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ONE-DIMENSIONAL TRAP MODELS.

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Contents

CHAPTER 1

Introduction

This thesis falls on the area of random walks in random media. More precisely, on the subject of one-dimensional trap models. The organization of the introduction is as follows. In section 1 we will give a quick overview of general random walks in random media. In section 2 we will briefly present some of the physical problems that motivated the introduction of trap models. The mathematical models used to study such physical problems are presented in section 3. Section 4 is devoted to the introduction of an important trap model, the *Bouchaud trap model.* In section 5 we will show some trap models which arise in relation with percolation on trees. The study of these trap models shows connections with the *Continuum random tree*, which is introduced in section 6. The results contained in this thesis are explained in section 2.

1. Random walks in random media.

Simple random walks are standard models to study various transport processes, as heat propagation or diffusion of matter through a medium. Sometimes, the medium in which the transport is taking place is disordered and irregular. Random walks on random media provide models for these phenomena. The disordered nature of the medium is taken into account by choosing it randomly. The mathematical study of these models has been very active during the past decades, and continues to attract a lot of attention from the mathematical community.

The definition of a random walk on a random medium is performed in two steps. First we choose randomly a medium w from an environment space Ω according to a law $\mathbb{P}$. Once the medium is chosen, we define a random walk X^w which is influenced by the medium in a certain way (which will depend on the particular model under study). Generally the medium is kept fixed during the time evolution of the random walk. We denote by P^w the law of the random walk X^w. These laws are referred as to the *quenched* laws. We can also consider the law obtained by averaging the quenched laws over the medium distribution to obtain the *annealed* law P. That is

$$P(A) := \int P^w(A)\mathbb{P}(dw). \tag{1.1}$$

Thus, the *annealed* process X will be a random walk having law P.

There are many ways to provide randomness to the medium where the motion is being performed. We now present some particular models.

1.1. Random walk in random environment (RWRE). We aim to define a walk on $\mathbb{Z}^d$. Let

$$S_d := \left\{ \boldsymbol{p} = (p_e)_{e\in\mathbb{Z}^d:|e|=1} : (p_e)_{e\in\mathbb{Z}^d:|e|=1} \subset [0,1] \text{ and } \sum_{e\in\mathbb{Z}^d:|e|=1} p_e = 1 \right\} \tag{1.2}$$

In this case the set of environments Ω will be $(S_d)^{\mathbb{Z}^d}$. Consider a probability measure $\mathbb{P}$ over Ω. First we choose an environment $\omega = (\boldsymbol{p}^z)_{z\in\mathbb{Z}^d} = \left((p_e^z)_{e\in\mathbb{Z}^d:|e|=1}\right)_{z\in\mathbb{Z}^d} \in \Omega$ according to $\mathbb{P}$. Once ω is chosen we define the random walk on the environment ω as a Markov chain $(X_k^\omega)_{k\in\mathbb{N}}$, taking values in $\mathbb{Z}^d$ which jumps from $x \in \mathbb{Z}^d$ to $x+e$ with probability p_e^x.

An important case is when the environment law $\mathbb{P}$ equals $\nu^{\otimes\mathbb{Z}}$, where ν is a probability measure on S_d. That is, the law $\mathbb{P}$ is the one of an i.i.d. family distributed according to ν. In this case we say that the environment is i.i.d. Another widely studied case is when the environment law is ergodic with respect to the translations on $\mathbb{Z}^d$.

The one-dimensional RWRE was introduced by Chernov [**Che62**] and Temkin [**Tem72**] and it was first regarded as a model for the replication of DNA chains. Despite its simplicity, the one-dimensional RWRE displays some striking features as subdiffusivity and localization. In the high-dimensional case, much less is known about the behavior of the model. For a general account on RWRE we refer to the notes of Zeitouni, [**Zei06**].

1.2. Random walk in a percolation cluster. In this case, the random media is taken from percolation on $\mathbb{Z}^d$. The space of environments Ω is $\{0,1\}^{\mathbb{Z}^d}$. The environment distribution $\mathbb{P}$ will be the law of supercritical percolation on $\mathbb{Z}^d$ conditioned on $|C_o| = \infty$ (where C_o is the connected component of the origin and $|\cdot|$ indicates cardinality). For each fixed realization of the medium $w \in \Omega$, $(X_k^w)_{k\in\mathbb{N}}$ will be a symmetric random walk on C_o starting from the origin.

Different results (see [**GKZ93**], [**Bar04**], [**SS04**], [**BB07**] and [**MP07**]) show that X^w behaves like a simple random walk on $\mathbb{Z}^d$. These results indicate that the infinite connected component C_o can,in some sense, be regarded as $\mathbb{Z}^d$ with "holes".

1.3. Other models. There is a wide variety of random walks on random media. We can mention random walks with random conductances, random walks among random obstacles, among many others. For an account on different models we refer to the books of Hughes [**Hug95**], [**Hug96**] and the book of Bolthausen and Sznitman [**BS02**]. Moreover, there exist a diverse family of continuous counterparts of random walks in random environment, for example Brownian motion among random obstacles. We refer to the book of Sznitman, [**Szn98**].

1.4. Trap models. As we have said, this thesis is focused on the *trap models*. Trap models are random walks on random media where the random media influences the walk through time that it spends on a given site. First we randomly choose an environment composed of traps. Once the traps are chosen, we perform a random walk which stays at a given site during a random time, whose distribution will depend on the trap attached to that site. So the walk can be regarded as a random motion among random traps. An account on the study of trap models can be found on the notes of Ben Arous and Černý, [**BČ06**].

Next we will briefly discuss some of the physical problems that motivated the introduction of trap models.

2. Metastability on physical systems

Out of equilibrium physical systems appear when it is not possible to reach a stable state. In the cases we will discuss (glass, spin glasses), the impossibility to reach equilibrium will be caused by the inherent disorder of such systems. The system is always stressed by its inability to equilibrate. Thus, it will keep changing its state and its physical properties will depend on time. These systems display very interesting physical features (*aging, memory, rejuvenation, failure of the fluctuation disipation theorem*). We may say that we are in the presence of *metastability*, i.e. a system fails to stabilize, though its dynamics favors the evolution towards the "more stable" states. We now briefly present some physical systems displaying such phenomena.

2.1. Structural glasses. Glass is the most common example of a metastable system. It is prepared by heating a crystal above its melting temperature T_m. The crystal is then suddenly cooled down below its melting temperature. If it is cooled fast enough, the molecules will not have enough time to form the crystal lattice. Instead, they will re-configure in a disordered way. At temperatures slightly smaller than T_m, the crystal behaves like a viscous liquid, even though it is below its melting temperature. For this reason, this state is called the *super-cooled liquid phase*. The viscosity of this super-cooled liquid increases as the temperature decreases until it reaches a temperature T_g where the viscosity becomes infinite and the super-cooled liquid solidifies, becoming a glass. The glassy state is a solid-like state in which the molecules are arranged in a disordered way, in sharp contrast to a crystallized solid. Sometimes it is said that glass is an *amorphous solid*. In the glassy phase, there is a mechanical stress due to the disorder on the positions of the particles. Thus, in the glassy phase, the system is out of equilibrium. The glass transition has not been completely understood. Furthermore, there is no agreement on whether it is a "real" phase transition or not. The explanation of the glass transition is one of the most challenging tasks on theoretical physics.

2.2. Spin glasses. Spin glasses are disordered magnetic alloys. More precisely, they are materials composed mostly of a magnetic-inert component and a small amount of magnetic *impurities*. These impurities occupy random positions in the alloy. The magnetic interaction between the magnetic impurities depend on their distance. Some pairs of magnetic impurities might have a tendency to be aligned (*ferromagnetic*), while others might prefer to be anti-aligned (*antiferromagnetic*). Generally, there does not exist a spin configuration that satisfies all the pairs. This phenomenon is called *frustration*. The sign of the magnetic interaction is very sensitive to the distance between the particles, thus, the disorder on the locations of the magnetic impurities leads to a disorder on the magnetic interactions. At high temperatures, the spins are changing fast and observation of the system at two different times yields highly uncorrelated results. As in the case of structural glasses, there exists a temperature T_g under which the spin glass enters the glassy phase, where the system freezes on a disordered configuration.

Study of spin glasses is important for at least two reasons. First of all, it is believed that understanding of spin glasses will shed light on the more challenging study of structural glasses. Secondly, spin glasses are important in their own right, because they present many interesting phenomena and because new and interesting techniques have been developed with the aim of understanding them. In general they provide a paradigmatic example of models which present *order in the presence of disorder*. For an account on the physical study of spin glasses we refer to [**BCKM98**].

2.3. Aging. One of the characteristic features of the dynamics of systems out of equilibrium is that of *aging*. It is a peculiar memory effect observed on experiments on spin glasses and other disordered physical systems: The system is initiated at high temperature and then suddenly cooled down. It then evolves from the disordered initial condition at a low temperature for a time t_w (the system ages), and is latter observed at a time $t_w + t$ when a quantity $R(t_w, t)$ is measured. $R(t_w, t)$ is usually a response function at time $t_w + t$ of a change of parameters (temperature or magnetic field) at t_w. The signature of aging is the following scaling behavior

$$\lim_{\substack{t_w, t \to \infty \\ t^\eta / t_w \to \theta}} R(t_w, t_w + t) = \mathcal{R}(\theta) \tag{2.1}$$

where $\mathcal{R}$ is a non-trivial function of θ. When $\eta = 1$ we say that we are in presence of normal aging, if $\eta < 1$ we are in the presence sub-aging, and if $\eta > 1$ is we are in presence of super-aging.

The aging is a phenomenon that can only occur out of equilibrium, since in equilibrium $R(t_w, t)$ would be independent of t_w and thus the limit in (2.1) would not depend on θ. Heuristically, aging refers to the ever longer delay to observe changes in the system. The older the system, the longer it takes to forget its state. That is, the system is more and more frozen as it ages.

3. Mathematical models of spin glasses

Here we will present some of the mathematical models used to study spin glasses. We start by recalling some elementary concepts of statistical mechanics.

3.1. Notions of statistical mechanics. Suppose we have a finite collection of particles denoted by Λ. Each particle can assume a spin $+1$ or -1. Our configuration space is then $\Gamma_\Lambda := \{-1, +1\}^\Lambda$. For each configuration $\sigma \in \Gamma_\Lambda$ we define an energy or Hamiltonian $H_\Lambda(\sigma)$. The equilibrium measure (or *Gibbs measure*) of a system at temperature T is

$$\mu_\Lambda(\sigma) := \frac{1}{Z_\Lambda} \exp(-\beta H_\Lambda(\sigma)) \tag{3.1}$$

where $\beta := 1/T$ is the inverse temperature and $Z_\Lambda := \sum_{\sigma \in \Gamma_\Lambda} \exp(-\beta H_\Lambda(\sigma))$ is a normalizing constant.

This framework is used to study a wide variety of models, depicting different interesting phenomena. We will start by recalling some models for ferromagnetic materials and then we will present the models for spin glasses.

One of the more important examples is the celebrated *Ising model*. In this case Λ will be a large box contained in $\mathbb{Z}^d$. To each configuration $\sigma := (x_i)_{i\in\Lambda} \in \Gamma_\Lambda$, we define its energy (or Hamiltonian) $H_\Lambda(\sigma)$ as

$$H_\Lambda(\sigma) := \sum_{i \sim j; i,j \in \Lambda} x_i x_j \tag{3.2}$$

where $\sim$ denotes the relation of being nearest neighbors. The Ising model is used to study spontaneous magnetization on a ferromagnetic material. It was introduced by Lenz in [**Len20**] and was studied by Ising in [**Isi25**].

Another, simpler model to study a ferromagnetic material is the *Curie-Wiess model*. This is a *mean-field* model. That is, we will assume that all pair of particles are in interaction (not only nearest neighbors). In this case Λ will simply be a collection of N particles. The Hamiltonian for a configuration $\sigma = (x_i)_{i\in\Lambda}$ is

$$H_\Lambda(\sigma) := \sum_{i,j\in\Lambda} x_i x_j. \tag{3.3}$$

This model was introduced in 1907 by Weiss, [**Wei07**].

Another example is the *Edwards-Anderson model*. In this case one wants to describe the magnetic properties of a spin glass. As we have said, the disordered positions of the magnetic impurities on a spin glass result on disordered magnetic interactions between the particles. Edwards and Anderson proposed in [**EA75**] to take into account this disorder by randomly choosing the magnetic interactions. Let $(J_{i,j})_{i,j\in\Lambda}$ be a collection of i.i.d. random variables which will represent the random magnetic interactions. As in the Ising model, here Λ will be a large box contained in $\mathbb{Z}_d$. For a fixed realization of $(J_{i,j})_{i,j\in\Lambda}$, we define a Hamiltonian which, for each configuration $\sigma := (x_i)_{i\in\Lambda} \in \Gamma_\Lambda$, assumes the form

$$H_\Lambda(\sigma) := \sum_{i\sim j; i,j\in\Lambda} J_{i,j} x_i x_j. \tag{3.4}$$

Note that in this case the Hamiltonian H_Λ is random because it depends on $(J_{i,j})_{i,j\in\Lambda}$. The distribution of $J_{i,j}$ is usually chosen to be a Bernoulli on $\{-1,1\}$ of parameter $1/2$ or uniform over $[-1,1]$.

A simplification of the Edwards-Anderson model can be obtained by replacing the nearest neighbor interaction by a global one. The model obtained is the *Sherrington-Kirkpatrick model*. As in the Curie-Weiss model, Λ will simply be a collection of N particles. The Hamiltonian for a configuration $\sigma = (x_i)_{i\in\Lambda}$ is

$$H_\Lambda(\sigma) := \sum_{i,j\in\Lambda} J_{i,j} x_i x_j \tag{3.5}$$

where the $(J_{i,j})_{i,j\in\Lambda}$ are chosen as in the Edwards Anderson model. This model was proposed by Sherrington and Kirkpatrick in [**SK75**]. The Sherrington-Kirkpatrick model is the mean-field version of the Edwards-Anderson model.

A further and radical simplification was proposed by Derrida in [**Der81**]. There he introduced the *Random Energy model* (REM). Here Λ is a collection of N particles. The Hamiltonian of the configurations are chosen randomly and independent. More precisely $(H_\Lambda(\sigma))_{\sigma\in\Gamma_\Lambda}$ is an i.i.d. collection of centered Gaussian variables with variance N.

3.2. Dynamical models. The static study of spin glasses amounts to the study of the properties of the (random) probability measure μ_Λ as the size of Λ becomes large. For an account on the mathematical study of static of spin glasses we refer to the book of Talagrand, [**Tal03**]. Here we will be concerned with the dynamics of spin glasses. There are many ways to endow the previous models with a dynamics. Usually one wants to have a dynamics under which the equilibrium measure μ_Λ is reversible. The one that concerns us is the Random Hoping Time dynamics (RHT). This is a Markovian dynamics over the configuration space Γ_Λ. The transition rates for the RHT are

$$c(\sigma_1, \sigma_2) := \exp(-\beta H_\Lambda(\sigma_1)). \tag{3.6}$$

That is, the spin glass changes from state σ_1 to σ_2 at a rate which depends only on the energy of σ_1.

4. Bouchaud trap model (BTM)

The *Bouchaud trap model* (BTM) is a random walk in a random medium that was introduced by Bouchaud in [**Bou92**] and by Bouchaud and Dean in [**BD95**] as a simplification of the dynamics of a spin glass at low temperature. We will start by providing the physical ansatz that justifies its introduction.

Due to the disordered nature of the spin glasses, their *energy landscape* presents a huge complexity. Confronted with this, it becomes very useful to have an heuristic idea of the overall structure of this landscape. It is widely accepted that the energy landscape of a disordered system is extremely irregular, with many local minima (pits) corresponding to metastable states. These metastable states are connected by "saddles"", and the energy barriers that the system must overcome to pass from one metastable state to another are rather high. Thus, the process should spend most of the time on the bottom of the pits and jumps quickly between them. This indicates that the probability of finding the system between two metastable states is negligible. The metastable states acts as "traps" which hold the system for a certain time. The "depth"" of the trap should be the energy barrier that the system must cross to leave that trap.

The Bouchaud trap model is a toy model that uses the previous heuristics. We replace the configuration space of the spin glass by a graph, where the vertices of the graph represent the metastable states. We label each vertex by the mean waiting time that the spin glass takes to leave the corresponding metastable state. The edges of our graph should approximate the structure of the saddles connecting the vertices. As very little was known about that structure, Bouchaud proposed to simply use a large complete graph. He also proposed to choose the energy barriers of the traps as i.i.d. exponential random variables of parameter 1. As the energy barriers of the traps are being chosen randomly, we are in the presence of a random walk in a random media, and as such, its definition will be made in two steps. First we will define the medium, and then we will define the walk.

Nothing restricts us to consider the model on more general graphs. Thus we will define the BTM on a generic graph $\mathcal{G}$. To each vertex x of $\mathcal{G}$ we assign a positive number τ_x where $(\tau_x)_{x\in\mathcal{G}}$ is an i.i.d. family of random variables. Thus, the space of environments Ω will be $\mathbb{R}_+^{\mathcal{G}}$ and the environment distribution $\mathbb{P}$ will be the one induced from $(\tau_x)_{x\in\mathcal{G}}$. For each realization $\boldsymbol{\tau} := (\tau_x)_{x\in\mathcal{G}} \in \Omega$ of the medium, we define the Bouchaud trap model as a continuous time random walk $X^{\boldsymbol{\tau}}$ on $\mathcal{G}$ with random jump rates. Each visit of $X^{\boldsymbol{\tau}}$ to $x \in \mathcal{G}$ lasts an exponentially distributed time with mean τ_x. We will usually consider the symmetric BTM. That is, where the walk jumps from a vertex to each of its neighbors with equal probabilities. Nevertheless, we will sometimes add a drift to the BTM. As we have said, the energy landscape of a disordered system is highly inhomogeneous. To incorporate that feature into the model we will assume that the distribution of the depth of the traps has heavy tails. More precisely, we assume that

$$\lim_{u\to\infty} u^\alpha \mathbb{P}[\tau_x \geq u] = 1 \tag{4.1}$$

with $\alpha \in (0,1)$. This assumption is satisfied at low temperatures for the standard choice of the statical mechanics: As we have said, Bouchaud proposed to choose the energy barriers of the traps as exponential r.v.'s of parameter one. Moreover, due to the rough nature of the landscape, we can approximate the energy barrier of a trap by the energy at its bottom. Let x be a vertex and E_x be the energy at the bottom of the corresponding trap. If E_x is an exponential r.v. of parameter one, its corresponding Gibbs measure at inverse temperature β will satisfy

$$\mathbb{P}(\tau_x \geq u) = \mathbb{P}(\exp(-\beta E_x) \geq u) = u^{-1/\beta}. \tag{4.2}$$

Hence, if we assume that the mean waiting time to exit a trap is proportional to the Gibbs measure at its bottom, we will have that our assumption is satisfied at low temperatures ($T < 1$).

The range of application of the BTM largely exceeds the study of dynamics of spin glasses. It is also is relevant for the study of fragile glasses, soft glassy and granular material and also pinning of extended defects.

If we consider the BTM on an n-dimensional hypercube, it corresponds to the Random Hoping time Dynamics in the REM. It is not easy to prove that the BTM is a good approximation of the more complex dynamics and more complex models of spin glasses. The previous task was partially achieved by Ben Arous, Bovier and Černỳ (see [**BBČ08**]).

Let X be a BTM. Let $S(k)$ be the time of the k-th jump of X. $(S(k), k \in \mathbb{N})$ is called the *clock process* of X. Let $Y_k := X(S(k))$ be the position of X after the k-th jump. $(Y_k : k \in \mathbb{N})$ is called the *embedded discrete time random walk* associated to X.

4.1. Aging on the Bouchaud Trap Model (BTM). The BTM keeps a rather sketchy image of the real dynamics of a disordered system. Nevertheless it displays some very peculiar features which have been observed experimentally on the real world. In fact, one can prove aging for some functions defined using the BTM. More specifically, let X be a BTM and define

$$\Pi(t_w, t) := \mathbb{P}(X_t = X_s \forall s \in [t, t + t_w]) \tag{4.3}$$

the probability that X does not jump between times t_w and $t_w + t$. Also define

$$R(t_w, t) := \mathbb{P}(X_t = X_{t+t_w}) \tag{4.4}$$

the probability of finding the walk on the same site at times t_w and $t_w + t$. Ben-Arous and Cěrný proved an almost universal aging scheme for these quantities on the BTM on a wide variety of graphs (see [**BČ08**]). For aging results on the REM with the random hoping time dynamics, see [**BBG03a**], [**BBG03b**], [**BČ08**] and [**ČG08**].

