightharpoonup 8\pi e \sum_{j=1}^{M} \delta_{\xi_i}.$$

Furthermore, for any $\varepsilon > 0$*,*

$$u_p \to 0 \quad \textit{uniformly in } \Omega \setminus \left(\cup_{j=1}^{M} B(\xi_j, \varepsilon)\right)$$

and

$$\sup_{x \in B(\xi_j, \varepsilon)} u_p(x) \to \sqrt{e}.$$

In this case, we prescribe the number M *of the point* ξ *of concentration.*

The paper is organized as follows; In section 2, we describe exactly the Ansatz for the solution we are looking for. We rewrite the problem in term of a linear operator L for which a solvability theory is performed in Section 3. In section 4 we solve an auxiliary nonlinear problem. We reduce (2.1) to solve a finite system $c_{ij}(\xi) = 0$, as we will see in section 5, and in section 6 we are going to prove the existence of ξ such that $c_{ij}(\xi) = 0$.

2.2 The first approximation of the solution

In this section we construct an approximate solution to Problem (2.1). To do so, we need to introduce several functions. Fix a point $\xi \in \Omega$ and a positive number δ. We start with the definition of the function

$$U_{\delta,\xi}(x) = \log\left(\frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2 a(\xi)}\right)$$

and consider the projection $PU_{\delta,\xi}(x)$ of $U_{\delta,\xi}(x)$ in $H_0^1(\Omega)$, namely the unique solution of

$$\begin{aligned} \Delta\left(PU_{\delta,\xi}(x) - U_{\delta,\xi}(x)\right) &= 0 \text{ in } \Omega \\ PU_{\delta,\xi}(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since

$$PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log\frac{8\delta^2}{a(\xi)} - 8\pi\left(G(x,\xi) - \frac{1}{2\pi}\log\frac{1}{|x-\xi|}\right) = O(\delta^2)$$

uniformly for $x \in \partial\Omega$ as $\delta \to 0$, a direct application of the Maximum Principle thus gives

$$PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log\frac{8\delta^2}{a(\xi)} - 8\pi H(x,\xi) = O(\delta^2) \text{ in } C^1(\bar{\Omega}), \tag{2.8}$$

and thus also

$$PU_{\delta,\xi}(x) - 8\pi G(x,\xi) = O(\delta^2) \text{ in } C^1_{\text{loc}}(\bar{\Omega} \setminus \{\xi\}. \tag{2.9}$$

In $\mathbb{R}^2$, we also define

$$v_\infty(y) = \log\frac{8}{(1+|y|^2)^2 a(\xi)},$$

and

$$\hat{v}(y) = v_\infty(y) + \frac{1}{p}\omega_0(y) + \frac{1}{p^2}\omega_1(y),$$

where ω_0 y ω_1 solve

$$\Delta\omega_i + \frac{8}{(1+|y|^2)^2}\omega_i = \frac{1}{(1+|y|^2)^2} f_i(y) \text{ in } \mathbb{R}^2. \tag{2.10}$$

The functions f_i are defined respectively as

$$f_0 = 4v_\infty^2,$$

and

$$f_1 = 8(\omega_0 v_\infty - \frac{1}{3}v_\infty^3 - \frac{1}{2}\omega_0^2 - \frac{1}{8}v_\infty^4 + \frac{1}{2}\omega_0 v_\infty^2).$$

Consequence of Lemma 2.1 in [20] is that there exists a radial solution $\omega_i(r)$ for (2.10) satisfying

$$w_i(y) = C_i \log|y| + O\left(\frac{1}{|y|}\right) \text{ as } |y| \to \infty. \tag{2.11}$$

where

$$C_i = \int_0^{+\infty} t\frac{t^2-1}{t^2+1} f_i(t)dt, \text{ provided } \int_0^{+\infty} t|f|(t)dt < \infty.$$

We compute C_i when $i = 0$. Observe that

$$\begin{aligned}
C_0 =& 4\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3} \log^2\left(\frac{8}{(1+t^2)^2 a(\xi)}\right) \\
=& 4\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3} \left[\log^2\left(\frac{8}{(1+t^2)^2}\right) - 2\log a(\xi)\log\left(\frac{1}{(1+t^2)^2}\right) + \log^2 a(\xi)\right] \\
=& I_1 - I_2 + I_3.
\end{aligned}$$

We have

$$I_1 = 12 - 4\log 8$$

and

$$\begin{aligned}
I_2 =& 8\log a(\xi)\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3} \log\frac{1}{(1+t^2)^2}\, dt \\
=& -16\log a(\xi)\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3} \log(1+t^2)\, dt \\
=& 16\log a(\xi)\frac{2t^2 + 2t^2\log(t^2+1) + 1}{4(t^2+1)^2}\bigg|_0^{\infty} = 4\log a(\xi).
\end{aligned}$$

On the other hand, a direct computation shows that

$$\begin{aligned}
I_3 =& 4\log^2 a(\xi)\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\, dt \\
=& -2\log^2 a(\xi)\frac{t^2}{(t^2+1)^2}\bigg|_0^{+\infty} = 0.
\end{aligned}$$

Thus we coclude that $C_0 = 12 - 4\log 8 - 4\log a(\xi)$.

Let us now consider the projection in $H_0^1(\Omega)$ of the functions $w_i(\frac{x-\xi}{\delta})$, namely the unique solutions $Pw_i(\frac{x-\xi}{\delta})$ of the problems $\Delta Pw_i(\frac{x-\xi}{\delta}) = \Delta w_i(\frac{x-\xi}{\delta})$ in Ω, with $Pw_i(\frac{x-\xi}{\delta}) = 0$ on

$\partial\Omega$. Since

$$w_i\left(\frac{x-\xi}{\delta}\right)-2\pi C_i\left(G(x,\xi)-\frac{1}{2\pi}\log\frac{1}{|x-\xi|}\right)+C_i\log\delta=O(\delta)$$

uniformly for $x\in\partial\Omega$, we have

$$P\left(\omega_i\left(\frac{x-\xi}{\delta}\right)\right)=\omega_i\left(\frac{x-\xi}{\delta}\right)-2\pi C_iH(x,\xi)+C_i\log(\delta)+O(\delta)\text{ in }C^1(\bar{\Omega}),\quad (2.12)$$

and also

$$P\left(\omega_i\left(\frac{x-\xi}{\delta}\right)\right)=-2\pi C_iG(x,\xi)+O(\delta)\text{ in }C^1_{\text{loc}}(\overline{\Omega}\setminus\{\xi\}).\quad (2.13)$$

We have now all the ingredients to define an approximate solution for Problem (2.1). Let $\varepsilon>0$ be fixed and consider a M-tuple $\hat{\xi}=(\xi_1,...\xi_M)\in\mathcal{O}_\varepsilon$ where

$$\mathcal{O}_\varepsilon=\{\hat{\xi}=(\xi_1,...,\xi_M)\in(\Omega\setminus Z)^M:\text{dist}(\xi_i,\partial(\Omega\setminus Z))\geq 2\varepsilon,\quad |\xi_i-\xi_j|\geq 2\varepsilon,\,i\neq j\}.$$

The set Z is defined by (2.6). For $\hat{\xi}\in\mathcal{O}_\varepsilon$, we define

$$U_{\hat{\xi}}(x)=\sum_{j=1}^{M}\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\left[PU_{\delta_j,\xi_j}(x)+\frac{1}{p}P\left(\omega_0\left(\frac{x-\xi_i}{\delta_i}\right)\right)+\frac{1}{p^2}P\left(\omega_1\left(\frac{x-\xi_i}{\delta_i}\right)\right)\right],$$

where

$$\gamma=p^{\frac{p}{p-1}}e^{-\frac{p}{2(p-1)}}\text{ and }\delta_j=\mu_je^{-\frac{p}{4}},\,\varepsilon\leq\mu_j\leq\varepsilon^{-1}.\quad (2.14)$$

Fix $i\in\{1,\dots,M\}$ and consider $|x-\xi_i|\leq\varepsilon$. If $j\neq i$, by the expansions (2.9) and (2.13), we get

$$
\begin{aligned}
&\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\left[PU_{\delta_j,\xi_j}(x)+\frac{1}{p}P\left(\omega_0\left(\frac{x-\xi_j}{\delta_j}\right)\right)+\frac{1}{p^2}P\left(\omega_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)\right]\\
&=\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\left[8\pi G(x,\xi_i)+\frac{-2\pi}{p}C_0G(x,\xi_j)+\frac{-2\pi}{p^2}C_1G(x,\xi_j)+O(\delta_j)\right]\\
&=\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}8\pi G(\xi_i,\xi_j)\left[1-\frac{C_0}{4p}-\frac{C_1}{4p^2}+O\left(e^{-\frac{p}{4}}+|x-\xi_i|\right)\right].
\end{aligned}
$$

and hance, for $|x-\xi_i|\le\varepsilon$, we have

$$
\begin{aligned}
U_{\hat{\xi}}(x)=&\frac{1}{\gamma\mu_i^{\frac{2}{p-1}}}\left[U_{\delta_i,\xi_i}(x)-\log\frac{8\delta_i^2}{a(\xi_i)}+8\pi H(\xi_i,\xi_i)+O(\delta_i^2+|x-\xi_i|)\right]\\
&+\frac{1}{p\gamma\mu_i^{\frac{2}{p-1}}}\left[w_0\left(\frac{x-\xi_i}{\delta_i}\right)-2\pi C_0H(\xi_i,\xi_i)+C_0\log\delta_i+O(\delta_i+|x-\xi_i|)\right]\\
&+\frac{1}{p^2\gamma\mu_i^{\frac{2}{p-1}}}\left[w_1\left(\frac{x-\xi_i}{\delta_i}\right)-2\pi C_1H(\xi_i,\xi_i)+C_1\log\delta_i+O(\delta_i+|x-\xi_i|)\right]\\
&+\sum_{j\neq i}\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}8\pi G(\xi_i,\xi_j)\left[1-\frac{C_0}{4p}-\frac{C_1}{4p^2}+O\left(e^{-\frac{p}{4}}+|x-\xi_i|\right)\right]\\
=&\frac{1}{\gamma\mu_i^{\frac{2}{p-1}}}\left[U_{\delta_i,\xi_i}(x)+\log(\delta_j^2e^p)+\frac{1}{p}w_0\left(\frac{x-\xi_i}{\delta_i}\right)+\frac{1}{p^2}w_1\left(\frac{x-\xi_i}{\delta_i}\right)-\log\frac{8\mu_j^4}{a(\xi_i)}\right.\\
&+8\pi H(\xi_i,\xi_i)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)+\frac{\log\delta_i}{p}\left(C_0+\frac{C_1}{p}\right)\\
&\left.+\sum_{j\neq i}\frac{\mu_i^{\frac{2}{p-1}}}{\mu_j^{\frac{2}{p-1}}}8\pi G(\xi_i,\xi_j)\left[1-\frac{C_0}{4p}-\frac{C_1}{4p^2}+O\left(e^{-\frac{p}{4}}+|x-\xi_i|\right)\right]\right]
\end{aligned}
$$

We now choose μ_i to solve the system

$$
\log\frac{8\mu_i^4}{a(\xi_i)}=8\pi H(\xi_i,\xi_i)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)+\frac{\log\delta_i}{p}\left(C_0+\frac{C_1}{p}\right)+8\pi\sum_{j\neq i}\frac{\mu_i^{\frac{2}{p-1}}}{\mu_j^{\frac{2}{p-1}}}G(\xi_j,\xi_i)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right).
$$

Equivalently, for p large, μ_i satisfies

$$\mu_i = e^{-\frac{3}{4}} e^{2\pi\left(H(\xi_i,\xi_i)+\sum_{j\neq i} G(\xi_j,\xi_i)+\frac{1}{4\pi}\log(a(\xi_i))\right)}(1+O(p^{-1})), \tag{2.15}$$

With this choice of the parameters μ_j, we see that, around each ξ_i, the function $U_{\hat{\xi}}(x)$ can be described as

$$U_{\hat{\xi}}(x) = \frac{1}{\gamma\mu_i^{\frac{2}{p-1}}}\left(p + v_\infty(y) + \frac{1}{p}\omega_0(y) + \frac{1}{p^2}\omega_1(y) + O\left(e^{-\frac{p}{4}}|y| + e^{-\frac{p}{4}}\right)\right) \tag{2.16}$$

for $x = \delta_i y + \xi_i$ and $|y| \leq \varepsilon\delta_i$. We finally observe that $U_{\hat{\xi}}$ is positive. Notice that there exists a positive constant C such that $|v_\infty(y_i) + \frac{1}{p}w_0(y_i) + \frac{1}{p^2}w_1(y_i)| \leq C$ for $|y_i| \leq \frac{\varepsilon}{\delta_i}$, then $U_{\hat{\xi}}$ is positive in $B(\xi_i, \varepsilon)$. Moreover, from Hopf lemma $\frac{\partial G}{\partial\eta}(\cdot, \xi_i) < 0$ in $\partial\Omega$ and

$$V_{\delta_i,\xi_i}(x) = PU_{\delta_j,\xi_j}(x) + \frac{1}{p}P\left(\omega_0\left(\frac{x-\xi_i}{\delta_i}\right)\right) + \frac{1}{p^2}P\left(\omega_1\left(\frac{x-\xi_i}{\delta_i}\right)\right) \to 8\pi G(\cdot, \xi_i)$$

in C^1-norm on $|x - \xi_i| \geq \varepsilon$. Hence $\frac{\partial V_{\delta_i,\xi_i}}{\partial\eta}(x) < 0$ in $\partial\Omega$, then V_{δ_i,ξ_i} is a positive function in Ω.

We are going to construct a solution u of the problem (2.1) in the form $U_{\hat{\xi}} + \phi$ where ϕ will represent a higher order term in the expansion of u. We define

$$W_{\hat{\xi}}(x) = pU_{\hat{\xi}}^{p-1}(x)$$

In terms of ϕ the problem (2.1) becomes

$$\begin{aligned} L(\phi) &= -(R + N(\phi)) \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where

$$L(\phi) = \Delta\phi + a(\cdot)W_{\hat{\xi}}\phi,$$

and

$$R_{\hat{\xi}} := \Delta U_{\hat{\xi}} + a(\cdot)U_{\hat{\xi}}^p,$$

$$N(\phi) := a(\cdot)\left[(U_{\hat{\xi}} + \phi)^p - (U_{\hat{\xi}})^p - pU_{\hat{\xi}}^{p-1}\phi\right].$$

The main objective of the next section will be to analyze the solvability of operator L in a weighted L^∞ space, to do that, we are going to use the invariance, under translation and dilations, of the problem $\Delta v + e^v = 0$ in $\mathbb{R}^2$ as in [20]. For $h \in L^\infty(\Omega)$ we define

$$\|h\|_* = \sup_{x\in\Omega}\left|\left(\sum_{j=1}^{M}\frac{\delta_j}{(\delta^2 + |x-\xi_j|^2)^{\frac{3}{2}}}\right)^{-1} h(x)\right|. \tag{2.17}$$

We conclude this section with an estimate of R in $\|\cdot\|_*$.

Lemma 2.2. *Let $\varepsilon > 0$. There exists a constant $C > 0$ and $p_0 > 0$ such that for any $\hat{\xi} \in \mathcal{O}_\varepsilon$ and $p > p_0$ we have that*

$$\|\Delta U_{\hat{\xi}} + a(x)U_{\hat{\xi}}^p\|_* \leq \frac{C}{p^4} \tag{2.18}$$

Proof. A direct computation gives

$$\begin{aligned}
\Delta U_{\hat{\xi}}(x) &= \sum_{j=1}^{M}\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\left(\Delta U_{\delta_j,\xi_j}(x) + \frac{1}{p\delta_j^2}\Delta\omega_0\left(\frac{x-\xi_j}{\delta_j}\right) + \frac{1}{p^2\delta_j^2}\Delta\omega_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)\\
&= \sum_{j=1}^{M}\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\left(-a(\xi_j)e^{U_{\delta_j,\xi_j}} + \frac{a(\xi_j)}{8p}\frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2 a(\xi_j)}f_0\left(\frac{x-\xi_j}{\delta_j}\right)\right.\\
&\quad + \frac{a(\xi_j)}{8p^2}\frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2 a(\xi_j)}f_1\left(\frac{x-\xi_j}{\delta_j}\right) - \frac{a(\xi_j)}{p}\frac{8}{(\delta_j^2+|x-\xi|^2)^2 a(\xi_j)}\omega_0\left(\frac{x-\xi_j}{\delta_j}\right)\\
&\quad \left. - \frac{a(\xi_j)}{p^2}\frac{8}{(\delta_j+|x-\xi_j|^2)^2 a(\xi_j)}\omega_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)
\end{aligned} \tag{2.19}$$

If $|x-\xi_i| \leq \varepsilon$ for some $i = 1, ..., M$, for $y_i = \dfrac{x-\xi_i}{\delta_i}$, we have that

$$
\begin{aligned}
\Delta U_{\hat{\xi}}(x) &= -\sum_{j=1}^{M} \frac{a(\xi_j)e^{U_{\delta_j,\xi_j}(x)}}{\gamma\mu_j^{\frac{2}{p-1}}}\left(1-\frac{1}{8p}f_0\left(y_i\right)-\frac{1}{8p^2}f_1(y_i)\ +\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)\right)\\
&= -\frac{a(\xi_i)e^{U_{\delta_i,\xi_i}(x)}}{\gamma\mu_i^{\frac{2}{p-1}}}\left(1-\frac{1}{8p}f_0\left(y_i\right)-\frac{1}{8p^2}f_1(y_i)\ +\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)\right)\\
&\quad -\sum_{j\neq i}^{M} \frac{a(\xi_j)e^{U_{\delta_j,\xi_j}}}{\gamma\mu_j^{\frac{2}{p-1}}}\left(1-\frac{1}{8p}f_0\left(y_i\right)-\frac{1}{8p^2}f_1(y_i)\ +\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)\right)\\
&= -\frac{a(\xi_i)e^{U_{\delta_i,\xi_i}(x)}}{\gamma\mu_i^{\frac{2}{p-1}}}\left(1-\frac{1}{8p}f_0\left(y_i\right)-\frac{1}{8p^2}f_1(y_i)\ +\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)\right)+O(p^{-1}e^{-\frac{p}{2}})
\end{aligned}
$$

then, using also (2.16),

$$
\begin{aligned}
\left|\Delta U_{\hat{\xi}}+a(x)U_{\hat{\xi}}^p\right| &= \left|-\frac{a(\xi_i)e^{U_{\delta_i,\xi_i}}}{\gamma\mu_i^{\frac{2}{p-1}}}\left(1-\frac{1}{8p}f_0\left(\frac{x-\xi_i}{\delta_i}\right)-\frac{1}{8p^2}f_1\left(\frac{x-\xi_i}{\delta_i}\right)\right.\right.\\
&\quad \left.\left.+\frac{1}{p}\omega_0\left(\frac{x-\xi_i}{\delta_i}\right)+\frac{1}{p^2}\omega_1\left(\frac{x-\xi_i}{\delta_i}\right)\right)+a(x)U_{\hat{\xi}}^p(x)+O(p^{-1}e^{-\frac{p}{2}})\right|
\end{aligned}
$$

We assume first that $|x-\xi_i|\leq\varepsilon\sqrt{\delta_i}$, where $y_i=\dfrac{x-\xi_i}{\delta_i}$. A Taylor expansion provides that

$$
\begin{aligned}
U_{\hat{\xi}}^p(x) &= \left(\frac{p}{\gamma\mu_i^{\frac{p}{p-1}}}\right)^p\left(1+\frac{1}{p}v_\infty(y_i)+\frac{1}{p^2}\omega_0(y_i)+\frac{1}{p^3}\omega_1(y_i)+O\left(\frac{e^{-\frac{p}{4}}}{p}|y_i|+\frac{e^{-\frac{p}{4}}}{p}\right)\right)^p\\
&= \frac{1}{\gamma\delta_i^2\mu_i^{\frac{2}{p-1}}}e^{v_\infty(y_i)}\left[1+\frac{1}{p}\left(\omega_0(y_i)-\frac{1}{2}v_\infty^2(y_i)\right)\right.\\
&\quad +\frac{1}{p^2}\left(\omega_1(y_i)-v_\infty\omega_0(y_i)+\frac{1}{3}v_\infty^3(y_i)+\frac{\omega_0^2(y_i)}{2}+\frac{1}{8}v_\infty^4-\frac{1}{2}\omega_0(y_i)v_\infty^2(y_i)\right)\\
&\quad \left.+O\left(\frac{\log^6(|y_i|+2)}{p^3}+p^2e^{-\frac{p}{4}}|y_i|+p^2e^{-\frac{p}{4}}\right)\right]
\end{aligned}
$$

$$= \frac{e^{U_{\delta_i,\xi_i}(y_i)}}{\gamma\mu_i^{\frac{2}{p-1}}}\left[1-\frac{1}{8p}4v_\infty^2(y_i)-\frac{1}{8p^2}8\left(v_\infty\omega_0-\frac{1}{3}v_\infty^3-\frac{1}{2}\omega_0^2-\frac{1}{8}v_\infty^4+\frac{1}{2}\omega_0v_\infty^2\right)(y_i)\right.$$
$$\left.+\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)+O\left(\frac{\log^6(|y_i|+2)}{p^3}+p^2e^{-\frac{p}{4}}|y_i|+p^2e^{-\frac{p}{4}}\right)\right]$$
$$= \frac{e^{U_{\delta_i,\xi_i}(y_i)}}{\gamma\mu_i^{\frac{2}{p-1}}}\left[1-\frac{1}{8p}f_0(y_i)-\frac{1}{8p^2}f_1(y_i)+\frac{1}{p}\omega_0(y_i)+\frac{1}{p^2}\omega_1(y_i)\right.$$
$$\left.+O\left(\frac{\log^6(|y_i|+2)}{p^3}+p^2e^{-\frac{p}{4}}|y_i|+p^2e^{-\frac{p}{4}}\right)\right]$$

