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On Monotonicity of Eigenvalues

por Julio Roberto Rincón Cristancho

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Contents

Introduction

The present work is a contribution to the Spectral Theory of Differential Operators and arises from the conjunction of two different subjects.

The first one covers a summary of works published in the recent years whose purpose is to develop a formulation that express in an abstract manner some classical results related with eigenvalues of second order differential operators with boundary conditions on bounded domains, and self-adjoint operators with nonempty discrete spectrum in general. This formulation introduces the use of commutators for trace identities and inequalities of self-adjoint operators, and allows to generate an abstract writing for the developments involving the specifical result of the calculus of these commutators in some typical cases. A common example is given when we consider the operators H and G where H is the Laplace operator or a Schrödinger operator on a bounded domain $\Omega \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions, and G is an operator of multiplication by the i-th coordinate x_i, $1 \leq i \leq d$. By means of the first and second commutators $[H, G]$ and $[[H, G], G]$, the procedures to derive several relationships for the eigenvalues and the discrete spectrum of H is recovered; this in an useful tool to obtain some important results on the eigenvalues, such as universal inequalities, differential inequalities on the Riesz means of eigenvalues, sharp Lieb-Thirring estimates on the moments of eigenvalues, and monotonicity of the eigenvalue moments considered as mappings depending on a parameter on which the operator H may depend. See [11], [12], [15] for reference.

The concrete example $H := -\triangle_D$, the Laplace operator defined on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with Dirichlet boundary conditions (or briefly the Dirichlet Laplacian on Ω), and $G := x_i$, is treated in [12] as a particular case of a general second order operator with a positive matrix of variable coefficients and Dirichlet boundary conditions. In this case the commutator trace identities turn to be universal inequalities for the second order operator, i.e., inequalities that do not depend on the bounded domain in which the operator is defined. The Laplace operator is the particular case in which the coefficient matrix is equal to the identity matrix, and the corresponding inequality is just the Yang inequality for the eigenvalues of the Dirichlet Laplacian, a well known universal inequality that has been obtained by applying the Rayleigh-Ritz variational criterion for the eigenvalues ([1]).

On the other hand, in [10] the trace identities can be applied to derive differential inequalities and difference inequalities for the Riesz means of the set of eigenvalues of the Dirichlet Laplacian. Moreover, the principal results presented in [11] provide a generalization of the trace identities that can be applied to nonempty continuous spectrum operators, including particularly the Schrödinger operators on the whole Euclidean space with nonpositive potentials vanishing at infinity. Subsequently to [11], J. Stubbe presented in [15] an application of the main trace inequality, in which he considers a Schrödinger operator $H = H(\alpha)$ with the presence of a positive parameter α, and derives the monotonicity behaviour with respect to α of the trace of $f(H(\alpha))$, being f a function belonging to a suitable class of trace-controllable functions ([11], [15]).

One of the specific objectives of the present work is to look for an extension to the results presented in [10], [11], [12], [15], and obtain monotonicity results for further kinds of one-parameter operators $H(\alpha)$ in addition to the one considered in [15].

The second subject covers the study of one-dimensional Schrödinger operators with symmetric double-wells potential. Speaking in general terms, the n-dimensional Schrödinger double wells operator have been a matter of particular attention in several distinct science topics; while they are ubiquitous in phisics, their main applications may be found in Quantum Field Theory, Statistical Mechanics and Molecular Chemistry. They are applied to model several problems of interaction between particles, in which there have been an interest to establish the asymptotic behaviour of these interactions when the particles move away in opposite directions to infinity. A general discussion about these operators can be found in [9]. One of the most common features that have been obtained in the study of the spectrum of operators with this kind of potentials is described as a "two-fold degeneracy"; namely, consecutive couples of eigenvalues can be associated in such a way that its spectral gap decreases to zero as the wells in the potential move far away. Some developments describing this behaviour can be found in [6] and [14].

In the particular case of the present work, our objective is to consider a one-dimensional operator of the form

$$Hu := -\frac{d^2u}{dx^2} + (V(-x-a) + V(x-a))u \tag{1}$$

in order to decide the monotonicity of the functions defined by the eigenvalues of H with respect to the parameter a, in the same spirit of the results presented in [15]. However, unlike the results of J. Stubbe, our developments involve techniques related to the Theory of Ordinary Differential Equations, instead of adopting methods with trace identities. A presentation of both methods is made in this work with the purpose of contrast them and state the posibility of combining them to extend as much as it is possible the information that may be obtained about monotonicity of traces with respect to a parameter.

The main content of this work has been subdivided into three chapters. In the first chapter we present an overview of some preliminary facts referring to the one-dimensional Schrödinger operator, with the aim of establishing a theoretical framework in order to provide some of the necessary tools for the developments in the third chapter. Most of the theory we overviewed in chapter 1 is a summary of several topics presented in [2] and we do not present any new result improving this theory in our work; however we emphasize the application of the theory to the case of Schrödinger operators with negative potential vanishing at infinity, the main case of our interest. The results we present in the initial sections of this chapter refer to the self-adjointness of the one-dimensional Schrödinger operator, and the sets of zeros of its eigenvalues. Concretely we introduce the Sturm Oscillation Theorem, that provides some information of comparison between solutions of two different Schrödinger operators, assuming information about comparison between its corresponding potentials. This result allows to conclude several facts on the structure of the sets of zeros for eigenfunctions and its relationships with the structure of the eigenvalues of the given Schrödinger operator. In the final stage of the chapter, we present some important results on the estimation for the number of eigenvalues in terms of the potential $v(x)$. This presentation involves some Perturbation Theory techniques and the Glazman variational lemma. In this approach we introduce as well the study of the Schrödinger operator over the half-line $[0, \infty)$ with Dirichlet and Neumann conditions, and we remark some facts about the contrast on the behaviour of the eigenvalues for each case.

The second chapter is focused on the main results of [11] and [15]. These are the most relevant results about trace inequalities, bearing in mind the purposes of chapter 3. The initial part of the chapter is devoted to prove the Theorem 2.1 of [11], that states a trace inequality for a self-adjoint operator having the discrete part of its spectrum below. Therefore it may be applied to operators of purely discrete spectrum, as well as Schrödinger operators with nonpositive potential vanishing at infinity. We present this result and its proof by considering distinct features from the references about trace identities we mentioned above, and we modify slightly the development of the proof in order to emphasize some important details about the properties of the function for which the inequality is applied. Then we describe the application of this trace identity to derive that the trace for an operator of the form $H(\alpha)$ is a factor for a monotone mapping that is obtained by the product of the trace with $\alpha^{\frac{d}{2}}$. This is one of the main results in [15]. Once this result has been presented, we introduce in the last part of the chapter an application of the result in [15] to study perturbations by multiplying the independent variable by the parameter α. In this case it is considered the Dilation Generator corresponding to a family of Hamiltonians related by the Unitary Dilation Group.

Finally, in the third chapter we come into the main purpose of this work. We develop here the solution for the problem of deciding the monotonicity for the trace (the sum of the eigenvalues) of the operator given by (1). In our presentation we assume that V is a piecewise continuous function with support contained in $[0, 1]$. We remark the contrast with the monotonicity problems of chapter 2 and explain the technical difficulties that does not allow to carry out the same procedures to solve this problem. Instead we make an elementary analysis of the eigenfunctions as solutions of the second order initial value problem associated to the operator. The fact that this solution belongs to C^1 implies a determined relationship between the eigenvalue λ and the parameter a. We find that in fact λ and a must satisfy an implicit relation of the form $F(\lambda) = a$ for a certain function $F(\lambda)$. Thus the eigenvalues λ as functions of a are the local inverses of the functions $F(\lambda)$, and their monotonicity with respect to a is obtained from the monotonicity of the function $F(\lambda)$. Henceforth, the core of the chapter consist on decide the intervals of monotonicity of $F(\lambda)$. Once we have determined this monotonicity, we provide some other essential facts in order to complete the description for the structure of the eigenvalues $\lambda = \lambda(a)$ for this Schrödinger operator.

In consequence, the essential results developed in the present work improve our knowledge about monotonicity with respect to a parameter that has been motivated by the results of J. Stubbe. Some natural questions that could rise from this work are referred to the possibilities of combining the monotonicity developments of chapters 2 and 3 to derive other essentially distinct monotonicity results for Schrödinger operators depending on a parameter. Another natural question can be formulated, referring to the generalization of the results of chapter 3 to the case of n-dimensional Schrödinger double-wells operators. We prospect to devote the study of this generalization in future research projects.

Finally, we refer the reader to [7] as a closely related work that allows to make an immediate extension of the results of Chapter 2 to other kind of functions different from the trace-controllable functions proposed in [11]. On the other hand, the results developed in Chapter 3 constitute an improvement to the Inverse Lieb-Thirring inequalities that are proved in [5]. The formal presentation of these comparisons belongs as well to the future research.

Chapter 1

The One-dimensional Schrödinger Operator

In this chapter we summarize some principles for the spectral theory of one-dimensional Schrödinger operators. Their study is obviously connected with the study of the Schrödinger equation, which relevance is fundamental in quantum physics; we shall be specially interested in this operator along the present work.

In many circumstances, the spectrum of a Schrödinger operator has a nonempty discrete set of eigenvalues, and the fundamental theory provides several facts describing the distribution of these eigenvalues and the behavior and structure of its eigenfunctions. The approach we adopt in the current chapter is the one developed in [2], and we quote and review its most relevant results according to our purposes. For most of these results we either ommit the proofs or provide merely outlines of them, in order to describe the procedures. The reader can find the complete details in [2], Chapter 2.

We start by considering a real-valued function $v(x)$, referred as *the potential function* or just the potential, which is supposed to be measurable and locally bounded. We introduce the operator H_0 given by

$$H_0 y(x) = -y''(x) + v(x)y(x) \tag{1.1}$$

defined on the domain $C_0^\infty(\mathbb{R})$. This is a symmetric operator in $L^2(\mathbb{R})$, but it is not necessarily an essentially self-adjoint operator on the provided domain. There exist some conditions that should be imposed on the potential $v(x)$ to conclude that H_0 is essentially self-adjoint. We provide them by means of the following theorem.

Theorem 1.0.1 (Sears) *Suppose that there exist a function $Q(x)$ such that*

$$v(x) \geq -Q(x), \quad x \in \mathbb{R} \tag{1.2}$$

$Q(x)$ being continuous, positive, even, increasing on $[0, \infty)$, and such that

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{Q(2x)}} = \infty. \tag{1.3}$$

Then the operator defined by (1.1) is essentially self-adjoint on $C_0^\infty(\mathbb{R})$.

We briefly describe the proof of this theorem, that requires to show two preliminary propositions. These propositions provide some necessary conditions for a function f to belong to the domain of H_0^*, and may be needed for other purposes on our work.

Proposition 1.0.2 *For every $f \in D_{H_0^*}$, it holds that f' is absolutely continuous and f'' is locally square-integrable.*

Proof: Consider $g = H_0^* f$. Then the equality

$$\int_{-\infty}^{\infty} \overline{f(x)}\phi''(x)\,dx = \int_{-\infty}^{\infty} (v(x)\overline{f(x)} - \overline{g(x)})\phi(x)dx, \quad \phi \in C_0^\infty(\mathbb{R})$$

holds. Let F be a second primitive function of $v \cdot f - g$; by integrating by parts we get:

$$\int_{-\infty}^{\infty} \overline{f}\phi''\,dx = \int_{-\infty}^{\infty} \overline{F}\phi''\,dx$$

It follows that the second distributional derivative of $F - f$ vanishes, and therefore, $F - f$ is a linear function, and hence $f = F - (F - f)$ should clearly satisfy the assertions of the proposition. ∎

Proposition 1.0.2 has a remarkable importance in our work, as we shall see in a subsequent chapter; nevertheless, most of the proof of Sears Theorem is covered by the following proposition:

Proposition 1.0.3 *Let v satisfy* (1.2)*, Q being a positive even function that is non-decreasing on the non-negative half axis. Then $f \in D_{H_0^*}$ implies*

$$\int_{-\infty}^{\infty} \frac{|f'(x)|^2}{Q(2x)}\,dx < \infty.$$

Outline of Proof: From the facts that f and $g := -f'' + vf$ are square-integrable functions, it can be derived elementarily the inequality

$$\frac{1}{4}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f'(x)|^2\,dx \leq \int_{-T}^{T} Q(x)|f(x)|^2\,dx + c \tag{1.4}$$

for any $T > 0$, where c is a constant that does not depend on T.

Now we take into account a result known as Bonnet Mean Value Theorem; namely, if f is continuous and K is continuous, decreasing and nonnegative on $[a, b]$, then it holds the inequality

$$\int_a^b f(x)K(x)\,dx = K(a)\int_a^\xi f(x)\,dx$$

for some value $\xi \in [a, b]$. If we denote by $\omega(T)$ the left-hand side of (1.4), and by $\chi(T)$ the right-hand side, then Bonnet Theorem gives us that

$$\begin{aligned} \int_0^T \frac{\omega' - \chi'}{Q}\,dx = \frac{1}{Q(0)}\int_0^\xi (\omega' - \chi')\,dx = \\ = \frac{1}{Q(0)}[\omega(\xi) - \chi(\xi) - \omega(0) + \chi(0)] \leq \\ \leq \frac{\chi(0) - \omega(0)}{Q(0)} = \text{const.} \end{aligned}$$

Since $\omega'(x) = \frac{1}{8}[|f'(\frac{x}{2})|^2 + |f'(-\frac{x}{2})|^2]$, $\chi'(x) = Q(x)[|f(x)|^2 + |f(-x)|^2]$, then the estimate obtained yields

$$\frac{1}{8}\int_0^T \frac{|f(\frac{x}{2})|^2 + |f'(-\frac{x}{2})|^2}{Q(x)}\,dx \le \int_0^T (|f(x)|^2 + |f(-x)|^2)\,dx + c.$$

Substituting x for $\frac{x}{2}$ and since $f \in L^2(\mathbb{R})$, we obtain the proposition. ∎

Outline of Proof for Theorem 1.0.1: The procedure to complete the proof of Sears Theorem consist on proving that the operator H_0^* is a symmetric operator, namely, for every $f_1,\, f_2 \in D_{H_0^*}$, $g_i = -f_i'' + v(x)f_i$, $i = 1, 2$, the equality

$$\int_{-\infty}^{\infty} f_1\overline{g}_2\,dx = \int_{-\infty}^{\infty} g_1\overline{f}_2\,dx$$

holds. This is accomplished as follows: set $\rho(t) = \frac{1}{\sqrt{Q(2t)}}$, $P(x) = \int_0^x \rho(\xi)\,d\xi$. Then some computations together with Proposition 1.0.3 allow to show that there exist a bound c verifying

$$\left|\int_{-T}^{T} (P(T) - P(x))[f_1\overline{g}_2 - g_1\overline{f}_2]\,dx\right| < c$$

for every $T > 0$. But by condition (1.3) it holds $P(T) \to +\infty$ as $T \to \infty$; therefore we can divide the former inequality on both sides by $P(T)$ and obtain, as $T \to \infty$,

$$\lim_{T\to\infty}\left|\int_{-T}^{T}\left(1 - \frac{P(x)}{P(T)}\right)[f_1\overline{g}_2 - g_1\overline{f}_2]\,dx\right| = 0.$$

Now let $\epsilon > 0$ and apply the square-integrability of f_i, g_i to find ω such that

$$\int_{|x|\ge\omega} (|f_1|\cdot|g_2| + |g_1|\cdot|f_2|)\,dx < \epsilon.$$

Then for every $T \ge \omega$ it holds

$$\left|\int_{-\omega}^{\omega}\left(1 - \frac{P(|x|)}{P(T)}\right)[f_1\overline{g}_2 - g_1\overline{f}_2]\,dx\right| \le \left|\int_{-T}^{T}\left(1 - \frac{P(|x|)}{P(T)}\right)[f_1\overline{g}_2 - g_1\overline{f}_2]\,dx\right| + \epsilon.$$

hence taking $T \to \infty$ on both sides and applying Dominated Convergence Theorem we have

$$\left|\int_{-\omega}^{\omega} (f_1\overline{g}_2 - g_1\overline{f}_2)\,dx\right| \le \epsilon,$$

and taking the limit as $\omega \to \infty$ we obtain $|\int_{-\infty}^{\infty} (f_1\overline{g}_2 - g_1\overline{f}_2)\,dx| \le \epsilon$ for arbitrary $\epsilon > 0$, implying $\int_{-\infty}^{\infty} (f_1\overline{g}_2 - g_1\overline{f}_2)\,dx = 0$ as desired. ∎

Remark Sufficient conditions in Sears Theorem are not necessary in general; however, if we consider the two-parameter potential given by $v(x) = a(1+|x|)^\alpha$, then the conditions of Sears Theorem characterize the values for a and α such that the corresponding operator H_0 is essentially self-adjoint. In fact, hypotheses of Sears Theorem are satisfied when $a \ge 0$ (α arbitrary), or when $a < 0$ and $\alpha \le 2$. Otherwise it can be proved that the operator H_0 is not essentially self-adjoint; this can be obtained from the asymptotic behavior of the eigenfunctions for the operator by means of Liouville's transform applied to its associated second order differential equation ([2], section 2.4, paragraph 2).

The one-dimensional self-adjoint Schrödinger operator is defined now as the closure of an operator H_0 of the form (1.1), defined on the domain C_0^∞, densely contained in the Hilbert space $L^2(\mathbb{R})$, provided that the potential $v(x)$ satisfies the hypothesis of Sears Theorem. This self-adjoint extension is denoted as H; we write $Hy(x) = -y''(x) + v(x)y(x)$, whenever the sense of this equality is clear.

1.1 Discrete Spectrum and Zeros of Eigenfunctions

We have described how the definition of H as a self-adjoint operator depends on the form of the potential. Now we present an overview of several results that are applied in [2] for the cases in which the operator H has nonempty discrete spectrum, and study some features of the relationship between the potential and the eigenvalues with their associated eigenfunctions. In most of these results we consider linear differential equations and its solutions, and we will apply frequently the existence and uniqueness of the solutions provided a fixed initial condition. This existence and uniqueness is assured by the Picard-Lindelöf Theorem in the case of $v(x)$ being a continuous potential; if $v(x)$ is merely a measurable, locally bounded potential, existence of solutions still can be assured in an extended sense by means of Carathéodory Theorem, while uniqueness can be derived in the particular case of linear differential equations by a Lipschitz condition satisfied by the first-order system associated to the equation. A detailed treatise on these topics can be found in chapters 1 and 2 of [4]. For the sake of simplicity, we assume that $v(x)$ is piecewise continuous along this section; thus existence and uniqueness is provided by Picard-Lindelöf Theorem.

We start by introducing some comparison results for solutions of the differential equations of the form

$$-y'' + vy = 0 \tag{1.5}$$

Our first presented result is called Sturm Oscillation Theorem.

Theorem 1.1.1 *Let y_1, y_2 be non-zero solutions of the differential equations*

$$-y_1'' + v_1 y_1 = 0, \quad -y_2'' + v_2 y_2 = 0. \tag{1.6}$$

If $v_1(x) \geq v_2(x)$ on a segment $[a, b]$ such that $y_1(a) = y_1(b) = 0$, then there exists $x_0 \in [a, b]$ such that $y_2(x_0) = 0$. In other words, between any two zeros of y_1 there is a zero of y_2. If, however, we additionally assume that $v_1(x) > v_2(x)$ on a subset $M \subset [a, b]$ of positive Lebesgue measure, then a point x_0 such that $y_2(x_0) = 0$ can be found even in an open interval (a, b) between two zeros of the function $y_1(x)$.

Proof: As y_1 does not identically vanishes, it may be assumed without loss that $y_1 > 0$ in (a, b). This implies that $y_1'(a) \geq 0$ and $y_1'(b) \leq 0$, but in fact, the inequalities are strict because either $y_1'(a) = 0$ or $y_1'(b) = 0$ would imply $y_1 \equiv 0$ by the Uniqueness Theorem. Reasoning by contradiction, suppose for example that $y_2(x)$ has no zeros on $[a, b]$ and, for example, $y_2(x) > 0$ for $x \in [a, b]$. It follows that

$$0 < y_1'(a)y_2(a) - y_1'(b)y_2(b).$$

On the other hand, if we multiply the first equation of (1.6) by y_2, the second equation by y_1, and integrate over $[a, b]$ the difference of the obtained equations, we get

$$0 = \int_a^b (y_1 y_2'' - y_1'' y_2)\, dx + \int_a^b (v_1 - v_2) y_1 y_2 \, dx.$$

Hence

$$0 \geq \int_a^b (y_1 y_2'' - y_1'' y_2)\,dx = \int_a^b \frac{d}{dx}(y_1 y_2' - y_1' y_2)\,dx =$$
$$= (y_1 y_2' - y_1' y_2)\Big|_a^b = y_1'(a) y_2(a) - y_1'(b) y_2(b),$$

a contradiction. For the second part of the theorem, we repeat the contradiction reasoning, considering $y_2(x) > 0$ on (a, b) instead of $[a, b]$. We will obtain now $0 \leq y_1'(a)y_2(a) - y_1'(b)y_2(b)$ and at the same time $0 > y_1'(a)y_2(a) - y_1'(b)y_2(b)$ by the integration. ∎

Corollary 1.1.2 *If $v(x) \geq 0$ for $x \in [a, b]$ in* (1.5)*, then any non-vanishing solution y has at most one zero on $[a, b]$.*

Proof: This corollary follows by using Theorem 1.1.1 to compare equation (1.5) with $-y'' = 0$, whose solution $y \equiv 1$ has no zeros at all. ∎

Corollary 1.1.2 has an important consequence for the solutions of (1.5). It will be applied in the derivation of the main propositions of our work in the chapters below.