The aging is generally proved by first showing that the rescaled clock processes converges to an α-stable subordinator. Then the arcsine law for α-stable subordinators yields the aging properties.

4.2. The BTM on high dimensions. Now we turn to the BTM on the lattice $\mathbb{Z}^d$ with $d \geq 2$. This is, $\mathcal{G} = \mathbb{Z}^d, d \geq 2$. In this graph, the symmetric model BTM has been studied by Ben Arous and

Černý in [**BČ07**], and by Ben Arous, Černý and Mountford in [**BČM06**]. In these papers, a scaling limit and several aging results where obtained. The scaling limit is the so called *fractional kinetics process* (FK), which is a time-change of a d-dimensional Brownian motion through the inverse of an α-stable subordinator.

More precisely, let V_α be an α stable subordinator. Denote by Ψ the right continuous generalized inverse of V_α and let B be a d-dimensional Brownian motion independent of V_α. Then, the FK process of index α is defined as B_{Ψ_t}. Its name comes from the fact that the evolution of its density satisfies a fractional kinetics equation. The FK process has a smooth density. Specifically, the fixed-time distribution of the FK follows the Mittag-Leffer distribution. The FK is a self similar process. It is also non-Markovian.

To be more specific, the result proved in [**BČ07**] is that, for $(d \geq 3)$, and $(X_t)_{t\geq 0}$ a d-dimensional BTM, we have that $(\epsilon^{\alpha/2} X_{\epsilon^{-1}t})_{t\geq 0}$ converges to the d-dimensional FK process as $\epsilon \to 0$. For $d = 2$ the scaling limit is also a FK process, but the scaling has a logarithmic correction (see [**BČM06**]).

It is worth mentioning that the convergence of the BTM to the FK is quenched. That is, it holds almost surely over the environments.

4.3. The one-dimensional BTM. Here we will focus on the one-dimensional BTM. This case is radically different from the model on higher dimensions. This difference is displayed by the fact that the scaling limit of the one-dimensional BTM is not the one-dimensional FK process but a process from a completely different class. This fact was proved in [**FIN02**] by Fontes, Isopi and Newman, where they proved that the scaling limit is a singular diffusion, called the *Fontes, Isopi, Newman diffusion* (FIN). To define it precisely, first we need to define speed measure changes of a Brownian motion: Let B_t be a one-dimensional Brownian motion and $l(t,x)$ be a bi-continuous version of its local time. Given any locally finite measure μ in $\mathbb{R}$ we define

$$\phi_\mu(s) := \int_{\mathbb{R}} l(s,y)\mu(dy),$$

and its generalized right-continuous inverse by

$$\psi_\mu(t) := \inf\{s > 0 : \phi_\mu(s) > t\}.$$

Then we define $X(\mu)_t$ (the speed measure change of B with speed measure μ) by

$$X(\mu)_t := B_{\psi_\mu(t)}. \tag{4.5}$$

Now we aim to define a random measure that will be the speed measure of the FIN diffusion. Let (x_i, v_i) be an inhomogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}^+$, independent of B, with intensity measure $\alpha v^{-1-\alpha} dx dv$. We define the random measure ρ as

$$\rho := \sum v_i \delta_{x_i}. \tag{4.6}$$

The diffusion $(Z_t; t \in [0,T])$ defined as $Z_s := B_{\psi_\rho(s)}$ is the FIN diffusion.

One of the interesting features of the one-dimensional BTM is that it presents *quenched localization* (see [**FIN02**]). That is, if X_t is a one-dimensional BTM, then, for almost every environment $\tau := (\tau_x)_{x\in\mathbb{Z}}$, we have that

$$\limsup_{t\to\infty} \sup_{i\in\mathbb{Z}} P_\tau(X_t = i) > 0, \tag{4.7}$$

where P_τ denotes que quenched law on the environment τ. In other words, no matter how much time the walk undergoes its random motion, there always exists a site which carries a positive proportion of the distribution.

In [**FIN02**], the one-dimensional BTM was studied in relation with the one-dimensional voter model with random rates. More precisely, it was analyzed in relation to the phenomenon of *chaotic time dependence*. There they also introduced a type of convergence called *point process convergence*. It was proved that convergence of fixed time distributions of the rescaled BTM to the fixed time distributions of the FIN diffusion holds in the point process sense. That allows to deduce localization of the BTM from the localization of the FIN diffusion (the usual weak convergence is not enough for this purposes). Suppose $(X_t)_{t\geq 0}$ is a one-dimensional BTM. Then the proper scaling for the convergence above is $\epsilon X_{\epsilon^{-(1+\alpha)/\alpha}t}$.

Aging properties can be deduced from the convergence of the BTM to the FIN diffusion (see [**FIN02**]). Furthermore, in [**BČ05**] Ben Arous and Černý proved that the one-dimensional BTM displays aging for the quantity $R(t, t_w)$ and sub-aging for $\Pi(t, t_w)$.

5. Geometric trap models

In this thesis, we will study "geometric" trap models. That is, models which are constructed as follows: Attach a graph $\mathcal{G}_z$ to each vertex of $\mathbb{Z}$. Then, consider a simple random walk Z on this enlarged graph. We can project Z to $\mathbb{Z}$ to obtain a one-dimensional, symmetric random walk with random jump times, denoted W. The random jump times of W at site $z \in \mathbb{Z}$ will be given by the time that Z spends on each visit to $\mathcal{G}_z$. We will see that models of this kind arise naturally in relation with percolation on regular trees. We start by recalling the definition of percolation.

5.1. Percolation. Let $\mathcal{G}$ be an infinite graph. We denote by $V_\mathcal{G}$ and $E_\mathcal{G}$ its corresponding sets of vertices and edges respectively. To define bond percolation on $\mathcal{G}$ of parameter $p \in [0, 1]$, we declare each edge in $E_\mathcal{G}$ "open" with probability p and "closed" with probability $1 - p$. This is done independently over the set edges. Similarly, we can define site percolation on $\mathcal{G}$ by declaring each vertex in $V_\mathcal{G}$ open with probability p and closed with probability $1 - p$. We say that two vertices are connected if there exists an open path between them. The connected component C_x of a vertex $x \in \mathcal{G}$ is the set composed of all the vertices which are connected to x. Percolation on $\mathbb{Z}^d$ $(d \geq 2)$ displays a very interesting transition phase. Let $\theta(p)$ be the probability (under percolation of parameter p) that the cluster of the origin is infinite. Then, there exists a critical value $p_c \in (0, 1)$ such that

- If $p < p_c$, then $\theta(p) = 0$.
- If $p > p_c$, then $\theta(p) > 0$.

When $p < p_c$ the connections given by the open edges (or vertices) are of a local character. When $p > p_c$ this connections become global. The sub-critical and super-critical regimes are well understood. On the other hand, percolation at the critical value is less understood and harder to study. For an account on percolation theory, we refer to the book of Grimmett, [**Gri99**].

5.2. The Incipient Infinite Cluster (IIC). One of the trap models studied in this thesis comes from the incipient infinite cluster on a regular tree. The *incipient infinite cluster* (IIC) is a random graph which emerges from the study of percolation. We will first recall some facts about the IIC on the lattice $\mathbb{Z}^d$.

Suppose we have critical percolation on $\mathbb{Z}^d$. It has been proved, for $d = 2$ or $d \geq 19$, that $\theta(p_c) = 0$ i.e. there are no infinite clusters under critical percolation. Nevertheless, the model displays its criticality by showing connected components at any scale. To be more precise, it is believed that, for each box of order n there exists, with high probability, a connected component whose diameter is of order n (see [**BCKS01**]). The IIC is a random infinite graph displaying a geometry similar to the one of a large percolation cluster. For $d = 2$, Kesten defined the IIC as the cluster of the origin conditioned on intersecting the boundary of a ball of radius n and then letting n increase to infinity (see [**Kes86a**]). For large d van der Hofstad and Járai defined the IIC using the *lace expansion* (see [**vdHJ04**]). It is believed that the global properties of the IIC are the same for all d big enough. On [**vdHJ04**] is proved that, for “spread out” models, the IIC has one end (any two paths to infinity intersect infinitely often). The results on [**vdHdHS02**] and [**vdHJ04**] support the conjecture that, for large d, the geometry of the IIC is close to the geometry when “$d = \infty$” i.e. the IIC on a regular tree.

The construction of the IIC for regular trees is more simple and it is related to critical branching processes conditioned on non-extinction. It is easy to see that the percolation cluster (of the root) on a regular tree corresponds to the Galton-Watson tree of a branching process whose offspring distribution is Binomial. Kesten gave the construction of the IIC for critical branching processes in [**Kes86b**] and it is performed in the same fashion that for $\mathbb{Z}^2$. That is, one consider the law a of critical Galton-Watson tree conditioned on surviving up to generation n, and then let $n \to \infty$. In that paper it is also shown that the IIC on a regular tree possesses a single path to infinity. This path is called the *backbone*. The backbone is isomorphic to the graph $\mathbb{N}$. Thus, the IIC can be seen as $\mathbb{N}$ adorned with finite branches. The description of the IIC on a regular tree given on [**Kes86b**] is even more detailed. In fact, for each $i \in \mathbb{N}$, let $\mathcal{L}_i$ be the branch attached to the i-th vertex of the backbone, then the sequence $(\mathcal{L}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of critical percolation clusters (on a regular tree).

5.3. Random walk on the IIC. One of the most interesting and challenging models of a random walk in random media is the random walk on a critical percolation cluster. One of the motivations for its study is to gain insight into the conductivity properties of a critical percolation cluster, about which very little is known. An approximation is to consider a random walk on the IIC. For this reasons, the study of a random walk on the IIC is very important. The random walk on the IIC on $\mathbb{Z}^2$ was studied by Kesten in [**Kes86b**]. There he proved that it is subdiffusive. We will focus on the IIC on a regular tree, which, as we have said, it is believed to be a good approximation for the IIC on $\mathbb{Z}^d$ for d big enough (see [**vdHdHS02**] and [**vdHJ04**]).

Consider a simple, discrete time random walk $(Z_n^{IIC})_{n\in\mathbb{N}}$ on the IIC (on a regular tree). Once Z^{IIC} is on vertex $x \in$ IIC, it jumps to each of its neighbors with probability $deg(x)^{-1}$, where $deg(x)$ is the degree of x in the IIC. Kesten studied this process on [**Kes86b**], where he proved subdiffusivity. Quenched and annealed properties for the transition kernel of this process where studied in [**BK06**]. We can use this random walk to construct a geometric trap model on $\mathbb{N}$, denoted by $(W_t^{IIC})_{t\geq 0}$, by stating that $W_t^{IIC} = z$ for all $t \geq 0$ such that $Z_{\lfloor t \rfloor}^{IIC} \in \mathcal{L}_z$. This thesis contains the theorem that gives the scaling limit for that geometric trap model.

5.4. The Invasion Percolation Cluster (IPC). This thesis also includes results about a geometric trap model which is constructed by means of the invasion percolation cluster on a regular tree. The *invasion percolation cluster* (IPC) is a random graph obtained by a stochastic growth process and it was introduced by Wilkinson and Willemsen on [**WW83**]. We pass to recall its construction. Suppose we have an infinite, connected graph $\mathcal{G}$ with a distinguished vertex o. We randomly assign, to each edge e, a weight w_e. We assume the family $(w_e)_{e \text{ vertex of } \mathcal{G}}$ to be i.i.d. and uniformly distributed over $[0,1]$. Then define C_0 as o. C_1 will be obtained from C_o by adding the neighbor x_1 of o whose corresponding edge has smaller weight. That is $w_{x_1} = \min\{w_y : x \sim o\}$. Generally, C_n is constructed from C_{n-1} by attaching the vertex on the outer boundary of C_{n-1} with smaller weight. The invasion percolation cluster is $C_\infty := \cup_{n\in\mathbb{N}} C_n$.

The IPC is closely related to critical percolation. As we have previously said, it is believed that on $\mathbb{Z}^d$ critical percolation clusters appear on all scales. Then we expect the invasion process to invade ever larger critical clusters. Note that, once a vertex of a connected component is invaded, the process must invade all the vertices on that cluster before leaving it. These two remarks indicate that the invasion process should spend most of the time inside these big connected components. In fact, in [**CCN85**], Chayes, Chayes and Newman proved that, for the IIC on $\mathbb{Z}^d$, for each $\epsilon > 0$, just a finite number of edges with weight above $p_c + \epsilon$ are invaded by the IPC. In [**HPS99**], Häggström, Peres and Schonmann proved that this fact holds for the IPC on a much wider variety of graphs.

Note that no parameter appears in the definition of the IPC. Nevertheless, the critical percolation probability p_c shows up spontaneously on the model. Hence we might say that the IPC displays "self organized criticality".

The IPC has been conjectured to be very similar to the IIC. In [**Jár03**] Járai showed (for $\mathcal{G} = \mathbb{Z}^2$) that the probability of an event E under the IIC is identical to the probability of the translation of E by $x \in \mathbb{Z}^2$ under the IPC measure conditional on x being invaded and in the limit as x to ∞.

The IPC on a regular tree was studied by Nickel and Wilkinson in [**NW83**]. There they computed the probability generating function of the weight and the height of the vertex added to C_n to form C_{n+1}. Their results suggested that the IPC has a different scaling limit than the IIC. This fact was proved by Angel, Goodman, den Hollander and Slade in [**AGdHS08**]. It that article they also provide a structural theorem for the IPC on a regular tree. Specifically, they shown that the IPC possesses a single path to infinity. From each vertex i of this backbone there emerges a subcritical percolation cluster with a parameter that depends on i and tends to the critical value as $i \to \infty$.

As in the case of the IIC, we can define a geometric trap model W^{IPC} as a simple random walk on the IPC projected to the backbone.

6. The Continuum random tree

To achieve the task of describing the scaling limits of W^{IIC} and W^{IPC}, it will be necessary to understand the behavior of a large percolation cluster on a tree. This understanding will rely on the *Continuum Random Tree* (CRT). The CRT was introduced by Aldous in [**Ald91a**] as a scaling limit for some families of random trees.

We say that a topological space is a *dendrite* if it is arc-connected and contains no subspace homeomorphic to the circle, i.e. a dendrite is the continuous analogous of a tree. The CRT is a random dendrite and it is constructed using a Brownian excursion.

Let $\mathcal{T}^n$ be a Galton-Watson tree obtained from a critical branching process whose offspring distribution has a finite variance conditioned on having n vertices. Aldous proved in [**Ald93**] that the scaling limit of $\mathcal{T}^n$ is the CRT. As we have said, a critical percolation cluster (of the root) on a regular tree is a critical Galton-Watson tree with a Binomial offspring distribution. So we can apply the mentioned result to obtain the geometrical structure of the "deep traps" for W^{IIC}. It turns out that the same analysis can be done to obtain the geometrical structure of the deep traps of W^{IPC}.

We will be interested on the behavior of a random walk on a large trap. So we will be concerned with the large-scale behavior of a random walk on a large percolation cluster. To identify this behavior we will use the results proved by Croydon in [**Cro08**], where the scaling limit of that walk is identified. This scaling limit is the so-called *Brownian motion on the Continuum Random Tree.*

7. Results

In subsection 7.1 we consider a sequence of drifted, one-dimensional BTM's. We will rescale this sequence of processes. The drift is fixed on the time evolution of each drifted BTM. Nevertheless, the drift of the BTM considered decays as we rescale the processes. We find a phase transition in terms of the scaling limit of the walks and the speed of decay the drift. In subsection 7.2 we obtain bounds for the annealed transition kernel of the one-dimensional BTM. Subsection 7.3 contains joint work with G. Ben Arous, J. Černý and R. Royfman. There, the one-dimensional BTM is generalized to a class of processes called *Randomly trapped random wals.* We establish several convergence results for these processes. New processes appear as scaling limits. We will also show phase transitions for some particular examples. In subsection 7.4 we establish the scaling limit some geometric trap models which arise in relation with percolation on regular trees. These results are joint work with G. Ben Arous.

7.1. Bouchaud walks with variable drift. As we have seen, the BTM on $\mathbb{Z}$ has radically different behavior from the BTM on $\mathbb{Z}^d$ $(d \geq 2)$. The scaling limit for the one-dimensional BTM is a speed measure changed Brownian motion, with a random speed measure which plays the role of a random environment. The scaling limit for the BTM on higher dimensions is the FK process. In both cases the limit is a random time change of a Brownian motion, but in the one-dimensional case we have that the clock process and the Brownian motion are dependent. On the contrary, the FK process is random time change of a Brownian motion where the clock process is independent from the Brownian motion. This difference can be understood as follows: The increments in the clock process of the BTM are the depths of the traps as sampled by the embedded discrete time random walk. On high dimensions, the embedded random walk is transient. Thus, each trap is sampled a finite number of times. This indicates that the clock process will not have long-range interactions with its past. Hence, its scaling limit will be a Markovian process, more precisely, it is an $\alpha-$stable subordinator. On the other hand, the one-dimensional random walk is recurrent. Thus, the embedded random walk Y_k will sample each trap an infinite number of times. As a result, the clock process will have long-range interactions with its past and the scaling limit of the clock process will be non-Markovian. Furthermore, the scaling limit of the clock process will be the local time of a Brownian motion integrated against a random measure.

A natural task is to search for intermediate behaviors between the transient and the recurrent cases. Chapter 2 will be devoted to this. We will do it by considering drifted BTM's. First note that if we introduce a drift to the one-dimensional BTM, its embedded discrete time random walk becomes transient. Thus, intermediate behaviors between the recurrent and the transient case might appear if we analyze a sequence of one-dimensional BTM's with a drift that decreases to 0. We will show that the speed of decay of the drift sets the long time behavior of this (sequence) of BTM's.

For each $\epsilon > 0$, denote by X^ϵ the BTM on $\mathbb{Z}$ where the transition probabilities of its embedded discrete time random walk are $\frac{1+\epsilon}{2}$ to the right and $\frac{1-\epsilon}{2}$ to the left. We will call this process the BTM with drift ϵ. For $a \geq 0$, consider a rescaled sequence of BTM's with drift n^{-a}, $(h_a(n)X^{n^{-a}}(tn); t \geq 0)$, indexed by n, where $h_a(n)$ is an appropriate space scaling depending on a. We will see that as the drift decays slowly (small a), the sequence of walks converges to the inverse of an α-stable subordinator, whereas if the drift decays fast (large a) the limiting process is the FIN diffusion. As these two processes are qualitatively different, we are led to think that there is either, a gradual interpolation between these two behaviors as the speed of decay changes, or a sharp transition between them as the speed of decay changes. We establish that there is a sharp transition between the two scaling limits, that there is a critical speed of decay where a new process appears and that the transition happens at $a = \alpha/(\alpha+1)$. More precisely, we prove that, depending on the value of a, there are three different scaling limits:

- **Supercritical case** $(a < \alpha/(\alpha+1))$. The sequence of walks converges to the inverse of an α-stable subordinator.
- **Critical case** $(a = \alpha/(\alpha+1))$. The sequence of walks converges to a process which is a speed measure change of a Brownian motion with drift that we will call the *drifted FIN diffusion*.
- **Subcritical case** $(a > \alpha/(\alpha+1))$. The sequence of walks converges to the FIN diffusion.

The case $a = 0$ (contained in the supercritical case), which corresponds to a constant drift, was already addressed by Zindy in [**Zin09**]. When preparing the final version of the article which contains this theorem we learned that it was obtained independently by Gantert, Mörter and Wachtel in [**GMW10**].

7.2. Gaussian lower bound for the one-dimensional BTM. A basic question is to describe the behavior in space and time of the annealed transition kernel of the one-dimensional, symmetric version of the BTM. In Chapter 3 we will establish a sub-Gaussian bound on the annealed transition kernel of the model which provides a positive answer to the behavior conjectured by E.M. Bertin and J.-P. Bouchaud in [**BB03**]. That article contains numerical simulations and non-rigorous arguments which support their claim. A first step on establishing the conjecture was given by J. Černý in [**Čer06**] where he proved the sub-Gaussian bound. We provide the proof for the corresponding lower bound. More specifically we prove that, if X is a one-dimensional BTM

THEOREM 1.1. *There exists positive constants C_1, c_1, C_2, c_2 and ϵ_1 such that*

$$C_1 \exp\left(-c_1\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right) \leq \mathbb{P}(|X_t| \geq x) \leq C_2 \exp\left(-c_2\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$ such that $x/t \leq \epsilon_1$.