Hence, if $|x-\xi_i|\le\varepsilon\sqrt{\delta_i}$ we get

$$|\Delta U_{\hat\xi}+a(x)U_{\hat\xi}^p|\le \frac{1}{\gamma\mu_i^{\frac{2}{p-1}}}|a(\xi_i)-a(x)|e^{U_{\delta_i,\xi_i}}\left[1-\tfrac{1}{8p}f_0(y_i)-\tfrac{1}{8p^2}f_1(y_i)+\tfrac{1}{p}\omega_0(y_i)+\tfrac{1}{p^2}\omega_1(y_i)\right]$$
$$+|a(x)|\frac{1}{\gamma\mu_i^{\frac{2}{p-1}}}e^{U_{\delta_i,\xi_i}}O\left(\frac{\log^6(|y_i|+2)}{p^3}+p^2e^{-\frac{p}{4}}|y_i|+p^2e^{-\frac{p}{4}}\right).$$

But, for $|y_i|\le\frac{\varepsilon}{\sqrt{\delta_i}}$ we have that $\left|\dfrac{1}{p}\omega_k(y_i)\right|\le C$ and $\left|\dfrac{1}{p^k}f_k(y_i)\right|\le C$; then

$$|\Delta U_{\hat\xi}+a(x)U_{\hat\xi}^p|\le\frac{C}{\gamma\delta_i^2}O\left[\frac{\sqrt{\delta_i}+\frac{1}{p^3}\log^6(|y|+2)}{(1+|y|^2)^2}\right].$$

Calculating $\|\cdot\|_*$ in the first region, we obtain

$$\left|\left(\sum_{j=1}^M\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_{\hat\xi}+a(x)U_{\hat\xi}^p\right)\right|\le\left|\frac{(\delta_i^2+|x-\xi_i|^2)^{\frac{3}{2}}}{\delta_i}\left(\Delta U_{\hat\xi}+a(x)U_{\hat\xi}^p\right)\right|$$
$$\le\frac{C}{\gamma}(1+|y_i|^2)^{\frac{3}{2}}O\left[\frac{\sqrt{\delta_i}+\frac{1}{p^3}\log^6(|y_i|+2)}{(1+|y_i|^2)^2}\right]\le\frac{C}{p^4}.$$

If $\varepsilon\sqrt{\delta_i} \le |x-\xi_i| \le \varepsilon$

$$U_{\hat{\xi}}^p(x) = O\left(\frac{\delta_i^2}{\gamma(\delta_i^2+|x-\xi_i|^2)^2}\right) \tag{2.20}$$

and

$$\Delta U_{\hat{\xi}}(x) = O\left(\frac{\delta_i^2}{\gamma(\delta_i^2+|x-\xi_i|^2)^2}\right) \tag{2.21}$$

Hence, in that region

$$\begin{aligned}\left|\left(\sum_{j=1}^{M}\frac{\delta_j}{\delta_j^2+|x-\xi_j|^2)^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_{\hat{\xi}}+a(x)U_{\hat{\xi}}^p\right)\right| &\le C\frac{1}{\gamma}\frac{\delta_i^2}{(\delta_i^2+|x-\xi_i|^2)^2}\frac{(\delta_i^2+|x-\xi_i|^2)^{\frac{3}{2}}}{\delta_i}\\ &\le C\frac{\delta_i}{\gamma(\delta_i^2+|x-\xi_i|^2)^{\frac{1}{2}}}\\ &\le C\frac{1}{\gamma(1+|y_i|^2)^{\frac{1}{2}}}\\ &\le C\frac{e^{-\frac{p}{8}}}{\gamma}\end{aligned}$$

Finally, if $|x-\xi_j| \ge \varepsilon$ for any $j = 1, ..., M$

$$\left|\left(\sum_{j=1}^{M}\frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{\frac{3}{2}}}\right)^{-1}\left(\Delta U_{\hat{\xi}}+a(x)U_{\hat{\xi}}^p\right)\right| \le Ce^{\frac{p}{4}}\left(p^{-1}e^{-\frac{p}{2}}+\left(\frac{C}{p}\right)^p\right).$$

This concludes the proof of our estimate. □

2.3 Linear Problem

Let us recall that $L(\phi) = \Delta\phi + a(x)W_{\hat{\xi}}(x)\phi$ where $W_{\hat{\xi}}(x) = pU_{\hat{\xi}}^{p-1}(x)$. For simplicity we are going to omit the dependence of $W_{\hat{\xi}}$ on $\hat{\xi}$ and $U_{\delta_j,\xi_j}(x) = U_j(x)$.

We are going to consider the operator $\tilde{L}$

$$\tilde{L}(\phi) = \Delta\phi + \left(\sum_{j=1}^{M}a(\xi_j)e^{U_j}\right)\phi,$$

to "approximate" the operator L because, after dilation and translation, the main important part of the operator, as $p \to \infty$ become,

$$\phi \to \Delta\phi + \frac{8}{(1+|y|^2)^2}\phi \text{ in } \mathbb{R}^2 \tag{2.22}$$

One of the important properties we want to use to develop the solvability of L is the non-degeneracy property of the equation (2.22) around the radial solution $\log \frac{8}{(1+|y|^2)^2}$.

We set

$$Z_{ij}(x) := z_i\left(\frac{x-\xi_j}{\delta_j}\right) = \begin{cases} \dfrac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2} & \text{if } i=0 \\ \dfrac{4\delta_j(x-\xi_j)_i}{\delta_j^2+|x-\xi_j|^2} & \text{if } i=1,2. \end{cases} \tag{2.23}$$

Let $h \in C(\bar{\Omega})$ we consider the linear problem to find a function $\phi \in W^{2,2}(\Omega)$ such that

$$\begin{aligned} L(\phi) &= h + \sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij}e^{U_j}Z_{ij}, \text{ in } \Omega \\ \phi &= 0 \quad \text{ on } \partial\Omega \\ \int_\Omega e^{U_j}Z_{ij}\phi &= 0 \text{ for all } i=1,2 \text{ for all } j=1,...,M \end{aligned} \tag{2.24}$$

for some coeficients $c_{ij},\ i=1,2,\ j=1,...,M$.

Lemma 2.3. *There exists $R>0$ large, $p_0>0$ such that for all $p>p_0$, the operator L satisfies the maximum principle in $\tilde{\Omega} = \Omega \setminus \cup_{j=1}^{M} B(\xi_j, R\delta_j)$,*

Proof. To proof Lemma 2.3 we need to find a function $Z>0$ in $\tilde{\Omega}$ such that $L(Z)<0$. Let $b>0$ we define Z like

$$Z(x) = \sum_{j=1}^{M} z_0\left(\frac{b(x-\xi_j)}{\delta_j}\right)$$

For $bR>2$ and p large enough

$$z_0\left(\frac{b(x-\xi_j)}{\delta_j}\right) = \frac{b^2|x-\xi_j|^2-\delta_j^2}{\delta_j^2+b^2|x-\xi_j|^2} > 0,$$

then $Z(x) > 0$ in $\tilde{\Omega}$.

As we can see in [20] there exists a positive constant D_0 such that

$$W(x) \leq D_0 \sum_{j=1}^{M} e^{U_j} \tag{2.25}$$

and if $|x - \xi_i| \leq \varepsilon\sqrt{\delta_i}$ a Taylor expansion give us

$$W(x) = \frac{8}{\delta_i^2(1+|y|^2)^2 a(\xi_i)} \left(1 + \frac{1}{p}(w_0 - v_\infty - \frac{1}{2}v_\infty^2) + O(\frac{\log^4(|y|+2)}{p^2})\right). \tag{2.26}$$

Then

$$a(x)W(x)Z(x) \leq D_0 \left(\sum_{j=1}^{M} e^{U_j(x)}\right) Z(x) \leq D_0 Z(x) \sum_{j=1}^{M} \frac{8\delta_j^2}{|x-\xi_j|^4}.$$

By the definition of z_0 we have

$$-\Delta Z(x) = \sum_{j=1}^{M} \frac{8b^2\delta_j^2(b^2|x-\xi_j|^2 - \delta_j^2)}{(b^2|x-\xi_j|^2 + \delta_j^2)^3} \geq \frac{4}{27} \sum_{j=1}^{M} \frac{8\delta_j^2}{b^2|x-\xi_j|^4}.$$

Hence

$$LZ(x) \leq \left(-\frac{4}{27b^2} + D_0\right) \sum_{j=1}^{M} \frac{8\delta_j^2}{|x-\xi_j|^4} < 0$$

for b small enough.

□

Let R from Lemma 2.3. We define the inner norm of ϕ

$$\|\phi\|_i = \sup_{\cup_{j=1}^{M} B(\xi_j, R\delta_j)} |\phi|(x) \tag{2.27}$$

Lemma 2.4. *There exists a constant $C > 0$ such that, if $L(\phi) = h$ in Ω with $h \in C^{0,\alpha}(\bar{\Omega})$ then*

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*] \tag{2.28}$$

Proof. Let $N = 2\text{diam}\Omega$ and ψ_j be a radial solution of the equation

$$-\Delta\psi_j = \frac{2\delta_j}{|x-\xi_j|^3} \quad \text{in } R\delta_j \leq |x-\xi_j| \leq N \tag{2.29}$$

$$\psi_j(x) = 0 \qquad \text{in } |x-\xi_j| = R\delta_j \tag{2.30}$$

$$\psi_j(x) = 0 \qquad \text{in } |x-\xi_j| = N \tag{2.31}$$

a direct computation give us

$$\psi_j(r) = -\frac{2\delta_j}{r} + C_0 \log r + C_1$$

where

$$C_0 = \frac{\frac{2}{R} - \frac{2\delta_j}{N}}{\log\left(\frac{R\delta_j}{N}\right)}$$

$$C_1 = \frac{2}{R} - C_0 \log(R\delta_j).$$

We notice that the function $\psi(r)$ attain the maximum value in $r = -\dfrac{2\delta_j}{C_1}$, hence the function ψ_j is uniformly upper-bounded, actually we have

$$\psi_j(r) \leq \psi\left(-\frac{2\delta_j}{C_0}\right) \leq \frac{2}{R}$$

Now, let us consider the function

$$\tilde{\phi}(x) = 2\|\phi\|_i Z(x) + \|h\|_* \sum_{j=1}^{M} \psi_j(x)$$

we notice that

$$\tilde{\phi}(x) \geq 2\|\phi\|_i Z(x) \geq \|\phi\|_i \geq \phi(x) \text{ in } |x| = R\delta_j,$$

$$\tilde{\phi}(x) \geq 0 = \phi(x) \text{ on } \partial\Omega. \tag{2.32}$$

Moreover, since

$$|h(x)| \leq \|h\|_* \left(\sum_{j=1}^{M} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{\frac{3}{2}}} \right), \tag{2.33}$$

we get

$$\begin{aligned} L\tilde{\phi} &\leq \|h\|_* \sum_{j=1}^{M} L\psi_j(x) = \|h\|_* \sum_{j=1}^{M} \left(-\frac{2\delta_j}{|x - \xi_j|^3} + a(x)W(x)\psi_j(x) \right) \\ &\leq \|h\|_* \sum_{j=1}^{M} \left(-\frac{2\delta_j}{|x - \xi_j|^3} + \frac{2mD_0}{R} e^{U_j} \right) \\ &\leq -\|h\|_* \left(\sum_{j=1}^{M} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{\frac{3}{2}}} \right) \leq -|h(x)| \leq |L(\phi)(x)| \end{aligned}$$

for R larger enough. By Lemma 2.3 we have that

$$|\phi(x)| \leq \tilde{\phi}(x) \text{ en } \tilde{\Omega}$$

and since $Z \leq 1$ and $\psi_j \leq \frac{2}{R}$

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*]$$

□

Lemma 2.5. *Consider the problem*

$$\begin{aligned} L(\phi) &= h \quad \textit{in } \Omega, \\ \phi &= 0 \quad \textit{on } \partial\Omega, \\ \int_\Omega e^{U_j} Z_{ij} \phi &= 0 \quad i = 0, 1, 2 \quad j = 1, ..., M, \end{aligned} \tag{2.34}$$

then there exist p_0 *and* $C > 0$ *such that for any* $\xi \in \mathcal{O}_\varepsilon$ *and any* $h \in C^{0,\alpha}(\bar{\Omega})$

$$\|\phi\|_\infty \leq C\|h\|_* \tag{2.35}$$

for all $p > p_0$.

Proof. We proceed by contradiction, suppose that there exists a sequence of $p_n \to \infty$, points $\xi^n \in \mathcal{O}_\varepsilon$ functions h_n and solutions associated ϕ_n such that $\|h_n\|_* \to 0$ and $\|\phi_n\| = 1$.

By Lemma 2.4 we have that $\liminf \|\phi_n\|_i > 0$. Let $\hat{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi_j^n)$. Cause (2.33) we have that $h_n(x) \to 0$ uniformly. Hence, by (2.25) and elliptic estimates ($h \in C^{0,\alpha}$), we get that $\hat{\phi}_j^n$ converge uniformly over compact to a bounded solutions $\hat{\phi}_j^\infty$ to the equation

$$\Delta\phi + \frac{8}{(1+|y|^2)^2}\phi = 0 \text{ in } \mathbb{R}^2.$$

An ortogonality argument give us $\hat{\phi}_j^\infty = 0$ which is a contradiction with $\liminf \|\phi_n\|_i > 0$ □

Now, we are going to prove the existence of a constant $C > 0$ such that for any solution of

$$L(\phi) = h \quad \text{in } \Omega$$

$$\phi = 0 \quad \text{in } \partial\Omega$$

$$\int_\Omega e^{U_j} Z_{ij}\phi = 0 \quad i =, 1, 2 \quad j = 1, ..., M$$

one has

$$\|\phi\|_\infty \leq Cp\|h\|_*$$

when $h \in C^{0,\alpha}(\bar{\Omega})$.

We proceed by contradiction again. Let us assume that

$$p_n\|h_n\|_* \to 0$$

since we have lost the orthogonality condition over to the solution z_0, we have that

$$\hat{\phi}_j^n \to C_j\frac{|y|^2-1}{|y|^2+1} \quad \text{in } C^0_{loc}(\mathbb{R}^2),$$

and we are going to show that $C_j = 0$ for all j.

We find the radial solution in $\mathbb{R}^2$ of

$$\Delta w + \frac{8}{(1+|y|^2)^2} w = \frac{8}{(1+|y|^2)^2} z_0(y),$$

and

$$\Delta t + \frac{8}{(1+|y|^2)^2} t = \frac{8}{(1+|y|^2)^2}.$$

By Lemma 2.1 in [20] we have that for $|y| \to \infty$

$$w(y) = \frac{4}{3}\log|y| + O(|y|^{-1}) \text{ y } t(y) = O(|y|^{-1}).$$

We set

$$u_j(x) = w\left(\frac{x-\xi_j}{\delta_j}\right) + \frac{4}{3}\log\delta_j Z_{0j}(x) + \frac{8\pi}{3}H(\xi_j,\xi_j)t\left(\frac{x-\xi_j}{\delta_j}\right) \tag{2.36}$$

and denote by Pu_j the projection of u_j in $H_0^1(\Omega)$. Using the fact that

$$u_j - Pu_j - \frac{8\pi}{3}\left(G(x,\xi_j) - \frac{1}{2\pi}\log\frac{1}{|x-\xi_j|}\right) = O(\delta_j)$$

in $\partial\Omega$, hamonicity and maximum principle give us

$$Pu_j = u_j - \frac{8\pi}{3}H(\cdot,\xi_j) + O(e^{-\frac{p}{4}}) \quad \text{in } C^1(\bar{\Omega}) \tag{2.37}$$

$$Pu_j = -\frac{8\pi}{3}G(\cdot,\xi_j) + O(e^{-\frac{p}{4}}) \quad \text{in } C^1_{loc}(\bar{\Omega}\setminus\{\xi_j\}) \tag{2.38}$$

A direct computation show us that Pu_j solves

$$\Delta Pu_j + a(x)W(x)Pu_j = a(\xi_j)e^{U_j}Z_{0j} + \left(a(x)W(x) - a(\xi_j)e^{U_j}\right)Pu_j + R_j$$

where

$$R_j = a(\xi_j)e^{U_j}\left(Pu_j - u_j + \frac{8\pi}{3}H(\xi_j,\xi_j)\right) = O(|x-\xi_j| + e^{-\frac{p}{4}})$$

$$\int_\Omega (\Delta P u_j + a(x)W(x)Pu_j)\,\phi = \int_\Omega a(\xi_j)e^{U_j}Z_{0j}\phi + \left(a(x)W(x) - a(\xi_j)e^{U_j}\right)Pu_j\phi + R_j\phi$$

integrating by parts

$$\int_\Omega Pu_j h - R_j\phi = a(\xi_j)\int_\Omega e^{U_j}Z_{0j}\phi + \int_\Omega \left(a(x)W(x) - a(\xi)e^{U_j}\right)Pu_j\phi \tag{2.39}$$

For the first term in the right hand side we have

$$a(\xi_j)\int_\Omega e^{U_j}Z_{0j}\phi \to C_j\int_{R^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4} = \frac{8\pi}{3}C_j.$$

and, for the second term, by (2.38) and

$$\int_{B(\xi_j,\varepsilon\sqrt{\delta_j})^c} e^{U_j}\,dx = O(\sqrt{\delta_j}), \quad \int_{B(\xi_j,\varepsilon\sqrt{\delta_j})^c} e^{U_j}\log(|x-\xi_j|) = O(\sqrt{\delta_j})$$

we get

$$\begin{aligned}\int_\Omega \left(a(x)W(x) - a(\xi_j)e^{U_j}\right)Pu_j\phi = & \int_{B(\xi_j,\varepsilon\sqrt{\delta_j})} \left(a(x)W(x) - a(\xi_j)e^{U_j}\right)Pu_j\phi \\ & -\frac{8\pi}{3}\sum_{k\neq j}G(\xi_k,\xi_j)\int_{B(\xi_k,\varepsilon\sqrt{\delta_j})} a(x)W(x)\phi + O(e^{-\frac{p}{8}}) \\ & = I_1 + I_2 + O(e^{-\frac{p}{8}})\end{aligned}$$

Now, by (2.26)

$$\begin{aligned} I_1 &= \sqrt{\delta_j}\int_{B(\xi_j,\varepsilon\sqrt{\delta_j})} W(x)Pu_j\phi + a(\xi_j)\int_{B(\xi_j,\varepsilon\sqrt{\delta_j})}\left(W(x) - e^{U_j}\right)Pu_j\phi \\ &= a(\xi_j)\int_{B(\xi_j,\varepsilon\sqrt{\delta_j})}\left(W(x) - e^{U_j}\right)Pu_j\phi + O(pe^{-\frac{p}{8}}) \\ &= \frac{4\log\delta_j}{3p}\int_{B\left(0,\frac{\varepsilon}{\sqrt{\delta_j}}\right)}\frac{8}{(1+|y|^2)^2}(w_0 - v_\infty - \frac{1}{2}v_\infty^2)z_0(y)\hat{\phi}_j \\ &= -\frac{C_j}{3}\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(|y|^2+1)^4}(w_0 - v_\infty - \frac{1}{2}v_\infty^2) + o(1),\end{aligned}$$

and

$$I_2 = \frac{8\pi}{3}\sum_{k\neq j} G(\xi_k,\xi_j)\int\limits_{B(\xi_k,\varepsilon\sqrt{\delta_j})} a(\xi_k)W(x)\phi + O(\frac{1}{p})$$
$$= -\frac{8\pi}{3}\sum_{k\neq j}\int_{B(0,\frac{\varepsilon}{\sqrt{\delta_j}})}\frac{8}{(1+|y|^2)^2}\hat{\phi}_k + O(p^{-1})$$

But

$$\int_{B(0,\frac{\varepsilon}{\sqrt{\delta_j}})}\frac{8}{(1+|y|^2)^2}\hat{\phi}_k \to C_k\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2}\frac{|y|^2-1}{|y|^2+1} = 0$$

In summary we have that there exists $K\neq 0$ such that

$$\int\limits_{\Omega}\left(a(x)W(x) - a(\xi_j)e^{U_j}\right)Pu_j\phi = KC_j + o(1)$$

The left hand side in (2.39) follows as

$$\left|\int_\Omega Pu_jh\right| = \|h\|_*\int_\Omega\sum_{k=1}^{M}\frac{\delta_k}{(\delta_k^2+|x-\xi_k|^2)^{\frac{3}{2}}}|Pu_j| = O(p\|h\|_*)$$

$$\int_\Omega R_j\phi = O\left(\int_\Omega e^{U_j}(|x-\xi| + e^{-\frac{p}{4}})\right) = O(e^{-\frac{p}{4}})$$

Hence, we have the equality

$$\hat{K}C_j = o(1)$$

then $C_j = 0$ for all j which is a contradiction.

