RANDOM WALKS IN STATIC AND MARKOVIAN TIME-DEPENDENT RANDOM ENVIRONMENT

DISSERTATION

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ABSTRACT

Firstly, we introduce ellipticity criteria for random walks in i.i.d. random environments, under which we can extend the ballisticity conditions of Sznitman's and the polynomial effective criteria of Berger, Drewitz and Ramírez originally defined for uniformly elliptic random walks. We prove under these ellipticity criteria the equivalence of Sznitman's (T') condition with the polynomial effective criterion $(P)_M$, for M large enough. We furthermore give ellipticity criteria under which a random walk satisfying $(P)_M$ for M large enough, is ballistic, satisfies the annealed central limit theorem or the quenched central limit theorem.

Secondly, we consider a random walk in a time-dependent random environment on the lattice $\mathbb{Z}^d$. Recently, Rassoul-Agha, Seppäläinen and Yilmaz [RSY11] proved a general large deviation principle under mild ergodicity assumptions on the random environment for such a random walk, establishing first level 2 and 3 large deviation principles. Here we present two alternative short proofs of the level 1 large deviations under mild ergodicity assumptions on the environment: one for the continuous time case and another one for the discrete time case. Both proofs provide the existence, continuity and convexity of the rate function. Our methods are based on the use of the sub-additive ergodic theorem as presented by Varadhan in [Var03].

To my wife Saidé

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INTRODUCTION

1.1 Random walks in static random environment

1.1.1 Introduction

A random walk in random environment (RWRE) is one of the fundamental models of random motions in random media. The first works which can be related in a direct way to this model were done in 1967 by Chernov [Che67] and in 1972 by Temkin [Tem72]. Chernov's work can be described as a model of replication of DNA, while Temkin gave a computer simulation about investigations concerning phase transitions in alloys. In 1982, Sinai in [Sin82b] showed that the RWRE model can be mathematically derived from turbulence phenomena as a simplified version of the so called Lorentz gas, which is a dynamical system of statistical mechanics where non-interacting particles move with constant velocities between elastic collisions from fixed scatterers. In Sinai's words, the mathematical formalism that he develops can be explained as follows: *in this kind of system each particle moves under the action of forces of interaction with neighboring particles. One can imagine such motion as a sequence of transitions between collisions which look like transitions of random walk in random media due to the randomness of configurations of particles.*

Let us now introduce the RWRE model. For $x \in \mathbb{R}^d$, denote by $|x|_1$ and $|x|_2$ its l_1 and l_2 norm respectively. Now, we consider the lattice $\mathbb{Z}^d$ and the set

$$\mathcal{P} := \left\{ (p(e))_{|e|=1, e\in\mathbb{Z}^d} : p(e) \in (0,1), \text{ and } \sum_{|e|_1=1} p(e) = 1 \right\}, \tag{1.1.1}$$

which is the space of probability measures on the set $\{e \in \mathbb{Z}^d : |e|_1 = 1\}$. Let $\Omega := \mathcal{P}^{\mathbb{Z}^d}$

be the *environment space* and for each *environment* $\omega = (\omega(x,.))_{x\in Z^d} \in \Omega$ we define the random walk in the environment ω as the Markov chain $(X_n)_{n\in\mathbb{N}}$ on Z^d with law $P_{x,\omega}$ defined by

$$P_{x,\omega}(X_0 = x) = 1, \quad P_{x,\omega}(X_{n+1} = z + e | X_n = z) = \omega(z,e), \quad |e|_1 = 1, z \in \mathbb{Z}^d, \tag{1.1.2}$$

whenever $P_{x,\omega}(X_n = z)$ has positive probability. Furthermore, let $\mathbb{P}$ be a probability measure on Ω. Whenever the coordinates $(\omega(x,.))_{x\in\mathbb{Z}^d}$ of the environment ω are i.i.d. under $\mathbb{P}$ we will say that the environment is i.i.d. Thus if μ is a probability measure on $\mathcal{P}$ then $\mathbb{P} = \mu^{\otimes Z^d}$.

The random environment (or also $\mathbb{P}$) is called *elliptic* if

$$\mathbb{P}\left(\min_{|e|=1} \omega(0,e) > 0\right) = 1. \tag{1.1.3}$$

In the case that there is a constant $\kappa \in (0,1)$ such that

$$\mathbb{P}(\omega(0,e) \geq \kappa, \quad \forall e \in \mathbb{Z}^d, |e| = 1) = 1, \tag{1.1.4}$$

it is called *uniformly elliptic.*

We call $P_{x,\omega}$ the *quenched law* of the random walk in random environment (RWRE) starting from x and correspondingly we define the *averaged* (or *annealed*) law of the RWRE as the semi-direct product on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$:

$$P_x := \mathbb{P} \times P_{x,\omega}, \quad \text{for } x \in \mathbb{Z}^d,$$

i.e, $P_x(.) = \int_\Omega P_{x,\omega}(.)d\mathbb{P} = \mathbb{E}P_{x,\omega}(.)$, where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. Note that the RWRE is not a Markov chain anymore under the annealed law. This fact is really an important handicap in the study of this model.

Let us define some basic concepts, which will be very important in the development of the thesis. Let $\ell \in \mathbb{S}^{d-1}$. We say that the random walk X_n is *transient in direction* ℓ if

$$P_0(A_\ell) = 1, \tag{1.1.5}$$

where

$$A_\ell := \{\omega : \lim_{n\to\infty} X_n \cdot \ell = \infty\}.$$

The random walk X_n is *ballistic in direction* ℓ if P_0-a.s it satisfies

$$\varliminf_{n\to\infty} \frac{X_n \cdot \ell}{n} > 0. \tag{1.1.6}$$

As we will see in the next subsection, in dimension 1 transience in one direction does not imply necessarily ballisticity in the same direction. However, it is conjectured that in dimensions greater than 1, there is a positive answer to the following open problem

Open problem 1. *Given a RWRE in a uniformly elliptic i.i.d. environment in dimensions $d \geq 2$, does transience in direction ℓ imply ballisticity in the same direction ℓ?*

According to this problem, a heuristic explanation is derived from a slowdown phenomena, which suffers the walk during its trajectory. This is explained by the existence of traps in the medium, which can be defined as zones where the walk spends an important amount of time with a relatively low cost. For instance, one of these traps is a ball of radius $(\log n)^{\frac{1}{d}}$, where the walk spends a fraction of time n. The cost of this event is of order $e^{-c(\log n)^d}$. So, in particular in dimension 1, this kind of event has a polynomial decay. This is very different compared to the situation in $d \geq 2$.

1.1.2 Random walk in random environment in $d = 1$

In [Sol75], Solomon proved that when $d = 1$ the RWRE can be either recurrent or transient according to a specific property of the law of the environment. To simplify notation we will write ω_x^+ instead of $\omega(x,1)$ and ω_x^- for $\omega(x,-1)$. Define for $x \in \mathbb{Z}$,

$$\rho_x = \frac{\omega_x^-}{\omega_x^+}.$$

In [Al99], Alili generalized Solomon's work.

Theorem 1.1.1. (Solomon and Alili) *Assume that*

*(**A1**) $\mathbb{P}$ is stationary and ergodic.*

*(**A2**) $\mathbb{E}(\log \rho_0)$ is well defined (with $+\infty$ or $-\infty$ as possible values).*

*(**A3**) $\mathbb{P}(\omega_0^+ + \omega_0^- > 0) = 1$.*

Then

(a) $\mathbb{E}(\log \rho_0) < 0 \Rightarrow$ P_0*-a.s.,* $\quad \lim_{n\to\infty} X_n = +\infty,$

(b) $\mathbb{E}(\log \rho_0) > 0 \Rightarrow$ P_0*-a.s.,* $\quad \lim_{n\to\infty} X_n = -\infty$

(c) $\mathbb{E}(\log \rho_0) = 0 \Rightarrow$ P_0*-a.s.,* $\quad -\infty = \varliminf_{n\to\infty} X_n < \varlimsup_{n\to\infty} X_n = +\infty.$

Note that (c) implies that P_0-a.s X_n is recurrent. In the particular case where $(\omega_x^+)_{x\in\mathbb{Z}}$ is a collection of i.i.d. random variables such that the support of the law of ω_0^+ is contained in $(0, 1)$, our chain is called Sinai's random walk [Sin82a]. Sinai proved that in contrast to the ordinary random walk for large n, the walk takes values of order $\log^2 n$ (extremely subdiffusive behavior). In a more concrete way, he showed that

$$\frac{\sigma^2}{(\log n)^2} X_n \overset{law}{\to} b_\infty, \tag{1.1.7}$$

where $\sigma^2 := \mathbb{E}(\log \rho_0)^2$ and b_∞ is a random variable. Then Golosov in [Gol86] and Kesten in [Kes86] obtained independently the distribution of b_∞, which is described by

$$\mathbb{P}(b_\infty \in dx) = z(x)dx := \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{\frac{-(2k+1)^2\pi^2}{8}|x|} dx. \tag{1.1.8}$$

(Note that $2z(t)$, $t > 0$, is the probability that standard Brownian motion starting at 0 does not exit the interval $[-1, 1]$ prior to time t. See for example [DH00] and [MP10]).

The second part of Solomon's work [Sol75] was concerned with the law of large numbers. Again in [Al99], Alili extended Solomon's result to ergodic and stationary environments.

Theorem 1.1.2. (Solomon and Alili) *Assume that*

*(**A1**) $\mathbb{P}$ is stationary and ergodic.*

*(**A2**) $\mathbb{E}(\log \rho_0)$ is well defined (with $+\infty$ or $-\infty$ as possible values).*

*(**A3**)* $\mathbb{P}(\omega_0^+ + \omega_0^- > 0) = 1$.

Then,

(a) $\mathbb{E}(\bar{S}) < \infty \Rightarrow$ *P_0-a.s.,* $\quad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}(\bar{S})}$,

(b) $\mathbb{E}(\bar{F}) < \infty \Rightarrow$ *P_0-a.s.,* $\quad \lim_{n\to\infty} \frac{X_n}{n} = -\frac{1}{\mathbb{E}(\bar{F})}$,

(c) $\mathbb{E}(\bar{S}) = \infty$ *and* $\mathbb{E}(\bar{F}) = \infty \Rightarrow$ *P_0-a.s.,* $\quad \lim_{n\to\infty} \frac{X_n}{n} = 0$.

where

$$\bar{S} := \sum_{i=1}^{\infty} \frac{1}{\omega_{(-i)}^+} \prod_{j=0}^{i-1} \rho_{(-j)} + \frac{1}{\omega_0^+}$$

and

$$\bar{F} := \sum_{i=1}^{\infty} \frac{1}{\omega_i^-} \prod_{j=0}^{i-1} \rho_j^{-1} + \frac{1}{\omega_0^-}.$$

In the case that $\mathbb{P}$ is i.i.d. Theorem 1.1.2 becomes

(a') $\mathbb{E}(\rho_0) < 1 \Rightarrow$ P_0-a.s., $\quad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - \mathbb{E}\rho_0}{\mathbb{E}\left(\frac{1}{\omega_0^+}\right)}$,

(b') $\mathbb{E}(\rho_0^{-1}) < 1 \Rightarrow$ P_0-a.s., $\quad \lim_{n\to\infty} \frac{X_n}{n} = -\frac{1 - \mathbb{E}\left(\frac{1}{\rho_0}\right)}{\mathbb{E}\left(\frac{1}{\omega_0^-}\right)}$,

(c') $\frac{1}{\mathbb{E}\rho_0} \leq 1 \leq \mathbb{E}\rho_0^{-1} \Rightarrow$ P_0-a.s,. $\quad \lim_{n\to\infty} \frac{X_n}{n} = 0$.

By Jensen's inequality $\mathbb{E}\log\rho_0 \leq \log\mathbb{E}\rho_0$, where a strict inequality holds whenever $\mathbb{P}$ is non-degenerate. Using this and both theorems, one can find examples, where the walk is transient but not ballistic (see the last paragraph of the previous subsection). This behavior is quite unusual as compared to the ordinary random walk and indicates some kind of slowdown in the transient case. In Solomon's words: *"randomizing the environment" is in some sense "slowdown" the random walk.*

1.1.3 Random walk in random environment in $d \geq 2$

In general, the situation in higher dimensions is poorly understood. Simple questions are yet unanswered. In particular, there is no result as described in Theorems 1.1.1 and 1.1.2. The following is an example of an open problem.

Open problem 2. *For a RWRE in elliptic i.i.d. environment in dimensions* $d \geq 3$*,*

$$\text{does } P_0(A_l) \in \{0, 1\} \text{ for all } \ell \in \mathbb{S}^{d-1}?$$

In the case $d = 2$, Zerner and Merkl gave an affirmative answer to this question in [ZeM01]. On the other hand, under the assumption that the environment is uniformly elliptic and i.i.d. Sznitman and Zerner [SZ99] proved the *Kalikow's zero-one law* when $d \geq 2$, which is described by the condition

$$P_0(A_\ell \cup A_{-\ell}) \in \{0, 1\}.$$

The content of the proof basically can be founded in [Kal81]. Subsequently in [ZeM01], Zerner and Merkl extended this result to elliptic i.i.d. environments.

A second example of a basic question which remains unsolved is open problem 1, which we restate here

Open problem 1 *Given a RWRE in a uniformly elliptic i.i.d. environment in dimensions* $d \geq 2$*, does transience in direction* ℓ *imply ballisticity in the same direction* ℓ*?*

The main focus of the first half of this thesis is on the concept of ballisticity for random walks in elliptic random environments which are not necessarily uniformly elliptic, and turns around versions of open problem 1 for non-uniformly elliptic random walks. To present a summary of the progress that has been made about ballisticity questions for RWRE, we subdivide the rest of this section in seven subsections. In the subsection 1.1.3.1 we will introduce the asymptotic direction, which will be useful later. In subsection 1.1.3.2 we will discuss the special case of random walks in random Dirichlet environment. The random Dirichlet environment is an example of an elliptic but non-uniformly elliptic environment where many explicit computations can be performed. Also, it is an example of a RWRE, where a lack of uniform ellipticity on the environment produces cases of transience in a given direction without ballisticity. In subsection 1.1.3.3 we will present the first results which gave some light about open problem 1 within the context of RWRE satisfying a condition known as Kalikow's criterion, which is an hypothesis stronger than transience. In subsection 1.1.3.4 we define the Sznitman's ballisticity conditions with respect to one fixed direction $\ell \in \mathbb{S}^{d-1}$ and a parameter $\gamma \in (0,1]$. These conditions are equivalent to transience in direction ℓ plus the finiteness of an exponential moment of the maximum displacement between two regeneration times of the random walk. In subsection 1.1.3.5 we introduce Sznitman's effective criterion, which is the most important tool to prove several central statements about ballisticity of the walk. One of them is the possible equivalence between Sznitman's ballisticity conditions. In subsection 1.1.3.6 we give a brief presentation about recent developments on this equivalence. Finally in subsection 1.1.3.7, we introduce the effective polynomial condition defined in [BDR12] by Berger, Drewitz and Ramírez, which is an a priori weaker condition than the ballisticity conditions of Sznitman. Berger, Drewitz and Ramírez proved in [BDR12] that in fact the effective polynomial condition is equivalent to the so called $(T')|\ell$ condition of Sznitman.

1.1.3.1 The asymptotic direction

We say that there is an *asymptotic direction* $\hat{v}$ if P_0-a.s. the limit

$$\hat{v} := \lim_{n\to\infty} \frac{X_n}{|X_n|_2},$$

exists. Simenhaus in [Sim07] proved the following theorem on an elliptic i.i.d. environment.

Theorem 1.1.3. *(***Simenhaus***)* *The following are equivalent:*

(a) There exists a non-empty open set $O \subset \mathbb{R}^d$ such that

$$P_0(A_\ell) = 1 \quad \ell \in O.$$

(b) There exists $\hat{v} \in \mathbb{S}^{d-1}$ such that P_0-a.s.

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = \hat{v}.$$

(c) There exists $\hat{v} \in \mathbb{S}^{d-1}$ *such that* $P_0(A_\ell) = 1$ *for all* $\ell \in \mathbb{R}^d$ *with* $\ell \cdot \hat{v} > 0$.

Thus, the existence of the asymptotic direction $\hat{v}$ can be defined in terms of transience in an open neighborhood of $\mathbb{R}^d$. A natural question is if transience in one direction is enough to conclude the existence of $\hat{v}$:

Open problem 3. *Does transience in at least one direction* $\ell \in \mathbb{S}^{d-1}$ *imply the existence of the asymptotic direction* $\hat{v}$*?*

We will now introduce a random walk in an i.i.d. environment where the distribution at each site is given by the Dirichlet distribution. This is an important example of a random environment which is elliptic but not uniformly elliptic, where several computations and results can be done in an explicit way.

1.1.3.2 Random Dirichlet Environment

The random walk in the random Dirichlet environment (RWDE) is a random walk in an i.i.d. environment where the law of the environment at each site is given by a Dirichlet distribution. More precisely, given a family $(\alpha_1, \ldots, \alpha_{2d})$ of positive numbers, the *Dirichlet distribution of parameter* $(\alpha_i)_{i\in\{1,\ldots,2d\}}$ is the probability distribution $\mathcal{D}\left((\alpha_i)_{i\in\{1,\ldots,2d\}}\right)$ on $\mathcal{P}$ of density

$$(x_i)_{i\in 1,\ldots,2d} \mapsto \frac{\Gamma\left(\sum_{i=1}^{2d} \alpha_i\right)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \prod_{i=1}^{2d} x_i^{\alpha_i - 1},$$

with respect to the Lebesgue measure $\prod_{i=1}^{2d-1} dx_i$ on the simplex $\mathcal{P}$. Note that this distribution does not satisfy the uniform ellipticity condition.

Denote $e_j = -e_{j-d}$, for $j = d+1, \ldots, 2d$ and let

$$\lambda := 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\ldots,d} (\alpha_i + \alpha_{i+d}). \tag{1.1.9}$$

The following theorem was proved by Sabot [Sa12], building up on previous results of Sabot [Sa11] and Sabot and Tournier [ST11].

Theorem 1.1.4. (Sabot and Tournier) *Consider a random walk in a random Dirichlet environment with parameters* $(\alpha_i)_{1\leq i\leq 2d}$. *Assume that there is an* $1 \leq i \leq d$ *such that* $\alpha_i < \alpha_{i+d}$. *Then there is an asymptotic direction* $\hat{v} = e_i$. *In particular, the random walk is transient in direction* e_i. *Furthermore, the random walk is ballistic if and only if* $\lambda > 1$.

The above theorem displays already examples of random walks which are transient in a given direction but not ballistic. Similar examples have been constructed within the context of random walks in random conductances (see [F11]).

We will now discuss the state of the art regarding open problem 1, beginning with the Kalikow's ballisticity criterion.

1.1.3.3 Kalikow's ballisticity criterion

In order to deal with the open problem 1, Sznitman considered the *Kalikow criterion with respect to a vector* $\ell \in S^{d-1}$. Consider an environment ω with an elliptic law. This criterion is defined in terms of the *Kalikow random walk*, which is defined on $U \cup \partial U$, where U is any connected subset of $\mathbb{Z}^d$ containing 0, and has a transition probability given by

$$\hat{P}_U(x, x+e) = \frac{\mathbb{E}\left[E_{0,\omega}\left[\sum_{n=0}^{T_U} 1_{\{X_n=x\}}\omega(x,e)\right]\right]}{\mathbb{E}\left[E_{0,\omega}\left[\sum_{n=0}^{T_U} 1_{\{X_n=x\}}\right]\right]}, \tag{1.1.10}$$

when $x \in U$ and $e \in Z^d$, with $|e| = 1$, while $\hat{P}_U(x,x) = 1$ when $x \in \partial U$. Here T_U stand the exit time of X_n in U. The canonical law of this Markov chain starting from $x \in U \cup \partial U$ is denoted by $\hat{P}_{x,U}$. Note that (1.1.10) is well-defined by the fact that the environment is elliptic and the fact that U is connected. The most relevant properties of this Markov chain, in relation to ballisticity of the RWRE, are

$$\hat{P}_{0,U}(T_U < \infty) = 1 \Rightarrow P_0(T_U < \infty) = 1 \quad \text{and}$$

$$P_0(X_{T_U} \in .) = \hat{P}_{0,U}(X_{T_U} \in .).$$

In particular, these properties establish a relation between a Markov chain and our RWRE under the annealed law, which is not markovian.

Now the *Kalikow's criterion relative to* $\ell \in \mathbb{S}^{d-1}$ is defined by the following property:

$$\exists\ \epsilon > 0, \text{ such that } \inf_{U, x \in U} \sum_{|e|=1} \ell \cdot e\ \hat{P}_U(x, x+e) \geq \epsilon; \tag{1.1.11}$$

where U runs over all possible connected strict subsets of $\mathbb{Z}^d$, which contain 0. As a first important result, Kalikow showed in [Kal81] that (1.1.11) implies $P_0(A_l) = 1$.

Regarding the ballisticity problem, Sznitman and Zerner considered a certain regeneration time τ_1 with respect to a vector $\ell \in S^{d-1}$ (see ([SZ99]). For arbitrary $a > 0$, its definition in words can be summarized as follows: it is the first time where $X_n \cdot \ell$ increases by an amount of at least a above its previous local maxima and never goes below this level from then on.

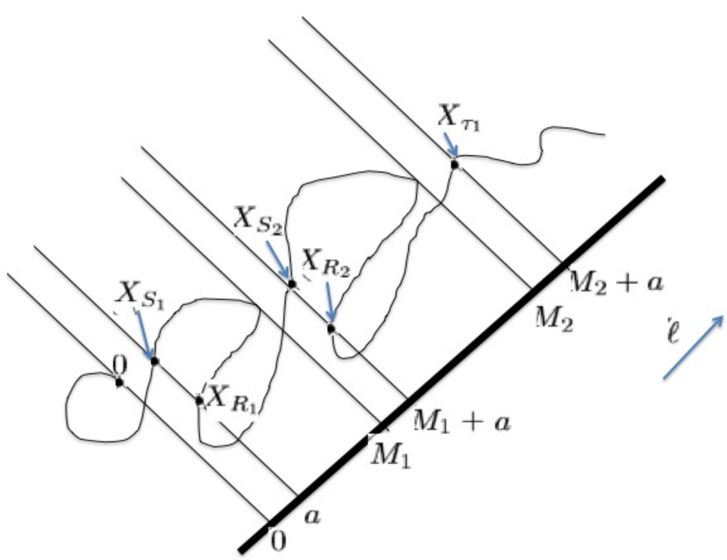

Figure 1.1: A realization of a trajectory of a RWRE with the stopping times which define the regeneration time τ_1.

In 1999, Sznitman and Zerner in [SZ99] proved that this regeneration time is integrable if (1.1.11) holds. One year later, Sznitman improved this result showing in [Sz00] that τ_1 has moments of any order. More precisely, the following theorem was deduced by them

Theorem 1.1.5. (Sznitman-Zerner)

Assume that (1.1.11) holds relative to ℓ, then

$$P_0 - a.s. \quad \frac{X_n}{n} \to v = \frac{E_0(X_{\tau_1}|D=\infty)}{E_0(\tau_1|D=\infty)}, \tag{1.1.12}$$

with $v \cdot \ell > 0$. Moreover,

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right) \tag{1.1.13}$$

converges in law on $D(\mathbb{R}_+, \mathbb{R}^d)$ under P_0 to the law of a non-degenerate Brownian motion with covariance matrix:

$$A = \frac{E_0\left((X_{\tau_1} - \tau_1 v)^t (X_{\tau_1} - \tau_1 v)|D=\infty\right)}{E_0(\tau_1|D=\infty)},$$

where $D = D^\ell := \min\{n \geq 0 : X_n \cdot \ell < X_0 \cdot \ell\}$.

In dimension one, the Kalikow's criterion with respect to $\ell = 1$, or $\ell = -1$, precisely characterizes the case of a non vanishing limit velocity. This is not the case in dimensions $d \geq 2$. Indeed, in [Sz01] and [Sz02] Sznitman introduced a family of ballisticity conditions which turn out to be weaker than Kalikow's condition.

1.1.3.4 Sznitman's ballisticity conditions

In order to improve his results in [Sz00], Sznitman introduced a family of ballisticity conditions that are weaker than Kalikow's condition, and which imply for dimensions $d \geq 2$ a ballistic behavior, an annealed central limit theorem and a quenched central limit theorem.

Definition 1.1.1. *Let $\gamma \in (0,1]$ and $\ell \in \mathbb{S}^{d-1}$. Condition $(T)_\gamma$ holds relative to ℓ (and we write $(T)_\gamma|\ell)$ if for all $\ell' \in \mathbb{S}^{d-1}$ in some neighborhood of ℓ,*

$$\overline{\lim_{L\to\infty}} L^{-\gamma} \log P_0\left(X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0\right) < 0 \quad \forall b > 0, \tag{1.1.14}$$

where $U_{\ell',b,L} := \{x \in \mathbb{Z}^d : -bL < x \cdot \ell' < L\}$ *denotes a slab and*

$$T_{U_{\ell',b,L}} := \inf\{n \in \mathbb{N}_0 : X_n \notin U_{\ell',b,L}\}$$

is the first exit time of this slab. Furthermore, we say that $(T')|$ *holds relative to* ℓ *(and write* $(T')|\ell$*) when* $(T)_\gamma|\ell$ *holds for each* $\gamma \in (0,1)$*, while we say that condition* $(T)|\ell$ *holds relative to* ℓ *(and write* $(T)|\ell$*)* $(T_\gamma)|\ell$ *for* $\gamma = 1$.

In other words, condition $(T)_\gamma|_\ell$ means that the walk exits by the back part of the slab $U_{\ell',b,L}$ with an annealed probability bounded from above by e^{-L^γ}. Sznitman studied these conditions for uniformly elliptic random walks. He showed that $(T)_\gamma|_\ell$ is equivalent to the conditions

$$P_0(A_l) = 1, \quad \text{and for some } c > 0,\ \ E_0\left(e^{c \max_{0\le k\le \tau_1} |X_k|^\gamma}\right) < \infty. \tag{1.1.15}$$

In [Sz02], Sznitman gave an example for which (T') holds but Kalikow's criterion breaks down. Hence, (T') is a genuinely weaker condition than Kalikow's condition. On the other hand, he also proved that condition (T') implies that all moments of the regeneration time τ_1 are integrable. In particular, this shows that Theorem 1.1.5 is valid under (T'). Furthermore, one can prove a quenched central limit theorem with the help of the following theorem, proved by Rassoul-Agha and Seppäläinen in [RAS09].

Theorem 1.1.6. (Rassoul-Agha and Seppäläinen) *Consider a RWRE in an elliptic i.i.d. environment. Let* $l \in \mathbb{S}^{d-1}$ *and let* τ_1 *be the corresponding regeneration time. Assume that*

$$E_0[\tau_1^p] < \infty,$$

for some $p > 176d$*. Then* $\mathbb{P}$*-a.s. we have that*

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right)$$

converges in law under $P_{0,\omega}$ *to a Brownian motion with non-degenerate covariance matrix.*

Neither this theorem nor (1.1.15) require the environment to be uniformly elliptic. Berger and Zeitouni in [BZ08] also proved a quenched central limit theorem, where assumptions on the moments of the regeneration times are weaker, but the environment is assumed to be uniformly elliptic.

Within the context of the way that Sznitman's ballisticity conditions are defined, it is natural to wonder if the required exit estimates could be assumed only with respect to one direction. This leads to the following open problem.

Open problem 4. *Is* $(T)_\gamma|\ell$ *equivalent to*

$$\overline{\lim_{L\to\infty}} L^{-\gamma}\log P_0\left(X_{T_{U_{\ell,b,L}}}\cdot\ell<0\right)<0\quad\forall b>0? \tag{1.1.16}$$

1.1.3.5 Sznitman's Effective Criterion

The main result in [Sz02] (recall that under the assumption that the environment is uniformly elliptic) is that when $d\geq 2$, (T') is equivalent to an effective criterion, which is defined by the property

$$\inf_{B,a\in[0,1]}\left\{c(d)\log\left(\frac{1}{\kappa}\right)^{3(d-1)}\tilde{L}^{d-1}L^{3(d-1)+1}\mathbb{E}(\rho_B^a)\right\}<1, \tag{1.1.17}$$

with

$$\rho_B:=\frac{P_{0,\omega}(X_{T_B}\notin\partial_+B)}{P_{0,\omega}(X_{T_B}\in\partial_+B)}=\frac{q_B}{p_B},$$

where $q_B:=P_{0,\omega}(X_{T_B}\notin\partial_+B)$, $p_B:=P_{0,\omega}(X_{T_B}\in\partial_+B)$, $T_B:=\inf\{n\geq 0: X_n\notin B\}$ and B is a finite box determined by a rotation R of $\mathbb{R}^d$ such that $R(e_1)=\ell$ and defined by

$$B=B(L,\tilde{L}):=R\left((2-L,L+2)\times(-\tilde{L},\tilde{L})^{d-1}\right) \tag{1.1.18}$$

with $\tilde{L}<L^3$. Note that (1.1.17) means a certain control on a potential small value of p_B. The effective character of this criterion comes from the fact it can be checked in boxes that satisfy a description as (1.1.18). Furthermore, the constant κ which appears in (1.1.17), is lower bound for the jump probabilities appearing in the definition of the uniform ellipticity condition.

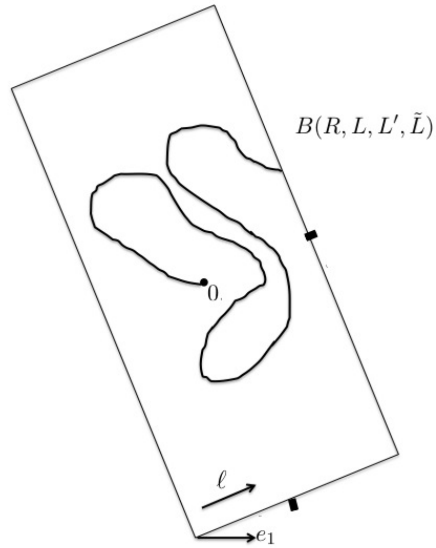

Figure 1.2: The effective criterion implies a control on a potential small value of p_B

With the help of the effective criterion, Sznitman [Sz02] proved the following theorem

Theorem 1.1.7. *Let $d \geq 2$ and $\mathbb{P}$ be uniformly elliptic. Then for each $\gamma \in (0.5, 1)$ and $\ell \in \mathbb{S}^{d-1}$, $(T)_\gamma|\ell$ is equivalent to $(T')|\ell$.*

More precisely, Sznitman proved in [Sz02], with the help of an induction argument along a sequence of boxes B_k with growing scales, that for $\gamma \in (0.5, 1)$, $(T)_\gamma|_\ell$ implies the effective criterion, which in turn implies something stronger than $(T')|\ell$, but weaker than $(T)|\ell$,

$$\overline{\lim_{L\to\infty}} L^{-1} e^{c(logL)^{1/2}} \log P_0 \left(X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0 \right) < 0 \quad \text{for } b > 0, \tag{1.1.19}$$

for all $\ell' \in \mathbb{S}^{d-1}$ in a neighborhood of ℓ and for some $c > 0$. Note that the exponential factor in the above inequality leaves condition $(T)|_\ell$ out of reach. Thus, we can state a new open problem

Open problem 5. *Let $d \geq 2$ and $\mathbb{P}$ be uniformly elliptic. Then for each $\ell \in \mathbb{S}^{d-1}$, $(T)|\ell$ is equivalent to $(T')|\ell$.*

The next subsection deals with extensions of Proposition 1.1.7.

1.1.3.6 Equivalence between $(T')|\ell$ and $(T)_\gamma|\ell$ for $\gamma \in (0,1)$

In [DR11] and [DR12], Drewitz and Ramírez improved Sznitman's Proposition 1.1.7. In the first of these papers, they showed that there is a $\gamma_d \in (0.37, 0.39)$ such that if $\gamma \in (\gamma_d, 1)$ then $(T)_\gamma$ implies (T'). In that opportunity, they based their approach in a method created by Sznitman [Sz01]. Regarding [DR12], they used a multi-scale method development by Berger in [Ber12], which is more sophisticated than Sznitman's, to prove that for $\gamma \in (0,1)$ and for $d \geq 4$, $(T)_\gamma|\ell$ is equivalent to $(T')|\ell$.

Recently, Berger, Drewitz and Ramírez in [BDR12] introduced a new family of ballisticity conditions, which are polynomial in nature, a priori more general than any of Sznitman's ballisticity condition and which in addition can in principle for each environment be checked on finite boxes.

1.1.3.7 The polynomial ballisticity conditions

The progress that was made in [DR11, DR12] about the equivalence between the different Sznitman's ballisticity conditions was not totally satisfactory. Nevertheless, this situation changed when a weaker family of ballisticity conditions was introduced with a new approach for dealing with this equivalence problem. These conditions correspond to changing the strechted exponential e^{-L^γ} which implicitly appears in the definition of condition $(T)_\gamma$, by a polynomial. Furthermore, these conditions will have an effective character, as it is the case for the effective criterion.

Let us consider the boxes

$$B(R, L, L', \tilde{L}) := R\left((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}\right) \cap \mathbb{Z}^d.$$

Let $M > 0$. Assuming $L' \leq \frac{5}{4}L$ and $\tilde{L} \leq 72L^3$ with L large enough, the *polynomial condition* $(P)_M$ *in direction* ℓ is defined by

$$P_0(X_{T_{B(R,L,L',\tilde{L})}} \cdot \ell < L) \leq \frac{1}{L^M}. \tag{1.1.20}$$

The equivalence between $(T)_\gamma|\ell$ and $(T')|\ell$ for $\gamma \in (0,1)$ in any dimension $d \geq 2$ is deduced from the following theorem, which was proved by Berger, Drewitz and Ramírez in [BDR12]

Theorem 1.1.8. (Berger-Drewitz-Ramírez) *Let $d \geq 2$ and $\mathbb{P}$ be uniformly elliptic. Assume that $(P)_M$ in direction ℓ holds for some $M \geq 15d + 5$. Then $(T')|\ell$ holds.*

To prove this theorem, Berger, Drewitz and Ramírez, show that $(P)_M$ implies the effective criterion. To do this, they introduce a new ballisticity condition which is defined as follows: condition $(T)_0|_\ell$ is satisfied if there exists a neighborhood $V \subset \mathbb{S}^{d-1}$ of ℓ such that for all $\ell' \in V$

$$\overline{\lim_{L\to\infty}} \frac{1}{L^{\gamma_L}} \log P_0(X_{T_{U_{\ell,\beta,L}}} \cdot \ell' < 0) < 0, \tag{1.1.21}$$

where

$$\gamma_L := \frac{\log 2}{\log\log L}. \tag{1.1.22}$$

Berger, Drewitz and Ramírez first prove that $(P)_M|\ell$ implies $(T)_0|\ell$. Then, they derive from $(T)_0|_\ell$ the following weak atypical quenched exit estimate.

Proposition 1.1.1. *Consider a random walk in a uniformly elliptic i.i.d. environment. Furthermore, assume that $(T)_0$ is satisfied. Then, for $\epsilon(L) := \frac{1}{(\log\log)^2}$, $c > 0$ and each function $\beta : (0,\infty) \to (0,\infty)$, there exists $C > 0$ such that*

$$\mathbb{P}\left(P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-cL^{\beta_L + \epsilon_L}}\right) \leq \frac{1}{C} e^{-CL^{\beta_L}}. \tag{1.1.23}$$

where B is a box which satisfies the specification given by (1.1.18).

If Proposition 1.1.1 holds then the effective criterion is satisfied ((1.1.23) controls small values of p_B, see subsection 1.1.3.5). In the proof of Proposition 1.1.1, they consider a collection of mesoscopic boxes $D(x)$ with their corresponding central parts $\tilde{D}(x)$. A box $D(x)$ is called *good box* if

$$\inf_{z \in \tilde{D}(x)} P_{z,\omega}(X_{T_{D(x)}} \in \partial_+ D(x)) \geq 1 - L^{\epsilon(L)^{-1}}. \tag{1.1.24}$$

At this point, they used a very simple strategy: they compute the quenched probability that appears in (1.1.23) for configurations which have a number of bad boxes intersecting the box B which is at most $L^{\beta(L)}$. In this case, if the walk exits all mesoscopic boxes encountered through their frontal part then it leaves B through $\partial_+ B$ (see Figure 1.3). But by (2.3.34) the probability of doing this satisfies the inequality

$$P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-cL^{\beta_L+\epsilon_L}}.$$

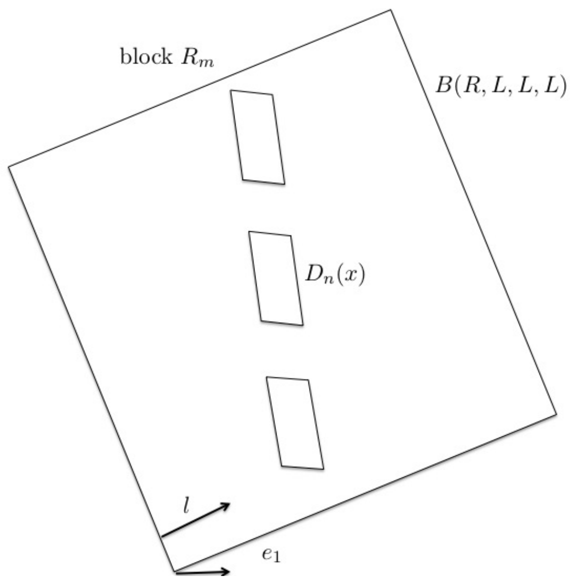

Figure 1.3: The walk exits each mesoscopic box through its frontal part

Thus, (1.1.23) comes from computing the probability of having a number of bad boxes intersecting B greater than $L^{\beta(L)}$.

The other half of the proof of Theorem 1.1.8, is to show that $(P)_M$ implies $(T)_0$. Berger, Drewitz and Ramírez in [BDR12] build another collection of mesoscopic boxes $B(x,k)$, indexed by their center $x \in \mathbb{Z}^d$ and by their scale k. Each mesoscopic box has a *middle front part* $\tilde{B}(x,k)$ which are the possible starting points of the random walk. A box of scale $k = 0$ is defined to be *good* if the quenched probability of leaving $B(x,0)$ through $\partial_+ B(x,0)$ starting from $\tilde{B}(x,0)$ is bigger than $1 - N_0^{-5}$, where N_0 is a fixed number chosen large enough. A box of scale $k \geq 1$ is defined to be *good* if essentially it has at most one box of scale $k-1$ intersecting it which is bad. It is then necessary to show that whenever a box of scale $k \geq 1$ is good, there is a high enough probability to exit it through $\partial_+ B(x,k)$ whenever the walk starts from $\tilde{B}(x,k)$. This probability turns out to be basically of the order $1 - e^{-k!}$. Then, it is shown through the binomial theorem that the probability that a box of scale k is bad is of the order $1 - e^{-2^k}$. Since $k!$ is the size of the side of a box of scale k, choosing k so that $k! = L$ one obtains the bound $e^{-L^{\gamma_L}}$ in the ballisticity condition $(T)_0$.

1.2 Random walks in Markovian time-dependent random environment

1.2.1 Introduction

Here we will consider a class of random walks which evolve in time-dependent random environments which have a Markovian dynamics. In general terms, the model of a random walk in a dynamic environment changes drastically in comparison with the static case. The reason which explains this fact is that somehow the medium will be much more disordered than in the static case. For instance, the phenomena of traps encountered for RWRE, which determines in a crucial way the velocity of the walk, will not play a predominant role in the dynamic case. Within a more general context, where the random walk could affect the dynamics of the environment so that it is not necessarily Markovian, one of the first works in dynamic environment was made by Harris in [H65], who studied interactions between particles that move independently as long as they are not in contact and according to some diffusion process (for instance the Wiener process). Subsequently, within the context of interacting particle systems, Spitzer [Sp70] studied tagged particles, which are essentially random walks in a special kind of dynamic random environment which influence each other (an extended bibliography can be found for example in Liggett [L99, L05]). Other examples of random walks in dynamic random environment can appear within the context of self-interacting random walks. For example, the excited or cookie random walk introduced by Benjamini and Wilson [BW03], which moves on the lattice $\mathbb{Z}^d$ and has a bias whenever it visits a site for the first time. The focus in the second half of this thesis is on the quenched large deviation principles for random walk in Markovian time-dependent random environments at continuous time as well as at discrete time.

Let us now define the class of models of random walk in Markovian time-dependent random environment (RWMRE) which will be considered. They will be defined both in continuous and discrete time.