As we have previously stated, the upper bound in theorem 1.1 has been already obtained in [**Čer06**].

The techniques used to prove theorem 1.1 can also be applied to obtain the corresponding result for the FIN singular diffusion, i.e., we will prove

THEOREM 1.2. *There exists positive constants C_3, c_3, C_4 and c_4 such that*

$$C_3 \exp\left(-c_3\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right) \leq \mathbb{P}(|Z_t| \geq x) \leq C_4 \exp\left(-c_4\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$.

Again, the upper bound of theorem 1.2 was obtained in [**Čer06**].

The main difficulty to obtain the lower bound in theorem 1.2 is to be able to take advantage of the independence between the Brownian motion B and the random measure ρ which appear in the construction of the FIN diffusion. As we will see, the Ray-Knight description of the local time of B allow us to overcome that difficulty.

7.3. Randomly trapped random walks. In chapter 4 we will define a class of one-dimensional random walks which we will call *Randomly trapped Random walks* (RTRW). These processes can be considered as an enrichment of the one-dimensional, symmetric Bouchaud trap model. This generalization will consist on allowing the traps to hold the walk by times that are not-necessarily exponentially distributed.

Consider a sequence of probability measures $\boldsymbol{\nu} := (\nu_z)_{z\in\mathbb{Z}} \in M_1(\mathbb{R}_+)^{\mathbb{Z}}$. Let Z be a continuous time, symmetric random walk on $\mathbb{Z}$ with $Z_0 = 0$. We denote by $\tau(i,z)$ the time that Z spends on its i-th visit to $z \in \mathbb{Z}$. Assume that for all $i \in \mathbb{N}$ and $z \in \mathbb{Z}$, $\tau(i,z)$ is distributed according to ν_z. Moreover, assume $(\tau(i,z))_{i\in\mathbb{N},z\in\mathbb{Z}}$ to be an independent family of random variables. We will say that Z is a *trapped random walk* (TRW) on the *trapping landscape* $\boldsymbol{\nu}$. When we consider a TRW on a random trapping landscape, we are in the presence of a randomly trapped random walk. Between the models which can be seen as particular instances of the RTRW we can mention Continuous time random walk a la Montroll Weiss, the symmetric BTM, and the walks on the IIC and the IPC projected to the backbone. i.e. W^{IPC}, and W^{IIC}. When the environment of a RTRW is chosen on an independent and identically distributed way, we will say that the RTRW is i.i.d. We will mostly work with i.i.d. RTRW's. Nevertheless, the methods applied here work on more general cases. In fact, they will work for W^{IPC}, which, as we will see, is not i.i.d.

We will also define the continuous counterpart of the TRW and RTRW. First we will define the object which play the role of the environment. It will be called *trap measure.* The introduction of this object comes from the necessity of generalizing the speed measure changed Brownian motion. Trap measures will play the role of the measure. Trap measures are similar to ordinary measures, but instead of assigning numbers to sets, trap measures will assign certain type of stochastic processes to sets. Formally speaking,

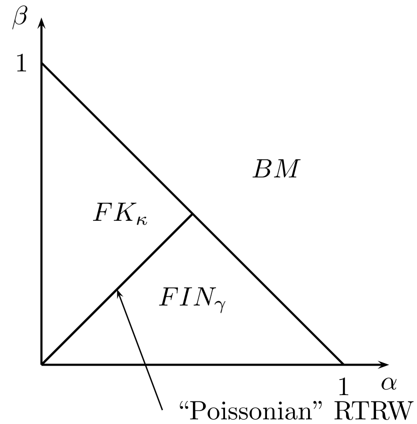

FIGURE 1. The simplest case of a phase transition.

trap measures will be certain type of random measures over $\mathbb{R} \times \mathbb{R}_+$. We use trap measures to define a one-dimensional process that we call *trapped Brownian motion* (TBM). TBM is our generalization of a speed measure changed Brownian motion. TBM can be regarded as the continuous version of TRW. We will also consider random trap measures to define the *Randomly trapped Brownian motion* (RTBM), which will be the continuous analogous of the RTRW. RTBM can be seen, in some sense as an enrichment of the FIN diffusion.

The RTBM's and RTBM's introduced here provide a general and suitable framework to study one-dimensional trap models. For instance, all the scaling limits of symmetric, one-dimensional trap models known so far are the FIN diffusion, the one-dimensional FK process and the Brownian motion, and all these processes can be expressed as RTBM's. Furthermore, we will see that family of RTBM's is much richer than that. We will also show that trap measures provides a general method to show convergence of rescaled one-dimensional trap models to their scaling limits: The space of trap measures naturally endowed with a topology. As a general theorem, we will show that convergence of our trapped processes can be deduced from convergence of their respective trap measures. All the convergence result we present here will rely on this fact.

One of the things that motivates our study is to try to identify the scaling limits for W^{IPC} or W^{IIC}. These questions are particular instances of the broader query: what kind of processes arise as scaling limits of the RTRW's?. We will present a classification theorem describing all the possible scaling limits of i.i.d's RTRW's. This proof will be based on a classification theorem for the *exchangeable measures* on a quadrant. This theorem is given by Kallenberg in [**Kal05**].

We will give a criterion to prove convergence of rescalings of an i.i.d. RTRW to its respective scaling limit. In Chapter 5 we will see that this criterion hods for the walk on the IIC projected to the backbone, W^{IIC}, and thus, we will identify its scaling limit as a particular case of the RTBM. The case of the IPC is different. As we have said, in the IPC the percolation parameter of the percolation cluster we attach to a given vertex of the backbone will vary depending of the vertex. This implies that the environment of W^{IPC} is not identically distributed (it will also fail to be independent). Thus we present a theorem to prove convergence of a given RTRW to its scaling limit where the i.i.d. hypothesis is relaxed. In chapter 5 we will use that theorem to identify the scaling limit of W^{IPC}.

We will provide criteria to deduce convergence of RTRW to the one-dimensional FK process. We use a multidimensional ergodic to provide criteria for convergence of RTRW to Brownian motion. We will also give criteria for the convergence to the FIN diffusion.

We will also analyze RTRW‘s which depend on some parameters. We will find phase transitions for these processes in terms of their scaling limits. The first model is defined as follows: Take α and β in $(0,1)$. Let $(\tau_z)_{z\in\mathbb{Z}}$ be a i.i.d. sequence of positive random variables defined on the space $(\Omega,\mathcal{F},\mathbb{P})$ such that

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}(\tau_0 > u) = 1 \tag{7.1}$$

and $\mathbb{P}(\tau_z > 1) = 1$. For each $z \in \mathbb{Z}$, consider the random probability distribution $\pi_z(\omega) := (1-\tau_z(\omega)^{\beta})\delta_0 + \tau_z(\omega)^{\beta}\delta_{\tau_z(\omega)}$. We construct a RTRW by letting the time the walk spends on a site $z \in \mathbb{Z}$ be distributed as π_z. We will see that this RTRW presents a phase transition in terms of its scaling limit as the values of α and β change. The phase diagram will include, one one hand, the FK process, on the other hand, the FIN diffusion, and a third process as a critical case. This third process is a RTBM which is neither FIN, nor FK, neither Brownian motion. The phase diagram for this model are depicted on figure 1.1.

We will also analyze the *comb model*. The comb model is a geometric trap model constructed as follows: Let G_z denote a line segment of length N_z with nearest-neighbor edges. Let G_{comb} be the tree-like graph with leaves $(G_z)_{z\in\mathbb{Z}}$. We refer to the geometric trap model on G_{comb} as the Comb model. We call the leaves of G_{comb} “teeth”. The comb graph has a great advantage as opposed to the trap models on the IIC or the IPC, since the distribution of the time spent in the teeth is easier to compute. We choose the depth of the teeth on an independent and identically distributed way. We assume that the distribution of the depth satisfies

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}(N_0 > u) = 1 \tag{7.2}$$

for some $\alpha > 0$.

As we will see, the scaling limit of the comb model is either Brownian motion or FK depending on the value of α. When $\alpha > 1$, the teeth are “short” and the mean time spent in traps has finite expectation. Thus the comb model is diffusive and Brownian motion is the scaling limit. If, on the other hand, $\alpha < 1$

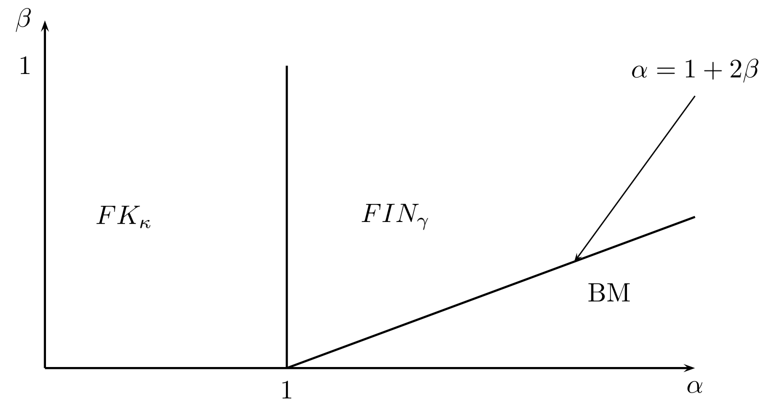

FIGURE 2. Phase diagram for the comb model.

the teeth are "long" and the expectation of the mean time spent in traps is infinite. However, the comb model does not explore deep traps and, therefore, does not "remember" the environment. Hence FK is the limit.

The comb model can be enriched further. In order to do so, we write each vertex of G_{comb} as a pair (n, z) with $z \in \mathbb{Z}$ and $n \in [0, N_z]$ in a straightforward fashion so that the points on the backbone have vanishing second coordinate. We (re-)define X^{comb} as follows. Whenever X^{comb} is not on the backbone, it performs a drifted random walk on $(z, 0) \cup G_z$ with a drift $g(N_z) \geq 0$ pointing away from the backbone and reflecting wall at the end of the tooth, i.e. for any $0 < n < N_z$

$$\mathbb{P}(X^{comb}(k+1) = (z, n+1) | X^{comb}(k) = (z, n)) = \frac{1 + g(N_z)}{2}, \tag{7.3}$$

$$\mathbb{P}(X^{comb}(k+1) = (z, n-1) | X^{comb}(k) = (z, n)) = \frac{1 - g(N_z)}{2} \tag{7.4}$$

and

$$\mathbb{P}(X^{comb}(k+1) = (z, N_z - 1) | X^{comb}(k) = (z, N_z)) = 1. \tag{7.5}$$

Otherwise, if $X^{comb} = (z, 0)$, it jumps to one of the three vertices $(z-1, 0), (z+1, 0)$ and $(z, 1)$ with equal probability. We will find in subsection 5.2 that the appropriate choice of g is

$$g(N) = \beta \frac{\log(N)}{N} \tag{7.6}$$

for some $\beta \geq 0$. The presence of the drift might force X^{comb} to explore even the deepest traps. The phase diagram for the comb-model, containing the FK process, the FIN diffusion and Brownian motion is depicted in figure 1.2.

7.4. Geometric trap models. As we have seen, some interesting and natural examples of RTRW arise in relation to percolation on regular trees, namely, W^{IIC} and W^{IPC}. It is natural to wonder what kind of scaling limits do the processes W^{IIC}, W^{IPC} have, in case they exist. In Chapter 5 we will use the results presented in Chapter 4 to identify their scaling limits. In particular we will see that they are RTBM's which are neither the FIN diffusion, neither FK nor Brownian motion. Furthermore, we will show that the scaling limit of W^{IIC} is different from the one for W^{IPC}. We will briefly explain the nature of these scaling limits by comparing them with the FIN diffusion.

The FIN diffusion can be regarded as a Brownian motion moving among traps. These traps are given by an atomic random measure. Each atom of this measure corresponds to a trap, the location of the atom represents the location of the trap and the weight of the atom represents the depth of the trap. In the case of the IIC the scaling limit can also be seen as a Brownian motion moving among random traps. Nevertheless, in this case the traps will be much more complex. In the FIN diffusion, the traps are described by their location and their weight. In our case, the traps will described by their location and a stochastic process (a subordinator). Here is where the Continuum Random Tree appears. The process characterizing each trap will be the inverse of the local time at the root of the Brownian motion on a CRT. More specifically, the construction of this environment can be performed as follows: First, we randomly choose a measure ρ in exactly the same way as we did in the definition of the FIN diffusion of parameter $\alpha = 1/2$. That is, a random measure associated to a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $\alpha y^{-1-1/2} dx dy$. Now, for each atom $y_i \delta_{x_i}$ of ρ, we randomly choose a realization of the CRT conditioned on having "size" y_i. Then, the trap located at x_i will be characterized by the inverse local time process (at the root) of the Brownian motion on that realization of the CRT. Then, the scaling limit of W^{IIC} can be seen as a Brownian motion moving among these complex traps.

The difference between the scaling limits of W^{IIC} and W^{IPC} stems on the random measure used to construct the traps. In the IIC case, we use a random atomic measure ρ to set the locations and "sizes" of the traps. The measure ρ was constructed by means of a Poisson point process. It is a known fact that ρ is the Lebesgue-Stieltjes measure associated to an α-stable subordinator. In the IPC, this measure will not be related to an α-stable subordinator, but to an *inverse Gaussian subordinator* with changing parameters. Thus, the way we select the "sizes" of our traps is different. The rest of the construction of the environment is the same for both cases. This differences comes from the fact that the percolation parameter of the attached percolation clusters to the backbone is always critical in the IIC. Whereas, in the IPC, this percolation parameter is subcritical and varies depending on the vertex of the backbone that it corresponds.

CHAPTER 2

One-dimensional BTM with varying drift

1. Introduction

The *Bouchaud trap model* (BTM) is a continuous time random walk X on a graph $\mathcal{G}$ with random jump rates. To each vertex x of $\mathcal{G}$ we assign a positive number τ_x where $(\tau_x)_{x\in\mathcal{G}}$ is an i.i.d. sequence such that

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}[\tau_x \geq u] = 1 \tag{1.1}$$

with $\alpha \in (0,1)$. This means that the distribution of τ_x has heavy tails. Each visit of X to $x \in \mathcal{G}$ lasts an exponentially distributed time with mean τ_x. Let $S(k)$ be the time of the k-th jump of X. $(S(k), k \in \mathbb{N})$ is called the *clock process* of X. Let $Y_k := X(S(k))$ be the position of X after the k-th jump. $(Y_k : k \in \mathbb{N})$ is called the *embedded discrete time random walk* associated to X. This model was introduced by J.-P. Bouchaud in [**Bou92**] and has been studied by physicists as a toy model for the analysis of the dynamics of some complex systems such as spin-glasses. More precisely, each vertex x of $\mathcal{G}$ corresponds to a metastable state of the complex system, and X represents the trajectory of the system over its phase space. One of the phenomena that this model has helped to understand is that of aging, a characteristic feature of the slow dynamics of many metastable systems. For an account of the physical literature on the BTM we refer to [**BCKM98**].

The model has also been studied by mathematicians on different graphs, exhibiting a variety of behaviors. In [**FIN02**], Fontes, Isopi and Newman analyze the one-dimensional case ($\mathcal{G} = \mathbb{Z}$)and where the walk X is symmetric. They obtain a scaling limit for X which is called the *Fontes-Isopi-Newman* (FIN) singular diffusion. This diffusion is a speed measure change of a Brownian motion by a random, purely atomic measure ρ, where ρ is the Stieltjes measure associated to an α-stable subordinator. Different aging regimes for the one-dimensional case where found by Ben-Arous and Černý in [**BČ05**]. In higher dimensions ($\mathcal{G} = \mathbb{Z}^d, d \geq 2$), the symmetric model has a behavior completely different to the one-dimensional case, as shown by Ben Arous and Černý in [**BČ07**], and by Ben Arous, Černý and Mountford in [**BČM06**]. In these papers, a scaling limit and aging results where obtained for X. The scaling limit is called *fractional kinetic process* (F.K.P) which is a time-change of a d-dimensional Brownian motion by the inverse of an α-stable subordinator. In [**BBG03a**] and [**BBG03b**] Ben Arous, Bovier and Gayrard obtained aging properties of the model on the complete graph. A study of this walk for a wider class of graphs can be found on [**BČ08**]. For a general account on the mathematical study of the model, we refer to [**BČ06**].

The difference between the one dimensional case and the model in higher dimensions can be understood as follows. We can express the clock process $S(k)$ of X as $S(k) = \sum_{i=0}^{k-1} \tau_{Y_i} e_i$, where the e_i are standard i.i.d. exponential random variables. Thus, the increments of $S(k)$ are the depths of the traps $(\tau_x)_{x\in\mathcal{G}}$ as sampled by Y_k. In the model in dimensions higher than two, the embedded discrete time random walk Y_k is transient (the case $d = 2$ is more delicate). Thus Y_k will sample each trap τ_x a finite number of times. That implies that $S(k)$ does not have long range interactions with its past and its scaling limit will be a Markovian process, which is an α-stable subordinator. On the other hand, in the one-dimensional symmetric BTM, we have that the embedded discrete time random walk Y_k is recurrent. Thus Y_k will sample each trap τ_x an infinite number of times. In this case, $S(k)$ has long range interactions with its past and its scaling limit will be non-Markovian. Furthermore, the clock process $S(k)$ will converge to the local time of a Brownian motion integrated against the random measure ρ. Here ρ plays the role of a scaling limit for the environment $(\tau_x)_{x\in\mathbb{Z}}$.

It is natural to ask if we can find intermediate behaviors between the transient case ($d \geq 1$) and the recurrent case ($d = 1$): if we introduce a drift to the one-dimensional BTM, note that the embedded discrete random walk becomes transient. Thus, intermediate behaviors between the transient and the recurrent case might appear when one analyzes a sequence of one-dimensional BTM's with a drift that decreases to 0 as we rescale the walks. In this chapter we study this question, showing that the speed of decay of the drift sets the long-term behavior of the model and exhibiting a sharp phase transition in terms of the type of limiting processes obtained. We next describe with more precision the way in which we define the BTM with drift and the results that are obtained in this chapter.

For each $\epsilon > 0$, denote by X^ϵ the BTM on $\mathbb{Z}$ where the transition probabilities of the embedded discrete time random walk are $\frac{1+\epsilon}{2}$ to the right and $\frac{1-\epsilon}{2}$ to the left. We will call this process the BTM with drift ϵ. For $a \geq 0$, consider a rescaled sequence of B.T.M's with drift n^{-a}, $(h_a(n)X^{n^{-a}}(tn); t \geq 0)$, indexed by n, where $h_a(n)$ is an appropriate space scaling depending on a. We will see that as the drift decays slowly (small a), the sequence of walks converges to the inverse of an α-stable subordinator, whereas if the drift decays fast (large a) the limiting process is the FIN. diffusion. As these two posibilities are qualitatively different, we are led to think that there is either, a gradual interpolation between these two behaviors as the speed of decay changes, or a sharp transition between them as the speed of decay changes. We establish that there is a sharp transition between the two scaling limits, that there is a critical speed of decay where a new, previously, process appears and that the transition happens at $a = \alpha/(\alpha + 1)$. As the main theorem of this chapter, we prove that, depending on the value of a, there are three different scaling limits:

- **Supercritical case** ($a < \alpha/(\alpha+1)$). The sequence of walks converges to the inverse of an α-stable subordinator.
- **Critical case** ($a = \alpha/(\alpha+1)$). The sequence of walks converges to a process which is a speed measure change of a Brownian motion with drift that we will call the *drifted FIN diffusion*.
- **Subcritical case** ($a > \alpha/(\alpha+1)$). The sequence of walks converges to the FIN diffusion.

The case $a = 0$ (contained in the supercritical case), which corresponds to a constant drift, was already addressed by Zindy in [**Zin09**].