Lemma 2.6. *There exists a constant $C>0$ such that for any solution ϕ of the problem* (2.24) *with $h\in C^{0,\alpha}(\bar{\Omega})$*

$$\|\phi\|_\infty \leq Cp\|h\|_* \tag{2.40}$$

Proof. Since

$$\|e^{U_j}Z_{ij}\|_* \leq 2\|e^{U_j}\|_* \leq 16.$$

and using the previous step

$$\|\phi\|_\infty \le Cp\left(\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{M}|c_{ij}|\right).$$

Hence, arguing by contradiction of (2.40) and proceeding as in Lemma 2.5 we suppose

$$p_n\|h_n\|_* \to 0, \quad p_n\sum_{i=1}^{2}\sum_{j=1}^{M}|c_{ij}^n| \ge \delta > 0$$

We omit the dependence on n. To estimate the coefficients c_{ij}, Multiply by PZ_{ij} the original equation and integrate by parts

$$\sum_{l=1}^{2}\sum_{h=1}^{M} c_{jh}(PZ_{lh}, PZ_{ij})_{H_0^1(\Omega)} + \int_\Omega hPZ_{ij} = \int_\Omega a(x)W(x)\phi PZ_{ij} - \int_\Omega e^{U_j}Z_{ij}\phi \quad (2.41)$$

Since $\Delta PZ_{ij} = \Delta Z_{ij} - e^{U_j}Z_{ij}$ we obtain

$$PZ_{ij} = Z_{ij} - 8\pi\delta_j\frac{\partial H}{\partial(\xi_j)_i}(\cdot,\xi_j) + O(\delta_j^3) \quad (2.42)$$

$$PZ_{0j} = Z_{0j} - 1 + O(\delta_j^2) \quad (2.43)$$

in $C^1(\bar\Omega)$ and

$$PZ_{ij} = -8\pi\delta_j\frac{\partial G(\cdot,\xi_j)}{\partial(\xi_j)_i} + O(\delta_j^3)$$

$$PZ_{0j} = O(\delta_j^2)$$

in $C^1_{loc}(\bar\Omega \setminus \{\xi_j\})$. From this we obtain the orthogonality relations, for $i, l = 1, 2$ and $j, h =$

$1, ..., M$ with $j \neq h$. See [20]

$$
\begin{aligned}
(PZ_{ij}, PZ_{lj})_{H_0^1(\Omega)} &= \left(64 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^4} \delta_{il}\right) + O(\delta_j^2) \\
(PZ_{ij}, PZ_{lh})_{H_0^1(\Omega)} &= O(\delta_j \delta_h) \\
(PZ_{0j}, PZ_{lj})_{H_0^1(\Omega)} &= O(\delta_j^2) \\
(PZ_{0j}, PZ_{lh})_{H_0^1(\Omega)} &= O(\delta_j \delta_h)
\end{aligned}
\tag{2.44}
$$

uniformly on $\hat{\xi} \in \mathcal{O}_\varepsilon$. Moreover

$$
\left| \int_\Omega hPZ_{ij} \right| \leq D \int_\Omega |h| \leq C\|h\|_*
$$

By (2.44), we can proof that

$$
\sum_{l=1}^{2} \sum_{h=1}^{M} c_{jh} (PZ_{lh}, PZ_{ij})_{H_0^1(\Omega)} + \int_\Omega hPZ_{ij} = Dc_{ij} + O(e^{-\frac{p}{2}} \sum_{l=1}^{2} \sum_{h=1}^{M} |c_{lh}|) + O(\|h\|_*)
$$

And for the right hand side in (2.41) by (2.42)-(2.43) we get

$$
\begin{aligned}
\int_\Omega a(x)W(x)\phi PZ_{ij} - \int_\Omega e^{U_j} Z_{ij}\phi &= a(\xi_j) \int_{B(\xi_j,\varepsilon)} W(x)\phi PZ_{ij} - a(\xi_j) \int_\Omega e^{U_j}\phi Z_{ij} + O(\sqrt{\delta_j}\|\phi\|_\infty) \\
&= a(\xi_j) \int_{B(\xi_j,\varepsilon\sqrt{\delta_j})} \left(W(x) - e^{U_j}\phi PZ_{ij}\right) + \int_\Omega e^{U_j}\phi(PZ_{ij} - Z_{ij}) \\
&\qquad + O(\sqrt{\delta_j}\|\phi\|_\infty) \\
&= \frac{1}{p} \int_{B(0,\frac{\varepsilon}{\sqrt{\delta_j}})} \frac{32y_i}{(1+|y|^2)^2} (w_0 - v_\infty - \frac{1}{2}v_\infty^2)\hat{\phi} + O(\frac{1}{p^2}\|\phi\|_\infty).
\end{aligned}
$$

We obtain

$$
Dc_{ij} + O(e^{-\frac{p}{2}} \sum_{l=1}^{2} \sum_{h=1}^{M} |c_{lh}|) = O(\|h\|_* + \frac{1}{p}\|\phi\|_\infty)
$$

hence

$$\sum_{l=1}^{2}\sum_{h=1}^{M}|c_{lh}| = O(\|h\|_* + \frac{1}{p}\|\phi\|_\infty) \tag{2.45}$$

To improve the estimate of RHS in (2.41) we use the convergence of

$$\int_{B(0,\frac{\varepsilon}{\sqrt{\delta_j}})} \frac{32y_i}{(1+|y|^2)^3}(w_0 - v_\infty - \frac{1}{2}v_\infty^2)(y)\hat{\phi}_j(y)$$

to

$$C_j\int_{\mathbb{R}^2}\frac{32y_i(|y|^2-1)}{(1+|y|^2)^4}(w_0 - v_\infty - \frac{1}{2}v_\infty^2) = 0$$

Therefore $RHS = o(p^{-1})$.

It follows that

$$\sum_{l=1}^{2}\sum_{h=1}^{M}|c_{lh}| = O(\|h\|_* + o(p^{-1}))$$

which contradicts our assumption

$$p_n\sum_{i=1}^{2}\sum_{j=1}^{M}|c_{ij}^n| \geq \delta > 0$$

To probe the solvability of (2.24) we consider the space

$$K_{\hat{\xi}} = \left\{\sum_{i=1}^{2}\sum_{j=1}^{M}c_{ij}PZ_{ij}; c_{ij}\in\mathbb{R},\ i=1,2,\ j=1,...,M\right\}$$

with

$$K_{\hat{\xi}}^* = \left\{\phi\in L^2(\Omega):\int_\Omega e^{U_j}z_{ij}\phi = 0 \text{ for } i=1,2\ j=1,...,M\right\}$$

Let $\Pi_{\hat{\xi}}: L^2 \to K_{\hat{\xi}}$ defined as

$$\Pi_{\hat{\xi}}\phi = \sum_{i=1}^{2}\sum_{j=1}^{M}c_{ij}PZ_{ij} \tag{2.46}$$

where c_{ij} is uniquely determined by the system (see (2.44))

$$\int_\Omega e^{U_h} Z_{lh}\left(\phi - \sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij} P Z_{ij}\right) = 0 \quad \text{for any } l = 1, 2,\ h = 1, ..., M$$

Let $\Pi^*_{\hat{\xi}} = Id - \Pi_{\hat{\xi}} : L^2(\Omega) \to K^*_{\hat{\xi}}$. The weak version of the original problem (2.24) is equivalent to find $\phi \in K^*_{\hat{\xi}} \cap H^1_0(\Omega)$ such that

$$(\phi, \psi)_{H^1_0(\Omega)} = \int_\Omega (aW\phi - h)\psi \ \text{ for all } \psi \in K^*_{\hat{\xi}} \cap H^1_0(\Omega).$$

By Riez representation theorem this equation can be written in $K^*_{\hat{\xi}} \cap H^1_0(\Omega)$

$$(Id - K)\phi = \tilde{h} \tag{2.47}$$

where $\tilde{h} = \Pi^*_{\hat{\xi}} \Delta^{-1} h$ and $K\phi = -\Pi^*_{\hat{\xi}} \Delta^{-1}(aW\phi)$ is a compact lineal operator in $K^*_{\hat{\xi}} \cap H^1_0(\Omega)$. By estimate (2.40) we have that the unique solution of homogeneous problem in (2.24) is the trivial solution. Then, Fredholm alternative theorem implies that for all $\tilde{h} \in K^*_{\hat{\xi}}$ there exists unique solution ϕ to (2.24). Standard elliptic regular theory give us that $\phi \in W^{2,2}(\Omega)$. The density of $C^{0,\alpha}(\bar{\Omega})$ in $C(\bar{\Omega})$ extends the result to $\tilde{h} \in C(\bar{\Omega})$. □

2.4 The Nonlinear Problem

We want to solve

$$\Delta(U_{\hat{\xi}} + \phi) + a(x)\left(U_{\hat{\xi}} + \phi\right)^p = \sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij} e^{U_j} Z_{ij} \quad \text{in } \Omega \tag{2.48}$$

$$U_{\hat{\xi}} + \phi > 0 \qquad \text{in } \Omega, \tag{2.49}$$

$$\phi = 0 \qquad \text{on } \partial\Omega, \tag{2.50}$$

$$\int_\Omega e^{U_j} Z_{ij}\phi = 0 \text{ for all } i = 1, 2 \quad j = 1, ..., M, \tag{2.51}$$

for some coefficients c_{ij} which depend on ξ.

Lemma 2.7. *Let $\varepsilon > 0$ be fixed. There exists $C > 0$ and $p_0 > 0$ such that for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$, the problem (2.48)-(2.51) has a solution $\phi_{\hat{\xi}}$ which satisfies*

$$\|\phi_{\hat{\xi}}\|_\infty \leq \frac{C}{p^3};$$

moreover

$$\sum_{i=1}^{2}\sum_{j=1}^{M} |c_{ij}(\xi)| \leq \frac{C}{p^4}, \quad \|\phi_{\hat{\xi}}\|_{H_0^1(\Omega)} \leq \frac{C}{p^3}. \tag{2.52}$$

Proof. Let us denote by C_* the space function $\left(C(\bar{\Omega}), \|\cdot\|_*\right)$. The existence of solutions of the linear problem (2.24) and Lemma 2.6 implies the existence of the operator between the Banach space C_* and $C_0(\Omega)$ with norm bounded by Cp. Then, the nonlinear problem is expressed as

$$\phi = A(\phi) := -T(R + N(\phi))$$

Let $\kappa > 0$, we consider

$$\mathcal{F} := \left\{\phi \in C_0(\Omega) : \|\phi\|_\infty \leq \frac{\kappa}{p^3}\right\} \tag{2.53}$$

Having account the following estimates

$$\|N(\phi)\|_* \leq Cp\|\phi\|_\infty^2 \tag{2.54}$$

$$\|N(\phi_1) - N(\phi_2)\|_\infty \leq Cp(\max_{i=1,2} \|\phi_i\|_\infty)\|\phi_1 - \phi_2\|_\infty \tag{2.55}$$

for any ϕ, $\phi_1, \phi_2 \in \mathcal{F}$, one has

$$\|A(\phi)\|_\infty \leq D'p\left(\|N(\phi)\|_* + \|R\|_*\right) \leq O\left(p^2\|\phi\|_\infty\right) + \frac{D}{p^3}$$

and

$$\|A(\phi_1) - A(\phi_2)\|_\infty \leq C'p\|N(\phi_1) - N(\phi_2)\|_* \leq Cp^2\left(\max\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

Then, if $\|\phi\|_\infty \le \dfrac{2D}{p^3}$ we have that

$$\|A(\phi)\|_\infty \le \frac{2D}{p^3}$$

Taking $\kappa = 2D$ we have that A is a contraction in $\mathcal{F}_\kappa$

$$\|A(\phi_1) - A(\phi_2)\|_\infty \le \frac{1}{2}\|\phi_1 - \phi_2\|_\infty$$

Hence, there exists a unique fixed point $\phi_{\hat\xi}$ to A in $\mathcal{F}_\kappa$ Moreover, using (2.45) we have

$$\sum_{i=1}^{2}\sum_{j=1}^{M}|c_{ij}(\hat\xi)| = O\left(\|N(\phi_{\hat\xi})\|_* + \|R\|_* + \frac{1}{p}\|\phi_{\hat\xi}\|_\infty\right) \le \frac{C}{p^4}$$

Multiplying (2.24) by the solution ϕ and applying (2.40) we get, for any $p > p_0$ fixed

$$\|\phi\|_{H_0^1(\Omega)} \le C\left(\|\phi\|_\infty + \|h\|_*\right) \tag{2.56}$$

then

$$\|\phi_{\hat\xi}\|_{H_0^1(\Omega)} = O(\|\phi_{\hat\xi}\|_\infty + \|N(\phi)\|_* + \|R\|_*) \le \frac{C}{p^3} \tag{2.57}$$

□

Let $\hat\xi_1, \hat\xi_2 \in \mathcal{O}_\varepsilon$, since

$$\begin{aligned}\Delta(\phi_{\hat\xi_1} - \phi_{\hat\xi_2}) + pa(\cdot)U_{\hat\xi}^{p-1}(\phi_{\hat\xi_1} - \phi_{\hat\xi_2}) &= a(\cdot)\left((U_{\hat\xi_2} + \phi_{\hat\xi_2})^p - (U_{\hat\xi_1} + \phi_{\hat\xi_2})^p\right)\\ +a(\cdot)\left((U_{\hat\xi_1} + \phi_{\hat\xi_2})^p - (U_{\hat\xi_1} + \phi_{\hat\xi_1})^p - pU_{\hat\xi_1}^{p-1}(\phi_{\hat\xi_2} - \phi_{\hat\xi_1})\right) &+ \Delta(U_{\hat\xi_2} - U_{\hat\xi_1})\\ + \sum_{i=1}^{2}\sum_{j=1}^{M}&(c_{ij}(\hat\xi_1) - c_{ij}(\hat\xi_2))e^{U_j}(\hat\xi_1)Z_{ij}(\hat\xi_1)\ ,\\ + \sum_{i=1}^{2}\sum_{j=1}^{M}c_{ij}(\hat\xi_2)&\left(e^{U_j}(\hat\xi_1)Z_{ij}(\hat\xi_1) - e^{U_j}(\hat\xi_2)Z_{ij}(\hat\xi_2)\right)\end{aligned}$$

and

$$\begin{aligned}
&\|a(\cdot)\left((U_{\hat{\xi}_1}+\phi_{\hat{\xi}_2})^p-(U_{\hat{\xi}_1}+\phi_{\hat{\xi}_1})^p-pU_{\hat{\xi}_1}^{p-1}(\phi_{\hat{\xi}_2}-\phi_{\hat{\xi}_1})\right)\|_*\\
&\le \frac{C}{p^2}\|\phi_{\hat{\xi}_1}-\phi_{\hat{\xi}_2}\|_\infty \|p(U_{\hat{\xi}_1}+O(p^{-3}))^{p-2}\|_*\\
&\le \frac{C}{p^2}\|\phi_{\hat{\xi}_1}-\phi_{\hat{\xi}_2}\|_\infty \left\|\sum_{j=1}^{M} e^{U_j(x)}\right\|_* \qquad ,\\
&= o\left(\tfrac{1}{p}\left\|\phi_{\hat{\xi}_1}-\phi_{\hat{\xi}_2}\right\|_\infty\right).
\end{aligned}$$

(2.40) and (2.45) implies, for any $p > p_0$

$$\begin{aligned}
&\|\phi_{\hat{\xi}_1}-\phi_{\hat{\xi}_2}\|_\infty \le Cp\|(U_{\hat{\xi}_2}+\phi_{\hat{\xi}_2})^p-(U_{\hat{\xi}_1}+\phi_{\hat{\xi}_2})^p\|_*\\
&+\frac{C}{p^3}\sum_{i=1}^{2}\sum_{j=1}^{M}\|e^{U_j}(\hat{\xi}_1)Z_{ij}(\hat{\xi}_1)-e^{U_j}(\hat{\xi}_2)Z_{ij}(\hat{\xi}_2)\|_* + Cp\|\Delta(U_{\hat{\xi}_1}-U_{\hat{\xi}_2})\|_*
\end{aligned}$$

Then, for $p > p_0$ fixed, the function $\hat{\xi} \to \phi_{\hat{\xi}}$ is continuous in $C_0(\bar{\Omega})$ and, in view of (2.56), in $H_0^1(\Omega)$. By the implicit function theorem applied to the equation

$$G(\hat{\xi},\phi) := \Pi^*_{\hat{\xi}}\left[U_{\hat{\xi}}+\Pi^*_{\hat{\xi}}\phi+\Delta^{-1}\left(a(\cdot)(U_{\hat{\xi}}+\phi)^p\right)\right]+\Pi_{\hat{\xi}}\phi=0 \quad \phi\in C_0(\bar{\Omega}) \tag{2.58}$$

we have that $\hat{\xi} \to \phi_{\hat{\xi}}$ is a function C^1 in $C_0(\bar{\Omega})$. In fact, for p large enough, the operator

$$\frac{\partial G}{\partial \phi}(\hat{\xi},\phi_{\hat{\xi}}) = \Pi^*_{\hat{\xi}}\left[\mathrm{Id}+p\Delta^{-1}\left(a(\cdot)(U_{\hat{\xi}}+\phi_{\hat{\xi}})^{p-1}\mathrm{Id}\right)\right] \tag{2.59}$$

is invertible. To see this, considerate the equation

$$\frac{\partial G}{\partial \phi}(\hat{\xi},\phi_{\hat{\xi}})[\phi]=0 \tag{2.60}$$

o equivalent

$$L\phi = pa(\cdot)(U_{\hat{\xi}}^{p-1}-(U_{\hat{\xi}}+\phi_{\hat{\xi}})^{p-1})\phi+\sum_{i=1}^{2}\sum_{j=1}^{M}c_{ij}e^{U_j}Z_{ij} \tag{2.61}$$

and applying (2.40)

$$\|\phi\|_\infty \leq C'p\| \left(U_{\hat{\xi}}^{p-1} - (U_{\hat{\xi}} + \phi_{\hat{\xi}})^{p-1}\right) \phi\|_*$$
$$\leq C'p^2\|\phi\|_\infty\|\phi_{\hat{\xi}}\|_\infty\|p(U_{\hat{\xi}} + O(p^{-3}))^{p-2}\|_* \leq \|\phi\|_\infty$$

then $\phi = 0$. Fredholm alternative theorem give us the invertibility and with this the regularity.