We will start with the continuous time case. Let $G := \{e_1, e_{-1}, \dots, e_d, e_{-d}\}$ the set of unit vectors in Z^d and $\mathcal{Q} := \{v = \{v(e) : e \in G\} : v(e) \in (0, \infty)\}$. Consider a continuous time Markov process $\omega := \{\omega(t) : t \geq 0\}$ with state space $\Omega_c := \mathcal{Q}^{\mathbb{Z}^d}$, so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x, e) : e \in G\} \in \mathcal{Q}$. We assume that for each initial condition ω_0, the process ω defines a probability measure Q_{ω_0} on the Skorokhod space $D([0, \infty[, \Omega_c)$. Furthermore, we assume that the process ω has an invariant measure μ and then define $Q_\mu(.) := \int_{\Omega_c} Q_\omega d\mu$, where with a slight abuse of notation, $\omega \in \Omega_c$. Assume that μ is also invariant under the action of space-translations. We will call each realization $\omega \in D([0, \infty[, \Omega_c)$ an *environment*. Now, for each environment ω we define formally a process $\{X_t : t \geq 0\}$ by the generator

$$L_s f(x) := \sum_{e \in G} \omega_s(x, e)(f(x + e) - f(x)) \tag{1.2.1}$$

for $s \geq 0$. For each $x \in \mathbb{Z}^d$ we define $P^c_{x,\omega}$ the law on $D([0, \infty[; \mathbb{Z}^d)$ of this random walk with initial condition $X_0 = x$. We call $P^c_{x,\omega}$ the *quenched law* of the *continuous time random walk in Markovian time-dependent environment* starting from x.

For each $s > 0$ and $x \in \mathbb{Z}^d$, let $T_{s,x} : D([0, \infty[; \Omega_c) \to D([0, \infty[; \Omega_c)$ be defined by $(T_{s,x}\omega)_t(y) := \omega_{t+s}(y + x)$. We assume that $\{T_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ is an *ergodic family of transformations* acting on the space $\big(D([0, \infty[; \Omega_c), \mathcal{B}(D([0, \infty[; \Omega_c)), Q^c_\mu\big)$, i.e. whenever $A \in \mathcal{B}(D([0, \infty[; \Omega_c))$ is such that $T^{-1}_{s,x} A = A$ for every $s > 0$ and $x \in \mathbb{Z}^d$, then $Q^c_\mu(A) \in \{0, 1\}$. As usual, for a set E, $\mathcal{B}(E)$ is the σ-algebra generated by Borel sets in E. We say that the law of the environment ω is *uniformly elliptic* if there exists constants κ_1, κ_2 with $0 < \kappa < \kappa_2$ such that for each $t \geq 0$ and $x \in \mathbb{Z}^d$ one has that

$$\mu\left(\kappa_1 \leq \inf_{e \in G} \omega_t(x, e) \leq \sup_{e \in G} \omega_t(x, e) \leq \kappa_2\right) = 1.$$

For the discrete time case, let $R \subset \mathbb{Z}^d$ finite and

$$\mathcal{P} := \{v = \{v(e) : e \in R\} : v(e) \in (0,1), \sum_{e \in R} v(e) = 1\}.$$

We consider a discrete time Markov process $\omega := \{\omega_n : n \geq 0\}$ with state space $\Omega_d := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x,e) : e \in R\} \in \mathcal{P}$. Let $Q_d^{\mathbb{N}}$ be the corresponding law of this process defined on the space $\Omega_d^{\mathbb{N}}$ and assume that this process ω has an invariant measure μ. Assume that μ is also invariant under the action of space-translations. We will call a realization $\omega \in \Omega_d^{\mathbb{N}}$ of this process an *environment*. Define $Q_\mu^d(.) := \int_{\Omega_d} Q_\omega(.) d\mu$, where again with a slight abuse of notation, $\omega \in \Omega_d$. Now, for each environment ω and $x \in \mathbb{Z}^d$, we define a discrete time random walk $\{X_n : n \geq 0\}$ with a law $P_{x,\omega}^d$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ given by the condition $P_{x,\omega}^d(X_0 = x) = 1$ and the transition probabilities

$$P_{x,\omega}^d = (X_{n+1} = x + e | X_n = x) = \omega_n(x,e), \tag{1.2.2}$$

for $n \geq 0$ and $e \in R$. The *nearest neighbor case* corresponds to case where

$$R = \{e \in \mathbb{Z}^d : |e|_1 = 1\}.$$

We call $P_{x,\omega}^d$ the *quenched law* of the *discrete time random walk in Markovian time-dependent environment* starting from x.

For each $x \in \mathbb{Z}^d$, let $T_{1,x} : D([0,\infty[: \Omega_d) \to D([0,\infty[: \Omega_d)$ be defined by

$$(T_{1,x}\omega)_n(y) = \omega_{n+1}(y + x).$$

We assume that the set $\{T_{1,x} : x \in R\}$ is an ergodic family of transformations acting on the space $(\Omega_d^{\mathbb{N}}, \mathcal{B}(\Omega_d^{\mathbb{N}}), Q_\mu^d)$. It is straightforward to check that whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $A = T_{n,x}^{-1} A$ for every $x \in R$ and $n \in \mathbb{N}$ then $Q_u^d(A) \in \{0,1\}$. We say that the law of the environment ω is *uniformly elliptic* if there is a constant $\kappa \in (0,1)$ such that for each $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$ one has that

$$\mu\left(\inf_{e \in R} \omega_n(x,e) \geq \kappa\right) = 1.$$

Throughout the rest of this section we will review the principal results that have been obtained about quenched large deviations for random walks in both static and Markovian time-dependent random environments.

1.2.2 Quenched large deviations for random walks in a static random environment

We will begin reviewing the development of the quenched large deviation principle for static random environment. Some ideas from there have been applied for the case of a Markovian time-dependent random environment. In subsection 1.2.2.1 we will give a brief summary on the two main results in dimension one. Subsection 1.2.2.2 explains the Varadhan's quenched large deviation principle for RWRE based on the use of the subadditive ergodic theorem. The result of Varadhan does not give a lot information about the rate function. To deal with this handicap, a perspective of homogenization for Hamilton-Jacobi equations can be very useful. This topic is described in subsection 1.2.2.3. In subsection 1.2.2.4 Rosenbluth used as reference the so called point of view of the particle, applying some ideas developed by Kosygina, Rezakhanlou and Varadhan in [KRV06] within the context of homogenization for Hamilton-Jacobi equations. In subsection 1.2.2.5 we discuss the level 2 quenched large deviation principle proved by Yilmaz in [Yil09b].

1.2.2.1 Quenched large deviation principle in $d = 1$

In [GdH94] Greven and den Hollander proved the first quenched large deviation principle (QLDP) for RWRE in dimension $d = 1$. They assumed that the environment is uniformly elliptic and i.i.d. and that $\mathbb{E} \log \rho_0 \leq 0$, which implies that the random walk is either recurrent or is transient to the right (see Theorem 1.1.1). Furthermore they obtained a variational formula for the rate function and proved several properties of it.

In 2000, Comets, Gantert and Zeitouni improved this result, assuming that the environment is stationary and ergodic (see [CGZ00]). Their approach was based on Gärtner-Ellis theorem, being completely different to the one used in [GdH94].

1.2.2.2 First results about the multidimensional quenched large deviation principle

Zerner worked in [Z98] on the model described in Section 1.1.1 (an elliptic i.i.d. environment). He assumed that the walk is nestling (i.e. the convex hull of the support of the local drift contains the origin) and that the transition probabilities satisfy the moment condition,

$$\mathbb{E}(-\log \omega(0,e))^d < \infty. \tag{1.2.3}$$

Zerner deduced a QLDP for RWRE in $d \geq 2$ applying the subadditive ergodic theorem to passage times. His approach is based on Sznitman's work on Brownian motion in Poissonian obstacles (see [S98]).

Varadhan in [Var03] generalizes Zerner's work proving a QLDP for random walks $(X_n)_{n\in\mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P}, T_z)$ that satisfies

$$\mathbb{P} \text{ is ergodic and stationary under } \{T_z\}_{z\in\mathbb{Z}^d}. \tag{1.2.4}$$

Here T_z is a tranformation on Ω defined by

$$(T_z\omega)_x := \omega_{x+z}, \tag{1.2.5}$$

for each $z \in \mathbb{Z}^d$. Varadhan assumed in [Var03] an additional hypothesis: the environment is uniformly elliptic. His approach starts regularizing the n-th transition probability between sites x and y, $\pi^{(n)}(x,y)$, defining the expression

$$\sup_{n\geq 0}[\pi^{(n)}(\omega,x,y)e^{-c|n-t|}], \tag{1.2.6}$$

which is supermultiplicative and does not have singularities. Taking logarithms, one can apply the subadditive ergodic theorem. Varadhan took advantage of this fact and the equicontinuity properties of the regularization of the transition probabilities.

An important difference between Zerner's [Z98] and Varadhan's work [Var03] is that while Zerner applied the subadditive ergodic theorem to passage times, Varadhan applied it directly to the smoothed up transition probabilities (this approach is the one that we will use in our QLDP), which a priori can give stronger results. However, we will not get too much information about the rate function.

Theorem 1.2.1. (Varadhan 2003) *Consider a RWRE, whose random environment is stationary and ergodic with respect to the transformations T_z defined in (1.2.5). Moreover, assume that the environment is uniformly elliptic. Then there is a nonrandom convex rate function $h(\xi)$ such that for almost all ω with respect to $\mathbb{P}$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{\xi\in C} h(\xi) \quad \text{for closed } C \subset \mathbb{R}^d,$$

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \log P_{0,\omega}\left(\frac{X_n}{n} \in O\right) \geq -\inf_{\xi\in O} h(\xi) \quad \text{for open } O \subset \mathbb{R}^d.$$

1.2.2.3 Homogenization of the stochastic Hamilton-Jacobi-Bellman equation

To deal with the lack of information about the rate function, Rosenbluth in [Ros06] made a parallelism with homogenization of the stochastic Hamilton-Jacobi-Bellman equation. Previously Kosygina, Rezakhanlou and Varadhan in [KRV06] studied solutions u_ϵ of Hamilton-Jacobi-Bellman type equations,

$$\frac{\partial u_\epsilon}{\partial t} = \frac{\epsilon}{2}\triangle u_\epsilon + H\left(\nabla u_\epsilon, \frac{x}{\epsilon}, \omega\right), \quad (t,x) \in [0,\infty[\times\mathbb{R}^d, \quad u_\epsilon(0,x) = f(x). \tag{1.2.7}$$

Here $H(x,p,\omega)$ is a convex function of p that is a stationary random process in x. Under additional assumptions on H (see [KV08]) one can obtain a homogenization result, where with probability 1 the solution $u_\epsilon(t,x,\omega)$ converges locally uniformly in t and x to a nonrandom limit $u(t,x)$, which is the solution of

$$\frac{\partial u}{\partial t} = \bar{H}(\nabla u), \quad u(0,x) = f(x), \tag{1.2.8}$$

where $\bar{H}$ is a convex Hamiltonian function. The connection between the Hamilton-Jacobi-Bellman equation and RWRE, is given by the choice

$$\bar{H}(z) := \log E\left(e^{\langle z, X_1\rangle}\right)$$

where E is the expectation with respect to the law of X_1 and $\langle \cdot \rangle$ is the Euclidean inner product. The solution to (1.2.8) is given by the *Hopf-Lax-Oleinik formula* so that

$$u = \sup_v (f(x+vt) - t\bar{L}(v))$$

where $\bar{L}$ is the Fenchel-Legendre transform of $\bar{H}$. (see [KRV06], [KV08], [R11]). Then, following standard arguments one gets that

$$\lim_{n\to\infty} \frac{1}{n} \log E\left(e^{nf\left(\frac{1}{n}X_{[nt]}\right)}\right) = u(x,t),$$

which is valid for every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$.

1.2.2.4 Variational formula for the level 1 quenched rate function

In [Ros06] Rosenbluth weakened the hypothesis of Varadhan's quenched large deviation principle for RWRE. Following the approach of the previous subsection based on the Hamilton-Jacobi-Bellman equation he also obtained more information on the rate function, showing that it satisfies a variational problem (see subsection 1.2.2.3). He considered the point of view of the particle method: let Ω be the space of environments and consider an ergodic family of transformations $\{T_e\}_{e\in U}$ with $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$. Let $p : \Omega \times U \to [0,1]$ be a function such that $\sum_{e\in U} p(x,\omega) = 1$, for each $\omega \in \Omega$ (this is a more general assumption than each T_e is ergodic). Now, denote by

$$p(\omega, T_e\omega) := p(\omega, e) \tag{1.2.9}$$

and consider the Markov chain $\{\bar{\omega}_n\}$ with state space Ω, whose transition kernel is determined by (1.2.9) and the induced measure $P_\omega(\bar{\omega}_0 = \omega) = 1$. In his work, Rosenbluth dropped the uniform ellipticity condition on the environment and assumed that

$$\mathbb{E}(-\log \omega(0,e))^{d+\alpha} < \infty. \tag{1.2.10}$$

for some $\alpha > 0$. Note that (1.2.10) is a slightly stronger than (1.2.3), but Rosenbluth considered a more general model, where stationary and ergodic environments substitute the i.i.d. environments and the nestling hypothesis is dropped (α can be taken to be 0 in the case $d = 1$).

Rosenbluth proved the existence of the logarithmic moment generating function (LMGF), i.e. the limit

$$\lim_{n\to\infty} \frac{1}{n} \log E_{0,\omega}\left(e^{\langle \lambda, X_n \rangle}\right) \tag{1.2.11}$$

where $\langle \cdot \rangle$ is the Euclidean inner product. To do this, Rosenbluth considered a collection $\mathcal{K}$ of functions $F : \Omega \times U \to \mathbb{R}$ that satisfies three conditions:

(a) *Moment*: for each $e \in U$, $\mathbb{E}(F^{d+\alpha}) < \infty$, for some $\alpha > 0$.

(b) *Mean zero*: for each $e \in U$, $\mathbb{E}(F(\omega, e)) = 0$.

(c) *Closed loop*: for any finite sequence $\{x_i\}_{i=0}^{n-1} \in \mathbb{Z}^d$, such that $x_{i+1} - x_i \in U$ and $x_0 = x_n$

$$F(T_{x_i}\omega, x_{i+1} - x_i) = 0.$$

and showed that (1.2.11) is equal to

$$\Lambda(\lambda) := \inf_{F \in \mathcal{K}} \operatorname*{ess\,sup}_{\omega} \log \sum_{e \in G} \omega(0, e) e^{\langle \lambda, e \rangle + F(\omega, e)}.$$

With the LMGF well-defined, Rosenbluth followed standard arguments to show that the rate function is the *Fenchel-Legendre transform* of Λ:

Theorem 1.2.2. (Rosenbluth 2006) *Let $(\Omega, \mathcal{F}, \mathbb{P}, T_e)$ be a probability space and an ergodic family of commuting measure preserving transformations. Suppose that (1.2.10) holds for some $\alpha > 0$. Then $\mathbb{P}$-a.s. $\dfrac{X_n}{n}$ obeys a large deviation principle (LDP) with respect to its law $P_{0,\omega}$ with rate function*

$$I(x) = \sup_{\lambda}\{\langle \lambda, x \rangle - \Lambda(\lambda)\}. \tag{1.2.12}$$

1.2.2.5 Quenched univariate level 2 large deviation principle of Yilmaz

In [Yil09b], Yilmaz deduced more information than just the sample mean through the *pair empirical measure environmental* (PEME) of a random walk $(X_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}^d$ whose environment is stationary and ergodic and with bounded jumps. (PEME) is defined by

$$\nu_{n,X} := \frac{1}{n}\sum_{k=0}^{n-1} 1_{T_{X_k}\omega, X_{k+1}-X_k}. \tag{1.2.13}$$

Yilmaz followed the same approach than Rosenbluth (the point of view of the particle) and deduced that $\mathbb{P}$-a.s. $(P_{0,\omega}(\nu_{n,X}\in\cdot))_{n\in\mathbb{N}}$ satisfies a QLDP. Moreover, applying the contraction principle, this result implies a QLDP for the sample mean $\frac{X_n}{n}$, where in particular a variational formula for the corresponding Rosenbluth's rate function is deduced.

1.2.3 Quenched large deviations for random walks in a Markovian time-dependent random environment

Throughout the rest of this section we will review three different works on QLDP in a Markovian time-dependent random environment. In subsection 1.2.3.1, we describe a random walk in a space-time product environment, which can be considered as a random walk in a static random environment as well. Yilmaz deduced both quenched and annealed large deviations, but with the additional fact that both rate functions coincide in a neighborhood around the velocity vector of the walk. Subsection 1.2.3.2 describes an article written by Rassoul-Agha, Seppäläinen and Yilmaz [RSY11], which takes relevance by its high level of generality. They deduced a level 3 large deviation principle for several models, among them the random walk in a static random environment and the random walk in a Markovian time-dependent random environment. Finally, in subsection 1.2.3.3 we introduce the model of a random walk among a Poisson system of moving traps that satisfies a level 1 QLDP (see [DGRS12]). The proof of our level 1 QLDP for RWMRW in continuous time is inspired in techniques that Drewitz, Gärtner, Ramírez and Sun used to prove their level 1 QLDP in [DGRS12], namely: application of the subadditive ergodic theorem and an extension of the domain of a certain expression which in turn solves an equicontinuity issue.

1.2.3.1 Large deviations for random walk in a space-time product environment: Yilmaz's work at discrete time

We consider a random walk $(X_n)_{n\geq 0}$ in a time-dependent random environment such that the environment $\omega = \{\omega_n : n \geq 0\}$ with $\omega_n = \{\omega_n(x) : x \in \mathbb{Z}^d\}$ is such that the random variables

$$\{\omega_n(x)_{n,x} : n \in \mathbb{Z}, x \in \mathbb{Z}^d\} \tag{1.2.14}$$

are i.i.d. Recall that $\omega_n(x) := \{\omega_n(x,e) : e \in G\}$ where G is the range of the jumps and $\omega_n(x,e)$ is the probability that the walk jumps from site x at time n to site $x+e$ at time $n+1$. Note also that the time runs through all $\mathbb{Z}$. $(X_n)_{n\geq 0}$ can be viewed as a random walk on a static random environment in Z^{d+1}, defining a new walk as $Y_n := (n, X_n)$. However, if one wants to deduce a QLDP, Varadhan's result is not applicable here, since our new extended walk Y_n is not uniformly elliptic. So, it is necessary a new approach. Under this "restriction", Yilmaz in [Yil09a] obtained a QLDP for this model. In fact, he also proved an annealed large deviation principle with rate function $\mathcal{I}_a$ and a law of large numbers (LLN) for the mean velocity of the particle under $P_{0,0}$ that is, with a slightly abuse of notation, the annealed measure of the random walk starting at the space-time point (0,0). Assuming a uniform ellipticity condition (on X_n), Yilmaz in [Yil09a] took the point of view of the particle and focused on the environment Markov chain $(T_{n,X_n}, \omega)_{n\geq 0}$ to prove the following theorem

Theorem 1.2.3. (Yilmaz 2009) *If $d \geq 3$, there is a $\eta > 0$ such that the QLDP for the mean velocity of the particle holds in the η-neighborhood of v and the rate function is identically equal to the rate function I_a of the annealed LDP in this neighborhood.*

Here v is the limiting velocity, which comes from a law of large numbers (LLN).

1.2.3.2 Quenched free energy and large deviations for random walk in random potentials: discrete time

Rassoul-Agha, Seppäläinen and Yilmaz studied quenched distributions on random walks in a random potential on $\mathbb{Z}^d$ and with an arbitrary finite set of admissible steps in [RSY11]. The potential can be unbounded.

Let us introduce the model. Fix a finite subset $\mathcal{R} \subset \mathbb{Z}^d$. We consider $X_{0,\infty} := (X_n)_{n\geq 0}$ as a reference random walk, where P_x denotes the distribution of this walk starting at x and has jump probability $\hat{p}(z) = \frac{1}{|\mathcal{R}|}$ if $z \in \mathcal{R}$ and $\hat{p}(z) = 0$ otherwise. They considered $(\Omega, \mathfrak{G}, \mathbb{P}, \{T_z : z \in \mathcal{G}\})$ a measurable ergodic dynamical system, where $\{T_z : z \in \mathcal{G}\}$ is a group of measurable commuting bijections that satisfy $T_{x+y} = T_x T_y$ and T_0 is the identity and $\mathcal{G}$ is the additive subgroup of $\mathbb{Z}^d$ generated by $\mathcal{R}$. A potential is a measurable function $V : \Omega \times \mathcal{R}^l \to \mathbb{R}$, for some $l \in \mathbb{N}_0$. Given an environment ω and a starting point $x \in \mathbb{Z}^d$, for $n \geq 1$ define the *quenched polymer measures*

$$Q^{V,\omega}_{n,x}(X_{0,\infty} \in A) = \frac{1}{Z^{V,\omega}_{n,x}} E_x\left(e^{-\sum_{k=0}^{n-1} V(T_{X_k}\omega, Z_{k+1,k+l})} 1_A(X_{0,\infty})\right) \tag{1.2.15}$$

where $Z^{V,\omega}_{n,x}$ is called the *quenched partition function*, $Z_k = X_k - X_{k-1}$ is a step of the walk and $X_{i,j} = (X_i, X_{i+1}, \ldots, X_j)$.

RWRE is a special case of (1.2.15) with $V(\omega, z_{1,l}) = -\log \omega(0, z_1)$. Furthermore this model includes random walks in a dynamic random environment if $\mathcal{R} \subset \{x : x \cdot e_1 = 1\}$. They proved the $\mathbb{P}$-a.s existence of the *quenched free energy*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{V,\omega}_{n,0} \tag{1.2.16}$$

and derived two variational formulas for the limit (here the authors used a class of functions, which is a generalization of the class of functions $\mathcal{K}$ introduced by [Ros06]; see subsection 1.2.2.4).

On the other hand, they also deduced LDP for the quenched distributions

$$Q^{V,\omega}_{n,0}(R^\infty_n \in \cdot) \tag{1.2.17}$$

of the empirical process

$$R_n^\infty := n^{-1} \sum_{k=0}^{n-1} \delta_{T_{X_k}\omega, Z_{k+1,\infty}},$$

where $Z_{k+1,\infty} = (Z_i)_{k+1 \leq i < \infty}$ is the entire sequence of future steps. A level 2 QLDP means that the path component in the empirical measure has only one step: $\delta_{T_{X_k}\omega, Z_{k+1}}$. Therefore, a QLDP on (1.2.17) implies a level 2 QLDP as well as a level 1 QLDP through the contraction principle. Coming back to the particular case of RWRE, they assumed that

$$|\log \omega(0,z)| \in \mathcal{L}, \quad \text{for each } z \in \mathcal{R}, \tag{1.2.18}$$

where $\mathcal{L}$ is the following class of functions:

Definition 1.2.1. *A function $g : \Omega \to \mathbb{R}$ is in class $\mathcal{L}$ if $g \in L^1(\mathbb{P})$ and for any nonzero $z \in \mathbb{R}$*

$$\overline{\lim_{\epsilon \to 0}} \, \overline{\lim_{n \to \infty}} \max_{x \in \cup_{k=0} n D_k} \frac{1}{n} \sum_{0 \leq i \leq \epsilon n} |g \circ T_{x+iz}| = 0, \quad \mathbb{P} - a.s.,$$

where D_n is the set of points accessible from the origin in exactly n steps from $\mathcal{R}$.

In particular, a condition as (1.2.10) guarantees (1.2.18). Thus, it is straightforward to deduce that uniform ellipticity implies (1.2.18). Finally, Rassoul-Agha, Seppäläinen and Yilmaz obtained an explicitly formula for the rate function. In the case of RWRE, it can be expressed directly as the lower semicontinuous regularization of an entropy.

1.2.3.3 Random walks among a Poisson system of moving traps: continuous time

Drewitz, Gärtner, Ramírez and Sun in [DGRS12] considered a model of random walks among a Poisson system of moving traps at continuous time. Consider a system of independent simple random walks on Z^d with jump rate $\rho > 0$ and initial distribution being the product of Poisson distribution with intensity ν. For each $x \in Z^d$ and $t \geq 0$ call $\xi(t,x)$ the *number of walks at site x and time t*. This defines a field $\xi := \{\xi(t,x) : t \geq 0, x \in \mathbb{Z}^d\}$ which can be represented as

$$\xi(t,x) = \sum_{y\in\mathbb{Z}^d, 1\le j\le n_y} \delta_x(Y_j^y(t)), \tag{1.2.19}$$

where $\{n_y\}_{y\in\mathbb{Z}^d}$ are independent Poisson random variables with mean ν, and $\{Y_j^t\}_{y\in\mathbb{Z}^d,1\le j\le n_y}$ is a family of independent simple random walks each with jump rate ρ and n_y particles start from site y. Let us call $\mathbb{P}$ their joint law. Consider now a simple random walk $\{X_t : t \ge 0\}$ with jump rate $\kappa > 0$. Let us call P_x^X its law starting from x and E_x^X the corresponding expectation.

The collection of walks Y are interpreted as traps and at each time t so that the walk X is killed with rate $\gamma\xi(t, X(t))$ for some parameter $\gamma > 0$. Conditional on the realization of the field of traps ξ, the probability that the walk survives by time t is given by

$$Z_{t,\xi}^{\gamma} := E_0^X \left(e^{-\gamma\int_0^t \xi(s,X(s))ds} \right). \tag{1.2.20}$$

We call this the *quenched survival probability*, which depends on the random medium ξ. For $x \in \mathbb{Z}^d$ and $s \ge 0$ define $a(s,t,x,y,\xi) := -\log e(s,t,x,y,\xi)$ with

$$e(s,t,x,y) := E_{x,s}^X \left(e^{-\gamma\int_s^t \xi(s,X(u))du} 1_{X(t)=y} \right) \tag{1.2.21}$$

where $P_{x,s}^X$ and $E_{x,s}^X$ denote respectively probability and expectation for a jump rate κ simple symmetric random walk X, starting from x at time s.

To prove the level 1 QLDP, Drewitz, Gärtner, Ramírez and Sun in [DGRS12] applied the subadditive ergodic theorem to $a(s,t,x,y) := -\log e(s,t,x,y)$ to show the existence of a function α that can be defined on $\mathbb{Q}^d$. At this point, it is necessary to solve an equicontinuous issue. To deal with this, Drewitz, Gärtner, Ramírez and Sun used well known large deviation estimates of the simple symmetric random walk on $\mathbb{Z}^d$. From this, they extended the function α to all $\mathbb{R}^d$ and deduced a *shape theorem* (terminology used by Sznitman in [S98]), which is an uniform approximation on a compact K of $\mathbb{R}^d$ between the mean of $a(0,t,0,x)$ and $\alpha\left(\frac{x}{t}\right)$ for each $x \in \mathbb{Z}^d \cap tK$, which is in turn the most difficult part in the proof of their QLDP.

1.3 Our results

In this section we will state the main results of the thesis. The thesis is divided in two parts: (1) ballisticity for elliptic RWRE; (2) quenched large deviations for RWMRE. In the first part we will introduce a new class of ellipticity conditions on the environment which ensures the applicability of the classical ballisticity conditions. In the second part, our interest will be oriented to the establishment of a quenched large deviation principle for the mean of the walk, whose proofs will be short and direct. Several models are covered since our assumptions are very general.

1.3.1 Main results for RWRE

Let us first introduce the following class of ellipticity criteria.

Definition 1.3.1. *For each $V \subset U$ consider the set of non-negative numbers $\{\alpha(e) : e \in V\}$. Let*

$$\mathcal{E}_V := \left\{ \{\alpha(e) : e \in V\} : \mathbb{E}\left[e^{\sum_{e\in V} \alpha(e) \log \frac{1}{\omega(0,e)}} \right] < \infty \right\}.$$

For each $e \in U$, we will use the notation $\mathcal{E}_e := \mathcal{E}_{\{e\}}$ and define

$$F_e := \sup\{\alpha \geq 0 : \{\alpha\} \in \mathcal{E}_e\}.$$

Let $\beta \geq 0$. We say that the law of the environment satisfies the ellipticity condition $(E)_\beta$ if for every $e \in U$ we have that

$$\min_{e\in U} F_e > \beta. \tag{1.3.1}$$

Note that the polynomial ballisticity condition (cf. subsection 1.1.3.7) can be defined even if the environment is not uniformly elliptic. Our first main result is the following one:

Theorem 1.3.1. *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Assume that the environment satisfies the ellipticity condition $(E)_0$. Then the polynomial condition $(P)_M|l$ is equivalent to $(T')|l$.*

Let us now define a second class of ellipticity criteria. Let $\beta \geq 0$. We say that the law of the environment satisfies the *ellipticity condition* $(E')_\beta$ if there exists an $\bar{\alpha} := \{\alpha(e) : e \in U\}$ such that

(i) $\sum_e \alpha(e) > \beta$.

(ii) $\bar{\alpha} \in \mathcal{E}_U$.

(iii) Condition $(E)_0$ is satisfied.

(iv) $\min_{e \in U \cap H_{\hat{v}}} F_e \geq \max_{e \in U} \alpha(e)$,

where $H_{\hat{v}}$ is the half space determined by the asymptotic direction $\hat{v}$ and defined by

$$H_{\hat{v}} := \{l \in \mathbb{S}^{d-1} : l \cdot \hat{v} \geq 0\}.$$

Assuming our polynomial ballisticity condition $(P)_M$, $\hat{v}$ is well-defined (see [Sim07]). From this, our second main result on RWRE can be written in terms of either the ellipticity condition $(E)_{\frac{1}{2}}$ or the ellipticity condition $(E')_1$.

Theorem 1.3.2. (Law of large numbers) *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$. Also, assume that either $(E)_{1/2}$ or $(E')_1$ is satisfied. Then the random walk is ballistic in direction l. Furthermore, there is a $v \in \mathbb{R}^d$, $v \neq 0$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad P_0 - a.s.$$

Assuming greater values of β and β' in the ellipticity conditions $(E)_\beta$ and $(E')_{\beta'}$ respectively, we have a third main result.

Theorem 1.3.3. *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$. Assume that the environment satisfies the ellipticity condition $(E)_0$.*

a) **(Annealed central limit theorem)** *If $(E)_1$ or if $(E')_2$ is satisfied then*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v)$$

converges in law under P_0 as $\epsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

b) **(Quenched central limit theorem)** *If $(E)_{88d}$ or if $(E')_{176d}$ is satisfied, then $\mathbb{P}$-a.s. we have that*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v)$$

converges in law under $P_{0,\omega}$ as $\epsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

1.3.2 Main results for RWMRE

The main results deduced for RWMRE are quenched large deviation principles in continuous time as well as in discrete time. In both cases, the proofs are short and direct and take into consideration works which follow an approach to deal with quenched large deviations, which is different from the one presented in [Yil09b] and [RSY11], being really a development of Varadhan's original ideas using the subadditive ergodic theorem.

In the continuous case, we take advantage of the approach given in [DGRS12], where large deviation estimates of a simple symmetric random walk are very useful. In our case, the environmental process $\omega = (\omega_t)_{t\geq 0}$ has bounded rates and thus it is possible to find a Radon-Nikodym derivative which relates this process with the law of a continuous simple symmetric random walk.

Theorem 1.3.4. *Consider a continuous time random walk $\{X_t : t \geq 0\}$ in a uniformly elliptic time-dependent environment ω such that $\{T_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ is an ergodic family. Then, there exists a convex continuous rate function $I_c(x) : \mathbb{R}^d \to [0,\infty)$ such that the following are satisfied.*

(i) *For every open set $G \subset \mathbb{R}^d$ we have that Q^c_μ-a.s.*

$$\varliminf_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in G\right) \geq -\inf_{x\in G} I_c(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ we have that Q^c_μ-a.s.

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\right) \leq -\inf_{x\in C} I_c(x).$$

In the discrete case, we denote by R the set of admissible steps of the walk (cf. subsection 1.2.1). We will prove a large deviation principle assuming that either R is finite, convex, symmetric and there is a neighborhood of 0 which belongs to the convex hull of R or R corresponds to the nearest neighbor case. Let U be the set defined by

$$U := \left\{x \in \mathbb{R}^d : x = \lim_{n\to\infty} x_n, \text{ for some sequence } x_n \in U_n\right\} \tag{1.3.2}$$

where $U_n := \dfrac{R_n}{n}$, being R_n the set of sites that a random walk with jump rate R visits with probability positive at time n. Now, our discrete time version of a large deviation principle is the following theorem.

Theorem 1.3.5. *Consider a discrete time random walk $\{X_n : n \geq 0\}$ in a uniformly elliptic time-dependent environment ω such that $\{T_{1,x} : x \in R\}$ is an ergodic family with jump range R. Assume that either (i) R is finite, convex, symmetric and there is a neighborhood of 0 which belongs to the convex hull of R; (ii) or that R corresponds to the nearest neighbor case. Consider U defined in (1.3.2). Then U equals the convex hull of R and there exists a convex rate function $I_d(x) : \mathbb{R}^d \to [0,\infty]$ such that $I_d(x) \leq |\log\kappa|$ for $x \in U$, $I_d(x) = \infty$ for $x \notin U$, I is continuous for every $x \in U^o$ and the following are satisfied.*

(i) For every open set $G \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in G\right) \geq -\inf_{x\in G} I_d(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x\in C} I_d(x).$$

CHAPTER 2

ELLIPTICITY CRITERIA FOR BALLISTIC BEHAVIOR OF RANDOM WALKS IN RANDOM ENVIRONMENT

2.1 Introduction

We introduce ellipticity criteria for random walks in random environment which enable us to extend to environments which are not necessarily uniformly elliptic the ballisticity conditions for the uniformly elliptic case of Sznitman [Sz02] and of Berger, Drewitz and Ramírez [BDR12], their equivalences and some of their consequences [SZ99, Sz00, Sz01, Sz02, RAS09, BZ08].

For $x \in \mathbb{R}^d$, denote by $|x|_1$ and $|x|_2$ its l_1 and l_2 norm respectively. Call $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\} = \{e_1, \ldots, e_{2d}\}$ the canonical vectors with the convention that $e_{d+i} = -e_i$ for $1 \leq i \leq d$ and let $\mathcal{P} := \{p(e) : p(e) \geq 0, \sum_{e \in U} p(e) = 1\}$. An *environment* is an element ω of the *environment space* $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ so that $\omega := \{\omega(x) : x \in \mathbb{Z}^d\}$, where $\omega(x) \in \mathcal{P}$. We denote the components of $\omega(x)$ by $\omega(x, e)$. The *random walk in the environment* ω *starting from* x is the Markov chain $\{X_n : n \geq 0\}$ in $\mathbb{Z}^d$ with law $P_{x,\omega}$ defined by the condition $P_{x,\omega}(X_0 = x) = 1$ and the transition probabilities

$$P_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega(x, e)$$

for each $x \in \mathbb{Z}^d$ and $e \in U$. Let $\mathbb{P}$ be a probability measure defined on the environment space Ω endowed with its Borel σ-algebra. We will assume that $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under $\mathbb{P}$. We will call $P_{x,\omega}$ the *quenched law* of the random walk in random environment (RWRE) starting from x, while $P_x := \int P_{x,\omega} d\mathbb{P}$ the *averaged* or *annealed law* of the RWRE starting from x.

We say that the law $\mathbb{P}$ of the RWRE is *elliptic* if for every $x \in \mathbb{Z}^d$ and $e \in U$ one has that $\mathbb{P}(\omega(x,e) > 0) = 1$. We say that $\mathbb{P}$ is *uniformly elliptic* if there exists a constant $\kappa > 0$ such that for every $x \in \mathbb{Z}^d$ and $e \in U$ it is true that $\mathbb{P}(\omega(x,e) \geq \kappa) = 1$. Given $l \in \mathbb{S}^{d-1}$ we say that the RWRE is *transient in direction* l if

$$P_0(A_l) = 1,$$

where

$$A_l := \{\lim_{n\to\infty} X_n \cdot l = \infty\}$$

We say that it is *ballistic in direction* l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

The following is conjectured (see for example [Sz04]).

Conjecture 2.1.1. *Let $l \in \mathbb{S}^{d-1}$. Consider a random walk in a uniformly elliptic i.i.d. environment in dimension $d \geq 2$, which is transient in direction l. Then it is ballistic in direction l.*

Some partial progress towards the resolution of this conjecture has been made in [Sz01, Sz02, DR11, DR12, BDR12]. In 2001 and 2002 Sznitman in [Sz01, Sz02] introduced a class of ballisticity conditions under which he could prove the above statement. For each subset $A \subset \mathbb{Z}^d$ define the first exit time from the set A as

$$T_A := \inf\{n \geq 0 : X_n \notin A\}. \tag{2.1.1}$$

For $L > 0$ and $l \in \mathbb{S}^{d-1}$ define the slab

$$U_{l,L} := \{x \in \mathbb{Z}^d : -L \leq x \cdot l \leq L\}. \tag{2.1.2}$$

Given $l \in \mathbb{S}^{d-1}$ and $\gamma \in (0,1)$, we say that condition $(T)_\gamma$ in direction l (also written as $(T)_\gamma|l)$ is satisfied if there exists a neighborhood $V \subset \mathbb{S}^{d-1}$ of l such that for all $l' \in V$

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log P_0(X_{T_{U_{l',L}}} \cdot l' < 0) < 0.$$

Condition $(T')|l$ is defined as the fullfilment of condition $(T)_\gamma|l$ for all $\gamma \in (0,1)$. Sznitman [Sz02] proved that if a random walk in an i.i.d. uniformly elliptic environment satisfies $(T')|l$ then it is ballistic in direction l. He also showed that if $\gamma \in (0.5,1)$, then $(T)_\gamma$ implies (T'). In 2011, Drewitz and Ramírez [DR11] showed that there is a $\gamma_d \in (0.37, 0.39)$ such that if $\gamma \in (\gamma_d, 1)$, then $(T)_\gamma$ implies (T'). In 2012, in [DR12], they were able to show that for dimensions $d \geq 4$, if $\gamma \in (0,1)$, then $(T)_\gamma$ implies (T'). Recently in [BDR12], Berger, Drewitz and Ramírez introduced a polynomial ballisticity condition, weakening further the conditions $(T)_\gamma$. The condition is effective, in the sense that it can a priori be verified explicitly for a given environment. To define it, for each $L, L', \tilde{L} > 0$ and $l \in \mathbb{S}^{d-1}$ consider the box

$$B_{l,L',L,\tilde{L}} := R\left((-L', L) \times \left(-\tilde{L}, \tilde{L}\right)^{d-1}\right) \cap \mathbb{Z}^d,$$

where R is a rotation of $\mathbb{R}^d$ defined by the condition

$$R(e_1) = l. \tag{2.1.3}$$

Let also

$$L_0 := \frac{2}{3} 3^{29d}. \tag{2.1.4}$$

Given $M \geq 1$, we say that condition $(P)_M$ in direction l is satisfied (also written as $(P)_M|l$) if for every $L \geq L_0$, $L' \leq \frac{5}{4}L$ and $\tilde{L} \leq 72L^3$ one has the following upper bound for the probability that the walk does not exit the box $B_{l,L',L,\tilde{L}}$ through its front side

$$P_0(X_{T_{B_{l,L',L,\tilde{L}}}} \cdot l < L) \leq \frac{1}{L^M}.$$

In [BDR12], Berger, Drewitz and Ramírez prove that every random walk in an i.i.d. uniformly elliptic environment which satisfies $(P)_M$ for $M \geq 15d+5$ is necessarily ballistic.

On the other hand, it is known (see for example Sabot-Tournier [ST11]) that in dimension $d \geq 2$, there exist elliptic random walks which are transient in a given direction but not ballistic in that direction. The purpose of this chapter is to investigate to which extent can the assumption of uniform ellipticity be weakened. To do this we introduce several classes of ellipticity conditions on the environment. For each $V \subset U$ consider the set of non-negative numbers $\{\alpha(e) : e \in V\}$. Let

$$\mathcal{E}_V := \left\{ \{\alpha(e) : e \in V\} : \mathbb{E}\left[e^{\sum_{e\in V} \alpha(e) \log \frac{1}{\omega(0,e)}} \right] < \infty \right\}.$$

For each $e \in U$, we will use the notation $\mathcal{E}_e := \mathcal{E}_{\{e\}}$ and define

$$F_e := \sup\{\alpha \geq 0 : \{\alpha\} \in \mathcal{E}_e\}.$$

Let $\beta \geq 0$. We say that the law of the environment satisfies the *ellipticity condition* $(E)_\beta$ if for every $e \in U$ we have that

$$\min_{e\in U} F_e > \beta.$$

The first main result of this chapter is the following one.