Proposition 1.1.3 *If $y(x)$ is a non-zero solution of* (1.5)*, and if there exist N such that v satisfies $v(x) \geq 0$ for every $|x| \geq N$, then $y(x)$ has a finite number of zeros (possibly none at all).*

Proof: The set of zeros for a solution is an isolated set, for if there exist an accumulation point x_0, then Rolle's Theorem and continuity of the solution and its derivative would give that $y(x_0) = 0$ and $y'(x_0) = 0$, i.e., $y \equiv 0$ by Uniqueness Theorem. Now, Corollary 1.1.2 and the hypothesis on v implies that there exist at most one zero on each of the intervals $(-\infty, -N]$ and $[N, \infty)$. This proves the proposition. ∎

A particular case of solutions for the equation (1.5) occurs when we want to study the eigenfunctions of a Schrödinger operator; indeed, λ is an eigenvalue of the operator $Hy = -y'' + vy$ if and only if the equation

$$-y''(x) + (v(x) - \lambda)y(x) = 0 \tag{1.7}$$

has a nonvanishing solution $y(x) \in D_H \subseteq L^2(\mathbb{R})$, which is an eigenfunction of H. We focus the principal results of the present work on Schrödinger operators with bounded, negative, compact-supported potentials, i.e, potentials $v(x)$ satisfying

$$M \leq v(x) \leq 0, \;\; v(x) = 0 \;\; \text{for every} \;\; |x| \geq N \tag{1.8}$$

for some M, N. It can be shown that in this case the operator H has no positive eigenvalues (see [2]), and hence, in equation (1.7) it may be assumed $\lambda < 0$. Therefore Proposition (1.1.3) is verified for this equation, i.e., the eigenfunctions of H have a finite number of zeros. Furthermore, Proposition 1.1.5 below provides an important relationship between the number of zeros of the eigenfunctions and its associated eigenvalues. Let us introduce first a preliminary lemma:

Lemma 1.1.4 *Let $v(x)$ be such that there exist $a > 0$ verifying $v(x) \geq 0$ for every $x \geq a$; if $y(x)$ is a solution for the differential equation $-y'' + vy = 0$ such that $y(x) \geq 0$ for every $x \geq a$, and $y(x) \to 0$ as $x \to \infty$, then $y'(x) \leq 0$ for every $x \geq a$.*

Proof: It holds that $y''(x) = v(x)y(x) \geq 0$ for every $x \geq a$; hence $y'(x)$ is nondecreasing. If the claimed statement is not true, then there exist some $b \geq a$ such that $y'(b) > 0$. But on the other hand,

$$y(x) = y(b) + \int_b^x y'(t)\,dt \geq y(b) + (x-b)y'(b),$$

and this would imply $y(x) \to \infty$ when $x \to \infty$, in opposition to the hypothesis. This proves the lemma. ∎

Remark We fixed as an hypothesis $y(x) \to 0$ as $x \to \infty$ in the former lemma, but we have applied only that it does not hold $y(x) \to \infty$ when $x \to \infty$, a considerably weaker fact. Nevertheless, in [2] it is proved that if a positive solution y that satisfies $-y'' + vy = 0$, with v verifying the conditions of the lemma, then either $y(x) \to 0$ or $y(x) \to \infty$. In particular, if $y \in L^2(\mathbb{R})$, then $y(x) \to 0$ as $x \to \infty$.

Proposition 1.1.5 *Let v be such that* (1.8) *holds. If y_1, y_2 are eigenfunctions in D_H satisfying* (1.7) *with the eigenvalues λ_1, λ_2 and n_1, n_2 are its respective numbers of zeros, then $\lambda_2 > \lambda_1$ implies $n_2 > n_1$.*

Proof: Suppose that $n_1 > 0$. Let $\alpha_1 < \alpha_2 < \ldots < \alpha_{n_1}$ be all the zeros of y_1. By Theorem 1.1.1, there exist a zero of y_2 in each of the open intervals (α_i, α_{i+1}), $i = 1, \ldots, n_1 - 1$. We show further that at least one zero can be found in each of the intervals $(-\infty, \alpha_1)$ and (α_{n_1}, ∞).

Consider the interval $(\beta, +\infty)$ and, arguing by contradiction, assume that $y_2(x) > 0$ for $x \in (\beta, +\infty)$, $y_1(\beta) = 0$. We may also assume, without loss of generality, that $y_1(x) > 0$ for $x \in (\beta, \infty)$. As in the proof of Theorem 1.1.1, we obtain for any N,

$$\begin{aligned} 0 &= \int_\beta^N (y_1 y_2'' - y_1'' y_2)\,dx + \int_\beta^N (\lambda_2 - \lambda_1) y_1 y_2\,dx \\ &= (y_1 y_2' - y_1' y_2)\Big|_\beta^N + \int_\beta^N (\lambda_2 - \lambda_1) y_1 y_2\,dx \end{aligned}$$

Since by hypothesis $y_1(\beta)y_2'(\beta) - y_1'(\beta)y_2(\beta) = -y_1'(\beta)y_2(\beta) \leq 0$, it follows that

$$y_1(N)y_2'(N) - y_1'(N)y_2(N) \leq -\epsilon < 0 \tag{1.9}$$

for any $N \geq \beta + 1$ (we can take as ϵ, for example, $\int_\beta^{\beta+1} (\lambda_2 - \lambda_1) y_1 y_2\,dx$). Note that since y_1, y_2 are decaying positive solutions of the corresponding equations and $v(x) - \lambda_i \geq 0$ up to some value N_0, by Lemma 1.1.4 it follows $y_1'(N) \leq 0$ and $y_2'(N) \leq 0$ for $N \geq N_0$. Therefore (1.9) now implies that

$$y_1(N)y_2'(N) \leq -\epsilon < 0, \quad N > N_0,$$

which is impossible because $y_1(N) \to 0$, $y_2'(N) \to 0$ as $N \to \infty$ (the decay of $y_2'(N)$ follows immediatly from $y_2'(N) \leq 0$ and $y_2(N) \to 0$ as $N \to \infty$).

It remains to complete the proof for the case $n_1 = 0$, i.e., y_1 has no zeros at all. Then y_2 must have at least one zero, for if this does not holds, then the eigenfunction y_2 is not orthogonal to y_1, but at the same time they are eigenfunctions associated to distinct eigenvalues of a self-adjoint operator, and hence should have to be orthogonal. This completes the proof. ∎

Propositions 1.1.3 and 1.1.5 implies obviously the following

Corollary 1.1.6 *Let v be such that* (1.8) *holds. For every eigenvalue λ_0 of H, there exist at most a finite number of eigenvalues λ such that $\lambda < \lambda_0$.*

We conclude this overview with the following result about the multiplicity of the eigenvalues.

Proposition 1.1.7 *With the same assuptions of 1.1.6, every eigenvalue λ of H is simple.*

Proof: Let a be such that $v(x) - \lambda > 0$ for every $x > a$. Then the solution for the initial value problem with the equation (1.7) and the initial conditions $y(a) = 1$, $y'(a) = 1$ has a solution that satisfies $y(x) \to \infty$ as $x \to \infty$ (see proof of Lemma (1.1.4)); on the other hand, the space of general solutions of (1.7) is spanned by two linearly independent solutions. If there were two linearly independent eigenfunctions $y_i(x) \in L^2(\mathbb{R})$ of H, $i = 1, 2$, then $y_i(x) \to 0$ as $x \to \infty$. But these eigenfunctions must span the whole space of solutions, and this would imply that every solution $y(x)$ satisfies $y(x) \to 0$ as $x \to \infty$, which is an absurd since it does not hold for the solution obtained above. ∎

Remark One of the main purposes of the above results in the approach of [2] is to provide a proof for the fact that a Schrödinger operator has a purely discrete spectrum consisting of a monotonically increasing sequence of eigenvalues when $v(x)$ is a confining potential, i.e., satisfies $v(x) \to \infty$ as $x \to \infty$. The most common argument that may be found in literature to prove this result is to show that in this case the operator is a compact-resolvent operator, i.e., its inverse operator is a compact operator. In fact, this argument is developed in a first proof presented in [2]. However, it is obtained alternatively by the application of several results such as the ones presented above, with the advantage of getting as much information as possible about the relationship between the sets of zeros and the eigenvalues distribution. In the same spirit we employ these results to describe the structure of the discrete spectrum in the case of compact-supported potentials. Following [2], we show now how the number of eigenvalues of a Schrödinger operator H can be estimated, applying the theory overviewed above.

1.2 Estimates on the number of eigenvalues in the case of compact-supported negative potentials

We have concluded in Corollary 1.1.6 and Proposition 1.1.7 an important fact about the set of negative eigenvalues for a Schrödinger operator satisfying (1.8). These statements tell us that the mentioned set should be either finite or a monotone increasing sequence; furthermore, it can be proved that the essential spectrum of the operator H is contained in $[0, \infty)$, and hence in the case of a monotone increasing sequence, this sequence converges to zero. Now we consider the relationship between the potential and the finiteness and/or the quantity of negative eigenvalues. We begin this consideration by introducing several concepts related with variational principles and perturbation theory. Recall that given a self-adjoint operator A in a Hilbert space $\mathcal{H}$, the Spectral Theorem guarantees the existence of a spectral family of self-adjoint projection operators $\{E_\lambda : -\infty < \lambda < \infty\}$ in $\mathcal{H}$ that satisfies several properties (cf. Theorem 1.1', section 1, supplement 1 of [2]). Particularly it holds that $E_{\lambda+0} := \lim_{\mu \to \lambda^+} E_\mu = E_\lambda$ in the strong operator topology. Define the *distribution function of the spectrum* of A by

$$N(\lambda) = \dim(E_\lambda \mathcal{H}),$$

which may be sometimes written as $N(\lambda; A)$ when the involved operator has to be specified. Notice that $N(\lambda)$ may be infinite. This function verifies $N(\lambda - 0) := \lim_{\mu \to \lambda^-} N(\mu) = \dim(E_{\lambda-0}\mathcal{H})$, and if $N(\lambda + 0) := \lim_{\mu \to \lambda^+} N(\mu)$ is finite, then $N(\lambda + 0) = N(\lambda)$. The distribution function of eigenvalues verify an important variational formula provided by the following

Lemma 1.2.1 (Glazman) *Let $\mathcal{D}$ be a subspace contained in the domain of A, $\mathcal{D}_A$, such that A is essentially self-adjoint on $\mathcal{D}$. Then for any $\lambda \in \mathbb{R}$ we have*

$$N(\lambda - 0) = \sup\{\dim L : L \textit{ is subspace of } \mathcal{D},\ \langle Au, u\rangle < \lambda\langle u, u\rangle, u \in L \setminus \{0\}\}. \tag{1.10}$$

If A is a semibounded operator, sometimes it is necessary to consider the quadratic or sesquilinear associated form of A, which is considered on a wider domain than $\mathcal{D}_A$. Indeed, let us assume first that $A \geq I$, that is, $\langle Au, u\rangle \geq \langle u, u\rangle$ for any $u \in \mathcal{D}_A$. Then set

$$A(u, v) = \langle Au, v\rangle \text{ for } u,\, v \in \mathcal{D}_A.$$

The form $A(\cdot,\cdot)$ defines a scalar product on $\mathcal{D}_A$ and the corresponding norm $\|\cdot\|_A$ satisfies $\|u\|_A \geq \|u\|$, where $\|\cdot\|$ is the norm in $\mathcal{H}$. Denote by $\mathcal{H}_A$ the completion of $\mathcal{D}_A$ with respect to $\|\cdot\|_A$. Then it can be proved that $\mathcal{H}_A$ is naturally embedded in $\mathcal{H}$ and the image of this embedding is dense in $\mathcal{H}$. If we do not have $A \geq I$ but we have $A \geq -\alpha I$ instead for some $\alpha \in \mathbb{R}$, then we define $\hat{A} := A + (\alpha + 1)I$ and apply the same construction to obtain a domain $\mathcal{H}_{\hat{A}}$, which we denote as $\mathcal{H}_A$ as well. We define on this domain the sesquilinear form

$$A(u, v) = \hat{A}(u, v) - (\alpha + 1)(u, v)$$

obtained from the form $A(\cdot,\cdot)$ defined on $\mathcal{D}_A$ by extension by continuity. This allows to formulate a modification of Lemma 1.2.1 given by the following:

Lemma 1.2.2 (Glazman modified) *Let A be a self-adjoint operator that is semibounded from below, and $\mathcal{D}$ a dense subspace of $\mathcal{H}_A$ (with respect to the norm in $\mathcal{H}_A$). Then for any $\lambda \in \mathbb{R}$ we have*

$$N(\lambda - 0) = \sup\{\dim L : L \textit{ is subspace of } \mathcal{D},\ A(u, u) < \lambda\langle u, u\rangle, u \in L \setminus \{0\}\}. \tag{1.11}$$

For the proof of these lemmas we refer to [2], section S1.3 in supplement 1.

Another important fact we apply in the statements below is a theorem on a perturbation of an isolated eigenvalue of finite multiplicity for a self-adjoint operator, namely, Theorem 3.1 from section S1.3 in supplement 1 of [2]. In this theorem it is considered an operator-valued function $A(\epsilon)$, $\epsilon \in (a, b)$, where $A(\epsilon)$ is a linear operator on a fixed Hilbert space $\mathcal{H}$ for every ϵ, and it may be written $A(\epsilon) = A(\epsilon_0) + G(\epsilon)$ for some $\epsilon_0 \in (a, b)$, where $G(\epsilon)$ is a symmetric operator that satisfy an analicity condition. It is supposed as well that there is given an isolated eigenvalue $\lambda_0 \in \mathbb{R}$ of $A(\epsilon_0)$. The conclusion of the Theorem provides, among several properties, that there exist analytic vector functions $\psi_j(\epsilon)$, $j = 1, \ldots, m$ and scalar analytic functions $\lambda_j(\epsilon)$, $j = 1, \ldots, m$ defined on a neighbourhood $U \subseteq \mathbb{R}$ of ϵ_0 such that

$$A(\epsilon)\psi_j(\epsilon) = \lambda_j(\epsilon)\psi_j(\epsilon), \;\; j = 1, \ldots, m, \;\; \epsilon \in U, \tag{1.12}$$

and $\{\psi_1(\epsilon), \ldots, \psi_m(\epsilon)\}$ is an orthonormal basis in the space $E_{(\lambda_0-\delta,\lambda_0+\delta)}\mathcal{H}$ for some fixed $\delta > 0$ for any $\epsilon \in U$. Here $E_{(s,t)} = E_t - E_s$, where $\{E_\lambda\}_\lambda$ is the spectral family of $A(\epsilon)$ for each $\epsilon \in U$

fixed. Moreover, if $\psi(\epsilon)$ is an arbitrary normalized function satisfying (1.12), then the derivative $\lambda_j'(\epsilon) = \frac{d\lambda_j(\epsilon)}{d\epsilon}$ satisfies the equation

$$\lambda_j'(\epsilon) = \langle A'(\epsilon)\psi_j(\epsilon), \psi_j(\epsilon)\rangle, \quad j = 1, \ldots, m, \quad \epsilon \in U \tag{1.13}$$

(this formula is also known as Feynman-Hellmann Theorem). The definition of the operator derivative $A'(\epsilon)$ in this theorem allows to calculate it formally as usual, at least in the situations we consider below.

Now we carry out the procedure of estimate for the function $N(\lambda)$ in our cases. We need to consider at first operators defined just on the half-line $\mathbb{R}^+ := [0, \infty)$ with Dirichlet or Neumann boundary conditions at zero. The result for an operator in the whole line is derived by splitting it in operators on half-lines.

Consequently, let us consider the operator

$$Hy = -y'' + v(x)y \tag{1.14}$$

defined on functions $y(x)$ on the interval $\mathbb{R}^+$ with the Dirichlet boundary condition

$$y(0) = 0. \tag{1.15}$$

Specifically, it is defined as the closure operator of an essentially self-adjoint operator H_0 in the domain

$$D := \{y \in C^\infty(\mathbb{R}^+) : y \text{ satisfies (1.15) and has compact support contained in } \mathbb{R}^+\}. \tag{1.16}$$

All the propositions in the last section are verified with obvious modifications for this case; In particular, if $v(x)$ is bounded from below, then the operator H defined by (1.14) is a self-adjoint operator by a straightforward adaption of Sears Theorem taking into account the domain of definition $\mathbb{R}^+$ and the boundary condition (1.15). Let $N\_(H)$ be the number of negative eigenvalues of H, i.e., $N\_(H) = N(-0; H)$. Then we have the following

Theorem 1.2.3 *Let v be negative and bounded from below, and suppose that the function $v(x)$ is continuous on $[0, \infty)$. Then*

$$N\_(H) \leq \int_0^\infty x|v(x)|\,dx. \tag{1.17}$$

Proof (Outline): Assume $v \not\equiv 0$; otherwise the assertion is trivial. We introduce the parameter τ, $0 \leq \tau \leq 1$, and consider a family of operators

$$H_\tau y = -y'' + \tau v(x)y.$$

By Corollary 1.1.6 and Proposition 1.1.7, it holds that for any τ, the spectrum of H_τ consists of discrete simple eigenvalues which we enumerate in ascending order:

$$\lambda_1(\tau) < \lambda_2(\tau) < \ldots < \lambda_n(\tau) < \ldots$$

By our preliminaries on perturbation theory and relationship (1.13), it follows that $\lambda_n(\tau)$ is an analytic function of τ, and

$$\lambda_n'(\tau) = \int_0^\infty v(x)|f_n(\tau, x)|^2 dx < 0,$$

where $f_n(\tau, x)$ is the normalized eigenfunction of H with eigenvalue $\lambda_n(\tau)$. Note that $\lambda_n(\tau)$ is defined just in a half-open interval $(\tau_0(n), 1]$, passing into the positive part of the spectrum for $\tau = \tau_0(n)$. For a fixed $\mu_0 < 0$ we consider the number N_{μ_0} of eigenvalues of H that are less than μ_0. It is clear that $N_{\mu_0} \to N\_(H)$ as $\mu_0 \to 0$. Letting τ decrease from 1 to 0 we see that for any n, τ_n can be found such that $\lambda_n(\tau_n) = \mu_0$ provided that $\lambda_n(1) < \mu_0$ (and only in this case). We see that the eigenvalues $\lambda_n = \lambda_n(1)$ such that $\lambda_n < \mu_0$ are in one-to-one correspondence with numbers τ such that $0 < \tau \leq 1$ and there exist a function $y(x) \in L^2(\mathbb{R})$ satisfying

$$-y'' - \mu_0 y = -\tau v(x) y \tag{1.18}$$

Let L denote the differential operator $-d^2/dx^2 - \mu_0$ defined on $L^2(\mathbb{R}^+)$ with the boundary condition (1.15) and with the natural domain corresponding to (1.15). Then L is invertible and L^{-1} is expressed in the form

$$L^{-1} f = \int_0^\infty K(x, \xi) f(\xi) \, d\xi$$

where $K(x, \xi)$ can be found by determining $y = L^{-1} f$ from the equation $-y'' - \mu_0 y = f$ by the method of variation of parameters, taking into account (1.15) and the requirement that $y(x) \in L^2(\mathbb{R}^+)$. We obtain the following expression for $K(x, \xi)$:

$$K(x, \xi) = \theta(\xi - x) \frac{\sinh \sqrt{-\mu_0} x}{\sqrt{-\mu_0}} e^{-\sqrt{-\mu_0} \xi} + \theta(x - \xi) \frac{\sinh \sqrt{-\mu_0}}{-\mu_0} e^{-\sqrt{-\mu_0} x},$$

where $\theta(x) = 1$ for $z \geq 0$ and $\theta(z) = 0$ for $z < 0$. We can now rewrite (1.18) in the form

$$y = \tau L^{-1}[(-v(x)) y],$$

from which we see that the required τ_k are the reciprocals of the eigenvalues of the integral operator K_1 with kernel

$$K_1(x, \xi) = -K(x, \xi) v(\xi).$$

Now we state without proof the following auxiliary lemma, whose proof can be found in [2]

Lemma 1.2.4 *If v is integrable in $[0, \infty)$, then the operator K_1 has not more than a countable set of non-zero eigenvalues λ_k, $k = 1, 2, \ldots$, all of them positive and simple, and*

$$\sum_{k=1}^{\infty} \lambda_k = \int_0^\infty K_1(x, x) \, dx.$$

Using this lemma we finish the proof of Theorem 1.2.3. For N_{μ_0} the following estimate is valid

$$N_{\mu_0} \leq \sum_{\tau_k \leq 1} \frac{1}{\tau_k} = \sum_{\lambda_k \geq 1} \lambda_k \leq \sum_{k=1}^{\infty} \lambda_k = \int_0^\infty \frac{\sinh \sqrt{-\mu_0} x}{\sqrt{-\mu_0}} e^{-\sqrt{-\mu_0} x} |v(x)| \, dx =$$

$$= \int_0^\infty \frac{1 - e^{-2\sqrt{-\mu_0} x}}{2\sqrt{-\mu_0}} |v(x)| \, dx \leq \int_0^\infty x |v(x)| \, dx,$$

where we have used the obvious concavity inequality $1 - e^{-y} \leq y$ for $y \geq 0$. Since the right hand side of this estimate does not depend on μ_0, we have proved (1.17). ∎

By means of Theorem 1.2.3, we have obtained a result of estimate for the number of eigenvalues in the case of Dirichlet boundary condition. We have to obtain now the corresponding results for the case of Neumann boundary condition and the operator in the whole line. The strategy to derive this results consist of applying Theorem 1.2.3 and the Glazman Lemma; in this sense, the estimate in the Dirichlet case is our most essential result, while the results of the other cases are merely consequences of the Dirichlet case by means of the Glazman Lemma.