2.5 Variational Reduction

After solving the nonlinear projected problem, we have to find points of concentration with the properties

$$c_{ij}(\hat{\xi}) = 0 \quad \text{for all } \ i = 1, 2\, j = 1, ..., M \tag{2.62}$$

Associated to the original problem we have a functional of energy

$$J_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega a(\cdot)|u|^{p+1}dx,\ u \in H_0^1(\Omega) \tag{2.63}$$

and the finite dimensional restriction

$$F(\hat{\xi}) = J_p(U_{\hat{\xi}} + \phi_{\hat{\xi}}) \tag{2.64}$$

where ϕ_ξ is the unique solution of equations (2.48)-(2.51)

Lemma 2.8. *The functional $F(\hat{\xi})$ is C^1. Moreover, for p larger enough, if $D_{\hat{\xi}}F(\hat{\xi}) = 0$ then $\hat{\xi}$ satisfies* (2.62)

Proof. Equivalent to $D_{\hat{\xi}}F(\hat{\xi}) = 0$ we have

$$
\begin{aligned}
0 &= -\int_\Omega \left(\Delta(U_{\hat\xi}+\phi_{\hat\xi}) + (U_{\hat\xi}+\phi_{\hat\xi})^p\right)(D_{\hat\xi}U_{\hat\xi} + \mathcal{D}_{\hat\xi}\phi_{\hat\xi}) \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij}(\hat\xi)\int_\Omega e^{U_j} Z_{ij}(D_{\hat\xi}U_{\hat\xi} + \mathcal{D}_{\hat\xi}\phi_{\hat\xi}) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij}(\hat\xi)\int_\Omega e^{U_j} Z_{ij} D_{\hat\xi}U_{\hat\xi} + \sum_{i=1}^{2}\sum_{j=1}^{M} c_{ij}(\hat\xi)\int_\Omega D_{\hat\xi}\left(e^{U_j}Z_{ij}\right)\phi_{\hat\xi},
\end{aligned}
$$

due to the orthogonality condition $\int_\Omega e^{U_j} Z_{ij}\phi_{\hat\xi} = 0.$

We recall that the projection

$$
P : L^\infty(\Omega) \to L^\infty(\Omega)
$$

is a continuous operator, and by Lemma 2.1 in [20]

$$
\frac{1}{p-1}U_{\delta_s,\xi_s},\ Z_{0,s},\ \frac{1}{p}w_j\left(\frac{x-\xi_s}{\delta_s}\right), \nabla w_j\left(\frac{x-\xi_s}{\delta_s}\right)\cdot\left(\frac{x-\xi_s}{\delta_s}\right)
$$

are a bounded function in Ω, then we have

$$
\begin{aligned}
\partial U_{\hat\xi}(x) = &-\sum_{s=1}^{m}\frac{1}{\gamma\mu_s^{2/(p-1)}}P\left[\frac{2}{p-1}U_{\delta_s,\xi_s}(x) - 2Z_{0,s}(x) + \left(\frac{2}{p(p-1)}\omega_0\left(\frac{x-\xi_s}{\delta_s}\right)\right.\right. \\
&+\left.\left. \frac{2}{p^2(p-1)}\omega_1\left(\frac{x-\xi_s}{\delta_s}\right) + \frac{1}{p}\nabla\omega_0\cdot\frac{x-\xi_s}{\delta_s} + \frac{1}{y}\nabla\omega_1\cdot\frac{x-\xi_s}{\delta_s}\right)\right]\partial\log(\mu_s) \\
&+\frac{1}{\gamma\delta_j\mu_j^{2/(p-1)}}P\left(Z_{ij} - \frac{1}{p}\partial_i\omega_0\left(\frac{x-\xi_s}{\delta_s}\right) - \frac{1}{p^2}\partial_i\omega_1\left(\frac{x-\xi_s}{\delta_j}\right) - \partial\ln a(\xi_h)\right) \\
=&\frac{1}{\gamma\delta_j\mu_j^{2/(p-1)}}P\left(Z_{ij} - \frac{1}{p}\partial_i\omega_0\left(\frac{x-\xi_s}{\delta_s}\right) - \frac{1}{p^2}\partial_i\omega_1\left(\frac{x-\xi_s}{\delta_j}\right) - \partial\ln a(\xi_h)\right) + O\left(\frac{1}{\gamma}\right),
\end{aligned}
$$

$$
\partial U_h = \quad 2\partial\ln(\mu_h)Z_{0h} - \tfrac{4(x-\xi_h)_i}{\delta_h^2+|x-\xi_h|^2} - \partial\ln(a(\xi_h)), \tag{2.65}
$$

$$\partial Z_{lh} = Z_{lh}\partial \ln \mu_h + \frac{4\delta_h \delta_{il}}{\delta_h^2 + |x-\xi_h|^2} - \frac{8\delta_h^2 (x-\xi_h)_l \partial\delta_h}{(\delta_h^2 + |x-\xi_h|^2)^2} - \frac{8\delta_h (x-\xi_h)_l (x-\xi_j)_i}{(\delta_h^2 + |x-\xi_h|^2)^2}$$

and

Hence

$$\begin{aligned}\partial(e^{U_h} Z_{lh}) = \;& 4e^{U_h}\delta_h \left[\frac{\delta_{il}}{\delta_h^2+|x-\xi_h|^2} - 6\frac{(x-\xi_h)_l (x-\xi_j)_i}{(\delta_h^2+|x-\xi_h|^2)^2}\right]\delta_{hj} \\ & +3e^{U_h} Z_{0h} Z_{lh}\partial \ln\mu_h - Z_{lh} e^{U_h}\partial \ln(a(\xi_h)),\end{aligned} \tag{2.66}$$

Where δ_{hj} Kronecker's delta. Hence, by (2.65) and (2.66)

$$\begin{aligned}\partial_{(\xi_j)_i} F(\hat{\xi}) = & -\frac{1}{\gamma\delta_j \mu_j^{\frac{2}{p-1}}} \sum_{l=1}^{2}\sum_{h=1}^{M} c_{lh}(\hat{\xi})\left(PZ_{ij}, PZ_{lh}\right)_{H_0^1(\Omega)} \\ & + O\left(\frac{1}{p\gamma\delta_j} + \|\phi_\xi\|_\infty \int_\Omega |\partial_{(\xi_j)_i}(e^{U_h} Z_{lh})|\right)\sum_{l=1}^{2}\sum_{h=1}^{M} |c_{lh}(\xi)|\end{aligned}$$

Finally by (2.44), (2.7) and (2.66) we have that

$$0 = \frac{64}{\gamma\delta_j \mu_j^{\frac{2}{p-1}}}\left(\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}\right) c_{ij}(\hat{\xi}) + O\left(\frac{1}{p\gamma\delta_j}\sum_{i=1}^{2}\sum_{j=1}^{M}|c_{lh}(\xi)|\right)$$

Which implies that $c_{ij}(\hat{\xi}) = 0$ for all $i = 1, 2$ and all $j = 1, 2, .., M$

□

Lemma 2.9. *Let $\varepsilon > 0$. $F(\hat{\xi})$ it satisfies the following finite dimensional expansion*

$$\begin{aligned}F(\hat{\xi}) = & \frac{4\pi M p}{\gamma^2} - \frac{32\pi^2}{\gamma^2}\varphi_M(\xi_1, ..., \xi_M) + \frac{4\pi M}{\gamma^2} \\ & + \frac{M}{2\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2} v_\infty - \Delta w_0\right) + O(p^{-3})\end{aligned}$$

uniformly for $\xi \in \mathcal{O}_\varepsilon$

Proof. Multiplying (2.48) by $U_{\hat\xi}+\phi_{\hat\xi}$ and integrating by parts

$$\int_\Omega a(\cdot)(U_{\hat\xi}+\phi_{\hat\xi})^{p+1}=\int_\Omega |\nabla(U_{\hat\xi}+\phi_{\hat\xi})|^2+\sum_{i=1}^{2}\sum_{j=1}^{2}c_{ij}\int_\Omega e^{U_j}(U_{\hat\xi}+\phi_{\hat\xi})$$

since $U_{\hat\xi}$ is a bounded function, by (2.52) we get

$$\int_\Omega a(\cdot)(U_{\hat\xi}+\phi_{\hat\xi})^{p+1}=\int_\Omega |\nabla(U_{\hat\xi}+\phi_{\hat\xi})|^2+O(p^{-4})$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$. Using (2.57), we write $F(\hat\xi)$ as

$$\begin{aligned}F(\hat\xi)&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega|\nabla(U_{\hat\xi}+\phi_{\hat\xi})|^2+O(p^{-4})\\&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega|\nabla U_{\hat\xi}|^2+2\int_\Omega\nabla U_{\hat\xi}\nabla\phi_{\hat\xi}+\int_\Omega|\nabla\phi_{\hat\xi}|^2+O(p^{-4})\\&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega|\nabla U_{\hat\xi}|^2+O(p^{-3}).\end{aligned}$$

Using (2.16) and (2.19) we get that

$$\begin{aligned}\int_\Omega|\nabla U_{\hat\xi}|^2=&\sum_{j=1}^{M}\frac{1}{\gamma\mu_j^{\frac{2}{p-1}}}\int_{B(\xi_j,\varepsilon)}\left(a(\xi_j)e^{U_j}-\frac{1}{p\delta_j^2}\Delta w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right.\\&\left.-\frac{1}{p^2\delta_j^2}\Delta w_1\left(\frac{x-\xi_j}{\delta_j}\right)+O(p^2e^{-\frac{p}{2}})\right)U_\xi+O(e^{-\frac{p}{2}})\\=&\sum_{j=1}^{M}\frac{1}{\gamma^2\mu_j^{\frac{4}{p-1}}}\int_{B(0,\frac{\varepsilon}{\delta_j})}\left(\frac{8}{(1+|y|^2)^2}-\frac{1}{p}\nabla w_0-\frac{1}{p^2}\nabla w_1+O(p^2e^{-p})\right)\times\\&\left(p+v_\infty+\frac{1}{p}w_0+\frac{1}{p^2}w_1+O\left(e^{-\frac{p}{4}}|y|+e^{-\frac{p}{4}}\right)\right)+O(e^{-\frac{p}{2}})\\=&\sum_{j=1}^{M}\frac{1}{\gamma^2\mu_j^{\frac{4}{p-1}}}\left(8\pi p+\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O(p^{-1})\right)\\=&\frac{8\pi Mp}{\gamma^2}-\frac{32\pi}{\gamma^2}\sum_{j=1}^{M}\log\mu_j+\frac{M}{\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O(p^{-3}).\end{aligned}$$

Let us notice that $\mu_j^{-\frac{4}{p-1}} = 1 - \frac{4}{p}\log\mu_j + O(p^{-2})$ and having into account (2.15) we have that

$$\int_\Omega |\nabla U_{\hat\xi}|^2 = \frac{8\pi Mp}{\gamma^2} - \frac{64\pi^2}{\gamma^2}\varphi_M(\xi_1,...,\xi_M) + \frac{24\pi M}{\gamma^2} + \frac{M}{\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty - \Delta w_0\right) + O(p^{-3}) \tag{2.67}$$

uniformly for $\hat\xi \in \mathcal{O}_\varepsilon$.

Finally we obtain the expansion of $F(\hat\xi)$

$$\begin{aligned} F(\hat\xi) =& \frac{4\pi Mp}{\gamma^2} - \frac{32\pi^2}{\gamma^2}\varphi_M(\xi_1,...,\xi_M) + \frac{4\pi M}{\gamma^2} \\ &+ \frac{M}{2\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty - \Delta w_0\right) + O(p^{-3}) \end{aligned}$$

□

Lemma 2.10. *Let $\varepsilon > 0$. The following C^1 expansion holds*

$$\nabla_{(\xi_j)_i}F(\hat\xi) = -\frac{32\pi^2}{\gamma^2}\nabla_{(\xi_j)_i}\varphi_M(\xi_1,...,\xi_M) + o(p^{-2}).$$

uniformly for $\hat\xi \in \mathcal{O}_\varepsilon$

Proof. Let us take a cut-off function η, $0 \le \eta \le 1$, $\quad \eta \equiv 1$ in $|x| \le \epsilon$ and $\eta \equiv 0$ in $|x| \ge 2\epsilon$

$$\begin{aligned} \int_\Omega (\Delta u_{\hat\xi} + au_{\hat\xi}^p)\partial_{(\xi_j)_i}\phi_{\hat\xi} &= \sum_{k=1}^{2}\sum_{l=1}^{M} c_{k,l}\int_\Omega e^{U_l}Z_{kl}\partial_{(\xi_j)_i}\phi_{\hat\xi} \\ &= -\sum_{k=1}^{2}\sum_{l=1}^{M} c_{k,l}\int_\Omega \partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right)\phi_{\hat\xi} \end{aligned}$$

$$= \sum_{k=1}^{2}\sum_{l=1}^{M} c_{k,l}\int_\Omega \partial_{x_i}\left(e^{U_l}Z_{kl}\right)\eta(x-\xi_j)\phi_{\hat\xi} - \sum_{k=1}^{2}\sum_{l=1}^{M} c_{k,l}\int_\Omega \left[\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right) + \eta(x-\xi_j)\partial_{x_i}(e^{U_l}Z_{kl})\right]\phi_{\hat\xi}$$

$$= -\sum_{k=1}^{2}\sum_{l=1}^{M} c_{kl}\int_\Omega e^{U_l}Z_{kl}\partial_{x_i}(\eta(x-\xi_j)\phi_{\hat\xi}) - \sum_{k=1}^{2}\sum_{l=1}^{M} c_{k,l}\int_\Omega \left[\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right) + \eta(x-\xi_j)\partial_{x_i}(e^{U_l}Z_{kl})\right]\phi_{\hat\xi}.$$

Having in mind (2.66), we obtain

$$\begin{aligned}\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right)+\eta(x-\xi_j)\partial_{x_i}(e^{U_l}Z_{kl}) &= 4\delta_l e^{U_l}\frac{\eta(x-\xi_j)-\delta_{lj}}{\delta^2+|x-\xi_l|^2}\delta_{lk}\\ &+24\delta_l e^{U_l}\frac{(x-\xi_l)_k(x-\xi_l)_i}{(\delta^2+|x-\xi_l|^2)^2}(\delta_{lj}-\eta(x-\xi_j)))\\ &+3e^{U_l}Z_{0l}Z_{kl}\partial_{(\xi_j)_i}\log(\mu_l-e^{U_l}Z_{kl}\partial_{(\xi)_i}\log(a(\xi_j))\\ &=3e^{U_l}Z_{0l}Z_{kl}\partial_{(\xi_J)_i}\log(\mu_l)-e^{U_l}Z_{kl}\partial_{(\xi_j)_i}\log(a(\xi_l))+O(e^{-\frac{3}{4}p})\end{aligned}$$

then

$$\begin{aligned}&\left|\sum_{k=1}^{2}\sum_{l=1}^{M}c_{kl}\int_\Omega\left[\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right)+\eta(x-\xi_j)\partial_{x_i}(e^{U_l}Z_{kl})\right]\phi_{\hat\xi}\right|\\ &\quad\le C\|\phi_{\hat\xi}\|_\infty\max_{k,l}|c_{kl}|\int_\Omega(e^{U_l}+O(e^{-\frac{3}{4}p}))=O\left(\frac{1}{p^7}\right)\end{aligned}$$

Using $|Z_{0l}Z_{kl}|<2$ and the orthogonality condition we get

$$\int_\Omega\left(\Delta u_{\hat\xi}+au_{\hat\xi}^p\right)\partial_{(\xi_j)_i}\phi_{\hat\xi}=-\sum_{k=1}^{2}\sum_{l=1}^{M}c_{kl}\int_\Omega e^{U_l}Z_{kl}\partial_{x_i}(\eta(x-\xi_j)\phi_{\hat\xi})+O(\frac{1}{p^7}) \tag{2.68}$$

since we have the behavior $p\phi_{\hat\xi}\to 0$ in C^1-norm away to the points $\xi_1,\xi_2,...\xi_M$ we obtain

$$\begin{aligned}\int_\Omega\left(\Delta u_{\hat\xi}+au_{\hat\xi}^p\right)\partial_{(\xi_j)_i}\phi_{\hat\xi}&=-\sum_{k=1}^{2}\sum_{l=1}^{M}\int_{B(\xi_j,\epsilon)}e^{U_l}Z_{kl}\partial_{x_i}\phi_{\hat\xi}+O(\frac{1}{p^7})\\ &=-\int_{B(\xi_j,\epsilon)}(\Delta u_{\hat\xi}+au_{\hat\xi}^p)\partial_{x_i}\phi_{\hat\xi}+O(\frac{1}{p^7}).\end{aligned}$$

Having account that the equality in (2.8) and (2.12) are in C^1, for $|x-\xi_j|\le 2\epsilon$ we have that

$$
\begin{aligned}
\partial_{x_i} U_{\hat{\xi}} &= \sum_{s=1}^{M} \frac{1}{\gamma \mu_s^{\frac{2}{p-1}}} \left(\partial_{x_i} P U_{\delta_s, \xi_s} + \frac{1}{p} \partial_{x_i} P \left(w_0 \left(\frac{x - \xi_s}{\delta_s} \right) \right) + \frac{1}{p^2} \partial_{x_i} P \left(w_1 \left(\frac{x - \xi_s}{\delta_s} \right) \right) \right) \\
&= \sum_{s=1}^{M} \frac{1}{\gamma \mu_s^{\frac{2}{p-1}}} \left(\partial_{x_i} U_{\delta_s \xi_s} + \frac{1}{p \delta_s} \partial_{x_i} w_0 \left(\frac{x - \xi_s}{\delta_s} \right) + \frac{1}{p^2 \delta_s} w_1 \left(\frac{x - \xi_s}{\delta_s} \right) \right) + O(\gamma^{-1}) \\
&= -\frac{1}{\gamma \delta_j \mu_j^{\frac{2}{p-1}}} \left(Z_{ij} - \frac{1}{p} \partial_{x_i} w_0 (\frac{x - \xi_j}{\delta_j}) - \frac{1}{p^2} \partial_{x_i} w_1 (\frac{x - \xi_j}{\delta_j}) \right) + O(\gamma^{-1})
\end{aligned}
$$

We recall that from lemma 2.1 in [20] we have $\partial_{x_i} w_l(\frac{x-\xi_j}{\delta_j}) = O(\delta_j)$ uniformly away from the points ξ_j. Then

$$
\partial_{x_i} U_{\hat{\xi}} = O(\gamma^{-1})
$$

for $\varepsilon \leq |x - \xi_j| \leq 2\varepsilon$.

$$
\partial_{(\xi_j)_i} U_{\hat{\xi}} + \eta(x - \xi_j) \partial_{x_i} U_{\hat{\xi}} = \frac{1}{\gamma \delta_j \mu_j^{\frac{2}{p-1}}} (P Z_{ij} - Z_{ij}) + O(\gamma^{-1}) = O(\gamma^{-1})
$$

$$
\begin{aligned}
\int_\Omega (\Delta u_{\hat{\xi}} + a u_{\hat{\xi}}^p) \partial_{(\xi_j)_i} U_{\hat{\xi}} - \sum \sum c_{kl} \int_\Omega e^{U_l} Z_{kl} \eta(x - \xi_j) \partial_{x_i} U_{\hat{\xi}} & \\
+ \sum \sum c_{kl} \int_\Omega e^{U_l} Z_{kl} (\partial_{(\xi_j)_i} U_{\hat{\xi}} + \eta(x - \xi_j) \partial_{x_i} U_{\hat{\xi}}) & \\
= -\sum \sum c_{kl} \int_{B(\xi_j, \epsilon)} e^{U_l} Z_{kl} \partial_{x_i} U_\xi + O(p^{-5}) & \\
= -\int_{B(\xi_j, \epsilon)} (\Delta u_\xi + u_\xi^p) \partial_{x_i} U_\xi + O(p^{-5}) &
\end{aligned}
$$

$$\partial_{(\xi_j)_i} F(\xi) = -\int_\Omega (\Delta u_\xi + a u_\xi^p)(\partial_{(\xi_j)_i} U_\xi + \partial_{(\xi_j)_i} U \phi_\xi)$$
$$= \int_{B(\xi_j,\epsilon)} (\Delta u_\xi + a u_\xi^p)(\partial_{x_i} U_\xi + \partial_{x_i} \phi_\xi) + O(p^{-5})$$
$$= \int_{B(\xi_j,\epsilon)} (\Delta u_\xi + a u_\xi^p)(\partial_{x_i} u_\xi) + O(p^{-5})$$

We consider the following identities

$$\int_B \Delta u \nabla = \int_{\partial B} \left(\partial_n u \nabla u - \frac{1}{2} |\nabla u|^2 n \right) \tag{2.69}$$
$$\int_B a(\cdot) u^p \nabla u = \frac{1}{p+1} \int_{\partial B} a u^{p+1} n - \frac{1}{p+1} \int_B \nabla a(x) u^{p+1} \tag{2.70}$$

where n(x) is the normal vector to ∂B, $x \in \partial B$. Let us denote

$$\varphi_j'(x) = H(x, \xi_j) + \sum_{l \neq j} G(x, \xi_l),$$

which satisfies

$$\sum_{l=1}^{M} G(x, \xi_l) = -\frac{1}{2\pi} \log |x - \xi_j| + \varphi_j'(x).$$

We notice the asymptotic property

$$p u_\xi(x) \to 8\pi \sqrt{e} \sum_{l=1}^{M} G(x, \xi_l), \quad \text{in } C^1_{\text{loc}} \bar{\Omega} \setminus \{\xi_1, ..., \xi_M\} \tag{2.71}$$

and the harmonicity of $\nabla \varphi_j'$ near to ξ_j the mean value property give us

$$\frac{1}{2\pi\epsilon} \int_{\partial B} \nabla \varphi_j' = \varphi_j'(\xi_j)$$

We applying the identities (2.69),(2.70) for $B = B(\xi_j, \varepsilon)$ and by (2.71) we get

$$\begin{aligned}
\int_B (\Delta u_\xi + a u_\xi^p)\nabla u_\xi &= \int_{\partial B}\left(\partial_n u_\xi \nabla u_\xi - \frac{1}{2}|\nabla u_\xi|^2 n + \frac{1}{p+1} a u_\xi^{p+1} n\right) - \frac{1}{p+1}\int_B \nabla a\, u^{p+1} \\
&\quad \frac{64\pi^2 e}{p^2}\int_{\partial B}\left[\left(-\frac{1}{2\pi\epsilon} + \partial_n \psi_j\right)\left(-\frac{1}{2\pi}\frac{x-\xi_i}{|x-\xi_j|^2} + \nabla\psi\right)\right. \\
&\quad \left. -\frac{1}{2}\left|-\frac{1}{2\pi}\frac{x-\xi_j}{|x-\xi_j|^2} + \nabla\psi_j\right|^2\right] - \frac{1}{p+1}\int_B \nabla a\, u^{p+1} + o(p^{-2}) \\
&= -\frac{64\pi e}{p^2}\nabla\varphi_j'(\xi_j) - \frac{1}{p+1}\int_B \nabla a\, u^{p+1} + o(p^{-2}) \\
&= -\frac{32\pi^2}{\gamma^2}\nabla\varphi_M'(\xi) - \frac{1}{p+1}\int_B \nabla a\, u^{p+1} + o(p^{-2})
\end{aligned} \tag{2.72}$$