Let us now make a few remarks concerning the proof of our main theorem. The strategy of the proof for the supercritical case is a generalization of the method used in [**Zin09**] and relies on the analysis of the sequence of processes of first hitting times $(H_b^n(x); x \in [0, nS])$ (S is fixed, $b > 0$) defined as

$$H_b^n(x) := \inf\{t : X^{n^{-b}}(t) \geq x\}. \tag{1.2}$$

We show that these processes (properly rescaled) converge to an α-stable subordinator. From that, it follows that the maximum of the walks converges to the inverse of an α-stable subordinator. This part of the proof requires some care, because, as we are working with a sequence of walks with variable drift, we cannot apply directly the methods used in [**Zin09**]. It turns out that we have to choose b properly to obtain a sequence of walks with the desired drift as we invert the hitting time processes. Then, it is easy to pass from the maximum of the walk to the walk itself. In [**FIN02**] The proof corresponding to the critical case follows the arguments used by [**FIN02**]. There they express rescaled, symmetric one-dimensional BTM's as speed measure changes of a Brownian motion trough a random speed measure. But here we are working with asymmetric walks, so we cannot work with the expression used there. To treat the asymmetry of the walks, we use a Brownian motion with drift instead of a Brownian motion. That is, we express each walk $X^{n^{-\alpha/(\alpha+1)}}$ as a speed measure change of a Brownian motion with drift, and then prove convergence of the sequence of speed measures to ρ. The latter is achieved by means of a coupling of the environments. In the subcritical case, although we obtain the same scaling limit as in [**FIN02**] (a FIN diffusion), again, because of the asymmetry of the model, we cannot work with the expression used there. We deal with this obstacle using, besides a random speed measure, a scaling function. That is, we express the rescaled walks as time-scale changes of a Brownian motion. Then we prove that the scale change can be neglected and show convergence of the sequence of speed measures to the random measure ρ.

The organization of the chapter is as follows. In section 2 we give the definition of the model and state our main results. There we also give simple heuristic arguments to understand the transition at $a = \alpha/(\alpha+1)$. In section 3 we obtain the behavior for the supercritical case, and in section 4 we obtain the scaling limit for the critical case. The behavior for the subcritical case is obtained in section 5.

Finally, we would like to mention that while preparing the final version of this article we have learned that Theorem (3.1) has been independently obtained by Gantert, Mörters and Wachtel [**GMW10**]. There, they also obtain aging results for the BTM with vanishing drift.

2. Notations and Main Results

A Bouchaud trap model on $\mathbb{Z}$ with drift ϵ, $(X^\epsilon(t); t \in [0,\infty])$ is a homogeneous Markov process with jump rates:

$$c(x,y) := \begin{cases} (1+\epsilon)\tau_x^{-1}/2 \text{ if } y = x+1 \\ (1-\epsilon)\tau_x^{-1}/2 \text{ if } y = x-1 \end{cases}, \tag{2.1}$$

where $\tau = (\tau_x)_{x\in\mathbb{Z}}$ are positive, i.i.d. under a measure P and satisfy

$$\lim_{u\to\infty} u^\alpha P[\tau_x \geq u] = 1. \tag{2.2}$$

For any topological space E, $\mathcal{B}(E)$ will stand for the σ-algebra of Borelians of E. $\mathbb{P}^x_\tau$ and $\mathbb{E}^x_\tau$ will denote the probability and expectation conditioned on the environment $\tau = (\tau_x)_{x\in\mathbb{Z}}$ and with $X^\epsilon(0) = x$. These probabilities are often referred as quenched probabilities. We define $\mathbb{P}^x$ on $\mathbb{Z}^\mathbb{N} \times \mathbb{R}^{+\mathbb{Z}}$ stating that for every $A \in \mathcal{B}(\mathbb{Z}^\mathbb{N})$ and $B \in \mathcal{B}(\mathbb{R}^{+\mathbb{Z}})$, $\mathbb{P}^x[A\times B] := \int_B \mathbb{P}^x_\tau[C_\tau]P(d\tau)$, where $C_\tau := \{x \in \mathbb{Z}^\mathbb{N} : (x,\tau) \in A\times B\}$.

$\mathbb{P}^x$ is called the annealed probability. Note that X^ϵ is Markovian w.r.t. $\mathbb{P}^x_\tau$ but non-Markovian w.r.t. $\mathbb{P}^x$. $\mathbb{E}^x$ is the expectation associated to $\mathbb{P}^x$. $\mathbb{P}^0$ and $\mathbb{E}^0$ will be simply denoted as $\mathbb{P}$ and $\mathbb{E}$. Also $\mathbb{P}_\tau$ and $\mathbb{E}_\tau$ will stand for $\mathbb{P}^0_\tau$ and $\mathbb{E}^0_\tau$ respectively. These notations will be used with the same meaning for all the processes appearing in this chapter.

We have to make some definitions in order to state our main result: let $B(t)$ be a standard one dimensional Brownian motion starting at zero and $l(t,x)$ be a bi-continuous version of his local time. Given any locally finite measure μ on $\mathbb{R}$, denote

$$\phi_\mu(s) := \int_\mathbb{R} l(s,y)\mu(dy),$$

and its right continuous generalized inverse by

$$\psi_\mu(t) := inf\{s > 0 : \phi_\mu(s) > t\}.$$

The right continuous generalized inverse exists by definition, is increasing and, as its name indicates, it is a right continuous function. Then we define the speed measure change of B with speed measure μ, $X(\mu)(t)$ as

$$X(\mu)(t) := B(\psi_\mu(t)). \tag{2.3}$$

We also need to define speed measure changes of a drifted Brownian motion. Let $C(t) := B(t) + t$. We know that $C(t)$ has a bi-continuous local time $\tilde{l}(t, y)$. Given any locally finite measure μ in $\mathbb{R}$ we define

$$\tilde{\phi}_\mu(s) := \int_{\mathbb{R}} \tilde{l}(s, y)\mu(dy),$$

and its generalized right-continuous inverse by

$$\tilde{\psi}_\mu(t) := \inf\{s > 0 : \tilde{\phi}_\mu(s) > t\}.$$

Then we define $\tilde{X}(\mu)(t)$ (the speed measure change of C with speed measure μ) by

$$\tilde{X}(\mu)(t) := C(\tilde{\psi}_\mu(t)). \tag{2.4}$$

By changing the starting point of our underlying Brownian motion B, we can change the starting point of $\tilde{X}(\mu)$ and $X(\mu)$.

Let (x_i, v_i) be an inhomogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}^+$, independent of B with intensity measure $\alpha v^{-1-\alpha} dx dv$. We define the random measure ρ as

$$\rho := \sum v_i \delta_{x_i}. \tag{2.5}$$

The diffusion $(Z(t); t \in [0, T])$ defined as $Z(s) := B(\psi_\rho(s))$ is called the *FIN diffusion*. We also define the *drifted FIN diffusion* $\tilde{Z}(t)$ as $\tilde{Z}(t) := C(\tilde{\psi}_\rho(t))$.

$D[0, T]$ will denote the space of cadlag functions from $[0, T]$ to $\mathbb{R}$. $(D[0, T], M_1)$, $(D[0, T], J_1)$ and $(D[0, T], U)$ will stand for $D[0, T]$ equipped with the Skorohod-M_1, Skorohod-J_1, and uniform topology respectively. We refer to [**Whi02**] for an account on these topologies. We define $(X^{(n,a)}; t \in [0, T])$, a rescaling of a walk with drift n^{-a}, by

$$X^{(n,a)}(t) := \begin{cases} \frac{X^{n^{-a}}(tn)}{n^{\alpha(1-a)}} \text{if } a < \frac{\alpha}{1+\alpha} \\ \\ \frac{X^{n^{-a}}(tn)}{n^{\alpha/(\alpha+1)}} \text{if } a \geq \frac{\alpha}{1+\alpha} \end{cases}. \tag{2.6}$$

Let V_α be an α-stable subordinator started at zero. That is, V_α is the increasing Levy process with Laplace transform $\mathbb{E}[\exp(-\lambda V_\alpha(t))] = \exp(-t\lambda^\alpha)$. Now we are in conditions to state the main result of this chapter.

THEOREM 2.1. *For all $T > 0$:*

(i) *If $a < \alpha/(\alpha + 1)$ we have that $(X^{(n,a)}(t); t \in [0, T])$ converges in distribution to $(V_\alpha^{-1}(t); t \in (0, T))$ in $(D[0, T], U)$ where V_α^{-1} is the right continuous generalized inverse of V_α.*

(ii) *If $a = \alpha/(\alpha + 1)$ we have that $(X^{(n,a)}(t); t \in [0, T])$ converges in distribution to the drifted FIN diffusion $(\tilde{Z}(t); t \in [0, T])$ on $(D[0, T], U)$.*

(iii) *If $a > \alpha/(\alpha+1)$ we have that $(X^{(n,a)}(t); t \in [0; T])$ converges in distribution to the FIN diffusion $(Z(t); t \in [0, t])$ on $(D[0, T], U)$.*

We present heuristic arguments to understand the transition at $a = \frac{\alpha}{1+\alpha}$. First we analyze a sequence of discrete time random walks. Let $(S^\epsilon(i), i \in \mathbb{N})$ be a simple asymmetric random walk with drift ϵ, $S^\epsilon(i) := \sum_{k=1}^{i} b_k^\epsilon$, where $(b_k^\epsilon)_{i\in\mathbb{N}}$ is an i.i.d. sequence of random variables with: $\mathbb{P}[b_k^\epsilon = 1] = \frac{1+\epsilon}{2}$; $\mathbb{P}[b_k^\epsilon = -1] = \frac{1-\epsilon}{2}$. We want to find the possible scaling limits of $(S^{\epsilon(n)}(in); i \in [0,T])$, depending on the speed of decay of $\epsilon(n)$ to 0 as $n \to \infty$.

We couple the sequence of walks $S^{\epsilon(n)}$ in the following way: Let $(U_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of uniformly distributed random variables taking values on $[0,1]$. We require that $S^{\epsilon(n)}$ takes his i-th step to the right ($b_i^{\epsilon(n)} = 1$) if $U_i > \frac{1-\epsilon(n)}{2}$ and to the left otherwise. For each walk, we can decompose the steps into two groups: the first group is given by the steps i such that $\frac{1-\epsilon(n)}{2} < U_i < \frac{1+\epsilon(n)}{2}$ and the second group consists of the remaining steps. We can think that the first group of steps takes account of the drift effect and the second one takes account of the symmetric fluctuations of the walk.

If the walk has given n steps, then the first group has about $n\epsilon(n)$ steps, and the second group has fluctuations of order $\sqrt{n}$. It is obvious that the drift effect will dominate the behavior if $\sqrt{n} = o(\epsilon(n))$. In this case we will have a ballistic (deterministic) process as a scaling limit. If $\epsilon(n) = o(\sqrt{n})$ the fluctuations will dominate and we will have a Brownian motion as scaling limit. Finally the two behaviors will be of the same order if $\epsilon(n) \approx \sqrt{n}$, and a Brownian motion with drift will be the scaling limit.

The same reasoning can now be used to understand the change of behavior at $a = \alpha/(\alpha+1)$ for the sequence of walks $(X^{n^{-a}}(tn), t \in [0,T])_{n\in\mathbb{N}}$. In order to apply the precedent arguments we first have to estimate the number of steps that $X^{n^{-a}}$ has given up to time Tn. To simplify we take $T = 1$. First, suppose that $X^{n^{-a}}(n)$ is of order n^u, where u is to be found. We know that after k steps, a walk with drift n^{-a} is approximately on site kn^{-a}, so, it takes about n^{u+a} steps to be on site n^u. Thus, we can also deduce that at time n, $X^{n^{-a}}$ has visited approximately n^a times each site. As the distribution of τ_i satisfies (5.1), then the sum $\sum_{i=0}^{n^u} \tau_i$ is of the same order that $\max_{\{0\le i\le n^u\}} \tau_i$, and both are of order $n^{u/\alpha}$. We can estimate the time needed to arrive at n^u as the depth of the deepest trap found ($\approx n^{u/\alpha}$) multiplied by the number of visits to that trap ($\approx n^a$). This gives that $n \approx n^{\frac{u}{\alpha}+a}$. But, we know, by definition, that $X^{n^{-a}}$ arrives at the site $n^{n/u}$ approximately at time n. It follows that $1 = (u/\alpha) + a$, which yields $u = (1-a)\alpha$. This means that the number of steps that $X^{n^{-a}}$ has given up to time n is of order $n^{(1-a)\alpha+a}$.

Again, we can decompose the steps of $X^{n^{-a}}$ into two groups. The first group accounts for the drift effect, and the second one accounts for the fluctuations. The first group will have approximately $n^{-a+[(1-a)\alpha+a]}$ steps and the second group will give a contribution to the position of order $n^{\frac{(1-a)\alpha+a}{2}}$. Now it is easy to see that the ballistic behavior and the fluctuations will be of the same order i.f.f. $[(1-a)\alpha+a]/2 = (1-a)\alpha$ or $a = \alpha/(1+\alpha)$.

3. The Supercritical Regime

The proof for the constant drift case ($a = 0$) in [**Zin09**] is roughly as follows: first he prove that the sequence of rescaled first hitting times, $(n^{-1/\alpha} \inf\{s \geq 0 : X^{\epsilon}(ns) \geq x\} : x \geq 0)$, converges to an α-stable subordinator. Then, using that the right continuous generalized inverse of the process of first hitting times is the maximum of $X^{\epsilon}(t)$, he can deduce that $(\max\{n^{-1} X^{\epsilon}(n^{1/\alpha} s) : s \leq t\} : t \geq 0)$ converges to the inverse of an α-stable subordinator. Finally he shows that the walk and its maximum are close.

For the proof of part (i) of theorem 3.1 we cannot follow the proof of [**Zin09**] in a straightforward way: suppose we show that a properly rescaled sequence of first hitting time processes $(p_a(n) H_a^n(nx) : x \in \mathbb{R}_+)$ (where $p_a(n)$ is an appropriate scaling) converges to an α-stable subordinator. Then, by inverting the processes, we get that the sequence $(\max\{n^{-1} X^{n^{-a}}(p_a(n)^{-1} s) : s \leq t\} : t \in \mathbb{R}_+)$ converges to the inverse of an α-stable subordinator. But we are searching a limit for $(\max\{d_a(n) X^{n^{-a}}(tn) : t \in \mathbb{R}_+\})$ (where $d_a(n)$ is appropriate space scaling). That is, we want to obtain the limit of a sequence of rescaled walks where the drift decays as n^{-a} when the time is rescaled by n. But when we invert $(p_a(n) H_a^n(nx) : x \in \mathbb{R}_+)$, we obtain the sequence $(\max\{n^{-1} X^{n^{-a}}(p_a(n)^{-1} s) : s \leq t\} : t \geq 0)$, which is a sequence of maximums of rescaled walks in which the drift decays as n^{-a} when the time is rescaled as $p_a(n)^{-1}$.

To solve this, we will prove that the limit of $(q_a(n) H_{b^*}^n(nx) : x \in \mathbb{R}_+)$ is an α-stable subordinator, where $q_a(n)$ is an appropriate scaling and b^* sets an appropriate drift decay and depends on a. Inverting, we will obtain that $(\max\{n^{-1} X^{n^{-b^*}}(q_a(n)^{-1} s) : s \leq t\} : t \geq 0)$ converges to an α-stable subordinator. As we have said, we want the limit of a sequence of rescaled walks with a drift that decays as n^{-a} as the time parameter is rescaled by n. Hence, when the time parameter is rescaled as $q_a(n)^{-1}$, the drift should rescale as $q_a(n)^a$. Thus we need to choose b^* so that $n^{-b^*} = q_a(n)^a$. But we know that $q_a(n)$ is the appropriate scaling for $(H_{b^*}^n(nx) : x \in \mathbb{R}_+)$. Hence, $q_a(n)$ must be the order of magnitude of $H_{b^*}^n(n)$. That is $q_a(n)$ is of the order of the time that the walk $X^{n^{-b^*}}$ needs to reach n.

We now give a heuristic argument to find $q_a(n)$ and b^*. When $X^{n^{-b^*}}(t)$ has given k steps, it has an order kn^{-b^*}. So it takes about n^{b^*+1} steps to be on site n. We can think that the number of visits to each site x is evenly distributed. Then each site is visited about n^{b^*} times before $X^{n^{-b^*}}$ hits n. The time that the walks needs to reach n is of the order of the time spent in the largest trap. Thus we can estimate the total time spent by the walk as the depth of the deepest trap (which is of order $n^{-1/\alpha}$) multiplied by the number of visits to that trap. This gives a time of order $n^{1/\alpha+b^*}$. What the previous arguments show is that $X^{n^{-b^*}}(t)$ arrives at n at time $t \approx n^{1/\alpha+b^*}$ $(q_a(n) \approx n^{1/\alpha+b^*})$. But at that time we want to analyze a walk of drift $(n^{1/\alpha+b^*})^{-a}$. That is, we need that $a(1/\alpha + b^*) = b^*$. In this way we find that $b^* := a/[(1-a)\alpha]$.

3.1. The embedded discrete time walk. For each natural n, the *clock processes* S^n is defined as $S^n(0) := 0$. Furthermore $S^n(k)$ is the time of the k-th jump of $X^{n^{-b^*}}$. S^n is extended to all $\mathbb{R}^+$ by setting $S^n(s) := S^n(\lfloor s \rfloor)$. To each drifted walk $X^{n^{b^*}}(t)$ we associate its corresponding *embedded discrete time random walk* $(Y_i^{n^{-b^*}} : i \in \mathbb{N})$ defined as $Y_i^{n^{-b^*}} := X^{n^{-b^*}}(t)$ where t satisfies: $S^n(i) \leq t < S^n(i+1)$.

Obviously $Y_i^{n^{-b^*}}$ is a discrete time random walk with drift n^{-b^*}. We can write

$$S(k) = \sum_{i=0}^{k-1} \tau_{Y_i} e_i,$$

where $(e_i)_{i \geq 0}$ is an i.i.d. sequence of exponentially distributed random variables with mean 1.

Define

$\epsilon = \epsilon(n) := n^{-b^*}$

$p = p(n) := (1 + \epsilon(n))/2$

$q = q(n) := (1 - \epsilon(n))/2$ and

$\nu(n) := \lfloor c \log(n) n^{b^*} \rfloor$ with $c > 2$.

Let $\Xi(x,k) = \Xi(x,k,n)$ be the probability that $Y_i^{\epsilon(n)}$ hits x before k starting at $x+1$. Then we have that $\Xi(x,k) = q + p\Xi(x+1,k)\Xi(x,k)$ and that $\Xi(k-2,k) = q$. These observations give a difference equation and an initial condition to compute $\Xi(x,k)$. Then we get that

$$\Xi(x,k) = r \frac{1 - r^{k-x-1}}{1 - r^{k-x}}, \tag{3.1}$$

where $r = r(n) := q(n)/p(n)$. Using that formula we can see that the probability that the walk $Y_i^{\epsilon(n)}$ ever hits $x-1$ starting at x is r. We now present a backtracking estimate.

Lemma 2.2. *Let* $\mathcal{A}(n) := \{\min_{i \leq j \leq \zeta_n(n)} (Y_j^{\epsilon(n)} - Y_i^{\epsilon(n)}) \geq -\nu(n)\}$ *where* $\zeta_n(i) := \min\{k \geq 0 : Y_k^{\epsilon(n)} = i\}$*, then* $\lim_{n\to\infty} \mathbb{P}[\mathcal{A}(n)] = 1$.

Proof: We can write

$$\mathcal{A}^c(n) = \bigcup_{x=0}^{n-1} \left\{ \min_{\zeta_n(x) \leq i \leq \zeta_n(n)} (Y_i^{\epsilon(n)} - Y_{\zeta_n(x)}^{\epsilon(n)}) < -\nu(n) \right\}.$$

Hence

$$\mathcal{A}^c(n) \subseteq \bigcup_{x=0}^{n-1} \left\{ \min_{\zeta_n(x) \leq i} (Y_i^{\epsilon(n)} - Y_{\zeta_n(x)}^{\epsilon(n)}) < -\nu(n) \right\}.$$

But, in order to arrive from x to $x - \nu(n)$, for each $j = x-1, \ldots, x-\nu(n)$, starting from $j+1$ the random walk $Y_i^{\epsilon(n)}$ needs to hit j in a finite time. Hence, it takes $\nu(n)$ realizations of independent events (strong Markov property) of probability $r(n)$. In other words $\mathbb{P}[\mathcal{A}^c(n)] \leq nr(n)^{\nu(n)} = n(1 - \frac{2}{1+n^{b^*}})^{\nu(n)}$, which can be bounded by $n(1 - \frac{1}{n^{b^*}})^{\nu(n)}$. Replacing $\nu(n)$ we obtain $n((1 - \frac{1}{n^{b^*}})^{n^{b^*}})^{c\log(n)}$. We can see that $(1 - \frac{1}{n^{b^*}})^{n^{b^*}} \to e^{-1}$ when $n \to \infty$. Now, for n big enough $\left(1 - \frac{1}{n^{b^*}}\right)^{n^{b^*}} \leq e^{-\frac{1}{2}}$. Then

$$\mathbb{P}[\mathcal{A}^c(n)] \leq n n^{-\frac{1}{2}c}.$$

But $c > 2$, so we get the result.