Indeed, let us consider an operator H of the form (1.14) on $\mathbb{R}^+ = [0, \infty)$, with the Neumann boundary condition at zero, given by

$$y'(0) = 0. \tag{1.19}$$

As well as in the Dirichlet case, it can be verified by a modification of Sears Theorem that H is an essentially self-adjoint operator in the domain

$$D_N := \{y \in C^\infty(\mathbb{R}^+) : y \text{ satisfies (1.19) and has compact support contained in } \mathbb{R}^+\}, \tag{1.20}$$

and hence, its closure is a self-adjoint operator. Let us denote this self-adjoint operator by H_N, and reserve the notation H for the self-adjoint operator with Dirichlet boundary condition. We consider now the number of eigenvalues $N\_(H_N)$, having this notation the same meaning as $N\_(H)$, replacing H by H_N.

Theorem 1.2.5 *Let v be bounded from below, and suppose that the function $v\_(x)$ is continuous on $[0, \infty)$. Then the estimate*

$$N\_(H_N) \leq 1 + \int_0^\infty x|v\_(x)|\,dx \tag{1.21}$$

holds.

Proof: By Theorem 1.2.3 it suffices to demonstrate that $N\_(H_N) \leq 1 + N\_(H)$, where H is the same operator but with the Dirichlet boundary condition (1.15). A more general inequality is actually true:

$$N(\lambda; H_N) \leq 1 + N(\lambda; H) \text{ for all } \lambda \in \mathbb{R}, \tag{1.22}$$

where $N(\lambda; H_N)$, $N(\lambda; H)$ are the spectrum distribution functions of H_N and H respectively. It suffices to verify (1.22), substituting $\lambda - 0$ for λ (for any λ). Let D and D_N be given by (1.16) and (1.20). Then according to Lemma 1.2.1,

$$N(\lambda - 0; H_N) = \sup\{\dim L : L \text{ is subspace of } D_N,\ \langle Hu, u\rangle < \lambda\langle u, u\rangle, u \in L \setminus \{0\}\}, \tag{1.23}$$

and

$$N(\lambda - 0; H) = \sup\{\dim \tilde{L} : \tilde{L} \text{ is subspace of } D,\ \langle Hu, u\rangle < \lambda\langle u, u\rangle, u \in \tilde{L} \setminus \{0\}\}. \tag{1.24}$$

Therefore it suffices to prove that for any L subspace of D_N such that

$$\langle Hu, u\rangle < \lambda\langle u, u\rangle, \;\; u \in L \setminus \{0\}$$

there exist $\tilde{L}$ subspace of D such that

$$\langle Hu, u\rangle < \lambda\langle u, u\rangle, \;\; u \in \tilde{L} \setminus \{0\}$$

and $\dim L \leq 1 + \dim \tilde{L}$. But this can be done setting

$$\tilde{L} = \{u : u \in L, u(0) = 0\}.$$

Since $\tilde{L} \subseteq L$ and $\dim L/\tilde{L} \leq 1$, $\tilde{L}$ evidently satisfies all the necessary requirements. ∎

We included the proof of the former theorem as it has been presented in [2] in order to illustrate the application of Glazman Lemma for the development of results involving numbers of eigenvalues. Further conclusions can be obtained by the application of this result; for example, one could get the inequality $N(\lambda; H) \leq 1 + N(\lambda; H_N)$ by interchanging the roles of Dirichlet and Neumann in the proof of Theorem 1.2.5. Nevertheless it may be applied the modified Glazman Lemma (Lemma 1.2.2) to obtain the stronger inequality

$$N(\lambda; H) \leq N(\lambda; H_N). \tag{1.25}$$

See [2] for its proof. Inequalities (1.22) and (1.25) imply in particular an important relationship between the eigenvalues of the operator H with the boundary conditions (1.15) and (1.19) which we call respectively the Dirichlet and Neumann eigenvalues of H; indeed, if we consider the Dirichlet eigenvalues as a sequence of the form

$$\lambda_1(H) \leq \lambda_2(H) \leq \lambda_3(H) \leq \dots,$$

and the Neumann eigenvalues as a sequence of the form

$$\lambda_1(H_N) \leq \lambda_2(H_N) \leq \lambda_3(H_N) \leq \dots,$$

then (1.22), (1.25) hold if and only if

$$\lambda_1(H_N) \leq \lambda_1(H) \leq \lambda_2(H_N) \leq \lambda_2(H) \leq \lambda_3(H_N) \leq \lambda_3(H) \leq \dots \tag{1.26}$$

These inequalities do not have to be strict if we consider the eigenvalues in a generalized sense; nevertheless, in the case of our Schrödinger operators, the eigenvalues are simple by Proposition 1.1.7 and hence the inequalities are strict.

We conclude this section by presenting a third theorem of estimate for the number of eigenvalues, applied to operators in the whole line.

Theorem 1.2.6 *Let H be an operator of the form* (1.14) *on* $(-\infty, \infty)$ *and suppose that the potential v is locally bounded, bounded from below, and has continuous negative part $v_-(x)$. Let $N_-(H)$ be the number of its negative eigenvalues. Then*

$$N_-(H) \leq 1 + \int_{-\infty}^{\infty} |x||v_-(x)|\, dx. \tag{1.27}$$

As usual, the thechniques developed for the proof of this theorem (see [2]) involve Glazman Lemma, perhaps in a subtler sense than the former results that required its application. Nevertheless this result can be easily recovered if one has that the potential $v(x)$ is an even function. In fact, we prove in Chapter 3 that in this case the operator H may be taken as the direct sum of a Dirichlet and a Neumann operators on $[0, \infty)$; thus the eigenvalue number $N_-(H)$ in this case is not greater than the sum of the right hand sides of equations (1.17) and (1.21), which by the assumtion on $v(x)$ is clearly equal to the right hand side of (1.27).

Chapter 2

The Harrell-Stubbe Trace Inequality for Commutators of Self-adjoint Operators

In this chapter we shall review the main theorem in [11], which appears below as theorem 2.0.7. This is one of the E. Harrell and J. Stubbe's publications about trace inequalities for self-adjoint operators; it is an abstract inequality involving commutators of self-adjoint operators which can be applied on concrete examples of differential operators in order to recover spectral bounds for eigenvalue gaps and Riesz means, and it is also the principal tool in [15] to derive the main monotonicity results for eigenvalue moments of one-parameter Schrödinger operators that we consider in our work and shall be treated in detail in the next chapter.

In what follows, it will be assumed that H is a self-adjoint operator with domain $\mathcal{D}_H$ on a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$; also, it will be supposed that H has nonempty point spectrum, and $\mathcal{J}$ is a finite-dimensional subspace of $\mathcal{H}$ spanned by an orthonormal set $\{\phi_j\}$ of eigenfunctions of H. We denote E_A the spectral projector associated to H and the Borel set A, and $J := \{\lambda_j : H\phi_j = \lambda_j\phi_j\}$. With this conditions we have the following:

Theorem 2.0.7 *Let H and G be self-adjoint operators with domains $\mathcal{D}_H$ and $\mathcal{D}_G$ such that $G(\mathcal{J}) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_G$. Then, for any real-valued C^1-function f defined on the smallest closed interval containing J such that its derivative f' is a concave function,*

$$\begin{aligned}\frac{1}{2}\sum_{\lambda_j\in J} & f(\lambda_j)\,\langle[G,[H,G]]\phi_j,\phi_j\rangle + f'(\lambda_j)\|[H,G]\,\phi_j\|^2 \\ &\leq \sum_{\lambda_j\in J}\int (f(\lambda_j)+\frac{1}{2}f'(\lambda_j)(\kappa-\lambda_j))(\kappa-\lambda_j)\,|\langle G\phi_j, dE_\kappa E_{J^c}G\phi_j\rangle|^2 .\end{aligned} \tag{2.1}$$

The proof we present for this theorem emphasizes the structures from the proofs of the corresponding theorems in [11], [12] and [15]; we also carry out some straightforward details that were omitted in [11].

Proof: Let us note first that $[H,G]\,\phi_j = (H-\lambda_j)G\phi_j$ for any eigenfunction ϕ_j. This implies

$$\langle G\,[H,G]\,\phi_j,\phi_j\rangle = \langle G(H-\lambda_j)G\phi_j,\phi_j\rangle = \langle (H-\lambda_j)G\phi_j, G\phi_j\rangle \tag{2.2}$$

By the spectral resolution of the last term in (2.2) we have

$$\begin{aligned}\langle G\,[H,G]\,\phi_j,\phi_j\rangle = \langle (H-\lambda_j)G\phi_j, G\phi_j\rangle &= \int\limits_{\sigma(H)} (\kappa-\lambda_j) dG^2_{j\kappa} \\ &= \sum_{\lambda_k\in J} (\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2 + \int\limits_{J^c} (\kappa-\lambda_j)\, dG^2_{j\kappa}\end{aligned} \tag{2.3}$$

where $dG^2_{j\kappa} = |\langle dE_\kappa G\phi_j, G\phi_k\rangle|$. In particular this implies that $\langle G\,[H,G]\,\phi_j,\phi_j\rangle$ is real.
On the other hand we have:

$$\begin{aligned}\langle G\,[H,G]\,\phi_j,\phi_j\rangle &= \langle [G,[H,G]]\,\phi_j,\phi_j\rangle + \langle [H,G]\,G\phi_j,\phi_j\rangle \\ &= \langle [G,[H,G]]\,\phi_j,\phi_j\rangle + \langle \phi_j,[H,G]\,G\phi_j\rangle \qquad (2.4)\\ &= \langle [G,[H,G]]\,\phi_j,\phi_j\rangle - \langle G\,[H,G]\,\phi_j,\phi_j\rangle \qquad (2.5)\end{aligned}$$

The step (2.4) is due to the conjugation of the real term according to (2.3), and the step (2.5) is due to the skew-symmetry of $[H,G]$ and the symmetry of G. Thus by summing on both sides the last term in (2.5) and replacing with (2.3), we have

$$\frac{1}{2}\,\langle [G,[H,G]]\,\phi_j,\phi_j\rangle = \sum_{\lambda_k\in J} (\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2 + \int\limits_{J^c} (\kappa-\lambda_j)\, dG^2_{j\kappa}$$

Multiplying on both sides of this equation by $f(\lambda_j)$ and summing over $j\in J$ we get

$$\begin{aligned}&\frac{1}{2}\sum_{\lambda_j\in J} f(\lambda_j)\,\langle [G,[H,G]]\,\phi_j,\phi_j\rangle = \\ &\qquad \sum_{\lambda_j\in J}\sum_{\lambda_k\in J} f(\lambda_j)\,(\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2 + \sum_{\lambda_j\in J}\int\limits_{J^c} f(\lambda_j)\,(\kappa-\lambda_j)\, dG^2_{j\kappa}\end{aligned} \tag{2.6}$$

but the term $(\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2$ is skew-symmetric in j and k, so we can symmetrize the double sum to get

$$\begin{aligned}\sum_{\lambda_j\in J}\sum_{\lambda_k\in J} f(\lambda_j)\,(\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2 &= -\frac{1}{2}\sum_{\lambda_j\in J}\sum_{\lambda_k\in J} (f(\lambda_k)-f(\lambda_j))\,(\lambda_k-\lambda_j)\,|\langle G\phi_j,\phi_k\rangle|^2 \\ &= -\frac{1}{2}\sum_{\lambda_j\in J}\sum_{\lambda_k\in J} \frac{f(\lambda_k)-f(\lambda_j)}{\lambda_k-\lambda_j}\,(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2\end{aligned} \tag{2.7}$$

By the Fundamental Theorem of Calculus, the ratio can be written as

$$\frac{f(\lambda_k)-f(\lambda_j)}{\lambda_k-\lambda_j} = \int\limits_0^1 f'(s\lambda_k+(1-s)\lambda_j)\,ds,$$

so we replace it in (2.7) and then in (2.6) to get

$$\frac{1}{2}\sum_{\lambda_j\in J} f(\lambda_j)\,\langle[G,[H,G]]\,\phi_j,\phi_j\rangle+\frac{1}{2}\sum_{\lambda_j\in J}\sum_{\lambda_k\in J}\left(\int_0^1 f'(s\lambda_k+(1-s)\lambda_j)\,ds\right)(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2$$
$$=\sum_{\lambda_j\in J}\int_{J^c} f(\lambda_j)\,(\kappa-\lambda_j)\,dG^2_{j\kappa}. \tag{2.8}$$

Now we apply the concavity of f' to estimate the definite integral for below; a term which is the discrete part of the spectral resolution for $\|[H,G]\,\phi_j\|^2$ appears. Then we add the remaining continuous term on both sides and this will give us the inequality. Indeed, because of the concavity of f', we have

$$\begin{aligned}
&\sum_{\lambda_j\in J}\sum_{\lambda_k\in J}\left(\int_0^1 f'(s\lambda_k+(1-s)\lambda_j)\,ds\right)(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2\\
&\quad\geq\sum_{\lambda_j\in J}\sum_{\lambda_k\in J}\frac{1}{2}\{f'(\lambda_k)+f'(\lambda_j)\}\,(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2\\
&\quad=\sum_{\lambda_j\in J} f'(\lambda_j)\sum_{\lambda_k\in J}(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2
\end{aligned}\tag{2.9}$$

The last equality in (2.9) is due to the symmetry of the term $(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2$.

By considering the inequality (2.9), from (2.8) we derive:

$$\begin{aligned}
&\frac{1}{2}\sum_{\lambda_j\in J}\{f(\lambda_j)\,\langle[G,[H,G]]\,\phi_j,\phi_j\rangle+f'(\lambda_j)\sum_{\lambda_k\in J}(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_k\rangle|^2\}\\
&\quad\leq\sum_{\lambda_j\in J}\int_{J^c} f(\lambda_j)\,(\kappa-\lambda_j)\,dG^2_{j\kappa}.
\end{aligned}\tag{2.10}$$

Now we consider the spectral resolution of $\|[H,G]\,\phi_j\|^2$:

$$\begin{aligned}
\|[H,G]\,\phi_j\|^2&=\langle[H,G]\,\phi_j,[H,G]\,\phi_j\rangle\\
&=\langle(H-\lambda_j)G\phi_j,(H-\lambda_j)G\phi_j\rangle\\
&=\int_{\kappa\in\sigma(H)}(\kappa-\lambda_j)^2\,dG^2_{j\kappa}\\
&=\sum_{\lambda_k\in J}(\lambda_k-\lambda_j)^2\,|\langle G\phi_j,\phi_j\rangle|^2+\int_{\kappa\in J^c}(\kappa-\lambda_j)^2\,dG^2_{j\kappa}
\end{aligned}$$

so, by adding on both sides of (2.10) the remaining term

$$\frac{1}{2}\sum_{\lambda_j\in J}\int_{\kappa\in J^c} f'(\lambda_j)(\kappa-\lambda_j)^2\,dG^2_{j\kappa},$$

we obtain the conclusion of the theorem. ∎

Remark Formula (2.1) involves the whole spectrum of H. Recall that the set J is a set of eigenvalues; the main purpose of this formula is to derive trace inequalities for differential operators that involves only the discrete part of the spectrum. It often happens that the right-hand side of (2.1) is nonpositive; one can get it by proving that the factor $f(\lambda_j) + \frac{1}{2}f'(\lambda_j)(\kappa - \lambda_j)$ at the integrand is nonpositive. Nevertheless there is not an explicit argument in [11] or [15] for this last fact, although there is an isolated proposition presented in [11] from which it can be easily derived. We recall first the proposition 1.2 from [11].

Proposition 2.0.8 *If the function $h(x)$ is concave for $0 < x < x_0$, then*

$$xh(x) - 2\int_0^x h(s)\,ds$$

is concave on the same interval.

Although we ommit the general proof of this fact, we observe that it turns out immediatly in the case of $h \in C^2([0, x_0])$ by calculating the second derivative.

This allows us to prove the following

Lemma 2.0.9 *Let f be a C^1 real function with support on the negative real axis such that f' is concave. Then for every $\kappa > 0$ and every every $x \leq 0$ one has*

$$f(x) + \frac{1}{2}f'(x)(\kappa - x) \leq 0$$

Proof: Since f has support on the negative real axis, one has $f(0) = f'(0) = 0$. But f' is concave, and from this and $f'(0) = 0$ it turns out that $f'(x) \leq 0$ on the real line. Thus, it suffices to show that

$$f(x) - \frac{1}{2}xf'(x) \leq 0. \tag{2.11}$$

Since $f(0) = 0$, (2.11) can be written as

$$2\int_0^x f'(t)dt - xf'(x) \leq 0. \tag{2.12}$$

As a function of x, the left-hand side of this inequality has support on the negative real axis; we claim this function is concave on $(-\infty, 0]$, so it will be non-positive also, because it is C^1 over $\mathbb{R}$ (defining it as 0 over the positive real axis). Indeed, by taking $h(x) = f'(-x)$ and making the change of variables $t = -s$ at the integral, the right hand side of (2.12) turns to be

$$-2\int_0^{-x} h(s)\,ds - xh(-x)$$

Hence our claim is equivalent to proposition 2.0.8, by replacing $-x$ by $x \geq 0$ (Note that this last change of variables does not affect the concavity). ∎

Example: We consider in this example an application of theorem 2.0.7 to derive the main results of [15]. This is a work of J. Stubbe, in which several results about monotonicity of eigenvalue moments are presented and provides a new approch to obtain sharp Lieb-Thirring inequalities for such eigenvalue moments.

Let $\alpha > 0$ be a fixed parameter, and let $V(x)$ a nonpositive measurable bounded potential on $\mathbb{R}^d$ that vanishes at infinity. Consider the operators given by $H(\alpha) = -\alpha\triangle + V(x)$ and $G_i = x_i$ the operator of multiplication by the i-th coordinate, $i = 1, \dots, d$, defined on the underlying Hilbert space $L^2(\mathbb{R}^d)$. The set J of eigenvalues of the operator $H(\alpha)$ is either a finite set of negative eigenvalues, or a negative sequence of eigenvalues that converges to zero (for the case $d = 1$ we overviewed the theory that assures this fact in chapter 1, but this also holds for $d > 1$; see chapter 3 of [2]). In both cases this set lies in the bottom of the complete spectrum. Let us apply theorem 2.0.7 taking $H(\alpha)$ as H and G_i as G. The commutator formulas in this case are given by

$$[H, G]\phi_j = -2\alpha \frac{\partial \phi_j}{\partial x_i}$$

and

$$[G, [H, G]] = 2\alpha.$$

Therefore, if f is a function that satisfies the hypothesis of lemma 2.0.9, then equation (2.1) turns to be

$$\sum_{\lambda_j \in J} \alpha f(\lambda_j) + 2\alpha^2 f'(\lambda_j) \int \left| \frac{\partial \phi_j}{\partial x_i} \right|^2 dx \leq \sum_{\lambda_j \in J} \int (f(\lambda_j) + \frac{1}{2} f'(\lambda_j)(\kappa - \lambda_j))(\kappa - \lambda_j) \left| \langle G\phi_j, dE_\kappa E_{J^c} G\phi_j \rangle \right|^2$$

but in the right-hand side integral, the spectral measure has support only for $\kappa \in J^c$, and since $\lambda_j \in J$ for every j, it can be assumed that $\kappa - \lambda_j \geq 0$ for every j. Therefore, lemma 2.0.9 implies that the left-hand side integral is nonpositive. Making explicit the dependence on α, we have obtained for every $i = 1, \dots, d$,

$$\sum_{\lambda_j(\alpha) \in J} \alpha f(\lambda_j(\alpha)) + 2\alpha^2 f'(\lambda_j(\alpha)) \int_{\mathbb{R}^d} \left| \frac{\partial \phi_j}{\partial x_i} \right|^2 dx \leq 0.$$

We sum this equation on i and multiply on both sides by $\frac{1}{2}\alpha^{\frac{d}{2}-2}$ to obtain

$$\frac{d}{2} \alpha^{\frac{d}{2}-1} \sum_{\lambda_j(\alpha)<0} f(\lambda_j(\alpha)) + \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} f'(\lambda_j(\alpha)) T_j(\alpha) \leq 0 \tag{2.13}$$

where

$$T_j(\alpha) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial \phi_j}{\partial x_i} \right|^2 dx = \int_{\mathbb{R}^d} |\nabla \phi_j|^2 dx.$$

Taking into account the Feynman-Hellmann theorem (see equation (1.13) in chapter 1 and remarks about it), it holds the formula $\frac{d\lambda_j(\alpha)}{d\alpha} = T_j(\alpha)$ if there is no degeneracy of the eigenvalue. Therefore, formally speaking, the left-hand side of (2.13) is the derivative of the mapping

$$\alpha \mapsto \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} f(\lambda_j(\alpha)) \tag{2.14}$$

and hence (2.13) implies that this mapping is non-increasing. The remaining details can be found in [15]

2.1 Dilation Generator

We have finished the former section with an example related to the main results in [15]. In this section we present an extension of these results replacing the operator

$$H(\alpha) = -\alpha\triangle + V(x) \tag{2.15}$$

by a one-parameter operator of the form

$$K(\alpha) := -\triangle + V\left(\frac{x}{\alpha}\right) \tag{2.16}$$

where $V(x)$ is a measurable bounded nonpositive potential that vanishes at infinity on $\mathbb{R}^d$, and $K(\alpha)$ is defined on the underlying Hilbert space $L^2(\mathbb{R}^d)$ for every $\alpha > 0$. For explicit reference we recall the mentioned results of [15] in the following theorems. We denote by $\lambda_j(\alpha)$ the eigenvalues of the operator (2.15).