For the missing term

$$\begin{aligned}
\int_B a_{x_i} u_\xi^{p+1} &= \int_{B(\xi_j,\sqrt{\delta_j}\epsilon)} a_{x_i} u_\xi^{p+1} + \int_{B(\xi_j,\epsilon)\setminus B(\xi_j,\sqrt{\delta_j}\epsilon)} a_{x_i} u_\xi^{p+1} \\
&= a'(\xi_j)\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} + \sqrt{\delta_j}\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} + \int_{B(\xi_j,\epsilon)\setminus B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} \\
&= a'(\xi_j)\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} + \int_{B(\xi_j,\epsilon)\setminus B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} + O(\sqrt{\delta_j}) \\
&= I_1 + I_2 + O(\sqrt{\delta_j})
\end{aligned}$$

with

$$\begin{aligned}
I_1 &= a'(\xi_j)\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} = a'(\xi_j)\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} \frac{p}{\gamma^2\mu^{\frac{2}{p-1}}} e^{U_j}\left[1 + \frac{z(y)}{p}\right] \\
&= a'(\xi_j)\int_{B(\xi_j,\sqrt{\delta_j}\varepsilon)} \frac{p}{\gamma^2\mu^{\frac{2}{p-1}}} e^{U_j} + O(p^{-2}) \\
&= \frac{a'(\xi_j)}{a(\xi_j)}\frac{p}{\gamma^2\mu^{\frac{2}{p-1}}}\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2} + O(p^{-2}) \\
&= \frac{a'(\xi_j)}{a(\xi_j)}\frac{p}{\gamma^2\mu^{\frac{2}{p-1}}} 8\pi + O(p^{-2}) = \frac{32\pi^2 p}{\gamma^2}\frac{\partial_{(\xi_j)_i}\ln(a(\xi_j))}{4\pi} + O(p^{-2})
\end{aligned}$$

Having into account (2.20)

$$
\begin{aligned}
I_2 &= \int_{B(\xi_j,\epsilon)\setminus B(\xi_j,\sqrt{\delta_j}\varepsilon)} u_\xi^{p+1} \\
&\leq C\int_{B(0,\frac{\varepsilon}{\delta_j})\setminus B(0,\frac{\varepsilon}{\sqrt{\delta_j}})} \left(\left(\frac{1}{\gamma(1+|y|^2)^2}\right)^{\frac{1}{p+1}} + \frac{\delta_j^2}{p^3}\right)^{p+1} \\
&\leq \frac{1}{p}\left(\left(\int_A \frac{1}{(1+|y|^2)^2}\right)^{\frac{1}{p+1}} + \frac{\delta_j}{p^2}\right)^{p+1} \\
&\leq C\frac{1}{p}\left((o(p^{-1})^{\frac{1}{p+1}}) + \frac{\delta_j}{p^2}\right)^{p+1} = o(p^{-2})
\end{aligned}
$$

Finally we have

$$
\frac{1}{p+1}\int_B \nabla a(x)u^{p+1} = \frac{32\pi^2}{\gamma^2}\frac{\nabla\log(a(\xi_i))}{4\pi} + o(p^{-2})
$$

and

$$
\partial_{(\xi_j)_i}F(\hat{\xi}) = -\frac{32\pi^2}{\gamma}\partial_{(\xi_j)_i}\varphi_M(\hat{\xi}) + o(p^{-2})
$$

□

2.6 Proof Of the Main results

According to the result of Lemma 2.8 and Lemma 2.9, the proof of Theorem 2.1 reduces to show that, for any $M \geq 1$, the function $\varphi_M(\xi_1, ..., \xi_M)$ has a non trivial critical value in some open set $\mathcal{M}$, compactly contained in $(\Omega\setminus Z)^M$. This fact has already been established in [10]. For the sake of completeness, we recall here the principal ingredients employed to characterize a topological nontrivial critical value of $\varphi_M(\xi_1, ..., \xi_M)$ in some set $\mathcal{M}$, compactly contained in $(\Omega\setminus Z)^M$. We refer the reader to [10] for a complete proof of each step.

From the assumptions (A_1) and (A_2), without loss of generality we write

$$a(x) = \prod_{s=1}^{k} |x - q_s|^{2\alpha_s}.$$

Then we have

$$\begin{aligned}\varphi_M(\hat{\xi}) &= \sum_{j=1}^{M} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i\neq j}^{M} G(\xi_j, \xi_i) + \frac{1}{4\pi} \sum_{j=1}^{M} \log(a(\xi_j)) \\ &= \sum_{j=1}^{M} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i\neq j}^{M} G(\xi_j, \xi_i) + \sum_{s=1}^{k} \alpha_s \sum_{j=1}^{M} G(\xi_j, q_s) + \underbrace{\sum_{s=1}^{k} \sum_{j=1}^{M} \alpha_s H(\xi_j, q_s)}_{O(1)}.\end{aligned}$$

Let us use the following notation

$$\mathcal{M} := \left\{ \hat{\xi} = (\xi_1, \cdots, \xi_M) \in (\Omega \backslash Z)^M \ : \ \xi_i \neq \xi_j \text{ if } i \neq j \right\}$$

and the subset of $\mathcal{M}$

$$\begin{aligned}\mathcal{D} = \Bigg\{ \hat{\xi} \in \mathcal{M}, \quad & \sum_{j=1}^{M} H(\xi_j, \xi_j) + \sum_{i,j=1,\, i\neq j}^{M} G(\xi_j, \xi_i) \\ & + \sum_{s=1}^{k} \alpha_s \sum_{j=1}^{M} G(\xi_j, q_s) + O(1) > -T \Bigg\}\end{aligned}$$

where $T > 0$ is a sufficiently large number to be chosen. We have that $\mathcal{D}$ is compactly contained in $\mathcal{M}$.

From A_2, we write $\{1,2,\ldots,M\} = I_1 \cup I_2 \cup \ldots \cup I_m$ where

$$
\begin{aligned}
I_1 &= \{1,\ldots,M_1\},\\
I_2 &= \{M_1+1,\ldots,M_1+M_2\},\\
&\ldots\\
I_s &= \{M_1+\ldots+M_{s-1}+1,\ldots,M_1+\ldots+M_{s-1}+M_s\},\\
&\ldots\\
I_m &= \{M_1+\ldots+M_{m-1}+1,\ldots,M\}.
\end{aligned}
$$

Let us take θ_q be fixed angles $(q \in Z)$ and $\delta \in (0, \frac{\pi}{2})$ be a number small enough such that the cones

$$\left\{q+\rho e^{i(\theta_q+\theta)} \;:\; \rho \geq 0,\; \theta \in [-\delta,\delta]\right\}, \quad q \in Z \tag{2.73}$$

are disjoint from one another. Moreover, we assume

$$dist(q,\partial\Omega) > 2\delta \;\; \forall\, q \in Z, \qquad |q_i - q_j| > 4\delta \;\; \forall\, q_i, q_j \in Z, i \neq j. \tag{2.74}$$

Now we define the $M-$tuple

$$\xi_0 = (\xi_1^0,\ldots,\xi_M^0)$$

by

$$\xi_j^0 = q_s + \frac{3}{2}\delta e^{i(\theta_{q_s}+j\frac{\delta}{M})} \quad \forall j \in I_s, \;\; s = 1,\ldots,m.$$

Let us set an annulus with radius δ and 2δ centered in q_s, that is

$$U_s := \left\{\xi \in \mathbb{R}^2 \;:\; \delta < |\xi - q_s| < 2\delta\right\},$$

and consider the $M-$tuple $\hat{\xi} = (\xi_1, \ldots, \xi_M)$ belongs to the open set

$$\left\{ \hat{\xi} \in U_1^{K_1} \times \ldots \times U_m^{K_m} \ : \ |\xi_i - \xi_j| > T^{-1} \ \ \forall\, i \neq j \right\}. \tag{2.75}$$

The choice of δ in (2.73) and (2.74) implies that $\xi_i^0 \neq \xi_j^0$ for $i \neq j$, then we have that ξ_0 belongs to (2.75) provided that T is large enough. Then we define

$$W := \text{the connected of (2.75) containing } \xi_0$$
$$\mathcal{K} := \bar{W}, \qquad \mathcal{K}_0 = \left\{ \hat{\xi} \in \mathcal{K} : \min_{i \neq j} |\xi_i - \xi_j| = T^{-1} \right\}.$$

From these facts, we get that

(P1) $\mathcal{D}$ is an open set, $\mathcal{K}$ and $\mathcal{K}_0$ are compact sets, $\mathcal{K}$ is connected and

$$\mathcal{K}_0 \subset \mathcal{K} \subset \mathcal{D} \subset \bar{\mathcal{D}} \subset \mathcal{M}.$$

We denote $\mathcal{F}$ to be the class of all continuous maps $\eta : \mathcal{K} \to \mathcal{D}$ with the property that there exists a continuous homotopy $\Gamma : [0,1] \times \mathcal{K} \to \mathcal{D}$ such that

$$\Gamma(0,\cdot) = id, \ \ \Gamma(1,\cdot) = \eta, \ \ \Gamma(t,\hat{\xi}) = \hat{\xi} \ \ \forall t \in [0,1], \ \forall \xi \in \mathcal{K}_0.$$

In [10], the following facts are proved:

(P2):

$$\Phi^* := \sup_{\eta \in \mathcal{F}} \min_{\hat{\xi} \in \mathcal{K}} \varphi_M(\eta(\hat{\xi})) < \min_{\hat{\xi} \in \mathcal{K}_0} \varphi_M(\hat{\xi})$$

(P3): for every $\hat{\xi} \in \partial\mathcal{D}$ such that $\varphi_M(\hat{\xi}) = \Phi^*$, $\partial\mathcal{D}$ is smooth at $\hat{\xi}$ and there exists a vector $\tau_{\hat{\xi}}$ tangent to $\partial\mathcal{D}$ at $\hat{\xi}$ so that $\tau_{\hat{\xi}} \cdot \nabla\varphi_M(\hat{\xi}) \neq 0$.

Under (P1), (P2) and (P3), there exists a critical point $\hat{\xi} \in \mathcal{D}$ of $\varphi_M(\hat{\hat{\xi}})$ with $\varphi_M(\hat{\xi}) = \Phi^*$, as a standard deformation argument involving the gradient flow of φ_M shows. This finishes the proof of Theorem 2.1.

CHAPTER 3

Entire Sign Changing Solutions whit finite energy to the Fractional Yamabe Equation

Danilo Garrido, Monica Musso

3.1 Introduction

We are interested in the existence of finite-energy sign-changing solutions to the fractional Yamabe type equation in $\mathbb{R}^n$,

$$(-\Delta)^s u = \gamma\, |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^n \tag{3.1}$$

where $n \geq 3$ and p is the fractional critical Sobolev exponent $p = \frac{n+2s}{n-2s}$. In (3.1), $\gamma > 0$ is a constant chosen for normalization purposes as

$$\gamma = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})}.$$

For any $s \in (0,1)$, $(-\Delta)^s$ is the nonlocal operator defined as

$$(-\Delta)^s u(x) = c(n,s)\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = c(n,s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy, \tag{3.2}$$

where P.V. stands for *the principal value* and $c(n,s) = \pi^{-(2s+\frac{n}{2})} \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(-s)}$. This nonolocal operator in $\mathbb{R}^n$ can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) \,:\, x \in \mathbb{R}^n,\, t > 0\}$, as we learn from Caffarelli and Silvestre [5]: given a solution $u = u(x)$ of $(-\Delta)^s u = f$ in $\mathbb{R}^n$, one can equivalently consider the dimensionally extended problem for $u = u(x,t)$ which solves

$$\text{div}(t^{1-2s}\nabla u) = 0, \quad \text{in} \quad \mathbb{R}^{n+1}_+, \quad -\lim_{t\to 0} d_s t^{1-2s} \partial_t u(x,t) = f, \quad \text{on} \quad \partial\mathbb{R}^{n+1}_+$$

where d_s is the positive constant $d_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}$.

By finite energy solutions of Problem (3.1), we mean the following. Consider the Schwartz space $\mathcal{S}$ of rapidly decaying C^∞ functions on $\mathbb{R}^n$, and for any $\varphi \in \mathcal{S}$ we denote by

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x)\, dx$$

the Fourier transformation of φ. We look for solutions u of Problem (3.1) in the energy space

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \,:\, \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} < \infty\}$$

where $\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$ is defined by $\left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$, endowed with the norm $\|u\|_{\mathcal{D}^s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$. These solutions correspond to critical points of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 - \gamma \frac{n-2s}{2n} \int_{\mathbb{R}^n} |u|^{\frac{2n}{n+2s}}, \quad u \in \mathcal{D}^s(\mathbb{R}^n).$$

It has been known after the work by Lieb [29] (see also [23, 24, 6] for alternative proofs) that positive solutions to Equation (3.1) are given by the family of functions defined by

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2s}{2}}, \quad \text{and} \quad \mu^{-\frac{n-2s}{2}}\, U(\frac{x-\xi}{\mu}) \tag{3.3}$$

for any $\mu > 0$ and $\xi \in \mathbb{R}^n$. Indeed these functions realize the Hardy-Littlewood-Sobolev inequality, which states the existence of a positive number S such that for all $u \in C^\infty(\mathbb{R}^n)$, one has

$$S\,\|u\|_{L^{2^*}(\mathbb{R}^n)} \le \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$$

where $2^* = p+1 = \frac{2n}{n-2s}$. Indeed, these functions are the only positive solutions to Equation (3.1) under some decay conditions, we refer to [9, 26, 28]. In particular, this is true if $u \in L^{\frac{2n}{n-2s}}_{loc}(\mathbb{R}^n)$ as shown in [9].

On the other hand, Problem (3.1) can be read on the sphere $S^n \subset \mathbb{R}^{n+1}$, after a stereographic projection. Indeed, the inverse of the stereographic projection $\pi : \mathbb{R}^n \to S^n \setminus \{S\}$, where, $S = (0,\ldots,0,-1) \in \mathbb{R}^{n+1}$, defined by

$$\pi(y) = \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2}\right)$$

is a conformal map and $\pi^* g_0 = U^{\frac{4s}{n-2s}}(y)\,dy$ where g_0 is the standard metric on S^n and U is defined in (3.3). In S^n, the fractional Laplacian $(-\Delta)^s$ reduces to a an elliptic pseudodifferential operator $P^{g_0}_s$ of order $2s$ with principal symbol $\sigma_{2s}(P^{g_0}_s) = |\xi|^2_{g_0} s$. In [8] a relation between this operator and a Dirichlet-to-Neumann operator of uniformly non degenerate elliptic boundary value problems in the spirit of [5] is established. We have $\pi^*(L^{g_0}_s v) = U^{-\frac{n+2s}{n-2s}}(-\Delta)^s(U\,\pi^*\,v)$ for any v defined on S^n. Thus u is a solution to (3.1) if and only if w, defined by $u = U\,\pi^* w$, solves

$$\Delta_{g_0} w + \gamma(|w|^{\frac{4s}{n-2s}} w - w) = 0 \quad \text{in } S^n \tag{3.4}$$

Positive solutions to (3.4) solves the so-called fractional Yamabe problem on the sphere S^n.

We refer to [25] for a general formulation of the fractional Yamabe problem and results concerning its solvability.

Finite energy sign-changing solutions to (3.1), or equivalently (3.4), are poorly understood. The purpose of this paper is to give a first example of finite-energy sign-changing solutions to (3.1), in all dimensions $n \geq 3$, and for $s \in (\frac{1}{2}, 1)$: we build a solution to Equation (3.1) which looks like the soliton U sorrounded by k negative copies U properly scaled and distributed along the vertices of a regular polygon with radius 1. Our main result is the following

Theorem 3.1. *Let $n \geq 3$ and $s \in (\frac{1}{2}, 1)$. Write $\mathbb{R}^n = \mathcal{C} \times \mathbb{R}^{n-2}$ and let $\xi_j^k = (e^{\frac{2j\pi i}{k}}, 0)$, $j = 1, \ldots, k$. Then for any sufficiently large k there is a finite energy solution to Problem* (3.1) *of the form*

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2s}{2}} U\left(\mu_k^{-1}(x - \xi_j)\right) + o(1),$$

where

$$\mu_k = \frac{[2^{\frac{n-2s}{2}} \sum_{j=1}^{\infty} j^{2s-n}]^{-1}}{k^2}(1 + o(1))$$

Moreover,

$$J(u_k) = (k+1)J(U) + O(1). \tag{3.5}$$

Here $O(1)$ remains bounded and $o(1) \to 0$ uniformly as $k \to +\infty$.

The proof of the result consists in defining a first approximation and then to show that a small perturbation of this approximation provides an actual solution to the Problem. This is done linearizing the equation around the approximation and performing an invertibility theory for the linearized operator. In this step, we use the non degeneracy property of U proved in [11], which states that all bounded solutions of the linear problem

$$(-\Delta)^s \phi - \gamma \frac{n+2s}{n-2s} U^{\frac{4s}{n-2s}} \phi = 0$$

are linear combinations of

$$\partial_{x_j} U(x), \quad \text{for} \quad j=1,\dots,n, \quad \text{and} \quad \frac{n-2s}{2}U(x)+x\cdot\nabla U(x).$$

Indeed, the above functions belong to the kernel of the linearized operator, due to the corresponding rigid motion under which Equation (3.1) is invariant. The result in [11] says that these are the only non trivial elements of the kernel.

A second ingredient we take advantage to perform an invertibility theory is the symmetry of the configuration. This reflects into the fact that our approximation, as well as our final solution, satisfy the symmetries

$$u(\overline{y},y')=u(e^{\frac{2\pi j}{k}\imath}\overline{y},y'), \quad j=1,\dots,k-1 \tag{3.6}$$

$$u(y_1,y_2,\dots,y_j\dots,y_n)=u(y_1,y_2,\dots,-y_j,\dots,y_n) \quad j=2,\dots,n \tag{3.7}$$

Furthermore, they are invariant under Kelvin transform, namely

$$u(y)=|y|^{2s-n}u(\frac{y}{|y|^2}).$$

The final step in the proof consists in adjusting properly the parameter μ_k. A detailed description of the scheme of the proof is given in Section 3.2.

Let us mention that a very similar construction for finite-energy sign-changing solutions to the classic Yamabe type problem in $\mathbb{R}^n$

$$\Delta u+|u|^{\frac{4}{n-2}}u=0 \quad \text{in} \quad \mathbb{R}^n,$$

namely when $s=1$ in Problem (3.1), has been performed in [14], and [15]. Indeed, our result extends to the case $s\in(\frac{1}{2},1)$ the construction performed in [14], from which we are inspired.

We learnt recently of the paper [22], where the author constructs solutions to (3.1) similar to ours, covering the whole range $s \in (0,1)$. Nevertheless, we believe that the construction in [22] is wrong: in their case indeed the concentration parameter μ_k is of order k^{-3} (see formula (2.4) in [22]), while our concentration parameter is $\mu_k \sim k^{-2}$, as $k \to \infty$. It is not clear to us how this choice of the parameter's rate provides a real solution to Problem (3.1). Indeed, it is this choice of the parameter's rate, in terms of k, that allows the author of [22] to cover the whole range $s \in (0,1)$.

Our restriction on s is consequence of two inequalities: we need a certain power of integrability q to be $q < n$ in order to have a good first approximation when estimated in proper norms, and at the same time we need $q > \frac{n}{2s}$ to guarantee enough regularity. These constraints restrict us to $s \in (\frac{1}{2}, 1)$. We believe that our construction should work in the whole range $s \in (0,1)$, and in fact we think that $\mu_k \sim k^{-2}$, as $k \to \infty$, for the whole range $s \in (0,1)$, but an invertibility theory on different weighted Sobolev spaces is needed. We will treat this problem in a forthcoming paper.

The rest of the paper will be devoted to the proof of Theorem 3.1.

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3.2 Ansatz for the solution and scheme of the proof

This section is devoted to define a first approximation for a solution to Problem (3.1) and to describe the scheme of the proof of our result.

We start reminding that U defined in (3.3) is invariant under Kelvin transform, namely

$$U(y) := |y|^{2s-n}\, U(|y|^{-2}y).$$

Even more, it can be proved that also the family of solutions $\mu^{-\frac{n-2s}{2}}(\frac{y-\xi}{\mu})$ is invariant under

Kelvin transform if and only if

$$|\xi|^2 + \mu^2 = 1.$$

Let k be a positive integer and define, for any $j = 1, \dots, k$, the k points

$$\xi_j = \sqrt{1-\mu^2}(e^{\frac{2\pi i(j-1)}{k}}, 0, \dots, 0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$$

where $\mu > 0$ is a positive number of the form

$$\mu = \frac{\delta}{k^2}, \quad \text{with} \quad c < \delta < c^{-1} \tag{3.8}$$

for a certain constant $c > 0$, independent of k, as $k \to \infty$. Define

$$U_*(y) = U(y) - \sum_{j=1}^{k} U_j(y), \tag{3.9}$$

where

$$U_j(y) = \mu^{-\frac{n-2s}{2}} U(\mu^{-1}(y - \xi_j)).$$

For large values of k, which at the same time makes the scaling parameters μ very small, we shall show that U_* is a good approximate solution for Problem (3.1). Observe that the function U_* satisfies the symmetry properties (3.6), (3.7). Furthermore, U_* is invariant under Kelvin transform

$$U_*(y) = |y|^{2-n} U_*(\frac{y}{|y|^2}).$$

This is consequence of a straighforward computation, using the fact that $\mu^2 + |\xi_j|^2 = 1$ for any $j = 1, \dots, k$.