□

Now we state the convergence result for the hitting time processes.

LEMMA 2.3. *Let*

$$H^{(n)}(t) := \frac{H^n_{b^*}(tn)}{n^{(1/\alpha)+b^*}}. \tag{3.2}$$

Then $(H^{(n)}(t); t \in [0,T])$ *converges weakly to* $((\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(t); t \in [0,T])$ *on* $(D[0,T], M_1)$*, where* $V_\alpha(t)$ *is an* α*-stable subordinator.*

The proof of this lemma will be given in subsection 3.5. We present the proof of part (i) of Theorem 3.1 using lemma 2.3 and devote the rest of the section to the proof of lemma 2.3.

3.2. Proof of (i) of Theorem 3.1. Let us denote

$$\bar{X}^n(t) := n^{-1} \max\{X^{n^{-b^*}}(sn^{(1/\alpha)+b^*}); s \in [0,t]\}.$$

First we will prove convergence in distribution of $\bar{X}^n$ to the (right continuous generalized) inverse of $(\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha$ in the uniform topology. That is, we want to to prove convergence in distribution of the inverse of $(H^{(n)}(t); t \in [0,T])$ to the inverse of $((\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(t); t \in [0,T])$ in the uniform topology. Define

$$\mathcal{C}(T,S)_n := \{H^{(n)}(S) \geq Tn^{(1/\alpha)+b^*}\}.$$

Then, we have that, on $\mathcal{C}(T,S)_n$, the right continuous generalized inverse of $(H^{(n)}(s); s \in [0,S])$ is $(\bar{X}^n(t); t \in [0,T])$. Let $T > 0$ be fixed, by Lemma 2.3, we know that we can choose S big enough so that $\lim_{n\to\infty} \mathbb{P}[\mathcal{C}(T,S)_n]$ is as close to 1 as we want. Let $D^{\uparrow}[0,T]$ be the subset of $D[0,T]$ consisting of the increasing functions. By corollary 13.6.4 of [**Whi02**], the inversion map from $(D^{\uparrow}[0,T], M_1)$ to $(D^{\uparrow}[0,T], U)$ is continuous at strictly increasing functions. Lemma (2.3) gives convergence in distribution of $(H^{(n)}(t); t \in [0,S])$ to $((\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(t); t \in [0,S])$ in the Skorohod M_1 topology. We know that V_α is a. s. strictly increasing, that is $((\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(t); t \in [0,S]) \in D^{\uparrow}[0,T]$ almost surely. So we can apply corollary 13.6.4 of [**Whi02**] and deduce convergence in distribution of $\bar{X}^n$ to the inverse of $(\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha$ in the uniform topology. As we have said previously, the inverse of $(H^{(n)}(s); s \in [0,S])$ is $(\bar{X}^n(t); t \in [0,T])$ in $\mathcal{C}(T,S)_n$. This proves convergence of the maximum of the walk. To deduce convergence of the walk itself it suffices to show that the walk is close enough to its maximum in the uniform topology. That is, to prove the theorem, it is enough to show that for all $\gamma > 0$:

$$\mathbb{P}\left[\sup_{0\leq t\leq T} |n^{-1}X^{n^{-b^*}}(tn^{(1/\alpha)+b^*}) - \bar{X}^n(t)| \geq \gamma\right] \to 0.$$

Again, by Lemma 2.3 we know that $\mathbb{P}[H^n_{b^*}(n\log(n)) \geq Tn^{(1/\alpha)+b^*}] \to 1$. Hence, we just have to prove that

$$\mathbb{P}\left[\sup_{0\leq t\leq H^n_{b^*}(n\log(n))} |n^{-1}X^{n^{-b^*}}(t) - n^{-1}\max\{X^{n^{-b^*}}(s); s\in[0,t]\}| \geq \gamma\right] \to 0.$$

Which is to say,

$$\mathbb{P}\left[\sup_{0\leq k\leq \zeta_n(\lfloor n\log(n)\rfloor)} |Y^{\epsilon(n)}_k - \bar{Y}^{\epsilon(n)}_k| \geq n\gamma\right] \to 0.$$

where $\bar{Y}^{\epsilon(n)}$ is the maximum of $Y^{\epsilon(n)}$. But, we can apply Lemma 2.2 to see that this is the case.

3.3. The environment. Here we give estimates concerning the environment. For each $n\in\mathbb{N}$ define

$$g(n) := \frac{n^{1/\alpha}}{(\log(n))^{\frac{2}{1-\alpha}}}.$$

Now, for each site $x\in\mathbb{N}$, we say that x is an *n-deep trap* if $\tau_x \geq g(n)$. Otherwise we will say that x is an *n-shallow trap*. We now order the set of n-deep traps according to their position from left to right. Then call $\delta_1(n)$ the leftmost n-deep trap and in general call for $j\geq 1$, $\delta_j(n)$ the j-th n-deep trap. The number of n-deep traps in $[0,n]$ is denoted by $\theta(n)$. Let us now define

$$\mathcal{E}_1(n) := \left\{n\varphi(n)\left(1-\frac{1}{\log(n)}\right) \leq \theta_n \leq n\varphi(n)\left(1+\frac{1}{\log(n)}\right)\right\},$$

$$\mathcal{E}_2(n) := \{\delta_1 \wedge (\min_{1\leq j\leq \theta_n - 1}(\delta_j - \delta_{j-1})) \leq \rho(n)\},$$

$$\mathcal{E}_3(n) := \{\max_{-\nu(n)\leq x\leq 0} \tau_x < g(n)\}, \text{ and}$$

$$\mathcal{E}(n) := \mathcal{E}_1(n)\cap\mathcal{E}_2(n)\cap\mathcal{E}_3(n)$$

where $\rho(n) := n^{\kappa} \quad \kappa < 1$ and $\varphi(n) := \mathbb{P}[\tau_x \geq g(n)]$.

LEMMA 2.4. *We have that* $\lim_{n\to\infty}\mathbb{P}[\mathcal{E}(n)] = 1$.

PROOF: $\theta(n)$ is binomial with parameters $(n,\varphi(n))$. $\mathcal{E}_1$ is estimated using the Markov inequality. To control $\mathcal{E}_2$ it is enough to see that in $0,..,n$ there are $O(n\rho(n))$ pairs of points at a distance less than $\rho(n)$. The estimate on $\mathcal{E}_3$ is trivial.

□

3.4. Time control. In this subsection we prove results about the time spent by the walk on the traps.

3.4.1. *Shallow traps.* Here we will show that the time that the walks spend in the shallow traps is negligible.

LEMMA 2.5. *Let* $\mathcal{I}(n) := \left\{\sum_{i=0}^{\zeta_n(n)} \tau_{Y_i^{\epsilon(n)}} e_i 1_{\{\tau_{Y_i^{\epsilon(n)}} \leq\, g(n)\}} \leq \frac{n^{1/[(1-a)\alpha]}}{\log(n)}\right\}$. *Then*

$$\mathbb{P}[\mathcal{I}(n)] \to 1 \text{ as } n \to \infty. \tag{3.3}$$

PROOF: We have that $\mathbb{P}[\mathcal{I}(n)^c] = \mathbb{P}[\mathcal{I}(n)^c \cap \mathcal{E}(n)] + o(1)$. Using the Markov inequality it suffices to show that

$$\mathbb{E}\left[\sum_{i=0}^{\zeta_n(n)} \tau_{Y_i^{\epsilon(n)}} e_i 1_{\{\tau_{Y_i^{\epsilon(n)}} < g(n)\}} 1_{\{Y_i^{\epsilon(n)} \geq -\nu(n)\}}\right] = o\left(\frac{n^{1/[(1-a)\alpha]}}{\log(n)}\right).$$

The number of visits of $Y_i^{\epsilon(n)}$ to x before time $\zeta_n(n)$ is $1 + G(x,n)$, where $G(x,n)$ is a geometrically distributed random variable of parameter $1-(q+p\Xi(x,n))$ (the parameter is the probability that, $Y_i^{\epsilon(n)}$, starting at x, hits n before returning to x). Also

$$\mathbb{E}_\tau\left[\sum_{i=0}^{\zeta_n(n)} \tau_{Y_i^{\epsilon(n)}} e_i 1_{\{\tau_{Y_i^{\epsilon(n)}} < g(n)\}} 1_{\{Y_i^{\epsilon(n)} \geq -\nu(n)\}}\right] \leq \sum_{x=-\nu(n)}^{n} \tau_x (1 + \mathbb{E}_\tau[G(x,n)]) 1_{\{\tau_x < g(n)\}}. \tag{3.4}$$

Using (3.1) we can deduce that $(1 + \mathbb{E}[G(x,n)]) \leq \frac{(1-r(n))}{p} \leq cn^{-b^*}$. So, averaging with respect to the environment in (3.4) we get

$$\mathbb{E}\left[\sum_{i=0}^{\zeta_n(n)} \tau_{Y_i^{\epsilon(n)}} e_i 1_{\{\tau_{Y_i^{\epsilon(n)}} < g(n)\}} 1_{\{Y_i^{\epsilon(n)} \geq -\nu(n)\}}\right] \leq Cn^{1+b^*} \mathbb{E}[\tau_0 1_{\{\tau_0 < g(n)\}}].$$

Also

$$\mathbb{E}[\tau_0 1_{\{\tau_0 < g(n)\}}] \leq \sum_{j=0}^{\infty} (1/2)^j g(n) \mathbb{P}[\tau_0 > (1/2)^{j+1} g(n)].$$

Now, using (1.2) there exists a constant C such that the righthand side of the above inequality is bounded above by

$$Cg(n)^{1-\alpha} \sum_{j=0}^{\infty} ((1/2)^{1-\alpha})^j.$$

Furthermore, since $1-\alpha > 0$ this expression is bounded above by $Cg(n)^{1-\alpha}$. This finishes the proof.

□

3.4.2. *Deep traps.* Here we will estimate the time spent in deep traps. We define the occupation time for $x \in \mathbb{Z}$ as

$$T_x = T_x(n) := \sum_{i=0}^{\zeta_n(n)} \tau_{Y_i^{\epsilon(n)}} e_i \ \mathbf{1}_{\{Y_i^{\epsilon(n)} = x\}}.$$

The walk visits x, $G(x,n)+1$ times before $\zeta_n(n)$, and each visit lasts an exponentially distributed time. This allows us to control the Laplace transform of T_x. For any pair of sequences of real numbers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $a_n \sim b_n$ will mean that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

LEMMA 2.6. *Let* $\lambda > 0$. *Define* $\lambda_n := \frac{\lambda}{n^{1/[(1-a)\alpha]}}$. *Then we have that*

$$\mathbb{E}^x[1-\exp(-\lambda_n T_x)|\tau_x \geq g(n)] \sim \frac{\mathbb{P}[\tau_x \geq g(n)]^{-1}\alpha\pi\lambda^{-\alpha}}{n\sin(\alpha\pi)}.$$

PROOF: We must perform an auxiliary computation about the asymptotic behavior of the parameter $1-(q+p\Xi(x,n))$ of $G(x,n)$:

$$(1-(q+p\Xi(x,n)))n^{b^*} = p\frac{1-r}{1-r^{n-x}}$$

$$= \frac{2p(1+n^{-b^*})^{n-x}}{(1+n^{-b^*})((1+n^{-b^*})^{n-x}-(1-n^{-b^*})^{n-x})}$$

$$= \frac{2p}{(1+n^{-b^*})(1-(1-\frac{2n^{-b^*}}{1+n^{-b^*}})^{n-x})}$$

which converges to 1. Thus we have showed that

$$(1-(q+p\Xi(x,n)))n^{b^*} \overset{n\to\infty}{\to} 1 \tag{3.5}$$

We have

$$\mathbb{E}^x_\tau[\exp(-\lambda_n T_x)] = \mathbb{E}^x_\tau\left[\exp\left(-\lambda_n\sum_{i=0}^{G(x,n)}\tau_x\tilde{e}_i\right)\right]$$

where $\tilde{e}_i$ are i.i.d. end exponentially distributed with $\mathbb{E}(\tilde{e}_i)=1$. Let $\tilde{\lambda}_n := \frac{\lambda}{n^{1/\alpha}}$. Then

$$\mathbb{E}^x_\tau[\exp(-\lambda_n T_x)] = \frac{1}{1+\tilde{\lambda}_n\frac{\tau_x}{n^{b^*}(1-(q+p\Xi(x,n)))}}.$$

Using (3.5) we get that the above expression equals

$$= \frac{1}{1+\tilde{\lambda}_n\tau_x(1+o(1))} = \frac{1}{1+\tilde{\lambda}_n\tau_x} + o(n^{-1/\alpha}).$$

Averaging with respect to the environment

$$\mathbb{E}^x[1-\exp(-\lambda_n T_x)1_{\{\tau_x\geq g(n)\}}] = \int_{g(n)}^\infty 1-\frac{1}{1+\tilde{\lambda}_n z}\tau_0(dz) + o(n^{-1/\alpha})$$

where the notation $\tau_0(dz)$ denotes integration with respect the distribution of τ_0. Integrating by parts $\int_{g(n)}^\infty 1-\frac{1}{1+\tilde{\lambda}_n z}\tau_0(dz)$ we get that the above display equals

$$\left[-\frac{\tilde{\lambda}_n z}{1+\tilde{\lambda}_n}\mathbb{P}[\tau_0 \geq z]\right]_{g(n)}^\infty + \int_{g(n)}^\infty \frac{\tilde{\lambda}_n}{(1+\tilde{\lambda}_n z)^2}\mathbb{P}[\tau_0 \geq z]dz + o(n^{-1/\alpha}).$$

The first term is smaller than $C\tilde{\lambda}_n g(n)^{1-\alpha} = o(n^{-1})$. To estimate the second term, note that for all $\eta > 0$ we have

$$(1-\eta)z^{-\alpha} \leq \mathbb{P}[\tau_0 \geq z] \leq (1+\eta)z^{-\alpha}$$

for z large enough. Then we must compute $\int_{g(n)}^{\infty} \frac{\tilde{\lambda}_n}{(1+\tilde{\lambda}_n z)^2} z^{-\alpha} dz$. Changing variables with $y = \frac{\tilde{\lambda}_n z}{1+\tilde{\lambda}_n z}$ we obtain

$$\tilde{\lambda}_n^{-\alpha} \int_{\frac{\tilde{\lambda}_n g(n)}{1+\tilde{\lambda}_n g(n)}}^{1} y^{-\alpha}(1-y)^{\alpha} dy.$$

But we know that this integral converges to $\Gamma(\alpha+1)\Gamma(\alpha-1) = \frac{\pi\alpha}{\sin(\pi\alpha)}$.

$\square$

3.5. Proof of Lemma 2.3. We will show the convergence of the finite dimensional Laplace transforms of the rescaled hitting times to the corresponding expression for an α-stable subordinator. This will prove finite dimensional convergence.

Let $0 = u_0 < \cdots < u_K \leq T$ and $\beta_i, i = 1..K$ be positive numbers. We know that

$$\mathbb{E}\left[\exp \sum_{i=1}^{K} -\beta_i((\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(u_i) - (\frac{\pi\alpha}{\sin(\pi\alpha)})^{-1/\alpha} V_\alpha(u_{i-1}))\right] \tag{3.6}$$

$$= \exp\left(\sum_{i=1}^{K} -\frac{\alpha\pi\beta_K^{-\alpha}}{\sin(\alpha\pi)}(u_K - u_{K-1})\right). \tag{3.7}$$

So, it only suffices to show that

$$\mathbb{E}\left[\exp \sum_{i=1}^{K} -\beta_i(H^{(n)}(u_i) - H^{(n)}(u_{i-1}))\right] \overset{n\to\infty}{\to} \exp\left(\sum_{i=1}^{K} -\frac{\alpha\pi\beta_K^{-\alpha}}{\sin(\alpha\pi)}(u_K - u_{K-1})\right)$$

where $H^{(n)}$ is as in (3.2). We can decompose the trajectory of $Y^{\epsilon(n)}$ up to $\zeta_n(\lfloor nu_K \rfloor)$ into three parts. The first one is the trajectory up to the time $\zeta_n(\lfloor nu_{K-1} - \nu(Tn)\rfloor)$, the second one is the trajectory between times $\zeta_n(\lfloor nu_{K-1} - \nu(Tn)\rfloor)$ and $\zeta_n(\lfloor nu_{K-1}\rfloor)$, finally, the third part is the trajectory starting from time $\zeta_n(\lfloor nu_{K-1}\rfloor)$ up to time $\zeta_n(\lfloor nu_K\rfloor)$. First we will show that the time spent in the second part of the trajectory is negligible. We have that $\mathbb{P}[\max_{y\in B_{\nu(Tn)}(x)} > g(Tn)] = o(1)$, which is to say that the probability of finding an n-deep trap in a ball of radius $\nu(Tn)$ is small. Indeed Lemma 2.5 implies that there exists a constant $C > 0$ such that

$$\mathbb{P}\left[\sum_{i=0}^{\zeta_{\lfloor u_K n\rfloor}} \tau_{Y_i^{\epsilon(n)}} e_i 1_{\left\{\tau_{Y_i^{\epsilon(n)}} \in B_{\nu(Tn)}(\lfloor u_{K-1} n\rfloor)\right\}} < Cn^{\frac{1}{(1-a)\alpha}} (\log(n))^{-1}\right] \to 1.$$

Hence, the time that the walk spends in $B_{\nu(Tn)}(\lfloor u_{K-1}n\rfloor)$ is negligible. But in $\mathcal{A}(Tn)$ the walk never backtracks a distance larger than $\nu(Tn)$, so, the time spent in the second part of the decomposition is negligible. The fact that in $\mathcal{A}(Tn)$ the walk never backtracks a distance larger than $\nu(Tn)$ also implies that, conditional on $\mathcal{A}(Tn)$, the first and the third parts of the decomposition of the trajectory corresponds to independent walks in independent environments.

So $\mathbb{E}[\exp(\sum_{i=1}^{K} -\beta_i(H^{(n)}(u_i) - H^{(n)}(u_{i-1})))]$ can be expressed as

$$\mathbb{E}\left[\exp \sum_{i=1}^{K-1} -\beta_i(H^{(n)}(u_i) - H^{(n)}(u_{i-1}))\right] \mathbb{E}^{\lfloor nu_{K-1}\rfloor}\left[\exp -\beta_K(H^{(n)}(u_i) - H^{(n)}(u_{i-1}))\right] + o(1)$$

where $o(1)$ is taking account of the time spent in the second part of the decomposition of the trajectory and of $\mathcal{A}(Tn)^c$.