Theorem 2.1.1. *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Assume that the environment satisfies the ellipticity condition $(E)_0$. Then the polynomial condition $(P)_M|l$ is equivalent to $(T')|l$.*

In this chapter we go further from Theorem 2.1.1, and we obtain assuming (T'), good enough tail estimates for the distribution of the regeneration times of the random walk. Let us recall that there exists an *asymptotic direction* if the limit

$$\hat{v} := \lim_{n\to\infty} \frac{X_n}{|X_n|_2}$$

exists P_0-a.s. Simenhaus in [Sim07], shows that if there is a neighborhood V of direction l such that for every $l' \in V$ the random walk is transient, then the asymptotic direction exists (cf. Subsection 1.1.3.1). It follows that under the polynomial condition $(P)_M$ the asymptotic direction exists. In this case, let us define the half space

$$H_{\hat{v}} := \{l \in \mathbb{S}^{d-1} : l \cdot \hat{v} \geq 0\}.$$

Let $\bar{\alpha} := \{\alpha(e) : e \in U\} \in [0,\infty)^{2d}$. Let $\beta \geq 0$. We say that the law of the environment satisfies the *ellipticity condition* $(E')_\beta$ if there exists an $\bar{\alpha} := \{\alpha(e) : e \in U\}$ such that

(i) $\sum_e \alpha(e) > \beta$.

(ii) $\bar{\alpha} \in \mathcal{E}_U$.

(iii) Condition $(E)_0$ is satisfied.

(iv) $\min_{e \in U \cap H_{\hat{v}}} F_e > \max_{e \in U} \alpha(e)$.

The second main result of this chapter is the following theorem.

Theorem 2.1.2. (Law of large numbers) *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$. Also, assume that either $(E)_{1/2}$ or $(E')_1$ is satisfied. Then the random walk is ballistic in direction l. Furthermore, there is a $v \in \mathbb{R}^d$, $v \neq 0$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad P_0 - a.s.$$

It should be noted that condition $(E)_{1/2}$ does not imply $(E')_1$, nor vice-versa.

On the other hand, the the value 1/2 of condition $(E)_{1/2}$ in Theorem 2.1.2 is optimal. Indeed, in analogy to the random conductance model studied by Fribergh in [F11], it is easy to construct an environment such that for every $\epsilon > 0$ one has that

$$\sup_e \mathbb{E}\left[\left(\frac{1}{\omega(0,e)}\right)^{1/2-\epsilon}\right] < \infty,$$

but the walk is transient in direction e_1 but not ballistic in direction e_1. Let ϕ be any random variable taking values on the interval $(0, 1/4)$ and such that the expected value of $\phi^{-1/2}$ is infinite, while for every $\epsilon > 0$, the expected value of $\phi^{-(1/2-\epsilon)}$ is finite. Let X be a Bernoulli random variable of parameter 1/2. We now define $\omega_1(0,e_1) = 2\phi$, $\omega_1(0,-e_1) = \phi$, $\omega_1(0,-e_2) = \phi$ and $\omega_1(0,e_2) = 1-4\phi$ and $\omega_2(0,e_1) = 2\phi$, $\omega_2(0,-e_1) = \phi$, $\omega_2(0,e_2) = \phi$ and $\omega_2(0,-e_2) = 1-4\phi$. We then let the environment at site 0 be given by the random variable $\omega(0) := 1_X(1)\omega_1(0) + 1_X(0)\omega_2(0)$. This environment has the property that traps can appear, where the random walk gets caught in an edge, as shown shown in Figure 2.1. Furthermore, as we will show, it is not difficult to check that the random walk in this random environment is transient in direction e_1 but not ballistic. We cannot say at this moment if this environment satisfies or not the polynomial condition $(P)_M$ for $M \geq 15d + 5$.

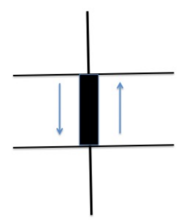

Figure 2.1: A trap produced by an elliptic environment which does not satisfy $(E)_{\frac{1}{2}}$.

Similar examples of random walks in elliptic i.i.d. random environment which are transient in a given direction but not ballistic have been exhibited within the context of the Dirichlet environment. Here, the environment is chosen i.i.d. with a Dirichlet distribution at each site $D(\beta_1, \ldots, \beta_{2d})$ of parameters $\beta_1, \ldots, \beta_{2d} > 0$ (see for example [Sa11, Sa12, ST11]), the parameter β_i being associated with the direction e_i. For a random walk in Dirichlet random environment (RWDRE), condition $(E)_{1/2}$ is equivalent to the condition

$$\min_{1 \leq i \leq 2d} \beta_i > \frac{1}{2}.$$

On the other hand, condition $(E')_1$ is equivalent to

$$d_* \beta_* + \sum_{j : e_j \notin H_{\hat{v}}} \beta_j \wedge \beta_* > 1,$$

where β_* is the minimal value of β_i among those directions which lie in the half space $H_{\hat{v}}$ and d_* is the number of directions which lie in the half spcae $H_{\hat{v}}$. These conditions are stronger than the ballisticity condition for RWDRE given by Tournier in [T11]. Furthermore, they are also stronger than the characterization of ballisticity for RWDRE given by Sabot in [Sa12] for random walks in random Dirichlet environments in dimension $d \geq 3$. Indeed, Sabot defines the parameter

$$\lambda := 2 \sum_{i=1}^{2d} \beta_i - \max_{1 \leq i \leq d} (\beta_i + \beta_{i+d}). \tag{2.1.5}$$

Tournier in [T11] proved that if $\lambda \leq 1$, then the RWDRE is not ballistic in any direction. Sabot in [Sa12], showed that if $\lambda > 1$, and if there is an $i = 1, \ldots, d$ such that $\beta_i \neq \beta_{i+d}$, then the random walk is ballistic. It is thus natural to wonder to what general condition corresponds (not restricted to random Dirichlet environments), the characterization of Sabot and Tournier. In section 2.2, we will see that there are several formulations of the condition of Sabot and Tournier for RWDRE, but which are not equivalent for general RWRE. Among these formulations, the following one is the weakest one in general. We say that condition (ES) is satisfied if

$$\max_{i:1 \leq i \leq d} \mathbb{E}\left[\frac{1}{1 - \omega(0, e_i)\omega(e_i, -e_i)}\right] < \infty.$$

We have furthermore the following proposition whose proof will be presented in section 2.2.

Proposition 2.1.1. *Consider a random walk in a random environment. Assume that condition (ES) is not satisfied. Then the random walk is not ballistic.*

We will see in the proof of Proposition 2.1.1 how important is the role played by certain edges which play the role of traps.

Another consequence of Theorem 2.1.1 and the machinery that we develop to estimate the tails of the regeneration times, is the following theorem.

Theorem 2.1.3. *Consider a random walk in an i.i.d. environment in dimensions $d \geq 2$. Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$. Assume that the environment satisfies the ellipticity condition $(E)_0$.*

a) **(Annealed central limit theorem)** *If $(E)_1$ or if $(E')_2$ is satisfied then*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v)$$

converges in law under P_0 as $\epsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

b) **(Quenched central limit theorem)** *If $(E)_{88d}$ or if $(E')_{176d}$ is satisfied, then $\mathbb{P}$-a.s. we have that*

$$\epsilon^{1/2}(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v)$$

converges in law under $P_{0,\omega}$ as $\epsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

Part (b) of the above Theorem is based on a result of Rassoul-Agha and Seppäläinen [RAS09], which gives as a condition so that an elliptic random walk satisfies the quenched central limit theorem that the regeneration times have moments of order higher than $176d$. As they point out in their paper, this particular lower bound on the moment should not have any meaning and it is likely that it could be improved. For example, Berger and Zeitouni in [BZ08], also prove the quenched central limit theorem under lower order moments for the regeneration times but under the assumption of uniform ellipticity. It should be possible to extend their methods to elliptic random walks in order to improve the moment condition of part (b) of Theorem 2.1.3.

The proof of Theorem 2.1.1 requires extending the methods that have already been developed within the context of random walks in uniformly elliptic random environments. Its proof is presented in section 2.3. To do this, we first need to show as in [BDR12], that the polynomial condition $(P)_M$ for $M \geq 15d+5$, implies the so called effective criterion, defined by Sznitman in [Sz02] for random walks in uniformly elliptic environments, and extended here for random walks in random environments satisfying condition $(E)_0$. Two renormalization methods are employed here, which need to take into account the fact that the environment is not necessarily uniformly elliptic. These are developed in subsections 2.3.1 and 2.3.2. In subsection 2.3.4 it is shown, following [Sz02], that the effective criterion implies condition (T'). The adaptation of the methods of [BDR12] and [Sz02] from uniformly elliptic environments to environments satisfying some of the ellipticity conditions that have been introduced is far from being straightforward.

The proof of Theorems 2.1.2 and 2.1.3, is presented in sections 2.4 and 2.5. In section 2.4, an atypical quenched exit estimate is derived which requires a very careful choice of the renormalization method, and includes the definition of an event which we call the *confinement event*, which ensures that the random walk will be able to find a path to an exit column where it behaves as if the environment was uniformly elliptic. In section 2.5, we derive the moments estimates of the regeneration time of the random walk using the atypical quenched exit estimate of section 2.4. Here, the conditions $(E)_{1/2}$ or $(E')_1$ are required, and appear as the possibility of finding either two or $2d$ different paths, respectively, connecting two points in the lattice.

2.2 Notation and preliminary results

Here we will fix up the notation of the chapter and will introduce the main tools that will be used. In subsection 2.2.2 we will prove Proposition 2.1.1. Its proof is straightforward, but instructive.

2.2.1 Setup and background

Throughout the whole chapter we will use letters without subindexes like c, ρ or κ to denote any generic constant, while we will use the notation $c_{3,1}, c_{3,2}, \ldots, c_{4,1}, c_{4,2}, \ldots$ to denote the specific constants which appear in each section of the chapter. Thus, for example $c_{4,2}$ is the second constant of section 4. On the other hand, we will use c_1, c_2, c_3, c_4, c'_1 and c'_2 for specific constants which will appear several times in several sections. Let $c_1 \geq 1$

be any constant such that for any pair of points $x, y \in \mathbb{Z}^d$, there exists a nearest neighbor path between x and y with less than

$$c_1 \max\{|x-y|_2, 1\} \tag{2.2.1}$$

sites. Given $U \subset \mathbb{Z}^d$, we will denote its outer boundary by

$$\partial U := \{x \notin U : |x-y|_1 = 1, \text{ for some } y \in U\}.$$

We define $\{\theta_n : n \geq 1\}$ as the canonical time shift on $\mathbb{Z}^{d^{\mathbb{N}}}$. For $l \in \mathbb{S}^{d-1}$ and $u \geq 0$, we define the times

$$T_u^l := \inf\{n \geq 0 : X_n \cdot l \geq u\} \tag{2.2.2}$$

and

$$\tilde{T}_u^l := \inf\{n \geq 0 : X_n \cdot l \leq u\}.$$

Throughout, we will denote any nearest neighbor path with n steps joining two points $x, y \in \mathbb{Z}^d$ by $(x_1, x_2, \ldots, x_n)$, where $x_1 = x$ and $x_n = y$. Furthermore, we will employ the notation

$$\Delta x_i := x_{i+1} - x_i, \tag{2.2.3}$$

for $1 \leq i \leq n-1$, to denote the directions of the jumps through this path. Finally, we will call $\{t_x : x \in \mathbb{Z}^d\}$ the canonical shift defined on Ω so that for $\omega = \{\omega(y) : y \in \mathbb{Z}^d\}$,

$$t_x(\omega) = \bar{\omega} := \{\omega(x+y) : y \in \mathbb{Z}^d\}. \tag{2.2.4}$$

Let us now define the concept of *regeneration times* with respect to direction l. Let

$$a > 2\sqrt{d} \tag{2.2.5}$$

and

$$D^l := \min\{n \geq 0 : X_n \cdot l < X_0 \cdot l\}.$$

Define $S_0 := 0$, $M_0 := X_0 \cdot l$,

$$S_1 := T^l_{M_0+a}, \quad R_1 := D^l \circ \theta_{S_1},$$

$$M_1 := \sup\{X_n \cdot l : 0 \leq n \leq R_1\},$$

and recursively for $k \geq 1$,

$$S_{k+1} := T^l_{M_k+a}, \quad R_{k+1} := D^l \circ \theta_{S_{k+1}} + S_{k+1},$$

$$M_{k+1} := \sup\{X_n \cdot l : 0 \leq n \leq R_{k+1}\}.$$

Define the *first regeneration time* as

$$\tau_1 := \min\{k \geq 1 : S_k < \infty, R_k = \infty\}.$$

The condition (2.2.5) on a will be eventually useful to prove the non-degeneracy of the covariance matrix of part (a) of Theorem 2.1.3. Now define recursively in n the $(n+1)$-st regeneration time τ_{n+1} as $\tau_1(X_\cdot) + \tau_n(X_{\tau_1+\cdot} - X_{\tau_1})$. Throughout the sequel, we will occasionally write $\tau_1^l, \tau_2^l, \ldots$ to emphasize the dependence of the regeneration times with respect to the chosen direction. It is a standard fact (see for example Sznitman and Zerner [SZ99]) to show that the sequence $((\tau_1, X_{(\tau_1+\cdot)\wedge\tau_2} - X_{\tau_1}), (\tau_2 - \tau_1, X_{(\tau_2+\cdot)\wedge\tau_3} - X_{\tau_2}), \ldots)$ is independent and except for its first term also i.i.d. with the same law as that of τ_1 with respect to the conditional probability measure $P_0(\cdot|D^l = \infty)$. This implies the following theorem (see Zerner [Z02] and Sznitman and Zerner [SZ99] and Sznitman [Sz00]).

Theorem 2.2.1. (Sznitman and Zerner [SZ99], Zerner [Z02], Sznitman [Sz00]) *Consider a RWRE in an elliptic i.i.d. environment. Let $l \in \mathbb{S}^{d-1}$ and assume that there is a neighborhood V of l such that for every $l' \in V$ the random walk is transient in the direction l'. Then there is a deterministic v such that P_0-a.s. one has that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v.$$

Furthermore, the following are satisfied.

a) If $E_0[\tau_1] < \infty$, the walk is ballistic and $v \neq 0$.

b) If $E_0[\tau_1^2] < \infty$ we have that

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right)$$

converges in law under P_0 to a Brownian motion with non-degenerate covariance matrix.

Rassoul-Agha and Seppäläinen in [RAS09] and Berger and Zeitouni in [BZ08] were able to prove a quenched central limit theorem under good enough moment conditions on the regeneration times. The result of Rassoul-Agha and Seppäläinen which does not require a uniform ellipticity assumption is the following one.

Theorem 2.2.2. (Rassoul-Agha and Seppäläinen [RAS09]) *Consider a RWRE in an elliptic i.i.d. environment. Let $l \in \mathbb{S}^{d-1}$ and let τ_1 be the corresponding regeneration time. Assume that*

$$E_0[\tau_1^p] < \infty,$$

for some $p > 176d$. Then $\mathbb{P}$-a.s. we have that

$$\epsilon^{1/2}\left(X_{[\epsilon^{-1}n]} - [\epsilon^{-1}n]v\right)$$

converges in law under $P_{0,\omega}$ to a Brownian motion with non-degenerate covariance matrix.

We now define the *n-th regeneration radius* as

$$X^{*(n)} := \max_{\tau_{n-1} \le k \le \tau_n} |X_k - X_{\tau_{n-1}}|_1.$$

The following theorem was stated and proved without using uniform ellipticity by Sznitman as Theorem A.2 of [Sz02], and provides a control on the lateral displacement of the random walk with respect to the asymptotic direction. We need to define for $z \in \mathbb{R}^d$

$$\pi(z) := z - (z \cdot \hat{v})\hat{v}.$$

Theorem 2.2.3. (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment satisfying condition $(T)_\gamma|l$. Let $l \in \mathbb{S}^{d-1}$ and $\gamma \in (0,1)$. Then, for any $c > 0$ and $\rho \in (0.5, 1)$,*

$$\limsup_{u\to\infty} u^{-(2\rho-1)\wedge(\gamma\rho)} \log P_0\left(\sup_{0\le n\le T_u^l} |\pi(X_n)| \ge cu^\rho \right) < 0,$$

where T_u^l is defined in (2.2.2).

Define the function $\gamma_L : [2,\infty) \to \mathbb{R}$ as

$$\gamma_L := \frac{\log 2}{\log\log L}. \tag{2.2.6}$$

Given $l \in \mathbb{S}^{d-1}$, we say that condition $(T)_0$ in direction l (also written as $(T)_0|l$) is satisfied if there exists a neighborhood $V \subset \mathbb{S}^{d-1}$ of l such that for all $l' \in V$

$$\limsup_{L\to\infty} \frac{1}{L^{\gamma_L}} \log P_0(X_{T_{U_{l',L}}} \cdot l' < 0) < 0,$$

where the slabs $U_{l',L}$ are defined in (2.1.2). An important consequence of Theorem 2.2.3 is the following equivalence proved by Sznitman [Sz02], for the case $\gamma \in (0,1)$ and which does not use uniform ellipticity. It is easy to extend Sznitman's proof to include the case $\gamma = 0$.

Theorem 2.2.4. (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment. Let $\gamma \in [0,1)$ and $l \in S^{d-1}$. Then the following are equivalent.*

(i) Condition $(T)_\gamma|l$ is satisfied.

(ii) $P_0(A_l) = 1$ and if $\gamma > 0$ we have that $E_0[\exp\{c(X^{(1)})^\gamma\}] < \infty$ for some $c > 0$, while if $\gamma = 0$ we have that $E_0[\exp\{c(X^{(1)})^{\gamma_L}\}] < \infty$ for some $c > 0$.

(iii) There is an asymptotic direction $\hat{v}$ such that $l \cdot \hat{v} > 0$ and for every l' such that $l' \cdot \hat{v} > 0$ one has that $(T)_\gamma|l'$ is satisfied.

The following corollary of Theorem 2.2.4 will be important.

Corollary 2.2.1. (Sznitman [Sz02]) *Consider a RWRE in an elliptic i.i.d. environment. Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$. Assume that $(T)_\gamma|_l$ holds. Then there exists a constant c such that for every L and $n \geq 1$ one has that*

$$P_0\left(X^{*(n)} > L\right) \leq \frac{1}{c} e^{-cL^\gamma}. \tag{2.2.7}$$

2.2.2 Comments and proof of Proposition 2.1.1

Let us show that $(E)_{\frac{1}{2}}$ and $(E')_1$ are stronger conditions than (ES). We will do this passing through another ellipticity condition. We say that condition (ES') is satisfied if there exist nonnegative real numbers $\alpha_1, \ldots, \alpha_d$ and $\alpha'_1, \ldots, \alpha'_d$ such that

$$\min_{1\leq i\leq d}(\alpha_i + \alpha'_i) > 1$$

and

$$\max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_i)}\right)^{\alpha_i}\right] < \infty \quad \text{and} \quad \max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_{i+d})}\right)^{\alpha'_i}\right] < \infty. \tag{2.2.8}$$

We have the following lemma.

Lemma 2.2.1. *Consider a random walk in an i.i.d. random environment. Then either condition $(E)_{1/2}$ or $(E')_1$ imply (ES') which in turn implies (ES). Furthermore, for a random walk in a random Dirichlet environment, (ES) and (ES') are equivalent to $\lambda > 1$ (cf. (2.1.5)).*

Proof. It is easy to check that $(E)_{1/2}$ implies (ES') and that for random Dirichlet environments (ES') and (ES) are equivalent to $\lambda > 1$. We therefore first prove that (ES') implies (ES). Note first that by the independence between $\omega(0,e_i)$ and $\omega(e_i,-e_i)$, (2.2.8) is equivalent to

$$\max_{1\leq i\leq d} \mathbb{E}\left[\left(\frac{1}{1-\omega(0,e_i)}\right)^{\alpha_i}\left(\frac{1}{1-\omega(e_i,-e_i)}\right)^{\alpha'_i}\right] < \infty.$$

Then it is enough to prove that for each pair of real numbers u_1, u_2 in $(0,1)$ one has that

$$\frac{1}{1-u_1u_2} \leq \frac{1}{(1-u_1)^{\alpha}(1-u_2)^{\alpha'}} \tag{2.2.9}$$

for any α, $\alpha' \geq 0$ such that $\alpha + \alpha' > 1$. Now if we denote by $v_1 = 1-u_1$ and $v_2 = 1-u_2$ then (2.2.9) is equivalent to

$$v_1v_2 + v_1^{\alpha}v_2^{\alpha'} \leq v_1 + v_2. \tag{2.2.10}$$

But (2.2.10) follows easily by our conditions on v_1, v_2, α and α'. To prove that $(E')_1$ implies (ES'), we choose for each $1 \leq i \leq d$

$$\alpha_i = \sum_{\alpha(e)\neq\alpha(e_i)} \alpha(e), \quad \alpha_i' = \sum_{\alpha(e)\neq\alpha(e_{i+d})} \alpha(e).$$

Note in particular that

$$\alpha_i + \alpha_i' > 1, \quad \forall i \in \{1, \ldots, d\}. \tag{2.2.11}$$

Furthermore, by $(E')_1$ and the monotonocity of the function $\log x$ one has that

$$\mathbb{E}\left(e^{-\sum_{e\neq e_i}\alpha(e)\log\omega(0,e)}\right) < \infty, \quad \alpha_i \log \sum_{e\neq e_i}\omega(0,e) \geq \sum_{e\neq e_i}\alpha(e)\log\omega(0,e). \tag{2.2.12}$$

for each $1 \leq i \leq d$. Then (ES') follows by (2.2.11) and (2.2.12).

∎

Let us now prove Proposition 2.1.1. If the random walk is not transient in any direction, there is nothing to prove. So assume that the random walk is transient in a direction l and hence the corresponding regeneration times are well defined. Essentially, we will exhibit a trap as the one depicted in Figure 2.1, in the edge $(0, e_i)$. Define the first exit time of the random walk form the edge $(0, e_i)$, so that

$$F := \min\{n \geq 0 : X_n \notin (0, e_i)\}.$$

We then have for every $k \geq 0$ that

$$P_{x,\omega}(F = 2k+2) = \omega_1^{k+1}\omega_2^k(1-\omega_2),$$

and

$$P_{x,\omega}(F = 2k+1) = \omega_1^k\omega_2^k(1-\omega_1).$$

Hence,

$$P_{x,\omega}(F > 2k) = (\omega_1\omega_2)^k$$

and

$$\sum_{k=0}^{\infty} P_{x,\omega}(F_x > 2k) = \frac{1}{1-\omega_1\omega_2}. \tag{2.2.13}$$

This proves that under the annealed law,

$$E_0(F) = \infty.$$

We can now show using the strong Markov property under the quenched measure and the i.i.d. nature of the environment, that for each natural $m > 0$, the time $T_m := \min\{n \geq 0 : X_n \cdot l > m\}$ can be bounded from below by a sequence $F_1, \ldots, F_m$ of random variables which under the annealed measure are i.i.d. and distributed as F. This proves that P_0-a.s. $T_m/m \to \infty$ which implies that the random walk is not ballistic in direction l.

2.3 Equivalence between the polynomial ballisticity condition and (T')

Here we will prove Theorem 2.1.1, establishing the equivalence between the polynomial condition $(P)_M$ and condition (T'). To do this, we will pass through both the effective criterion and an version of condition $(T)_\gamma$ which corresponds to the choice of $\gamma = \gamma_L$ according to (2.2.6) (see [BDR12]). Now, to prove Theorem 2.1.1, we will first show in subsection 2.3.1 that $(P)_M$ implies $(T)_0$ for $M \geq 15d+5$. In subsection 2.3.2, we will prove that $(T)_0$ implies a weak kind of an atypical quenched exit estimate. In these first two steps, we will generalize the methods presented in [BDR12] for random walks satisfying condition $(E)_0$. In subsection 2.3.3, we will see that this estimate implies the effective criterion. Finally, in subsection 2.3.4, we will show that the effective criterion implies (T'), generalizing the method presented by Sznitman [Sz02], to random walks satisfying $(E)_0$.

Before we continue, we will need some additional notation. Let $l \in \mathbb{S}^{d-1}$. Let $L, L' > 0$, $\tilde{L} > 0$,

$$B(R, L, L', \tilde{L}) := R\left((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}\right) \cap \mathbb{Z}^d \tag{2.3.1}$$

and

$$\partial_+ B(R, L, L', \tilde{L}) := \partial B \cap \left\{x \in \mathbb{Z}^d : x \cdot l \geq L', |R(e_j) \cdot x| < \tilde{L}, \text{ for each } 2 \leq j \leq d\right\}. \tag{2.3.2}$$

Here R is the rotation defined by (2.1.3). When there is no risk of confusion, we will drop the dependence of $B(R, L, L', \tilde{L})$ and $\partial_+ B(R, L, L', \tilde{L})$ with respect to R, L, L' and $\tilde{L}$ and write B and $\partial_+ B$ respectively. Let also,

$$\rho_B := \frac{P_{0,\omega}(X_{T_B} \notin \partial_+ B)}{P_{0,\omega}(X_{T_B} \in \partial_+ B)} = \frac{q_B}{p_B},$$

where $q_B := P_{0,\omega}(X_{T_B} \notin \partial_+ B)$ and $p_B := P_{0,\omega}(X_{T_B} \in \partial_+ B)$ and for $0 < \alpha < \min_{e \in U} F_e$,

$$\eta_\alpha := \sup_{e \in U} \mathbb{E}\left[\left(\frac{1}{\omega(0, e)}\right)^\alpha\right]. \tag{2.3.3}$$

2.3.1 Polynomial ballisticity implies $(T)_0$

Here we will prove that the Polynomial ballisticity condition implies $(T)_0$. To do this, we will use a multi-scale renormalization scheme as presented in Section 3 of [BDR12]. Let us note that [BDR12] assumes that the walk is uniformly elliptic.

Proposition 2.3.1. *Let $M > 15d + 5$ and $l \in \mathbb{S}^{d-1}$. Assume that conditions $(P)_M|l$ and $(E)_0$ are satisfied. Then $(T)_0|l$ holds.*

Let us now to prove Proposition 2.3.1. Let $N_0 \geq \frac{3}{2}L_0$, where L_0 is defined in (2.1.4). For $k \geq 0$, define recursively the scales

$$N_{k+1} := 3(N_0 + k)^2 N_k. \tag{2.3.4}$$

Define also for $k \geq 0$ and $x \in \mathbb{R}^d$ the boxes

$$B(x,k) := \left\{ y \in \mathbb{Z}^d : -\frac{N_k}{2} < (y-x)\cdot l < N_k,\ |(y-x)\cdot R(e_i)| < 25N_k^3 \text{ for } 2 \leq i \leq d \right\} \tag{2.3.5}$$

and their *middle frontal part*

$$\tilde{B}(x,k) := \left\{ y \in \mathbb{Z}^d : N_k - N_{k-1} \leq (y-x)\cdot l < N_k,\ |(y-x)\cdot R(e_i)| < N_k^3 \text{ for } 2 \leq i \leq d \right\} \tag{2.3.6}$$

with the convention that $N_{-1} := 2N_0/3$. We also define the *the front side*

$$\partial_+ B(x,k) := \{ y \in \partial B(x,k) : (y-x)\cdot l \geq N_k \},$$

the *back side*

$$\partial_- B(x,k) := \{ y \in \partial B(x,k) : (y-x)\cdot l \leq -\frac{N_k}{2} \},$$

and the *lateral sides*

$$\partial_l B(x,k) := \{ y \in \partial B(x,k) : |(y-x)\cdot R(e_i)| \geq 25N_k^3 \text{ for } 2 \leq i \leq d \}.$$

We need to define for each $n, m \in \mathbb{N}$ the sub-lattices

$$\mathcal{L}_{n,m} := \{ x \in \mathbb{Z}^d : [x\cdot l] \in n\mathbb{Z},\ [x\cdot R(e_j)] \in m\mathbb{Z},\ \text{for } 2 \leq j \leq d \}$$

and refer to the elements of

$$\mathcal{B}_k := \left\{ B(x,k) : x \in \mathcal{L}_{N_{k-1}-1, N_k^3 - 1} \right\}$$

as *boxes of scale* k. When there is no risk of confusion, we will denote a typical element of this set by B_k or simply B and its middle part as $\tilde{B}_k$ or $\tilde{B}$. Furthermore, we have

$$\cup_{B\in\mathcal{B}_k}\tilde{B}=\mathbb{Z}^d,$$

which will be an important property that we will be useful. In this subsection, it is enough to assume a weaker condition than $(P)_M|l$. The following lemma is straightforward, so its proof will be omitted.

Lemma 2.3.1. *Let* $M>0$ *and* $l\in\mathbb{S}^{d-1}$. *Assume that condition* $(P)_M|l$ *is satisfied. Then, whenever* $N_0\geq\frac{2}{3}L_0$ *one has that*

$$\sup_{x\in\tilde{B}_0}P_x\left(X_{T_{B_0}}\notin\partial_+B_0\right)<N_0^{-M}. \tag{2.3.7}$$

We now say that box $B\in\mathcal{B}_0$ is *good* if

$$\sup_{x\in\tilde{B}_0}P_{x,\omega}\left(X_{T_{B_0}}\notin\partial_+B_0\right)<N_0^{-5}. \tag{2.3.8}$$

Otherwise, we say that the box $B\in\mathcal{B}_0$ is *bad*. The following lemma appears in [BDR12] as Lemma 3.3.

Lemma 2.3.2. *Let* $M>0$ *and* $l\in\mathbb{S}^{d-1}$. *Assume that* $(P)_M|_l$ *holds. Then for all* $B_0\in\mathcal{B}_0$ *and* $N_0\geq\frac{2}{3}L_0$,

$$\mathbb{P}(B_0\text{ is good})\geq 1-2^{d-1}N_0^{3d+3-M}.$$

Proof. Note that

$$\mathbb{P}(B_0\text{ is bad})\leq\sum_{x\in\tilde{B}_0}\mathbb{P}\left(P_{x,\omega}\left(X_{T_{B_0}}\notin\partial_+B_0\right)\geq N_0^{-5}\right). \tag{2.3.9}$$

Now by Markov's inequality we have for $x\in\tilde{B}_0$ that

$$\mathbb{P}\left(P_{x,\omega}\left(X_{T_{B_0}}\notin\partial_+B_0\right)\geq N_0^{-5}\right)\leq N_0^5\sup_{x\in\tilde{B}_0}P_x\left(X_{T_{B_0}}\notin\partial_+B_0\right). \tag{2.3.10}$$

Now, with the help of Lemma 2.3.1, (2.3.9), (2.3.10) and from a routine counting argument we obtain

$$\mathbb{P}(B_0 \text{ is bad}) \leq 2^{d-1} N_0^{3d+3-M}.$$

∎

Now, we want to extend the concept of good and bad boxes of scale 0 to boxes of any scale $k \geq 1$. To do this, due to the lack of uniform ellipticity, we need to modify the notion of good and bad boxes for scales $k \geq 1$ presented in Berger, Drewitz and Ramírez [BDR12]. Consider a box Q_{k-1} of scale $k-1 \geq 1$. For each $x \in \tilde{Q}_{k-1}$ we associate a natural number n_x and a self-avoiding path $\pi^{(x)} := (\pi_1^{(x)}, \ldots, \pi_{n_x}^{(x)})$ starting from x so that $\pi_1^{(x)} = x$, such that $(\pi_{n_x}^{(x)} - x) \cdot l \geq N_{k-2}$ and so that

$$c_{3,1} N_{k-2} \leq n_x \leq c_{3,2} N_{k-2},$$

for some pair of constants $c_{3,1}$ and $c_{3,2}$. Now, let

$$\xi := \frac{1}{2} e^{-\frac{c_{3,2} \log \eta_\alpha + 9d}{c_{3,1}}}. \tag{2.3.11}$$

We say that the box $Q_{k-1} \in \mathcal{B}_{k-1}$ is *elliptically good* if for each $x \in \tilde{Q}_{k-1}$ one has that

$$\sum_{i=1}^{n_x} \log \frac{1}{\omega(\pi_i^{(x)}, \Delta \pi_i^{(x)})} \leq n_x \log\left(\frac{1}{\xi}\right). \tag{2.3.12}$$

Otherwise the box is called *elliptically bad*. We can now recursively define the concept of good and bad boxes. For $k \geq 1$ we say that a box $B_k \in \mathcal{B}_k$ is *good*, if the following are satisfied:

(a) There is a box $Q_{k-1} \in \mathcal{B}_{k-1}$ which is elliptically good.

(b) Each box $C_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ satisfying $C_{k-1} \cap Q_{k-1} \neq \emptyset$ and $C_{k-1} \cap B_k \neq \emptyset$ is elliptically good.

(c) Each box $B_{k-1} \in \mathcal{B}_{k-1}$ of scale $k-1$ satisfying $B_{k-1} \cap Q_{k-1} = \emptyset$ and $B_{k-1} \cap B_k \neq \emptyset$, is good.

Otherwise, we say that the box B_k is *bad.* Now we will obtain an important estimate on the probability that a box of scale $k \geq 1$ is good, corresponding to Lemma 3.4 of [BDR12]. Nevertheless, note that here we have to deal with our different definition of good and bad boxes due to the lack of uniform ellipticity. Let

$$c_{3,3} := c_{3,1} \log \frac{1}{\xi} - c_{3,2} \log \eta_\alpha - 9d = c_{3,1} \log 2 > 0.$$

We first need the following estimate.

Lemma 2.3.3. *For each $k \geq 1$ we have that*

$$\mathbb{P}(B_k \text{ is not elliptically good}) \leq e^{-c_{3,3} N_{k-1}}. \tag{2.3.13}$$

Proof. By translation invariance and using Chebychev's inequality as well as independence, we have that for any $\alpha > 0$

$$\begin{aligned}\mathbb{P}(B_k \text{ is not elliptically good}) &\leq \sum_{x \in \tilde{B}_k} \mathbb{P}\left(\sum_{i=1}^{n_x} \log \frac{1}{\omega(\pi_i^{(x)}, \Delta\pi_i^{(x)})} > n_x \log\left(\frac{1}{\xi}\right)\right) \\ &\leq N_{k-1} N_k^{3(d-1)} e^{-N_{k-1}\left(c_{3,1}\alpha \log\left(\frac{1}{\xi}\right) - c_{3,2} \log \eta_\alpha\right)} \\ &\leq e^{-N_{k-1}\left(c_{3,1} \log\left(\frac{1}{\xi}\right) - c_{3,2} \log \eta_\alpha - 9d\right)}\end{aligned}$$

where $N_{k-1}N_k^{3(d-1)}$ is an upper bound for $|\tilde{B}_k|$ and we have used the inequality $N_k \leq 12N_{k-1}^3$. But this expression can be bounded by e^{9dN_k} due to our choice of N_0. Then for any $\alpha > 0$, using the definition of ξ in (2.3.11), we have that

$$\mathbb{P}(B_k \text{ is not elliptically good}) \leq e^{-c_{3,3} N_{k-1}}.$$

∎

We can now state the following lemma giving an estimate for the probability that a box of scale $k \geq 0$ is bad. We will use Lemma 2.3.1.

Lemma 2.3.4. *Let $l \in \mathbb{S}^{d-1}$, $M \geq 15d + 5$, and assume that $(P)_M|l$ is satisfied. Then for $N_0 \geq \frac{3}{2} L_0$ one has that for all $k \geq 0$ and all $B_k \in \mathcal{B}_k$,*

$$\mathbb{P}(B_k \text{ is good}) \geq 1 - e^{-2^k}. \tag{2.3.14}$$

Proof. By Lemma 2.3.2 we see that

$$\mathbb{P}(B_0 \text{ is bad}) \leq e^{-c_{3,0}},$$

where

$$c'_{3,0} := \log \frac{N_0^{M-3d-3}}{2^{d-1}}.$$

We will show that this implies for all $k \geq 1$ that

$$P_0(B_k \text{ is bad}) \leq e^{-c'_{3,k} 2^k}, \tag{2.3.15}$$

for a sequence of constants $\{c'_{3,k} : k \geq 0\}$ defined recursively by

$$c'_{3,k+1} := c'_{3,k} - \frac{\log\left(3^{16d}(N_0 + k)^{12d}\right)}{2^{k+1}}. \tag{2.3.16}$$

We will now prove (2.3.15) using induction on k. To simplify notation, we will denote by q_k for $k \geq 0$, the probability that the box B_k is bad. Assume that (2.3.15) is true for some k, $k \geq 0$. Let A be the event that all boxes of scale k that intersect B_{k+1} are elliptically good, and B the event that each pair of bad boxes of scale k have a non-empty intersection. Note that the event $A \cap B$ implies that the box B_{k+1} is good. Therefore, the probability q_{k+1} that the box B_{k+1} is bad is bounded by the probability that there are at least two bad boxes B_k which intersect B_{k+1} plus the probability that there is at least one elliptically bad box of scale k, so that by Lemma 2.3.3, for each $k \geq 0$ one has that

$$q_{k+1} \leq m_k^2 q_k^2 + m_k e^{-c_{3,3} N_k}, \tag{2.3.17}$$

where m_k is the total number of bad boxes of scale k that intersect B_{k+1}. Now note that

$$\sqrt{2} m_k \leq 3^{8d}(N_0 + k)^{6d}. \tag{2.3.18}$$

But by the the fact that $c_{3,3} N_k \geq c'_{3,k} 2^{k+1}$ for $k \geq 0$ we have that

$$e^{-c_{3,3} N_k} \leq e^{-c'_{3,k} 2^{k+1}}.$$

Hence, substituting this estimate and estimate (2.3.18) back into (2.3.17) and using the induction hypothesis, we conclude that

$$q_{k+1} \leq 3^{16d}(N_0+k)^{12d} e^{-c'_{3,k}2^{k+1}} = e^{-c'_{3,k+1}2^{k+1}}.$$

Now note that the recursive definition (2.3.16) implies that

$$c'_{3,k} \geq \log \frac{N_0^{M-3d-3}}{2^{d-1}} - \sum_{k=0}^{\infty} \frac{\log\left(3^{16d}(N_0+k)^{12d}\right)}{2^{k+1}}.$$

Using the inequality $\log(a+b) \leq \log a + \log b$ valid for $a, b \geq 1$, we see that

$$\sum_{k=0}^{\infty} \frac{\log\left(3^{16d}(N_0+k)^{12d}\right)}{2^{k+1}} \leq 16d\log 3 + 12d\log N_0 + 12d.$$

From these estimates we see that whenever $M \geq 15d+5$ and

$$\log N_0 - \log 2^{d-1}3^{16d}e^{12d+1} \geq 0, \tag{2.3.19}$$

then for every $k \geq 0$, one has that $c'_{3,k} \geq 1$. But (2.3.19) is clearly satisfied for $N_0 \geq 3^{29d}$.

∎

The next lemma establishes that the probability that a random walk exits a box B_k through its lateral or back side is small if this box is good.

Lemma 2.3.5. *There is a constant $c_{3,4} > 0$ such that for each $k \geq 0$ and $B_k \in \mathcal{B}_k$ which is good one has*

$$\sup_{x\in\tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \notin \partial_+ B_k\right) \leq e^{-c_{3,4}N_k}. \tag{2.3.20}$$

Proof. Let us first note that for each $k \geq 0$,

$$P_{x,\omega}\left(X_{T_{B_k}} \notin \partial_+ B_k\right) \leq P_{x,\omega}\left(X_{T_{B_k}} \in \partial_- B_k\right) + P_{x,\omega}\left(X_{T_{B_k}} \in \partial_l B_k\right).$$

We denote by $p_k := \sup_{x\in\tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \in \partial_l B_k\right)$ and $r_k := \sup_{x\in\tilde{B}_k} P_{x,\omega}\left(X_{T_{B_k}} \in \partial_- B_k\right)$. We will first show by induction on k that

$$p_k \le e^{-c''_{3,k} N_k} \quad \text{and} \tag{2.3.21}$$

$$r_k \le e^{-c''_{3,k} N_k}, \tag{2.3.22}$$

where

$$c''_{3,k} := \frac{5\log N_0}{N_0} - \sum_{j=1}^{k} \frac{\log 27(N_0+j)^4}{N_{j-1}} - \sum_{j=1}^{k} \frac{5N_{j-1} + \log 24 + 6d(\log\xi)^2 N_{j-1}}{N_j}, \tag{2.3.23}$$

and ξ is defined in (2.3.11). The case $k = 0$ follows easily by the definition of good box at scale 0 with

$$c''_{3,0} := \frac{5\log N_0}{N_0}.$$

Now, we assume that (2.3.21) and (2.3.22) hold for some $k \ge 0$ and will show that this implies that (2.3.21) is satisfied for $k+1$. Let κ_1 be the first time that the random walk exits some fixed box of scale k whose middle part frontal part contains the point x. Define recursively for every $n \ge 1$, κ_{n+1} as the first time after time κ_n such that the random walk exits some fixed box of scale k whose middle frontal part contains the point X_{κ_n}. We choose these fixed boxes arbitrarily. We now define the *rescaled random walk* $\{Y_n : n \ge 0\}$ as

$$Y_0 := x \quad \text{and} \quad Y_n := X_{\kappa_n},$$

for $n \ge 1$. Since the box B_{k+1} is good, we know that there exists a box $Q_k \in \mathcal{B}_k$ such that every box of scale k, intersecting B_{k+1} but not Q_k, is good. Let us now define for each $k \ge 1$ the collection of sets

$$\mathcal{S}_k := \begin{array}{l} \{B_k \in \mathcal{B}_k : B_k \cap B_{k+1} \neq \emptyset, \text{ and } \forall\, i \in \{2,\ldots,d\}, x \cdot R(e_i) = y \cdot R(e_i), \\ \quad \text{for some } x \in B_k,\ y \in Q_k\}. \end{array}$$

In words, this is the collection of boxes of scale k which have at least one point whose component orthogonal to l coincides with the component orthogonal to l of some point in Q_k. Now, define the strip

$$S_k := \bigcup_{B_k \in \mathcal{S}_k} B_k.$$

(See Figure 2.2)

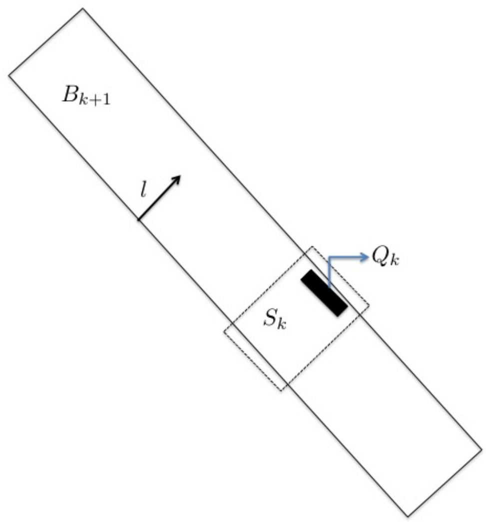

Figure 2.2: The bad box Q_k with its strip S_k.