We will show that Problem (3.1) admits a solution of the form

$$u(y) = U_*(y) + \phi(y)$$

where ϕ is small when compared with U_*, it satisfies the symmetry conditions (3.6), (3.7) and it is invariant under Kelvin transform. Then Equation (3.1) can be rewritten in terms of ϕ as

$$(-\Delta)^s\phi - p\gamma|U_*|^{p-1}\phi - E - \gamma N(\phi) = 0 \tag{3.10}$$

where E is

$$\gamma^{-1}E = |U - \sum U_j|^{p-1}(U - \sum U_j) - (U^p - \sum U_j^p) \tag{3.11}$$

and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1} - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi. \tag{3.12}$$

The size of the Error term E defined in (3.11) turns out to be relatively small, as the number k tends to infinity, when estimated with proper norms. Let us fix a number $q > \frac{n}{2s}$; we define the weighted L^q norm

$$\|h\|_{**} := \|(1 + |y|)^{n+2s-\frac{2n}{q}})h\|_{L^q(\mathbb{R}^n)}$$

Let $\eta > 0$ be a small and fixed number, independent of k. The error can be estimated separately in the *exterior region* $\cap_j\{|y - \xi_j| > \frac{\eta}{k}\}$ and then in each of the *inner regions* $\{|y - \xi_j| < \frac{\eta}{k}\}$. Indeed, we shall prove that there exists a constant C such that, for all k large enough,

$$\|(1 + |y|)^{n+2s-\frac{2n}{q}})E\|_{L^q(\cap_j\{|y-\xi_j|>\frac{\eta}{k}\})} \leq Ck^{1-\frac{n}{q}}. \tag{3.13}$$

Observe that, in order to have a small (in k) size for the error in the exterior domain, we need $q < n$. On the other hand, for regularity issue we will discuss later, we assume that $q > \frac{n}{2s}$. The set of possible values for q, $\frac{n}{2s} < q < n$, is not empty since we are considering s in the range $s \in (\frac{1}{2}, 1)$.

If we change scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}}E(\xi_j + \mu y)$, in $|y| < \dfrac{\eta}{\mu k}$, for any $j = 1, \ldots, k$, we have the following estimate for the error in each *interior domain*

$$\|(1 + |y|)^{n+2s-\frac{2n}{q}}\tilde{E}_j(y)\|_{L^q\{|y|<\frac{\eta}{k\mu}\}} \leq Ck^{-\frac{n}{q}}. \tag{3.14}$$

We shall prove the validity of estimates (3.13) and (3.14) at the end of this Section.

In order to solve in ϕ the non linear Equation (3.10), we use a *gluing method*. Let ζ be a cut-off function defined as follows: $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We also defined $\zeta^{-}(t) = \zeta(2t)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(k\eta^{-1}|y|^{-2}|(y - \xi_j|y|)|) & if \quad |y| > 1, \\ \zeta(k\eta^{-1}|y - \xi_j|) & if \quad |y| \leq 1. \end{cases}$$

Observe that

$$\zeta_j(y) = \zeta_j(|y|^{-2}y)$$

A function ϕ of the form

$$\phi = \sum_{j=1}^{k} \tilde{\phi}_j + \psi. \tag{3.15}$$

is a solution of the problem (3.10), provided that we can solve the following coupled system of elliptic equation in $(\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_k)$ and ψ:

$$(-\Delta)^s(\tilde{\phi}_j) - p\gamma|U_*|\zeta_j\tilde{\phi}_j \quad - \quad \zeta_j\left[p\gamma|U_*|^{p-1}\psi + E + \gamma N\left(\tilde{\phi}_j + \sum_{i\neq j}\tilde{\phi}_i + \psi\right)\right] = 0$$
$$j = 1, 2, \ldots, k \tag{3.16}$$

and

$$\begin{aligned}(-\Delta)^s\psi - p\gamma U^{p-1}\psi \quad &- \quad \left[p\gamma(|U_*|^{p-1} - U^{p-1})\left(1 - \sum_{j=1}^{k}\zeta_j\right) + p\gamma U^{p-1}\sum_{j=1}^{k}\zeta_j\right]\psi \\ &-p \quad \gamma|U_*|^{p-1}\sum_j(1-\zeta_j)\tilde{\phi}_j \qquad (3.17) \\ &- \quad \left(1 - \sum_{j=1}^{k}\zeta_j\right)\left(E + \gamma N\left(\sum_{j=1}^{k}\tilde{\phi}_j + \psi\right)\right) = 0\end{aligned}$$

To solve the above coupled system, we follow the following strategy. First we solve Problem

(3.17) in the unknown ψ, assuming that $\tilde{\phi}_j$ are fixed functions satisfying

$$\tilde{\phi}_j(\overline{y}, y') = \tilde{\phi}_1(e^{\frac{2\pi j}{k}\imath}\overline{y}, y'), \quad j = 1, 2. \dots, k-1, \tag{3.18}$$

$$\tilde{\phi}_1(y_1, y_2, \dots, y_j \dots, y_n) = \tilde{\phi}_1(y_1, y_2, \dots, -y_j, \dots, y_n) \quad j = 2, \dots, n \tag{3.19}$$

and the invariant condition under Kelvin's transform

$$\tilde{\phi}_1 = |y|^{2s-n}\tilde{\phi}_1(|y|^{-2}y). \tag{3.20}$$

Furthermore, we assume that

$$\|\phi_1\|_* < \rho \text{ where } \phi_1 = \mu^{\frac{n-2s}{2}}\tilde{\phi}_1(\xi_1 + \mu y). \tag{3.21}$$

We have the validity of the following result:

Proposition 3.2. *There exist constants k_0, C, ρ_0 such that for all $k \geq k_0$ the following holds: Let $\tilde{\phi}_j$ $j = 1, 2..., k$ satisfying conditions (3.18)-(3.19)-(3.20) and (3.21) with $\rho < \rho_0$. Then there exists a unique solution $\psi = \Psi(\phi_1)$ to Problem* (3.17) *that satisfies the symmetries*

$$\psi(\overline{y}, y') = \psi(e^{\frac{2\pi j}{k}\imath}\overline{y}, y'), \quad j = 1, 2. \dots, k-1,$$

$$\psi(\overline{y}, \dots, y_j \dots, y_n) = \psi(\overline{y}, \dots, -y_j, \dots, y_n), \quad j = 3, \dots, n$$

$$\psi = |y|^{2s-n}\psi(|y|^{-2}y) \quad \textit{and} \quad \|\psi\|_* \leq \frac{C}{k^{\frac{n}{q}-1}} + C\|\phi_1\|_*^2.$$

Moreover, the operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C\|\phi_1^1 - \phi_1^2\|_*.$$

Once we have the result of the above Proposition, under the assumption on $\tilde{\phi}_j$ we have that all equations (3.16) reduce to just one, say that for $\tilde{\phi}_1$. Then we will find a solution to our problem if we solve

$$(-\Delta)^s \tilde{\phi}_1 - p\gamma |U_1|^{p-1} \tilde{\phi}_1 - \zeta_1 E - \gamma \mathcal{N}(\phi_1) = 0 \quad \text{in } \mathbb{R}^n \tag{3.22}$$

where

$$\mathcal{N}(\phi_1) = p\left(|U_*|^{p-1}\zeta_1 - |U_1|^{p-1}\right)\phi_1 + \zeta_1\left[p|U_*|^{p-1}\Psi(\phi_1) + N(\tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1))\right]$$

Rather than solving (3.22) directly, we shall first solve the corresponding projected version of (3.22)

$$(-\Delta)^s \widetilde{\phi}_1 - p\gamma |U_1|^{p-1} \widetilde{\phi}_1 - \zeta_1 E + \gamma \mathcal{N}(\phi) = c_{n+1} \int_{\mathbb{R}^n} U_1^{p-1} \widetilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n \tag{3.23}$$

$$c_{n+1} = -\frac{\displaystyle\int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi)) \widetilde{Z}_{n+1}}{\displaystyle\int_{\mathbb{R}^n} U_1^{p-1} \widetilde{Z}_{n+1}^2}. \tag{3.24}$$

Proposition 3.3. *There exist constants k_0, C such that for all $k \geq k_0$ the following holds: Let $\Psi(\phi_1)$ the solution predicted by Proposition 3.2. Then there exists a unique solution $\phi_1 = \Phi(\delta)$, $c_{n+1} = c_{n+1}(\delta)$ to Problem (3.23)-(3.24), which depend continously on δ. Moreover, we have*

$$\|\Phi\|_* \leq Ck^{-\frac{n}{q}}, \quad \textit{and} \quad \|\mathcal{N}(\phi)\|_{**} \leq Ck^{-\frac{2n}{q}},$$

for some fixed positive constant C.

To conclude our argument, we shall show the existence of a number δ in the definition of μ in (3.8) so that the above constant c_{n+1} is equal to zero. In this way, we constructed a solution to Problem (3.1) with the qualitative properties predicted by Theorem 3.1.

Scheme of the paper. In Section 3.3 we prove some basic results on linear problems in $\mathbb{R}^n$. These results will be applied to prove Propositions 3.2 and 3.3 in Section 3.4. Section 3.5 is dedicated to show the existence of $\delta > 0$ so that $c_{n+1} = 0$, concluding in this way the proof of our Theorem.

We finish this Section with the proof of estimates (3.13) and (3.14).

Proof of (3.13). *In the exterior region* $\cap_j\{|y-\xi_j| > \frac{\eta}{k}\}$. For any y in this region, we have

$$|E(y)| \leq C\left[\frac{1}{(1+|y|^2)^{2s}} + \left|\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}\right|^{\frac{4s}{n-2s}}\right]\left(\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}\right),$$

for some positive constant $C>0$. Since for any j fixed and $|y-\xi_j| = \frac{\eta}{k}$ we have

$$\sum_{i=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_i|^{n-2s}} = \frac{1}{k^{n-2s}}k^{n-2s} + \sum_{i\neq j}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_i|^{n-2s}} \leq 1 + \frac{k-1}{ck^{n-2s}},$$

then we conclude that

$$|E| \leq C\frac{\mu^{\frac{n-2s}{2}}}{(1+|y|^2)^{2s}}\sum_{j=1}^{k}\frac{1}{|y-\xi_j|^{n-2s}}.$$

Thus a direct computation gives

$$\begin{aligned}
\|(1+|y|)^{n+2s-\frac{2n}{q}})E\|_{L^q(Ext)} &\leq C\mu^{\frac{n-2s}{2}}\left\|\frac{(1+|y|)^{n+2s-\frac{2n}{q}})}{(1+|y|^2)^{2s}}\sum_{j=1}^{k}\frac{1}{|y-\xi_j|^{n-2s}}\right\|_{L^q(Ext)} \\
&\leq C\mu^{\frac{n-2s}{2}}\sum_{j=1}^{k}\left(\int_{|y-\xi_j|>\frac{\eta}{k}}\frac{(1+|y|)^{(n+2s)q-2n}}{(1+|y|^2)^{2sq}}\frac{1}{|y-\xi_j|^{(n-2s)q}}dy\right)^{\frac{1}{q}} \\
&\leq C\mu^{\frac{n-2s}{2}}k\left(\int_{\frac{\eta}{k}}^{1}\frac{t^{n-1}}{t^{(n-2s)q}}dt\right)^{\frac{1}{q}} = C\mu^{\frac{n-2s}{2}}k\left(k^{(n-2s)q-n}-1\right)^{\frac{1}{q}} \\
&\leq C\mu^{\frac{n-2s}{2}}k^{(n-2s)+1-\frac{n}{q}}
\end{aligned}$$

Thus we get (3.13).

Proof of (3.14). *In the inner region* $|y-\xi_j| < \frac{\eta}{k}$*, for some j fixed.* Observe that, if y is close to ξ_j, we have that $U_j \sim O(\mu^{-\frac{n-2s}{2}})$. For any y in this region, there exits $t \in (0,1)$ such that

$$E = p\left(-U_j + t(-\sum_{i\neq j} U_j + U)\right)^{p-1}\left(-\sum_{i\neq j} U_j + U\right) - U^p + \sum_{i\neq j} U_j$$

We consider the change of scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y), \quad |y| < \dfrac{\eta}{\mu k}$. Therefore, we obtain that for some $t \in (0,1)$

$$\tilde{E}_j(y) = p\left(-U(y) + t\left(\sum_{i\neq j} U(y - \mu^{-1}(\xi_i - \xi_j)) + \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y)\right)\right)^{p-1} \times$$

$$\left(-\sum_{i\neq j} U(y - \mu^{-1}(\xi_i - \xi_j)) + \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y)\right) + \sum_{i\neq j} U^p(y - \mu^{-1}(\xi_i - \xi_j)) - \mu^{\frac{n+2s}{2}} U^p(\xi_j + \mu y).$$

Taking into account the configuration of the points ξ_j, we have that

$$|\xi_i - \xi_j| \sim \frac{|i-j|}{k};$$

Furthermore, for $i \neq j$ and $|y| < \frac{\eta}{k\mu}$,

$$U(y - \mu^{-1}(\xi_i - \xi_j)) \leq C\frac{\mu^{n-2s}}{|\xi_j - \xi_i|^{n-2s}}\left(\frac{|\xi_j - \xi_i|^2}{\mu^2 + |\mu y - (\xi_j - \xi_i)|^2}\right)^{\frac{n-2s}{2}} \leq C\frac{\mu^{n-2s} k^{n-2s}}{|i-j|^{n-2s}};$$

moreover

$$\left|\sum_{i\neq j} U(y - \mu^{-1}(\xi_i - \xi_j)\right| \leq C k^{n-2s}\mu^{n-2s}, \quad \text{and} \quad \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y)) \leq C\mu^{\frac{n-2s}{2}}.$$

for some constant $C > 0$. Thus we conclude that

$$|\tilde{E}_j(y)| \leq C\left[\frac{k^{n-2s}\mu^{n-2s}}{1 + |y|^{4s}} + \mu^{\frac{n+2s}{2}}\right],$$

and we have an estimate of the error in the inner region

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\tilde{E}_j(y)\right\|_{L^q\left\{|y|<\frac{\eta}{k\mu}\right\}}\leq C\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left[\frac{k^{n-2s}\mu^{n-2s}}{1+|y|^{4s}}+\mu^{\frac{n+2s}{2}}\right]\right\|_{L^q\left\{|y|<\frac{\eta}{k\mu}\right\}}.$$

Since

$$\begin{aligned}\|(1+|y|)^{n-2s-\frac{2n}{q}}\|^q_{L^q\left\{|y|<\frac{\eta}{k\mu}\right\}} &\leq C\int_0^{\frac{\eta}{k\mu}}(1+r)^{(n-2s)q-n-1}dr\\ &\leq C\left(\frac{1}{k\mu}\right)^{(n-2s)q-n}\end{aligned}$$

and

$$\begin{aligned}\|(1+|y|)^{n+2s-\frac{2n}{q}}\|^q_{L^q\left\{|y|<\frac{\eta}{k\mu}\right\}} &\leq C\int_0^{\frac{\eta}{k\mu}}(1+r)^{(n+2s)q-n-1}dr\\ &\leq C\left(\frac{1}{k\mu}\right)^{(n+2s)q-n}\end{aligned}$$

it follows that

$$\|(1+|y|)^{n+2s-\frac{2n}{q}}\tilde{E}_j(y)\|_{L^q\left\{|y|<\frac{\eta}{k\mu}\right\}}\leq Ck^{-\frac{n}{q}}(1+k^{-4s}).$$

This gives the proof of (3.14).

3.3 Some linear problems

Let L_0 be the linear operator defined by

$$L_0(\phi):=(-\Delta)^s(\phi)-p\,\gamma U^{p-1}\phi\ \text{ in }\mathbb{R}^n.$$

As we know from [11], the set of bounded solutions of the homogeneous equation $L_0(\phi)=0$ is spanned by the $n+1$ functions defined by

$$Z_i=\partial_{x_i}U\quad i=1,...,n,\quad Z_{n+1}=\frac{n-2s}{2}U+x\cdot\nabla U.$$

We now establish a solvability result for the linear problem

$$L_0(\phi) = h \quad \text{in} \quad \mathbb{R}^n,$$

under proper orthogonality conditions on h and ϕ. For this purpose, we introduce the norm

$$\|\phi\|_* := \|(1 + |y|^{n-2s})\phi\|_\infty. \tag{3.25}$$

We have the validity of the following

Lemma 3.4. *Assume $q \in (\frac{n}{2s}, \frac{n}{s})$. Let h be such that $\|h\|_{**} < \infty$ and*

$$\int_{\mathbb{R}^n} U^{p-1} Z_l h \, dx = 0 \quad \textit{for all } l = 1, 2, \ldots, n+1$$

Then the equation

$$(-\Delta)^s \phi - pU^{p-1}\phi = h \quad \textit{in } \mathbb{R}^n \tag{3.26}$$

has a unique solution ϕ with $\|\phi\|_* < +\infty$ such that

$$\int_{\mathbb{R}^n} U^{p-1} Z_l \phi \, dx = 0 \quad \textit{for all } l = 1, 2, \ldots, n+1.$$

Furthermore, there exists a positive constant C depending only on q, s and n such that

$$\|\phi\|_* \leq C\|h\|_{**}. \tag{3.27}$$

Proof. Let H^s be the completion of $C_0^\infty(\mathbb{R}^n)$ equipped with the norm

$$\|\phi\|_{H^s} = \sqrt{\int_{\mathbb{R}^n} |\phi|^2 + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dx\, dy}$$

$(H^s, \langle , \rangle_{H^s})$ is a Hilbert space with the product

$$\langle f, g\rangle_{H^s} = \int_{\mathbb{R}^{2n}} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+2s}} dx\, dy.$$

Let us consider the subspace

$$H = \{\phi \in H^s(\mathbb{R}^n) \text{ such that } \int U^{p-1} Z_l \phi dx = 0, \quad l = 1, 2, \dots, n+1\}.$$

We consider the problem of finding $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \varphi\, dx - p\gamma \int_{\mathbb{R}^n} U^{p-1} \phi \varphi + \int_{\mathbb{R}^n} h\varphi = 0 \text{ for all } \varphi \in H;$$

this variational formulation makes sense if we consider for instance $h \in L^{\frac{2n}{n+2s}}$, since $H^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ continuously, (see for instance [18]).

Let $f \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$. By Riesz's theorem there exist a unique $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \varphi\, dx + \int_{\mathbb{R}^n} f\varphi\, dx = 0 \text{ for all } \varphi \in H.$$

Thus $A(f) = \phi$ defines a linear operator between $L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$ and H. By the local compactness of Sobolev embeddings [18] and the decay at infinity of U^{p-1}, we have that the map $\phi \in H \to U^{p-1}\phi \in L^{\frac{2n}{n+2s}}$ is compact. Hence, Fredholm's alternative applies to problem

$$\phi - A(p\gamma U^{p-1}\phi) = A(h). \tag{3.28}$$

For $h = 0$, we have $L_0(\phi) = 0$ and $\phi \in H$. Thus $(-\Delta)^s \phi = pU^{p-1}\phi$ in $\mathbb{R}^n$ and hence

$$\phi(x) = \sigma_{n,s}\, p\, \gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi(y)}{|x-y|^{n-2s}},$$

for some explicit positive constant $\sigma_{n,s}$. We claim that ϕ is bounded. Indeed, let $\delta > 0$ be a fixed positive small number and write

$$\int_{\mathbb{R}^n} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} = \int\limits_{|x-y|<\delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} + \int\limits_{|x-y|>\delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} := I_1 + I_2. \tag{3.29}$$

We have

$$I_1 \leq C\|\phi\|_\infty \int\limits_{|x-y|<\delta} \frac{1}{|x-y|^{n-2s}}\, dy \leq C\delta^{2s}\|\phi\|_\infty \tag{3.30}$$

and, using repeatedly Holder inequality

$$\begin{aligned} I_2 &\leq \left(\int_{|x-y|>\delta} \left(\frac{1}{|x-y|^{n-2s}}\right)^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{2n}} \left(\int_{|x-y|>\delta} \left(U^{p-1}\phi\right)^{\frac{2n}{n+2s}}\right)^{\frac{n+2s}{2n}} \\ &\leq C\left(\int_{|x-y|>\delta} \phi^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{2n}} \left(\int_{|x-y|>\delta} U^{(p-1)\frac{2n}{4s}}\right)^{\frac{4s}{2n}} \leq C\|\phi\|_{L^{\frac{2n}{n-2s}}} \end{aligned}$$

Chosing δ properly small, we obtain that ϕ is bounded. We can now apply the result in [11] and conclude that ϕ is a linear combination of the functions Z_l, $l = 1, \dots, n+1$. Since $\phi \in H$ we have that $\phi \equiv 0$. Fredholm's alternative implies that, for any h satisfying the orthogonality condition, a function $\phi \in H$ solution to (3.28) exists.