The strong Markov property of $Y^{\epsilon(n)}$ applied at the stopping time $\zeta_n(\lfloor nu_{K-1}\rfloor)$ and translational invariance of the environment give that $H^n_{b^*}(nu_i) - H^n_{b^*}(nu_{i-1})$ is distributed as $H^n_{b^*}(ns_n(K))$ where $s_n(K) = \frac{\lfloor u_K n\rfloor - \lceil u_{K-1} n\rceil}{n}$. Iterating this procedure $K-2$ times we reduce the problem to the computation of one-dimensional Laplace transforms. Hence, we have to prove that, for each $k \leq K$

$$\mathbb{E}[\exp(-\beta_k n^{-(1/\alpha)-a} H^n_{b^*}(ns_n(k)))] \to \exp\left(-\frac{\pi\alpha}{\sin(\pi\alpha)}\beta_k^\alpha (u_k - u_{k-1})\right).$$

We have that $\mathbb{P}[\mathcal{E}(Tn) \cap A(Tn)] \to 1$, then we can write

$$\mathbb{E}[\exp(-\beta_k n^{-(1/\alpha)-a} H^n_{b^*}(ns_n(k)))] = \mathbb{E}[\exp(-\beta_k n^{-(1/\alpha)-a} H^n_{b^*}(ns_n(k)))1_{\{\mathcal{E}(Tn)\cup A(Tn)\}}] + o(1).$$

We know that the time spent in the shallow traps is negligible, so we only have to take into account the deep traps. We also know that on $A(Tn)$, the walk does not backtrack more than $\nu(Tn)$, and that, on $\mathcal{E}(Tn)$, the deep traps on $[0, Tn]$ are well separated. Then we can write

$$\mathbb{E}[\exp(-\beta_k n^{-(1/\alpha)-a} H^n_{b^*}(ns_n(k)))] = \mathbb{E}\left[\prod_{j=1}^{\theta(ns_n(k))} \mathbb{E}^{\delta_i}_\tau[\exp\left(-\beta_k n^{-(1/\alpha)-a} T_{\delta_i}\right)]\right] + o(1).$$

Also, in $\mathcal{E}(Tn)$ we have upper and lower bounds for $\theta(Tn)$. Using the upper bound we see that the righthand side of the above equality is bounded above by

$$\mathbb{E}\left[\prod_{j=1}^{ns_n(k)\varphi(ns_n(k))(1-\frac{1}{\log(ns_n(k))})} \mathbb{E}^{\delta_i}_\tau[\exp\left(-\beta_k n^{-(1/\alpha)-a} T_{\delta_i}\right)]\right] + o(1),$$

Applying again the translational invariance of the environment and the strong Markov property we get that that the above display is equal to

$$\mathbb{E}[\mathbb{E}^{\delta_i}_\tau[\exp\left(-\beta_k n^{-(1/\alpha)-a} T_{\delta_i}\right)]]^{ns_n(k)\varphi(ns_n(k))(1-\frac{1}{\log(ns_n(k))})} + o(1)$$

which in turn can be expressed as

$$\mathbb{E}[\exp\left(-\beta_k n^{-(1/\alpha)-a} T_0\right)|\tau_0 \geq g(ns_n(k))]^{ns_n(k)\varphi(ns_n(k))(1-\frac{1}{\log(ns_n(k))})} + o(1).$$

Using lemma (2.6) and the fact that $s_n(k) \xrightarrow{n} u_k - u_{k-1}$ we obtain

$$\limsup \mathbb{E}[\exp(-\beta_k n^{-(1/\alpha)-a} H^n_{b^*}(ns_n(k)))] \leq \exp\left(-\frac{\alpha\pi\beta_k^{-\alpha}}{\sin(\alpha\pi)}(u_k - u_{k-1})\right).$$

The lower bound can be obtained in an analogous fashion. For the tightness, the arguments are the same as in Chapter 5 of [**BBČ08**]

4. The Critical Case

We want to show that for $a = \frac{\alpha}{\alpha+1}$ the sequence of walks $(X^{(n,a)}(t); t \in [0,\infty])$ converges in distribution to a drifted FIN diffusion. We will mimic the arguments in [**FIN02**]. But to treat the asymmetry of the model we will use a Brownian motion with drift instead of a Brownian motion. We use the existence of a bi-continuous version of the local time for a Brownian motion with drift.

4.1. The construction of the walks. Recall the definition of $\tilde{X}(\mu)$ given in display (2.4). Let s be a real number and define

$$\mu := \sum_{i \in (Z)} v_i \delta_{si}.$$

Then $\tilde{X}(\mu)$ is a homogeneus Markov process with $s\mathbb{Z}$ as its state space. The transition probabilities and jump rates of $\tilde{X}(\mu)$ can be computed from the positions and weights of the atoms using the generator L of $C(t)$

$$Lf := \frac{1}{2}\frac{d^2 f}{dx^2} + \frac{df}{dx}. \tag{4.1}$$

The arguments we will give below are an adaptation of the reasoning used by Stone in [**Sto63**]. For each i let η_{si} be the time of the first jump of $\tilde{X}(\mu)$ started at si. By construction we will have that $\eta_{si} = v_i \tilde{l}(\sigma_s, 0)$, where σ_s is the hitting time of $(-s, s)$ by $C(t)$. Using the strong Markov property for $C(t)$ we can deduce that η_{si} is exponentially distributed. It is easy to see that its mean is $v_i \mathbb{E}[\tilde{l}(\sigma_s, 0)]$. Denote by $p_t(x)$ the density at site x of the distribution of $C(t)$ absorbed at $\{-s, s\}$. Using that $\tilde{l}(\sigma_s, 0) := \epsilon^{-1} \lim_{\epsilon \to 0} m(t \in [0, \sigma_s] : C(t) \in [-\epsilon, \epsilon])$ and applying Fubini‘s Theorem we find that $\mathbb{E}[\tilde{l}(\sigma_s, 0)] = \epsilon^{-1} \lim_{\epsilon \to 0} \int_0^{\sigma_s} \mathbb{P}[C(t) \in [-\epsilon, \epsilon]] dt$. Then we find that

$$\mathbb{E}[\tilde{l}(\sigma_s, 0)] = \int_0^\infty p_t(0) dt.$$

We also know that $\int_0^\infty p_t(0) dt = f(0)$, where f is the Green function of (4.1) with Dirichlet conditions on $\{-s, s\}$. That is, f is the continuous function that satisfies

$$\frac{1}{2}\frac{d^2 f}{dx^2} + \frac{df}{dx} = -\delta_0 \text{ and } f(s) = f(-s) = 0.$$

We know that the general solution to $\frac{1}{2}\frac{d^2 g}{dx^2} + \frac{dg}{dx} = 0$ is $g = C_1 \exp(-2x) + C_2$. This and the constraints on f give that

$$\mathbb{E}[\eta_{si}] = v_i^{-1} \frac{\exp(-2s) + 1}{1 - \exp(-2s)}. \tag{4.2}$$

For the computation of the respective transition probabilities we can use again the generator L. Let $g : [-s, s] \to \mathbb{R}$ be a continuous function such that $\frac{1}{2}\frac{d^2 g}{dx^2} + \frac{dg}{dx} = 0$ and $g(-s) = 0, g(s) = 1$. Using Itō's formula, we find that that $g(C(t))$ is a martingale. By the optional stopping theorem with the stopping time σ_s we find that the probability that the walk takes his first step to the right is $g(0)$. We can use the constraints on g to see that

$$\mathbb{P}[\tilde{X}(\mu)(\eta_{si}) = s(i+1)] = \frac{\exp(2s)}{1+\exp(2s)}. \tag{4.3}$$

The proof of part (ii) of Theorem (3.1) will rely strongly on the following proposition.

PROPOSITION 2.7. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures that converges vaguely to ν, a measure whose support is $\mathbb{R}$. Then the corresponding processes $(\tilde{X}(\nu_n)(t), 0 \le t \le T)$ converges to $(\tilde{X}(\nu)(t), 0 \le t \le T)$ in distribution in $(D[0,T], U)$.*

For the case where the underlying process is a Brownian motion, the proof of this fact can be found in [**Sto63**]. We will use the continuity properties for the local time $\tilde{l}$. For each fixed t, $\tilde{l}$ is continuous and of compact support in x. Then, the vague convergence of ν_n to ν implies the almost sure convergence of $\tilde{\phi}_{\nu_n}(t)$ to $\tilde{\phi}_\nu(t)$. As $\tilde{l}$ is continuous in t, we obtain continuity of $\tilde{\phi}_{\nu_n}$ and of $\tilde{\phi}_\nu$. That, plus the fact that the $\tilde{\phi}_{\nu_n}$ are non-decreasing implies that that $\tilde{\phi}_{\nu_n}$ converges uniformly to $\tilde{\phi}_\nu$. The function $\tilde{\phi}_\nu$ is almost surely strictly increasing, because the support of ν is $\mathbb{R}$. Now we can apply corollary 13.6.4 of [**Whi02**] to obtain that $\tilde{\psi}_{\nu_n}$ converges uniformly to $\tilde{\psi}_\nu$. That plus the continuity of the Brownian paths yields the lemma.

4.2. The coupled walks. To prove part (ii) of Theorem 3.1, we will use Proposition 2.7. That is we want to show that each walk $(X^{(n,a)}(t); t \in [0,\infty])$ can be expressed as a speed measure change of $C(t)$, and then use convergence of the measures to get convergence of the processes. The problem is that we are dealing with a sequence of random measures, and the proposition deals only with deterministic measures. To overcome this obstacle we can construct a coupled sequence of random measures $(\rho_n)_{n\in\mathbb{N}}$, such that $(\tilde{X}(\rho_n)(t); t \in [0,\infty])$ is distributed as $(X^{(n,a)}(t); t \in [0,\infty])$ and that $(\rho_n)_{n\in\mathbb{N}}$ converges almost surely vaguely to ρ, where ρ is the random measure defined in (1.6) such that $\tilde{Z} = \tilde{X}[\rho]$. This section is devoted to the construction of the coupled measures.

We recall that V_α is an α-stable subordinator. To make the construction clearer, we will first suppose that τ_0 is equidistributed with the positive α-stable distribution $V_\alpha(1)$. Let us consider the strictly increasing process $(\tilde{V}_\rho(t); t \in \mathbb{R})$ given by $\tilde{V}_\rho(t) := \rho[0,t]$ if $t \ge 0$ and $\tilde{V}_\rho(t) := -\rho[t,0)$ if $t < 0$. It is a known fact from the theory of Levy processes that $\tilde{V}_\rho(t)$ is a two sided α-stable subordinator. We now use this process to construct the coupled sequence of random measures $(\rho_n)_{n\in\mathbb{N}}$ as

$$\rho_n := \sum_i n^{-1/(1+\alpha)} \tau_i^n \delta_{s_n i},$$

where $s_n := \frac{1}{2}\log\frac{n^{-a}+1}{1-n^{-a}}$ and

$$\tau_i^n := n^{1/(1+\alpha)}(\tilde{V}_\rho(n^{-\alpha/(1+\alpha)}(i+1)) - \tilde{V}_\rho(n^{-\alpha/(1+\alpha)} i)). \tag{4.4}$$

Observe that $(\tau_i^n)_{i\in\mathbb{Z}}$ is an i.i.d. sequence distributed like τ_0, so that using (4.3) and (4.2) we see that $\tilde{X}(\rho_n)$ is a walk with drift $n^{-1/\alpha}$ taking values in $s_n\mathbb{Z}$. The latter means that $\tilde{X}(\rho_n)$ is distributed like $s_n n^{\frac{\alpha}{1+\alpha}} X^{(n,a)}$. The key observation here is that the scaling factor s_n satisfies

$$s_n n^{\alpha/(1+\alpha)} \to 1 \text{ as } n \to \infty. \tag{4.5}$$

So, we just have to show that $\tilde{X}(\rho_n)$ converges to $\tilde{X}(\rho)$, because (4.5) implies that if $\tilde{X}(\rho_n)$ converges to $\tilde{X}(\rho)$, so $s_n n^{\alpha/(1+\alpha)}\tilde{X}(\rho_n)$ does. With (4.5) in mind it is easy to prove that the sequence of measures (ρ_n) converges almost surely vaguely to ρ. Suppose that $a < b$ are real numbers and that V_ρ is continuous at a and b, then

$$\rho_n((a,b]) = V_\rho(n^{-\alpha/(1+\alpha)}\lfloor a/s_n\rfloor) - V_\rho(n^{-\alpha/(1+\alpha)}(\lfloor b/s_n\rfloor + 1)).$$

But using (4.5) it is clear that $n^{-\alpha/(1+\alpha)}\lfloor a/s_n\rfloor \overset{n\to\infty}{\to} a$ and $n^{-\alpha/(1+\alpha)}\lfloor b/s_n\rfloor \overset{n\to\infty}{\to} b$. Then the continuity of V_ρ at a and b implies that $\rho_n((a,b]) \overset{n\to\infty}{\to} \rho(a,b]$, and we have proves the vague convergence of ρ_n to ρ.

Suppose now that τ_0 is not a positive α-stable random variable. Then, we can follow Section 3 of [**FIN02**]. There they construct constants c_ϵ and functions g_ϵ such that $\tau_i^{(\epsilon)}$ is distributed like τ_0, where

$$\tau_i^{(\epsilon)} := c_\epsilon^{-1} g_\epsilon(\tilde{V}_\rho(\epsilon(i+1)) - \tilde{V}_\rho(\epsilon i)). \tag{4.6}$$

Lemma 3.1 of [**FIN02**] says that

$$g_\epsilon(y) \to y \text{ as } \epsilon \to 0. \tag{4.7}$$

As τ_0 satisfies (5.1) and using the construction of c_ϵ in Section 3 of [**FIN02**], we can deduce that

$$c_\epsilon \sim \epsilon^{1/\alpha} \tag{4.8}$$

Define

$$\tau_i^n := \tau_i^{(n^{(-\alpha/(1+\alpha))})} \tag{4.9}$$

and again

$$\rho_n := \sum_i n^{-1/(1+\alpha)} \tau_i^n \delta_{s_n i}.$$

Then, by definition (4.6), $\tilde{X}(\rho_n)$ is a walk with drift $n^{-1/\alpha}$ taking values in $s_n\mathbb{Z}$. Using (4.7), (4.8) and (4.5) we can see that $\mathbb{P}$-a.s. $\rho_n \to \rho$ vaguely.

5. The Subcritical Regime

We will prove that if $a > \alpha/(1+\alpha)$, then $(X^{(n,a)}; t \in [0,\infty])$ converges to a FIN diffusion. We obtain the same scaling limit that was obtained in [**FIN02**] for a symmetric BTM Nevertheless, here we have to deal with walks which are not symmetric, in contrast with the situation of [**FIN02**]. For this purpose we express each rescaled walk as a time scale change of a Brownian motion. The scale change is necessary to treat the asymmetry of the walk. Then we show that the scale change can be neglected. We now proceed to define a time scale change of a Brownian motion. Let μ be a locally finite discrete measure

$$\mu(dx) := \sum_{i\in\mathbb{Z}} w_i \delta_{y_i}(dx),$$

where $(y_i)_{i\in\mathbb{Z}}$ is an ordered sequence of real numbers so that $y_i < y_j$ i.i.f. $i < j$.

Let $S : \mathbb{R} \to \mathbb{R} \cup \{\infty, -\infty\}$ be a real valued, strictly increasing function, μ will be the speed-measure and S the scaling function of the time scale change of Brownian motion. Define the scaled measure $(S \circ \mu)(dx)$ as

$$(S \circ \mu)(dx) := \sum_i w_i \delta_{S(y_i)}(dx).$$

Let

$$\phi(\mu, S)(t) := \int_{\mathbb{R}} l(t, y)(S \circ \mu)(dy)$$

and $\psi(\mu, S)(s)$ be the right continuous generalized inverse of $\psi(\mu, S)$. Then, as shown in [**Sto63**]

$$X(\mu, S)(t) := S^{-1}(X(S \circ \mu)(t)))$$

is a continuous time random walk with $\{y_i\}$ as its state space. The mean of the exponentially distributed waiting time of $X(S \circ \mu)$ on y_i is

$$2w_i \frac{(S(y_{i+1}) - S(y_i))(S(y_i) - S(y_{i-1}))}{S(y_{i+1}) - S(y_{i-1})} \tag{5.1}$$

and the transition probabilities to the right and to the left respectively are

$$\frac{S(y_{i+1}) - S(y_i)}{S(y_{i+1}) - S(y_{i-1})} \text{ and } \frac{S(y_i) - S(y_{i-1})}{S(y_{i+1}) - S(y_{i-1})}. \tag{5.2}$$

As in the previous section, we need to define a sequence of measures $(\nu_n)_{n\in\mathbb{Z}}$ converging almost surely vaguely to ρ, and which can be used to express the sequence of rescaled walks $X^{(n,a)}$.

Let

$$\nu_n := \sum_{i\in\mathbb{Z}} \frac{r_n + 1}{2r_n^i} \tau_i^n \delta_{in^{\alpha/(\alpha+1)}},$$

where τ_i^n are defined in display (4.9), and $r_n := 1 - \frac{2n^{-a}}{1+n^{-a}}$. We will also use a sequence of scaling functions S^n (which will converge to the identity mapping) given by

$$S^n(in^{\alpha/(\alpha+1)}) := \sum_{j=0}^{i-1} \frac{r^j}{n^{\alpha/(\alpha+1)}}.$$

We extend the domain of definition of S^n to $\mathbb{R}$ by linear interpolation. Then, by (5.1) and (5.2), we have that $X(\nu_n, S^n)$ is distributed like $X^{(n,a)}$. We will use the following theorem proved by Stone in [**Sto63**].

PROPOSITION 2.8. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures that converges vaguely to ν. Then the corresponding processes $(X(\nu_n)(t), 0 \leq t \leq T)$ converges to $(X(\nu)(t), 0 \leq t \leq T)$ in distribution in $(D[0,T], J_1)$*

The proof of part (iii) of theorem 3.1 will rely in the following lemma. Let *id* denote the identity mapping on $\mathbb{R}$, then we have that

LEMMA 2.9. *$S^n(n^{-\alpha/(1+\alpha)}\lfloor n^{\alpha/(\alpha+1)}\cdot\rfloor)$ converges uniformly on compacts to id and ν_n to converges almost surely vaguely to ρ.*

PROOF. The convergence of the scaling functions is easily seen to be true under the assumption $a > \alpha/(\alpha+1)$ because

$$S^n(n^{-\alpha/(1+\alpha)}\lfloor n^{\alpha/(1+\alpha)}x\rfloor) = \sum_{j=0}^{\lfloor n^{\alpha/(1+\alpha)}x\rfloor} \frac{r_n^j}{n^{\alpha/(\alpha+1)}}$$

and

$$\frac{r_n^{\lfloor n^{\alpha/(1+\alpha)}x\rfloor}\lfloor n^{\alpha/(\alpha+1)}x\rfloor}{n^{\alpha/(1+\alpha)}} \leq \sum_{j=0}^{\lfloor n^{\alpha/(1+\alpha)}x\rfloor} \frac{r_n^j}{n^{\alpha/(\alpha+1)}} \leq \frac{\lfloor n^{\alpha/(\alpha+1)}x\rfloor}{n^{\alpha/(1+\alpha)}}.$$

Now we use the fact that

$$r_n^{\lfloor n^{\alpha/(1+\alpha)}x\rfloor} = \left(1 - \frac{2n^{-a}}{1+n^{-a}}\right)^{\lfloor n^{\alpha/(1+\alpha)}x\rfloor}$$

converges to 1, because $a > \alpha/(1+\alpha)$.

In a similar fashion it can be shown that the "correcting factors" $\frac{r_n+1}{2r_n^i}$ in the definition of ν_n converge uniformly to 1 in any bounded interval. Hence, we can show the convergence of ν_n to ρ as in the previous section.

□

Lemma 2.9 implies the vague convergence of $(S^n \circ \nu_n)$ to ρ. Then, by proposition 2.8 we can deduce that $X(S^n \circ \nu_n)$ converges to $X(\rho)$. Let $T > 0$, by lemma 2.9 we have that S^{-1} also converges uniformly to the identity. Thus, using the precedent observations, we get that $(X(\mu, S)(t) : 0 \leq t \leq T)$ converges to $(X(\rho)(t) 0 \geq t \geq T)$ in $D[0,T]$ with the Skorohod $J-1$ topology. We have proved that $(X^{(n,a)}(t); t \in [0;T])$ converges in distribution to the FIN diffusion $(Z(t); t \in [0,t])$ on $(D[0,T], J_1)$.

Thus, it remains to prove that the convergence takes place also in the uniform topology. Using the fact that the support of ρ is $\mathbb{R}$, we can show that $\phi(\rho, id)$ is strictly increasing. The almost sure vague convergence of $S \circ \nu_n$ to ρ implies that, for all $t \geq 0$, $\phi(\nu_n, S^n)(t)$ converges to $\phi(\rho, id)(t)$. As l is continuous in t, we obtain continuity of $\phi(\nu_n, S^n)$ and of $\phi(\rho, id)$. That, plus the fact that the $\phi(\nu_n, id)$ are non-decreasing implies that that $\phi(\nu_n, S^n)$ converges uniformly to $\phi(\rho, id)$. The function $\phi(\rho, id)$ is almost surely strictly increasing, because the support of ρ is $\mathbb{R}$. Now we can apply corollary 13.6.4

of [**Whi02**] to obtain that $\psi(\nu_n, S^n)$ converges uniformly to $\psi(\rho, id)$. That, plus the continuity of the Brownian paths yields that $X(S^n \circ \nu_n)$ converges uniformly to $X(\rho, id)$. Using that S^{n-1} converges to the identity, we finally get that $X(\nu_n, S^n)$ converges uniformly to $X(\rho)$.

CHAPTER 3

Sub-Gaussian bound for the one-dimensional BTM

1. Introduction

The *Bouchaud trap model* (BTM) was introduced by J.-P. Bouchaud in [**Bou92**] as a toy model for the analysis of the dynamics of some complex disordered systems such as spin glasses. This model is a great simplification of the actual dynamics of such models, nevertheless, it presents some interesting properties which had been observed in the real physical systems. For an account of the physical literature on the BTM we refer to [**BCKM98**].