Let m_1 be the first time that the random walk $\{Y_n\}$ is at a distance larger than $7N_{k+1}^3$ from the strip S_k and from the sides $\partial_l B_{k+1}$ of the box B_{k+1},

$$m_1 := \inf\left\{n \geq 0 : \mathrm{dist}(Y_n, S_k) \geq 7N_{k+1}^3 \quad \text{and} \quad \mathrm{dist}(Y_n, \partial_l B_{k+1}) \geq 7N_{k+1}^3\right\}.$$

Let m_2 be the first time that the random walk $\{Y_n\}$ exits the box B_{k+1} so that

$$m_2 := \inf\{n \geq 0 : Y_n \notin B_{k+1}\}$$

and note that on the event $\left\{X_{T_{B_{k+1}}} \in \partial_l B_{k+1}\right\}$ one has that

$$m_1 < m_2 < \infty.$$

Also, define

$$m_3 := \{n > m_1 : Y_n \in S_k\}.$$

Define

$$J_k := \frac{3N_{k+1}/2}{N_{k-1}} + 1.$$

This is the minimal number of steps needed by the random walk $\{Y_n\}$ to exit the box B_{k+1} through its front side. Then, we have that a.s. on the event $\left\{X_{T_{B_{k+1}}} \in \partial_l B_{k+1}\right\}$ the following inequality is satisfied

$$m_2 \wedge m_3 - m_1 \geq \frac{7N_{k+1}^3}{25N_k^3} \geq \frac{4}{25} J_k \frac{N_{k+1}}{N_k} + 1. \tag{2.3.24}$$

Now note that starting from Y_{m_1} if the random walk $\{Y_n\}$ consecutively exits J_k boxes of scale k through their front side, it would leave the box B_{k+1} through $\partial_+ B_{k+1}$. Therefore, by the induction hypothesis we have that

$$P_{Y_{m_1},\omega}(Y_j \in B_k, \text{ for all } 1 \leq j \leq J_k) \leq J_k e^{-c''_{3,k} N_k}.$$

Thus, by the Markov property we get that

$$P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_l B_{k+1}) \leq \left(e^{-c''_{3,k} N_k + \log J_k}\right)^{N_{k+1}/N_k} \leq e^{-c''_{3,k+1} N_{k+1}}.$$

This completes the proof of (2.3.21) for k.

Recall the definition of r_k. We will now assume that (2.3.21) and (2.3.22) hold for some $k \geq 0$ and will show that (2.3.22) is satisfied for $k+1$. Define

$$L_{Q_k} := \inf\{l \cdot z : z \in Q_k\} - N_{k-1} \qquad R_{Q_k} := \sup\{l \cdot z : z \in Q_k\} + \frac{3}{2} N_k,$$

where Q_k is a box of scale k in B_{k+1} with the property that any other box of scale k which does not intersect it but which intersect B_{k+1} is good, while any other box of scale k which does intersect it but which intersects B_{k+1} is elliptically good. We will define a one dimensional random walk which at most sites has a very strong drift to the right (towards the front side of the box) whenever it is at any site $x \in \mathbb{Z}\backslash([L_{Q_k}, R_{Q_k}] \cap \mathbb{Z})$: we define $\{Z_n : n \geq 0\}$ as a random walk which at each unit time, if it is at site $x \in \mathbb{Z}\backslash([L_{Q_k}, R_{Q_k}] \cap \mathbb{Z})$, it jumps N_{k-1} steps to the right with probability $1 - e^{-c''_{3,k}N_k}$ and $\frac{3}{2}N_k$ steps to the left with probability $e^{-c''_{3,k}N_k}$, while if it as a site $x \in \mathbb{Z} \cap [L_{Q_k}, R_{Q_k}]$ it jumps N_{k-1} steps to the right with probability $\xi^{N_{k-1}}$ and $\frac{3}{2}N_k$ steps to the left with probability $1 - \xi^{N_{k-1}}$. We will call P_z the law of this random walk starting from $z \in \mathbb{Z}$. Let us call H_k the first hitting time of the random walk to the strip defined by L_{Q_k} and R_{Q_k} so that

$$H_k := \inf \{n \geq 0 : X_n \cdot l \in [L_{Q_k}, R_{Q_k}]\}.$$

Coupling in the natural way the random walk $\{X_n\}$ with the random walk $\{Z_n\}$, now note that

$$\begin{aligned} &\sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_- B_{k+1}) \\ \leq &\sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \leq T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}) \times \sup_{z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}} P_z \left(T_{-\left(R_{Q_k} + \frac{N_{k+1}}{2}\right)} < T_{N_{k+1} - R_{Q_k}} \right) \end{aligned} \tag{2.3.25}$$

But,

$$\begin{aligned} &\sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \leq T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}) \\ \leq &\sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(X_{T_{B_{k+1}}} \in \partial_l B_{k+1}) + \sup_{x \in \tilde{B}_{k+1}} P_{x,\omega}(H_k \leq T_{\partial_l B_{k+1}} \wedge T_{\partial_+ B_{k+1}}, T_{\partial B_{k+1}} \neq T_{\partial_l B_{k+1}}). \end{aligned}$$

Now, by the estimate already done concerning the probability to exit the box B_{k+1} through the sides, we know that the first term is bounded from above by $e^{-c''_{3,k+1}N_{k+1}}$. For the second term, we couple the random walk to the random walk $\{Z_n\}$ previously defined. It is easy to see that $\{Z_n\}$ can be coupled to a random walk $\{Z'_n\}$ which jumps $\frac{3}{2}N_k$ steps to the right with probability $(1 - e^{-c''_{3,k}N_k})^{3N_k/(2N_{k-1})}$ and $\frac{3}{2}N_k$ steps to the left with probability $1 - (1 - e^{-c''_{3,k}N_k})^{3N_k/(2N_{k-1})}$. Now, the probability that a random walk which jumps one step to the right with probability p and one to the left with probability q to exit the interval $[-a, b] \cap \mathbb{Z}$ through $-a$, where $a, b \in \mathbb{Z}$ is given by

$$q^a \frac{p^b - q^b}{p^{b+a} - q^{b+a}}.$$

Applying the above formula with $a = (N_{k+1} - N_k - R_{Q_k})/(3N_k/2)$, $b = 2$, $p = (1 - e^{-c''_{3,k}N_k})^{3N_k/(2N_{k-1})}$ and $q = 1 - p$ we get that for $N_0 \geq \log \frac{1}{\xi}$,

$$\sup_{x \in \tilde{B}_k} P_{x,\omega}(H_k \leq T_{\partial_l B_k} \wedge T_{\partial_+ B_k}, T_{\partial B_k} \neq T_{\partial_l B_k}) \leq e^{-\frac{c''_{3,k}}{4}(N_{k+1} - N_k - R_{Q_k})}.$$

We will now find an upper bound for the second factor of (2.3.25). Let $z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}$ and define the events

$$D^+ := \left\{ T_{N_{k+1} - R_{Q_k}} < T_z \circ \theta_1(Z) \right\} \quad \text{and} \quad D^- := \left\{ T_{-\left(R_{Q_k} + \frac{N_{k+1}}{2}\right)} < T_z \circ \theta_1(Z) \right\}.$$

It is straightforward to see that

$$\sup_{z \in [L_{Q_k}, R_{Q_k}] \cap \mathbb{Z}} P_z \left(T_{-\left(R_{Q_k} + \frac{N_{k+1}}{2}\right)} < T_{N_{k+1} - R_{Q_k}} \right) \leq \sup_{z \in \left[L_{Q_k}, R_{Q_k}\right] \cap \mathbb{Z}} \frac{P_z(D^-)}{P_z(D^+)}. \qquad (2.3.26)$$

Now, by the fact that the box Q_k and those which intersect it are elliptically good, we conclude as in [BDR12] that for N_0 large enough,

$$P_z(D^+) \geq \frac{1}{2} \xi^{c_{3,2} 4 N_k},$$

where ξ is defined in (2.3.11). On the other hand, by the strong Markov property we conclude that

$$P_z(D^-) \leq 3 \left(e^{-c''_{3,k} N_k} \right)^{\frac{L_{Q_k} + N_{k+1}/2}{N_k}}.$$

From here we see that

$$\sup_{x \in \tilde{B}_k} P_{x,\omega} \left(X_{T_{B_k}} \in \partial_- B_k \right) \leq e^{-c''_{3,k} N_k}.$$

It is easy to check that

$$c_{3,4} := \inf_k c''_{3,k} > 0.$$

∎

We can now repeat the last argument of Proposition 2.1 of [BDR12], which does not require uniform ellipticity, to finish the proof of Proposition 2.3.1.

2.3.2 Condition $(T)_0$ implies a weak atypical quenched exit estimate

In this subsection we will prove that the condition $(T)_0$ implies a weak atypical quenched exit estimate. Throughout, we will denote by B the box

$$B := B(R, L, L, L), \tag{2.3.27}$$

as defined in (2.3.1), with R the rotation which maps e_1 to l. Let

$$\epsilon_L := \frac{1}{(\log \log L)^2}.$$

Proposition 2.3.2. *Let $l \in \mathbb{S}^{d-1}$. Assume that the ellipticity condition $(E)_0$ and that $(T)_0|l$ are fulfilled. Then, for each function $\beta_L : (0,\infty) \to (0,\infty)$ and each $c > 0$ there exists $c_{3,11} > 0$ such that*

$$\mathbb{P}\left(P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-cL^{\beta_L+\epsilon_L}}\right) \le \frac{1}{c_{3,11}} e^{-c_{3,11} L^{\beta_L}} \tag{2.3.28}$$

where B is the box defined in (2.3.27).

Let us now prove Proposition 2.3.2. Let $\rho > 0$. We will perform a one scale renormalization analysis involving boxes of side $\rho L^{\frac{\epsilon_L}{d+1}}$ which intersect the box B. Without loss of generality, we assume that e_1 belongs to the intersection of the half-spaces so that

$$e_1 \in \{x \in \mathbb{Z}^d : x \cdot l \ge 0\} \tag{2.3.29}$$

and

$$e_1 \in \{x \in \mathbb{Z}^d : x \cdot \hat{v} \ge 0\}. \tag{2.3.30}$$

Define the hyperplane perpendicular to direction e_1 as

$$H := \{x \in \mathbb{R}^d : x \cdot e_1 = 0\}. \tag{2.3.31}$$

We will need to work with the projection on the direction l along the hyperplane H defined for $z \in \mathbb{Z}^d$ as

$$P_l z := \left(\frac{z \cdot e_1}{l \cdot e_1}\right) l, \tag{2.3.32}$$

and the projection of z on H along l defined by

$$Q_l z := z - P_l z. \tag{2.3.33}$$

Let $r > 0$ be a fixed number which will eventually be chosen large enough. For each $x \in \mathbb{Z}^d$ and n define the *mesoscopic box*

$$D_n(x) := \{y \in \mathbb{Z}^d : -n < (y - x) \cdot e_1 < n, \ -rn \le |Q_l(y - x)|_\infty \le rn\},$$

and their front boundary

$$\partial^+ D_n(x) := \{y \in \partial D_n(x) : (y - x) \cdot e_1 \ge n\}.$$

Define the set of mesoscopic boxes intersecting B as

$$\mathcal{D} := \{D_n(x) \text{ with } x \in \mathbb{Z}^d : D_n(x) \cap B \neq \emptyset\}.$$

From now on, when there is no risk of confusion, we will write D instead of D_n for a typical box in $\mathcal{D}$. Also, let us set $n := \rho L^{\frac{\epsilon_L}{d+1}}$. We now say that a box $D(x) \in \mathcal{D}$ is *good* if

$$P_{x,\omega}(X_{T_{D(x)}} \in \partial_+ D(x)) \ge 1 - \frac{1}{L}. \tag{2.3.34}$$

Otherwise we will say that $D(x)$ is *bad*.

Lemma 2.3.6. *Let $l \in \mathbb{S}^{d-1}$ and $M > 15d + 5$. Consider a RWRE satisfying condition $(P)_M|l$ and the ellipticity condition $(E)_0$. Then, there is a $c_{3,5}$ such that for $r \ge c_{3,5}$ one has that*

$$\limsup_{L\to\infty} L^{-\frac{\epsilon_L \gamma_L}{d+1}} \log \mathbb{P}(D(0) \text{ is bad}) < 0. \tag{2.3.35}$$

Proof. By (2.3.34) and Markov inequality we have that

$$\mathbb{P}\,(D(0) \text{ is bad}) \leq \mathbb{P}\left(P_{0,\omega}(X_{T_{D(0)}} \notin \partial_+ D(0)) > \frac{1}{L}\right) \leq L P_0\left(X_{T_{D(0)}} \notin \partial_+ D(0)\right). \quad (2.3.36)$$

Now, by Proposition 2.3.1 of Section 2.3.1, we know that the polynomial condition $(P)_M|l$ and the ellipticity condition $(E)_\alpha$ imply $(T)_0|l$. But by Theorem 2.2.4, and the fact that e_1 is in the half spaces determined by l and $\hat{v}$ (see (2.3.29) and (2.3.30), we can conclude that $(T)_0|l$ implies $(T)_0|_{e_1}$. On the other hand, it is straightforward to check that there are constants $c_{3,5}$, $c_{3,6} > 0$ such that for $r \geq c_{3,5}$, $(T)_0|_{e_1}$ implies that

$$P_0\left(X_{T_{D(0)}} \notin \partial_+ D(0)\right) \leq \frac{1}{c_{3,6}} e^{-c_{3,6} L^{\frac{\epsilon_L \gamma_L}{d+1}}}.$$

Substituting this back into inequality (2.3.36) we see that (2.3.35) follows.

∎

For each m such that $0 \leq m \leq \left\lceil \frac{2L(l \cdot e_1)}{n} \right\rceil$ define the *block* R_m as the collection of mesoscopic boxes (see Figure 2.3)

$$R_m := \{D(x) \in \mathcal{D} : \text{ for some } x \text{ such that } x \cdot e_1 = nm\}. \quad (2.3.37)$$

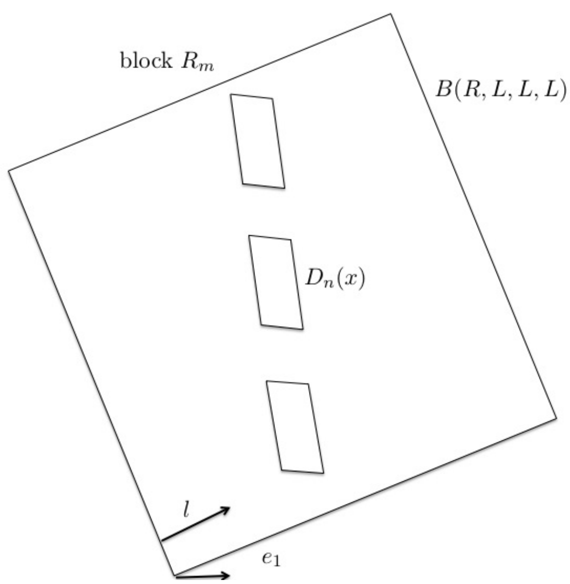

Figure 2.3: A box B with a set of inner boxes $D_n(x)$, which belong to a block R_m.

The collection of these blocks is denoted by $\mathcal{R}$. We will say that a block R_m is *good* if every box $D \in R_m$ is good. Otherwise, we will say that the block R_m is *bad*. Now, for each $x \in R_m$ we associate a self-avoiding path $\pi^{(x)}$ such that

(a) The path $\pi^{(x)} = (\pi_1^{(x)}, \ldots, \pi_{2n+1}^{(x)})$ has $2n$ steps.

(b) $\pi_1^{(x)} = x$ and the end-point $\pi_{2n+1}^{(x)} \in R_{m+1}$.

(c) Whenever $D(x)$ does not intersect $\partial_+ B$, the path $\pi^{(x)}$ is contained in B. Otherwise, the end-point $\pi_{2n+1}^{(x)} \in \partial_+ B$.

Define next J as the total number of bad boxes of the collection $\mathcal{D}$ and define

$$G_1 := \{\omega \in \Omega : J \leq L^{\beta_L + \frac{d}{d+1}\epsilon_L}\}. \tag{2.3.38}$$

We will now denote by $\{m_1, \ldots m_N\}$ a generic subset of $\{0, \ldots, |\mathcal{R}|-1\}$ having N elements. Let $\xi \in (0,1)$. Define

$$G_2 := \left\{ \omega \in \Omega : \sup_{N, \{m_1, \ldots, m_N\}} \sum_{j=1}^{N} \sup_{x_j \in R_{m_j}} \sum_{i=1}^{2n} \log \frac{1}{\omega(\pi_i^{(x_j)}, \Delta\pi_i^{(x_j)})} \leq 2n \log\left(\frac{1}{\xi}\right) L^{\beta_L + \frac{d}{d+1}\epsilon_L} \right\}, \tag{2.3.39}$$

where the first supremum runs over $N \leq L^{\beta_L + \frac{d}{d+1}\epsilon_L}$ and all subsets $\{m_1, \ldots, m_N\}$ of the set of blocks. Now, we can say that

$$\mathbb{P}\left(p_B \leq e^{-cL^{\beta_L + \epsilon_L}}\right) \leq \mathbb{P}\left(p_B \leq e^{-cL^{\beta_L + \epsilon_L}}, G_1 \cap G_2\right) + \mathbb{P}(G_1^c) + \mathbb{P}(G_2^c). \tag{2.3.40}$$

Let us now show that the first term on the right-hand side of (2.3.40) vanishes. Indeed, on the event $G_1 \cap G_2$, the probability p_B is bounded from below by the probability that the random walk exits every mesocopic box from its front side. Since $\omega \in G_1$, the random walk will have to do this for at most $L^{\beta_L + \frac{d}{d+1}\epsilon_L}$ bad boxes. On each bad box $D(x)$ it will follow the path $\pi^{(x)}$ defined above. But then on the event G_2, we have a control on the product of the probability of traversing all these paths through the bad boxes. Hence, applying the strong Markov property and using the definition of good box, we conclude that for fixed ξ there is a $c_{3,7} > 0$ such that for $0 < \rho \leq c_{3,7}$ and on the event $G_1 \cap G_2$,

$$p_B \geq e^{-2L^{\beta_L+\epsilon_L}\rho\log\left(\frac{1}{\xi}\right)}\left(1-\frac{1}{L}\right)^L > e^{-cL^{\beta_L+\epsilon_L}}. \tag{2.3.41}$$

Let us now estimate the term $\mathbb{P}(G_1^c)$ of (2.3.40). Note first that the set $\mathcal{D}$ of mesocopic boxes can be divided into less that $2^d r^{d-1}\rho^d L^{\frac{d\epsilon_L}{d+1}}$ collections of boxes, whose union is $\mathcal{D}$ and each collection has only disjoint boxes. Let us call M the number of such collections. We also denote by $\mathcal{D}_i$ and J_i, where $1 \leq i \leq M$, the i-th collection and the number of bad boxes in such a collection respectively. We then have that

$$\mathbb{P}(G_1^c) \leq \sum_{i=1}^{M}\mathbb{P}\left(J_i \geq \frac{1}{M}L^{\beta_L+\frac{d}{d+1}\epsilon_L}\right). \tag{2.3.42}$$

Now, by Chebychev inequality

$$\begin{aligned}\mathbb{P}\left(J_i \geq \frac{1}{M}L^{\beta_L+\frac{d}{d+1}\epsilon_L}\right) &\leq e^{-\frac{L^{\beta_L+\frac{d}{d+1}\epsilon_L}}{M}}\mathbb{E}[e^{J_i}] \\ = e^{-\frac{L^{\beta_L+\frac{d}{d+1}\epsilon_L}}{M}}\sum_{n=0}^{|\mathcal{D}_i|}\binom{|\mathcal{D}_i|}{n}(ep_L)^n(1-ep_L)^{|\mathcal{D}_i|-n}&\left(\frac{1-p_L}{1-ep_L}\right)^{|\mathcal{D}_i|-n}, \end{aligned}\tag{2.3.43}$$

where p_L is the probability that a box is bad. Now the last factor of each term after the summation of the right-hand side of (2.3.43) is bounded by

$$\left(\frac{1-p_L}{1-ep_L}\right)^{|\mathcal{D}_i|},$$

which clearly tends to 1 as $L \to \infty$ by the fact that $|\mathcal{D}_i| \leq c_{3,8}L^d$, the definition of ϵ_L and by Lemma 2.3.6 for some $c_{3,8} > 0$. Thus, there is a constant $c_{3,9} > 0$ such that

$$\mathbb{P}\left(J_i \geq \frac{1}{M}L^{\beta_L+\frac{d}{d+1}\epsilon_L}\right) \leq c_{3,9}e^{-\frac{L^{\beta_L+\frac{d}{d+1}\epsilon_L}}{M}}.$$

Substituting this back into (2.3.42) we hence see that

$$\mathbb{P}\left(G_1^c\right) \leq c_{3,9}(2\rho)^d r^{d-1}L^{\frac{d}{d+1}\epsilon_L}e^{-\frac{L^{\beta_L}}{(2\rho)^d r^{d-1}}}. \tag{2.3.44}$$

Let us now bound the term $\mathbb{P}(G_2^c)$ of (2.3.40). Define $\beta_L' := \beta_L + \dfrac{d}{d+1}\epsilon_L$. Note that for each $0 < \alpha < \min_e F_e$ one has that

$$\mathbb{P}\left(G_2^c\right) \leq \sum_{N=1}^{L^{\beta'_L}} \mathbb{P}\Big(\exists\left\{m_1,\ldots,m_N\right\} \text{ and } x_j \in R_{m_j} \text{ such that}$$

$$\sum_{j=1}^{N}\sum_{i=1}^{2n} \log \frac{1}{\omega\left(\pi_i^{(x_j)}, \Delta\pi_i^{(x_j)}\right)} > 2n\log\left(\frac{1}{\xi}\right) L^{\beta_L}\Bigg)$$

$$\leq \sum_{N=1}^{L^{\beta'_L}} \binom{|\mathcal{R}|}{N} r^{d-1}(2\rho)^{L^{\beta'_L}} e^{(\log L)\frac{\epsilon_L}{d+1}L^{\beta'_L}} e^{(\log\eta_\alpha)2nL^{\beta'_L}-2\alpha n\log\left(\frac{1}{\xi}\right)L^{\beta'_L}}$$

$$\leq L^{\beta'_L}\lceil\frac{2L(l\cdot e_1)}{n}\rceil^{L^{\beta'_L}} r^{d-1}(2\rho)^{L^{\beta'_L}} e^{(\log L)\frac{\epsilon_L}{d+1}L^{\beta'_L}} e^{(\log\eta_\alpha)2nL^{\beta'_L}-2\alpha n\log\left(\frac{1}{\xi}\right)L^{\beta'_L}}. \quad (2.3.45)$$

It now follows that for ξ such that $\log\left(\frac{1}{\xi^{2\alpha}\eta_\alpha^3}\right) > 0$ one can find a constant $c_{3,10}$ such that

$$\mathbb{P}\left(G_2^c\right) \leq \frac{1}{c_{3,10}} e^{-c_{3,10}L^{\beta_L+\epsilon_L}}. \quad (2.3.46)$$

Substituting back (2.3.44) and (2.3.46) into (2.3.40) we end up the proof of Proposition 2.3.2.

2.3.3 Condition $(T)_0$ implies the effective criterion

Here we will introduce a generalization of the effective criterion introduced by Sznitman in [Sz02] for RWRE, dropping the assumption of uniformly ellipticity and replacing it by the ellipticity condition $(E)_0$. Let $l \in \mathbb{S}^{d-1}$ and $d \geq 2$. We will say that the *effective criterion in direction* l holds if

$$c_2(d) \inf_{L\geq c_3, 3\sqrt{d}\leq\tilde{L}<L^3} \inf_{\alpha>0} \inf_{0<a\leq\alpha} \left\{\Upsilon^{3(d-1)}\tilde{L}^{d-1}L^{3(d-1)+1}\mathbb{E}[\rho_B^a]\right\} < 1, \quad (2.3.47)$$

where

$$B = B(R, L-2, L+2, \tilde{L}) \text{ and } \Upsilon := \max\left\{\frac{\alpha}{24}, \left(\frac{2c_1}{c_1-1}\right)\log\eta_\alpha^2\right\}, \quad (2.3.48)$$

while $c_2(d)$ and $c_3(d)$ are dimension dependent constants that will be introduced in subsection 2.3.4. Note that in particular, the effective criterion in direction l implies that condition $(E)_0$ is satisfied. Here we will prove the following proposition.

Proposition 2.3.3. *Let $l \in \mathbb{S}^{d-1}$. Assume that the ellipticity condition $(E)_0$ and that $(T)_0|l$ are fulfilled. Then, the effective criterion in direction l is satisfied.*

To prove Proposition 2.3.3, we begin defining the following quantities

$$\beta_1(L) := \frac{\gamma_L}{2} = \frac{\log 2}{2 \log\log L} \tag{2.3.49}$$

$$\sigma(L) := \frac{\gamma_L}{3} = \frac{\log 2}{3 \log\log L} \tag{2.3.50}$$

$$a := L^{-\sigma(L)}. \tag{2.3.51}$$

We will write ρ instead of ρ_B, where B is the box defined in (2.3.48) (see 2.3.1) with $\tilde{L} = L^2$. Following [BDR12], it is convenient to split $\mathbb{E}\rho^a$ according to

$$\mathbb{E}\rho^a = \mathcal{E}_0 + \sum_{j=1}^{n-1} \mathcal{E}_j + \mathcal{E}_n \tag{2.3.52}$$

where

$$n := n(L) := \left\lceil \frac{4(1-\gamma_L/2)}{\gamma_L} \right\rceil + 1,$$

$$\mathcal{E}_0 := \mathbb{E}\left(\rho^a, p_B > e^{-cL^{\beta_1}}\right),$$

$$\mathcal{E}_j := \mathbb{E}\left(\rho^a, e^{-cL^{\beta_{j+1}}} < p_B \leq e^{-cL^{\beta_j}}\right)$$

for $j \in \{1, \ldots, n-1\}$, and

$$\mathcal{E}_n := \mathbb{E}\left(\rho^a, p_B \leq e^{-cL^{\beta_n}}\right)$$

with parameters

$$\beta_j(L) := \beta_1(L) + (j-1)\frac{\gamma_L}{4}, \tag{2.3.53}$$

for $2 \leq j \leq n(L)$. We will now estimate each of the n terms appearing in (2.3.52). For the first $n-1$ terms, we now state two lemmas proved by Berger, Drewitz and Ramírez in [BDR12], whose proofs we omit. The following lemma is a consequence of Jensen's inequality.

Lemma 2.3.7. *Assume that $(T)_0$ is satisfied. Then*

$$\mathcal{E}_0 \leq e^{cL^{\frac{\gamma_L}{6}} - L^{\frac{2}{3}\gamma_L(1+o(1))}} \tag{2.3.54}$$

as $L \to \infty$.

The second lemma follows from Proposition 2.3.2.

Lemma 2.3.8. *Assume that the weak atypical quenched exit estimate (2.3.28) is satisfied. Then there exists a constant $c_{3,12} > 0$ such that for all L large enough and all $j \in \{1, \ldots, n-1\}$ one has that*

$$\mathcal{E}_j \leq \frac{1}{c_{3,12}} e^{cL^{\left(\frac{1}{6}+\frac{j}{4}\right)\gamma_L} - c_{3,12} L^{\left(\frac{1}{4}+\frac{j}{4}\right)\gamma_L - \epsilon(L)}}. \tag{2.3.55}$$

In [BDR12], where it is assumed that the environment is uniform elliptic, one has that $\mathcal{E}_n = 0$ for a suitable constant $c > 0$. Nevertheless, since here we are not assuming uniform ellipticity this is not the case.

Lemma 2.3.9. *Assume that $(E)_0$ and $(T)_0$ are satisfied. Then there exists a constant $c_{3,16} > 0$ such that for all L large enough we have*

$$\mathcal{E}_n \leq \frac{1}{c_{3,16}} e^{-c_{3,16} L^{1-\epsilon(L)}}. \tag{2.3.56}$$

Proof. Choose $0 < \alpha < \min_e F_e$. Consider a nearest neighbor self-avoiding path $(x_1, \ldots, x_m)$ from 0 to $\partial_+ B$, so that $x_1 = 0$ and $x_m \in \partial_+ B$, $x_1, \ldots, x_{m-1} \in B$ and which has the minimal number of steps m. Then,

$$\begin{aligned} \mathbb{E}\left[\rho^a, p_B \leq e^{-cL^{\beta_n}}\right] \leq \mathbb{E}\left[e^{\frac{\alpha}{2}\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}, \sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} > \frac{3m}{\alpha}\log\eta_\alpha\right] \\ + \mathbb{E}\left[e^{a\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}, \sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} \leq \frac{3m}{\alpha}\log\eta_\alpha, p_B \leq e^{-cL^{\beta_n}}\right], \end{aligned} \tag{2.3.57}$$

where in the first line, we have used that for any $\alpha > 0$, $a \leq \frac{\alpha}{2}$ for L large. Now, using Cauchy-Schwartz inequality, Chebyshev inequality and (2.3.28), we can see that the right-hand side of (2.3.57) is smaller than

$$\mathbb{E}\left[e^{\alpha\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2}\mathbb{P}\left(\sum_1^m \log\frac{1}{\omega(x_i,\Delta x_i)} > \frac{3m}{\alpha}\log\eta_\alpha\right)^{1/2} + e^{\frac{3am}{\alpha}\log\eta_\alpha}\mathbb{P}\left(p_B \leq e^{-cL^{\beta_n}}\right)$$
$$\leq e^{-m\log\eta_\alpha} + \frac{1}{c_{3,13}}e^{\frac{3am}{\alpha}\log\eta_\alpha - c_{3,13}L^{\beta_n(L)-\epsilon(L)}}, \tag{2.3.58}$$

for some constant $c_{3,13} > 0$. Now, using the fact that there are constants $c_{3,14}$ and $c_{3,15}$ such that

$$c_{3,14}L \leq m \leq c_{3,15}L,$$

we can substitute (2.3.58) into (2.3.57) to conclude that there is a constant $c_{3,16}$ such that

$$\mathbb{E}\left[\rho^a, p_B \leq e^{-cL^{\beta_n}}\right] \leq \frac{1}{c_{3,16}}e^{-c_{3,16}L^{1-\epsilon(L)}}.$$

∎

It is now straightforward to conclude the proof of Proposition 2.3.3 using the estimates of Lemmas 2.3.7, 2.3.8 and 2.3.9.

2.3.4 The effective criterion implies (T')

We will prove that the generalized effective criterion and the ellipticity condition $(E)_0$ imply (T'). To do this, it is enough to prove the following.

Proposition 2.3.4. *Throughout choose $0 < \alpha < \min_e F_e$. Let $l \in \mathbb{S}^{d-1}$ and $d \geq 2$. If the effective criterion in direction l holds then there exists a constant $c_{3,28} > 0$ and a neighborhood V_l of direction l such that for all $l' \in V_l$ one has that*

$$\overline{\lim_{L\to\infty}} L^{-1}e^{c_{3,28}(\log L)^{1/2}}\log P_0\left[\tilde{T}^{l'}_{-\tilde{b}L} < T^{l'}_{bL}\right] < 0, \ \textit{for all } b, \tilde{b} > 0. \tag{2.3.59}$$

In particular, if (2.3.47) is satisfied, condition $(T')|l$ is satisfied.

To prove this proposition, we will follow the same strategy used by Sznitman in [Sz02] to prove Proposition 2.3 of that paper under the assumption of uniform ellipticity. Firstly we need to define some constants. Let

$$c_1'(d,\alpha) := 13 + \frac{24d}{\alpha} + \frac{24d + 12\log\eta_\alpha}{2\log\eta_\alpha},$$

$$c_2'(d,\alpha) := c_1 c_1',$$

and

$$c_4(d,\alpha) := \frac{48c_2'}{\alpha},$$

where c_1 is defined in (2.2.1). Define for $k \geq 0$ the sequence $\{N_k : k \geq 0\}$ by

$$N_k := \frac{c_4}{u_0} 8^k, \tag{2.3.60}$$

where $u_0 \in (0,1)$. Let $L_0, \tilde{L}_0, L_1$ and $\tilde{L}_1$ be constants such that

$$3\sqrt{d} \leq \tilde{L}_0 \leq L_0^3, \quad L_1 = N_0 L_0 \quad \text{and } \tilde{L}_1 = N_0^3 \tilde{L}_0. \tag{2.3.61}$$

Now, for $k \geq 0$ define recursively the sequences $\{L_k : k \geq 0\}$ and $\{\tilde{L}_k : k \geq 0\}$ by

$$L_{k+1} := N_k L_k, \quad \text{and} \quad \tilde{L}_{k+1} := N_k^3 \tilde{L}_k. \tag{2.3.62}$$

It is straightforward to see that for each $k \geq 1$

$$L_k = \left(\frac{c_4}{u_0}\right)^k 8^{\frac{k(k-1)}{2}} L_0, \quad \tilde{L}_k = \left(\frac{L_k}{L_0}\right)^3 \tilde{L}_0. \tag{2.3.63}$$

Furthermore, we also consider for $k \geq 0$ the box

$$B_k := B(R, L_k - 1, L_k + 1, \tilde{L}_k), \tag{2.3.64}$$

and the positive part of its boundary $\partial_+ B_k$, and will use the notations

$$\rho_k = \rho_{B_k}, p_k = p_{B_k}, q_k = q_{B_k} \text{ and } n_k = [N_k].$$

Following Sznitman [Sz02], we introduce for each $i \in \mathbb{Z}$

$$\mathcal{H}_i := \{x \in \mathbb{Z}^d, \exists\, x' \in \mathbb{Z}^d, |x - x'| = 1, (x \cdot l - iL_0)(x' \cdot l - iL_0) \leq 0\}. \tag{2.3.65}$$

We also define the function $I : \mathbb{Z}^d \to \mathbb{Z}$ by

$$I(x) := i, \text{ for } x \text{ such that } x \cdot l \in \left[iL_0 - \frac{L_0}{2}, iL_0 + \frac{L_0}{2}\right).$$

Consider now the successive times of visits of the random walk to the sets $\{\mathcal{H}_i : i \in \mathbb{Z}\}$, defined recursively as

$$V_0 := 0, \quad V_1 := \inf\{n \geq 0 : X_n \in \mathcal{H}_{I(X_0)+1} \cup \mathcal{H}_{I(X_0)-1}\}$$

and

$$V_{k+1} := V_k + V_1 \circ \theta_{V_k}, \ k \geq 0.$$

For $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $i \in \mathbb{Z}$, let

$$\widehat{q}(x,\omega) := P_{x,\omega}[X_{V_1} \in \mathcal{H}_{I(x)-1}] \tag{2.3.66}$$

while $\widehat{p}(x,\omega) := 1 - \widehat{q}(x,\omega)$, and

$$\widehat{\rho}(i,\omega) := \sup\left\{\frac{\widehat{q}(x,\omega)}{\widehat{p}(x,\omega)} : x \in \mathcal{H}_i, \sup_{2\leq j\leq d} |R(e_j) \cdot x| < \tilde{L}_1\right\}. \tag{2.3.67}$$

We consider also the stopping time

$$\tilde{T} := \inf\left\{n \geq 0 : \sup_{2\leq j \leq d} |X_n \cdot R(e_j)| \geq \tilde{L}_1\right\},$$

and the function $f : \{n_0 + 2, n_0 + 1, \ldots\} \times \Omega \to \mathbb{R}$ defined by

$$f(n_0 + 2, \omega) := 0, \quad f(i,\omega) := \sum_{m=i}^{n_0+1} \prod_{j=m+1}^{n_0+1} \widehat{\rho}(j,\omega)^{-1}, \quad \text{for} \quad i \leq n_0 + 1. \tag{2.3.68}$$

We will frequently write $f(n)$ instead $f(n,\omega)$. Let us now proceed to prove Proposition 2.3.4. The following proposition corresponds to the first step in an induction argument which will be used to prove Proposition 2.3.4.

Proposition 2.3.5. *Let $\alpha > 0$. Let $L_0, L_1, \tilde{L}_0$ and $\tilde{L}_1$ be constants satisfying (2.3.61), with $N_0 \geq 7$. Then, there exist $c_{3,17}, c_{3,18}(d), c_{3,19}(d) > 0$ such that for $L_0 \geq c_{3,17}$, $a \in (0, \alpha]$, $u_0 \in [\xi^{L_0/d}, 1]$, $0 < \xi < \frac{1}{\eta_\alpha^{2/\alpha}}$ and*

$$N_0 \leq \frac{1}{L_0}\left(\frac{e}{\xi}\right)^{L_0}, \tag{2.3.69}$$

the following is satisfied

$$\mathbb{E}[\rho_1^{a/2}] \leq c_{3,18}\Bigg\{\xi^{-c_2' L_1}\left(c_{3,19}\tilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\tilde{L}_0\mathbb{E}[q_0]\right)^{\frac{\tilde{L}_1}{12N_0\tilde{L}_0}} + \sum_{m=0}^{N_0+1}\left(c_{3,19}\tilde{L}_1^{(d-1)}\mathbb{E}[\rho_0^a]\right)^{\frac{N_0+m-1}{2}} + e^{-c_1 L_1 \log\frac{1}{\xi^\alpha \eta_\alpha^2}}\Bigg\}. \tag{2.3.70}$$

Proof. The following inequality is stated and proved in [Sz02] by Sznitman without using any kind of uniform ellipticity assumption (inequality (2.18) in [Sz02]). For every $\omega \in \Omega$

$$P_{0,\omega}\left(\tilde{T}^l_{1-L_1} < \tilde{T} \wedge T^l_{L_1+1}\right) \leq \frac{f(0)}{f(1-n_0)}. \tag{2.3.71}$$

Consider now the event

$$G := \{\omega : P_{0,\omega}\left(\tilde{T} \leq \tilde{T}^l_{1-L_1} \wedge T^l_{L_1+1}\right) \leq \xi^{(c_1'-1)c_1 L_1}\}, \tag{2.3.72}$$

and write

$$\mathbb{E}[\rho_1^{a/2}] = \mathbb{E}[\rho_1^{a/2}, G] + \mathbb{E}[\rho_1^{a/2}, G^c]. \tag{2.3.73}$$

The first term $\mathbb{E}[\rho_1^{a/2}, G]$ of (2.3.73), can in turn be decomposed as

$$\mathbb{E}[\rho_1^{a/2}, G] = \mathbb{E}[\rho_1^{a/2}, G, A_1] + \mathbb{E}[\rho_1^{a/2}, G, A_1^c], \tag{2.3.74}$$

where we have defined

$$A_1 := \{\omega \in \Omega : f(2-n_0) - f(0) \geq f(1-n_0)\xi^{(c_1'-1)c_1 L_1}, f(0) \geq f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}.$$

Furthermore, note that

$$A_1^c \subset A_2 \cup A_3,$$

where

$$A_2 := \{\omega \in \Omega : f(2-n_0) - f(0) < f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}, \text{ while}$$

$$A_3 := \{\omega \in \Omega : f(0) < f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\}.$$

Therefore,

$$\mathbb{E}[\rho_1^{a/2}] \leq \mathbb{E}[\rho_1^{a/2}, G, A_1] + \mathbb{E}[\rho_1^{a/2}, A_2] + \mathbb{E}[\rho_1^{a/2}, G, A_3] + \mathbb{E}[\rho_1^{a/2}, G^c]. \tag{2.3.75}$$

We now subdivide the rest of the proof in several steps corresponding to an estimation for each one of the terms in inequality (2.3.75).