Theorem 2.1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with support on the negative half axis such that f' is concave. If $V \leq 0$ vanishes at infinity, then the mapping*

$$\alpha \mapsto \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} f(\lambda_j(\alpha)) \tag{2.17}$$

is non-increasing for $\alpha > 0$. In particular, for all $\sigma \geq 2$,

$$\alpha \mapsto \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} (-\lambda_j(\alpha))^{\sigma} \tag{2.18}$$

is non-increasing for $\alpha > 0$. Consequently,

$$\alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} (-\lambda_j(\alpha))^{\sigma} \leq L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} (-V(x))^{\sigma+\frac{d}{2}}\, dx. \tag{2.19}$$

Theorem 2.1.2 *Let $H(\alpha)$ be given by (2.15) and let $V(x)$ be a confining potential, such that for all $t > 0$,*

$$\int_{\mathbb{R}^d} e^{-tV(x)}\, dx < \infty.$$

Then for all $t > 0$ the mapping

$$\alpha \mapsto \alpha^{\frac{d}{2}}\, tr(e^{-tH(\alpha)}) \tag{2.20}$$

is non-increasing for all $\alpha > 0$. Consequently, for all $\alpha > 0$,

$$tr(e^{-tH(\alpha)}) \leq (4\pi\alpha t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-tV(x)} dx < \infty. \tag{2.21}$$

Up to now, these theorems can be applied only for eigenvalues of operators of the form (2.15). In the following, we will call any operator of this form a Stubbe-type operator.

The key to derive the Stubbe monotonicity results for the one-parameter operator given by (2.16) is to deduce that it is unitarily equivalent to an Stubbe-type operators family, by an application of one-parameter semigroups and Stone Theorem. Let G be the dilation generator on $L^2(\mathbb{R}^d)$, given formally by

$$G := \frac{i}{2}(x \cdot \nabla + \nabla \cdot x) \tag{2.22}$$

We shall first present the relationship with G and the "dilation" of functions, i.e., the change of parameter given by multiplying the independent variable by e^{-c}, $c \in \mathbb{R}$. This is given by the following lemma:

Lemma 2.1.3 *Let $\phi_c : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by $\phi_c(f) = f_c$, where*

$$f_c(x) = e^{-\frac{cd}{2}} f(e^{-c}x)$$

for every $c \in \mathbb{R}$, $x \in \mathbb{R}^d$. Then $\phi_c = e^{icG}$ for all $c \in \mathbb{R}$.

Proof: We want to apply Stone Theorem for the one-parameter group $\{\phi_c\}_{c\in\mathbb{R}}$, so we have to prove the following facts:

i) $\{\phi_c\}$ is a strongly-continuous family. This follows from the continuity at b of the application

$$a \mapsto \int_{\mathbb{R}^d} \left| e^{-\frac{ad}{2}} f(e^{-a}x) - e^{-\frac{bd}{2}} f(e^{-b}x) \right|^2 \, dx = \|(\phi_a - \phi_b)f\|^2$$

where $f \in C_0^\infty(\mathbb{R}^d)$ is fixed. Continuity holds for every $f \in L^2(\mathbb{R}^d)$ by the density of C_0^∞ in $L^2(\mathbb{R}^d)$.

ii) ϕ_c is an unitary operator. In fact, for all $c \in \mathbb{R}$,

$$\begin{aligned} \langle \phi_c f, g \rangle &= e^{-\frac{cd}{2}} \int_{\mathbb{R}^d} f(e^{-c}x) g(x) \, dx \\ \text{(by change of variables)} &= e^{-\frac{cd}{2}} e^{cd} \int_{\mathbb{R}^d} f(x) g(e^c x) \, dx \\ &= \int_{\mathbb{R}^d} f(x) (e^{\frac{cd}{2}} g(e^c x)) \, dx \\ &= \langle f, \phi_{-c} g \rangle, \end{aligned} \tag{2.23}$$

and it is easy to check by a change of variables that ϕ_c is a bounded operator on $L^2(\mathbb{R}^d)$ for all $c \in \mathbb{R}$; thus $(\phi_c)^* = \phi_{-c}$, and obviously $\phi_c \cdot \phi_{-c} = I$. Therefore ϕ_c is unitary.

Now, by Stone's theorem, there exist a self-adjoint operator G on $L^2(\mathbb{R}^d)$ such that $\phi_c = e^{icG}$. We have to conclude that G is the dilation generator.

Consider the derivative of the application $c \to \phi_c$ at $c = 0$, it should be equal to iG. In terms of the definition given for ϕ_c, we have for any $f \in C^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\begin{aligned} \frac{\partial}{\partial c}((\phi_c f)(x)) &= \frac{\partial}{\partial c}\left(e^{-\frac{cd}{2}} f(e^{-c}x)\right) \\ &= -\frac{d}{2} e^{-\frac{cd}{2}} f(e^{-c}x) - e^{-c\frac{(d-2)}{2}} x \cdot \nabla f(e^{-c}x) \end{aligned} \tag{2.24}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial c}((\phi_c f)(x))\Big|_{c=0} &= -\frac{d}{2}f(x) - x\cdot\nabla f(x) \\ &= \frac{i^2}{2}\sum_{k=1}^{d}\frac{\partial}{\partial x_k}(x_k f(x)) + x_k\frac{\partial f}{\partial x_k}(x).\end{aligned} \tag{2.25}$$

Therefore the operator G is forced to agree with the one at the right hand of (2.22), at least for every $f \in C^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. The density of $C^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ in the whole domain of the dilation generator gives the equality of G for every f in this domain. ∎

The reformulation of the change of variables by means of the operator (2.22) allows us to get an unitary equivalence of the family $K(\alpha)$ with a family of Stubbe-type operators in order to get the following

Theorem 2.1.4 *Let $\lambda_j(\alpha^{-2})$ be an eigenvalue of $H(\alpha^{-2})$ associated with the eigenfunction ϕ_j as in [15]. Then $\lambda_j(\alpha^{-2})$ is an eigenvalue of $K(\alpha)$ with the same multiplicity as an eigenvalue of $H(\alpha^{-2})$. It will be denoted by $\Lambda_j(\alpha)$ everytime we want to regard it as an eigenvalue of $K(\alpha)$.*

Proof: Let $\alpha = e^c$, $c\in\mathbb{R}$. For every $f \in Dom(V(\frac{x}{\alpha}))$ (as a multiplication operator) we have

$$\begin{aligned}V\left(\tfrac{x}{\alpha}\right)f(x) &= V(e^{-c}x)f(x) \\ &= e^{\frac{cd}{2}}\{e^{-\frac{cd}{2}}V(e^{-c}x)f(e^c e^{-c}x)\} \\ &= e^{\frac{cd}{2}}e^{icG}\{V(x)f(e^c x)\} \\ &= e^{icG}\{V(x)e^{\frac{cd}{2}}f(e^c x)\} \\ &= e^{icG}V(x)e^{-icG}f(x).\end{aligned} \tag{2.26}$$

Therefore $V\left(\frac{x}{\alpha}\right) \equiv e^{icG}Ve^{-icG}$; on the other hand, for every $c\in\mathbb{R}$ and $f\in Dom(\triangle)$,

$$\begin{aligned}(e^{-icG}\triangle e^{icG}f)(x) &= (e^{-icG}\triangle)(e^{-\frac{cd}{2}}f(e^{-c}x)) \\ &= e^{-\frac{cd}{2}}e^{-icG}e^{-2c}(\triangle f)(e^{-c}x) \\ &= e^{\left(-\frac{d}{2}-2\right)c}e^{-icG}(\triangle f)(e^{-c}x) \\ &= e^{\left(-\frac{d}{2}-2\right)c}e^{\frac{cd}{2}}(\triangle f)(x) \\ &= e^{-2c}(\triangle f)(x).\end{aligned} \tag{2.27}$$

Therefore we obtain the following conjugation for the operator $K(\alpha)$:

$$\begin{aligned}K(\alpha) &= -\triangle + V(\tfrac{x}{\alpha}) \\ &= -\triangle + e^{icG}Ve^{-icG} \\ &= e^{icG}\left\{-e^{-icG}\triangle e^{icG} + V\right\}e^{-icG} \\ &= e^{icG}\left\{-e^{-2c}\triangle + V\right\}e^{-icG} \\ &= e^{icG}\left\{-\alpha^{-2}\triangle + V\right\}e^{-icG}.\end{aligned} \tag{2.28}$$

Now, let ϕ_j be a normalized eigenfunction of $H(\alpha^{-2})$ associated with the eigenvalue $\lambda_j(\alpha^{-2})$. Henceforth the dependence on α for λ_j can be omitted. The eigenvalue satisfies

$$H\phi_j = \lambda_j \phi_j, \tag{2.29}$$

but $e^{-icG}e^{icG} = I$, which can be replaced in the last expression to get

$$He^{-icG}(e^{icG}\phi_j) = \lambda_j \phi_j, \tag{2.30}$$

and by operating on both sides by e^{icG} we get

$$\begin{aligned} e^{icG}He^{-icG}(e^{icG}\phi_j) = &\quad \lambda_j(e^{icG}\phi_j), \quad \text{or equivalently} \\ K(\alpha)\phi_j = &\quad \lambda_j(e^{icG}\phi_j), \end{aligned} \tag{2.31}$$

so λ_j is an eigenvalue of $K(\alpha)$ and their associated eigenfunctions are images of the eigenfunctions with respect to $H(\alpha^{-2})$ by an isometry. ∎

We are able now to get the corresponding monotonicity result established in theorems 2.1.1 and 2.1.2 for the operator $K(\alpha)$:

Corollary 2.1.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function with support on the negative half axis such that f' is concave, and let $V \leq 0$ vanish at infinity. Then the mapping*

$$\alpha \to \alpha^{-d} \sum_{\Lambda_j(\alpha)<0} f(\Lambda_j(\alpha)) \tag{2.32}$$

is non-decreasing for $\alpha > 0$. In particular, for all $\sigma \geq 2$,

$$\alpha \to \alpha^{-d} \sum_{\Lambda_j(\alpha)<0} (-\Lambda_j(\alpha))^{\sigma} \tag{2.33}$$

is non-decreasing for $\alpha > 0$, and

$$\alpha^{-d} \sum_{\Lambda_j(\alpha)<0} (-\Lambda_j(\alpha))^{\sigma} \leq L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} (-V(x))^{\sigma+\frac{d}{2}}\, dx \tag{2.34}$$

Proof: Let τ be the application at (2.17), i.e., $\tau(\alpha) := \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha)<0} f(\lambda_j(\alpha))$, and let $g(\alpha) := \alpha^{-2}$. Since g is decreasing on α and τ is non-increasing on α, then $\tau \circ g$ is non-decreasing on α. But

$$\begin{aligned} \tau \circ g = &\quad \alpha^{-d} \sum_{\lambda_j(\alpha^{-2})<0} f(\lambda_j(\alpha^{-2})) \\ = &\quad \alpha^{-d} \sum_{\Lambda_j(\alpha)<0} f(\Lambda_j(\alpha)), \end{aligned}$$

that is just the application at (2.32). Inequality (2.34) is obtained by taking the limit of (2.33) as $\alpha \to \infty$. ∎

Similarly we get

Corollary 2.1.6 *Let $V(x)$ be a confining potential, as in theorem 2.1.2. Then*

$$\alpha \mapsto \alpha^{-d}\, tr(e^{-tK(\alpha)}) \tag{2.35}$$

is non-decreasing for all $\alpha > 0$. Consequently, for all $\alpha > 0$,

$$tr(e^{-tK(\alpha)}) \leq (4\pi\alpha^{-2}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-tV(x)} dx < \infty. \tag{2.36}$$

Chapter 3

Monotonicity for the Eigenvalues of a Double-wells One-Parameter Schrödinger Operator

In this chapter we focus on studying the following problem:

Let $V(x)$ be a nonpositive, piecewise continuous real function with support contained in the interval $[0,1]$. Consider the one-parameter Schrödinger operator

$$H_a := -\frac{d^2}{dx^2} + V(x-a) + V(-x-a), \tag{3.1}$$

where $a > 0$, defined, according to the theory presented in Chapter 1, as a self-adjoint operator on the underlying Hilbert Space $L^2(\mathbb{R})$. Since $V(x-a)+V(-x-a)$ is bounded and compact-supported, we know from the results of Chapter 1 that the discrete spectrum of the operator H_a is contained in the interval $(-\infty, 0]$ and consist of a finite set of eigenvalues $\lambda_j(a)$, $j = 1, \dots n$. Our aim is to determine whether the eigenvalues $\lambda_j(a)$ are monotone functions of the parameter a.

The study of this problem is motivated by several facts; particularly, we recall the results from [15] discussed in the example of Chapter 2, providing a corresponding result that describes the monotonicity of the application

$$\alpha \to \mathrm{Tr}(f(\widetilde{H}(\alpha))),$$

where

$$\widetilde{H}(\alpha) := -\alpha\triangle + V(x),$$

in $L^2(\mathbb{R}^d)$, for nonpositive potentials $V(x)$ vanishing at infinity, and a suitable class of functions f for which the functional calculus $f(\widetilde{H}(\alpha))$ is defined (the detailed description of this class can be found in [11] and [15]). One of the key steps to show the monotonicity is to apply the Feynman-Hellmann Theorem; specifically, if $\widetilde{\lambda}_j(\alpha)$ is an eigenvalue for $\widetilde{H}(\alpha)$ associated to the eigenfunction ϕ_j, then

$$\frac{\partial \widetilde{\lambda}_j}{\partial \alpha} = \int |\phi_j|^2 \, dx \geq 0.$$

Therefore the applications $\alpha \to \widetilde{\lambda}_j(\alpha)$ are always increasing. However, such an approach cannot be developed for the eigenvalues of (3.1) in a straightforward manner, for the analogue application of Feynman-Hellmann in this case does not tells us any information about the sign of the derivative of $\lambda_j(a)$ with respect to a. Instead we deal with a more particular procedure to describe the monotonicity for $\lambda_j(a)$; we get an explicit formulae, which gives us the form of the functions $\lambda_j(a)$ in order to obtain the monotonicity directly.

Before going on the development to obtain the monotonicity, we will give a couple preliminary Lemmas to carry out a decomposition for the operator H_a into a direct sum of operators involving just the nonnegative half-axis of the real line. In what follows, we denote by $\mathfrak{D}$ the domain of the operator H_a, i.e., the Sobolev space $H^2(\mathbb{R})$.

3.1 Structure of H_a

Lemma 3.1.1 *Let $P_N : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator given by*

$$(P_N f)(x) := \frac{f(x)+f(-x)}{2},$$

i.e., the even projector of the function f, and let $P_D := 1 - P_N$ its orthogonal complement, which is the odd projector. Then $[H_a, P_N] = [H_a, P_D] = 0$.

Proof: Observe that clearly $P_N\mathfrak{D} \subseteq \mathfrak{D}$; hence the expression $[H_a, P_N]f$ makes sense for every $f \in \mathfrak{D}$. Actually, it is sufficient to show that $P_N H_a f = H_a P_N f$ for every $f \in \mathfrak{D}$. Since the potential $V(x-a) + V(-x-a)$ is an even function in x, it is easily seen that for every even function g, $H_a g$ is even as well. Therefore the even part of $H_a g$ is $H_a g$ itself, and this holds in particular when $g = P_N f$; thus

$$P_N H_a P_N f = H_a P_N f \quad \text{for every} \quad f \in \mathfrak{D}. \tag{3.2}$$

But P_N is bounded and symmetric; hence, for every $f, g \in \mathcal{D}$,

$$\begin{aligned}\langle P_N H_a f, g\rangle &= \langle f, H_a P_N g\rangle \\ &= \langle f, P_N H_a P_N g\rangle \\ &= \langle P_N H_a P_N f, g\rangle,\end{aligned}$$

so it follows that $P_N H_a f = P_N H_a P_N f$ for every $f \in \mathfrak{D}$, and by (3.2) this implies $P_N H_a f = H_a P_N f$. From the definition of P_D, it is clear that $H_a P_D f = P_D H_a f$ for every $f \in \mathfrak{D}$ as well. ∎

The subspace $P_\mu L^2(\mathbb{R})$ is the subspace of even functions in $L^2(\mathbb{R})$ if $\mu = N$, or odd functions if $\mu = D$. In both cases, every function $f \in P_\mu L^2(\mathbb{R})$ is uniquely determined by its restriction to the interval $[0,\infty)$. In particular, we notice that if $g = f\Big|_{[0,\infty)}$ with $f \in P_D L^2(\mathbb{R})$, then g satisfy the Dirichlet condition $g(0) = 0$, whereas if $f \in P_N L^2(\mathbb{R})$, then g satisfy the Neumann condition $g'(0) = 0$. Let ψ_μ be the mapping $f \to f\Big|_{[0,\infty)}$, for every $f \in P_\mu L^2(\mathbb{R})$, being $\mu = N$ or D.

Lemma 3.1.2 *The mapping ψ_μ induces the operator $H_a^\mu := \psi_\mu H_a \psi_\mu^{-1}$ from the space $\mathfrak{D}_\mu$ into $L^2_\mu([0,\infty))$, where $\mathfrak{D}_\mu := \psi_\mu(P_\mu \mathfrak{D})$. Furthermore, for every $f \in \mathfrak{D}_\mu$,*

$$H_a^\mu f = -\frac{d^2 f}{dx^2} + V(x-a)f \tag{3.3}$$

Proof: We need to show first that the composition $\psi_\mu H_a \psi_\mu^{-1}$ is well defined from $\mathfrak{D}_\mu$ to $L^2_\mu([0,\infty))$. Indeed, by Lemma 3.1.1,

$$\begin{aligned} H_a\psi_\mu^{-1}(\mathfrak{D}_\mu) &= \psi_\mu H_a \psi_\mu^{-1}(\psi_\mu(P_\mu \mathfrak{D})) \\ &= H_a(P_\mu \mathfrak{D}) \\ &= P_\mu(H_a \mathfrak{D}) \end{aligned}$$

and this last set is contained in the domain of ψ_μ, and hence $H_a^\mu := \psi_\mu H_a \psi_\mu^{-1}$ is well defined. Now, let $f \in \mathfrak{D}_\mu$. Then $f = \psi_\mu g$ for some $g \in P_\mu \mathfrak{D}$. From the definition of ψ_μ, it is clear that $f(x) = g(x)$ for every $x \geq 0$. Since (3.3) has to be shown over $[0,\infty)$, then it suffices to show:

$$H_a^\mu f = -\frac{d^2 g}{dx^2} + V(x-a)g \quad \text{over} \quad [0,\infty)$$

with the suitable boundary conditions. But $H_a^\mu f = \psi_\mu H_a \psi_\mu^{-1} f = \psi_\mu H_a g$, and

$$\begin{aligned} \psi_\mu H_a g &= H_a g\Big|_{[0,\infty)} \\ &= \left\{ -\frac{d^2 g}{dx^2} + V(x-a)g + V(-x-a)g \right\}\Big|_{[0,\infty)} \\ &= -\frac{d^2 g}{dx^2} + V(x-a)g, \end{aligned}$$

where the last step follows from the fact that the support of $V(-x-a)$ is contained in $(-\infty, 0)$. This completes our proof. ∎

Since the space $\mathfrak{D}$ may be decomposed as the direct sum $P_N\mathfrak{D} \oplus P_D\mathfrak{D}$, Lemma 3.1.2 implies that H_a is equivalent to $H_a^N \oplus H_a^D$ defined on $P_N L^2(\mathbb{R}) \oplus P_D L^2(\mathbb{R})$ and has the same spectrum. Thus we only need to study the spectrum of the operators H_a^μ, being $\mu = N$ or D.