A basic question is to describe the behavior in space and time of the annealed transition kernel of the one-dimensional, symmetric version of the BTM. In this chapter we establish a sub-Gaussian bound on the annealed transition kernel of the model which provides a positive answer to the behavior conjectured by E.M. Bertin and J.-P. Bouchaud in [**BB03**]. That article contains numerical simulations and non-rigorous arguments which support their claim. A first step on establishing the conjecture was given by J. Černý in [**Čer06**] where he proved the upper side of the sub-Gaussian bound. In this chapter we provide the proof for the corresponding lower bound.

The one-dimensional, symmetric BTM is a continuous time random walk $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ with a random environment. Let $(\tau_z)_{z\in\mathbb{Z}}$ be a family of i.i.d., non-negative random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Those random variables will stand for the environment. For $(\tau_x)_{x\in\mathbb{Z}}$ fixed, we define X as a homogeneous Markov process with jump rates:

$$c(x,y) := \begin{cases} (2\tau_x)^{-1} \text{ if } |x-y| = 1, \\ 0 \text{ otherwise.} \end{cases} \tag{1.1}$$

That is, when X is on site x, it waits an exponentially distributed time with mean τ_x until it jumps to one of its neighbors with equal probabilities. Thus τ_x should be regarded as the depth of the trap at x. The trapping mechanism becomes relevant in the large time behavior of the model only if $\mathbb{E}(\tau_0) = \infty$, i.e., when the environment is heavy tailed. Thus, we assume that

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}(\tau_x \geq u) = 1 \tag{1.2}$$

for some $\alpha \in (0,1)$. Under this assumption the BTM presents a sub-diffusive behavior.

Having defined precisely the one-dimensional, symmetric BTM, we can proceed to state the main result obtained in this chapter.

THEOREM 3.1. *There exists positive constants C_1, c_1, C_2, c_2 and ϵ_1 such that*

$$C_1 \exp\left(-c_1\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right) \leq \mathbb{P}(|X_t| \geq x) \leq C_2 \exp\left(-c_2\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$ such that $x/t \leq \epsilon_1$.

As we have previously stated, the lower bound in theorem 3.1 has been already obtained in [**Čer06**].

OBSERVATION 3.2. *We can take $x = at$ with $0 \leq a \leq \epsilon_1$ on theorem 3.1. Then we obtain exponential upper and lower bounds on $\mathbb{P}(|X_t| \geq at)$. This indicates that a large deviation principle for the BTM might hold.*

The proof that we will present for the corresponding lower bound relies heavily on the fact that $(X_t)_{t\geq 0}$ has a clearly identified scaling limit. This scaling limit is called the *Fontes, Isopi, Newman singular diffusion* (FIN). It was discovered by Fontes, Isopi and Newman in [**FIN02**] and it is a singular diffusion on a random environment. More accurately, this diffusion is a speed measure change of a Brownian motion through a random, purely atomic measure ρ, where ρ is the Stieltjes measure associated to an α-stable subordinator.

To define the FIN diffusion, first we recall the definition of a speed measure changed Brownian motion. Let B_t be a standard one dimensional Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$ and starting at zero. Let $l(t,x)$ be a bi-continuous version of its local time. Given any locally finite measure μ on $\mathbb{R}$, denote

$$\phi_\mu(s) := \int_{\mathbb{R}} l(s,y)\mu(dy),$$

and its right continuous generalized inverse by

$$\psi_\mu(t) := inf\{s > 0 : \phi_\mu(s) > t\}.$$

Then we define the **speed measure change of B with speed measure** μ, $(B[\mu]_t)_{t\geq 0}$ as

$$B[\mu]_t := B_{\psi_\mu(t)}. \tag{1.3}$$

Now, we proceed to define the random measure appearing on the definition of the FIN diffusion. Let $(V_x)_{x\in\mathbb{R}}$ be a two sided, α-stable subordinator with cadlag paths defined over $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of B. That is, V_x is the non-decreasing Levy process with cadlag paths and satisfying

$$V_0 = 0 \tag{1.4}$$

$$\begin{aligned}\mathbb{E}(\exp(-\lambda(V_{x+y} - V_x))) &= \exp(\alpha y \int_0^\infty (e^{-\lambda w} - 1) w^{-1-\alpha} dw) \\ &= \exp(-y\lambda^\alpha \Gamma(1-\alpha))\end{aligned} \tag{1.5}$$

for all $x, y \in \mathbb{R}$ and $\lambda > 0$.

Let ρ be the Lebesgue-Stieltjes measure associated to V, that is $\rho(a,b] = V_b - V_a$. It is a known fact that $(V_t)_{t\geq 0}$ is a pure-jump process. Thus we can write

$$\rho := \sum v_i \delta_{x_i}. \tag{1.6}$$

Moreover, it is also known that $(x_i, v_i)_{i\in\mathbb{N}}$ is an inhomogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $\alpha v^{-1-\alpha} dx dv$. The diffusion $(Z_t)_{t\geq 0}$ defined as $Z_s := B[\rho]_s$ is the **FIN diffusion**.

OBSERVATION 3.3. *It is easy to see that the measure ρ has scaling invariance in the sense that $\lambda^{-1/\alpha}\rho(0,\lambda)$ is distributed as $\rho(0,1)$ for all $\lambda > 0$. The Brownian motion B is scale invariant in the sense that $(\lambda^{-1/2}B_{\lambda t})_{t\geq 0}$ is distributed as $(B_t)_{t\geq 0}$. Those two facts imply that Z is scale invariant in the sense that $(\lambda^{-(\alpha+1)/\alpha}Z_{\lambda t})_{t\geq 0}$ is distributed as $(Z_t)_{t\geq 0}$ for all $\lambda > 0$. This fact reflects that the FIN diffusion is subdiffusive.*

The techniques used to prove theorem 3.1 can also be applied to obtain the corresponding result for the FIN singular diffusion, i.e., we will prove

THEOREM 3.4. *There exists positive constants C_3, c_3, C_4 and c_4 such that*

$$C_3 \exp\left(-c_3\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right) \leq \mathbb{P}(|Z_t| \geq x) \leq C_4 \exp\left(-c_4\left(\frac{x}{t^{\frac{\alpha}{1+\alpha}}}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$.

Again, the upper bound of theorem 3.4 was obtained by J. Černý in [**Čer06**].

The main difficulty to obtain the lower bound in theorem 3.4 is to be able to profit from the independence between the Brownian motion B and the random measure ρ which appear in the construction of the FIN diffusion Z. As we will see, the Ray-Knight description of the local time of B allow us to overcome that difficulty. The proof of theorem 3.1 follows the same line of reasoning than the proof of theorem 3.4, but the technical details are slightly more complicated.

We would like to point out some results concerning the BTM on other graphs as state space. In higher dimensions (when the state space is $\mathbb{Z}^d, d \geq 2$), the symmetric BTM has a behavior completely different from the one-dimensional case, as shown by Ben Arous and Cěrný in [**BČ07**], and by Ben Arous, Cěrný and Mountford in [**BČM06**]. In these papers it is shown that the scaling limit of that model is the *fractional kinetic process* (FK), which is a time-change of a d-dimensional Brownian motion through the inverse of an α-stable subordinator. In [**BBG03a**] and [**BBG03b**] Ben Arous, Bovier and Gayrard obtained *aging* properties of the model on the complete graph. A study of this walk for a wider class of graphs can be found on [**BČ08**]. For a general account on the mathematical study of the Bouchaud trap model and the FIN diffusion, we refer to [**BČ06**].

2. Proofs of the main theorems

We will first present the proof of theorem 3.4. Then we adapt the techniques used on that proof to obtain theorem 3.1.

2.1. Proof of theorem 3.4. We begin by stating the Ray-Knight theorem. Recall that B is a standard one-dimensional Brownian motion started at the origin and $l(t,x)$ is its local time. For any $b \in \mathbb{R}$ let $\tau_b := \inf\{t \geq 0 : B_t = b\}$. The Ray-Knight theorem ([**Ray63**] and [**Kni63**]) states that

THEOREM 3.5. *(Ray-Knight) For each $a > 0$, the stochastic process $(l(\tau_{-a}, t) : t \geq -a)$ is Markovian. Moreover*

$$(l(\tau_{-a}, t) : -a \leq t \leq 0) \tag{2.1}$$

is distributed as a squared Bessel process of dimension $d = 2$ started at 0 . Further

$$(l(\tau_{-a}, t) : t \geq 0) \tag{2.2}$$

is distributed as a squared Bessel process of dimension $d = 0$ started at $l(\tau_{-a}, 0)$ and killed at 0.

Thanks to the scaling invariance of Z, to prove the lower bound in theorem 3.4 it is enough to show that there exists positive constants C_4 and c_4 such that

$$\mathbb{P}(|Z_1| \geq x) \geq C_4 \exp(-c_4 x^{1+\alpha}) \text{ for all } x \geq 0. \tag{2.3}$$

Both $\mathbb{P}(|Z_1| \geq x)$ and $C_4 \exp(-c_4 x^{1+\alpha})$ are decreasing in x. Hence it will suffice to show that there exists positive constants C_5 and c_5 such that

$$\mathbb{P}(|Z_1| \geq n^{1/(1+\alpha)}) \geq C_5 \exp(-c_5 n) \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

For any $b \in \mathbb{R}$, we define $H_b := \inf\{t \geq 0 : Z_t = b\}$. Let $n \in \mathbb{N}$ be fixed, we define

$$G := \{Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}\} \tag{2.5}$$

$$G_1 := \{H_{n^{1/(1+\alpha)}(n+1)/n} \leq (n+2)/n\} \tag{2.6}$$

$$G_2 := \{H_{-2n^{1/(1+\alpha)}} - H_{-n^{1/(1+\alpha)}(n+1)/n} \geq (n+2)/n\} \tag{2.7}$$

$$G_3 := \{Z_t \leq -n^{1/(1+\alpha)} \text{ for all } t \in [H_{-n^{1/(1+\alpha)}(n+1)/n}, H_{-2n^{1/(1+\alpha)}}]\}. \tag{2.8}$$

Note that $G \subset G_1 \cap G_2 \cap G_3$. We will establish a sub-Gaussian lower bound for $\mathbb{P}(G)$. Then it will be easy to deduce display (2.4) (and hence theorem 3.4).

We will start by controlling the probability of G_1. Notice that, due to the fact that the set of atoms of ρ is $\mathbb{P}$-a.s. dense, the event $\{H_{-n^{1/(1+\alpha)}(n+1)/n} \leq 1\}$ is equivalent to $\{\min\{Z_t : t \in [0,1]\} \leq -n^{1/(1+\alpha)}(n+1)/n\}$. Let $\theta_1 := \inf\{t > 0 : B_t = -n^{1/(1+\alpha)}(n+1)/n\}$. We can express $H_{-n^{1/(1+\alpha)}(n+1)/n}$

as $\int_{-n^{1/(1+\alpha)}(n+1)/n}^{\infty} l(\theta_1,u)\rho(du)$. Let $H^- := \int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} l(\theta_1,u)\rho(du)$ and $H^+ := \int_0^\infty l(\theta_1,u)\rho(du)$. Thus

$$H^- := \sum_{i=1}^{n+1} \int_{-n^{1/(1+\alpha)}i/n}^{-n^{1/(1+\alpha)}(i-1)/n} l(\theta_1,u)\rho(du). \tag{2.9}$$

Clearly

$$\bigcap_{i=1}^{n+1} \left\{ \int_{-n^{1/(1+\alpha)}i/n}^{-n^{1/(1+\alpha)}(i-1)/n} l(\theta_1,u)\rho(du) \le 1/n \right\} \subset \{H^- \le (n+1)/n\}. \tag{2.10}$$

As the intervals $[-n^{1/(1+\alpha)}i/n, -n^{1/(1+\alpha)}(i-1)/n)$, $i = 1,..,n+1$ are disjoint and the process V has independent increments, we have that the random variables $\rho[-n^{1/(1+\alpha)}i/n, -n^{1/(1+\alpha)}(i-1)/n)$, $i = 1,..,n+1$ are independent between them. Also, the Ray-Knight theorem states that $(l(\theta_1,u) : u \ge -n^{1/(1+\alpha)}(n+1)/n)$ is a process with independent increments. We will profit of those independencies by finding a family of $n+1$ independent events with the same probability, whose intersection is contained in $\{H^- \le (n+1)/n\}$. Then, using the scaling invariance of the measure ρ and the scaling invariance of squared Bessel processes, we will show that all those events have the same probability for all n. Then we would have showed that there exists positive constants C_6 and c_6 such that

$$\mathbb{P}(H^- \le (n+1)/n) \ge C_6 \exp^{-c_6(n+1)}. \tag{2.11}$$

An similar argument can be used to control the probability of $\{H^+ \le 1/n\}$. Hence we will obtain

$$\mathbb{P}(H_{-n^{1/(1+\alpha)}(n+1)/n} \le (n+2)/n) \ge C_7 \exp^{-c_6(n+2)} \tag{2.12}$$

where C_7 is a positive constant. As $(n+2)/n \le 3$, we can use scaling invariance of Z to obtain that there exists positive constants C_8 and c_8 such that

$$\mathbb{P}(H_{-n^{1/(1+\alpha)}(n+1)/n} \le 1) \ge C_8 \exp^{-c_8 n}. \tag{2.13}$$

To obtain theorem 3.4 from (2.13) is not immediate, because the event $\{H_{-n^{1/(1+\alpha)}(n+1)/n} \le (n+2)/n\}$ is not independent of ρ. To overcome that obstacle we can make repeated use of the Ray-Knight theorem using the stopping times θ_1 and $\theta_2 := \inf\{t > 0 : B_t = -2n^{1/(1+\alpha)}\}$. That will allow us to control simultaneously the probabilities of G_1, G_1 and G_3.

Next, we give some definitions needed for the proof of theorem 3.4. Let $(W_t)_{t\ge0}$ be a Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$, independent of ρ and started at 0. Let $(\bar{\mathcal{Y}}_t : t \in [-n^{1/(1+\alpha)}(n+1)/n, \infty)$ be the Bessel process with $d = 2$ given by

$$\bar{\mathcal{Y}}_t := \int_{-n^{1/(1+\alpha)}(n+1)/n}^{t} \frac{1}{2\bar{\mathcal{Y}}_s} ds + W_t - W_{-n^{1/(1+\alpha)}(n+1)/n}. \tag{2.14}$$

Let $a > 0$. We also define $(\bar{\mathcal{Y}}^i_t : t \in [-n^{1/(1+\alpha)}(n-i)/n, \infty))$, $i = -1, 0, .., n-1$ as the Bessel processes with $d = 2$ given by

$$\bar{\mathcal{Y}}^i_t := (an^{-\alpha/(1+\alpha)})^{1/2} + \int_{-n^{1/(1+\alpha)}(n-i)/n}^{t} \frac{1}{2\bar{\mathcal{Y}}^i_s} ds + W_t - W_{-n^{1/(1+\alpha)}(n-i)/n}. \tag{2.15}$$

Note that we are using the same Brownian motion for the construction of the $\bar{\mathcal{Y}}^i$, $i=-1,0,..,n-1$ and $\bar{\mathcal{Y}}$. Let $(\bar{\mathcal{X}}_t)_{t\geq 0}$ be a Bessel process with $d=0$ given by

$$\bar{\mathcal{X}}_t := \bar{\mathcal{Y}}_0 - \int_0^t \frac{1}{2\bar{\mathcal{X}}_s} ds + W_t. \tag{2.16}$$

We also define

$$\bar{\mathcal{X}}^0_t := (an^{-\alpha/(1+\alpha)})^{1/2} - \int_0^t \frac{1}{2\bar{\mathcal{X}^0}_s} ds + W_t. \tag{2.17}$$

We aim to use the Ray-Knight theorem, which deals with squared Bessel processes. Thus we define $Y_t := (\bar{\mathcal{Y}}_t)^2; Y^i_t := (\bar{\mathcal{Y}}^i_t)^2; \mathcal{X}_t := (\bar{\mathcal{X}}_t)^2$ and $\mathcal{X}^0_t := (\bar{\mathcal{X}}^0_t)^2$.
We have that

$$G_1 = \left\{ \int_{\mathbb{R}} l(\theta_1, u)\rho(du) \leq (n+2)/n \right\}. \tag{2.18}$$

Thus, in view of the Ray-Knight theorem

$$\mathbb{P}(G_1) = \mathbb{P}\left(\int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} \mathcal{Y}_t\rho(dt) + \int_0^\infty \mathcal{X}_t\rho(dt) \leq (n+2)/n \right).$$

Let

$$A := \left\{ \int_{-n^{1/(1+\alpha)}(n+1)/n}^{0} \mathcal{Y}_t\rho(dt) + \int_0^\infty \mathcal{X}_t\rho(dt) \leq (n+2)/n \right\}. \tag{2.19}$$

For all $i=-1,..,n-1$, let

$$A_i := \left\{ \int_{-n^{1/(1+\alpha)}(n-i)/n}^{-n^{1/(1+\alpha)}(n-i-1)/n} \mathcal{Y}_t\rho(dt) \leq 1/n \right\} \tag{2.20}$$

and

$$B := \left\{ \int_0^\infty \mathcal{X}_t\rho(dt) \leq 1/n \right\}. \tag{2.21}$$

Thus, it is clear that

$$\left(\bigcap_{i=-1}^{n-1} A_i \right) \cap B \subset A. \tag{2.22}$$

We would like to have independence of the events in the L.H.S. of (2.22) to compute a lower bound for $\mathbb{P}(A)$. But they are not independent. Thus, for $i=-1,..,n-1$ we define

$$\tilde{A}_i := \left\{ \int_{-n^{1/(1+\alpha)}(n-i)/n}^{-n^{1/(1+\alpha)}(n-i-1)/n} \mathcal{Y}^i_t\rho(dt) \leq 1/n; \mathcal{Y}^i_{-n^{1/(1+\alpha)}(n-i-1)/n} \leq ax/n \right\} \tag{2.23}$$

and

$$\tilde{B} := \left\{ \int_0^\infty \mathcal{X}^0_t\rho(dt) \leq 1/n \right\}. \tag{2.24}$$

We can use independence on those events. The fact that all the Bessel processes appearing are defined using the same Brownian motion W implies that, conditioned on $\{\bar{\mathcal{Y}}_{-n^{1/(1+\alpha)}(n-i)/n} \leq (an^{-\alpha/(1+\alpha)})^{1/2}\}$, we have that $\bar{\mathcal{Y}}_t \le \bar{\mathcal{Y}}^i_t$ for all $t \in [-n^{1/(1+\alpha)}(n-i)/n, \infty)$. Also, conditioned on $\{\bar{\mathcal{Y}}_0 \le (an^{-\alpha/(1+\alpha)})^{1/2}\}$, we have that $\bar{\mathcal{X}}_t \le \bar{\mathcal{X}}^0_t$ for all $t \ge 0$. Thus, it is clear that

$$\left(\bigcap_{i=-1}^{n-1} \tilde{A}_i \right) \cap \tilde{B} \subset A. \tag{2.25}$$

For any $b \in \mathbb{R}_+$, $\mathbb{P}_b$ will denote probability conditioned on $\mathcal{Y}_0 = b$. When there is no risk of confusion, $\mathbb{P}_b$ will also denote probability conditioned on $\mathcal{X}_0 = b$. Note that, for all $1 = -1, .., n-1$ we have that

$$\mathbb{P}(\tilde{A}_i) = \mathbb{P}_{an^{-\alpha/(1+\alpha)}} \left(\int_0^{n^{-\alpha/(1+\alpha)}} \mathcal{Y}_t \rho(dt) \le 1/n; \mathcal{Y}_{n^{-\alpha/(1+\alpha)}} \le an^{-\alpha/(1+\alpha)} \right). \tag{2.26}$$

Now consider

$$\mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \le 1; \mathcal{Y}_1 \le a \right). \tag{2.27}$$

Let us perform a change of variables inside the integral. We obtain that (2.27) equals

$$\mathbb{P}_a \left(\int_0^{n^{-\alpha/(1+\alpha)}} \mathcal{Y}_{sn^{\alpha/(1+\alpha)}} \rho(n^{-\alpha/(1+\alpha)} ds) \le 1; \mathcal{Y}_1 \le a \right). \tag{2.28}$$

Using the scale invariance of the measure ρ we obtain

$$= \mathbb{P}_a \left(\int_0^{n^{-\alpha/(1+\alpha)}} \mathcal{Y}_{sn^{\alpha/(1+\alpha)}} \rho(ds) \le \left(n^{-\alpha/(1+\alpha)} \right)^{1/\alpha}; \mathcal{Y}_1 \le a \right). \tag{2.29}$$

Thus, the last expression equals

$$\mathbb{P}_a \left(\int_0^{n^{-\alpha/(1+\alpha)}} n^{-\alpha/(1+\alpha)} \mathcal{Y}_{sn^{\alpha/(1+\alpha)}} \rho(ds) \le \frac{1}{n}; \mathcal{Y}_1 \le a \right). \tag{2.30}$$

The scale invariance of the squared Bessel processes implies that, under $\mathbb{P}_a$, we have that $\tilde{\mathcal{Y}}_s := n^{-\alpha/(1+\alpha)} \mathcal{Y}_{n^{\alpha/(1+\alpha)} s}$ is distributed as $\mathcal{Y}_t$ but starting from $an^{-\alpha/(1+\alpha)}$. Also $\{\mathcal{Y}_1 \ge a\}$ is equivalent to $\{\tilde{\mathcal{Y}}_{n^{-\alpha/(1+\alpha)}} \ge an^{-\alpha/(1+\alpha)}\}$. Thus (2.27) equals

$$\mathbb{P}_{an^{-\alpha/(1+\alpha)}} \left(\int_0^{n^{-\alpha/(1+\alpha)}} \mathcal{Y}_s \rho(ds) \le 1/n; \mathcal{Y}_{n^{-\alpha/(1+\alpha)}} \le an^{-\alpha/(1+\alpha)} \right). \tag{2.31}$$

Hence

$$\mathbb{P}(\tilde{A}_i) = \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \le 1; \mathcal{Y}_1 \le a \right). \tag{2.32}$$

To control the time spent in the negative axis we perform a similar argument to show that

$$\mathbb{P}_{an^{-\alpha/(1+\alpha)}} \left(\int_0^\infty \mathcal{X}_s \rho(ds) \le 1/n \right) = \mathbb{P}_a \left(\int_0^\infty \mathcal{X}_s \rho(ds) \le 1 \right). \tag{2.33}$$

Which in turn equals $\mathbb{P}(\tilde{B})$. Thus we have showed that

LEMMA 3.6.

$$\mathbb{P}(G_1) \ge \mathbb{P}_a \left(\int_0^\infty \mathcal{X}_s \rho(ds) \le 1 \right) \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \le 1; \mathcal{Y}_1 \le a \right)^{n+1}.$$

Our lemma states that the probability of G_1 is big enough for our purposes. We aim to deduce an upper bound for $\mathbb{P}(\cap_{i=1}^{3} G_i)$. Recall that $\theta_1 = \inf\{t > 0 : B_t = -n^{1/(1+\alpha)}(n+1)/n\}$ and $\theta_2 = \inf\{t > 0 : B_t = -2n^{1/(1+\alpha)}\}$. The strategy will be to make repeated use of the Ray-Knight theorem using the stopping times θ_1 and θ_2. That will allow us to control the probabilities of G_1, G_2 and G_3 simultaneously.