Step 1: estimate of $\mathbb{E}[\rho_1^{a/2}, G, A_1]$. Here we estimate the first term of display (2.3.74). To do this, we can follow the argument presented by Sznitman in Section 2 of [Sz02], to prove that inequality (2.3.71) implies that there exist constant $c_{3,20}(d)$ such that

$$\mathbb{E}\left[\rho_1^{a/2}, G, A_1\right] \leq 2\sum_{m=0}^{n_0+1}\left(c_{3,20}(d)\tilde{L}_1^{(d-1)}\mathbb{E}[\rho_0^a]\right)^{\frac{n_0+m-1}{2}}. \tag{2.3.76}$$

Indeed on $G \cap A_1$ and with the help of (2.3.71) one gets that

$$\begin{aligned}
\rho_1 &= \frac{P_{0,\omega}[\tilde{T}^l_{-L_1+1} < \tilde{T} \wedge T^l_{L_1+1}] + P_{0,\omega}[\tilde{T} \leq \tilde{T}^l_{-L_1+1} \wedge T^l_{L_1+1}]}{1 - P_{0,\omega}[\tilde{T}^l_{-L_1+1} < \tilde{T} \wedge T^l_{L_1+1}] - P_{0,\omega}[\tilde{T} \leq \tilde{T}^l_{-L_1+1} \wedge T^l_{L_1+1}]} \\
&\leq \frac{f(0) + f(1-n_0)\xi^{(c_1'-1)c_1 L_1}}{\left(f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\right)_+} \\
&\leq \frac{2f(0)}{\left(f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1 L_1}\right)_+},
\end{aligned} \tag{2.3.77}$$

where in the first inequality we have used the fact that $\omega \in G$, while in the second that $\omega \in A_1$. Regarding the term in the denominator in the last expression, we can use the definition of the function f and obtain

$$\begin{aligned}
&f(1-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1 L_1} \\
&= \prod_{j=2-n_0}^{n_0+1}\widehat{\rho}(j,\omega)^{-1} + f(2-n_0) - f(0) - f(1-n_0)\xi^{(c_1'-1)c_1 L_1} \\
&\geq \prod_{j=2-n_0}^{n_0+1}\widehat{\rho}(j,\omega)^{-1},
\end{aligned}$$

where we have used that $\omega \in A_1$ in the last inequality. Substituting this estimate in (2.3.77), we conclude that for $\omega \in G \cap A_1$ one has that

$$\rho_1 \le 2 \prod_{j=2-n_0}^{n_0+1} \widehat{\rho}(j,\omega) f(0) = 2 \sum_{m=0}^{n_0+1} \prod_{j=2-n_0}^{m} \widehat{\rho}(j,\omega). \tag{2.3.78}$$

At this point, using (2.3.78), the fact that $(u+v)^{a/2} \le u^{a/2}+v^{a/2}$ for $u, v \ge 0$, the fact that $\{\widehat{\rho}(j,\omega), j \text{ even}\}$ and $\{\widehat{\rho}(j,\omega), j \text{ odd}\}$ are two collections of independent random variables and the Cauchy-Schwartz's inequality, we can assert that

$$\begin{aligned} & \mathbb{E}[\rho_1(\omega)^{a/2}, G, A_1] \\ \le\ & 2 \sum_{0\le m\le n_0+1} \mathbb{E}\left[\prod_{1-n_0<j\le m} \widehat{\rho}(j,\omega)^{a/2}\right] \\ \le\ & 2 \sum_{0\le m\le n_0+1} \mathbb{E}\left[\prod_{\substack{1-n_0<j\le m \\ j\, is\, even}} \widehat{\rho}(j,\omega)^{a}\right]^{1/2} \mathbb{E}\left[\prod_{\substack{1-n_0<j\le m \\ j\, is\, odd}} \widehat{\rho}(j,\omega)^{a}\right]^{1/2} \\ =\ & 2 \sum_{0\le m\le n_0+1} \prod_{1-n_0<j\le m} \mathbb{E}\left[\widehat{\rho}(j,\omega)^{a}\right]^{1/2}. \end{aligned}$$

In view of (2.3.66) one gets easily that for $i \in \mathbb{Z}$ and $x \in \mathcal{H}_i$,

$$\widehat{p}(x,\omega) \ge p_0 \circ t_x(\omega),$$

where the canonical shift $\{t_x : x \in \mathbb{Z}^d\}$ has been defined in (2.2.4). Hence, for $i \in \mathbb{Z}$ and $x \in \mathcal{H}_i$,

$$\frac{\widehat{q}(x)}{\widehat{p}(x)} \le \rho_0 \circ t_x.$$

Following Sznitman [Sz02] with the help of (2.3.65) the estimate (2.3.76) follows.

Step 2: estimate of $\mathbb{E}[\rho_1^{a/2}, A_2]$. Here we will prove the following estimate for the second term of inequality (2.3.75),

$$\mathbb{E}[\rho^{a/2}, A_2] \le 4e^{-c_1 L_1 \log \frac{1}{\xi^\alpha \eta_\alpha^2}}.$$

By the definition of c_1 (see (2.2.1)), we know that necessarily there exists a path with less than $c_1(L_1 + 1 + \sqrt{d})$ steps between the origin and $\partial_+ B_1$. Therefore, for $L_0 \ge 1 + \sqrt{d}$, there is a nearest neighbor self-avoiding path $(x_1,\dots,x_n)$ with n steps from the origin to $\partial_+ B_1$, such that $2c_1L_1 \le n \le 2c_1L_1+1$, $x_1,\dots,x_n \in B_1$ and $x_n \cdot l \ge L_1+1$. Thus, for every $r \ge 0$ we have that

$$\rho_1^r \le \frac{1}{p_1^r} \le e^{r\sum_{i=1}^n \log \frac{1}{\omega(x_i,\Delta x_i)}}, \tag{2.3.79}$$

where $\Delta x_i := x_{i+1} - x_i$ for $1 \le i \le n-1$ as defined in (2.2.3). We then have applying inequality (2.3.79) with $r = a/2$ that

$$\begin{aligned}\mathbb{E}\left[\rho_1^{a/2}, A_2\right] \le{}& \mathbb{E}\left[e^{\alpha/2\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)}}, A_2, \sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} \le n\log\left(\frac{1}{\xi}\right)\right] \\ &+\mathbb{E}\left[e^{\alpha/2\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)}}, \sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\left(\frac{1}{\xi}\right)\right].\end{aligned} \tag{2.3.80}$$

Regarding the second term of the right side of (2.3.80), we can apply the Cauchy-Schwarz inequality, the exponential Chebychev inequality and conclude that and use the fact that the jump probabilities $\{\omega(x_i,\Delta_i) : 1 \le i \le n-1\}$ are independent to conclude that

$$\begin{aligned}\mathbb{E}\left[e^{\alpha\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2} \mathbb{P}\left(\sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\left(\frac{1}{\xi}\right)\right)^{1/2} \\ \le e^{\left(2\log\eta_\alpha - \alpha\log\left(\frac{1}{\xi}\right)\right)n/2}.\end{aligned} \tag{2.3.81}$$

Meanwhile, note that the first term on the right side of (2.3.80) can be bounded by

$$e^{\frac{\alpha\log\left(\frac{1}{\xi}\right)n}{2}}\mathbb{P}(A_2). \tag{2.3.82}$$

Hence, we need an adequate estimate for $\mathbb{P}(A_2)$. Now,

$$\begin{aligned}\mathbb{P}(A_2) ={}& \mathbb{P}\left(f(2-n_0) - f(0) < e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}, A_2\right) \\ &+\mathbb{P}\left(f(2-n_0) - f(0) \ge e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}, A_2\right) \\ \le{}& \mathbb{P}\left(f(2-n_0) - f(0) < e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) \\ &+\mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right).\end{aligned} \tag{2.3.83}$$

The two terms in the rightmost side of display (2.3.83) will be estimated by similar methods: in both cases, we will use the fact that $\{\widehat{\rho}(j,\omega), j\,\text{even}\}$ and $\{\widehat{\rho}(j,\omega), j\,\text{odd}\}$ are two collections of independent random variables, the Cauchy-Schwartz's inequality and the Chebyshev inequality. Specifically for the first term of the rightmost side of (2.3.83) we have that

$$\begin{aligned}
\mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) &\leq \mathbb{P}\left(\prod_{j=0}^{n_0+1}\widehat{\rho}(j,\omega)^{-1}<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right)\\
&= \mathbb{P}\left(\prod_{j=0}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha/2}>e^{\frac{(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\right)\\
&\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\mathbb{E}\left[\prod_{\substack{j=1,\\ j\ odd}}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}\mathbb{E}\left[\prod_{\substack{j=0,\\ j\ even}}^{n_0+1}\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}\\
&= e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}\prod_{j=0}^{n_0+1}\mathbb{E}\left[\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2}. \qquad (2.3.84)
\end{aligned}$$

By an estimate analogous to (2.3.79), we know that for $L_0 \geq 1+\sqrt{d}$, for each $j \in \{0,\dots,n_0+1\}$ and each $x \in \mathcal{H}_j$, there exists a nearest neighbor self-avoiding path $(y_1,\dots,y_m)$ with m steps, such that $2c_1L_0 \leq m \leq 2c_1L_0+1$, between x and $\mathcal{H}_{j+1}$. Also, $y_1\cdot l,\dots,y_{m-1}\cdot l \in (1-L_0,L_0+1)$ and $y_m\cdot l \geq L_0+1$. Then, in view of (2.3.62), (2.3.63), (2.3.66) and (2.3.67), we have that for each $j \in \{0,\dots,n_0+1\}$

$$\begin{aligned}
\mathbb{E}\left[\widehat{\rho}(j,\omega)^{\alpha}\right]^{1/2} &\leq \sum \mathbb{E}\left[\widehat{p}(x,\omega)^{-\alpha}\right]^{1/2}\\
&\leq 2L_1^{3(d-1)}\mathbb{E}\left[e^{\alpha\sum_1^m\log\frac{1}{\omega(y_i,\Delta y_i)}}\right]^{1/2} \leq 2L_1^{3(d-1)}e^{\frac{m\log\eta_\alpha}{2}}, \qquad (2.3.85)
\end{aligned}$$

where the summation goes over all $x \in \mathcal{H}_j$ such that $\sup_{2\leq i\leq d}|R(e_i)\cdot x| < \widetilde{L}_1$. Substituting the estimate (2.3.85) back into (2.3.84) we see that

$$\begin{aligned}
\mathbb{P}\left(f(2-n_0)-f(0)<e^{\frac{-(c_1'-1)c_1L_1\log\left(\frac{1}{\xi}\right)}{2}}\right) &\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}}2^{(n_0+2)}L_1^{3(d-1)(n_0+2)}e^{\frac{(\log\eta_\alpha)m(n_0+2)}{2}}\\
&\leq e^{\frac{-(c_1'-1)c_1L_1\alpha\log\left(\frac{1}{\xi}\right)}{4}+\log 2(n_0+2)+3(d-1)\frac{\log L_0N_0}{L_0}L_0(n_0+2)+(\log\eta_\alpha)\frac{(2c_1L_0+1)(n_0+2)}{2}}\\
&\leq e^{-L_1\left(\frac{(c_1'-1)\alpha\log\left(\frac{1}{\xi}\right)c_1}{4}-1-6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right)-\log\eta_\alpha(3c_1+1)\right)}, \qquad (2.3.86)
\end{aligned}$$

where we have used the fact that for $L_0 \geq 2\log c_4$ it is true that $\dfrac{\log N_0L_0}{L_0} \leq 1+\log\left(\dfrac{1}{\xi}\right)$

for all $u_0 \in [\xi^{L_0/d}, 1]$. Meanwhile, for the second term of the rightmost side of (2.3.83), we have that

$$\begin{aligned}
\mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1 L_1 \log\left(\frac{1}{\xi}\right)}{2}}\right) &= \mathbb{P}\left(\sum_{k=1-n_0}^{1+n_0} \prod_{j=k+1}^{n_0+1} \widehat{\rho}(j,\omega)^{-1} > e^{\frac{(c_1'-1)c_1 L_1 \log\left(\frac{1}{\xi}\right)}{2}}\right) \\
&\leq \sum_{k=1-n_0}^{1+n_0} \mathbb{P}\left(\prod_{j=k+1}^{n_0+1} \widehat{\rho}(j,\omega)^{-\alpha/2} > \frac{e^{(c_1'-1)c_1 L_1 \alpha \log\left(\frac{1}{\xi}\right)/4}}{(2n_0+1)^{\alpha/2}}\right) \\
&= e^{\frac{-(c_1'-1)c_1 L_1 \alpha \log\left(\frac{1}{\xi}\right)}{4}} (2n_0+1)^{\alpha/2} \sum_{k=1-n_0}^{1+n_0} \prod_{j=k+1}^{n_0+1} \mathbb{E}\left[\widehat{\rho}(j,\omega)^{-\alpha}\right]^{1/2}
\end{aligned}$$

In analogy to (2.3.85), we can conclude that $\mathbb{E}\left[\widehat{\rho}(j,\omega)^{-\alpha}\right]^{1/2} \leq 2L_1^{3(d-1)} e^{\frac{(\log \eta_\alpha)m}{2}}$. Therefore, for $L_0 \geq 2\log c_4$ we see that

$$\begin{aligned}
&\mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1 L_1 \log\left(\frac{1}{\xi}\right)}{2}}\right) \\
&\leq e^{\frac{-(c_1'-1)c_1 L_1 \alpha \log\left(\frac{1}{\xi}\right)}{4}} (2n_0+1)^{\alpha/2} \sum_{k=1-n_0}^{1+n_0} 2^{n_0+1-k} L_1^{3(d-1)(n_0+1-k)} e^{(\log \eta_\alpha)(2c_1 L_0+1)(n_0+1-k)} \\
&\leq e^{\frac{-(c_1'-1)\alpha \log\left(\frac{1}{\xi}\right) c_1 L_1}{4} + \left(\frac{\alpha+2}{2}\right)\log(2n_0+1) + 2n_0 \log 2 + 6(d-1)n_0\left(\frac{\log L_0 N_0}{L_0}\right) L_0 + (4c_1 L_0 n_0 + 2n_0)\log \eta_\alpha} \\
&\leq e^{-L_1\left(\frac{(c_1'-1)\alpha \log\left(\frac{1}{\xi}\right) c_1}{4} - 1 - 6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right) - (4c_1+1)(\log \eta_\alpha)\right)}. \qquad (2.3.87)
\end{aligned}$$

Now, in view of (2.3.82), (2.3.83), (2.3.86) and (2.3.87) the first term on the right side of (2.3.80) is bounded by

$$2e^{-L_1\left(\frac{(c_1'-1)\alpha \log\left(\frac{1}{\xi}\right) c_1}{4} - 2\alpha c_1 \log\left(\frac{1}{\xi}\right) - 1 - 6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right) - (4c_1+1)(\log \eta_\alpha)\right)}. \qquad (2.3.88)$$

Now, since $c_1' \geq 13 + \dfrac{24d}{\alpha} + \dfrac{24d + 12\log \eta_\alpha}{\alpha \log \frac{1}{\xi}}$, we conclude that

$$\begin{aligned}
\frac{(c_1'-1)\alpha \log\left(\frac{1}{\xi}\right) c_1}{4} - 2\alpha c_1 \log\left(\frac{1}{\xi}\right) - 1 - 6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right) - (4c_1+1)(\log \eta_\alpha) \\
\geq \alpha c_1 \log\left(\frac{1}{\xi}\right) - 2c_1 \log \eta_\alpha,
\end{aligned}$$

and therefore, by (2.3.81) and (2.3.88) we have that

$$E\left[\rho_1^{a/2}, A_2\right] \leq 4e^{-c_1 L_1 \log \frac{1}{\xi^\alpha \eta_\alpha^2}}. \qquad (2.3.89)$$

Step 3: estimate of $\mathbb{E}[\rho_1^{a/2}, G, A_3]$. Here we will estimate the third term of the inequality (2.3.75). Specifically we will show that

$$\mathbb{E}[\rho_1^{a/2}, G, A_3] \leq 2e^{-c_1 L_1 \log \frac{1}{\xi^\alpha \eta_\alpha^2}}. \tag{2.3.90}$$

This upper bound will be almost obtained as the previous case, where we achieved (2.3.89). Indeed, in analogy to the development of (2.3.77) in Step 3, one has that for $\omega \in G$,

$$\rho_1 \leq \frac{f(0) + f(1-n_0)\xi^{(c_1'-1)c_1 L_1}}{(f(1-n_0) - f(0) - \xi^{(c_1'-1)c_1 L_1} f(1-n_0))_+}.$$

But, if $\omega \in A_3$ also, one easily gets that $0 < \rho_1 \leq 1$ if $L_0 \geq \dfrac{\alpha \log 4}{2 \log \eta_\alpha}$. Thus,

$$\mathbb{E}\left[\rho_1^{a/2}, G, A_3\right] \leq \mathbb{P}\left(A_3\right). \tag{2.3.91}$$

Therefore, since $c_1' \geq 13 + \dfrac{24d}{\alpha} + \dfrac{24d + 12\log \eta_\alpha}{\alpha \log \frac{1}{\xi}}$, it is enough to prove that

$$\mathbb{P}(A_3) \leq 2e^{-L_1\left(\frac{(c_1'-1)\alpha \log\left(\frac{1}{\xi}\right)c_1}{4} - 1 - 6(d-1)\left(1+\log\left(\frac{1}{\xi}\right)\right) - (4c_1+1)(\log \eta_\alpha)\right)}. \tag{2.3.92}$$

To justify this inequality, note that

$$\begin{aligned}\mathbb{P}(A_3) &\leq \mathbb{P}\left(f(0) < e^{\frac{-(c_1'-1)c_1 L_1 \log \frac{1}{\xi}}{2}}\right) + \mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1 L_1 \log \frac{1}{\xi}}{2}}\right) \\ &\leq \mathbb{P}\left(\prod_{j=1}^{n_0+1} \widehat{\rho}(j,\omega)^{-1} < e^{\frac{-(c_1'-1)c_1 L_1 \log\left(\frac{1}{\xi}\right)}{2}}\right) + \mathbb{P}\left(f(1-n_0) > e^{\frac{(c_1'-1)c_1 L_1 \log \frac{1}{\xi}}{2}}\right),\end{aligned}$$

and hence we are in a very similar situation as in (2.3.83) and development in (2.3.84) and (2.3.87), from where we derive (2.3.92).

Step 4: estimate of $\mathbb{E}[\rho_1^{a/2}, G^c]$. Here we will prove that there exist constants $c_{3,21}(d)$ and $c_{3,22}(d)$ such that

$$\mathbb{E}[\rho_1^{a/2}, G^c] \leq c_{3,21}\xi^{-c_1' c_1 L_1}\left(c_{3,22}\widetilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\widetilde{L}_0 \mathbb{E}[q(0)]\right)^{\frac{\widetilde{L}_1}{12 N_0 \widetilde{L}_0}} + e^{-c_1 L_1 \log \frac{1}{\xi^\alpha \eta_\alpha^2}}. \tag{2.3.93}$$

Firstly, we need to consider the event

$$A_4 := \left\{\omega \in \Omega : P_{0,\omega}\left(T_{L_1+1}^l \leq \widetilde{T} \wedge \widetilde{T}_{1-L_1}^l\right) \geq \xi^{2c_1 L_1}\right\}.$$

In the case that $\omega \in G^c \cap A_4$, the walk behaves as if effectively it satisfies a uniformly ellipticity condition with constant $\kappa = \xi$, so that we can follow exactly the same reasoning presented by Sznitman in [Sz02] leading to inequality (2.32) of that paper, showing that there exist constants $c_{3,21}(d)$, $c_{3,22}(d)$ such that whenever $\widetilde{L}_1 \geq 48N_0\widetilde{L}_0$ one has that

$$\mathbb{E}\left[\rho_1^{a/2}, G^c, A_4\right] \leq \xi^{-c_1L_1}\mathbb{P}(G^c) \leq c_{3,21}\xi^{-c_1'c_1L_1}\left(c_{3,22}\widetilde{L}_1^{(d-2)}\frac{L_1^3}{L_0^2}\widetilde{L}_0\mathbb{E}[q(0)]\right)^{\frac{\widetilde{L}_1}{12N_0\widetilde{L}_0}}. \quad (2.3.94)$$

The second inequality of (2.3.94) does not use any uniformly ellipticity assumption. It would be enough now to prove that

$$\mathbb{E}(\rho_1^{a/2}, A_4^c) \leq e^{-c_1L_1\log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \quad (2.3.95)$$

To do this we will follow the reasoning presented in Step 2. Namely, for $L_0 \geq 1+\sqrt{d}$, there is a nearest neighbor self-avoiding path $(x_1,\ldots,x_n)$ with n steps from 0 to $\partial_+ B_1$ such that $2c_1L_1 \leq n \leq 2c_1L_1+1$, $x_1,\ldots,x_n \in B_1$ and $x_n\cdot l \geq L_1+1$. Therefore

$$A_4^c \subset \left\{\omega\in\Omega : \prod_1^n \omega(x_i,\Delta x_i) < \xi^n\right\} = \left\{\omega\in\Omega : \sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\frac{1}{\xi}\right\},$$

so that

$$\begin{aligned}\mathbb{E}\left[\rho_1^{a/2}, A_4^c\right] &\leq \mathbb{E}\left[e^{\alpha/2\sum_1^n\log\frac{1}{\omega(x_i,\Delta x_i)}}, \sum_1^n \log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\frac{1}{\xi}\right]\\ &\leq \mathbb{E}\left[e^{\alpha\sum_1^n\log\frac{1}{\omega(x_i,\Delta x_i)}}\right]^{1/2}\mathbb{P}\left(\sum_1^n\log\frac{1}{\omega(x_i,\Delta x_i)} > n\log\frac{1}{\xi}\right)^{1/2}\\ &\leq e^{(2\log\eta_\alpha-\alpha\log\frac{1}{\xi})c_1L_1},\end{aligned} \quad (2.3.96)$$

which proves (2.3.95) and finishes Step 4.

Step 5: conclusion. Combining the estimates (2.3.76) of step 1, (2.3.89) of step 2, (2.3.90) of step 3 and (2.3.93) of step 4, we have (2.3.70). ∎

We will now prove a corollary of Proposition 2.3.5, which will imply Proposition 2.3.4. For this, it will be important to note that the statement of Proposition 2.3.5 is still valid if given $k \geq 1$ we change L_0 by L_k, L_1 by L_{k+1}, $\tilde{L}_0$ by $\tilde{L}_k$ and $\tilde{L}_1$ by $\tilde{L}_{k+1}$. In effect, to see this, it is enough to note that inequality (2.3.69) is satisfied with these replacements. Define

$$c_{3,23} := e^{-\frac{4c_1\log\eta_\alpha}{(c_1-1)\alpha}}.$$

Corollary 2.3.1. *Let $0 < \xi < \min\{c_{3,23}, e^{-1/24}\}$ and $\alpha > 0$. Let $\{L_k : k \geq 0\}$ and $\{\widetilde{L}_k : k \geq 0\}$ be sequences satisfying (2.3.60), (2.3.61) and (2.3.62). Then there exists $c_{3,25}(d,\alpha) > 0$, such that when for some $L_0 \geq c_{3,25}$, $a_0 \in (0,\alpha]$, $u_0 \in [\xi^{L_0/d}, 1]$, it is true that*

$$\phi_0 := c_{3,19}\widetilde{L}_1^{d-1}L_0\mathbb{E}[\rho_0^{a_0}] \leq \xi^{\alpha u_0 L_0}, \tag{2.3.97}$$

then for all $k \geq 0$,

$$\phi_k := c_{3,19}\widetilde{L}_{k+1}^{d-1}L_k\mathbb{E}[\rho_k^{a_k}] \leq (k+1)\xi^{\alpha u_k L_k}, \tag{2.3.98}$$

with $a_k := a_0 2^{-k}$, $u_k := u_0 8^{-k}$.

Proof. We will use induction in k to prove (2.3.98). By hypothesis we only need to show (2.3.98) for $n = k+1$ assuming that (2.3.98) holds for $n = k$. To do this, with the help of Proposition 2.3.5 we have that for any $k \geq 0$

$$\mathbb{E}[\rho_{k+1}^{a_{k+1}}] \leq c_{3,18}\left\{\xi^{-c_2' L_{k+1}}\left(c_{3,19}\widetilde{L}_{k+1}^{(d-2)}\frac{L_{k+1}^3}{L_k^2}\widetilde{L}_k\mathbb{E}[q_k]\right)^{\frac{\widetilde{L}_{k+1}}{12N_k\widetilde{L}_k}}\right.$$
$$\left. + \sum_{0\leq m\leq N_k+1}\left(c_{3,19}\widetilde{L}_{k+1}^{(d-1)}\mathbb{E}[\rho_k^{a_k}]\right)^{\frac{[N_k]+m-1}{2}} + e^{-c_1 L_{k+1}\log\frac{1}{\xi^\alpha \eta_\alpha^2}}\right\},$$

so that, for $k \geq 0$ and with the help of (2.3.62)

$$\phi_{k+1} \leq c_{3,18}c_{3,19}\widetilde{L}_{k+2}^{(d-1)}L_{k+1}\left\{\xi^{-c_2' L_{k+1}}\phi_k^{N_k^2/12} + \sum_{0\leq m\leq N_k+1}\phi_k^{\frac{N_k+m-1}{2}}\right\}$$
$$+c_{3,18}c_{3,19}\widetilde{L}_{k+2}^{(d-1)}L_{k+1}e^{-c_1 L_{k+1}\log\frac{1}{\xi^\alpha\eta_\alpha^2}}. \tag{2.3.99}$$

Since $\xi < c_{3,23}$, we can assert that $c_{3,18}c_{3,19}\widetilde{L}_{k+2}^{(d-1)}L_{k+1}e^{-c_1 L_{k+1}\log\frac{1}{\xi^\alpha\eta_\alpha^2}} \leq \xi^{\alpha u_{k+1}L_{k+1}}$. Hence, we only need to prove that

$$c_{3,18}c_{3,19}\widetilde{L}_{k+2}^{(d-1)}L_{k+1}\left\{\xi^{-c_2' L_{k+1}}\phi_k^{N_k^2/12} + \sum_{0\leq m\leq N_k+1}\phi_k^{\frac{N_k+m-1}{2}}\right\} \leq (k+1)\xi^{\alpha u_{k+1}L_{k+1}} \tag{2.3.100}$$

Firstly, note that for L_0 large enough by the induction hypothesis, (2.3.62) and the fact that $\xi < e^{-\frac{1}{24}}$

$$\xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/24} \le \xi^{-c_2' L_{k+1}} (k+1)^{\frac{N_k^2}{24}} \xi^{\frac{\alpha u_k N_k^2 L_k}{24}}$$
$$\le e^{c_2' \left(\log \frac{1}{\xi} + \frac{1}{24} - \frac{c_4 \log \frac{1}{\xi}}{24} \right) L_{k+1}} \le 1.$$

Substituting this estimate back into (2.3.100) and using the hypothesis induction again, we obtain that

$$c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \xi^{-c_2' L_{k+1}} \phi_k^{N_k^2/12} + \sum_{0 \le m \le N_k+1} \phi_k^{\frac{N_k+m-1}{2}} \right\}$$
$$\le c_{3,18} c_{3,19} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} \left\{ \phi_k^{N_k^2/24} + (N_k+2) \phi_k^{N_k/4} \right\}$$
$$\le c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+1} N_{k+1} \phi_k^{N_k/4}$$
$$\le c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} \phi_k^{N_k/8} (k+1)^{N_k/8} \xi^{\alpha u_k L_k N_k/8}$$
$$= c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} \phi_k^{N_k/8} (k+1)^{N_k/8-1} (k+1) \xi^{\alpha u_{k+1} L_{k+1}},$$

where $c_{3,24} := 2 c_{3,18} c_{3,19}$. Thus, in order to show that $\phi_{k+1} \le (k+2) \xi^{\alpha u_{k+1} L_{k+1}}$ it is enough to prove that

$$c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/8-1} \phi_k^{N_k/8} \le 1. \tag{2.3.101}$$

First, note that by the induction hypothesis,

$$c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/8-1} \phi_k^{N_k/8} \le c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}. \tag{2.3.102}$$

From (2.3.61), (2.3.62) and (2.3.63), we can say that

$$c_{3,24} \widetilde{L}_{k+2}^{(d-1)} L_{k+2} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}$$
$$= c_{3,24} \left(\frac{L_{k+2}}{L_0} \right)^{3(d-1)} \widetilde{L}_0^{(d-1)} L_{k+2} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}$$
$$\le c_{3,24} L_{k+2}^{3(d-1)+1} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}$$
$$= c_{3,24} (N_{k+1} N_k)^{3d-2} L_k^{3d-2} (k+1)^{N_k/4-1} \xi^{6 c_2' L_k}$$
$$\le c_{3,24} 8^{3d-2} L_k^{3d-2} (k+1)^{N_k/4-1} \xi^{c_1 L_k} N_k^{6d} \xi^{(6 c_1' - 1) c_1 L_k}. \tag{2.3.103}$$

But, note that

$$c_{3,24}8^{3d-2}L_k^{3d-2}(k+1)^{N_k/4-1}\xi^{c_1L_k}\leq 1.$$

for L_0 large enough. Hence, substituting this estimate back into (2.3.103) and (2.3.102) we deduce that

$$c_{3,24}\widetilde{L}_{k+2}^{(d-1)}L_{k+2}(k+1)^{N_k/8-1}\phi_k^{N_k/8}\leq N_k^{6d}\xi^{(6c_1'-1)c_1L_k}\leq N_k^{6d}\xi^{77c_1L_k},\tag{2.3.104}$$

by our choice of c_1'. Finally, choosing L_0 large enough, the expression $N_k^{6d}\xi^{77c_1L_k}\leq 1$ for all $k\geq 1$. In the case of $k=0$, we have that

$$\left(\frac{c_4}{u_0}\right)^{6d}\xi^{77c_1L_0}\leq u_0^{-6d}\xi^{6L_0}\leq 1$$

by our assumption on u_0. Then (2.3.101) follows and thus we get (2.3.98) by induction and choosing $L_0\geq c_{3,25}$ for some constant $c_{3,25}>0$.

∎

The following corollary implies Proposition 2.3.4. Since such a derivation follows exactly the argument presented by Sznitman in [Sz02], we omit it.

Corollary 2.3.2. *Let $l\in\mathbb{S}^{d-1}$, $d\geq 2$ and $\Upsilon=\max\left\{\frac{\alpha}{24},\left(\frac{2c_1}{c_1-1}\right)\log\eta_\alpha^2\right\}$. Then, there exist constants $c_{3,26}=c_{3,26}(d)>0$ and $c_{3,27}=c_{3,27}(d)>0$ such that if the following inequality is satisfied*

$$c_{3,26}(d)\inf_{L_0\geq c_{3,27},3\sqrt{d}\leq\tilde{L}_0<L_0^3}\inf_{0<a\leq\alpha}\left\{\Upsilon^{3(d-1)}\tilde{L}_0^{d-1}L_0^{3(d-1)+1}\mathbb{E}[\rho_B^a]\right\}<1,\tag{2.3.105}$$

where $B=B(R,L_0-1,L_0+1,\tilde{L}_0)$, then there exists a constant $c_{3,28}>0$ such that

$$\overline{\lim_{L\to\infty}}L^{-1}e^{c_{28}(\log L)^{1/2}}\log P_0\left[\tilde{T}^l_{-\tilde{b}L}<T^l_{bL}\right]<0,\ \ \textit{for all } b,\tilde{b}>0.\tag{2.3.106}$$

Proof. If (2.3.105) holds then there is a $\xi>0$ such that

$$c_{3,26}(d)\inf_{L_0\geq c_{3,27},3\sqrt{d}\leq\tilde{L}_0<L_0^3}\inf_{0<a\leq\alpha}\left\{\left(\alpha\log\frac{1}{\xi}\right)^{3(d-1)}\tilde{L}_0^{d-1}L_0^{3(d-1)+1}\mathbb{E}[\rho_B^a]\right\}<1,\tag{2.3.107}$$

with $\xi < \{c_{3,23}, e^{-1/24}\}$. Then, by (2.3.60) and (2.3.61),

$$\widetilde{L}_1^{d-1} L_0 = \left(\frac{c_4}{u_0}\right)^{3(d-1)} \widetilde{L}_0^{d-1} L_0.$$

Now, the maximum of $u_0^{3(d-1)} \xi^{\alpha u_0 L_0}$, as a function of u_0 for $u_0 \in [\xi^{\frac{L_0}{d}}, 1]$, is given by $c_{3,29}(d) \left(\alpha L_0 \log \frac{1}{\xi}\right)^{-3(d-1)}$ for $u_0 = \frac{3(d-1)}{\alpha L_0 \log \frac{1}{\xi}}$, when L_0 is large enough, where $c_{3,29}(d) := \left(\frac{3(d-1)}{e}\right)^{3(d-1)}$. Thus if (2.3.107) holds, (2.3.97) holds as well. Hence, applying Corollary 2.3.1 we can say that (2.3.98) is true for all $k \geq 0$.

Then, for each $b, \widetilde{b} > 0$ we consider the discrete truncated cylinder,

$$C = \left\{x \in \mathbb{Z}^d : |x|_\perp \leq \frac{bL}{L_k} \widetilde{L}_k,\ x \cdot l \in (-\widetilde{b}L, bL)\right\}$$

and the event $\mathcal{H}$ associated to C and defined by

$$\mathcal{H} = \left\{\text{for some}\, x \in C, q_k \circ t_x \geq \xi^{\frac{\alpha}{2} u_k L_k}\right\},$$

where for large L, we chose an unique k such that

$$L_k \leq \widetilde{b}L < L_{k+1}. \tag{2.3.108}$$

It is easy to see that

$$\mathbb{P}(\mathcal{H}) \leq \frac{|C|(k+1)\xi^{\frac{\alpha}{2} u_k L_k}}{c_{3,19} \widetilde{L}_{k+1}^{d-1} L_k} \leq |C| \xi^{\frac{\alpha}{2} u_k L_k} \tag{2.3.109}$$

with the help of (2.3.98) and the fact that $\mathbb{E}[q_k] \leq \mathbb{E}[\rho_k^{a_k}]$.

Meanwhile on $\mathcal{H}^c$ we can apply the strong Markov property $\left[\frac{bL}{L_k}\right]$ times and obtain that

$$P_{0,\omega}\left(T_{bL}^l < \tilde{T}_{-\tilde{b}L}^l\right) \geq \left(1 - \xi^{\frac{\alpha}{2} u_k L_k}\right)^{\left([\frac{bL}{L_k}+1]\right)}.$$

At this point, with the help of (2.3.60), (2.3.61), (2.3.62), (2.3.63) and (2.3.108) we deduce that

$$\begin{aligned} P_0, \left(\tilde{T}_{-\tilde{b}L}^l < T_{bL}^l\right) &\leq \left(|C| + \frac{bL}{L_k} + 1\right) \xi^{\frac{\alpha}{2} u_k L_k} \\ &\leq e^{-\widetilde{b}L e^{-c_{3,28}(\log \widetilde{b}L)^{\frac{1}{2}}}}, \end{aligned} \tag{2.3.110}$$

for some constant $c_{3,28} > 0$ and L large enough, and where we have chosen $u_0 = \dfrac{3(d-1)}{\alpha L_0 \log \frac{1}{\xi}}$. ∎

2.4 An atypical quenched exit estimate

Here we will prove a crucial atypical quenched exit estimate for tilted boxes, which will subsequently enable us in section 2.5 to show that the regeneration times of the random walk are integrable. Let us first introduce some basic notation.

Without loss of generality, we will assume that e_1 is contained in the open half-space defined by the asymptotic direction so that

$$\hat{v} \cdot e_1 > 0.$$

Recall the definition of the hyperplane perpendicular to direction e_1 in (2.3.31) so that

$$H := \{x \in \mathbb{R}^d : x \cdot e_1 = 0\}.$$

Let $P := P_{\hat{v}}$ (see (2.3.32)) be the projection on the asymptotic direction along the hyperplane H defined for $z \in \mathbb{Z}^d$

$$Pz := \left(\frac{z \cdot e_1}{\hat{v} \cdot e_1}\right) \hat{v},$$

and $Q := Q_l$ (see (2.3.33)) be the projection of z on H along $\hat{v}$ so that

$$Qz := z - Pz.$$

Now, for $x \in \mathbb{Z}^d$, $\beta > 0$, $\varrho > 0$ and $L > 0$, define the *tilted boxes with respect to the asymptotic direction* $\hat{v}$ as

$$B_{\beta,L}(x) := \left\{y \in \mathbb{Z}^d : -L^\beta < (y-x) \cdot e_1 < L;\ \|Q(y-x)\|_\infty < \varrho L^\beta\right\}. \tag{2.4.1}$$

and their *front boundary* by

$$\partial^+ B_{\beta,L}(x) := \{y \in \partial B_{\beta,L}(x) : y \cdot e_1 - x \cdot e_1 = L\}.$$

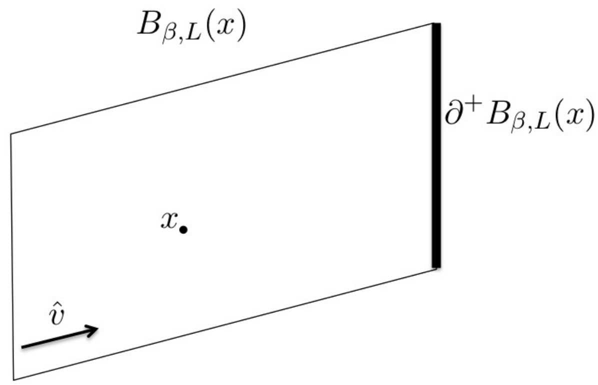

Figure 2.4: The box $B_{\beta,L}(x)$.

See Figure 2.4 for a picture of the box $B_{\beta,L}$ and its front boundary.

Proposition 2.4.1. *Let $\alpha > 0$ and assume that $\eta_\alpha < \infty$ as defined in (2.3.3). Let $M \geq 15d + 5$ and assume that $(P)_M|l$ is satisfied. Let $\beta_0 \in (1/2, 1)$, $\beta \in \left(\frac{\beta_0+1}{2}, 1\right)$ and $\zeta \in (0, \beta_0)$. Then, for each $\kappa > 0$ we have that*

$$\limsup_{L\to\infty} L^{-g(\beta_0,\beta,\zeta)} \log \mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \leq e^{-\kappa L^\beta}\right) < 0,$$

where

$$g(\beta_0, \beta, \zeta) := \min\{\beta + \zeta, 3\beta - 2 + (d-1)(\beta - \beta_0)\}. \tag{2.4.2}$$

We will now prove Proposition 2.4.1 following similar ideas to those presented by Sznitman in [Sz02].

2.4.1 Preliminaries

Firstly we need to define an appropriate mesoscopic scale to perform a renormalization analysis. Let $\beta_0 \in (0.5, 1)$, $\beta \in (\beta_0, 1)$ and $\chi := \beta_0 + 1 - \beta \in (\beta_0, 1]$. Define

$$L_0 := \frac{L - \varrho L^{\beta_0}}{[L^{1-\chi}]}.$$

Now, for each $x \in \mathbb{R}^d$ we consider the *mesoscopic box*

$$\tilde{B}(x) := \left\{ y \in \mathbb{Z}^d : -L^{\beta_0} < (y-x)\cdot e_1 < \varrho L_0;\ \|y - x - P(y-x)\|_\infty < (1+\varrho)L^{\beta_0} \right\},$$

and its *central part*

$$\tilde{C}(x) := \left\{ y \in \mathbb{Z}^d : 0 \le (y-x)\cdot e_1 < \varrho L_0;\ \|y - x - P(y-x)\|_\infty < L^{\beta_0} \right\}.$$

Define also

$$\partial^+ \tilde{B}(x) := \{ y \in \partial \tilde{B}(x) : y\cdot e_1 - x \cdot e_1 = \varrho L_0 \}$$

and

$$\partial^+ \tilde{C}(x) := \{ y \in \partial \tilde{C}(x) : y\cdot e_1 - x \cdot e_1 = \varrho L_0 \}.$$

We now say that a box $\tilde{B}(x)$ is *good* if

$$\sup_{x \in \tilde{C}(x)} P_{x,\omega}\left(X_{T_{\tilde{B}(x)}} \notin \partial_+ \tilde{B}(x) \right) < \frac{1}{2},$$

Otherwise the box is called *bad*. At this point, by Theorem 2.1.1 proved in section 2.3, we have the following version of Theorem 2.2.3 (Theorem A.2 of Sznitman [Sz02]).