Now we are ready to begin the main steps to get the desired monotonicity. This will be accomplished by the following steps:

1. Determine the relationships that should be satisfied for a pair (λ, a), where λ is an eigenvalue for H_a. We will derive formulas of the form $F(\lambda) = a$ for an explicit function $F(\lambda)$.

2. Since the eigenvalues, as applications in λ, are inverse functions of the function F, we show that F is monotone by differentiating it with respect to λ. This will give us a necessary condition involving the function $\lambda \to \rho(0,\lambda)$, where ρ is the solution for a Riccati equation.

3. We show that $\frac{d\rho}{d\lambda}(0,\lambda) \geq 0$ over its domain, by considering the approximation by the numerical Euler method for the solutions of the first-order Riccati differential equation associated to the Schrödinger eigenvalue equation of our operator.

3.2 Relationships

The relationship between the eigenvalues λ and the parameter a is given by the following

Theorem 3.2.1 *Let $\lambda \in (-\|V\|_\infty, 0)$ and $u(x,\lambda)$ be the unique solution to the well-posed IVP [1]:*

$$\begin{aligned} -\partial_x^2 u + (V(x)-\lambda)u = 0, &\quad 0 \le x \le 1, \\ u(1,\lambda) = 1 & \\ \partial_x u(1,\lambda) = -\sqrt{-\lambda} & \end{aligned} \tag{3.4}$$

a) If $\dfrac{\partial_x u(0,\lambda)}{\sqrt{-\lambda}u(0,\lambda)} > 1$, then λ is an eigenvalue of H_a^D if and only if

$$a = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\sqrt{-\lambda}u(0,\lambda)}{\partial_x u(0,\lambda)}\right). \tag{3.5}$$

b) If $0 < \dfrac{\partial_x u(0,\lambda)}{\sqrt{-\lambda}u(0,\lambda)} < 1$, then λ is an eigenvalue of H_a^N if and only if

$$a = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\partial_x u(0,\lambda)}{\sqrt{-\lambda}u(0,\lambda)}\right). \tag{3.6}$$

Moreover, if none of the conditions a) or b) hold for $\dfrac{\partial_x u(0,\lambda)}{\sqrt{-\lambda}u(0,\lambda)}$, then it does not exist $a > 0$ verifying that λ is an eigenvalue of H_a^μ for either $\mu = D$ or N.

Proof: Consider the case a). Recall that by Proposition 1.0.2, every function on the domain of a self-adjoint Schrödinger operator in $L^2(\mathbb{R})$ has absolutely continuous first derivative; this can be obviously extended to the operator H_a^D defined on $L_D^2([0,\infty))$, either by considering the boundary condition at zero and modifying the proof of 1.0.2, or by considering the equivalence with H_a given by Lemmas 3.1.1 and 3.1.2. Taking this fact into account, Let us prove the implication to the right, i.e., assume that λ is an eigenvalue of H_a^D for some $a > 0$ fixed. Then the problem

$$\begin{gathered} -\partial_x^2 w + (V(x-a)-\lambda)w = 0, \\ w(\cdot,\lambda) \in C^1([0,\infty)) \cap L^2(0,\infty), \;\; w(0,\lambda) = 0 \end{gathered} \tag{3.7}$$

has a nonidentically vanishing solution $w(\cdot,\lambda)$ in the domain of H_a^D; in particular, $w(\cdot,\lambda)$ is continuous and has continuous first derivative. These continuity and differentiability conditions are critical at $x = a$, $x = a+1$, since $w(\cdot,\lambda)$ must be joined at these points respecting the regularity; let us calculate $w(x,\lambda)$ over the intervals $[a+1,\infty)$, $[a,a+1]$, $[0,a]$ and check how the continuity conditions must be fixed:

- Over $[a+1,\infty)$, $V(x-a) \equiv 0$. Moreover, $w(\cdot,\lambda) \in L^2(0,\infty)$, so necessarily

$$w(x,\lambda) = c_1 e^{-\sqrt{-\lambda}(x-a-1)} \quad \text{wherever} \quad x \ge a+1 \tag{3.8}$$

  for some c_1 (we can write $x-a-1$ instead of x, since it implies just multiplying by a constant).

[1] Initial Value Problem. We fix this acronym henceforth.

- Over $[a, a+1]$, we should have a solution that agrees with the one obtained over $[a+1, \infty)$, up to its first derivative. Moreover, $V(x-a)$ on this interval is equal to the translation from $[0,1]$ to $[a, a+1]$ of the potential $V(x)$ of the problem (3.4). Hence $w(x, \lambda)$ must have the form

$$w(x, \lambda) = c_1 u(x-a, \lambda), \quad \text{wherever} \quad a \leq x \leq a+1 \tag{3.9}$$

In particular, $w(a+1, \lambda) = u(1, \lambda) = c_1$ and $\partial_x w(a+1, \lambda) = c_1 \partial_x u(1, \lambda) = -c_1\sqrt{-\lambda}$. Clearly, both w and $\partial_x w$ agree with the solution over $[a+1, \infty)$.

- Over $[0, a]$. We have to satisfy the Dirichlet condition $w(0, \lambda) = 0$; therefore the solution must have the form

$$w(x, \lambda) = c_2 \sinh(\sqrt{-\lambda}x), \quad \text{wherever} \quad 0 \leq x \leq a. \tag{3.10}$$

Note that the constants c_1, c_2 may depend on λ; nevertheless we do not need to consider this dependence and will be omitted.

The solution has to be differentiable at $x = a$; this condition reduces to the matrix equation

$$\begin{bmatrix} \sinh(\sqrt{-\lambda}a) & -u(0, \lambda) \\ \sqrt{-\lambda}\cosh(\sqrt{-\lambda}a) & -\partial_x u(0, \lambda) \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.11}$$

and this system of equations has a nontrivial solution for the vector $[c_2, c_1]^T$ if and only if the determinant of the associated matrix is equal to zero; that is,

$$-\sinh(\sqrt{-\lambda}a)\partial_x u(0, \lambda) + \sqrt{-\lambda}\cosh(\sqrt{-\lambda}a)u(0, \lambda) = 0,$$

and this implies, after working out a, the equation (3.5) as desired. Of course it would not be possible to work out a if the inequality required in the item a) of the statement did not holds.

For the implication to the left, let λ be such that (3.5) holds. Then the determinant of the matrix in the equation (3.11) vanishes, and we can find a nontrivial solution $[c_2, c_1]^T$ for the matrix equation (3.11). Now we define $w(\cdot, \lambda)$ by the equations (3.8), (3.9), (3.10). In order to conclude that $w(\cdot, \lambda)$ is an eigenfunction and λ its associated eigenvalue, it remains to show that $w(\cdot, \lambda) \in Dom(H_a^D)$; we consider for this purpose the core of the self-adjoint operator H_a^D, given by

$$\mathcal{D} := \{\phi \in C^\infty([0, \infty)) : \phi(0) = 0 \text{ and } \phi \text{ has a compact support}\}$$

(See Chapter 1, equation (1.14) and related paragraph). Let us denote $K := H_a^D\big|_{\mathcal{D}}$. By integration by parts it can be proved that for every $\phi \in \mathcal{D}$,

$$\langle w, K\phi \rangle = \int_0^\infty w(x, \lambda)(-\phi''(x) + V(x-a)\phi(x))\, dx = \int_0^\infty \lambda w(x, \lambda)\phi(x)\, dx = \langle \lambda w, \phi \rangle.$$

Hence $w(\cdot, \lambda)$ belongs to the domain of K^*, which is itself the domain of H_a^D due to the self-adjointness of H_a^D. This proves the left implication. The procedure for the case b) is analogue to the case a), but now the solution $v(x, \lambda)$ for $x \in [0, a]$ is of the form $w(x, \lambda) = c_2 \cosh(\sqrt{-\lambda}x)$, and its derivative is $\partial_x w(x, \lambda) = \sqrt{-\lambda}c_2 \sinh(\sqrt{-\lambda}x)$. The continuity condition at $x = a$ imply (3.6); in the left implication, the core for H_a^N is its restriction to

$$\mathcal{D}_N := \{\phi \in C^\infty([0, \infty)) : \phi'(0) = 0 \text{ and } \phi \text{ has a compact support}\}.$$

All the remaining steps are essentially the same steps as a). ∎

3.3 Differentiation

The former Theorem provides the relationships between a and λ for the operators H_a^D, H_a^N, and these relationships, as we stated, have the form $a = F_D(\lambda)$ or $a = F_N(\lambda)$ (F_D and F_N will denote the corresponding functions for H_a^D, H_a^N respectively). Now we want to deduce the monotonicity of λ with respect to a. In this case, λ is defined implicitly with respect to a by equations of the form $a = F(\lambda)$, and the functions $\lambda(a)$, whenever they are defined, are local inverses to the function $F(\lambda)$. Moreover, if it is assumed that F is differentiable on an interval and $F'(\lambda) \neq 0$, then by the Implicit Function Theorem, it holds that $\lambda(a)$ is differentiable as well, and $\dfrac{d\lambda}{da} = \dfrac{1}{F'(\lambda)}$. Therefore we can obtain the ranges of monotonicity of λ as a function of a by determining the sets of values λ such that $F'(\lambda) > 0$ or $F'(\lambda) < 0$.

It may be noticed obviously from the equations (3.5) and (3.6), that the differentiability information for the functions $F_\mu(\lambda)$ defined by the right hand sides of the equations depend directly on the knowledge we can obtain for the term $\dfrac{\partial_x u(0,\lambda)}{u(0,\lambda)}$. Actually, we focus most of the current chapter in the study of this term. Indeed, let λ be fixed, and $\rho(x,\lambda) := \dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ for every $x \in [0,1]$ such that $u(x,\lambda) \neq 0$, where $u(x,\lambda)$ is the unique solution of (3.4); it is well known that $\rho(x,\lambda)$ satisfies the Riccati IVP

$$\begin{aligned} \frac{d\rho}{dx} &= V(x) - \lambda - \rho^2, \;\; 0 \leq x \leq 1, \\ \rho(1,\lambda) &= -\sqrt{-\lambda}. \end{aligned} \tag{3.12}$$

In particular we concluded that there exist a solution for the IVP (3.12) in the set of all values of x on the interval $[0,1]$ such that $u(x,\lambda) \neq 0$ (recall that the set of zeros of $u(x,\lambda)$ is finite by Proposition 1.1.3). Actually, the Fundamental Existence and Uniqueness Theorem (see e.g. [13]) provides that this solution is unique on the biggest interval of the form $(t,1]$ such that $u(x,\lambda)$ is continuous on this interval; however, *a priori* the uniqueness of the solution on the other connected components in which $u(x,\lambda)$ is continuous in x cannot be assured by theoretical tools of Ordinary Differential Equations. Nevertheless we shall introduce a Lemma below that provides a resource to guarantee the uniqueness on the rest of the domain along $[0,1]$. Moreover, we state a result that provides the sign of the derivative for $\rho(0,\lambda)$, in which we assume the uniqueness of the solution.

Theorem 3.3.1 *For every λ with $-\|V\|_\infty < \lambda < 0$, let $\rho(x,\lambda)$ be the unique solution for the problem* (3.12). *Then $\dfrac{\partial\rho}{\partial\lambda}(0,\lambda) \geq 0$ for every λ in the domain of $\lambda \mapsto \rho(0,\lambda)$, i.e, $\lambda \mapsto \rho(0,\lambda)$ is increasing over each connected component of its domain.*

The proof of this Theorem is a delicate issue and it is the main purpose of section 3.4; now we obtain the sign of the derivatives for $F_\mu(\lambda)$ that Theorem 3.3.1 implies. Note that if λ is an eigenvalue of H_a^N or H_a^D for some $a > 0$, then $\rho(0,\lambda) > 0$ by Theorem 3.2.1. Thus we restrict the domain of the functions F_μ, $\mu = N$, D to the subset of $(-\|V\|_\infty, 0)$ such that $\rho(0,\lambda) > 0$.

Theorem 3.3.2

a) $\dfrac{dF_N}{d\lambda} > 0$ *on each connected component of its domain.*

b) $\dfrac{dF_D}{d\lambda} < 0$ *on each connected component of its domain.*

Proof: Writing (3.5) and (3.6) in terms of $\rho(0,\lambda)$, we obtain respectively $a = F_D(\lambda)$ and $a = F_N(\lambda)$, where

$$F_N(\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\rho(0,\lambda)}{\sqrt{-\lambda}}\right), \tag{3.13}$$

$$F_D(\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\sqrt{-\lambda}}{\rho(0,\lambda)}\right), \tag{3.14}$$

and F_N, F_D are positive since $\rho(0,\lambda) > 0$ is positive.

Proof of a): As $\rho(0,\lambda)$ is positive, and increasing by Theorem 3.3.1, it holds that $\dfrac{\rho(0,\lambda)}{\sqrt{-\lambda}}$ is also increasing because $\lambda \to \dfrac{1}{\sqrt{-\lambda}}$ is increasing and positive. This still holds after composing with the monotone increasing function $\operatorname{arctanh}(\cdot)$, and multiplying again by $\dfrac{1}{\sqrt{-\lambda}}$.

Proof of b): This case is not so straightforward. Indeed, by applying the Fundamental Theorem of Calculus for $\operatorname{arctanh}(\cdot)$,

$$\operatorname{arctanh} x = \int_0^x \frac{1}{1-u^2}\, du,$$

and replacing in this formula $x = \dfrac{\sqrt{-\lambda}}{\rho(0,\lambda)}$ and multiplying by $\dfrac{1}{\sqrt{-\lambda}}$ we get

$$\begin{aligned} F_D(\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\sqrt{-\lambda}}{\rho(0,\lambda)}\right) &= \frac{1}{\sqrt{-\lambda}} \int_0^{\frac{\sqrt{-\lambda}}{\rho(0,\lambda)}} \frac{1}{1-u^2}\, du \\ &= \int_0^{\frac{1}{\rho(0,\lambda)}} \frac{dt}{1+\lambda t^2}. \end{aligned} \tag{3.15}$$

Now we calculate the derivative of F_D with respect to λ, where differentiation of the integral (3.15) can be made by using the chain rule. We obtain

$$\begin{aligned} \frac{dF_D}{d\lambda} &= -\frac{\partial_\lambda \rho(0,\lambda)}{\rho(0,\lambda)^2}\left(\frac{1}{1+\lambda t^2}\bigg|_{t=\frac{1}{\rho(0,\lambda)}}\right) - \int_0^{\frac{1}{\rho(0,\lambda)}} \frac{t^2}{(1+\lambda t^2)^2}\, dt \\ &= -\frac{\partial_\lambda \rho(0,\lambda)}{\rho(0,\lambda)^2+\lambda} - \int_0^{\frac{1}{\rho(0,\lambda)}} \frac{t^2}{(1+\lambda t^2)^2}\, dt \end{aligned}$$

but $\rho(0,\lambda)^2 + \lambda > 0$, since $\dfrac{\rho(0,\lambda)}{\sqrt{-\lambda}} > 1$ (see condition a) in Theorem 3.2.1). Moreover, $\rho(0,\lambda) > 0$ and $\dfrac{d\rho(0,\lambda)}{d\lambda} \geq 0$; thus we get $\dfrac{dF_D}{d\lambda} < 0$ as desired. ∎

Remark Both parts of the proof of Theorem 3.3.2 depends strongly on the condition $\rho(0,\lambda) > 0$, a necessary condition for λ to be an eigenvalue of H_a^μ for some $a > 0$, $\mu = N, D$. However, the functions given by (3.14) and (3.13) are naturally defined in larger domains. Indeed, for every $\lambda \in \mathbb{R}$ (not necessarily an eigenvalue) such that $|\rho(0,\lambda)/\sqrt{-\lambda}| \neq 1$, one of the functions F_N or F_D is still defined. This gives rise to the natural question about whether the the monotonicity still holds in the intervals of the natural domains of $F_a^\mu(\lambda)$ where $\rho(0,\lambda) < 0$; for these values of λ, of course, $F_a^\mu(\lambda) < 0$ for $\mu = N, D$. The answer of this question is negative, for it is not difficult to construct potentials $V(x)$ such that the functions $F_N(\lambda)$ and $F_D(\lambda)$ may have local extrema at values with negative image. See example 2 in section 3.7 for an illustration.

3.4 Riccati Equations and the Euler Method

Now we focus on the proof of Theorem 3.3.1. This result has an interest on its own; indeed, there are several classical results in the Ordinary Differential Equations (ODE) Theory that study the behavior of the solutions for IVPs with the presence of a parameter λ that appears either in its initial condition or in the differential equation, i.e., problems of the form

$$\begin{aligned} \frac{dy}{dx} &= F(x,y;\lambda), \;\; 0 \leq x \leq L, \;\; y \in \mathbb{R}^n \\ y(0) &= y_0(\lambda). \end{aligned} \tag{3.16}$$

In the common literature about the study for this problem one can find a lot of results establishing the facts that can be concluded about the solutions $y = y(x,\lambda)$ (if there exist someone), provided some determined hypotheses on the function $F(x,y;\lambda)$ or the initial condition $y_0(\lambda)$. We are particularly interested in the issues about dependence on the initial condition $y(0)$ and/or the parameter λ in F. One of the most relevant theoretical facts known in this context is that in common situations, a solution $y(x,\lambda)$ of (3.16) is uniformly continuous with respect to λ; namely, with the suitable hypotheses on F and $y_0(\lambda)$, it holds that the supremum of the difference in norm between $y(x,\lambda_1)$ and $y(x,\lambda_2)$ can be made arbitrarily small, by taking λ_1 and λ_2 sufficiently near. The specific results we employ to assure this property are summarized in Appendix B. The interested reader can find the complete development of the involved topics in [3], [4] or [13], or analogue references on ODE Theory.

It also may be noticed that the IVPs we take into consideration have the form of problem (3.16) with $n = 1$. Thus the solution $y(x,\lambda)$ is a scalar function, for which natural questions of real-valued functions are applicable. It is a known result (Theorem 3.4.2 of [3]) that if F has continuous partial derivative with respect to y, then the solution $y(x,\lambda)$ is differentiable with respect to the initial condition (i.e. with respect to λ in our case, if we assume $y_0(\lambda)$ differentiable). However the property we want is monotonicity with respect to λ fixing x as a constant value. We have not found a general theoretical tool to derive this monotonicity provided some hypotheses on F or $y_0(\lambda)$, but instead we determined it in the particular case of the IVPs of our interests. The develop of such a theoretical tool in the general ODE Theory is an unanswered question to our knowledge.

Let us begin by recapitulating and fixing some notations; consider the second order IVP

$$\begin{aligned} -\partial_x^2 u + (V(x) - \lambda)u = 0, \;\; 0 \leq x \leq 1, \\ u(1,\lambda) = 1 \\ \partial_x u(1,\lambda) = \xi(\lambda) \end{aligned} \tag{3.17}$$

(in the case of problem (3.12) in the former section, $\xi(\lambda) := -\sqrt{-\lambda}$). Recall that by Proposition 1.1.3 it holds that given $\lambda \in (-\|V\|_\infty, 0)$, the solution to (3.17) has a finite number of zeros for x along $[0,1]$. Recall moreover that by 1.0.2, the solutions have absolutely continuous first order derivative, and hence, a second derivative which is unique up to a set of measure zero. If $V(x)$ is piecewise continuous, one can assume the second derivative of u (with respect to x) piecewise continuous as well; for simplicity we assume V to be continuous, since there is no difficulty to extend the results to the piecewise continuous case. We consider simultaneously the Riccati first order IVP associated to (3.17), that is given by

$$\begin{aligned} \frac{d\rho}{dx} &= V(x) - \lambda - \rho^2, \quad x \in [0,1] \\ \rho(1,\lambda) &= \xi(\lambda), \end{aligned} \tag{3.18}$$

whose solution satisfies $\rho(x,\lambda) = \dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$, and the *inverse-Riccati first-order IVP* given by

$$\begin{aligned} \frac{d\sigma}{dx} &= (\lambda - V(x))\sigma^2 + 1 \\ \sigma(1,\lambda) &= \xi(\lambda), \end{aligned} \tag{3.19}$$

whose solution satisfies $\sigma(x,\lambda) = \dfrac{u(x,\lambda)}{\partial_x u(x,\lambda)}$. The main Theorem giving the monotonicity results requires two preliminary Lemmas., which are based on the concept of Euler-continuous solution of an IVP; see Appendix A for more details.

Lemma 3.4.1 *Let $\omega = \omega(x,\lambda)$ be the solution to the IVP*

$$\begin{aligned} \frac{d\omega}{dx} &= F(x,\omega;\lambda), \quad x \in [a,b] \\ \omega(b,\lambda) &= \xi(\lambda), \end{aligned} \tag{3.20}$$

where $\xi(\lambda)$ is differentiable and either $F(x,\omega;\lambda) = V(x) - \lambda - \omega^2$ or $F(x,\omega;\lambda) = (\lambda - V(x))\omega^2 + 1$. Suppose that the following conditions are fulfilled:

i) There exist $\delta > 0$ such that for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $x \to \omega(x,\lambda)$ is an Euler-continuous solution of (3.20).

ii) There exist $M > 0$ such that for every $x \in (a,b)$ and every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $|\omega(x,\lambda)| \le M$.