Assume now that ϕ solves (3.26), we shall now show the a-priori bound (3.27). We first show that ϕ is bounded. First we have

$$\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq \|\phi\|_{H^s(\mathbb{R}^n)} \leq \|h\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} \leq \|(1+|y|)^{n+2s-\frac{2n}{q}} h\|_{L^q(\mathbb{R}^n)}. \tag{3.31}$$

Observe now that $\phi(x) = \sigma_{n,s}\, p \displaystyle\int_{\mathbb{R}^n} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}}$. Fixing a small $\delta > 0$, we get

$$\int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}}\, dy = \int_{|x-y|<\delta} \frac{h(y)}{|x-y|^{n-2s}}\, dy + \int_{|x-y|<\delta} \frac{h(y)}{|x-y|^{n-2s}}\, dy = J_1 + J_2$$

with

$$J_1 \leq \int_{|x-y|<\delta} \left(\frac{1}{|x-y|^{(n-2s)q'}} \right)^{q'} \|h\|_{L^q(\mathbb{R}^n)} \leq C\|h\|_{L^q(\mathbb{R}^n)} \quad \text{since } q > \frac{n}{2s}, \quad \text{and} \tag{3.32}$$

$$J_2 \leq \left(\int_{|x-y|>\delta} \frac{1}{|x-y|^{2n}} \right)^{\frac{n-2s}{2n}} \|h\|_{L^{\frac{2n}{n+2s}}} \leq C\|h\|_{L^{\frac{2n}{n+2s}}}. \tag{3.33}$$

Thus, thanks also to (3.29) and (3.30), for all $x \in \mathbb{R}^n$

$$|\phi(x)| \leq C\delta^{2s}\|\phi\|_\infty + C\left(\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{\frac{2n}{n+2s}}} \right).$$

Chosing δ small, we conclude that ϕ is bounded since

$$\|\phi\|_\infty \leq C\left(\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{\frac{2n}{n+2s}}} \right). \tag{3.34}$$

Next we show the decay rate at infinity of ϕ. Consider

$$\tilde{\phi}(y) = |y|^{2s-n}\phi(|y|^{-2}y) \quad \text{and} \quad \tilde{h}(y) = |y|^{-n-2s}h(|y|^{-2}y).$$

A direct computation shows that

$$(-\Delta)^s\tilde{\phi} - p\gamma U^{p-1}(y)\tilde{\phi} = \tilde{h} \text{ on } \mathbb{R}^n \setminus \{0\},$$

and

$$\|\tilde{\phi}\|_{H^s(\mathbb{R}^n)} + \|\tilde{\phi}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} = \|\phi\|_{H^s(\mathbb{R}^n)} + \|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)},$$

$$\|\tilde{h}\|_{L^q(\mathbb{R}^n)} = \|(1+|y|)^{n+2s-\frac{2n}{q}}h\|_{L^q(\mathbb{R}^n)} = \|h\|_{**}.$$

Applying the estimate (3.34) to to $\widetilde{\phi}$ we get

$$\|\widetilde{\phi}\|_{L^\infty(B(0,1))} \le \|\tilde{\phi}\|_{L^\infty(\mathbb{R}^n)} \le C\left(\|\widetilde{\phi}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|\widetilde{h}\|_{L^q(\mathbb{R}^n)} + \|\widetilde{h}\|_{L^{\frac{2n}{n+2s}}}\right)$$

$$\le C\left(\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|\widetilde{h}\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{\frac{2n}{n+2s}}}\right)$$
$$\le C(\|h\|_{**} + \|\widetilde{h}\|_{L^q(\mathbb{R}^n)}) = C\|h\|_{**}.$$

Since $\||y|^{n-2s}\phi\|_{L^\infty\{|y|>1\}} = \|\widetilde{\phi}\|_{L^\infty(B(0,1))}$, we conclude that $\|\phi\|_* \le C\|h\|_{**}$.

□

Under further symmetry conditions on h and ϕ, Problem (3.26) can be solved without the orthogonality conditions. For a general function ψ defined in $\mathbb{R}^n$, consider the following symmetries

$$\psi(\overline{y}, y') = \psi(e^{\frac{2\pi j}{k}\imath}\overline{y}, y'), \quad j = 1, 2 \ldots, k-1, \tag{3.35}$$

and

$$\psi(\overline{y}, \ldots, y_j \ldots, y_n) = \psi(\overline{y}, \ldots, -y_j, \ldots, y_n)\ j = 3, \ldots, n. \tag{3.36}$$

and it is invariant under Kelvin transform

$$\psi(y) = |y|^{2s-n}\psi(|y|^{-2}\, y). \tag{3.37}$$

We have the validity of the following result

Lemma 3.5. *Assume that h satisfies (3.35), (3.36), and $\|h\|_{**} < \infty$. Furthermore, we assume that*

$$h(y) = |y|^{-n-2s}h(|y|^{-2}y).$$

Then Equation (3.26) has a unique bounded solution $\phi = T(h)$ that satisfies symmetries

(3.35),(3.36),(3.37). *Moreover there exits C depending only on q, s and n such that*

$$\|\phi\|_* \leq \|h\|_{**}$$

The proof of this result is very close to the proof of (4.19) in [14]. We refer the interested reader to [14].

For a later purpose, we need to establish a result like the one in Lemma 3.4 for a linear operator more general then L_0.

Lemma 3.6. *Let $2s < \nu < n$. There exists numbers δ, C, depending on ν, n such that the following holds: If g, a and ϕ are functions such that $\|(1+|y|^{\nu})g\|_\infty < +\infty$, $\|(1+|y|^{\nu-2s})\phi\|_\infty < +\infty$ and $\|(1+|y|^{2s})a\|_\infty < \delta$, and*

$$L_0(\phi) + a(y)\phi = g(y) + \sum_{l=1}^{n+1} c_l U^{p-1} Z_l \quad \text{in } \mathbb{R}^n \tag{3.38}$$

where

$$\int_{\mathbb{R}^n} U^{p-1} Z_l \phi = 0 \quad \text{for all } l = 1, ..., n+1 \tag{3.39}$$

and

$$c_l \int_{\mathbb{R}^n} U^{p-1} Z_l^2 = \int_{R^n} (a(y)\phi - g(y)) Z_l \phi \quad \text{for all } l = 1, ...n+1, \tag{3.40}$$

then

$$\|(1+|y|^{\nu-2s})\phi\|_\infty \leq C\|(1+|y|^{\nu})g\|_\infty. \tag{3.41}$$

Proof. By contradiction, let us assume the existence of functions ϕ_n, a_n, g_n and constants c_l^n such that (3.38)-(3.40) hold, and

$$\|(1+|y|^{\nu})g_n\|_\infty \to 0, \quad \|(1+|y|^{\nu-2s})\phi_n\|_\infty = 1, \quad \|(1+|y|^{2s})a_n\|_\infty \to 0. \tag{3.42}$$

Clearly we have that $\|(1+|y|^{\nu})a_n g_n\|_\infty \to 0$ and also that $c_l^n \to 0$, so with no loss of generality we may assume that $a_n \equiv 0$, and $c_l^n = 0$. We claim first that

$$\|\phi_n\|_\infty \to 0.$$

Assume the opposite: there are numbers $\gamma, R > 0$ and points x_n such that

$$|\phi_n(x_n)| \geq \gamma, \quad |x_n| \leq R.$$

Passing to subsequence, and arguing like in the proof of Lemma 3.4, we find that ϕ_n converges in the energy space and locally uniformly over compact sets to a bounded function $\phi_0 \neq 0$ with

$$L_0(\phi_0) = 0, \quad \text{and} \quad \int_{\mathbb{R}^n} U^{p-1}\phi Z_l = 0, \quad \text{for all } l$$

which gives $\phi_0 = 0$. This is a contradiction due to the result in [11]. Thus we have that $\|\phi_n\|_\infty \to 0$.

Next we shall show that $\|(1 + |y|^{\nu-2s})\phi_n\|_\infty \to 0$, thus getting to a contradiction with (3.42), and the proof of the Lemma. Using the equation, we have that

$$\phi_n(x) = \sigma_{n,s}\, p\, \gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi_n(y)}{|x-y|^{n-2s}}\, dy + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{g_n(y)}{|x-y|^{n-2s}}\, dy \tag{3.43}$$

for some explicit positive constant $\sigma_{n,s}$. Since $2s < \nu < n$, and taking into account that $\|(1+|y|^\nu)g_n\|_\infty \to 0$, and the behavior of U^{p-1} at infinity, there exists a positive constant C, independent of n, such that

$$|\phi_n(x)| \leq C \left[\frac{\|\phi_n\|_\infty}{(1+|x|^{2s})} + \frac{o(1)}{(1+|x|^{\nu-2s})} \right]$$

for some $o(1) \to 0$, as $n \to \infty$. Replacing the above estimate in (3.43) and repeating the same procedure a finite number of time, we get that $|\phi_n(x)| \leq C \frac{\|\phi_n\|_\infty + o(1)}{(1+|x|^{\nu-2s})}$. This concludes the proof of the Lemma. □

3.4 Proof of Propositions 3.2 and 3.3

Proof of Proposition 3.2. Let us fix functions $\tilde{\phi}_j$ and we assume that they satisfy the symmetry assumptions (3.6), (3.7) and the invariance under Kelvin transform

$$\tilde{\phi}_1 = |y|^{2s-n}\tilde{\phi}_1(|y|^{-2}y).$$

Finally we assume

$$\|\phi_1\|_* < \rho \text{ where } \phi_1 = \mu^{\frac{n-2s}{2}}\tilde{\phi}_1(\xi_1 + \mu y). \tag{3.44}$$

for a small fixed $\rho > 0$.

We next solve equation (3.17). To do so, we write it in the form

$$(-\Delta)^s(\psi) - p\gamma U^{p-1}(y)\psi - \underbrace{\gamma V(y)\psi - p\gamma|U_*|^{p-1}\sum_{j=1}^{k}(1-\zeta_j)\tilde{\phi}_j - M(\psi)}_{:=h} = 0$$

where

$$V(y) := \underbrace{p(|U_*|^{p-1} - U^{p-1})\left(1 - \sum_{j=1}^{k}\zeta_j\right)}_{:=V_1} + \underbrace{pU^{p-1}\sum_{j=1}^{k}\zeta_j}_{:=V_2} := V_1 + V_2$$

and

$$M(\psi) := \left(1 - \sum_{j=1}^{k}\zeta_j\right)\left(E + \gamma N\left(\sum_{j=1}^{k}\tilde{\phi}_j + \psi\right)\right)$$

A basic observation is that the function h as defined above satisfies the conditions (3.35), (3.36), and $\|h\|_{**} < \infty$. Furthermore, we have that

$$h(y) = |y|^{-n-2s}h(|y|^{-2}y).$$

Hence, we can define the linear operator T in the Lemma 3.5 and we can write our problem

(3.17) in fixed point as

$$\psi = -T\left(V\psi + p\gamma|U_*|^{p-1}\sum_j (1-\zeta)\tilde{\phi}_j + M(\psi)\right) =: \mathcal{M}(\psi) \tag{3.45}$$

We notice that $\mathcal{M}$ is well defined in space X of continuous functions ψ with $\|\psi\|_* \leq \infty$, and satisfying

$$\psi(\overline{y}, y') = \psi(e^{\frac{2\pi j}{k}\imath}\overline{y}, y'), \quad j = 1, 2. \dots, k-1,$$

$$\psi(\overline{y}, \dots, y_j \dots, y_n) = \psi(\overline{y}, \dots, -y_j, \dots, y_n), \quad j = 3, \dots, n$$

$$\psi = |y|^{2s-n}\psi(|y|^{-2}y).$$

We claim that

$$\|V\psi(y)\|_{**} \leq Ck^{1-\frac{n}{q}}\|\psi\|_* \tag{3.46}$$

and

$$\|p\gamma|U_*|^{p-1}\sum_{j=1}^{k}(1-\zeta_j)\tilde{\phi}_j\|_{**} \leq Ck^{1-\frac{n}{q}}\|\psi\|_*. \tag{3.47}$$

We claim that, if

$$\|\psi\|_* + \|\phi_1\|_* \leq 2\rho,$$

then

$$\|M(\psi)\|_{**} \leq C\left[k^{1-\frac{n}{q}} + k^{1-\frac{n}{q}}\|\phi_1\|_*^2 + \|\psi\|_*^2\right]. \tag{3.48}$$

Furthermore, for ψ_1, ψ_2 in X, we have

$$\|M(\psi_1) - M(\psi_2)\|_{**} \leq C\rho\|\psi_1 - \psi_2\|_*$$

We can thus conclude that, for ρ small enough, the operator $\mathcal{M}$ defines a contraction map in the set of functions $\psi \in X$ with

$$\|\psi\|_* \leq C[\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}]. \tag{3.49}$$

From the estimate (3.49) we get the Lipschitz dependence

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C\|\phi_1^1 - \phi_1^2\|_*.$$

This concludes the proof of Proposition 3.2.

We shall next show the validity of (3.46), (3.47) and (3.48).

Proof of (3.46). If we consider $f(t) = \left|U - t\sum_{j=1}^{k} U_j\right|^{p-1}$, by the mean value theorem we get

$$|V_1| \leq p(p-1)\left|U - s\sum_{j=1}^{k} U_j\right|^{p-2}\left(\sum_{j=1}^{k} U_j\right) \leq CU^{p-2}\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}.$$

Thus, if for all j, $|y-\xi_j| > \frac{\eta}{k}$, then we have

$$|V_1\psi(y)| \leq C\|\psi\|_* U^{p-1}(y)\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}.$$

Since $\zeta_j \equiv 1$ on $|y-\xi_j| < \frac{\eta}{k}$,

$$\begin{aligned}
\left\|(1+|y|)^{n+2s-\frac{2n}{q}}V_1\psi\right\|_{L^q(\mathbb{R}^n)} &= \left\|(1+|y|)^{n+2s-\frac{2n}{q}}V_1\psi\right\|_{L^q(\mathbb{R}^n\setminus\bigcup_j B(\xi_j,\frac{\eta}{k}))} \\
&\leq Ck\left[\int_{B(\xi_1,\frac{\eta}{k})^c\cap B(0,2)}\frac{\mu^{\frac{(n-2s)q}{2}}}{|y-\xi_j|^{(n-2s)q}}\,dy\right]^{\frac{1}{q}}\|\psi\|_* \\
&\leq Ck\mu^{\frac{n-2s}{2}}k^{(n-2s)-\frac{n}{q}}\|\psi\|_*,
\end{aligned}$$

for some positive constant C. Thus $\|V_1\psi(y)\|_{**} \leq C\,k^{1-\frac{n}{q}}\,\|\psi\|_*$. On the other hand,

$$\|V_2\psi\|_{**} = \|(1+|y|)^{n+2s-\frac{2n}{q}} pU^{p-1}\sum_{j=1}^{k}\zeta_j\psi\|_{L^q(\mathbb{R}^n)}$$

$$\leq C\left[\int_{B(0,1)}\left((1+|y|)^{n+2s-\frac{2n}{q}}U^{p-1}\sum_{j=1}^{k}\zeta_j\psi\right)^q dy\right]^{\frac{1}{q}}$$

with

$$\left[\int_{B(0,1)}\left((1+|y|)^{n+2s-\frac{2n}{q}}U^{p-1}\sum_{j=1}^{k}\zeta_j\psi\right)^q dy\right]^{\frac{1}{q}} \leq C\sum_j\left[\int_{B(\xi_j,\frac{2\eta}{k})}\frac{U^{(p-1)q}(1+|y|)^{(n+2s)q-2n}}{(1+|y|)^{(n-2s)q}}dy\right]^{\frac{1}{q}}\|\psi\|_*$$
$$\leq Ck^{1-n}\|\psi\|_*$$

Thus we get the validity of (3.46).

Proof of (3.47). Estimate (3.47) can be obtained arguing as in the proof of estimate (3.46), after noticing that

$$|\tilde{\phi}_j(y)| \leq CU(y)\|\phi_1\|_*\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi|^{n-2s}}.$$

Proof of (3.48). For the moment we shall assume that

$$\|\psi\|_* + \|\phi_1\|_* \leq 2\rho$$

for a ρ sufficiently small. Let us assume that $|y-\xi_j| > \dfrac{\eta}{k}$ for all j. First we recall that

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left(1-\sum_{j=1}^{k}\zeta_j\right)E\right\|_{L^q(\mathbb{R}^n)} = \left\|(1+|y|)^{n+2s-\frac{2n}{q}}E\right\|_{L^q(ext)} \leq Ck^{1-\frac{n}{q}}$$

Then we find in this region

$$\left|N\left(\sum_{j=1}^{k}\tilde{\phi}_j+\psi\right)\right|\leq CU^{p-2}\left(\left|\sum_{j=1}^{k}\widetilde{\phi}_j\right|^2+|\psi|^2\right).$$

But

$$U^{p-2}\left|\sum_{j=1}^{k}\tilde{\phi}_j\right|^2\leq C\|\phi_1\|_*^2U^p\sum_{j=1}^{k}\frac{\mu^{n-2s}}{|y-\xi_j|^{2(n-2s)}},\quad U^{p-2}|\psi|^2\leq U^p\|\psi\|_*^2$$

Thus, we have

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left(1-\sum_{j=1}^{k}\zeta_j\right)\left(\gamma N\left(\sum_{j=1}^{k}\tilde{\phi}_j+\psi\right)\right)\right\|_{L^q(\mathbb{R}^n)}=\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left(\gamma N\left(\phi\right)\right)\right\|_{L^q(ext)}$$

$$\leq C\|\phi_1\|_*^2\left\|(1+|y|)^{n+2s-\frac{2n}{q}}U^p\left(\sum_{j=1}^{k}\frac{\mu^{n-2s}}{|y-\xi_j|^{2(n-2s)}}+\psi\right)\right\|_{L^q(ext)}\leq\frac{C\mu^{n-2s}}{k^{-2(n-2s)+\frac{n}{q}-1}}\|\phi_1\|_*^2+C\|\psi\|_*^2$$

Using the above inequalities we get

$$\|M(\psi)\|_{**}\leq Ck^{1-\frac{n}{q}}+k^{1-\frac{n}{q}}\|\phi_1\|_*^2+C\|\psi\|_*^2,$$

that is the validity of (3.48).

This concludes the proof of Proposition 3.2.