Theorem 2.4.1. *Let $l \in \mathbb{S}^{d-1}$ and $M \ge 15d + 5$. Consider an elliptic RWRE satisfying condition $(P)_M|l$. Then, for any $c > 0$ and $\rho \in (0.5, 1)$,*

$$\limsup_{u\to\infty} u^{-(2\rho-1)} \log P_0 \left(\sup_{0 \le n \le T_u^{e_1}} |X_n - P(X_n)| \ge c u^\rho \right) < 0,$$

where $T_u^{e_1}$ is defined in (2.2.2).

The following lemma is an important corollary of Theorem 2.4.1.

Lemma 2.4.1. *Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d+5$. Consider an elliptic RWRE satisfying condition $(P)_M|l$. Then*

$$\limsup_{L\to\infty} L^{-(\beta+\beta_0-1)} \log \mathbb{P}(\tilde{B}(0) \text{ is bad}) < 0. \tag{2.4.3}$$

Proof. By Chebyschev's inequality we have that

$$\mathbb{P}(\tilde{B}(0) \text{ is bad})$$
$$\leq 2^{d-1} L_0 L^{\beta_0(d-1)} \left(P_0 \left(\sup_{0\leq n \leq T^{\hat{v}}_{\varrho L_0}} |X_n - PX_n| \geq (1+\varrho)L^{\beta_0} \right) + P_0\left(\tilde{T}^{\hat{v}}_{-L^{\beta_0}} < \infty \right) \right).$$

By Theorem 2.4.1, the first summand can be estimated as

$$\limsup_{L\to\infty} L^{-(\beta+\beta_0-1)} \log P_0 \left(\sup_{0\leq n\leq T^{\hat{v}}_{\varrho L_0}} |X_n - PX_n| \geq (1+\varrho)L^{\beta_0} \right) < 0.$$

To estimate the second summand, since $(P)_M|l$ is satisfied, by Theorem 2.1.1 and the equivalence given by Theorem 2.2.4, we can chose γ close enough to 1 so that $\gamma\beta_0 \geq \beta_0 + \beta - 1$ and such that

$$\limsup_{L\to\infty} L^{-\gamma\beta_0} \log P_0 \left(\tilde{T}^{\hat{v}}_{-(1+\varrho)L^{\beta_0}} < \infty \right) < 0.$$

∎

Let $k_1, \ldots, k_d \in \mathbb{Z}$. From now on, we will use the notation $x = (k_1, \ldots, k_d) \in \mathbb{R}^d$ to denote the point

$$x = k_1 \frac{\varrho}{\hat{v}\cdot e_1} L_0 \hat{v} + \sum_{j=2}^{d} 2k_j(1+\varrho)L^{\beta_0} e_j.$$

Define the following set of points which will correspond to the centers of mesoscopic boxes.

$$\mathcal{L} := \left\{ x \in \mathbb{R}^d : x = (k_1, \ldots, k_d) \text{ for some } k_1, \ldots, k_d \in \mathbb{Z} \right\}.$$

We will use subsequently the following property of the lattice $\mathcal{L}$: there exist 2^d disjoint sub-lattices $\mathcal{L}_1, \ldots, \mathcal{L}_{2^d}$ such that $\mathcal{L} = \cup_{i=1}^{2^d} \mathcal{L}_i$ and for each $1 \leq i \leq 2^d$, the sub-lattice $\mathcal{L}_i$ corresponds to the centers of mesoscopic boxes which are pairwise disjoint. Let $\mathcal{L}_0$ be the set defined by

$$\mathcal{L}_0 := \{x = (k_1, \ldots, k_d) \in \mathcal{L} : k_1 = 0\}.$$

For each $x \in \mathcal{L}_0$ we define the *column* of mesoscopic boxes as

$$C_x := \bigcup_{k_1=-1}^{[L^{1-\chi}]} \tilde{B}\left(x + k_1 \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right)$$

See Figure 2.5 for a picture of the column C_x, for some $x \in \mathcal{L}_0$.

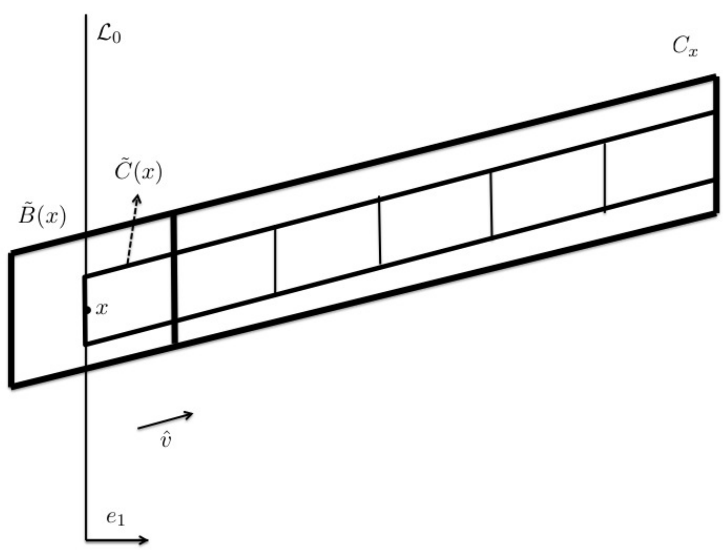

Figure 2.5: A box $\tilde{B}$ with its corresponding middle part $\tilde{C}$, which belongs to the column C_x.

The collection of these columns will be denoted by $\mathcal{C}$. Define now for each $C_x \in \mathcal{C}$ and $-1 \leq k \leq [L^{1-\chi}]$ define

$$\partial_{k,1} C_x := \partial_+ \tilde{C}\left(x + k \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right) \quad \text{and} \quad \partial_{k,2} C_x := \partial_+ \tilde{B}\left(x + k \frac{\varrho}{\hat{v} \cdot e_1} L_0 \hat{v}\right) \setminus \partial_{k,1} C_x.$$

For each point $y \in \partial_{k,1} C_x$ we assign a path $\pi^{(k)} = \{\pi_1^{(k)}, \ldots, \pi_{n_1}^{(k)}\}$ with $n_1 := \left[2c_1 \frac{\varrho}{\hat{v} \cdot e_1} L_0\right]$ steps from y to $\partial_{k+1,1} C_x$, so that $\pi_1^{(k)} = y$ and $\pi_{n_1}^{(k)} \in \partial_{k+1,1} C_x$. For each point $z \in \partial_{k,2} C_x$ we assign a path $\bar{\pi}^{(k)} = \{\bar{\pi}_1^{(k)}, \ldots, \bar{\pi}_{n_2}^{(k)}\}$ with $n_2 := \left[2c_1 \varrho L^{\beta_0}\right]$ steps from z to $\partial_{k,1} C_x$, so that $\bar{\pi}_1^{(k)} = z$ and $\bar{\pi}_{n_2}^{(k)} \in \partial_{k,1} C_x$. We will also use the notation $\{m_1, \ldots, m_N\}$ to denote some subset of $\{-1, \ldots, [L^{1-\chi}]\}$ with N elements.

Let $x \in \mathcal{L}_0$ and $\xi > 0$. A column of boxes $\mathcal{C}_x \in \mathcal{C}$ will be called *elliptically good* if it satisfies the following two conditions

$$\sup_{N \leq \left[\frac{L^\beta}{L_0}\right]} \sup_{\{m_1, \ldots, m_N\}} \sum_{j=1}^{N} \sup_{y_{m_j} \in \partial_{m_j,1} C_x} \sum_{i=1}^{n_1} \log \frac{1}{\omega\left(\pi_i^{(m_j)}, \Delta \pi_i^{(m_j)}\right)} \leq 2c_1 \frac{\varrho}{\hat{v} \cdot e_1} \log \left(\frac{1}{\xi}\right) L^\beta \quad (2.4.4)$$

and

$$\sum_{k=-1}^{[L^{1-\chi}]} \sup_{z_k \in \partial_{k,2} C_x} \sum_{i=1}^{n_2} \log \frac{1}{\omega\left(\bar{\pi}_i^{(k)} \Delta \bar{\pi}_i^{(k)}\right)} \leq 2c_1 \varrho \log \left(\frac{1}{\xi}\right) L^\beta. \quad (2.4.5)$$

If neither (2.4.4) nor (2.4.5) is satisfied, we will say that the column C_x is *elliptically bad.*

Lemma 2.4.2. *For any* $x \in \mathcal{L}_0$, $\beta \geq \frac{\beta_0 + 1}{2}$ *and* $\xi > 0$ *such that* $\log \frac{1}{\xi^{2\alpha} \eta_\alpha^3} > 0$ *we have that*

$$\limsup_{L \to \infty} L^{-\beta} \log \mathbb{P}\left(C_x \textit{ is elliptically bad}\right) < 0 \quad (2.4.6)$$

Proof. Let us first note that $\frac{L^\beta}{L_0} \geq 1$ by our condition on β. Now, it is clear that

$$\mathbb{P}\left(C_x \text{ is elliptically bad}\right) \leq \mathbb{P}\left((2.4.4) \text{ is not satisfied}\right) + \mathbb{P}\left((2.4.5) \text{ is not satisfied}\right) \quad (2.4.7)$$

Regarding the first term on the right of (2.4.7) and since $2\beta - \beta_0 - 1 < \beta - \beta_0 < \beta$ we have that

$$\mathbb{P}\left((2.4.4)\,\text{is not satisfied}\right) \leq \sum_{N=1}^{[\frac{L^\beta}{L_0}]} \mathbb{P}\Bigg(\exists\, \{m_1,\ldots,m_N\} \text{ and } y_{m_j} \in \partial_{m_j,1} C_x \text{ such that}$$

$$\sum_{j=1}^{N}\sum_{i=1}^{n_1} \log \frac{1}{\omega\left(\pi_i^{(m_j)}, \Delta\pi_i^{(m_j)}\right)} > 2c_1 \frac{\varrho}{\hat{v}\cdot e_1} \log\left(\frac{1}{\xi}\right) L^\beta\Bigg)$$

$$\leq \frac{L^\beta}{L_0} L^{(\beta-\beta_0)\frac{L^\beta}{L_0}} e^{(\log L)\beta_0 (d-1) L^{\beta-\beta_0}} e^{2(\log \eta_\alpha) c_1 \frac{\varrho}{\hat{v}\cdot e_1} L^\beta} e^{-2c_1 \frac{\varrho}{\hat{v}\cdot e_1}\left(\alpha \log \frac{1}{\xi}\right) L^\beta} \leq e^{-c_{4,1} L^\beta} \quad (2.4.8)$$

for some constant $c_{4,1} > 0$ if L is large enough and $\log \dfrac{1}{\xi^{2\alpha}\eta_\alpha^3} > 0$.

Similarly for the rightmost term of (2.4.7) we have that,

$$\mathbb{P}\left((2.4.5)\,\text{is not satisfied}\right)$$

$$\leq \mathbb{P}\left(\exists\, z_k \in \partial_{k,2} C_x \text{ such that} \sum_{k=-1}^{[L^{1-\chi}]}\sum_{i=1}^{n_2} \log \frac{1}{\omega(\bar{\pi}_i^{(k)} \Delta \bar{\pi}_i^{(k)})} > 2c_1 \varrho \log\left(\frac{1}{\xi}\right) L^\beta\right)$$

$$\leq e^{\log L(\varrho\beta_0(d-1)L^{1-\chi})} e^{2(\log\eta_\alpha)c_1\varrho L^\beta} e^{-2c_1\varrho\left(\alpha\log\frac{1}{\xi}\right)L^\beta} \leq e^{-c_{4,2}L^\beta} \quad (2.4.9)$$

for some constant $c_{4,2} > 0$ if L is large enough and $\log \dfrac{1}{\xi^{2\alpha}\eta_\alpha^3} > 0$. Substituting (2.4.8) and (2.4.9) back into (2.4.7), (2.4.6) follows. ∎

The proof Proposition 2.4.1 will reduced to the control of the probability of the three events: the first one, corresponding to subsection 2.4.2, gives a control on the number of bad boxes; the second one, corresponding to subsection 2.4.3, gives a control on the number of elliptically good columns; the third one, corresponding to subsection 2.4.4, gives a control on the probability that the random walk can find an appropriate path which leads to an elliptically good column.

2.4.2 Control on the number of bad boxes

We will need to consider only the mesoscopic boxes which intersect the box $B_{\beta,L}(0)$ and whose k_1 index is larger than or equal to -1. We hence define the collection of mesoscopic boxes

$$\mathcal{B} := \left\{\tilde{B}(x) : \tilde{B}(x) \cap B_{\beta,L}(0) \neq \emptyset, x = (k_1,\ldots,k_d),\ k_1,\ldots,k_d \in \mathbb{Z},\ k_1 \geq -1\right\}$$

In addition, we call the number of bad mesoscopic boxes in $\mathcal{B}$,

$$N(L) := \left|\left\{\tilde{B} \in \mathcal{B} : \tilde{B} \text{ is bad}\right\}\right|,$$

and for each $1 \le i \le 2^d$, call the number of bad mesoscopic boxes in $\mathcal{B}$ with centers in the sub-lattice $\mathcal{L}_i$ as

$$N_i(L) := \left|\left\{\tilde{B}(x) \in \mathcal{B} : \tilde{B}(x) \text{ is bad and } x \in \mathcal{L}_i\right\}\right|,$$

Define

$$G_1 := \left\{\omega \in \Omega : N(L) \le \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)} L^\beta}{2(1+\varrho)^{d-1} L_0}\right\}. \tag{2.4.10}$$

Lemma 2.4.3. *Assume that* $\beta > \dfrac{\beta_0+1}{2}$. *Then, there is a constant* $c_{4,3} > 0$ *such that for every* $L > 1$ *we have that*

$$\mathbb{P}(G_1^c) \le e^{-c_{4,3} L^{3\beta-2+(d-1)(\beta-\beta_0)}}.$$

Proof. Note that the number of columns intersecting the box $B_{\beta,L}(0)$ is equal to

$$\left\lceil \frac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}} \right\rceil.$$

Hence, whenever $\omega \in G_1$, necessarily there exist at least $\left\lceil \dfrac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}} \right\rceil$ columns each one with at most $\left\lceil \dfrac{L^\beta}{L_0} \right\rceil$ bad boxes. Let us take $m_1 := \left\lceil \dfrac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)} L^\beta}{2(1+\varrho)^{d-1} L_0} \right\rceil$ and $m_2 := |\mathcal{B}| = \left\lceil \dfrac{\varrho^{d-2} L^{d(\beta-\beta_0)}}{(1+\varrho)^{d-1}} + \dfrac{\varrho^{d-1} L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}} \right\rceil$. Now, using the fact that the mesoscopic boxes in each sub-lattice $\mathcal{L}_i$, $1 \le i \le 2^d$, are disjoint, and the estimate (2.4.3) of Lemma 2.4.1, we have by independence that there exists a constant $c_{4,3} > 0$ such that for every $L \ge 1$,

$$\mathbb{P}(G_1^c) = \mathbb{P}\left(N(L) \geq m_1\right) \leq \sum_{i=1}^{2^d} \mathbb{P}\left(N_i(L) \geq \frac{m_1}{2^d}\right)$$

$$\leq \sum_{i=1}^{2^d} \sum_{n=m_1/2^d}^{m_2} \binom{m_2}{n} \mathbb{P}\left(\tilde{B}(0) \text{ is bad}\right)^n \leq 2^d \sum_{n=m_1/2^d}^{m_2} m_2^n e^{-nL^{\beta+\beta_0-1}}$$

$$\leq e^{-c_{4,3}L^{\beta+\beta_0-1+(d-1)(\beta-\beta_0)+2\beta-\beta_0-1}} \leq e^{-c_{4,3}L^{3\beta-2+(d-1)(\beta-\beta_0)}}. \tag{2.4.11}$$

Note that in the second to last inequality we have used the fact that $2\beta + \beta_0 - 2 > 0$ which is equivalent to the condition $\beta > \frac{2-\beta_0}{2}$. Now, this last condition is implied by the requirement $\beta > \frac{\beta_0+1}{2}$.

∎

2.4.3 Control on the number of elliptically bad columns

Let $m_3 := \left[\frac{\varrho^{d-1}L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}}\right]$ and define the event that any sub-collection of the set of columns of cardinality less than or equal to m_3 has at least one elliptically good column

$$G_2 := \{\omega \in \Omega : \forall \mathcal{D} \subset \mathcal{C},\ |\mathcal{D}| \geq m_3,\ \exists C_x \in \mathcal{D} \text{ such that } C_x \text{ is elliptically good}\}. \tag{2.4.12}$$

Here we will prove the following lemma.

Lemma 2.4.4. *There is a constant $c_{4,4} > 0$ such that for every $L \geq 1$,*

$$\mathbb{P}(G_2^c) \leq e^{-c_{4,4}L^{\beta+(d-1)(\beta-\beta_0)}}. \tag{2.4.13}$$

Proof. Note that the total number of columns intersecting the box $B_{\beta,L}$ is equal to

$$m_4 := \left[\frac{\varrho^{d-1}L^{(d-1)(\beta-\beta_0)}}{(1+\varrho)^{d-1}}\right].$$

Using the fact that the events $\{C_x \text{ is elliptically bad}\}$, $\{C_y \text{ is elliptically bad}\}$ are independent if $x \neq y$, since these columns are disjoint, we conclude that there is a constant $c_{4,4} > 0$ such that for all $L \geq 1$,

$$\mathbb{P}(G_2^c) = \mathbb{P}\left(\exists\, \mathcal{D} \subset \mathcal{C},\ |\mathcal{D}| \geq m_3 \text{ such that } \forall C_x \in \mathcal{D},\ C_x \text{ is elliptically bad}\right)$$
$$\leq \sum_{n=m_3}^{m_4} m_4^n \mathbb{P}\left(C_x \text{ is elliptically bad}\right)^n \leq e^{-c_{4,4}L^{\beta+(d-1)(\beta-\beta_0)}},$$

where in the last inequality we have used the estimate (2.4.6) of Lemma 2.4.2 which provides a bound for the probability of a column to be elliptically bad.

∎

2.4.4 The confinement event

Here we will obtain an adequate estimate for the probability that the random walk hits an elliptically good column. We will need to introduce some notation, corresponding to the the box where the random walk will move before hitting the elliptically good column and a certain class of hyperplanes of this region. Let first $\zeta \in (0, \beta_0)$, a parameter which gives the order of width of the box $\bar{B}_{\zeta,\beta,L}$ where the random walk will be able to find a reasonable path to the elliptically good column, so that

$$\bar{B}_{\zeta,\beta,L} := \{x \in \mathbb{Z}^d : -L^\zeta \leq x \cdot e_1 \leq L^\zeta, \|x - Px\|_\infty < L^\beta\}.$$

Note that this box is contained in $B_{\beta,L}(0)$ and that it also contains the starting point 0 of the random walk. Define now for each $0 \leq z \leq L^\zeta$, the hyperplane

$$H_z := \left\{x \in \bar{B}_{\zeta,\beta,L} : x \cdot e_1 = z\right\},$$

and consider the two collection of hyperplanes defined as

$$\mathcal{H}^+ = \left\{H_z : z \in \mathbb{Z}, 0 \leq z \leq L^\zeta\right\} \quad \text{and} \quad \mathcal{H}^- = \left\{H_z : z \in \mathbb{Z}, -L^\zeta \leq z < 0\right\}.$$

Whenever there is no risk of confusion, we will drop the subscript from H_z writing H instead. Let $r := [2\varrho L^\beta]$. Now, for each $H \in \mathcal{H}^+ \cup \mathcal{H}^-$ and each j such that $e_j \neq \pm e_1$, we will consider the set of paths Π_j with r steps defined by $\pi = \{\pi_1, \ldots, \pi_r\} \in \Pi_j$ if and only if

$$\pi \subset H \quad \text{and} \quad \pi_{i+1} - \pi_i = e_j.$$

In other words, π is contained in the hyperplane H and it has steps which move only in the direction e_j. We now say that an hyperplane $H \in \mathcal{H}^+ \cap \mathcal{H}^-$ is *elliptically good* if for all paths $\pi \in \cup_{j \neq 1, d+1} \Pi_j$ one has that

$$\sum_{i=1}^{r} \log \frac{1}{\omega(\pi_i, \Delta\pi_i)} \leq 2\varrho \log\left(\frac{1}{\xi}\right) L^{\beta}. \tag{2.4.14}$$

Otherwise H will be called *elliptically bad* (See Figure 2.6).

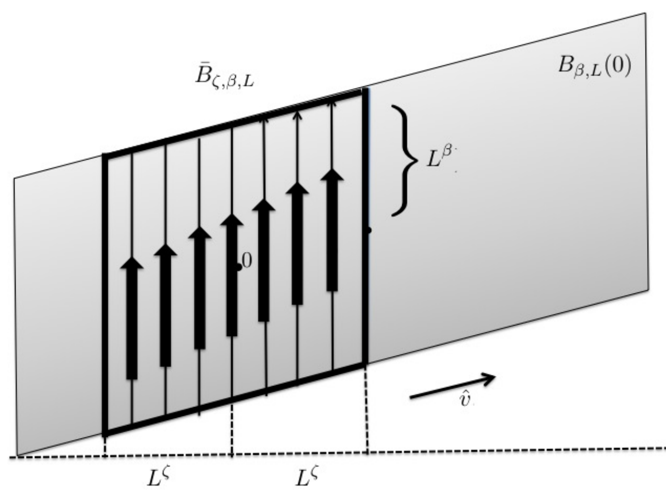

Figure 2.6: The box $\bar{B}_{\zeta,\beta,L}$. The arrows indicate the uniform ellipticity condition given by (2.4.14), which implies that each hyperplane is elliptically good.

From a routine counting argument and applying Chebyshev inequality, note that for each $H \in \mathcal{H}^+ \cup \mathcal{H}^-$ and $\xi > 0$ such that $\log \frac{1}{\xi^{\alpha}\eta_{\alpha}^2} > 0$ there is a constant $c_{4,5} > 0$ such that

$$\mathbb{P}\left(H \text{ is elliptically bad}\right) \leq e^{-c_{4,5} L^{\beta}}. \tag{2.4.15}$$

Now choose a rotation $\hat{R}$ such that $\hat{R}(e_1) = \hat{v}$. Let $\hat{v}_j := \hat{R}(e_j)$ for $j \geq 2$. We now want to make a construction analogous to the one which led to the concept of elliptically good hyperplane. But now, we would need to define hyperplanes perpendicular to the directions $\{\hat{v}_j\}$ which are not necessarily equal to a canonical vector. Therefore, we will work here with strips, instead of hyperplanes. For each $z \in \mathbb{Z}$ even and $k \in \{2, \ldots, d\}$ consider the strip $I_{k,z} := \{x \in \bar{B}_{\zeta,\beta,L}(0) : z-1 < x \cdot \hat{v}_j < z+1\}$. Consider also the two sets of strips, $\mathcal{I}_k^+$ and $\mathcal{I}_k^-$ defined by

$$\mathcal{I}_k^+ := \left\{ I_{k,z} : z \text{ even}, 0 \leq z \leq \varrho L^\beta \right\} \quad \text{and} \quad \mathcal{I}_k^- := \left\{ I_{k,z} : z \text{ even}, -\varrho L^\beta \leq z < 0 \right\}.$$

Whenever there is no risk of confusion, we will drop the subscripts from a strip $I_{k,z}$ writing I instead. We will need to work with the set of canonical directions which are contained in the closed positive half-space defined by the asymptotic direction, so that

$$U^+ := \{e \in U : e \cdot \hat{v} \geq 0\}.$$

Let $s := \left[2c_1 \frac{L^\zeta}{\hat{v} \cdot e_1} \right]$. For each $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ and each $y \in I$ we associate a path $\hat{\pi} = \{\hat{\pi}_1, \ldots, \hat{\pi}_n\}$, with $s \leq n \leq s+1$, which satisfies

$$\hat{\pi} \subset I_{j,z}$$

and

$$\hat{\pi}_{i+1} - \hat{\pi}_i \in U^+ \quad \text{for } 1 \leq i \leq n-1, \quad \hat{\pi}_n \in H_{[L^\zeta]}.$$

Note that by the fact that the strip I has a Euclidean width 1, it is indeed possible to find a path satisfying these conditions and also that such a path is not necessarily unique. We will call $\hat{\Pi}_k$ such a set of paths associated to all the points of the strip I. Now, a strip $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ will be called *elliptically good* if for all paths $\hat{\pi} \in \hat{\Pi}_k$ one has that

$$\sum_{i=1}^{n} \log \frac{1}{\omega\left(\hat{\pi}_i, \Delta\hat{\pi}_i\right)} \leq \log\left(\frac{1}{\xi}\right) n \tag{2.4.16}$$

Otherwise I will be called *elliptically bad* (See Figure 2.7).

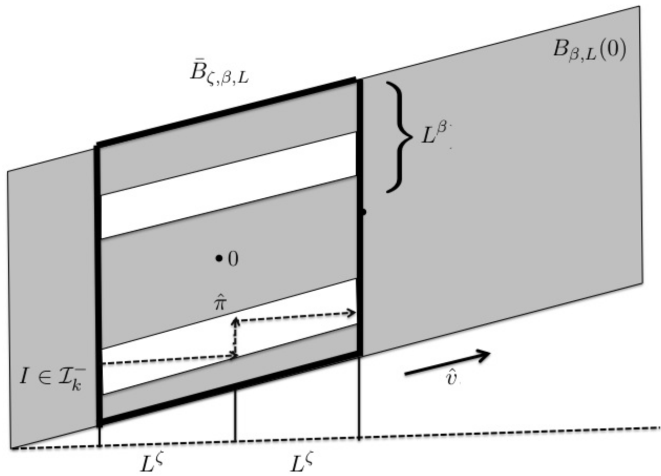

Figure 2.7: In each strip $\mathcal{I}$, every path π chosen previously satisfies the uniform ellipticity condition given by (2.4.16). Then $\mathcal{I}$ is elliptically good.

As before, from a routine counting argument and by Chebyshev inequality, note that for each $k \in \{2, \ldots, d\}$, $I \in \mathcal{I}_k^+ \cup \mathcal{I}_k^-$ and $\xi > 0$ which satisfies $\log \frac{1}{\xi^\alpha \eta_\alpha^2} > 0$, there exists a constant $c_{4,6} > 0$ such that

$$\mathbb{P}\left(I \,\text{is elliptically bad}\right) \leq e^{-c_{4,6} L^{\zeta}}. \tag{2.4.17}$$

We now define the *confinement event* as

$$G_3 := \{\omega \in \Omega : \exists\, H_+ \in \mathcal{H}_+, H_- \in \mathcal{H}_-, I_{+,2} \in \mathcal{I}_2^+, \ldots, I_{+,d} \in \mathcal{I}_d^+, I_{-,2} \in \mathcal{I}_2^-, \ldots, I_{-,d} \in \mathcal{I}_d^-$$
$$\text{such that } H_+, H_-, I_{+,2}, \ldots, I_{+,d}, I_{-,2}, \ldots, I_{-,d} \text{ are elliptically good}\}. \tag{2.4.18}$$

We can now state the following lemma which will eventually give a control on the probability that the random walk hits an elliptically good column.

Lemma 2.4.5. *There is a constant $c_{4,7} > 0$ such that for every $L \geq 1$,*

$$\mathbb{P}\left(G_3^c\right) \leq e^{-c_{4,7} L^{\beta+\zeta}}. \tag{2.4.19}$$

Proof. Note that

$$\mathbb{P}\left(G_3^c\right) \le \mathbb{P}\left(\bigcap_{H\in\mathcal{H}^+}\{H \text{ is elliptically bad}\}\right) + \mathbb{P}\left(\bigcap_{H\in\mathcal{H}^-}\{H \text{ is elliptically bad}\}\right)$$
$$+\sum_{k=2}^{d}\mathbb{P}\left(\bigcap_{I\in\mathcal{I}_k^+}\{I \text{ is elliptically bad}\}\right) + \sum_{k=2}^{d}\mathbb{P}\left(\bigcap_{I\in\mathcal{I}_k^-}\{I \text{ is elliptically bad}\}\right),$$

Now, inequality (2.4.19) follows using the estimate (2.4.15) for the probability that a hyperplane is elliptically bad, the estimate (2.4.17) for the probability that a strip is elliptically bad, applying independence and translation invariance.

∎

2.4.5 Proof of Proposition 2.4.1

Firstly, note that for any $\kappa > 0$,

$$\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \le e^{-\kappa L^\beta}\right) \le \mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c)$$
$$+\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) \le e^{-\kappa L^\beta}, G_1, G_2, G_3\right). \tag{2.4.20}$$

Let us begin bounding the first three terms of the right-hand side of (2.4.20). Let $\zeta \in (0,\beta_0)$ and $\beta > \frac{\beta_0+1}{2}$. By Lemma 2.4.3 of subsection 2.4.2, Lemma 2.4.4 of subsection 2.4.3 and Lemma 2.4.5 of subsection 2.4.4 we have that there is a constant $c_{4,8} > 0$ such that

$$\mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c) \le \frac{1}{c_{4,8}} e^{-c_{4,8}L^{3\beta-2+(d-1)(\beta-\beta_0)}} + \frac{1}{c_{4,8}} e^{-c_{4,8}L^{\beta+(d-1)(\beta-\beta_0)}} + \frac{1}{c_{4,8}} e^{-c_{4,8}L^{\beta+\zeta}}. \tag{2.4.21}$$

Since $\beta < 1$ is equivalent to $\beta + (d-1)(\beta-\beta_0) > 3\beta - 2 + (d-1)(\beta-\beta_0)$, the sum in (2.4.21) can be bounded as

$$\mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) + \mathbb{P}(G_3^c) \le \frac{1}{c_{4,9}} e^{-c_{4,9}L^{g(\beta,\beta_0,\zeta)}}, \tag{2.4.22}$$

for some constant $c_{4,9} > 0$ and where $g(\beta,\beta_0,\zeta) := \min\{\beta+\zeta, 3\beta-2+(d-1)(\beta-\beta_0)\}$.

We will now prove that the fourth term of the right-hand side of inequality (2.4.20) satisfies for L large enough

$$\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}}\in\partial^+B_{\beta,L}(0)\right)\le e^{-\kappa L^\beta},G_1,G_2,G_3\right)=0. \tag{2.4.23}$$

In fact, we will show that for L large enough on the event $G_1\cap G_2\cap G_3$ one has that

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}}\in\partial^+B_{\beta,L}(0)\right)>e^{-\kappa L^\beta}. \tag{2.4.24}$$

We will prove (2.4.24) showing that the walk can exit $B_{\beta,L}(0)$ through $\partial^+B_{\beta,L}(0)$ choosing a strategy which corresponds to paths which go through an elliptically good column. This implies, in particular, that the walk exit successively of boxes $\tilde{B}(x)$ through $\partial^+\tilde{B}(x)$. The event G_1 implies that there exist at least $m_3=\left[\frac{\varrho^{d-1}L^{(d-1)(\beta-\beta_0)}}{2(1+\varrho)^{d-1}}\right]$ columns each one with at most $\left[\frac{L^\beta}{L_0}\right]$ of bad boxes. Meanwhile, the event G_2 asserts that in any collection of columns with cardinality m_3 or more, there is at least one elliptically good column. Therefore, on the event $G_1\cap G_2$ there exists at least one elliptically good column D with at most L^β/L_0 bad boxes. Thus, on $G_1\cap G_2$ we have that for any point $y\in D$ and $\xi>0$,

$$P_{y,\omega}\left(X_{T_{B_{\beta,L}(0)}}\in\partial^+B_{\beta,L}(0)\right)\ge\left(\frac{1}{2}\right)^{L^{\beta-\beta_0}+1}\xi^{2c_1\frac{\varrho}{\hat{v}\cdot e_1}L^\beta}\xi^{2c_1\varrho L^\beta}, \tag{2.4.25}$$

where the first factor is a bound for the probability that the random walk exits all the good boxes of the column through their front side, while the second factor is a bound for the probability that the walk traverses each bad box (whose number is at most L^β/L_0) exiting through its front side and following a path with at most $\frac{2c_1\rho L_0}{\hat{v}\cdot e_1}$ steps and is given by the condition (2.4.4) for elliptically good columns, while the third factor is a bound for the probability that once the walk exits a box (whose number is at most $L^{\beta-\beta_0}+1$) it moves through its front boundary to the central point of this front boundary following a path with at most $[2c_1\rho L_0^\beta]$ steps and is given by the condition (2.4.5) for elliptically good columns.

Now, the confinement event G_3 ensures that with a high enough probability the random walk will reach the elliptically good column D which has at most L^β/L_0 bad boxes. More precisely, a.s. on G_3, the random walk reaches either an elliptically good hyperplane $H\in\mathcal{H}_+\cup\mathcal{H}_-$, an elliptically good strip $I\in\mathcal{I}_2^+\cup\cdots\cup\mathcal{I}_d^+$ or an elliptically good strip $I\in\mathcal{I}_2^-\cup\cdots\cup\mathcal{I}_d^-$ (recall the definitions of elliptically good hyperplanes and strips given in

(2.4.14) and (2.4.16) of subsection 2.4.3). Now, once the walk reaches either an elliptically good hyperplane or strip, we know by (2.4.14) or (2.4.16), choosing an appropriate path that the probability that it hits the column D is at least $\xi^{c_{4,10}\varrho L^\beta}$ for some constant $c_{4,10} > 0$. Thus, we know that there is a constant $c_{4,10} > 0$ such that

$$P_{0,\omega}\left(\text{the walk reaches } D \cap \bar{B}_{\zeta,\beta,L}(0)\right) \geq \xi^{c_{4,10}\varrho L^\beta}. \tag{2.4.26}$$

Therefore, combining (2.4.25) and (2.4.26), we conclude that there is a constant $c_{4,11} > 0$ such that for all $\varrho \in (0,1)$ on the event $G_1 \cap G_2 \cap G_3$ the following estimate is satisfied,

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) > e^{-c_{4,11}\rho L^\beta}.$$

Hence, choosing ϱ sufficiently small, we have that on $G_1 \cap G_2 \cap G_3$,

$$P_{0,\omega}\left(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\right) > e^{-\kappa L^\beta} \tag{2.4.27}$$

for L larger than a deterministic constant depending only on ϱ. This proves (2.4.23).

Finally, with the help of (2.4.20), (2.4.22) and (2.4.27) the Proposition 2.4.1 is proved.

2.5 Moments of the regeneration time

Here we will prove Theorem 2.1.2. Our method is inspired on some ideas used by Sznitman to prove Proposition 3.1 of [Sz01], which give tail estimates on the distribution of the regeneration times. Parts (a) and (b) of Theorem 2.1.2 will follow from Theorem 2.2.1, while part (c) from Theorem 2.2.2.

Proposition 2.5.1. *Let $l \in \mathbb{S}^{d-1}$ and $M \geq 15d + 5$. Assume that $(P)_M|l$ holds. Then, the following are satisfied.*

a) For every $0 < \alpha < \min_e F_e$ one has that

$$\limsup_{u\to\infty} (\log u)^{-1} \log P_0[\tau_1^{\hat{v}} > u] \leq -2\alpha. \tag{2.5.1}$$

b) Let $\alpha > 0$. Assume $(E')_\alpha$. Then, for every $\alpha' < \alpha$ one has that

$$\limsup_{u\to\infty}(\log u)^{-1}\log P_0[\tau_1^{\hat{v}} > u] \le -\alpha'. \tag{2.5.2}$$

The proof of the above proposition is based on the atypical quenched exit estimate corresponding to Proposition 2.4.1 of section 2.4. Some slight modifications in the proof of Proposition 2.4.1, would lead to a version of it, which could be used to show that Proposition 2.5.1 remains valid if the regeneration time $\tau_1^{\hat{v}}$ is replaced by τ_1^l for any direction l such that $l\cdot\hat{v} > 0$. Note also that Proposition 2.5.1 implies that whenever $(E)_{1/2}$ or $(E')_1$ are satisfied, then the first regeneration time is integrable. Through Theorem 2.2.1, this implies part (a) of Theorem 2.1.2. Similarly we can conclude part (b) of Theorem 2.1.2. Part (c) of Theorem 2.1.2 can be derived analogously through Theorem 2.2.2.

Let us now proceed with the proof of Proposition 2.5.1. Let us take a rotation $\hat{R}$ in R^d such that $\hat{R}(e_1) = \hat{v}$ and fix $\beta \in \left(\frac{5}{6}, 1\right)$ and $M > 0$. For each $u > 0$ define the scale

$$L = L(u) := \left(\frac{1}{4M\sqrt{d}}\right)^{\frac{1}{\beta}} (\log u)^{\frac{1}{\beta}},$$

and the box

$$C_L := \left\{x \in \mathbb{Z}^d : \frac{-L}{2(\hat{v}\cdot e_1)} \le x\cdot\hat{R}(e_i) \le \frac{L}{2(\hat{v}\cdot e_1)}, \text{for } 0 \le i \le 2d\right\}.$$

Throughout the rest of this proof we will continue writing τ_1 instead of $\tau_1^{\hat{v}}$. Now note that

$$P_0(\tau_1 > u) \le P_0\left(\tau_1 > u, T_{C_{L(u)}} \le \tau_1\right) + P_0\left(T_{C_{L(u)}} > u\right), \tag{2.5.3}$$

where $T_{C_{L(u)}}$ is the first exit time from the set $C_{L(u)}$ defined in (2.1.1). For the second term of the right-hand side of inequality (2.5.3), we can use Corollary 2.2.1, to conclude that for every $\gamma \in (\beta, 1)$ there exists a constant $c_{5,1}$ such that

$$P_0\left(\tau_1 > u, T_{C_{L(u)}} \le \tau_1\right) \le \frac{1}{c_{5,1}} e^{-c_{5,1}L^\gamma(u)}. \tag{2.5.4}$$

For the first term of the right-hand side of inequality (2.5.3), following Sznitman [Sz01] we introduce the event

$$F_1 := \left\{ \omega \in \Omega : t_\omega \left(C_{L(u)} \right) > \frac{u}{(\log u)^{\frac{1}{\beta}}} \right\},$$

where for each $A \subset \mathbb{Z}^d$ we define

$$t_\omega(A) := \inf \left\{ n \geq 0 : \sup_x P_{x,\omega} \left(T_A > n \right) \leq \frac{1}{2} \right\}.$$

Trivially,

$$P_0 \left(T_{C_{L(u)}} > u \right) \leq \mathbb{E} \left[F_1^c, P_{0,\omega} \left(T_{C_{L(u)}} > u \right) \right] + \mathbb{P}(F_1). \tag{2.5.5}$$

To bound the first term of the right-hand side of (2.5.5), on the event F_1^c we apply the strong Markov property $[(\log u)^{\frac{1}{\beta}}]$ times to conclude that

$$\mathbb{E} \left[F_1^c, P_{0,\omega} \left(T_{C_{L(u)}} > u \right) \right] \leq \left(\frac{1}{2} \right)^{[(\log u)^{\frac{1}{\beta}}]}. \tag{2.5.6}$$

To bound the second term of the right-hand side of (2.5.5), we will use the fact that for each $\omega \in \Omega$ there exists $x_0 \in C_{L(u)}$ such that

$$P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2|C_{L(u)}|}{t_\omega(C_{L(u)})} \tag{2.5.7}$$

where for $y \in \mathbb{Z}^d$,

$$\tilde{H}_y = \inf\{n \geq 1 : X_n = y\}.$$

(2.5.7) can be derived using the fact that for every subset $A \subset \mathbb{Z}^d$ and $x \in A$,

$$E_{x,\omega}(T_A) = \sum_{y \in A} \frac{P_{x,\omega}(H_y < T_A)}{P_{y,\omega}(\tilde{H}_y > T_A)}$$

(see for example Lemma 1.3 of Sznitman [Sz01]). Now note that (2.5.7) implies

$$\mathbb{P}(F_1) \leq \mathbb{P} \left(\omega \in \Omega : \exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}| \right). \tag{2.5.8}$$

Choose for each $x \in C_{L(u)}$ a point y_x as any point in $\mathbb{Z}^d$ which is closest to the point $x + \left(\frac{\log u}{2M\sqrt{d}(\hat{v}\cdot e_1)}\right)\hat{v} = x + 2\left(\frac{L^\beta}{\hat{v}\cdot e_1}\right)\hat{v}$. It is straightforward to see that

$$N - 1 \leq |y_x - x|_1 \leq N + 1, \tag{2.5.9}$$

where

$$N := \frac{|\hat{v}|_1 \log u}{2M\sqrt{d}(\hat{v}\cdot e_1)}.$$

We can now find $2d$ different paths $\{\pi^{(i)} : 1 \leq i \leq 2d\}$, each one with n_i steps, with $\pi^{(i)} := \{\pi^{(i)}_1, \ldots, \pi^{(i)}_{n_i}\}$ for each $1 \leq i \leq 2d$ such that the following conditions are satisfied:

(a) Each path $\pi^{(i)}$ goes from x to y_x, so that $\pi^{(i)}_1 = x$ and $\pi^{(i)}_{n_i} = y_x$.