Then one has the following:

a) $\lambda \to \omega(a,\lambda)$ is increasing on $(\lambda_0 - \delta, \lambda_0 + \delta)$, provided that $F(x,\omega;\lambda) = V(x) - \lambda - \omega^2$ and $\xi(\lambda)$ is an increasing function of $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$.

b) $\lambda \to \omega(a,\lambda)$ is decreasing on $(\lambda_0 - \delta, \lambda_0 + \delta)$, provided that $F(x,\omega;\lambda) = (\lambda - V(x))\omega^2 + 1$ and $\xi(\lambda)$ is a decreasing function of $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$.

Proof: We need to consider a slight modification of the functions $F(x,\omega;\lambda)$, since their partial derivative is not bounded with respect to ω; let us define the auxiliar application

$$\phi(y) := \begin{cases} y^2 & \text{if } \ |y| \le M \\ 2M|y| - M^2 & \text{if } \ |y| > M \end{cases}$$

and define $\bar{F}(x,\omega;\lambda) := F(x, \sqrt{\phi(\omega)};\lambda)$. Then ϕ is differentiable, positive, and agrees with y^2 if $|y| \le M$, but ϕ satisfies $|\phi'(y)| \le 2M$ for every $y \in \mathbb{R}$; moreover, $\bar{F}$ has bounded partial derivative with respect to ω, and hence we can apply Theorem A.0.1 of Appendix A to the IVP (3.20) replacing F by $\bar{F}$. From condition ii) it is clear that $\omega(x,\lambda)$ is a solution of (3.20) if and only if it is a solution of the modified problem

$$\begin{aligned} \frac{d\omega}{dx} &= \bar{F}(x,\omega;\lambda), \quad x \in [a,b] \\ \omega(b,\lambda) &= \xi(\lambda). \end{aligned}$$

Referring to Appendix A, we consider the numerical Euler approximation of the Euler-continuous solution $x \to \omega(x,\lambda)$. Let $\epsilon > 0$ be given by

$$\epsilon = \frac{1}{2}\min\left\{\frac{1}{2M}, \frac{1}{4M\|V\|_\infty}\right\}, \tag{3.21}$$

and let n be a fixed integer such that $h := \dfrac{b-a}{n} < \epsilon$, $x_k = b - kh$, $0 \le k \le n$. The corresponding Euler sequences for both equations have the form

$$\widetilde{\omega}_0 := \omega(x_0,\lambda) = \xi(\lambda) \tag{3.22}$$

$$\widetilde{\omega}_{k+1} := \widetilde{\omega}_k - h\bar{F}(x_k,\widetilde{\omega}_k;\lambda). \tag{3.23}$$

Notice that according to the statement of the Lemma, the function $\bar{F}$ has partial derivatives with respect to ω and λ, and these partial derivatives are continuous functions of λ, keeping x fixed. Therefore it holds that the right-hand side of (3.23) is a differentiable function of λ, provided that $\widetilde{\omega}_k$ is. Thus, taking into account that by hypothesis $\widetilde{\omega}_0$ is differentiable with respect to λ, it follows by induction that $\widetilde{\omega}_k$ is differentiable with respect to λ for every k with $0 \le k \le n$; moreover, differentiating on both sides of (3.23) we obtain

$$\frac{d\widetilde{\omega}_{k+1}}{d\lambda} = \frac{d\widetilde{\omega}_k}{d\lambda} - h\frac{d}{d\lambda}\left(\bar{F}(x_k,\widetilde{\omega}_k;\lambda)\right)$$

where

$$\frac{d}{d\lambda}\left(\bar{F}(x_k,\widetilde{\omega}_k;\lambda)\right) = \begin{cases} -1 - \phi'(\widetilde{\omega}_k)\dfrac{d\widetilde{\omega}_k}{d\lambda} & \text{if} \quad \bar{F}(x,\omega;\lambda) = V(x) - \lambda - \phi(\omega) \\ \phi(\widetilde{\omega}_k) + (\lambda - V(x_k))\phi'(\widetilde{\omega}_k)\dfrac{d\widetilde{\omega}_k}{d\lambda} & \text{if} \quad \bar{F}(x,\omega;\lambda) = (\lambda - V(x))\phi(\omega) + 1, \end{cases}$$

and therefore

$$\frac{d\widetilde{\omega}_{k+1}}{d\lambda} = \begin{cases} \dfrac{d\widetilde{\omega}_k}{d\lambda}(1 + h\phi'(\widetilde{\omega}_k)) + h & \text{if} \quad \bar{F}(x,\omega;\lambda) = V(x) - \lambda - \phi(\omega) \\ \dfrac{d\widetilde{\omega}_k}{d\lambda}(1 - h(\lambda - V(x_k))\phi'(\widetilde{\omega}_k)) - h\phi(\widetilde{\omega}_k) & \text{if} \quad \bar{F}(x,\omega;\lambda) = (\lambda - V(x))\phi(\omega) + 1. \end{cases}$$

But we have the estimates

$$\begin{aligned} |h\phi'(\widetilde{\omega}_k)| &\leq 2Mh < 2M\epsilon < \tfrac{1}{2} \\ |h(\lambda - V(x_k))\phi'(\widetilde{\omega}_k)| &\leq 2M(|\lambda| + |V(x_k)|)h < 4M\|V\|_\infty \epsilon < \tfrac{1}{2}. \end{aligned}$$

Therefore, for every k with $0 \leq k \leq n-1$, it follows that $\frac{d\widetilde{\omega}_{k+1}}{d\lambda} \geq 0$ provided that $\frac{d\widetilde{\omega}_k}{d\lambda} \geq 0$ and $\bar{F}(x,\omega;\lambda) = V(x) - \lambda - \phi(\omega)$. On the other hand, since ϕ is always positive, then $-h\phi(\widetilde{\omega}_k)$ is negative, so we get that $\frac{d\widetilde{\omega}_{k+1}}{d\lambda} \leq 0$ provided that $\frac{d\widetilde{\omega}_k}{d\lambda} \leq 0$ and $\bar{F}(x,\omega;\lambda) = (\lambda - V(x))\phi(\omega) + 1$. It turns out by induction on k that the finite sequence $\left\{\frac{d\widetilde{\omega}_k}{d\lambda} : 0 \leq k \leq n\right\}$ is always positive if $\bar{F}(x,\omega;\lambda) = V(x) - \lambda - \phi(\omega)$, or always negative if $\bar{F}(x,\omega;\lambda) = (\lambda - V(x))\phi(\omega) + 1$.

Recall that we have considered n (and hence h) fixed along all our formulation. Clearly the sequence defined by (3.22) and (3.23) depends on h, and each term depends on the subindex k as well, independently of h; let us briefly denote this double dependence by $\widetilde{\omega}_k(h)$, namely, $\widetilde{\omega}_k(h)$ is the k-th term corresponding to the step h. By fixing the constraint $h = (b-a)/n$, define the sequence $\{\omega_n\}_n$ given by $\omega_n := \widetilde{\omega}_n(h)$ for every positive integer n (actually, we need this definition just for n sufficiently large).

Then we conclude our proof as follows; for the case a), it is assumed that $F(x,\omega;\lambda) = V(x) - \lambda - \omega^2$ and $\xi(\lambda)$ is an increasing function of λ on $(\lambda_0 - \delta, \lambda_0 + \delta)$. We have proved from this hypothesis that for each positive integer $n > (b-a)/\epsilon$, where ϵ is given by (3.21), it holds $\frac{d\omega_n}{d\lambda} \geq 0$ on $(\lambda_0 - \delta, \lambda_0 + \delta)$. Therefore, for every $n > (b-a)/\epsilon$, $\omega_n(\lambda)$ is an increasing function of λ on $(\lambda_0 - \delta, \lambda_0 + \delta)$, i.e., if λ_1, $\lambda_2 \in (\lambda_0 - \delta, \lambda_0 + \delta)$ and $\lambda_1 \leq \lambda_2$, then $\omega_n(\lambda_1) \leq \omega_n(\lambda_2)$. Now, by Corollary A.0.2 of Appendix A, the maximum of the errors with respect to the exact solution, i.e. $\max\{|\widetilde{\omega}_k - \omega(x_k)| : 1 \leq k \leq n\}$ is a sequence depending on n that is estimated above for a bound that converges to zero as n tends to infinity. Therefore it follows that $\omega_n(\lambda) \to \omega(a,\lambda)$ as $n \to \infty$, and hence, taking the limit on both sides of $\omega_n(\lambda_1) \leq \omega_n(\lambda_2)$ as $n \to \infty$, we conclude that $\omega(a,\lambda_1) \leq \omega(a,\lambda_2)$, i.e., $\lambda \mapsto \omega(a,\lambda)$ is increasing. The conclusion for the case b) is similar. ∎

The former Lemma provides a result of monotonicity, given a family of Euler-continuous solutions. Now we introduce the following two Lemmas that explain how we obtain Euler-continuous solutions for problems (3.18) and (3.19) satisfying the hypotheses of Lemma 3.4.1.

Lemma 3.4.2 *Let $V(x) \in C^1([0,1])$ be a nonpositive potential. Then, given $\lambda_0 \in (-\|V\|_\infty, 0)$, there exists and a partition $0 = t_m < t_{m-1} < \ldots < t_1 < t_0 = 1$ of the interval $[0,1]$, such that $\rho(x,\lambda_0)$ is an Euler-continuous solution of (3.18) over each interval of the form $[t_{2i+1}, t_{2i}]$, with $0 \leq i \leq (m-1)/2$ and $\sigma(x,\lambda_0)$ is an Euler-continuous solution of (3.19) over each interval of the form $[t_{2i}, t_{2i-1}]$ with $1 \leq i \leq m/2$. If the potential $V(x)$ is only continuous on $[0,1]$, then we conclude that the solution is just continuous.*

Proof: We start by assuming that $V(x) \in C^1([0,1])$, and proving that if a solution of (3.18) or (3.19) is continuous on an interval $[p,q]$, then such a solution is Euler-continuous on $[p,q]$. Indeed, consider the solution $\rho(x,\lambda_0)$ of (3.18). If one has that $\rho(x,\lambda_0)$ is continuous in x for $p \leq x \leq q$, then from the fact that $V(x) \in C^1([0,1])$ and $\partial_x \rho = V(x) - \lambda_0 - \rho^2$ it follows that $\rho(x,\lambda_0)$ have continuous second derivative (with respect to x) on the interval $[p,q]$, and this second derivative is bounded in $[p, q]$ since $[p, q]$ is compact. But this agrees with the definition of being an Euler-continuous solution on $[p, q]$ (see remark 3 of appendix A). An analogue conclusion can be derived for the solution $\sigma(x, \lambda_0)$ of (3.19).

Thus the Euler-continuity requirement in the statement has been reduced to continuity, and now we assume only continuity on V. Now we construct the partition $\{t_0, t_1, \ldots, t_m\}$ as follows: if $u(x, \lambda_0)$ does not have zeros for $0 \leq x \leq 1$, then we fix $m = 1$ and $t_0 := 1$, $t_1 := 0$; in this case, clearly we have just that $\rho(x, \lambda_0)$ is continuous on $[0, 1]$.

If the set of zeros of $u(x, \lambda_0)$ is nonempty, let $0 \leq z_n < z_{n-1} < \ldots < z_1 < 1$ be these distinct zeros. Then they are also zeros for $\sigma(x, \lambda_0)$, and $\sigma(x, \lambda_0)$ is continuous at z_i for every i, $1 \leq i \leq n$. Along the rest of the proof, we denote by (a, b) the interior set of the closed interval $[a, b]$ in the topological space $[0, 1]$, considered as a subspace of $\mathbb{R}$ with the usual topology. Take for every integer i with $1 \leq i \leq n$ an interval of the form $(t_{2i}, t_{2i-1}) \subseteq [0, 1]$ in such a way that $t_1 < 1$, and the following conditions are satisfied:

i) $z_i \in (t_{2i}, t_{2i-1})$ for every i, $1 \leq i \leq n$

ii) $(t_{2i}, t_{2i-1}) \cap (t_{2j}, t_{2j-1}) = \emptyset$, whenever $i \neq j$.

It may be noticed that one should have $\partial_x u(z_i, \lambda_0) \neq 0$ for every $1 \leq i \leq n$. Otherwise, by existence and uniqueness, $u(x, \lambda_0)$ would identically vanish for every $x \in [0, 1]$. Therefore, since $x \to \partial_x u(x, \lambda_0)$ is continuous on $x = z_i$, $1 \leq i \leq n$, then the intervals (t_{2i}, t_{2i-1}) can be shortened to satisfy the additional property:

iii) No zero of $\partial_x u(x, \lambda_0)$ (and hence of $\rho(x, \lambda_0)$) lies inside $\bigcup_{i=1}^{n} [t_{2i}, t_{2i-1}]$.

We make the desired partition by taking $m = 2n$ if $t_{2n} = 0$, $m = 2n + 1$ if $t_{2n} > 0$; then define $t_0 := 1$, $t_m := 0$, and define the remaining elements t_k for $1 < k < m$ as the extremes of the intervals (t_{2i}, t_{2i-1}), $1 \leq i \leq n$.

Since $\partial_x u(x, \lambda_0)$ does not vanish over any interval of the form $[t_{2i}, t_{2i-1}]$ by condition iii), it follows that $\sigma(x, \lambda_0) = \frac{u(x,\lambda_0)}{\partial_x u(x,\lambda_0)}$ is continuous over each of these intervals. Moreover, since $\{z_1, \ldots z_n\} \subseteq \bigcup_{i=1}^{n} (t_{2i}, t_{2i-1})$ by condition i), then $u(x, \lambda_0)$ does not vanish over any interval in the complement inside $[0, 1]$, i.e., any interval of the form $[t_{2i+1}, t_{2i}]$, and therefore $\rho(x, \lambda_0) = \frac{\partial_x u(x,\lambda_0)}{u(x,\lambda_0)}$ is continuous over each interval of this form. Therefore, $\{t_0, \ldots, t_m\}$ is the desired partition. ∎

Lemma 3.4.3 *Let $V(x) \in C^1([0, 1])$ be a nonpositive potential. Given $\lambda_0 \in (-\|V\|_\infty, 0)$, let $0 = t_m < t_{m-1} < \ldots < t_1 < t_0 = 1$ be the partition of Lemma 3.4.2. Then there exist $\delta > 0$ such that for every λ with $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$, $\rho(x, \lambda)$ is an Euler-continuous solution of* (3.18) *over each interval of the form $[t_{2i+1}, t_{2i}]$, with $0 \leq i \leq (m - 1)/2$ and $\sigma(x, \lambda)$ is an Euler-continuous solution of* (3.19) *over each interval of the form $[t_{2i}, t_{2i-1}]$ with $1 \leq i \leq m/2$. If V is just continuous, then we have the same conclusion with continuity instead of Euler-continuity.*

Proof: We made the partition in Lemma 3.4.2 in such a way that $x \mapsto \partial_x u(x, \lambda_0)$ is nonvanishing on $K := \bigcup_{i=1}^{n} [t_{2i}, t_{2i-1}]$. Hence the continuity of $x \mapsto \partial_x u(x, \lambda_0)$ and the compactness of K imply

$$\min_{x \in K} |\partial_x u(x, \lambda_0)| > 0.$$

Similarly, since $x \mapsto u(x,\lambda_0)$ does not vanish on the compact $L := [0,1] \setminus \overset{\circ}{K}$, then $\min_{x\in L} |u(x,\lambda_0)| > 0$. But by Theorem B.0.6 and the proof of Theorem B.0.7 of Appendix B, it follows that the mappings

$$\lambda \mapsto \min_{x\in K} |\partial_x u(x,\lambda)|, \quad \text{and} \quad \lambda \mapsto \min_{x\in L} |u(x,\lambda)|$$

are continuous on $\lambda = \lambda_0$; thus, there exist some $\delta > 0$ such that both mappings are strictly positive for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Notice that in particular, the derivative $\partial_x u(x,\lambda)$ preserves the same sign on each interval $[t_{2i}, t_{2i-1}]$, $1 \le i \le n$. Therefore, fixed $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, the solution $u(x,\lambda)$ must have at most one zero in x on each of the open intervals (t_{2i}, t_{2i-1}), $1 \le i \le n$. But we made the choice of the partition $\{t_0, \ldots, t_m\}$ in such a way that $z_i \in (t_{2i}, t_{2i-1})$; thus we may take the value $\delta > 0$ in such a way that given $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $u(x,\lambda)$ has *exactly* one zero on each interval (t_{2i}, t_{2i-1}). Hence the solution $u(x,\lambda)$ and its first derivative in x, $\partial_x u(x,\lambda)$ satisfy the conditions i)-iii) at the beginning of the proof of Lemma 3.4.2, replacing λ_0 for λ, and clearly the same conclusion can be derived for λ. This proves the Lemma. ∎

We remark that Lemmas 3.4.2 and 3.4.3 have been proved for a potential defined over the interval $[0,1]$, but this interval may be replaced by any interval $[r,s]$. This is an important fact that allows us to assure the validity of Lemmas 3.4.2 and 3.4.3 if the potential V is piecewise C^1 on $[0,1]$ (or piecewise continuous, if it is the only property needed).

We apply now the Lemma 3.4.2 to determine that the IVPs (3.18) and (3.19) have a unique solution.

Lemma 3.4.4 *Assume that V is piecewise continuous, and let $\lambda \in (-\|V\|_\infty, 0)$ be fixed. If $\rho(x,\lambda)$ is a solution of the IVP* (3.18)*, defined for every $x \in [0,1]$ except perhaps for a finite set of singularities, then $\rho(x,\lambda)$ is defined for every x such that $u(x,\lambda) \neq 0$ and $\rho(x,\lambda) = \dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$.*

Proof: Consider the partition $\{t_0, \ldots, t_m\}$ of Lemma 3.4.2. Then by the Existence and Uniqueness Theorem, $\rho(x,\lambda)$ and $\dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ agree on $[t_1, 1]$, since they both satisfy the initial condition

$$\rho(1,\lambda) = \xi(\lambda) = \frac{\partial_x u(1,\lambda)}{u(1,\lambda)}.$$

Reasoning by contradiction, let $[t_{k+1}, t_k]$ be the first interval from the right to the left in which they do not agree; then they agree in $[t_k, t_{k-1}]$. But in particular, $\rho(t_k,\lambda) = \dfrac{\partial_x u(t_k,\lambda)}{u(t_k,\lambda)}$; then we have the following two possibilities:

i) $k = 2i$ for some integer i. Then $[t_{k+1}, t_k] = [t_{2i+1}, t_{2i}]$. But $\dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ is a continuous solution on this interval for the IVP

$$\begin{aligned} \frac{d\rho}{dx} &= V(x) - \lambda - \rho^2, \quad x \in [t_{2i+1}, t_{2i}] \\ \rho(t_{2i},\lambda) &= \frac{\partial_x u(t_{2i},\lambda)}{u(t_{2i},\lambda)}, \end{aligned} \tag{3.24}$$

and the continuity of this solution implies by the Existence and Uniqueness Theorem that must be unique. But since $\rho(x,\lambda)$ is a solution for (3.18), then clearly is also a solution for (3.24), and hence it must agree with $\dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$, a contradiction.

ii) $k = 2i-1$ for some integer i. Then $[t_{k+1}, t_k] = [t_{2i}, t_{2i-1}]$ and $\dfrac{u(x,\lambda)}{\partial_x u(x,\lambda)}$ is a continuous solution on this interval for the IVP

$$\begin{aligned} \frac{d\sigma}{dx} &= (\lambda - V(x))\sigma^2 + 1, \quad x \in [t_{2i}, t_{2i-1}] \\ \sigma(t_{2i-1},\lambda) &= \frac{u(t_{2i-1},\lambda)}{\partial_x u(t_{2i-1},\lambda)}, \end{aligned} \tag{3.25}$$

meanwhile $\dfrac{1}{\rho(x,\lambda)}$, being a solution of (3.19), must be also a solution of (3.25), which implies again that $\rho(x,\lambda)$ agrees with $\dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ and makes a contradiction as well.

Thus $\rho(x,\lambda)$ must agree with $\dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ on each interval of the form $[t_{k+1}, t_k]$, and hence on $[0,1]$. ∎

Now we prove Theorem 3.3.1 for the case of $V(x) \in C^1([0,1])$; its proof is a simple induction made by the succesive application of Lemmas 3.4.1 and 3.4.2. Recall that $\rho(x,\lambda) = \dfrac{\partial_x u(x,\lambda)}{u(x,\lambda)}$ for every (x,λ) such that $u(x,\lambda) \neq 0$; in particular, we consider the application $\lambda \to \rho(0,\lambda)$ as a function on the domain

$$D := \{\lambda \in [-\|V\|_\infty, 0) : u(0,\lambda) \neq 0\}.$$

Let us restate the Theorem for easy reference.