Proof of Proposition 3.3. In order to prove Proposition 3.3, we need to consider the linear problem

$$(-\Delta)^s\tilde{\phi}_1-p\gamma U_1^{p-1}\tilde{\phi}-\tilde{h}(y)=c_{n+1}U_1^{p-1}\tilde{Z}_{n+1}\text{ in }\mathbb{R}^n \tag{3.50}$$

for a general function $\tilde{h}$, where

$$\tilde{Z}_{n+1}(y)=\mu^{-\frac{n-2s}{2}}Z_{n+1}(\mu^{-1}(y-\xi_1))\quad\text{and}\quad c_{n+1}=\frac{\int_{\mathbb{R}^n}\widetilde{h}\widetilde{Z}_{n+1}}{\int_{\mathbb{R}^n}U_1^{p-1}\widetilde{Z}_{n+1}^2}.$$

Lemma 3.7. *Assume that $\widetilde{h}$ is even with respect to each variable $y_2, ...y_n$ and it satisfies the invariance*

$$\tilde{h}(y) = |y|^{-n-2s} h(|y|^{-2}y)$$

Assume in addition that

$$h(y) = \mu^{\frac{n+2s}{2}} \widetilde{h}(\xi_1 + \mu y)$$

satisfies $\|h\|_{**} \leq \infty$. Then the equation (3.50) has a unique solution $\widetilde{\phi} := \widetilde{T}(\widetilde{h})$ that is even with respect to each of the variables $y_2, ..., y_n$, invariant under Kelvin's transformations

$$\widetilde{\phi}(y) = |y|^{2s-n} \widetilde{\phi}(|y|^{-2}y)$$

where $\phi(y) = \mu^{\frac{n-2s}{2}} \widetilde{\phi}(\xi_1 + \mu y)$ and satisfies

$$\int_{\mathbb{R}^n} \phi U^{p-1} Z_{n+1} = 0.$$

Moreover, there exits C such that

$$\|\phi\|_* \leq C \|h\|_{**}.$$

Proof. We consider ϕ and h such that

$$(-\Delta)^s \phi - p\gamma |U|^{p-1} \phi = h(y) \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \widetilde{h} \widetilde{Z}_{n+1} = 0.$$

The evenness of h in the last $(n-1)$ coordinates guarantees that we have

$$\int_{\mathbb{R}^n} h Z_l = 0, \quad l = 2, ..., n, \; l = n+1.$$

We have that to prove that $\int_{\mathbb{R}^n} hZ_1 = 0$. Let

$$I(t) = \int_{\mathbb{R}^n} w_\mu(y - t\xi_1)\widetilde{h}(y)\, dy.$$

We notice that

$$(\xi_1)_1 \int_{\mathbb{R}^n} hZ_1 = \partial_t I(t)\Big|_{t=0} = -(\xi_1)_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(y - \xi_1)h(y)\, dy; \tag{3.51}$$

after a change of variable

$$I(t) = \int_{\mathbb{R}^n} w_\mu(|y|^{-2}y - t\xi_1)\widetilde{h}(|y|^{-2}y)|y|^{-2n} = \int_{\mathbb{R}^n} w_{\mu(t)}(y - a(t)\xi_1)\widetilde{h}(y)\, dy$$

where

$$\mu(t) = \frac{\mu t}{\mu^2 + |\xi_1|2t^2} \quad s(t) = \frac{t}{\mu^2 + |\xi_1|^2 t^2}.$$

Hence

$$\partial_t I(t)|_{t=1} = \mu'(1) \int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\Big|_{\mu=1} \widetilde{h}(y)dy - s'(1)\xi_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(t)(y - \xi_1)h(y)dy = 0. \tag{3.52}$$

We can check that

$$\int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\Big|_{\mu=1} \widetilde{h}(y)dy = \int_{\mathbb{R}^n} Z_{n+1}(y)h(y)dy = 0$$

and $s'(1) = 1 - 2|\xi_1|^2$. Hence, using (3.51)-(3.52) we obtain $\int_{\mathbb{R}^n} hZ_1 = 0$. It follows from Lemma 3.4 that there exist a unique solution ϕ_1 for equation (3.50) with

$$\int_{\mathbb{R}^n} hZ_l = 0 \quad l = 1, ...n + 1 \quad \text{and} \quad \|\phi\|_* \le C\|h\|_{**}$$

Arguing by uniqueness, as in proof of Lemma 3.5, we find that $\widetilde{\phi}$ satisfies the corresponding symmetries. $\square$

We use the above lemma to solve (3.23)-(3.24) We consider the operator $\widetilde{T}$ defined in the above Lemma. We are going to prove the existence of a solution to equation (3.23) by a fixed point argument

$$\widetilde{\phi}_1 = \widetilde{T}(\zeta_1 + \gamma \mathcal{N}(\phi_1)) =: \mathcal{M}(\phi_1). \tag{3.53}$$

For any f we set $\overline{f}(y) = \mu^{\frac{n+2s}{2}} f(\xi + \mu y)$. Let

$$f_1(y) = p\zeta_1(|U_*|^{p-1} - |U_1|^{p-1})\widetilde{\phi}_1.$$

For $|y| < \frac{\eta}{k\mu}$ we have

$$|\overline{f}_1(y)| \le C \left(\mu^{n-2s} k^{n-2s} \sum_{j=1}^{k-1} \frac{1}{j^{n-2s}} + \mu^{\frac{n-2s}{2}} \right) U^{p-1} \|\phi_1\|_*$$

and so

$$\|\overline{f}_1(y)\|_{**} \le C \left(\mu^{n-2s} k^{n-2s} + \mu^{\frac{n-2s}{2}} \right) (\mu k)^{-n+2s+\frac{n}{q}} \|\phi_1\|_* = C\mu^{\frac{2n}{q}} \|\phi_1\|_*.$$

Analogously for $f_2 = (\zeta_1 - 1)U_1^{p-1}\widetilde{\phi}_1$ in the region $|y| < \dfrac{\eta}{\mu k}$

$$|\overline{f}_2(y)| \le U^p \|\phi_1\|_*$$

hence we find $\|\overline{f}_2\|_{**} \le Ck^{-\frac{n}{q}} \|\phi_1\|_*$. Now we consider $f_3 = \zeta_1 p |U_*|^{p-1} \Psi(\phi_1)$ on $|y| < \frac{n}{\mu k}$,

$$|\overline{f}_3| \le CU^{p-1} \mu^{\frac{n-2s}{2}} \|\Psi(\phi_1)\|_\infty \le CU^{p-1} \mu^{\frac{n-2s}{2}} (\|\phi_1\|_* + k^{1-\frac{n}{q}});$$

thus

$$\|\overline{f}_3(y)\|_{**} \le C\mu^{\frac{n}{2q}} (\|\phi_1\|_* + k^{1-\frac{n}{q}}).$$

Now, for

$$f_4 = \zeta_1 N(\widetilde{\phi}_1 + \sum_{i=2} \widetilde{\phi}_i)\Psi(\phi_1)$$

we notice that

$$\overline{N}(\phi) = (V_* + \hat{\phi})^p - V_*^p - pV_*^{p-1}\hat{\phi}$$

where $\hat{\phi}(y) := \mu^{\frac{n-2s}{2}}\phi(\xi_1 + \mu y)$ and

$$V_*(y) = U(y) + \sum_{i=2}^{k} U(y + \mu^{-1}(\xi_1 - \xi_j)) - \mu^{\frac{n-2s}{2}} U(\xi_1 + \mu y)$$

with

$$\phi = \widetilde{\phi}_1 + \sum_{i=2}^{k} \widetilde{\phi}_i + \Psi(\phi_1)$$

therefore

$$|\overline{f}_4| \leq C[U^{p-1}\mu^{\frac{n-2s}{2}}\|\phi_1\|_* + U^{p-1}\mu^{\frac{n-2s}{2}}(\|\phi_1\|_* + k^{1-\frac{n}{q}})],$$

and hence

$$\|\overline{f}_4\|_{**} \leq C\left[\mu^{\frac{n}{2q}}\|\phi_1\|_* + \mu^{\frac{n}{2q}}[\|\phi_1\|_* + k^{1-\frac{n}{q}}]^2\right].$$

Concerning $f_5 = \zeta_1 E$, we recall that

$$\|\overline{f}_5\|_{**} \leq C\mu^{\frac{n}{2q}}.$$

The above estimates suggest that it is possible to perform a fixed point argument of contraction type in the set of all continuous functions $\phi_1 = \Phi(\delta)$ such that $\|\phi_1\|_* \leq C\mu^{\frac{n}{2q}}$. This gives the existence and the estimate for ϕ_1, satisfying

$$\|\Phi\|_* \leq Ck^{-\frac{n}{q}},$$

and

$$\mathcal{N}(\phi)\|_{**} \leq Ck^{-\frac{2n}{q}}.$$

Starighforward computations shows also the continuous dependence of $\phi_1 = \Phi(\delta)$ and c_{n+1} on the parameter δ. This concludes the proof of Proposition 3.3.

3.5 Conclusion

In this secction we show the existence of $\delta > 0$ such that $c_{n+1}(\delta) = 0$ in (3.23). Indeed this fact gives that the function

$$U_* + \phi,$$

where $U_* = U - \sum U_j$ is defined in (3.9) and $\phi = \sum_{j=1}^{k} \widetilde{\phi}_j + \psi$ is defined in (3.15), is a solution for the original problem (3.1). Let $\widetilde{Z}_{n+1} = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y - \xi_1))$. We recall that

$$Z_{n+1}(y) = y \cdot \nabla U + \frac{n-2s}{2} U$$

We need the existence of a δ such that

$$c_{n+1} = \int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi_1))\, \widetilde{Z}_{n+1} = 0. \tag{3.54}$$

Since we are assuming that $s > \frac{1}{2}$, we claim that

$$\int_{\mathbb{R}^n} \zeta_1 E\, \widetilde{Z}_{n+1} = A\, \delta\, k^{2s-n} \left[-2^{\frac{n-2s}{2}} (\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}})\, \delta + 1 \right] + k^{1-n} \Theta_k(\delta) \tag{3.55}$$

and

$$\int_{\mathbb{R}^n} \gamma \mathcal{N}(\phi_1) \widetilde{Z}_{n+1} = k^{-(n-2s)}\, k^{1-\frac{n}{q}}\, \Theta_k(\delta), \tag{3.56}$$

where $\Theta_k(\delta)$ denotes a continuous function of δ, which is uniformly bounded, as $k \to \infty$. Since $n - 2s > 1$ for any $s \in (\frac{1}{2}, 1)$, from (3.55) and (3.56) we obtain the existence of a unique δ solution to (3.54) with

$$\delta = \left[2^{\frac{n-2s}{2}} (\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}}) \right]^{-1} \left(1 + O(k^{1-2s})\right).$$

What is left of this Section is devoted to the proof of (3.55) and (3.56).

Proof of (3.55). We write

$$\int_{\mathbb{R}^n} \zeta_1 E \, \widetilde{Z}_{n+1} = \int_{\mathbb{R}^n} E \widetilde{Z}_{n+1} + \int_{\mathbb{R}^n} (\zeta_1 - 1) E \widetilde{Z}_{n+1}$$

Expanding the first term we get

$$\int_{\mathbb{R}^n} E \widetilde{Z}_{n+1} = \int_{B_1} E \widetilde{Z}_{n+1} + \int_{\mathbb{R}^n \setminus \cup B_j} E \widetilde{Z}_{n+1} + \sum_{j \neq 1} \int_{B_j} E \widetilde{Z}_{n+1} := I_1 + I_2 + I_3$$

where $B_j = B(\xi_j, \frac{\eta}{k})$. With the rescaling $x = \mu y + \xi_1$ and writing

$$\widetilde{E}(y) = \mu^{\frac{n+2s}{2}} E(\xi_1 + \mu y)$$

we get

$$\int_{B_1} E \widetilde{Z}_{n+1} = \int_{B(0, \frac{\eta}{\mu k})} \widetilde{E}_1(y) Z_{n+1}(y).dy$$

Thus

$$
\begin{aligned}
I_1 \;&= \int_{B(0,\frac{\eta}{\mu k})} \widetilde{E}_1 Z_{n+1}(y)\,dy = -\gamma p \sum_{j\neq 1} \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} U(y-\mu^{-1}(\xi_j-\xi_1)) Z_{n+1} \\
&\quad + \gamma p \mu^{\frac{n-2s}{2}} \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} U(\xi_1+\mu y) Z_{n+1}\,dy \\
&\quad + \gamma p \int_{B(0,\frac{\eta}{\mu k})} [(U(y)+sV(y))^{p-1} - U^{p-1}] V(y) Z_{n+1}\,dy \\
&\quad + \gamma \sum_{j\neq 1} \int_{B(0,\frac{\eta}{\mu k})} U^{p}(y-\mu^{-1}(\xi_j-\xi_1)) Z_{n+1} \\
&\quad - \mu^{\frac{n+2s}{2}} \gamma \int_{B(0,\frac{\eta}{\mu k})} U^{p}(\xi_j+\mu y) Z_{n+1}\,dy,
\end{aligned}
$$

where

$$
V(y) = \left(-\sum_{j\neq 1} U(y-\mu^{-1}(\xi_j-\xi_1)) + \mu^{\frac{n-2s}{2}} U(\xi_1+\mu y) \right).
$$

For $j \neq 1$, and by Tylor expansion

$$
U(y+\mu^{-1}(\xi_1-\xi_j)) = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} (1+O(\mu^2 k^2))
$$

where $\hat{\xi}_1 = (1, 0..., 0)$ and $\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}} \hat{\xi}_1$; thus

$$
\int_{B(0,\frac{\eta}{\mu k})} U^{p-1} U(y-\mu^{-1}(\xi_j-\xi_1)) Z_{n+1} = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \int_{B(0,\frac{\eta}{\mu k})} U^{p-1}(1+O(\mu^2 k^2)) Z_{n+1}
$$

$$
\begin{aligned}
&= \frac{2^{\frac{n-2s}{2}}\mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \left[\int_{\mathbb{R}^n} U^{p-1} Z_{n+1} - \int_{\mathbb{R}^n \setminus B(0,\frac{\eta}{\mu k})} U^{p-1} Z_{n+1} + O(\mu^2 k^2) \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} Z_{n+1} \right] \\
&= \frac{2^{\frac{n-2s}{2}}\mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \left[C_1 + O(\mu^{2s}k^{2s}) + O(\mu^2 k^2) \right] \\
&= \frac{2^{\frac{n-2s}{2}}\mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} C_1 (1 + O(\mu^{2s}k^{2s})) \qquad \text{where } C_1 = \int_{\mathbb{R}^n} U^{p-1} Z_{n+1}
\end{aligned}
$$

For the second term

$$
\mu^{\frac{n-2s}{2}} \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1}\, dy = \mu^{\frac{n-2s}{2}} C_1 (1 + O(\mu^{2s}k^{2s})).
$$

Furthermore,

$$
\begin{aligned}
\left| \int_{B(0,\frac{\eta}{\mu k})} [(U(y) + sV(y))^{p-1} - U^{p-1}] V(y) Z_{n+1}\, dy \right| &\le \left| \sum_{i \ne 1} \int_{B(0,\frac{\eta}{\mu k})} U^p (y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \right| \\
&\le C \sum_{i \ne 1} \frac{\mu^{n+2s}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2s}} \int_{B(0,\frac{\eta}{\mu k})} \frac{1}{(1+|y|)^{n-2s}} \\
&\le C(\mu k)^{-2s} \sum_{i \ne 1} \frac{\mu^{n+2s}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2s}}
\end{aligned}
$$

and

$$
\left| \mu^{\frac{n+2s}{2}} \gamma \int_{B(0,\frac{\eta}{\mu k})} U^p (\xi_j + \mu y) Z_{n+1}\, dy \right| \le C \mu^{\frac{n+2s}{2}} \int_{B(0,\frac{\eta}{\mu k})} \frac{1}{(1+|y|)^{n-2s}}\, dy \le C \mu^{\frac{n-2s}{2}} k^{-2s}.
$$

Therefore, we conclude that

$$
I_1 = A\, \delta\, k^{-(n-2s)} \left[-2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \right) \delta + 1 \right] + k^{-n} \Theta_k(\delta)
$$

where $\Theta_k(\delta)$ is a smooth function of δ, which is uniformly bounded as $k \to \infty$.

Now we are going to estimate I_2. Holder inequality gives

$$
\left| \int_{\mathbb{R}^n \setminus \cup B_j} E \widetilde{Z}_{n+1} \right| \le C \| (1+|y|)^{n+2s-\frac{2n}{q}} E \|_{L^q(\mathbb{R}^n \setminus \cup B_j)} \times
$$

$$\|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n\setminus\cup B_j)}.$$

A direct computation gives that

$$\|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n\setminus\cup B_j)} \leq Ck^{-n\frac{q-1}{q}}$$

for some constant $C > 0$. Thus we conclude that

$$|I_2| \leq Ck^{1-n}$$

since we have already proved that $\|(1+|y|)^{n+2s-\frac{2n}{q}}E\|_{L^q(\mathbb{R}^n\setminus\cup B_j)} \leq Ck^{1-\frac{n}{q}}$, see (3.13).

Let $j \neq 1$ fixed and $\widetilde{E}_j(y) = \mu^{\frac{n+2s}{2}}E(\xi_j+\mu y)$. After the change of variable $x = \mu y + \xi_j$ we obtain

$$\begin{aligned}\left|\int_{B_j} E\widetilde{Z}_{n+1}\right| &= \left|\mu^{\frac{n-2s}{2}}\int_{B(0,\frac{\eta}{\mu k})}\widetilde{E}_j\widetilde{Z}_{n+1}(\mu y+\xi_j)\right| \\ &\leq C\mu^{\frac{n-2s}{2}}\|(1+|y|)^{n+2s-\frac{2n}{q}}\widetilde{E}_j\|_{L^q(B(0,\frac{\eta}{\mu k}))}\times \\ &\quad \|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\|_{L^{\frac{q}{q-1}}(B(0,\frac{\eta}{\mu k}))}.\end{aligned}$$

We have

$$\left\|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\right\|_{L^{\frac{q}{q-1}}(B(0,\frac{\eta}{\mu k}))}$$

$$\leq C\frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}}\left(\int_1^{\frac{\eta}{\mu k}}\frac{t^{n-1}}{t^{(n+2s-\frac{2n}{q})\frac{q}{q-1}}}\,dt\right)^{\frac{q-1}{q}} \leq C\frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}}(\mu k)^{2s-\frac{n}{q}}$$

and

$$\|(1+|y|)^{n+2s-\frac{2n}{q}}\widetilde{E}_j\|_{L^q(B(0,\frac{\eta}{\mu k}))} \leq (\mu k)^{\frac{n}{q}}\left(1+k^{-(n+2s)}\mu^{-\frac{n-2s}{2}}\right).$$

Hence

$$|I_3| = \left|\sum_{j\neq 1}\int_{B_j} E\widetilde{Z}_{n+1}\right| \leq \mu^{\frac{n-2s}{2}}(\mu k)^{\frac{n}{q}}\left(1+k^{-(n+2s)}\mu^{-\frac{n-2s}{2}}\right)\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}}(\mu k)^{2s-\frac{n}{q}}$$
$$\leq C\mu^{\frac{n-2s}{2}}k^{-2s}$$

Finally, we conclude that

$$\int_{\mathbb{R}^n} E\widetilde{Z}_{n+1} = A\,\delta\,k^{-(n-2s)}\left[-2^{\frac{n-2s}{2}}(\sum_{j=1}^{\infty}\frac{1}{j^{n-2s}})\,\delta+1\right]+k^{1-n}\Theta_k(\delta) \qquad (3.57)$$

where $\Theta_k(\delta)$ is a smooth function of δ, which is uniformly bounded as $k\to\infty$.

In order to complete the proof of (3.55), we are left with the estimation of the integral $\int_{\mathbb{R}^n}(\zeta_1-1)E\widetilde{Z}_{n+1}$. We have

$$\left|\int_{\mathbb{R}^n}(\zeta_1-1)E\widetilde{Z}_{n+1}\right| \leq C\left|\int_{|y-\xi_1|>\frac{\eta}{k}} E\widetilde{Z}_{n+1}\right|$$

Then we split the domain of integration as follows

$$\int_{|y-\xi_1|>\frac{\eta}{k}} E\widetilde{Z}_{n+1} = \int_{\cap_j|y-\xi_j|>\frac{\eta}{k}} E\widetilde{Z}_{n+1}+\sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\widetilde{Z}_{n+1}$$

In the exterior region, we already proved that $\int_{\cap_j|y-\xi_j|>\frac{\eta}{k}} E\widetilde{Z}_{n+1} = k^{1-n}\Theta_k(\delta)$, for some smooth function Θ_k of δ, which is uniformly bounded as $k\to\infty$. On the another hand, to estimate $\sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\widetilde{Z}_{n+1}$ we can argue like in the estimate of the term I_3 above, thus concluding that

$$\left|\sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\widetilde{Z}_{n+1}\right| \leq Ck^{-n}$$

for some constant $C>0$. This concludes the proof of (3.55).

Proof of (3.56)**.** It is convenient to decompose

$$\mathcal{N}(\phi_1) = \tilde{\mathcal{N}}(\phi_1) + N(\tilde{\phi}_1)$$

where

$$\tilde{\mathcal{N}}(\phi_1) = p\left(|U_*|^{p-1}\zeta_1 - U_1^{p-1}\right)\tilde{\phi}_1 + p\zeta_1|U_*|^{p-1}\Psi(\phi_1) + N\left(\tilde{\phi}_1 + \sum_{j\neq 1}\widetilde{\phi}_j + \Psi(\phi_1)\right) - N(\widetilde{\phi}_1)$$

and

$$N(\widetilde{\phi}_1) = |U_* + \widetilde{\phi}_1|^{p-1}(U_* + \widetilde{\phi}_1) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\widetilde{\phi}_1$$

We have that

$$I := \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)\widetilde{Z}_{n+1} = \mu^{\frac{n+2}{2}} \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)(\xi_1 + \mu x) Z_{n+1}(x)dx$$

so that, from the estimates found we readily check

$$|I| \leq Ck^{2s-n}k^{1-\frac{n}{q}} \int_{R^n} U^{p-1}|Z_{n+1}|. \tag{3.58}$$

On the other hand, if we let

$$II := \int_{R^n} N(\widetilde{\phi}_1)\widetilde{Z}_{n+1}$$

we find that

$$|II| \leq \|\phi_1\|_* \int_{R^n} U^{p-1}|\phi_1||Z_{n+1}|.$$

Now, we notice that from equation (3.23), that we can write

$$L_0(\phi_1) + a\phi_1 = g + \sum_l c_l U^{p-1} Z_l \text{ where } a = \mu^{\frac{n+2s}{2}}\gamma N(\widetilde{\phi}_1)(\xi_1 + \mu y)$$

so that

$$|a| \leq CU^{p-1}\|\phi_1\|_*, \text{ and } |g| \leq C\mu^{\frac{n-2s}{2}}(1+|y|)^{-4s}$$

Thus, applying Lemma 3.6 with $\nu = 4s$ we find

$$|\phi_1| \leq C\mu^{\frac{n-2s}{2}}(1+|y|)^{-2s}$$

and we conclude that

$$|II| \leq C\|\phi_1\|_* \mu^{\frac{n-2s}{2}} \leq Ck^{2s-n-\frac{n}{q}}.$$

Combining this with (3.58) we find

$$\left|\int_{R^n} \mathcal{N}(\phi_1)\widetilde{Z}_{n+1}\right| \leq Ck^{2s-n}k^{1-\frac{n}{q}}.$$

We thus get the proof of (3.56).

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