(b) Except for the initial and last points, the paths are pairwise disjoint so that

$$\pi^{(i)} \cap \pi^{(j)} = \{x, y_x\} \text{ for all } 1 \leq i < j \leq 2d.$$

(c) The number of steps n_i of each path is bounded by $N + 4$.

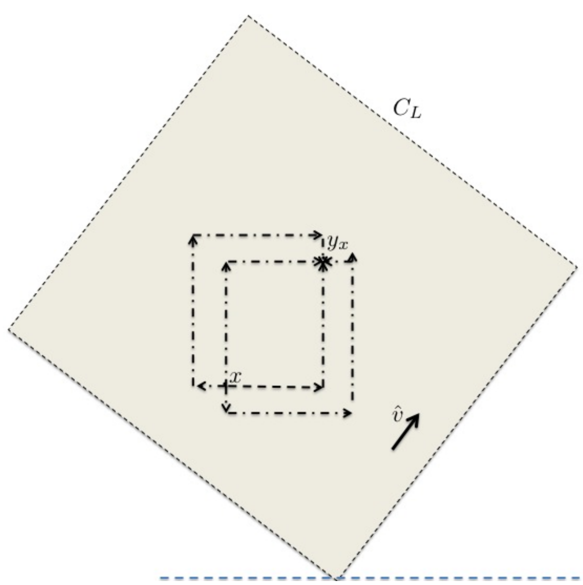

Figure 2.8: The $2d$ paths from x to y_x are represented by dashed lines.

In Figure 2.8 it is seen how can one construct such a set of paths for dimensions $d = 2$ (a similar construction works for dimensions $d \geq 3$). From Figure 2.8, note that the maximal number of steps of each path is given by $|y_x - x|_1 + 4$, where the 4 corresponds to the extra steps which have to be performed when a path exits the point x or enters the point y_x using a direction $e \in U$ such that $e \cdot \hat{v} < 0$. Let us now introduce the event

$$F_2 := \Big\{\omega \in \Omega : \text{for each } x \in C_{L(u)},\ \exists i \in \{1, \ldots, 2d\} \text{ such that} \\ \sum_{j=1}^{n_i} \log \frac{1}{\omega(\pi_j^{(i)}, \Delta\pi_j^{(i)})} \leq \frac{2(M-1)(\hat{v} \cdot e_1)\sqrt{d}}{|\hat{v}|_1} n_i \Big\}.$$

Then, with the help of (2.5.8) we have that

$$\mathbb{P}(F_1) \leq \mathbb{P}\left(\exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|, F_2\right) + \mathbb{P}(F_2^c). \tag{2.5.10}$$

Let us define

$$F_3 := \left\{\omega \in \Omega : \exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|, F_2\right\}.$$

Note that on the event F_3, which appear in the probability of the right-hand side of (2.5.10), we can use the definition of the event F_2 to join x_0 and y_{x_0} using one of the paths $\pi^{(i)}$ to conclude that

$$e^{-\frac{8(M-1)(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}} u^{-\left(1-\frac{1}{M}\right)} P_{y_{x_0},\omega}\left(T_{C_{L(u)}} < H_{x_0}\right) \leq P_{x_0,\omega}\left(T_{C_{L(u)}} < \tilde{H}_{x_0}\right) \leq \frac{2(\log u)^{\frac{1}{\beta}}}{u} |C_{L(u)}|.$$

In particular, on F_3 we can see that for u large enough $y_{x_0} \in C_{L(u)}$. As a result, on F_3 we have that for u large enough

$$P_{y_{x_0},\omega}\left(X_{T_{y_{x_0}+U_{\beta,L}}} \cdot e_1 > y_{x_0} \cdot e_1\right) \leq P_{y_{x_0},\omega}\left(T_{C_{L(u)}} < H_{x_0}\right) \leq \frac{1}{u^{\frac{1}{2M}}} = e^{-2\sqrt{d}L(u)^{\beta}},$$

where

$$U_{\beta,L} := \{x \in \mathbb{Z}^d : -L^\beta < x \cdot e_1 < L\}.$$

From this and using the translation invariance of the measure $\mathbb{P}$, we conclude that

$$\begin{aligned}\mathbb{P}&\left(\exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}[\tilde{H}_{x_0} > T_U] \le \frac{2(\log u)^{\frac{1}{\beta}}}{u}|C_{L(u)}|, F_2\right)\\ &\le |C_{L(u)}|\mathbb{P}\left(P_{0,\omega}\left(X_{T_{U_{\beta,L(u)}}} \cdot e_1 > 0\right) \le e^{-2\sqrt{d}L(u)^\beta}\right)\\ &\le |C_{L(u)}|\mathbb{P}\left(P_{0,\omega}\left(X_{T_{B_{\beta,L(u)}}} \cdot e_1 > 0\right) \le e^{-2\sqrt{d}L(u)^\beta}\right),\end{aligned}$$

where the titled box $B_{\beta,L}$ was defined in (2.4.1) of section 2.4. Therefore, we can estimate the first term of the right-hand side of (2.5.10) using Proposition 2.4.1 to conclude that there is a constant $c_{5,2} > 0$ such that for each $\beta_0 \in \left(\frac{1}{2}, 1\right)$ one has that

$$\mathbb{P}\left(\exists\, x_0 \in C_{L(u)} \text{ such that } P_{x_0,\omega}[\tilde{H}_{x_0} > T_U] \le \frac{2(\log u)^{\frac{1}{\beta}}}{u}|C_{L(u)}|, F_2\right) \le \frac{1}{c_{5,2}} e^{-c_{5,2}L(u)^{g(\beta_0,\beta,\zeta)}}, \tag{2.5.11}$$

where $g(\beta_0, \beta, \zeta)$ is defined in (2.4.2) of Proposition 2.4.1. On the event F_2^c, for each $x \in C_L$ we define $\omega^* := \max\{\omega(x, e_i) : i = 2, \dots, 2d\}$. Let i^* be the direction where this maximum is achieved. To be more precise, we write $\pi^* := \pi^{(i^*)}$. Thus, for instance, $\omega(\pi_1^*, \Delta\pi_1^*) = \omega^*(\pi_1, \Delta\pi_1)$. We will also define $n_* := n_{i^*}$. Now, applying Chebyshev inequality, we can say that for any $\alpha > 0$ one has that

$$\begin{aligned}\mathbb{P}(F_2^c) \le \mathbb{P}&\Bigg(\exists\, x \in C_{L(u)} \text{ such that } \log\frac{1}{\omega(\pi_1^1, \Delta\pi_1^1)} + \log\frac{1}{\omega(\pi_1^*, \Delta\pi_1^*)} +\\ &\sum_{j=2}^{n_1}\log\frac{1}{\omega(\pi_j^1, \Delta\pi_j^1)} + \sum_{j=2}^{n_*}\log\frac{1}{\omega(\pi_j^*, \Delta\pi_j^*)} > 2(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}(n_1 + n_*)\Bigg)\\ \le |C_{L(u)}|\mathbb{E}&\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)} + \alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}} e^{\alpha\sum_{j=2}^{n_1}\log\frac{1}{\omega(\pi_j^1,\Delta\pi_j^1)}} e^{\alpha\sum_{j=2}^{n_*}\log\frac{1}{\omega(\pi_j^*,\Delta\pi_j^*)}}\right] e^{-4n_1\alpha(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}},\end{aligned}$$

where we have used the inequality $n_j \ge n_1$ valid for every $1 \le j \le 2d$, which is a consequence of the fact that $e_1 \cdot \hat{v} > 0$. Now, by the construction of these paths, the exponentials that appear under the expectation are independent. Hence, with the help of the inequality $|C_{L(u)}| \le e^{(N+4)\log\eta_\alpha}$, valid for u large enough, we have that

$$\mathbb{P}\left(F_2^c\right) \leq \mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}\right] e^{3(N+4)\log\eta_\alpha} e^{-4n_1\alpha(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}}.$$

Meanwhile, the expectation in the above inequality can be estimated as follows

$$\begin{aligned}
\mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}\right] &= \mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}, \omega(\pi_1^1,\Delta\pi_1^1) > \omega(\pi_1^*,\Delta\pi_1^*)\right] \\
&+\mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}, \omega(\pi_1^1,\Delta\pi_1^1) \leq \omega(\pi_1^*,\Delta\pi_1^*)\right] \\
&\leq \mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}, \frac{1}{2d} \leq \omega(\pi_1^1,\Delta\pi_1^1)\right] \\
&+\mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}+\alpha\log\frac{1}{\omega(\pi_1^*,\Delta\pi_1^*)}}, \frac{1}{2d} \leq \omega(\pi_1^*,\Delta\pi_1^*)\right] \\
\leq (2d)^\alpha\mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^2,\Delta\pi_1^2)}}\right] &+ (2d)^\alpha\mathbb{E}\left[e^{\alpha\log\frac{1}{\omega(\pi_1^1,\Delta\pi_1^1)}}\right] \leq 2^{\alpha+1}d^\alpha\eta_\alpha < \infty,
\end{aligned}$$

where we have used in the last line the fact that $\omega(\pi_1^*,\Delta\pi_1^*) \geq \omega(\pi_1^2,\Delta\pi_1^2)$ by the definition of ω^*. As a result, using the inequality $N-1 \leq n_1$, we have that

$$\mathbb{P}\left(F_2^c\right) \leq 2^{\alpha+1}d^\alpha\eta_\alpha e^{4(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}} e^{3(N+4)\log\eta_\alpha} e^{-2\alpha\left(1-\frac{1}{M}\right)\log u}.$$

Using the definition of N, we see from here that for every $\alpha' < \alpha$, if we choose M such that

$$\alpha' < \alpha\left(1-\frac{1}{M}\right) - \frac{3|\hat{v}|_1\log\eta_\alpha}{4M\sqrt{d}(\hat{v}\cdot e_1)},$$

one has that for u large enough

$$\mathbb{P}(F_2^c) \leq c_{5,3}u^{-2\alpha'}. \tag{2.5.12}$$

for some constant $c_{5,3} > 0$. Now note that for each $\beta \in \left(\frac{5}{6}, 1\right)$ there exists a $\beta_0 \in \left(\frac{1}{2}, \beta\right)$ such that for every $\zeta \in \left(0, \frac{1}{2}\right)$ one has that

$$g(\beta,\beta_0,\zeta) > \beta. \tag{2.5.13}$$

Therefore, substituting (2.5.11) and (2.5.12) back into (2.5.10) and using (2.5.13) we can see that there is a constant $c_{5,4} > 0$ such that for u large enough

$$\mathbb{P}(F_1) \leq c_{5,4} u^{-2\alpha'}. \tag{2.5.14}$$

Now with the help of (2.5.5), (2.5.6) and (2.5.14) there exists a constant $c_{5,5} > 0$ such that for u large

$$P_0\left(T_{C_{L(u)}} > u\right) \leq c_{5,5} u^{-2\alpha'}. \tag{2.5.15}$$

Finally, since $\gamma \in (\beta, 1)$ in (2.5.4), using (2.5.3) we conclude the proof, since we see that for u large enough

$$P_0(\tau_1 > u) \leq c_{5,6} u^{-2\alpha'},$$

for a certain constant $c_{5,6} > 0$. This proves part (a) of Proposition 2.5.1 under the assumption that $(E)_\alpha$ is satisfied. Let us now take a $\bar{\alpha} = \{\alpha_e : e \in U\}$. Note that

$$\begin{aligned} &\mathbb{P}(F_2^c) = \\ &\mathbb{P}\left(\exists x \in C_{L(u)} \text{ such that } \forall\, i \ \log \frac{1}{\omega(\pi_1^{(i)}, \Delta\pi_1^{(i)})} + \sum_{j=2}^{n_j} \log \frac{1}{\omega(\pi_j^{(i)}, \Delta\pi_j^{(i)})} > 2(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1} n_j\right) \\ &\leq |C_{L(u)}| \mathbb{E}\left[e^{\sum_i \alpha(e_i) \log \frac{1}{\omega(0,e_i)}}\right] e^{-2(\sum_i n_i \alpha(e_i))(M-1)\frac{(\hat{v}\cdot e_1)\sqrt{d}}{|\hat{v}|_1}} \Pi_i \mathbb{E}\left[e^{\alpha(e_i)\sum_{j=2}^{n_i} \log \frac{1}{\omega\left(\pi_j^{(i)}, \Delta\pi_j^{(i)}\right)}}\right] \\ &\leq |C_{L(u)}| \eta_{\bar{\alpha}} e^{\sum_i n_i \log \eta_{\alpha(e_i)}} e^{-\sum_i \alpha(e_i)\left(1-\frac{1}{M}\right)\log u}, \end{aligned}$$

where $\eta_{\bar{\alpha}} := \mathbb{E}\left[e^{\sum_i \alpha(e_i) \log \frac{1}{\omega(0,e_i)}}\right]$. Choosing M large enough and following the argument leading to (2.5.14), we conclude that for every $\alpha' < \sum_i \alpha(e_i)$ one has that

$$\mathbb{P}(F_2^c) \leq c_{5,7} e^{-\alpha' \log u}$$

for some constant $c_{5,7} > 0$ and hence that

$$\mathbb{P}(F_1) \leq c_{5,8} e^{-\alpha' \log u},$$

for some constant $c_{5,8} > 0$. This proves part (b) of Proposition 2.5.1.

CHAPTER 3

LEVEL 1 QUENCHED LARGE DEVIATION PRINCIPLE FOR A RANDOM WALK IN MARKOVIAN TIME-DEPENDENT RANDOM ENVIRONMENT

3.1 Introduction

We consider uniformly elliptic random walks in time-space random environment both in continuous and discrete time. We present two alternative short proofs of the level 1 quenched large deviation principle under mild conditions on the environment, based on the use of the sub-additive ergodic theorem as presented by Varadhan in [Var03]. Previously, in the discrete time case, Rassoul-Agha, Seppäläinen and Yilmaz [RSY11], proved a level 2 and 3 large deviation principle, from which the level 1 principle can be derived via contraction.

Let $\kappa_2 > \kappa_1 > 0$. Denote by $G := \{e_1, e_{-1}, \ldots, e_d, e_{-d}\}$ the set of unit vectors in $\mathbb{Z}^d$. Define $\mathcal{Q} := \{v = \{v(e) : e \in G\} : \kappa_1 \leq \inf_{e\in G} v(e) \leq \sup_{e\in G} v(e) \leq \kappa_2\}$. Consider a continuous time Markov process $\omega := \{\omega_t : t \geq 0\}$ with state space $\Omega_c := \mathcal{Q}^{\mathbb{Z}^d}$, so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x, e) : e \in G\} \in \mathcal{Q}$. We call ω the *continuous time environmental process*. We assume that for each initial condition ω_0, the process ω defines a probability measure $Q^c_{\omega_0}$ on the Skorokhod space $D([0,\infty); \Omega_c)$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_c \to \mathbb{R}$ and $t \geq 0$ we have that

$$\int f(\omega_t) d\mu = \int f(\omega_0) d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q^c_\mu := \int Q^c_\omega d\mu$, where with a slight abuse of notation here $\omega \in \Omega_c$. For a given trajectory $\omega \in D([0,\infty);\Omega_c)$ consider the process $\{X_t : t \geq 0\}$ defined by the generator

$$L_s f(x) := \sum_{e\in G} \omega_s(x,e)(f(x+e) - f(x)),$$

where $s \geq 0$. We call this process a *continuous time random walk in a uniformly elliptic time-dependent random environment* and denote for each $x \in \mathbb{Z}^d$ by $P^c_{x,\omega}$ the law on $D([0,\infty);\mathbb{Z}^d)$ of this random walk with initial condition $X_0 = x$. We call $P^c_{x,\omega}$ the *quenched law* starting from x of the random walk.

For $x \in \mathbb{R}^d$, $|x|_2$, $|x|_1$ and $|x|_\infty$ denote respectively, their Euclidean, l_1 and l_∞-norm. Also, for $r > 0$, we define $B_r(x) := \{y \in \mathbb{Z}^d : |y - x|_2 \leq r\}$. Furthermore, given any topological space T, we will denote by $\mathcal{B}(T)$ the corresponding Borel sets.

We will also consider a discrete version of this model which we define as follows. Let $\kappa > 0$ and $R \subset \mathbb{Z}^d$ finite. Define $\mathcal{P} := \{v = \{v(e) : e \in R\} : \inf_{e\in R} v(e) \geq \kappa, \sum_{e\in R} v(e) = 1\}$. Consider a discrete time Markov process $\omega := \{\omega_n : n \geq 0\}$ with state space $\Omega_d := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x,e) : e \in R\} \in \mathcal{P}$. We call ω the *discrete time environmental process*. Let us denote by Q^d_ω the corresponding law of the process defined on the space $\Omega_d^{\mathbb{N}}$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_d \to \mathbb{R}$ and $n \geq 0$ we have that

$$\int f(\omega_n) d\mu = \int f(\omega_0) d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q^d_\mu := \int Q^d_\omega d\mu$. Given $\omega \in \Omega_d$ and $x \in \mathbb{Z}^d$, consider now the discrete time random walk $\{X_n : n \geq 0\}$ with a law $P^d_{x,\omega}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined through $P^d_{x,\omega}(X_0 = x) = 1$ and the transition probabilities

$$P^d_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega_n(x,e),$$

for $n \geq 0$ and $e \in R$. We call this process a *discrete time random walk in a uniformly elliptic time-space random environment with jump range* R and call $P^d_{x,\omega}$ the *quenched law* of the discrete time random walk starting from x. We will say that R *corresponds to the nearest neighbor case* if $R = \{e \in \mathbb{Z}^d : |e|_1 = 1\}$. We say that a subset $A \subset \mathbb{Z}^d$ is *convex* if there exists a convex subset $V \subset \mathbb{R}^d$ such that $A = V \cap \mathbb{Z}^d$, while we say that A is *symmetric* if $A = -A$. Throughout, we will assume that the jump range is R is finite, convex and symmetric or that it corresponds to the nearest neighbor case.

Throughout we will make the following ergodicity assumption. Note that we do not demand the environment to be necessarily ergodic under time shifts.

Assumption (EC). Consider the continuous time environmental process ω. For each $s > 0$ and $x \in \mathbb{Z}^d$ define the transformation $T_{s,x} : D([0,\infty); \Omega_c) \to D([0,\infty); \Omega_c)$ by

$$(T_{s,x}\omega)_t(y) := \omega_{t+s}(y + x).$$

We say that the environmental process ω satisfies *assumption (EC)* if $\{T_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ is an ergodic family of transformations acting on the space $(D([0,\infty); \Omega_c), \mathcal{B}(D([0,\infty); \Omega_c)), Q^c_\mu)$. In other words, the latter means that whenever $A \in \mathcal{B}(D([0,\infty); \Omega_c))$ is such that $T^{-1}_{s,x}A = A$ for every $s > 0$ and $x \in \mathbb{Z}^d$, then $Q^c_\mu(A)$ is 0 or 1.

Assumption (ED). Consider the discrete time environmental process ω. For $x \in \mathbb{Z}^d$ define the transformation $T_{1,x} : D([0,\infty); \Omega_d) \to D([0,\infty); \Omega_d)$ by

$$(T_{1,x}\omega)_n(y) := \omega_{n+1}(y + x).$$

We say that the environmental process ω satisfies *assumption (ED)* if $\{T_{1,x} : x \in R\}$ is an ergodic family of transformations acting on the space $(\Omega_d^{\mathbb{N}}, \mathcal{B}(\Omega_d^{\mathbb{N}}), Q^d_\mu)$. In other words, whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $T^{-1}_{1,x}A = A$ for every $x \in R$, then $Q^d_\mu(A)$ is 0 or 1.

It is straightforward to check that assumption (ED) is equivalent to asking that whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $A = T^{-1}_{n,x}A$ for every $x \in R$ and $n \in \mathbb{N}$ then $Q^d_\mu(A)$ is 0 or 1.

In this chapter we present a level 1 quenched large deviation principle for both the continuous and the discrete time random walk in time-space random environment. It should be noted that the discrete time version of our result can be derived via a contraction principle from results that have been obtained in Rassoul-Agha, Seppäläinen and Yilmaz [RSY11] establishing level 2 and 3 large deviations, for discrete time random walks on time-space random environments and potentials. There, the authors also derive variational expressions for the rate functions. Nevertheless, the proofs we present here of both Theorem 3.1.1 and 3.1.2, are short and direct.

Theorem 3.1.1. *Consider a continuous time random walk* $\{X_t : t \geq 0\}$ *in a uniformly elliptic time-dependent environment* ω *satisfying assumption (EC). Then, there exists a convex continuous rate function* $I_c(x) : \mathbb{R}^d \to [0, \infty)$ *such that the following are satisfied.*

(i) For every open set $G \subset \mathbb{R}^d$ *we have that* Q^c_μ*-a.s.*

$$\underline{\lim}_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in G\right) \geq -\inf_{x\in G} I_c(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ *we have that* Q^c_μ*-a.s.*

$$\overline{\lim}_{t\to\infty} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\right) \leq -\inf_{x\in C} I_c(x).$$

To state the discrete time version of Theorem 3.1.1, we need to introduce some notation. Let $R_0 := \{0\} \subset \mathbb{Z}^d$, $R_1 := R$ and for $n \geq 1$ define

$$R_{n+1} := \{y \in \mathbb{Z}^d : y = x + e \text{ for some } x \in R_n \text{ and } e \in R\},$$

and $U_n := R_n/n$. Note that R_n is the set of sites that a random walk with jump range R visits with positive probability at time n. We then define U as the set of limit points of the sequence of sets $\{U_n : n \geq 1\}$, so that

$$U := \{x \in \mathbb{R}^d : x = \lim_{n\to\infty} x_n \text{ for some sequence } x_n \in U_n\}. \tag{3.1.1}$$

Theorem 3.1.2. *Consider a discrete time random walk* $\{X_n : n \geq 0\}$ *in a uniformly elliptic time-dependent environment* ω *satisfying assumption (ED) with jump range* R*. Assume that either (i)* R *is finite, convex, symmetric and there is a neighborhood of* 0 *which belongs to the convex hull of* R*; (ii) or that* R *corresponds to the nearest neighbor case. Consider* U *defined in (3.1.1). Then* U *equals the convex hull of* R *and there exists a convex rate function* $I_d(x) : \mathbb{R}^d \to [0, \infty]$ *such that* $I_d(x) \leq |\log \kappa|$ *for* $x \in U$*,* $I_d(x) = \infty$ *for* $x \notin U$*,* I *is continuous for every* $x \in U^o$ *and the following are satisfied.*

(i) For every open set $G \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\varliminf_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in G\right) \geq -\inf_{x\in G} I_d(x).$$

(ii) For every closed set $C \subset \mathbb{R}^d$ we have that Q^d_μ-a.s.

$$\varlimsup_{n\to\infty} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x\in C} I_d(x).$$

Both quenched and annealed large deviations for discrete time random walks on random environments which do not depend on time, have been thoroughly studied in the case in which $d = 1$ (see the reviews of Sznitman [Sz04] and Zeitouni [Zei04] for both the one-dimensional and multi-dimensional cases). The first quenched multidimensional result was obtained by Zerner in [Z98] under the so called plain nestling condition, concerning the law of the support of the quenched drift (see also [Zei04] and [Sz04]). In [Var03], Varadhan established both a general quenched and annealed large deviation principle for discrete time random walks in static random environments via the use of the subadditive ergodic theorem. In the quenched case, he assumed uniform ellipticity and the ergodicity assumption (ED). Subsequently, in his Ph.D. thesis [Ros06], Rosenbluth extended the quenched result of Varadhan under a condition weaker than uniform ellipticity, along with a variational formula for the rate function (see also Yilmaz [Y08, Yil09a, Yil09b]). The method of Varadhan based on the subadditive ergodic theorem and of Rosenbluth [Ros06], Yilmaz [Yil09b] and Rassoul-Agha, Sepäläinen, Yilmaz [RSY11], are closely related to the use of the subadditive ergodic theorem in the context of non-linear stochastic homogenization (see for example the paper of dal Maso, Modica [DMM86]). Closer and more recent examples of stochastic homogenization for the Hamilton-Jacobi-Bellman equation with static Hamiltonians via the subadditive ergodic theorem are the work of Rezakhanlou and Tarver [RT00] and of Souganidis [So99] and in the context of the totally asymmetric simple K-exclusion processes and growth processes the works of Seppäläinen in [S99] and Rezakhanlou in [R02]. Stochastic homogenization for the Hamilton-Jacobi-Bellman equation with respect to time-space shifts was treated by Kosygina and Varadhan in [KV08] using change of measure techniques giving variational expressions for the effective Hamiltonian.

A particular case of Theorem 3.1.1 is the case of a random walk which has a drift in a given direction on occupied sites and in another given direction on unoccupied sites, where the environment is generated by an attractive spin-flip particle system or a simple exclusion process (see Avena, den Hollander and Redig [ADHR10] for the case of a one-dimensional attractive spin-flip dynamics, and also [ADHR11, ADSV11, DHDSS11]). This case is also included in the results presented in [RSY11]. Another particular case of Theorem 3.1.1 is a continuous time random walk in a static random environment with a law which is ergodic under spatial translations: two of these cases are the Bouchaud trap random walk with bounded jump rates (see for example [BC06]) and the continuous time random conductances model (see for example [DFGW89]). Our proof would also apply to the polymer measure defined by a continuous time random walk in time-dependent random environment and bounded random potential (see [RSY11]). Note that Theorem 3.1.2 does include the classical nearest neighbor case (a nearest neighbor case example is the random walk on a time-space i.i.d. environment studied by Yilmaz [Yil09a]).

Our proofs are obtained by directly establishing the level 1 large deviation principle and is based on the sub-additive ergodic theorem as used by Varadhan in [Var03]. Let us note, that in [Var03], Varadhan applies sub-additivity directly to the logarithm of a smoothed up version of the inverse of the transition probabilities of the random walk, as opposed to the earlier approach of Zerner [Z98] (see also Sznitman [S98]), where sub-additivity is applied to a generalized Laplace transform of the hitting times of sites of the random walk forcing to assume the so called nestling property on the random walk. While our methods do not give any explicit information about the rate function, besides its convexity and continuity, the proofs are short and simple.

We do not know how to define a smoothed up version of the transition probabilities as is done by Varadhan in [Var03]. We therefore have to prove directly an equicontinuity estimate for the transition probabilities of the random walk, which is the main difficulty in the proofs of Theorems 3.1.1 and 3.1.2. In the case of Theorem 3.1.1 we follow the method presented in [DGRS12]: we first express the transition probabilities of the walk in terms of those of a simple symmetric random walk through a Radon-Nykodym derivative, then through the use of Chapman-Kolmogorov equation we rely on standard large deviation estimates for the continuous time simple symmetric random walk.

In section 3.2 we present the proof of Theorem 3.1.1 using the methods developed in [DGRS12]. In section 3.3 we continue with the proof of Theorem 3.1.2 in the case in which the jump range of the walk R is convex, symmetric and a neighborhood of 0 is contained in its convex hull. In section 3.4 we prove Theorem 3.1.2 for the discrete time nearest neighbor case. Throughout the rest of the chapter we will use the notations c, C, C', C'' to refer to different positive constants.

3.2 Proof of Theorem 3.1.1

For each $s \geq 0$, let $\theta_s : D([0,\infty);\Omega_c) \to D([0,\infty);\Omega_c)$ denote the canonical time shift. As in [DGRS12], we first define for each $0 \leq s < t$ and $x, y \in \mathbb{Z}^d$ the quantities

$$e(s,t,x,y) := P^c_{x,\theta_s\omega}\left(X_{t-s} = y\right),$$

and

$$a_c(s,t,x,y) := -\log e(s,t,x,y),$$

where the subscript c in a_c is introduced to distinguish this quantity from the corresponding discrete time one. Note that these functions still depend on the realization of ω. We call $a_c(s,t,x,y)$ the point to point passage function from x to y between times s and t. Due to the fact that we are considering a continuous time random walk, here we do not need to smooth out the point to point passage functions (see [Var03]). Nevertheless, there is an equicontinuity issue that should be resolved. Theorem 3.1.1 will follow directly from the following shape theorem. A version of this shape theorem for a random walk in random potential has been established as Theorem 4.1 in [DGRS12] (see also Theorem 2.5 of Chapter 5 of Sznitman [S98]).

Theorem 3.2.1. [Shape theorem] *There exists a deterministic convex function I_c : $\mathbb{R}^d \to [0,\infty)$ such that $Q^c_\mu - a.s.$, for any compact set $K \subset \mathbb{R}^d$*

$$\lim_{t\to\infty} \sup_{y\in tK\cap\mathbb{Z}^d} \left| t^{-1} a_c(0,t,0,y) - I_c\left(\frac{y}{t}\right)\right| = 0. \tag{3.2.1}$$

Furthermore, for any $M > 0$, we can find a compact $K \subset \mathbb{R}^d$ such that $Q^c_\mu - a.s.$

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \notin K\right) \leq -M. \tag{3.2.2}$$

Let us first see how to derive Theorem 3.1.1 from Theorem 3.2.1. We will first prove the upper bound of part (ii) of Theorem 3.1.1. By (3.2.2) of Theorem 3.2.1, we know that we can choose a compact set $K \subset \mathbb{R}^d$ such that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \notin K\right) < -\inf_{x\in C} I_c(x),$$

where C is a closed set. It is therefore enough to prove that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\cap K\right) \leq -\inf_{x\in C} I_c(x).$$

Now,

$$\begin{aligned}\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\cap K\right) &\leq \overline{\lim_{t\to\infty}} \frac{1}{t} \sup_{y\in(tC\cap tK)\cap\mathbb{Z}^d} \log P^c_{0,\omega}\left(X_t = y\right)\\ &= \overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(X_t = y_t\right),\end{aligned}$$

where $y_t \in (tC \cap tK) \cap \mathbb{Z}^d$, is a point that maximizes $P^c_{0,\omega}(X_t = \cdot)$. Now, by compactness, there is a subsequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{y_{t_n}}{t_n} =: x^* \in C \cap K,$$

and $\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(X_t = y_t\right) = \overline{\lim_{n\to\infty}} \frac{1}{t_n} \log P^c_{0,\omega}\left(X_{t_n} = y_{t_n}\right)$. Thus, by the continuity of I_c and by (3.2.1) we see that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log P^c_{0,\omega}\left(\frac{X_t}{t} \in C\cap K\right) \leq -I_c(x^*) \leq -\inf_{x\in C} I_c(x).$$

To prove the lower bound, part (i) of Theorem 3.1.1, note that by (3.2.1) we have that

$$\varliminf_{t\to\infty}\frac{1}{t}\log P^c_{0,\omega}\left(\frac{X_t}{t}\in G\right)\geq \varliminf_{t\to\infty}\frac{1}{t}\sup_{y\in(tG)\cap\mathbb{Z}^d}\log P^c_{0,\omega}\left(X_t=y\right)\geq-\inf_{x\in G}I_c(x).$$

Let us now continue with the proof of Theorem 3.2.1. Display (3.2.2) of Theorem 3.2.1 follows from standard large deviation estimates for the process $\{N_t : t \geq 0\}$, where N_t is the total number of jumps up to time t of the random walk $\{X_t : t \geq 0\}$, which can be coupled with a Poisson process of parameter $2d\kappa_2$. To prove the first statement (3.2.1) of Theorem 3.2.1 we first observe that for every $0 \leq t_1 < t_2 < t_3$ and $x_1, x_2, x_3 \in \mathbb{Z}^d$ one has that Q^c_μ-a.s.

$$a_c(t_1,t_3,x_1,x_3)\leq a_c(t_1,t_2,x_1,x_2)+a_c(t_2,t_3,x_2,x_3). \tag{3.2.3}$$

We will also need to obtain bounds on the point to point passage functions which will be eventually used to prove some crucial equicontinuity estimates. To prove these bounds, we first state Lemma 4.2 of [DGRS12] with its respective proof, which is a large deviation estimate for the simple symmetric random walk.

Lemma 3.2.1. *Let X be a simple symmetric random walk on $\mathbb{Z}^d$ with jump rate κ and starting point $X(0) = 0$. For each $x \in \mathbb{Z}^d$ and $t > 0$ let $p(t, 0, x)$ be the probability that this random walk is at position x at time t starting from 0. Then for every $t > 0$ and $x \in \mathbb{Z}^d$, we have*

$$p(t,0,x)=\frac{e^{-J(\frac{x}{t})\,t}}{(2\pi t)^{\frac{d}{2}}\Pi_{i=1}^d\left(\frac{x_i^2}{t^2}+\frac{\kappa^2}{d^2}\right)^{1/4}}\left(1+o(1)\right), \tag{3.2.4}$$

where

$$J(x):=\sum_{i=1}^d\frac{\kappa}{d}j\Big(\frac{dx_i}{\kappa}\Big)\qquad \textit{with}\quad j(y):=y\sinh^{-1}y-\sqrt{y^2+1}+1,$$

and the error term $o(1)$ tends to zero as $t \to \infty$ uniformly in $x \in tK \cap \mathbb{Z}^d$, for any compact $K \subset \mathbb{R}^d$. Furthermore the function j is increasing with $|y|$ and $j \geq 0$.

Proof. Since the coordinates of X are independent, it suffices to consider the case X is a rate $\dfrac{\kappa}{d}$ simple symmetric random walk on $\mathbb{Z}$. Let $\sigma := \dfrac{t}{\lceil t\rceil}$. Let $Z^\lambda_1, \ldots, Z^\lambda_{\lceil t\rceil}$ be i.i.d. with common law

$$P(Z^\lambda_1=y)=p(\sigma,0,y)e^{\lambda y-\Phi(\lambda)},\quad y\in\mathbb{Z}$$

where

$$\Phi(\lambda) := \log E(e^{\lambda X(\sigma)}) = \frac{\sigma\kappa}{d}(\cosh\lambda - 1).$$

Here E is the expectation with respect to P. Note that

$$E(Z_1^\lambda) = \frac{d\Phi}{d\lambda}(\lambda) = \frac{\sigma\kappa}{d}\sinh\lambda \quad \text{and} \quad \mathrm{Var}(Z_1^\lambda) = \frac{d^2\Phi}{d^2\lambda}(\lambda) = \frac{\sigma\kappa}{d}\cosh\lambda.$$

We will set $\lambda := \sinh^{-1}\left(\frac{dx}{\kappa t}\right)$ so that $E(Z_1^\lambda) = \frac{x}{\lceil t\rceil}$. If we let $S_{\lceil t\rceil} := \sum_{i=1}^{\lceil t\rceil} Z_1^\lambda$, then observe that

$$p(t,0,x) = P\left(S_{\lceil t\rceil} = x\right) e^{-\lambda x + \lceil t\rceil\Phi(\lambda)} = P\left(S_{\lceil t\rceil} = x\right) e^{-\frac{\kappa}{d} j\left(\frac{dx}{\kappa t}\right)t}.$$

Note that $S_{\lceil t\rceil} - x$ has mean 0, variance $t\sqrt{\frac{x^2}{t^2} + \frac{\kappa^2}{d^2}}$, and characteristic function

$$e^{\lceil t\rceil(\Phi(ik+\lambda) - \Phi(\lambda)) - ikx} = e^{ix(\sin k - k) - t\sqrt{\frac{x^2}{t^2} + \frac{\kappa^2}{d^2}}(1-\cos k)}.$$

Finally (3.2.4) is deduced applying Fourier inversion.

∎

We will need the following estimates for the transition probabilities.

Lemma 3.2.2. *Consider the transition probabilities of a random walk on a uniformly elliptic time-dependent environment. The following hold Q_μ^c-a.s.*

(i) Let $C_3 > 0$. There exists a $t_0 > 0$ and constants C_1, C_1' and C_2 such that for $\epsilon > 0$ small enough and every $t \geq t_0$, $y, z \in \mathbb{Z}^d$ such that $|y - z|_2 \leq \epsilon t + \frac{tC_3}{|\log\epsilon|}$ we have that

$$C_1 e^{-C_1' t \frac{1}{|\log\epsilon|^{1/2}}} p(\epsilon t, z, y) \leq e(t(1-\epsilon), t, z, y) \leq C_2 e^{C_2 t \frac{1}{|\log\epsilon|^{1/2}}} p(\epsilon t, z, y).$$

(ii) Let $r > 0$. There exists a $t_0 > 0$ and a constant $C > 0$ such that for each $t \geq t_0$ and $x \in B_{tr}(0)$ one has that

$$e(0,t,0,x) \geq e^{-Ct} p(t,0,x).$$

(iii) There is a function $\alpha : (0,\infty) \times [0,\infty) \to (0,\infty)$ such that for each $x, y \in \mathbb{Z}^d$ and $t > s \geq 0$ one has that

$$e(s,t,x,y) \geq \alpha(t-s, |x-y|_1) > 0. \tag{3.2.5}$$

Proof. Part (i). Note that

$$e(t(1-\epsilon),t,z,y) = E_{z,t(1-\epsilon)}\left[e^{\int_{t(1-\epsilon)}^t \log(2d\omega_s(Y_{s\_},Y_s-Y_{s\_}))dN_s - \int_{t(1-\epsilon)}^t(\omega_s(Y_s,G)-1)ds} 1_{Y_t}(y)\right], \tag{3.2.6}$$

where $E_{z,s}$ is the expectation with respect to the law of a continuous time simple symmetric random walk $\{Y_t : t \geq 0\}$ of jump rate 1 starting from z at time s, N_t is the number of jumps up to time t of the walk, while for each $x \in \mathbb{Z}^d$ and $s > 0$, $\omega_s(x,G) := \sum_e \omega_s(x,e)$ is the total jump rate at site x and time s (see for example Proposition 2.6 in Appendix 1 of Kipnis-Landim [KL99]). Using the fact that the jump rates are bounded from above and from below, it is clear that there is a constant $C > 0$ such that

$$e^{\int_{t(1-\epsilon)}^t \log(2d\omega_s(Y_{s\_},Y_s-Y_{s\_}))dN_s - \int_{t(1-\epsilon)}^t(\omega_s(Y_s,G)-1)ds} \leq e^{C(N_t - N_{t(1-\epsilon)})+C\epsilon t}.$$

Substituting this bound in (3.2.6), we see that

$$e(t(1-\epsilon),t,z,y) \leq e^{C\epsilon t} E\left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z,y)\right], \tag{3.2.7}$$

where now E is the expectation with respect to a Poisson process $\{N_t : t \geq 0\}$ of rate 1 and p_n is the n-step transition probability of a discrete time simple symmetric random walk. Let now $R_\epsilon := \frac{1}{\epsilon|\log\epsilon|^{1/2}}$. Note that

$$\begin{aligned} E\left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z,y)\right] &\leq e^{CR_\epsilon t\epsilon} p(\epsilon t,z,y) + E[e^{N_{\epsilon t}C}, N_{\epsilon t} > R_\epsilon t\epsilon] \\ &\leq e^{CR_\epsilon t\epsilon} p(\epsilon t,z,y) + E[e^{2N_{\epsilon t}C}]^{1/2} P(N_{\epsilon t} > R_\epsilon t\epsilon)^{1/2}. \end{aligned}$$

Now, using the exponential Chebychev inequality with parameter $\log R_\epsilon$, we get

$$P(N_{\epsilon t} > R_\epsilon \epsilon t) \leq e^{-\epsilon t(R_\epsilon \log R_\epsilon - (R_\epsilon - 1))} \tag{3.2.8}$$

and we compute $E[e^{2N_{\epsilon t}C}] = e^{\epsilon t(e^{2C}-1)}$. Hence,

$$E\left[e^{CN_{\epsilon t}}p_{N_{\epsilon t}}(z,y)\right] \leq e^{CR_\epsilon t\epsilon}p(\epsilon t,z,y) + e^{\epsilon\frac{t}{2}(e^{2C}-1)}e^{-\epsilon\frac{t}{2}(R_\epsilon \log R_\epsilon-(R_\epsilon-1))}. \tag{3.2.9}$$

Now, by Lemma 3.2.1 we know that $j(y)$ is increasing with $|y|$, so that

$$\sup_{y,z:|y-z|_2\leq \epsilon t+\frac{C_3 t}{|\log\epsilon|}} \epsilon t j\left(\frac{|z-y|}{\epsilon t}\right) \leq \epsilon t j\left(\frac{C_3}{\epsilon|\log\epsilon|}+1\right) \leq t\left(\frac{C_3}{|\log\epsilon|}+\epsilon\right)\log\left(3+\frac{2C_3}{\epsilon|\log\epsilon|}\right)$$

for $t\geq 1$. Hence, again by Lemma 3.2.1 with $\kappa=1$, we see that for any constant $c>0$ we can choose ϵ small enough such that

$$\lim_{t\to\infty}\frac{e^{\epsilon\frac{t}{2}(e^{2C}-1)}e^{-\epsilon tc(R_\epsilon\log R_\epsilon-(R_\epsilon-1))}}{\inf_{y,z}p(\epsilon t,z,y)} = 0, \tag{3.2.10}$$

where the infimum is taken over y,z as in the previous display. Applying (3.2.10) with $c=1/2$, we see that the second term of the right-hand side of inequality (3.2.9), after taking the supremum over y,z such that $|y-z|_2\leq \epsilon t+\dfrac{C_3 t}{|\log\epsilon|}$, is negligible with respect to the first one. Hence, for ϵ small enough, there is a constant C and a $t_0>0$ such that for y,z such that $|y-z|_2\leq\epsilon t+\dfrac{C_3 t}{|\log\epsilon|}$ and $t\geq t_0$ one has

$$e(t(1-\epsilon),t,z,y)\leq Ce^{(R_\epsilon+1)Ct\epsilon}p(\epsilon t,z,y).$$

Similarly, using the fact that the jump rates are bounded from above and from below it can be shown that for y,z such that $|y-z|_2\leq\epsilon t+\dfrac{C_3 t}{|\log\epsilon|}$ and t large enough

$$\begin{aligned}e(t(1-\epsilon),t,z,y) &\geq e^{-C'\epsilon t}E[e^{-C'N_{\epsilon t}}p_{N_{\epsilon t}}(z,y)1_{N_{\epsilon t}\leq R_\epsilon\epsilon t}]\\ \geq e^{-(R_\epsilon+1)\epsilon tC'}E[p_{N_{\epsilon t}}(z,y)1_{N_{\epsilon t}\leq R_\epsilon \epsilon t}] &\geq e^{-(R_\epsilon+1)\epsilon tC'}\left(p(\epsilon t,z,y)-P(N_{\epsilon t}>R_\epsilon\epsilon t)\right)\\ &\geq C''e^{-(R_\epsilon+1)\epsilon tC'}p(\epsilon t,z,y),\end{aligned}$$

where we have used (3.2.8) and (3.2.10) with $c=1$.