Theorem 3.4.5 *Let $\rho = \rho(x,\lambda)$ be the unique solution to the IVP (3.18); suppose that $V \in C^1([0,1])$ and $\xi(\lambda)$ is increasing. Then $\lambda \to \rho(0,\lambda)$ is increasing over any open interval contained in D.*

Proof: Let $0 = t_m < t_{m-1} < \ldots < t_1 < t_0 = 1$ be the partition of $[0,1]$ given by Lemma 3.4.2. Then, given λ_0 in $\mathring{D}$, $\rho(x,\lambda_0)$ is continuous as an x-function over the intervals of the form $[t_{2i+1}, t_{2i}]$, $0 \leq i \leq (m-1)/2$, and $\sigma(x,\lambda_0)$ is continuous as an x-function over the intervals of the form $[t_{2i}, t_{2i-1}]$, $1 \leq i \leq m/2$. Then we can define, at least in $\lambda = \lambda_0$, the function

$$\Phi(\lambda) := \sum_{i \leq (m-1)/2} \sup_{x \in [t_{2i+1}, t_{2i}]} |\rho(x,\lambda)| + \sum_{i \leq m/2} \sup_{x \in [t_{2i}, t_{2i-1}]} |\sigma(x,\lambda)|,$$

which of course takes a real finite value in λ_0. But by Lemma 3.4.3, there exists a value $\delta > 0$ such that $(\lambda_0 - \delta, \lambda_0 + \delta) \subseteq \mathring{D}$, and the function $\Phi(\lambda)$ is well defined and finite-valued for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$; and by Theorem B.0.7, it holds that $\Phi(\lambda)$ is continuous in $\lambda = \lambda_0$. Then, by shorten δ if necessary, it can be found $M > 0$ such that $\Phi(\lambda) = |\Phi(\lambda)| < M$ for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$; in particular, it holds that $|\rho(x,\lambda)| \leq M$ for every $x \in \bigcup_i [t_{2i+1}, t_{2i}]$, and $|\sigma(x,\lambda)| \leq M$ for every

$x \in \bigcup_i [t_{2i}, t_{2i-1}]$. Now, the statement follows by induction; each induction step consist of applying a couple of times the Lemma 3.4.1. In the first "induction half-step", we apply the Lemma 3.4.1 to the obtained constant M, and the IVP

$$\begin{aligned} \frac{d\rho}{dx} &= V(x) - \lambda - \rho^2, \;\; x \in [t_{2i+1}, t_{2i}] \\ \sigma(t_{2i+1}, \lambda) &= \frac{1}{\rho(t_{2i+1}, \lambda)}, \end{aligned} \tag{3.26}$$

where $\xi_0(\lambda) = \xi(\lambda)$ and $\xi_i(\lambda) = \dfrac{1}{\sigma(t_{2i}, \lambda)}$ if $i > 0$; In the second half-step, we apply the Lemma 3.4.1 to the constant M, and the IVP

$$\begin{aligned} \frac{d\sigma}{dx} &= (\lambda - V(x))\sigma^2 + 1, \;\; x \in [t_{2i+2}, t_{2i+1}] \\ \sigma(t_{2i+1}, \lambda) &= \frac{1}{\rho(t_{2i+1}, \lambda)}. \end{aligned} \tag{3.27}$$

Notice that by the solutions ρ and σ in the right hand side of (3.26), (3.27) we refer to the ones obtained in the last induction step; these initial conditions have been taken to agree the solutions over the subintervals with the solution over the whole interval $[0, 1]$ of the IVPs (3.18) and (3.19). ∎

3.5 Complementary Remarks on the monotonicity of $F_\mu(\lambda)$

We have proved that $F_N(\lambda)$ and $F_D(\lambda)$ are respectively increasing and decreasing over each interval on its domain. The previous monotonicity result allows us to note further facts of the form of these functions. Indeed, let us write formulas (3.13) and (3.14) in terms of the solution $u(x, \lambda)$ to the IVP (3.17). We have that

$$F_N(\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\partial_x u(0, \lambda)}{\sqrt{-\lambda} u(0, \lambda)} \right), \tag{3.28}$$

$$F_D(\lambda) = \frac{1}{\sqrt{-\lambda}} \operatorname{arctanh}\left(\frac{\sqrt{-\lambda} u(0, \lambda)}{\partial_x u(0, \lambda)} \right). \tag{3.29}$$

It should be noticed that in the context of Theorem 3.2.1 the domain of F_μ is not the natural domain that corresponds to the formulas (3.28) and (3.29), for λ is an eigenvalue of the operator H_a^μ if and only if $F_\mu(\lambda) > 0$, and hence the domain is restricted to the set $\{\lambda \in [-\|V\|_\infty, 0) : F_\mu(\lambda) > 0\}$, $\mu = N, D$. Let us drop off for the moment this restriction and consider, for example, the natural domain of $F_N(\lambda)$ defined by (3.28). Then a value $\lambda \in [-\|V\|_\infty, 0)$ belongs to this natural domain if and only if $\sqrt{-\lambda}|u(0, \lambda)| > |\partial_x u(0, \lambda)|$ (in particular, $\sqrt{-\lambda} u(0, \lambda) \neq 0$). Now, Theorem 3.4.5 can be applied to conclude that the mapping

$$g(\lambda) := \frac{\partial_x u(0, \lambda)}{\sqrt{-\lambda} u(0, \lambda)}$$

is increasing on any interval contained in $[-\|V\|_\infty, 0)$ in which $g(\lambda) \geq 0$. Moreover, a value $\lambda_0 \in [-\|V\|_\infty, 0)$ is a boundary value for the domain of $F_N(\lambda)$ if and only if $|g(\lambda_0)| = 1$. In particular, suppose that λ_0 is a boundary point of the set

$$\{\lambda \in [-\|V\|_\infty, 0) : F_N(\lambda) \geq 0\}. \tag{3.30}$$

Then the monotonicity of $g(\lambda)$ implies that such value λ_0 is the right-hand extreme for a connected component in the domain of $F_N(\lambda)$. Consequently, since $\operatorname{arctanh}(x)$ tends to $+\infty$ as x tends to 1 for the left, it follows that

$$\lim_{\lambda \to \lambda_0^-} F_N(\lambda) = +\infty. \tag{3.31}$$

Moreover, if λ_1 is a boundary point of the domain of $F_N(\lambda)$ but not of the set (3.30), then we have in this case $g(\lambda_1) = -1$ and hence λ_1 satisfies

$$\lim_{\lambda \to \lambda_1} F_N(\lambda) = -\infty.$$

However, nothing can be concluded about whether λ_1 is a left-hand side or a right-hand side extreme, since despite Theorem 3.4.5, $g(\lambda)$ need not to be increasing when it is negative. We shall see an example in the section 3.7 for which both cases occur. The same reasoning can be made for $F_D(\lambda)$, to conclude that if λ_0 is a boundary point to $\{\lambda \in [-\|V\|_\infty, 0) : F_D(\lambda) \geq 0\}$, then λ_0 is a left-hand side extreme that satisfies

$$\lim_{\lambda \to \lambda_0^+} F_D(\lambda) = +\infty, \tag{3.32}$$

and we have similar conclusions values λ_1 that are boundary points on the negative range of $F_D(\lambda)$.

Now let us consider the case $\lambda_0 = 0$; it also fails to belong to the natural domain of $F_N(\lambda)$, but it might occur that $\lim_{\lambda \to 0^-} F_N(\lambda)$ exists. If this is the case, then necessarily $\rho(0,0) = 0$ by (3.28), but by Theorem 3.4.5 $\rho(0,\lambda)$ is increasing in λ, and hence $\rho(0,\lambda)$ must be negative for small negative values of λ implying the same for $F_N(\lambda)$. Therefore such a finite limit has to be nonpositive and hence $\lambda_0 = 0$ is not a boundary point for «positive range part» of the domain of $F_N(\lambda)$. On the other hand, we cannot derive the same conclusion for $F_D(\lambda)$, since in this case it might happen that $\rho(0,0) > 0$ and still $\lim_{\lambda \to 0^-} F_D(\lambda)$ might exist. Indeed we will see in the examples below that this case is possible.

These facts will allow us to provide most of the features on the behavior of the functions $F_\mu(\lambda)$, especially in its positive range, which is the relevant part of the range to our purposes. In the following lemma we show another important fact about the sign of $\rho(0,\lambda)$ (which is the same sign of $F\mu(\lambda)$, $\mu = N, D)$ in $\lambda := -\|V\|_\infty$.

Lemma 3.5.1 *Let $\rho(x,\lambda)$ be the unique solution of the IVP* (3.18)*, with $\xi(\lambda) = -\sqrt{-\lambda}$. Then $\lambda = -\|V\|_\infty$ is in the domain of $\lambda \mapsto \rho(0,\lambda)$ and $\rho(0,\lambda) < 0$.*

Proof: Let $\epsilon > 0$. Consider the solution ρ_ϵ of the IVP

$$\begin{aligned} \partial_x \rho_\epsilon &= V(x) - \lambda + \epsilon - \rho_\epsilon^2, \;\; x \in [0,1] \\ \rho_\epsilon(1,\lambda) &= -\sqrt{-\lambda}, \end{aligned}$$

It can be found in the literature (see e.g. Theorem 1.11.1 of [3]) that the dependence on the parameter ϵ is uniformly continuous, i.e. the solution ρ_ϵ of (3.5) converges uniformly to the solution of (3.18) as $\epsilon \to 0$. Bearing in mind this fact, we compare ρ_ϵ with the solution of the IVP

$$\begin{aligned} \partial_x \hat{\rho} &= -\hat{\rho}^2, \quad x \in [0,1] \\ \hat{\rho}(1,\lambda) &= -\sqrt{-\lambda}. \end{aligned}$$

Specifically, let $x_0 \in (0,1]$ be such that $\rho_\epsilon(x_0,\lambda) = \hat{\rho}(x_0,\lambda)$. Then $\partial_x \rho_\epsilon(x_0,\lambda) - \partial_x \hat{\rho}(x_0,\lambda) = V(x_0) - \lambda + \epsilon$, and therefore the difference between both solutions $\hat{\rho}_\epsilon := \rho_\epsilon - \hat{\rho}$ satisfy $\hat{\rho}_\epsilon(x_0,\lambda) = 0$ and $\partial_x \hat{\rho}_\epsilon(x_0,\lambda) > 0$. Hence, since the derivatives of the solutions are continuous, there exist $\delta > 0$ such that $\partial_x \hat{\rho}_\epsilon(x,\lambda) > 0$ for every $x \in (x_0 - \delta, x_0]$; thus $\hat{\rho}_\epsilon(x,\lambda)$ must be increasing in $x \in (x_0 - \delta, x_0]$, but since $\hat{\rho}_\epsilon(x_0,\lambda) = 0$, then it must be negative for $x \in (x_0 - \delta, x_0]$. It follows that $\rho_\epsilon(x,\lambda) < \hat{\rho}(x,\lambda)$ for every $x \in (x_0 - \delta, x_0]$.

Now, actually the problem (3.5) can be solved explicitly, and is solution is given by

$$\hat{\rho}(x,\lambda) = \frac{1}{x - \left(1 + \frac{1}{\sqrt{-\lambda}}\right)} = \frac{\sqrt{-\lambda}}{\sqrt{-\lambda}x - \left(\sqrt{-\lambda} + 1\right)}.$$

In particular, $\hat{\rho}(x,\lambda)$ is continuous in $x \in [0,1]$, and $\hat{\rho}(0,\lambda) < 0$. Since $\hat{\rho}(1,\lambda) = \rho_\epsilon(1,\lambda) = -\sqrt{-\lambda}$, the above facts imply that $\rho_\epsilon(x,\lambda)$ is bounded above by $\hat{\rho}(x,\lambda)$ on the maximum interval of the form $(t,1] \subseteq [0,1]$ in which $\rho_\epsilon(x,\lambda)$ is continuous. But if $t > 0$, we would have that $\rho_\epsilon(x,\lambda)$ is not bounded from below in $[0,1]$, implying that its derivative $\partial_x \rho_\epsilon = V(x) - \lambda + \epsilon - \rho_\epsilon^2$ is not bounded from above in $[0,1]$, which is an absurd. Hence $t = 0$ and we have concluded that $\rho_\epsilon(0,\lambda) \le \hat{\rho}(0,\lambda) < 0$. Taking $\epsilon \to 0$, it follows that $\rho(0,\lambda) < 0$. ∎

In order to avoid ambiguity, we refer as the *eigenvalue domains* for the domains of the functions $F_N(\lambda)$ and $F_D(\lambda)$ in the context of Theorem 3.2.1, i.e. the subsets of the natural domains consisting of the values λ such that $F_\mu(\lambda) > 0$, $\mu = N, D$. We apply the former observations, and Lemma 3.5.1 to conclude the following

Theorem 3.5.2

a) Let (α,β) be a connected component in the eigenvalue domain of $F_N(\lambda)$. Then $\lim_{\lambda\to\beta^-} F_N(\lambda) = \infty$, and $\lim_{\lambda\to\alpha^+} F_N(\lambda) = 0$.

b) Let (γ,δ) be a connected component in the eigenvalue domain of $F_D(\lambda)$. Then $\lim_{\lambda\to\gamma^+} F_D(\lambda) = \infty$, and $\lim_{\lambda\to\delta^-} F_D(\lambda) = 0$ if $\delta < 0$.

Proof: Consider the case a). Since $F_N(\lambda)$ is increasing on (α,β), then it is bounded below and is naturally defined for $\lambda = \alpha$. But this implies $F_N(\alpha) = 0$; otherwise, $F_N(\lambda)$ can be defined positively on an interval of the form (α_1,β) with $\alpha_1 < \alpha$, in contradiction with the fact that (α,β) is a connected component of the eigenvalue domain. On the other hand, again since $F_N(\lambda)$ is increasing on (α,β), it follows that $F_N(\lambda)$ cannot be evaluated in β by a similar reason than the one provided for α. Therefore β is a point in the boundary of the natural domain of $F_N(\lambda)$, and hence $\lim_{\lambda\to\beta^-} F_N(\lambda) = \infty$ according to (3.31) and the fact that $F_N(\lambda) > 0$.

The case b) is identical, with the difference that $F_D(\lambda)$ is a decreasing function. ∎

3.6 Estimates on the Number of Eigenvalues

Another feature that can be improved for describing the functions $F_\mu(\lambda)$ and the point spectra of the operator H_a is to estimate from above the number of eigenvalues, in the same spirit as the inequalities (1.17) and (1.21) in Chapter 1. The key to obtain a corresponding result for our case is to take into account that there exist one and only one eigenvalue for each connected component in the eigenvalues domain of the functions $F_\mu(\lambda)$. Hence the counting of eigenvalues is made by counting the connected components. Indeed we have the following

Theorem 3.6.1 *Let $N_\_(H_a)$ be the number of negative eigenvalues of the operator H_a given by* (3.1). *Then it holds the inequality*

$$N_\_(H_a) \le 2 + 2\int_0^1 x|V(x)|\,dx \tag{3.33}$$

for every $a > 0$.

Proof: Let $a > 0$ be fixed. We know that the spectrum of the operator H_a is the same as the spectrum of the equivalent operator $H_a^N \oplus H_a^D$, and hence the number of its eigenvalues is the sum of the number of eigenvalues of H_a^N and H_a^D. But by Theorems 1.2.3 and 1.2.5, the Neumann and Dirichlet eigenvalue numbers $N_\_(H_a^N)$ and $N_\_(H_a^D)$ are estimated by

$$N_\_(H_a^D) \le \int_0^\infty x|V(x-a)|\,dx,$$

$$N_\_(H_a^N) \le 1 + \int_0^\infty x|V(x-a)|\,dx.$$

Therefore, the total number of eigenvalues satisfy the inequality

$$N_\_(H_a) \le 1 + 2\int_0^\infty x|V(x-a)|\,dx.$$

now, take the limit as $a \to 0$ by the right on both sides; we get

$$\lim_{a\to 0^+} N_\_(H_a) \le 1 + 2\int_0^\infty x|V(x)|\,dx = 1 + 2\int_0^1 x|V(x)|\,dx, \tag{3.34}$$

since the support of the potential $V(x)$ is contained in the interval $[0,1]$. Now it is clear that the limit of the left-hand side of (3.34) is equal to the number of elements in the union of sets of zeros of $F_N(\lambda)$ and $F_D(\lambda)$. But each zero corresponds injectively with a connected component for the union of eigenvalue domains of the functions $F_N(\lambda)$ and $F_D(\lambda)$, except perhaps for the first connected component at the right, i.e., the component with right-hand extreme $\lambda = 0$, for which its corresponding eigenvalue may arise from a positive value of a. Thus the right-hand side of (3.34) may be increased by 1; This proves the desired estimate. ∎

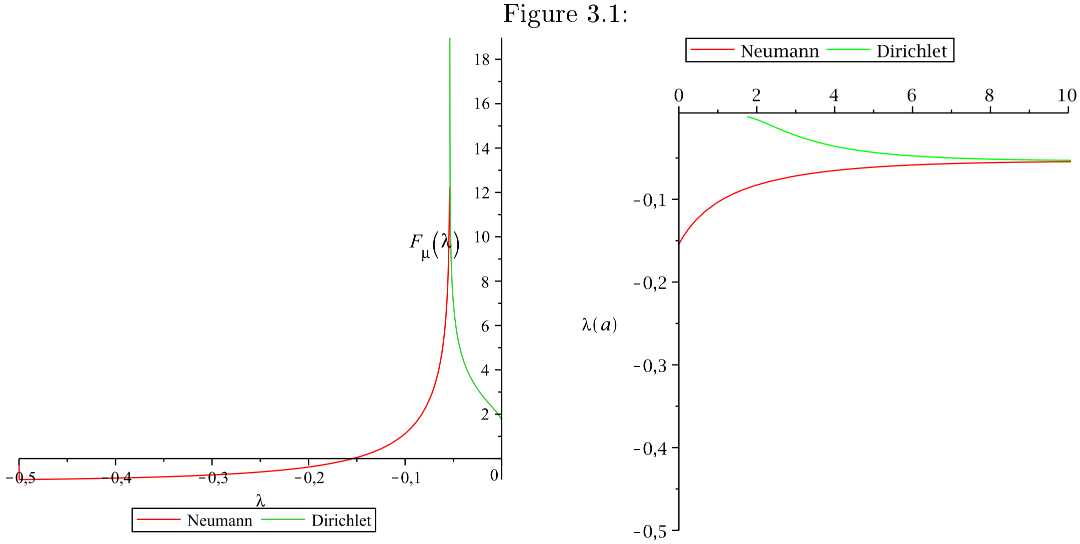
3.7 Examples

1. *Constant potential; sharpness of the estimate* (3.33): Fix $M > 0$ and let $V(x) \equiv -M$ for $0 \leq x \leq 1$. The Riccati IVP for this potential is $\frac{d\rho}{dx} = -M - \lambda - \rho^2$, $\rho(1,\lambda) = -\sqrt{-\lambda}$, $0 \leq x \leq 1$. This equation can be easily solved by separation of variables to obtain

$$\rho(x,\lambda) = -\tan\left(x\sqrt{M+\lambda} - \sqrt{M+\lambda} + \arctan\left(\frac{\sqrt{-\lambda}}{\sqrt{M+\lambda}}\right)\right)\sqrt{M+\lambda}$$

and we have in this case an explicit expression for $\rho(0,\lambda)$ and for the functions $F_\mu(\lambda)$ as well. Let us take in particular $M = 0.5$, and consider the plot of the graphs of the functions $F_N(\lambda)$, $F_D(\lambda)$, on its natural domains; this is given by the left graph in figure 3.1.

Figure 3.1:



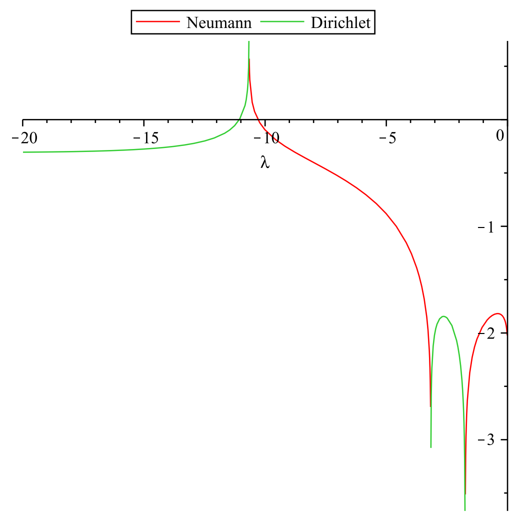
Since the eigenvalues, as functions of the parameter a, are local inverses of the functions $F_\mu(\lambda)$, then the graphs of $\lambda_1(a)$, $\lambda_2(a)$ are given by the inverse graph of the left figure 3.1, which is given by the right figure 3.1. The eigenvalue below is a Neumann eigenvalue and starts from $a = 0$, and the eigenvalue above is a Dirichlet eigenvalue that starts from a positive value of a, which corresponds to the limit as λ tends to zero by the right side of $F_D(\lambda)$ (see remarks at the beginning of section 3.5). Moreover it can be noticed that the right-hand side of (3.33) equals 2.5, implying that the maximum number of possible eigenvalues should be 2, which is exactly the number of eigenvalues that arise as $a \to \infty$. Therefore the estimate (3.33) is as sharp as the the estimates provided by Theorems 1.2.3 and 1.2.5.