Part (ii). The proof of part (ii) is analogous to the proof of the lower bound of part (i).

Part (iii). By the same argument as the last part of the proof of part (i), there is a constant $C'>0$ such that

$$e(s,t,x,y)\geq e^{-C'(t-s)}E[e^{-C'N_{t-s}}p_{N_{t-s}}(x,y),N_{t-s}=|x-y|_1]$$

But $P(N_{t-s} = |x-y|_1) > 0$ (there is, with positive probability, a trajectory from 0 to x such that $N_{t-s} = |x-y|_1$). Thus,

$$\begin{aligned} e(s,t,x,y) &\geq e^{-C'(t-s)-C'|x-y|_1} p_{|x-y|_1}(x,y) P(N_{t-s} = |x-y|_1) \\ &\geq e^{-C'(t-s)-C'|x-y|_1} \frac{1}{(2d)^{|x-y|_1}} P(N_{t-s} = |x-y|_1) > 0. \end{aligned}$$

∎

We can now apply Kingman's sub-additive ergodic theorem (see for example Liggett [L85]), to prove the following lemma.

Lemma 3.2.3. *There exists a deterministic function $I_c : \mathbb{Q}^d \to [0,\infty)$ such that for every $y \in \mathbb{Q}^d$, Q^c_μ-a.s. we have that*

$$\lim_{\substack{t\to\infty \\ ty\in\mathbb{Z}^d}} \frac{a_c(0,t,0,ty)}{t} = I_c(y). \tag{3.2.11}$$

Proof. Assume first that $y \in \mathbb{Z}^d$. Let $q \in \mathbb{N}$. We will consider for $m > n \geq 1$ the random variables

$$X_{n,m}(y) := a_c(nq, mq, ny, my).$$

By (3.2.3), we have

$$X_{0,m}(y) \leq X_{0,n}(y) + X_{n,m}(y).$$

By part (iii) of Lemma 3.2.2, we see that the random variables $\{X_{n,m}(y)\}$ are integrable. Hence, by Kingman's sub-additive ergodic theorem (see Liggett [L85]) we can then conclude that the limit

$$\hat{I}(q,y,\omega) := \lim_{m\to\infty} \frac{a_c(0,mq,0,my)}{m} \tag{3.2.12}$$

exists for $y \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. We have to show that it is deterministic. For this reason, let $r > 0$, $z \in \mathbb{Z}^d$ be arbitrary. It suffices to prove that

$$\hat{I}(q,y,\omega) \leq \hat{I}(q,y,T_{r,z}\omega) = \lim_{m\to\infty} \frac{a_c(r, mq+r, z, my+z)}{m}.$$

First, we have that

$$\frac{a_c(0,mq,0,my)}{m} \leq \frac{a_c(0,r,0,z)}{m} + \frac{a_c(r,mq,z,my)}{m}.$$

By part (*iii*) of Lemma 3.2.2, the first term of the right-hand side of the last equation tends to 0 as $m \to \infty$. Therefore,

$$\hat{I}(q,y,\omega) = \lim_{m\to\infty} \frac{a_c(0,mq,0,my)}{m} \leq \varliminf_{m\to\infty} \frac{a_c(r,mq,z,my)}{m}. \tag{3.2.13}$$

On the other hand, for $u \in \mathbb{N}$ such that $m > u > r$ we have that

$$\begin{aligned} \frac{a_c(r,mq,z,my)}{m} &\leq \frac{a_c(r,(m-u)q+r,z,(m-u)y+z)}{m} \\ &+ \frac{a_c((m-u)q+r,mq,(m-u)y+z,my)}{m}. \end{aligned}$$

Again, by part (*iii*) of Lemma 3.2.2, the last term tends to 0 as $m \to \infty$. Therefore

$$\varliminf_{m\to\infty} \frac{a_c(r,mq,z,my)}{m} \leq \lim_{m\to\infty} \frac{a_c(r,(m-u)q+r,z,(m-u)y+z)}{m} = \hat{I}(q,y,T_{r,z}\omega). \tag{3.2.14}$$

Hence $\hat{I}(q,y,\omega) \leq \hat{I}(q,y,T_{r,z}\omega)$. Since $r > 0$ and $z \in \mathbb{Z}^d$ are arbitrary, $\hat{I}(q,y)$ is shift-invariant under each transformation $T_{r,z}$. By assumption (EC), $\hat{I}(q,y)$ is Q^c_μ-a.s equal to a constant for each y. Now, if $y \in \mathbb{Q}^d$, choose the smallest $q \in \mathbb{N}$ such that $qy \in \mathbb{Z}^d$. Then by (3.2.12), we conclude that

$$\lim_{m\to\infty} \frac{a_c(0,mq,0,mqy)}{mq} = \frac{1}{q}\hat{I}(q,qy,\omega) =: I_c(y), \tag{3.2.15}$$

exists (and is well-defined) and is Q^c_μ-a.s. equal to a constant. ∎

We now need to extend the definition of the function $I_c(x)$ for all $x \in \mathbb{R}^d$ and prove the uniform convergence in (3.2.1). To do this, we will prove that for each compact K there is a $t_0 > 0$ such that the family of functions $\{t^{-1}a_c(0,t,0,ty) : t \geq t_0\}$ defined on K is equicontinuous. We can now proceed to the main step of the proof of Theorem 3.2.1.

Lemma 3.2.4. *Let K be any compact subset of $\mathbb{R}^d$. There exist deterministic $\phi_K : (0,\infty) \to (0,\infty)$ with $\lim_{r\downarrow 0} \phi_K(r) = 0$, and $t_0 > 0$ such that for any $\epsilon > 0$ and $t \geq t_0$, Q_μ^c-a.s., we have*

$$\sup_{\substack{x,y\in tK\cap\mathbb{Z}^d \\ |x-y|_2\leq \epsilon t}} t^{-1}|a_c(0,t,0,x) - a_c(0,t,0,y)| \leq \phi_K(\epsilon). \tag{3.2.16}$$

Proof. Let us note that for every $\epsilon > 0$, t and $x \in \mathbb{Z}^d$ one has that

$$e(0,t,0,x) = \sum_{z\in\mathbb{Z}^d} e(0,t(1-\epsilon),0,z)e(t(1-\epsilon),t,z,x).$$

Let $R_K := \sup\{|x|_2 : x \in K\}$ be the maximal distance to 0 for any point in K and $r_K = \dfrac{C_K}{|\log \epsilon|}$, where C_K is a constant that will be chosen large enough. From part (i) of Lemma 3.2.2 and Lemma 3.2.1, note that for $t \geq t_0$ (where t_0 is given by part (i) of Lemma 3.2.2)

$$e(0,t,0,x) \leq \sum_{z\in B_{r_K t}(x)} e(0,t(1-\epsilon),0,z)e(t(1-\epsilon),t,z,x) + Ce^{\frac{1}{|\log\epsilon|^{1/2}}tC - \epsilon t\frac{1}{d}j\left(d\frac{r_K}{\epsilon}\right)}. \tag{3.2.17}$$

On the other hand by part (ii) of Lemma 3.2.2 we have that for $t \geq t_0$

$$e(0,t,0,x) \geq e^{-C't - tJ\left(\frac{x}{t}\right)}.$$

Using the upper bound $J\left(\dfrac{x}{t}\right) \leq dR_K \log(1+2dR_K)$ we see that if

$$\epsilon\frac{1}{d}j\left(d\frac{r_K}{\epsilon}\right) > C + C' + dR_K\log\left(1+2dR_K\right), \tag{3.2.18}$$

the second term of (3.2.17) is negligible. But (3.2.18) is satisfied for $C_K > 2(C + C' + dR_K\log(1+2dR_K))$ and $\epsilon > 0$ small enough. Hence, it is enough to prove that, Q_μ^c-a.s. we have that

$$\sup_{\substack{x,y\in tK\cap\mathbb{Z}^d \\ |x-y|_2\leq \epsilon t}} \sup_{z\in B_{r_K t}(x)} \frac{e(t(1-\epsilon),t,z,x)}{e(t(1-\epsilon),t,z,y)} \leq Ce^{t\phi_K(\epsilon)}. \tag{3.2.19}$$

To this end, by Lemmas 3.2.1 and 3.2.2

$$\frac{e(t(1-\epsilon),t,z,x)}{e(t(1-\epsilon),t,z,y)} \leq Ce^{2tC\frac{1}{|\log\epsilon|^{1/2}}} e^{-\epsilon t\left(J\left(\frac{x-z}{\epsilon t}\right) - J\left(\frac{y-z}{\epsilon t}\right)\right)}. \tag{3.2.20}$$

But,

$$J\left(\frac{z-x}{t\epsilon}\right)-J\left(\frac{z-y}{t\epsilon}\right)=\sum_{i=1}^{d}\frac{1}{d}\left[j\left(d\frac{z_i-x_i}{t\epsilon}\right)-j\left(d\frac{z_i-y_i}{t\epsilon}\right)\right]$$
$$\leq\sum_{i=1}^{d}\left|\frac{1}{d}\int_{d\frac{z_i-x_i}{t\epsilon}}^{d\frac{z_i-y_i}{t\epsilon}}\log\left(1+2|u|\right)du\right|\leq d\log\left(1+\frac{2dC_K}{\epsilon|\log\epsilon|}\right).$$

Substituting this estimate back into (3.2.20) we obtain (3.2.19) with $\phi_K(\epsilon)=C\frac{1}{|\log\epsilon|^{1/2}}$.

∎

Using this lemma, we can extend I_c to a continuous function on $\mathbb{R}^d$. It remains to show the convexity of I_c. For this purpose, let $\lambda\in(0,1)$, $x,y\in\mathbb{R}^d$ and let $(\lambda_n)\subset(0,1)\cap\mathbb{Q}$, $(x_n),(y_n)\subset\mathbb{Q}^d$ such that $\lambda_n\to\lambda$, $x_n\to x$, and $y_n\to y$. In addition let $r_n\in\mathbb{N}$ be such that $r_n(\lambda_n x_n+(1-\lambda_n)y_n)$, $\lambda_n m r_n$, and $\lambda_n m r_n x_n$, are contained in $\mathbb{Z}^d$. Then for any $n\in\mathbb{N}$ one has

$$\begin{aligned}I_c(\lambda_n x_n+(1-\lambda_n)y_n) &= \lim_{m\to\infty}\frac{a_c(0,mr_n,0,mr_n(\lambda_n x_n+(1-\lambda_n)y_n))}{mr_n}\\ &\leq \lim_{m\to\infty}\frac{a_c(0,\lambda_n m r_n,0,\lambda_n m r_n x_n)}{mr_n}\\ &\quad+\lim_{m\to\infty}\frac{a_c(\lambda_n m r_n,mr_n,\lambda_n m r_n x_n,mr_n(\lambda_n x_n+(1-\lambda_n)y_n))}{mr_n}.\end{aligned}$$

Now taking $n\to\infty$, the continuity of I_c yields that the left-hand side converges to $I_c(\lambda x+(1-\lambda)y)$. Taking advantage of the continuity of I_c and (3.2.15), the first summand on the right-hand side converges to $\lambda I_c(x)$ a.s., while in combination with the fact that the transformations $T_{\lambda_n m r_n,\lambda_n m r_n x_n}$ are measure preserving, the second summand converges in probability to $(1-\lambda)I_c(y)$; from the last fact we deduce a.s. convergence along an appropriate subsequence and hence the convexity of I_c.

3.3 Proof of Theorem 3.1.2 for the convex case

Here we consider the case in which the jump range R of the walk is convex, symmetric and a neighborhood of 0 is contained in the convex hull of R. Let us call $\pi_{n,m}(x,y)$, the probability that the discrete time random walk in time-space random environment jumps from time n to time m from site x to site y. Define

$$a_d(n,m,x,y) := -\log \pi_{n,m}(x,y).$$

As in the continuous time case, we have the following sub-additivity property for $n \le p \le m$ and $x, y, z \in \mathbb{Z}^d$,

$$a_d(n,m,x,y) \le a_d(n,p,x,z) + a_d(p,m,z,y). \tag{3.3.1}$$

We first need to define some concepts that will be used throughout this section. An element (n,z) of the set $\mathbb{N} \times \mathbb{Z}^d$ will be called a *time-space point*. The time-space points of the form $(1,z)$, with $z \in R$, will be called *steps*. Furthermore, given two time-space points $(n_1, x^{(1)})$ and $(n_2, x^{(2)})$ a sequence of steps $(1, z^{(1)}), \ldots, (1, z^{(k)})$, with $k = n_2 - n_1$ will be called an *admissible path from* $(n_1, x^{(1)})$ *to* $(n_2, x^{(2)})$, if $x^{(2)} = x^{(1)} + z^{(1)} + \ldots + z^{(k)}$ and

$$\begin{aligned}\pi_{n_1,n_1+1}(x^{(1)}, x^{(1)} + z^{(1)})\pi_{n_1+1,n_1+2}(x^{(1)} + z^{(1)}, x^{(1)} + z^{(1)} + z^{(2)}) \times \cdots \\ \cdots \times \pi_{n_2-1,n_2}(x^{(1)} + z^{(1)} + \cdots + z^{(k-1)}, x^{(1)} + z^{(1)} + \cdots + z^{(k)}) > 0.\end{aligned} \tag{3.3.2}$$

In other words, there is a positive probability for the time-space random walk (n, X_n) to jump through the sequence of time-space points $(n_1, x^{(1)}), (n_1+1, x^{(1)}+z^{(1)}), \ldots, (n_2, x^{(2)}) = (n_2, x^{(1)} + z^{(1)} + \cdots + z^{(k)})$. Note that the sequence of steps $(1, z^{(1)}), \ldots, (1, z^{(k)})$, is an admissible path if and only if $z^{(j)} \in R$ for all $1 \le j \le k$. Let us note that by uniform ellipticity asking that the left-hand side of (3.3.2) be positive is equivalent to asking that it be larger than or equal to $\kappa^{n_2-n_1}$. With a slight abuse of notation, we will adopt the convention that for $u \in \mathbb{R}$, $[u]$ is the integer closest to u that is between u and 0. Furthermore, we introduce for $x \in \mathbb{R}^d$, the notation $[x] := ([x_1], \ldots, [x_d]) \in \mathbb{Z}^d$. Throughout, given $A \subset \mathbb{R}^d$ we will call A^o its interior.

Lemma 3.3.1. *Consider a discrete time random walk in a uniformly elliptic time-dependent environment ω with finite, convex and symmetric jump range R such that a neighborhood* *of* 0 *belongs to its convex hull. Then,* U *equals the convex hull of* R *and for every* $n \geq 1$ *we have that*

$$R_n = (nU) \cap \mathbb{Z}^d. \tag{3.3.3}$$

Proof. It is straightforward to check that U equals the convex hull of R in $\mathbb{R}^d$. On the other hand, note that if $x \in R_n$, we have that for every $m \in \mathbb{N}$, $mx \in R_{nm}$, which implies that $\frac{x}{n} \in U_{nm}$. This proves that $R_n \subset (nU) \cap \mathbb{Z}^d$. Finally, using the fact that R is convex, we can prove that $(nU) \cap \mathbb{Z}^d \subset R_n$. ∎

For each $x \in \mathbb{Z}^d$ define $s(x)$ as the minimum number n of steps such that there is an admissible path between $(0,0)$ and (n,x). Alternatively,

$$s(x) = \min\{n \geq 0 : x \in R_n\}.$$

Let us now define a norm in $\mathbb{R}^d$ which will be a good approximation for the previous quantity. For each $y \in \partial U$ define $||y|| = 1$. Then, for each $x \in \mathbb{R}^d$ which is of the form $x = ay$ for some real $a \geq 0$, we define $||x|| = a$. Note that since U is convex, symmetric and there is a neighborhood of 0 which belongs to its interior, this defines a norm in $\mathbb{R}^d$ (see for example Theorem 15.2 of Rockafellar [R97]) and that $x \in U^o$ if and only if $||x|| < 1$. Furthermore, note that for every $x \in \mathbb{R}^d$ we have that

$$||x|| \leq s(x) \leq ||x|| + 1. \tag{3.3.4}$$

Lemma 3.3.2. *Let* $z \in U$ *and* $x \in U^o$. *Then, for each natural* n *there exists an* n_2 *such that*

$$n \leq n_2 \leq n + 1 + \frac{4d+1}{1-||x||} + n\frac{||x-z||}{1-||x||}. \tag{3.3.5}$$

and there is an admissible path between (n,z) *and* (n_2,x) *so that*

$$a_d(0,n_2,0,[n_2x]) \leq a_d(0,n,0,[nz]) - \log \kappa^{n_2-n}. \tag{3.3.6}$$

Similarly, for each natural n *there exists an* n_1 *such that*

$$n - 1 - \frac{4d+1}{1-||x||} - n\frac{||x-z||}{1-||x||} \leq n_1 \leq n \tag{3.3.7}$$

and there is an admissible path between (n_1, x) *and* (n, z) *so that*

$$a_d(0, n, 0, [nz]) \leq a_d(0, n_1, 0, [n_1 x]) - \log \kappa^{n-n_1} \tag{3.3.8}$$

Proof. Assume that $n_2 \geq n$. It is enough to prove that for n and n_2 satisfying (3.3.5) and (3.3.6) it is true that

$$s\left([n_2 x] - [nz]\right) \leq n_2 - n. \tag{3.3.9}$$

Now, by (3.3.4) and the fact that $||x - [x]|| \leq d$ we have that

$$s\left([n_2 x] - [nz]\right) \leq ||[n_2 x] - [nz]|| + 1 \leq ||[n_2 x] - [nx]|| + ||[nx] - [nz]|| + 1$$
$$\leq ||(n_2 - n)x|| + ||n(x - z)|| + 4d + 1 = (n_2 - n)||x|| + n||x - z|| + 4d + 1.$$

It follows that to prove (3.3.9) it is enough to show that

$$(n_2 - n)||x|| + n||x - z|| + 4d + 1 \leq n_2 - n, \tag{3.3.10}$$

which is equivalent to

$$n_2 \geq n + \frac{4d+1}{1 - ||x||} + n\frac{||x - z||}{1 - ||x||}.$$

This proves (3.3.5). Now assume that $n_1 \leq n$. We have to show that

$$s\left([nz] - [n_1 x]\right) \leq n - n_1.$$

Now,

$$s\left([nz] - [n_1 x]\right) \leq ||[nz] - [n_1 x]|| + 1 \leq n||z - x|| + (n - n_1)||x|| + 4d + 1.$$

Hence, it is enough to show that

$$n||z - x|| + (n - n_1)||x|| + 4d + 1 \leq n - n_1,$$

which is equivalent to

$$n_1 \leq n - \frac{4d+1}{1-||x||} - n\frac{||z-x||}{1-||x||}.$$

∎

We are now ready to prove the following proposition.

Proposition 3.3.1. *For each $x \in \mathbb{R}^d$ we have that Q^d_μ-a.s. the limit*

$$I(x) := -\lim_{n\to\infty} \frac{1}{n} \log \pi_{0,n}(0, [nx]),$$

exists, is convex and deterministic. Furthermore, $I(x) < \infty$ if and only if $x \in U$.

Proof. From Lemma 3.3.1, it follows that for $x \notin U$ it is true for $n \geq 1$, that $nx \notin nU$ and hence from Lemma 3.3.1 that $nx \notin R_n$ so that $\pi_n(0, [nx]) = 0$. Thus, $I(x) = \infty$. We divide the rest of the proof in four steps. In step 1 for each $x \in \mathbb{Q}^d \cap U^o$ we define a function $\tilde{I}(x)$. In step 2 we will show that $\tilde{I}$ is deterministic for $x \in \mathbb{Q}^d \cap U^o$. In step 3 we will show that $I(x)$ is well-defined for $x \in \mathbb{Q}^d \cap U^o$ and that $I(x) = \tilde{I}(x)$ and in step 4, we extend the definition of $I(x)$ to $x \in U$.

Step 1. Here we will define for each $x \in \mathbb{Q}^d \cap U^o$ a function $\tilde{I}(x)$. Given $x \in \mathbb{Q}^d \cap U^o$, there exist a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap kU^o$ such that $x = k^{-1}y$. Now, by display (3.3.3) of Lemma 3.3.1 we know that $y \in R_k$. Then, by the convexity of R and the sub-additive ergodic theorem and (3.3.1) we can define Q^d_μ-a.s.

$$\tilde{I}(k^{-1}y) := -\lim_{m\to\infty} \frac{1}{mk} \log \pi_{0,mk}(0, my).$$

This definition is independent of the representation of x. Indeed, assume that $x = k^{-1}y_1 = l^{-1}y_2$ for some $k, l \in \mathbb{N}$, $y_1 \in \mathbb{Z}^d \cap kU^o$ and $y_2 \in \mathbb{Z}^d \cap lU^o$. Then, passing to subsequences,

$$\begin{aligned}
\tilde{I}(k^{-1}y_1) &= -\lim_{n\to\infty} \frac{1}{nlk} \log \pi_{0,nlk}(0, nly_1) \\
&= -\lim_{n\to\infty} \frac{1}{nlk} \log \pi_{0,nlk}(0, nky_2) = \tilde{I}(l^{-1}y_2).
\end{aligned}$$

Step 2. Here we will show that $\tilde{I}$ is deterministic in $\mathbb{Q}^d \cap U^o$. Let $x \in \mathbb{Q}^d \cap U^o$. We know that there exists a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap kU^o$ such that $x = k^{-1}y$. Let us now fix $z \in R$. It suffices to prove that

$$\tilde{I}(x,\omega) \leq \tilde{I}(x, T_{1,z}\omega) = \lim_{m\to\infty} \frac{a_d(1, mk+1, z, my+z)}{mk}.$$

First, for each $n \in \mathbb{N}$, we have that

$$\frac{a_d(0, mnk, 0, mny)}{mnk} \leq \frac{a_d(0,1,0,z)}{mnk} + \frac{a_d(1, mnk, z, mny)}{mnk}.$$

By uniform ellipticity, the first term of the right-hand side of the last inequality tends to 0 as $m \to \infty$. Therefore,

$$\tilde{I}(x,\omega) = \lim_{m\to\infty} \frac{a_d(0, mnk, 0, mny)}{mnk} \leq \varliminf_{m\to\infty} \frac{a_d(1, mnk, z, mny)}{mnk}. \tag{3.3.11}$$

On the other hand,

$$\begin{aligned} \frac{a_d(1, mnk, z, mny)}{mnk} &\leq \frac{a_d(1, (m-1)nk+1, z, (m-1)ny+z)}{mnk} \\ &+ \frac{a_d((m-1)nk+1, mnk, (m-1)ny+z, mny)}{mnk}. \end{aligned} \tag{3.3.12}$$

Let us now assume that there is an admissible path from $(0, z+(m-1)ny)$ to $(nk-1, mny)$. This is equivalent to asking that z satisfies the following condition:

$$\pi_{0,nk-1}(z+(m-1)ny, mny) > 0 \qquad \text{for some } n \in \mathbb{N}. \tag{3.3.13}$$

Then, by uniform ellipticity, the last term of (3.3.12) tends to 0 as $m \to \infty$. Therefore, if $z \in R$ satisfies condition (3.3.13), by (3.3.11) and (3.3.12) we have that

$$\tilde{I}(x,\omega) \leq \tilde{I}(x, T_{1,z}\omega). \tag{3.3.14}$$

Hence, to finish the proof it is enough to show that every $z \in R$ satisfies (3.3.13). Now, z satisfies (3.3.13) if and only if there exists an $n \in \mathbb{N}$ such that

$$z - ny \in R_{nk-1}. \tag{3.3.15}$$

We will show by contradiction that every $z \in R$ satisfies (3.3.15). Indeed, assume that for each n it is true that

$$z - ny \notin R_{nk-1}.$$

Then,

$$\frac{z}{nk-1} - y\frac{n}{nk-1} \notin U_{nk-1}.$$

Therefore, taking the limit $n \to \infty$, we conclude that $\frac{y}{k} \notin U^o$, which is a contradiction. This proves that for every $z \in R$ condition (3.3.13) is satisfied and hence (3.3.14) is also valid. It follows now by the ergodicity assumption (ED), that for each $x \in \mathbb{Q}^d \cap U^o$, $\tilde{I}(x)$ is Q_μ^d-a.s equal to a constant.

Step 3. Here we will show that I is well-defined in $\mathbb{Q}^d \cap U^o$ and hence equals $\tilde{I}$ there. Let $x \in \mathbb{Q}^d \cap U^o$. Let k be such that $kx \in \mathbb{Z}^d$. Given n, choose m so that $mk \leq n < (m+1)k$. Note that there exists a sequence of increments $z^{(j)} \in R$, $1 \leq j \leq n - mk$, such that

$$[nx] = mkx + z^{(1)} + \cdots + z^{(n-mk)}.$$

Hence, by sub-additivity and considering that by uniform ellipticity the path $(1, z^{(1)}), \ldots,$ $(1, z^{(n-mk)})$ from $[nx]$ to mkx is admissible, we conclude that

$$\frac{a_d(0, n, 0, [nx])}{n} \leq \frac{a_d(0, mk, 0, mkx)}{n} - \frac{\log \kappa^{n-mk}}{n}.$$

It follows that

$$\overline{\lim_{n\to\infty}} \frac{a_d(0, n, 0, [nx])}{n} \leq \tilde{I}(x).$$

For the upper bound, first note that similarly there exists an admissible path of $(m+1)k-n$ steps from $[nx]$ to $(m+1)kx$. Hence,

$$\frac{a_d(0, (m+1)k, 0, (m+1)kx)}{n} \leq \frac{a_d(0, n, 0, [nx])}{n} - \frac{\log \kappa^{(m+1)k-n}}{n}.$$

Taking the limit when $n \to \infty$ we obtain

$$\underline{\lim}_{n\to\infty} \frac{a_d(0,n,0,[nx])}{n} \geq \tilde{I}(x).$$

Step 4. Here we will show that I is well-defined in the set $(\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U^o$. Let $z \in (\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U^o$. Pick a rational point x such that

$$\frac{1}{1-||x||} \leq 2\frac{1}{1-||z||}. \tag{3.3.16}$$

For each n, from Lemma 3.3.2, we can find n_1, n_2 such that $n_1 \leq n \leq n_2$,

$$\frac{n_2}{n} \cdot \frac{1}{n_2} a_d(0,n_2,0,[n_2 x]) \leq \frac{1}{n} a_d(0,n,0,[nz]) + b\left(\frac{n_2}{n} - 1\right)$$

and

$$\frac{1}{n} a_d(0,n,0,[nz]) \leq \frac{n_1}{n} \cdot \frac{1}{n_1} a_d(0,n_1,0,[n_1 x]) + b\left(1 - \frac{n_1}{n}\right),$$

where $b = -\log\kappa \in (0,\infty)$. Take $n \to \infty$. From (3.3.5) and (3.3.7) and taking $C(z) = 2\frac{1}{1-||z||}$, the limit points of $\frac{n_2}{n} - 1$ and $1 - \frac{n_1}{n}$ lie in the interval $[0, C(z)||x-z||]$ because x satisfies (3.3.16). Consequently from the last two inequalities we see that

$$I(x) \leq \underline{\lim}_{n\to\infty} \frac{1}{n} a_d(0,n,0,[nz]) + C(z)b||x-z|| \tag{3.3.17}$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{n} a_d(0,n,0,[nz]) \leq I(x) + C(z)b||x-z||. \tag{3.3.18}$$

Letting $x \to z$, we conclude that I is well-defined in the set $(\mathbb{R}^d \backslash \mathbb{Q}^d) \cap U^o$.

∎

We are now in a position to introduce the rate function of Theorem 3.1.2. We define, for each $x \in U$,

$$I_d(x) := \begin{cases} I(x) & \text{for } x \in U^o \\ \underline{\lim}_{U^o \ni y \to x} I(y) & \text{for } x \in \partial U \\ \infty & \text{for } x \notin U. \end{cases} \tag{3.3.19}$$

We will now prove that I_d satisfies the requirements of Theorem 3.1.2. By uniform ellipticity, it is clear that $I(x) \leq |\log \kappa|$ when $x \in U$. From (3.3.17) and (3.3.18), we see that I is continuous in the interior of R (in fact, Lipschitz continuous in any compact contained in U^o). These observations imply that I_d defined in (3.3.19) is bounded by $|\log \kappa|$ in U, is continuous in U^o, and is lower semi-continuous in U. The convexity of I_d is derived in a manner similar to the continuous time case. We now prove parts (i) and (ii) of Theorem 3.1.2.

Part (i) of Theorem 3.1.2 follows immediately from the definition of I_d and the fact that for open sets G, $\inf_{x\in G} I(x) = \inf_{x\in G} I_d(x)$. To prove part (ii) we first consider a compact set C contained in U^o. In this case, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq \overline{\lim_{n\to\infty}} \sup_{x\in C} \frac{1}{n} \log \pi_{0,n}(0,[nx])$$
$$= \inf_n \sup_{m\geq n} \sup_{x\in C} \frac{1}{m} \log \pi_{0,m}(0,[mx]) = \inf_n \sup_{x\in C} \sup_{m\geq n} \frac{1}{m} \log \pi_{0,m}(0,[mx])$$
$$= \inf_n \sup_{x\in C} a_n(x),$$

where we have defined for $x \in U^o$,

$$a_n(x) := \sup_{m\geq n} \frac{1}{m} \log \pi_{0,m}(0,[mx]).$$

Hence, the upper bound follows if we can show that, for any given $\epsilon > 0$,

$$\sup_{x\in C} a_n(x) \leq -\inf_{x\in C} I(x) + \epsilon$$

for large enough n. If we assume the opposite, we can find points $z_m \in C$ which have a subsequence converging to $z \in C$ and such that along this subsequence one also has that

$$\frac{1}{m} \log \pi_{0,m}(0,[mz_m]) > -I(z) + \epsilon.$$

Applying the first part of Lemma 3.3.2 gives an index $m_2 > m$ such that

$$\frac{1}{m_2} \log \pi_{0,m_2}(0,[m_2 z]) \geq \frac{m}{m_2}(-I(z)+\epsilon) - b\left(1 - \frac{m}{m_2}\right).$$

Now, since $\lim_{m\to\infty} \frac{m}{m_2} = 1$ and since by Proposition 3.3.1 $\lim_{m_2\to\infty} \frac{1}{m_2} \log \pi_{0,m_2}(0,[m_2 z]) = -I(z)$, we obtain that $-I(z) \geq -I(z) + \epsilon$, which is a contradiction.

In the general case, let $C \subset U$ be a compact set. Fix $\delta > 0$ and let $C_1 = \frac{1}{1+\delta} C$. Now C_1 is a compact set contained in U^o. Pick $\epsilon > 0$ small enough so that the closed $\epsilon-$fattening $C_2 = \overline{C_1^{(\epsilon)}}$ is still a compact set contained in U^o. Let $n_2 = \lfloor (1+\delta)n \rfloor$. Then for large enough n, $\frac{x}{n} \in C$ implies $\frac{x}{n_2} \in C_2$. By uniform ellipticity, we have that

$$P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right)\kappa^{n_2-n} = \sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x)\kappa^{n_2-n}$$
$$\leq \sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x)\pi_{n,n_2}(x,x) = \sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_n = x, X_{n_2} = x)$$
$$\leq \sum_{x \in nC\cap\mathbb{Z}^d} P^d_{0,\omega}(X_{n_2} = x) \leq P^d_{0,\omega}\left(\frac{X_{n_2}}{n_2} \in C_2\right),$$

where the last inequality is satisfied for n large enough. Then, from the first step of the proof of part (ii) of Theorem 3.1.2

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x \in C_2} I(x) + \delta b.$$

By taking $\epsilon \searrow 0$ and using compactness and the continuity of I

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x \in C_1} I(x) + \delta b.$$

Take $\delta \searrow 0$ along a subsequence δ_j. This takes C_1 to C. For each δ_j, let $z_j \in C_1 = C_1(\delta_j)$ satisfy $I(z_j) = \inf_{C_1(\delta_j)} I$. Pass to a further subsequence such that $\lim_{j\to\infty} z_j = z \in C$. Then regardless of whether z lies in the interior of U or not, by (3.3.19)

$$\underline{\lim}_{j\to\infty} I(z_j) \geq I_d(z) \geq \inf_C I_d,$$

and we get the final upper bound

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P^d_{0,\omega}\left(\frac{X_n}{n} \in C\right) \leq -\inf_{x \in C} I_d(x).$$

3.4 Proof of Theorem 3.1.2 for the nearest neighbor case

Here we consider the case in which the jump range R of the random walk $\{X_n : n \geq 0\}$ is nearest neighbor. Define the *even lattice* as $\mathbb{Z}^d_{even} := \{x \in \mathbb{Z}^d : x_1 + \ldots + x_d \,\text{is even}\}$.

Note that $\mathbb{Z}^d_{even}$ is a free Abelian group which is isomorphic to $\mathbb{Z}^d$. It therefore has a basis $f_1, \dots f_d \in \mathbb{Z}^d_{even}$ and there is an isomorphism $h : \mathbb{Z}^d_{even} \to \mathbb{Z}^d$ such that $h(f_i) = e_i$ for $1 \leq i \leq d$. It is obvious that h can be extended as an automorphism defined in $\mathbb{R}^d$. Now, note that the random walk $\{Y_n : n \geq 0\}$ defined as

$$Y_n := h(X_{2n}),$$

is a random walk in $\mathbb{Z}^d$ with finite, convex and symmetric jump range $Q = h(R)$ and such that a neighborhood of the origin is contained in its convex hull. From Theorem 3.1.2 for this class of random walks proved in section 3, it follows that $\{Y_n : n \geq 0\}$ satisfies a large deviation principle with a rate function I. From this and the linearity of h we conclude that the limit

$$I_{even}(x) := I(h(x)) = -\lim_{n\to\infty} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)])), \tag{3.4.1}$$

exists Q^d_μ-a.s, where $\pi_{n,m}(x, y)$ is the probability that the random walk $\{X_n : n \geq 0\}$ jumps from time n to time m from site x to site y. Furthermore, if $U := \{x \in \mathbb{R}^d : |x| \leq 1\}$, as in (3.3.19), one can define

$$I_{d,even}(x) := \begin{cases} I_{even}(x) & \text{for } x \in U^o \\ \underline{\lim}_{U^o \ni y \to x} I_{even}(y) & \text{for } x \in \partial U \\ \infty & \text{for } x \notin U, \end{cases} \tag{3.4.2}$$

and $\{X_{2n} : n \geq 0\}$ satisfies a large deviation principle with rate function I_{even}.

At this point, we need to extend the above large deviation principle for the walk at even times, to all times taking into account the odd number of steps of the random walk. The next lemma will be very useful for this objective. To do this, we first prove that for each $x \in \mathbb{R}^d$ and each $g \in \mathcal{H} := \left\{ \sum_{i=1}^{d} c_i x : c_i \in \{-1, 0, 1\}, x \in R \right\}$ we have that,

$$I_{even}(x) := -\lim_{n\to\infty} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)]) + g) \qquad Q^d_\mu\text{−a.s.} \tag{3.4.3}$$

Note that to prove (3.4.3), it is enough to show that for every $g \in \mathcal{H}$ we have that,

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\pi}_{0,n}(0, [nh(x)] + h(g)) = \lim_{n\to\infty} \frac{1}{n} \log \tilde{\pi}_{0,n}(0, [nh(x)]), \tag{3.4.4}$$

where $\tilde{\pi}_{n,m}(x,y)$ is the probability that the random walk $\{Y_n : n \geq 1\}$ jumps from time n to time m from site x to site y. The proof that the limit in the right-hand side of (3.4.4) exists, is a repetition of the proofs of Lemma 3.3.2 and Proposition 3.3.1, so we omit it. We just point out here that in the proof of Lemma 3.3.2 we need to replace the points $[nz]$, $[n_1x]$ and $[n_2x]$ by $[nz]+h$, $[n_1x]+h$ and $[n_2x]+h$ respectively. On the other hand, the equality in (3.4.4) is established using the uniform ellipticity of the walk and the Markov property.

Let us now see how to derive from (3.4.3) the large deviation principle for a random walk with a nearest neighbor jump range R. Note that for any subset $A \subseteq \mathbb{R}^d$ one has that

$$P_{0,\omega}\left(\frac{X_{2n+1}}{2n+1} \in A\right) = \sum_{i=1}^{2d} \pi_{0,1}(0,e_i) P_{e_i,\omega}\left(\frac{X_{2n}}{2n} \in A\right) = \sum_{i=1}^{2d} \pi_{0,1}(0,e_i) P_{0,\bar{\omega}}\left(\frac{X_{2n}}{2n} \in A - \frac{e_i}{2n}\right)$$

where $\bar{\omega} = \{\omega_n : n \geq 1\}$ and $e_{i+d} = -e_i$ for $i = 1, \ldots, d$. We will show that $P_{e_i,\omega}\left(\frac{X_{2n}}{2n} \in A\right)$ does not depend on e_i, regardless of whether A is an open subset or a closed subset of $\mathbb{R}^d$ and we will use the result obtained in the even case. It is important to note that this argument can be used, even with $\bar{\omega}$, because the limit depends only on the distribution of ω.

Now, when $A = G$, where G is an open subset of $\mathbb{R}^d$, we can follow the arguments used in the convex case, observing that for any $x \in G$ and any $i \in \{1, \ldots, d\}$, $[nx] + e_i \in nG$, for n large enough. On the other hand, if $A = C$, where C is a compact subset of U_2°, note that

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \frac{1}{2n} \log P_{0,\bar{\omega}}\left(\frac{X_{2n}}{2n} \in C - \frac{e_i}{2n}\right) &\leq \overline{\lim_{n\to\infty}} \sup_{x \in C - \frac{e_i}{2n}} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)])) \\
&= \overline{\lim_{n\to\infty}} \sup_{x\in C} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x) - h(e_i)])) \\
&\leq \overline{\lim_{n\to\infty}} \sup_{x\in C} \max_{g\in\mathcal{H}} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)]) + g)
\end{aligned}$$

However, by (3.4.3) the last expression is independent of g.

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