2. *Monotonicity may fail on the negative range:* Consider the potential

$$V(x) = \begin{cases} -4.04 & \text{if} \quad 0 \leq x < 0.5 \\ -20 & \text{if} \quad 0.5 \leq x \leq 1 \end{cases}$$

This is a piecewise constant potential and as well as the first example, the solution of its associated Riccati IVP can be obtained by finding the solution $\rho(x, \lambda)$ on the piece of the potential in the interval $[0.5, 1]$, and then finding the solution on $[0, 0.5]$ where the initial conditions are provided to fit the former piece of the solution at $x = 0.5$. This is an useful example to illustrate why the condition $\rho(\lambda, 0) \geq 0$ was essential in the proof of Theorem 3.3.2. Indeed, although the functions $F_\mu(\lambda)$ are monotone in the positive part of the range, when $F_\mu(\lambda)$, $\mu = N, D$ are considered on its natural domain we get that there exist values $\lambda \in [-20, 0]$ for which $F_\mu(\lambda) < 0$ and the functions attain local extremas in this values (See figure 3.2). Hence condition $\rho(\lambda, 0) \geq 0$ cannot be dropped in Theorem 3.3.2.

Figure 3.2:



Appendix A

The Euler Method

We consider solutions of differential equations of the form

$$y' = F(x, y), \quad y(0) = \alpha \tag{A.1}$$

in our work. For the monotonicity results on the dependence of a parameter in the initial condition (either α or another one from which α depends on), numerical approximations by Euler Method to the unique solution of (A.1) are easier to handle than the exact solution. Hence we state without proof the following theorem from [8] describing the Euler approximations and its convergence:

Theorem A.0.1 *Let I be the open interval $(0, L)$ and $\overline{I}$ the closed interval $[0, L]$. Assume the initial value problem (A.1) has the unique solution $Y(x)$ on $\overline{I}$, i.e.,*

$$Y'(x) \equiv F(x, Y(x))$$

and

$$Y(0) = \alpha.$$

Assume that $Y'(x)$ and $Y''(x)$ are continuous, and there exist positive constants B, N such that

$$|Y''(x)| \leq N, \;\; 0 \leq x \leq L \tag{A.2}$$

$$\left|\frac{\partial F}{\partial y}\right| \leq B, \;\; 0 \leq x \leq L, \;\; -\infty < y < \infty. \tag{A.3}$$

Next, let n be a fixed positive integer, and let $\overline{I}$ be subdivided into n equal parts by the grid points $x_0 < x_1 < x_2 < \ldots < x_n$, where $x_0 = 0$, $x_n = L$. Denote the grid size h by

$$h = L/n. \tag{A.4}$$

Let y_k be the numerical solution of (A.1) by Euler method on the grid points, so that

$$\begin{aligned} y_{k+1} &= y_k + hF(x_k, y_k), \quad k = 0, 1, 2, \ldots, n-1 \\ y_0 &= \alpha \end{aligned} \tag{A.5}$$

and define the error E_k at each grid point x_k by

$$E_k = Y_k - y_k, \;\; k = 0, 1, 2, \ldots, n.$$

Then

$$|E_k| \leq \frac{[(1+Bh)^k - 1]Nh}{2B}, \;\; k = 0, 1, 2, \ldots, n. \tag{A.6}$$

This theorem has a straightforward corollary that we have modified from its original presentation in [8] to our purposes.

Corollary A.0.2 *Under the assumptions of theorem A.0.1, for every positive value $h \in \{L/n : n \in \mathbb{Z}^+\}$ it follows that*

$$S(h) := \max\{|E_k| : 1 \leq k \leq n\} \leq \frac{(e^{BL} - 1)Nh}{2B} \tag{A.7}$$

Notice that taking h fixed is equivalent to take n fixed, but in our conventions we prefer to make explicit the dependence on h for simplicity of notations.

Proof: Since $(1 + Bh) > 1$, then the largest value of $(1 + Bh)^k$ results when $k = n$. Thus, from (A.6),

$$|E_k| \leq \frac{[(1+Bh)^n - 1]Nh}{2B},$$

which, since $n = L/h$, can be written as

$$\begin{aligned} |E_k| &\leq \frac{\left\{\left[(1+Bh)^{\frac{1}{Bh}}\right]^{BL} - 1\right\}Nh}{2B} \\ &\leq \frac{\left\{e^{BL} - 1\right\}Nh}{2B}, \end{aligned}$$

where we employed the fact that $(1 + x)^{1/x} \leq e$ for every $x > 0$. Since the right hand side does not depend on k, then (A.7) follows. ∎

Remark 1 If *a priori* (A.3) does not necessarily holds, but there exist an $M > 0$ such that the solution $Y(x)$ satisfies:

$$|Y(x)| \leq M \quad \text{for every} \quad x \in \overline{I},$$

then theorem A.0.1 may be adapted by replacing condition (A.3) with:

$$\text{There exist } B > 0 \text{ such that } \left|\frac{\partial F}{\partial y}\right| \leq B, \;\; 0 \leq x \leq L, \;\; -M \leq y \leq M. \tag{A.8}$$

Indeed, given a function $F(x, y)$ satisfying condition (A.8), we can define the function $F_M(x, y)$ for every $(x, y) \in [0, L] \times [-M, M]$ by:

$$F_M(x,y) = F(x,0) + \int_0^y G(x,t)\,dt, \;\; \text{where } G(x,t) = \begin{cases} F_y(x,M) & \text{for every } \;\; t > M \\ F_y(x,t) & \text{for every } \;\; |t| \leq M \\ F_y(x,-M) & \text{for every } \;\; t < -M \end{cases}$$

It is clear that if $Y(x)$ is the unique solution for (A.1), then it is also the unique solution for

$$y' = F_M(x, y), \;\; y(0) = \alpha. \tag{A.9}$$

In this case, we can apply equivalently theorem A.0.1 and corollary A.0.2 to the problem (A.9) instead of (A.1). Hence the corresponding Euler aproximation will be

$$\begin{aligned} y_{k+1} &= y_k + hF_M(x_k, y_k), \qquad k = 0, 1, 2, \ldots, n-1 \\ y_0 &= \alpha. \end{aligned} \tag{A.10}$$

Remark 2 All these facts are extended in an obvious way to the initial value problems of the form

$$y' = F(x, y), \qquad y(x_0) = \alpha$$

over intervals of the form $[x_0, x_1]$ or $[x_1, x_0]$. In particular we are mainly interested in problems with initial condition at the right hand side. In this case, the Euler method is made from the right to the left, so the recursion step (A.10) should be replaced by

$$y_{k+1} = y_k - hF_M(x_k, y_k), \qquad k = 0, 1, 2, \ldots, n-1$$

Remark 3 If there exist constants B, M, N, such that the solution $Y(x)$ of the initial value problem (A.1) (or a general initial value problem) satisfies the hypotheses (A.2) and (A.8), we say that $Y(x)$ that it is an *Euler-continuous* solution. In other words, an Euler-continuous solution is a solution such that its numerical Euler approximation converges as the step h tends to zero.

Appendix B

Some Facts of Ordinary Differential Equations Theory

We summarize in this appendix some of the main facts from ordinary differential equations used to derive the results in chapter 2. We adopt here the presentation made in [3]. See [13] and [4] for further reference.

Consider the differential equation

$$\frac{du}{dx} = f(x, u) \tag{B.1}$$

It may be assumed in general that f is a continuous function defined on a subset U of $\mathbb{R} \times E$, which is often supposed to be open but might be closed sometimes. In [3], E is taken to be a real Banach space, but we shall be interested in the special case of $E = \mathbb{R}^n$, and we impose as well some stronger conditions over f in the propositions below.

Let I be a real interval. A solution of the differential equation (B.1) is defined to be a C^1 function $\varphi : I \mapsto R^n$ that satisfies the conditions

I) $(x, \varphi(x)) \in U$ for every $x \in I$

II) $\varphi'(x) = f(x, \varphi(x))$ for every $x \in I$.

Moreover, let $\epsilon > 0$; we say that a piecewise C^1 function $\varphi : I \mapsto \mathbb{R}^n$ is an approximate solution of (B.1) with accuracy less than ϵ, or an ϵ-approximate solution, if the following conditions are verified:

i) $(x, \varphi(x)) \in U$ for every $x \in I$

ii) $\|\varphi'(x) - f(x, \varphi(x))\| \leq \epsilon$ for every $x \in I$

By a piecewise C^1 function we refer to a function that is continuous on the interval I and has continuous derivative on I except on a finite set of points, where it has just left-hand and right-hand sided derivatives. Of course, it has to be noticed that condition ii) is satisfied in the generalized sense of sided derivatives for the piecewise C^1 function φ.

It is clear that a solution of the differential equation (B.1) is an ϵ-approximate solution for every $\epsilon > 0$. In order to emphasize, we refer to a solution, satisfying the conditions I) and II) above, as an *exact* solution.

These definitions are the basic elements in the development of subsequent propositions in [3]. We present now the statement of one of those propositions, which is presented in [3] as «The Fundamental Lemma» and works as the essential tool to derive the local existence an uniqueness theorem for initial value problems with the differential equation (B.1), as well as other several classic propositions in ODE theory.

Lemma B.0.3 *Let $\varphi_1 : I \mapsto \mathbb{R}^n$ an ϵ_1-approximate solution, and $\varphi_2 : I \mapsto \mathbb{R}^n$ an ϵ_2-approximate solution of the equation* (B.1)*; let $u_1 = \varphi_1(x_0)$, and $u_2 = \varphi_2(x_0)$ its initial values for $x_0 \in I$. Suppose that f is k-lipschitzian in u; then, for every $x \in I$,*

$$\|\varphi_1(x) - \varphi_2(x)\| \leq \|u_1 - u_2\| e^{k|x-x_0|} + (\epsilon_1 + \epsilon_2)\frac{e^{k|x-x_0|} - 1}{k} \tag{B.2}$$

This lemma implies obviously the following

Corollary B.0.4 *If in the last lemma, φ_1 and φ_2 are exact solutions, then the formula* (B.2) *is replaced by*

$$\|\varphi_1(x) - \varphi_2(x)\| \leq \|u_1 - u_2\| e^{k|x-x_0|}$$

A fact that may be simply noticed from this corollary is that if u_1 and u_2 are the respective initial values for the solutions $\varphi_1(x)$ and $\varphi_2(x)$, and the interval of definition I for both the initial value problems is compact, then $\varphi_1(x)$ converges uniformly to $\varphi_2(x)$ as u_1 tends to u_2. This would allow us to consider as the initial condition, a continuous function of an extra parameter λ, and we could derive that a solution $\varphi(x;\lambda)$ depends on the parameter λ «in the sense of uniform convergence»; nevertheless, we are interested in initial value problems that have a dependence on λ not just in the initial condition, but in the differential equation as well. Therefore we need a corresponding result that describes the behavior of the solution of initial value problems with differential equations of the form (B.1), but with an extra dependence of a parameter in the function f, i.e. $f(x, u; \lambda)$ instead of $f(x, u)$. We state without proof the main result about this issue from [3].

Theorem B.0.5 *Consider the differential equation*

$$\frac{du}{dx} = f(x, u, \lambda) \tag{B.3}$$

where f is a continuous function

$$f : I \times \overline{B}(u_0, r) \times L \mapsto \mathbb{R}^n,$$

being I a compact interval, $\overline{B} = \overline{B}(u_0, r)$ the closed ball $\|u - u_0\| \leq r$ in $\mathbb{R}^n$ and L is a metric space. Suppose that the following conditions are fulfilled:

i) $\|f(x, u, \lambda)\| \leq M$ *for every* $(x, u, \lambda) \in I \times \overline{B} \times L$,

ii) $\|f(x, u_1, \lambda) - f(x, u_2, \lambda)\| \leq k\|u_1 - u_2\|$ *for any* $x \in I$, $\|u_1 - u_0\| \leq r$, $\|u_2 - u_0\| \leq r$, $\lambda \in L$.

Then, for every $\lambda \in L$, the equation (B.3) *has an unique solution $u_\lambda(x)$ defined over the interval*

$$J = I \cap \left[x_0 - \frac{r}{M}, x_0 + \frac{r}{M}\right]$$

such that $u_\lambda(x_0) = u_0$; furthermore, it verifies that u_λ converges uniformly to u_{λ_0} whenever λ converges to λ_0.

Remark Theorem B.0.5 has been stated for a general function $f : I \times \overline{B}(u_0, r) \times L \mapsto \mathbb{R}^n$ satisfying several hypotheses. In our case, we are interested in consider a function

$$f : I \times \mathbb{R}^n \times L \mapsto \mathbb{R}^n$$

of the form

$$f(x, u, \lambda) = A(x, \lambda)u,$$

i.e., a function that is linear in u. Of course, $A(x, \lambda)$ is a real $n \times n$ matrix for any $(x, \lambda) \in I \times L$ which varies continuously with respect to (x, λ) in the Banach space of real $n \times n$ matrices, with the usual matrix norm. This function does not exactly fit in the hypotheses of theorem B.0.5, but given any $r > 0$, the restriction of f to $I \times \overline{B}(u_0, r) \times L$ do satisfy these hypotheses. Henceforth it raises the natural question about whether the steps of the proof given in [3] for the theorem work as well in the case of $f(x, u, \lambda) = A(x, \lambda)u$; a priori, it might be possible that the solution has not a bounded range of values and it is defined only in a proper subinterval of the interval I, which may be even noncompact. Nevertheless, the existence and uniqueness theorem for first-order linear initial value problems guarantees the definition of the solution along the whole compact interval I, in contrast with the case of nonlinear functions, for which we have the results only locally. Thus, exceptionally we can carry out the proof of the theorem for the u-linear function $f(x, u, \lambda) = A(x, \lambda)u$, to conclude that the solution $u_\lambda(x)$ converges uniformly to $u_{\lambda_0}(x)$ on I, as λ tends to λ_0, and there is not essentially further work to do for deriving this conclusion; see [3].

Now, by means of corollary B.0.4 and theorem B.0.5, one gets the following

Theorem B.0.6 *Let I, $L \subseteq \mathbb{R}$ be real compact intervals, and let $A(x, \lambda)$ be a continuous function that maps $I \times L$ into the space of real $n \times n$ matrices. Consider the initial value problem*

$$\begin{aligned} \frac{du}{dx} &= A(x, \lambda)u(x), \qquad x \in I \\ u(x_0) &= u_0(\lambda) \end{aligned} \tag{B.4}$$

where $u_0(\lambda)$ is a continuous function defined in L into $\mathbb{R}^n$. Then for every $\lambda \in L$, there exist an unique solution $u_\lambda(x)$ for (B.4)*, defined for every $x \in I$, and if λ converges to λ_0 in L, then u_λ converges to u_{λ_0} uniformly on I.*

Proof: Let us consider at first the initial value problem

$$\begin{aligned} \frac{du}{dx} &= f(x, u, \lambda) \\ u(x_0) &= u_0(\mu), \end{aligned} \tag{B.5}$$

where $f(x, u, \lambda) = A(x, \lambda)u$, and $\lambda, \mu \in L$. For every $(\lambda, \mu) \in L \times L$, denote the solution for (B.5) as $u(x, \lambda, \mu)$. Our aim is to show that $u(\cdot, \lambda, \lambda)$ converges uniformly to $u(\cdot, \lambda_0, \lambda_0)$.

Consider the initial value problem (B.5), assuming $\mu \in L$ is fixed; by theorem B.0.5 and the subsequent remark, we have that $u(\cdot, \lambda, \mu)$ converges uniformly to $u(\cdot, \lambda_0, \mu)$ on I as λ tends to λ_0. Therefore, given $\epsilon > 0$, and given $\lambda_0 \in L$, there exist $\delta_1 > 0$ such that for every $\lambda \in L$ with $|\lambda - \lambda_0| < \delta_1$, it holds that

$$\sup_{x \in I} \|u(x, \lambda, \mu) - u(x, \lambda_0, \mu)\| < \frac{\epsilon}{2}.$$

On the other hand, the mapping $(x,\lambda) \mapsto \|A(x,\lambda)\|$ is continuous over $I \times L$, which is compact. This implies that there exist a constant $k > 0$ such that $\|A(x,\lambda)\| \leq k$ for every $(x,\lambda) \in I \times L$; thus

$$\begin{aligned} \|f(x,u_1,\lambda) - f(x,u_2,\lambda)\| &= \|A(x,\lambda)u_1 - A(x,\lambda)u_2\| \\ &= \|A(x,\lambda)(u_1 - u_2)\| \\ &\leq \|A(x,\lambda)\|\|u_1 - u_2\| \\ &\leq k\|u_1 - u_2\|, \end{aligned}$$

i.e., f is k-lipschitzian with respect to u. Hence corollary B.0.4 tells us as well that given μ_1, $\mu_2 \in L$, one has the inequality

$$\|u(x,\lambda_0,\mu_1) - u(x,\lambda_0,\mu_2)\| \leq \|u_0(\mu_1) - u_0(\mu_2)\| e^{k|x-x_0|} \quad \text{for every} \quad x \in I.$$

but I is a compact interval and u_0 is a continuous function, so there exist a constant $K > 0$ such that $e^{k|x-x_0|} \leq K$ for every $x \in I$, and there exist $\delta_2 > 0$ such that

$$\|u_0(\mu_1) - u_0(\mu_2)\| \leq \frac{\epsilon}{2K} \quad \text{whenever} \quad |\mu_1 - \mu_2| < \delta_2.$$

Then, by taking $\delta = \min\{\delta_1, \delta_2\}$, $|\lambda - \lambda_0| < \delta$, and $\mu_1 = \lambda$, $\mu_2 = \lambda_0$, we conclude that

$$\begin{aligned} \|u(x,\lambda_0,\lambda_0) - u(x,\lambda,\lambda)\| &\leq \|u(x,\lambda_0,\lambda_0) - u(x,\lambda_0,\lambda)\| + \|u(x,\lambda_0,\lambda) - u(x,\lambda,\lambda)\| \\ &\leq K\frac{\epsilon}{2K} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and hence we have proved that $u(\cdot,\lambda,\lambda)$ converges to $u(\cdot,\lambda_0,\lambda_0)$ uniformly on I as λ tends to λ_0, as desired. ∎

By means of this theorem, we easily derive the concluding proposition of this appendix.

Theorem B.0.7 *Let $u(x,\lambda)$ be the solution of the initial value problem:*

$$\begin{aligned} \frac{d^2u}{dx^2} + a_1(x,\lambda)\frac{du}{dx} + a_0(x,\lambda)u(x) = 0, \qquad x \in I \\ u(x_0) = \xi(\lambda), \\ u'(x_0) = \zeta(\lambda). \end{aligned} \tag{B.6}$$

where a_0, a_1 are continuous functions in (x,λ), and $\xi(\lambda)$, $\zeta(\lambda)$ are continuous in λ.

a) Let $\lambda_0 \in \mathbb{R}$ be fixed, and suppose that $u(x,\lambda_0)$ does not vanish for every $x \in [a,b]$. Then there exist $\epsilon > 0$ such that $u(x,\lambda)$ does not vanish for every $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, and for every $x \in [a,b]$. For every $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, define the application

$$\rho_\lambda : [a,b] \ni x \mapsto \frac{u'(x,\lambda)}{u(x,\lambda)} \tag{B.7}$$

Then ρ_λ converges uniformly to ρ_{λ_0} if λ tends to λ_0.

b) *The same conclusion as in a) is obtained if we interchange everywhere u by u', and the application σ_λ by*

$$\sigma_\lambda : [a,b] \ni x \mapsto \frac{u(x,\lambda)}{u'(x,\lambda)}$$

Proof: By taking $u_1 = u$, $u_2 = \frac{du}{dx}$, we get that the initial value problem (B.6) is equivalent to the first-order linear IVP:

$$\begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(x,\lambda) & -a_1(x,\lambda) \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$$

$$\begin{bmatrix} u_1(x_0) \\ u_2(x_0) \end{bmatrix} = \begin{bmatrix} \xi(\lambda) \\ \zeta(\lambda) \end{bmatrix}.$$

Hence we are in the situation of theorem B.0.6, and therefore, the solution

$$u(x,\lambda) = \begin{bmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{bmatrix}$$

satisfies that $u(\cdot,\lambda)$ converges uniformly to $u(\cdot,\lambda_0)$; Moreover, for every $F : \mathbb{R}^2 \mapsto \mathbb{R}$ that is continuous over the image by u of the compact $I \times [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, it holds that $F(u(\cdot,\lambda))$ converges uniformly to $F(u(\cdot,\lambda_0))$. Hence the theorem is proved by taking $F(x,y) = \frac{y}{x}$ in a) and $F(x,y) = \frac{x}{y}$ in b) ∎

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