By the strong Markov property, $B_{\theta_1+t} + n^{1/(1+\alpha)}(n+1)/n$ is distributed as a Brownian motion starting from the origin and has local time

$$\tilde{l}(t,u) := l(\theta_1 + t, u + n^{1/(1+\alpha)}(n+1)/n) - l(\theta_1, u + n^{1/(1+\alpha)}(n+1)/n). \tag{2.34}$$

Thus, $\tilde{l}(t,u)$ has the distribution of $l(t,u)$. Let us apply the Ray-Knight theorem to $\tilde{l}$ using the stopping time θ_2. Then the time that Z spends between its first visit to $-n^{1/(1+\alpha)}(n+1)/n$ and its first visit to $-2n^{1/(1+\alpha)}$ is represented as the integral of a squared Bessel process with respect to ρ. Let $(W_t^*)_{t\geq 0}$ be a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $W^*_{-n^{1/(1+\alpha)}(n+1)/n} = 0$ and independent of W. Let $(\bar{\mathcal{Y}}_t^* : t \in [-2n^{1/(1+\alpha)}, \infty))$ be the Bessel process with $d = 2$ given by

$$\bar{\mathcal{Y}}_t^* = \int_{-2n^{1/(1+\alpha)}}^{t} \frac{1}{2\bar{\mathcal{Y}}_s^*} ds + W_t^* - W^*_{-2n^{1/(1+\alpha)}}. \tag{2.35}$$

We also define $(\bar{\mathcal{Y}}_t^{(i,*)} : t \in [-2n^{1/(1+\alpha)}(2n-i)/n, \infty))$, $i = 0,..,n-2$ as the Bessel process with $d = 2$ given by

$$\bar{\mathcal{Y}}_t^{(i,*)} = (an^{-\alpha/(1+\alpha)})^{1/2} + \int_{-2n^{1/(1+\alpha)}(2n-i)/n}^{t} \frac{1}{2\bar{\mathcal{Y}}_s^{(i,*)}} ds + W_t^* - W^*_{-(2n-i)n^{1/(1+\alpha)}/n}. \tag{2.36}$$

Let $(\bar{\mathcal{X}}_t^* : t \in [-n^{1/(1+\alpha)}(n+1)/n, \infty))$ be a Bessel process with $d = 0$ given by

$$\bar{\mathcal{X}}_t^* = \bar{\mathcal{Y}}^*_{-n^{1/(1+\alpha)}(n+1)/n} - \int_{-n^{1/(1+\alpha)}(n+1)/n}^{t} \frac{1}{2\bar{\mathcal{X}}_s^*} ds + W_t^*. \tag{2.37}$$

We also define

$$\bar{\mathcal{X}}_t^{(0,*)} = (an^{-\alpha/(1+\alpha)})^{1/2} - \int_{-n^{1/(1+\alpha)}(n+1)/n}^{t} \frac{1}{2\bar{\mathcal{X}}_s^*} ds + W_t^*. \tag{2.38}$$

We define $\mathcal{Y}_t^* := (\bar{\mathcal{Y}}_t^*)^2; \mathcal{Y}_t^{(i,*)} := (\bar{\mathcal{Y}}_t^{(i,*)})^2; \mathcal{X}_t^* := (\bar{\mathcal{X}}_t^*)^2$ and $\mathcal{X}_t^{(0,*)} := (\bar{\mathcal{X}}_t^{(0,*)})^2$.
Note that

$$G_2 = \left\{ \int_{\mathbb{R}} l(\theta_2, u) - l(\theta_1, u)\rho(du) \geq (n+2)/n \right\}. \tag{2.39}$$

Thus, in view of the Ray-Knight theorem

$$\mathbb{P}(G_2) = \mathbb{P}\left(\int_{-2n^{1/(1+\alpha)}}^{-n^{1/(1+\alpha)}(n+1)/n} \mathcal{Y}_t^* \rho(dt) + \int_{-n^{1/(1+\alpha)}(n+1)/n}^{\infty} \mathcal{X}_t^* \rho(dt) \geq (n+2)/n \right). \tag{2.40}$$

Let

$$C := \left\{ \int_{-2n^{1/(1+\alpha)}}^{-n^{1/(1+\alpha)}(n+1)/n} \mathcal{Y}_t^* \rho(dt) + \int_{-n^{1/(1+\alpha)}(n+1)/n}^{\infty} \mathcal{X}_t^* \rho(dt) \geq (n+2)/n \right\}. \tag{2.41}$$

Let

$$C_0 := \left\{ \int_{-2n^{1/(1+\alpha)}}^{-n^{1/(1+\alpha)}(2n-1)/n} \mathcal{Y}_t^{(0,*)} \rho(dt) \geq 4/n \right\}. \tag{2.42}$$

For $i = 1, .., n-2$, let

$$C_i := \left\{ \int_{-n^{1/(1+\alpha)}(2n-i)/n}^{-n^{1/(1+\alpha)}(2n-i-1)/n} \mathcal{Y}_t^{(i,*)} \rho(dt) \geq 1/n \right\} \tag{2.43}$$

and $D := \{\mathcal{X}^{(0,*)}_{-n^{1/(1+\alpha)}} = 0\}$. Thus, it is clear that

$$\left(\bigcap_{i=0}^{n-1} C_i \right) \subset C. \tag{2.44}$$

We can apply here the same argument leading to lemma 3.7 to obtain

$$\mathbb{P}(C_0) = \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \geq 4; \mathcal{Y}_1 \geq a \right). \tag{2.45}$$

Similarly, for $i = 1, .., n-2$, we have that

$$\mathbb{P}(C_i) = \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \geq 1; \mathcal{Y}_1 \geq a \right). \tag{2.46}$$

Thus we have proved

$$\mathbb{P}(G_2) \geq \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \geq 4; \mathcal{Y}_1 \geq a \right) \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_t \rho(dt) \geq 1; \mathcal{Y}_1 \geq a \right)^{n-2}. \tag{2.47}$$

The argument leading to lemma 3.7 can be applied once more to get

$$\mathbb{P}(D) = \mathbb{P}_a(\tilde{\mathcal{X}}_1 = 0). \tag{2.48}$$

The event G_3 is equal to $\{B_t \leq n^{1/(1+\alpha)}$ for all $t \in [\theta_1, \theta_2]\}$. Which, in turn, is equivalent to $\{\tilde{l}(-1/n, \theta_2) = 0\}$. This, in turn is equivalent to D. Moreover G_1 is equivalent to A and G_2 is equivalent to C. Thus G is equivalent to $A \cap C \cap D$. But

$$\bigcap_{i=-1}^{n-1} \tilde{A}_i \cap \tilde{B} \bigcap_{i=0}^{n-2} C_i \cap D \subset A \cap C \cap D. \tag{2.49}$$

Furthermore, the L.H.S. of the inclusion (2.49) is an intersection of independent events. The independence of the events can easily be seen because they are defined in terms of disjoint intervals of ρ and independent processes. Note that the events D and $\tilde{A}_{-1}$ are defined in terms of events that occur on the same interval $[-n^{1/(1+\alpha)}(n+1), -n^{1/(1+\alpha)})$, but the event D does not depend upon ρ so that independence holds. Thus we have deduced that

$$\mathbb{P}(Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}) \geq \mathbb{P}(\tilde{A}_{-1})^{n+1} \mathbb{P}(\tilde{B}) \mathbb{P}(C_1)^{n-2} \mathbb{P}(D) \mathbb{P}(C_0). \tag{2.50}$$

To check that $\mathbb{P}(\tilde{A}_{-1})^{n+1} \mathbb{P}(\tilde{B}) \mathbb{P}(C_1)^{n-2} \mathbb{P}(D) \mathbb{P}(C_0) > 0$ we recall (2.48), (2.46), (2.45), (2.32) and (2.33). Thus the fact that those probabilities are non zero can be easily checked using the facts that, for each $\epsilon > 0$, $\mathbb{P}(\rho(0,1) \leq \epsilon) > 0$, for each $M > 0$, $\mathbb{P}(\rho(0,1) \geq M) > 0$ and that the Bessel processes can be bounded below and above with positive probability. We need also to use the fact that

0-dimensional Bessel process hits the origin before time 1 with positive probability. Thus we find that there exists positive constants C_9 and c_9 such that

$$\mathbb{P}(Z_{(n+2)/n} \leq -n^{1/(1+\alpha)}) \geq C_9 \exp(-c_9 n). \tag{2.51}$$

The scaling invariance of Z can be used to deduce theorem 3.4.

2.2. Proof of theorem 3.1. The strategy to prove theorem 3.1 will be to mimic the arguments leading to theorem 3.4 using the fact that the FIN diffusion is the scaling limit of the one-dimensional, symmetric version of the BTM. The main tool used in [**FIN02**] to prove that the FIN diffusion is the scaling limit of the BTM is a coupling between different time scales of the BTM. We will make use of this coupling for the proof of theorem 3.1, so we proceed to recall it.

For each $\epsilon > 0$, we define a family of random variables $(\tau_z^\epsilon)_{z\in\mathbb{Z}}$ as follows. Let $G : [0, \infty) \to [0, \infty)$ be the function defined by the relation

$$\mathbb{P}(V_1 > G(u)) = \mathbb{P}(\tau_0 > u). \tag{2.52}$$

The function G is well defined since V_1 has a continuous distribution function. Moreover, G is non-decreasing and right continuous. Thus G has a right continuous generalized inverse $G^{-1}(s) := \inf\{t : G_t \geq s\}$. Now, for all $\epsilon > 0$ and $z \in \mathbb{Z}$, we define the random variables τ_z^ϵ as

$$\tau_z^\epsilon := G^{-1}(\epsilon^{-1/\alpha}\rho(\epsilon z, \epsilon(z+1)]) \tag{2.53}$$

For all $\epsilon > 0$, we have that $(\tau_z^\epsilon)_{z\in\mathbb{Z}}$ is an i.i.d. family of random variables distributed according to τ_0. For a proof of that fact we refer to [**FIN02**].

We define a coupled family of random measures as

$$\rho^\epsilon := \sum_{z\in\mathbb{Z}} \epsilon^{1/\alpha}\tau_z^\epsilon \delta_{\epsilon z}. \tag{2.54}$$

for all $\epsilon > 0$. Using these measures we can express the rescalings of X as speed measure changed Brownian motions. That is

LEMMA 3.7. *For all $\epsilon > 0$ the process $(\epsilon X_{t\epsilon^{-(1+\alpha)/\alpha}})_{t\geq 0}$ has the same distribution that $(B[\rho^\epsilon]_t)_{t\geq 0}$. Moreover, we have that*

$$\rho^\epsilon \xrightarrow{v} \rho \ \mathbb{P}\text{-}a.s. \ as \ \epsilon \to 0 \tag{2.55}$$

where $\xrightarrow{v}$ denotes vague convergence of measures.

For the proof of this statement we refer to [**FIN02**]. Lemma 3.7 implies in particular that $(X_t)_{t\geq 0}$ is distributed as $(B[\rho^1]_t)_{t\geq 0}$. Thus

$$\mathbb{P}(|X_t| \geq x) = \mathbb{P}(|B[\rho^1]_t| \geq x) \tag{2.56}$$

for all $x \geq 0$ and $t \geq 0$.

We will proceed as in the proof of lemma 3.7. Let $t \geq 0$ be fixed. It will suffice to establish the lower bound of theorem 3.1 for $x = m^{1/(1+\alpha)} t^{\alpha/(1+\alpha)}$, where $m \in \mathbb{N}$ (and $x/t \leq \epsilon_1$). Then, using the fact that for fixed $t \geq 0$, $\mathbb{P}(|X_t| \geq x)$ is decreasing on x we can extend our result to all $x \geq 0$ (with $x/t \leq \epsilon_1$).
Let $H_b^0 := \inf\{t \geq 0 : B[\rho^1]_t = b\}$. We define a collection of events analogous to G, G_1, G_2 and G_3 defined on the displays (2.5),(2.6),(2.7) and (2.8). Let

$$G^0 := \{B[\rho^1]_{t(m+2)/m} \leq -x\} \tag{2.57}$$

$$G_1^0 := \{H^1_{x(m+1)/m} \leq t(m+2)/m\} \tag{2.58}$$

$$G_2^0 := \{H^1_{-2x} - H^1_{-x(m+1)/m} \geq t(m+2)/m\} \tag{2.59}$$

$$G_3^0 := \{B[\rho^1]_t \leq -x \text{ for all } t \in [H^1_{-x(m+1)/m}, H^1_{-2x}]\}. \tag{2.60}$$

Note that $G^0 \subset G_1^0 \cap G_2^0 \cap G_3^0$.
First we will control the probability of G_1^0 in the same way that we controlled the probability of G_1 in the proof of lemma 3.7. Recall that W_t is a Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$, independent of ρ and started at 0. Let $(\bar{\mathcal{Y}}_t^0 : t \in [-x(m+1)/m, \infty)$ be the Bessel process with $d = 2$ given by

$$\bar{\mathcal{Y}}_t^0 := \int_{-x(m+1)/m}^{t} \frac{1}{2\bar{\mathcal{Y}}_s^0} ds + W_t - W_{-x(m+1)/m}. \tag{2.61}$$

Let $a > 0$. We also define $(\bar{\mathcal{Y}}_t^{(0,i)} : t \in [-x(m-i)/m, \infty))$, $i = -1, 0, .., m-1$ as the Bessel processes with $d = 2$ given by

$$\bar{\mathcal{Y}}_t^{(0,i)} := (ax/m)^{1/2} + \int_{-x(m-i)/m}^{t} \frac{1}{2\bar{\mathcal{Y}}_s^{(0,i)}} ds + W_t - W_{-x(m-i)/m}. \tag{2.62}$$

Let $(\bar{\mathcal{X}}_t^0)_{t \geq 0}$ be a Bessel process with $d = 0$ given by

$$\bar{\mathcal{X}}_t^0 := \bar{\mathcal{Y}}_0^0 - \int_0^t \frac{1}{2\bar{\mathcal{X}}_s^0} ds + W_t. \tag{2.63}$$

We also define

$$\bar{\mathcal{X}}_t^{(0,0)} := (ax/m)^{1/2} - \int_0^t \frac{1}{2\bar{\mathcal{X}^{(0,0)}}_s} ds + W_t. \tag{2.64}$$

We aim to use the Ray-Knight theorem, which deals with squared Bessel processes. Thus we define $Y_t^0 := (\bar{\mathcal{Y}}_t^0)^2; Y_t^{0,i)} := (\bar{\mathcal{Y}}_t^{(0,i)})^2; \mathcal{X}_t^0 := (\bar{\mathcal{X}}_t^0)^2$ and $\mathcal{X}_t^{(0,0)} := (\bar{\mathcal{X}}_t^{(0,0)})^2$.
Using the squared Bessel processes constructed above, we define a family of events analogous to the events $A, \tilde{A}_i$ and $\tilde{B}$ appearing on displays (2.19),(2.23) and (2.24). Let

$$A^0 := \left\{ \int_{-x(m+1)/m}^{0} \mathcal{Y}_t^0 \rho^1(dt) + \int_0^\infty \mathcal{X}_t^0 \rho^1(dt) \leq t(m+2)/m \right\}. \tag{2.65}$$

For $i = -1, .., m-1$ let

$$\tilde{A}_i^0 := \left\{ \int_{-x(m-i)/m}^{-x(m-i-1)/m} \mathcal{Y}_t^{(0,i)} \rho^1(dt) \leq t/m; \mathcal{Y}^{(0,i)}_{-x(m-i-1)/m} \leq ax/m \right\}. \tag{2.66}$$

Also let

$$\tilde{B}^0 := \left\{ \int_0^\infty \mathcal{X}_t^{(0,0)} \rho^1(dt) \le t/m \right\}. \tag{2.67}$$

Thus, it is clear that

$$\left(\bigcap_{i=-1}^{m-1} \tilde{A}_i^0 \right) \cap \tilde{B}^0 \subset A^0. \tag{2.68}$$

Using the Ray-Knight theorem we see that G_1^0 is equivalent to A^0.

From now on, $\mathbb{P}_b$ will denote probability conditioned on $\mathcal{Y}_0^0 = b$, and when there is no risk of confusion, it will also denote probability conditioned on $\mathcal{X}_0^0 = b$. Let $d = x/m$. As in the proof of theorem 3.4 we have that, for $i = -1, .., m-1$

$$\mathbb{P}(\tilde{A}_i^0) = \mathbb{P}_{ad} \left(\int_0^d \mathcal{Y}_s \rho^1(ds) \le \frac{t}{m}; \mathcal{Y}_d \le ad \right) \tag{2.69}$$

We define

$$Sc(r)(\mu) := r^{1/\alpha} \mu(r^{-1} A) \tag{2.70}$$

Performing the change of variables $u = sd^{-1}$ inside the integral, we obtain

$$\mathbb{P}(\tilde{A}_i^0) = \mathbb{P}_{ad} \left(\int_0^1 \mathcal{Y}_{du} d^{1/\alpha} Sc(d^{-1}) \rho^1(du) \le \frac{t}{m}; \mathcal{Y}_d \le ad \right) \tag{2.71}$$

Using the scaling invariance of the squared Bessel process $\mathcal{Y}$ we obtain

$$\mathbb{P}(\tilde{A}_i^0) = \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_u Sc(d^{-1})(\rho^1)(du) \le 1; \mathcal{Y}_1 \le a \right) \tag{2.72}$$

But $Sc(d^{-1})(\rho^1)$ is distributed as $\rho^{d^{-1}}$. Thus, we can replace to obtain

$$\mathbb{P}(\tilde{A}_i^0) = \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_u \rho^{d^{-1}}(du) \le 1; \mathcal{Y}_1 \le a \right) \tag{2.73}$$

Similar arguments can be applied to get

$$\mathbb{P}(\tilde{B}^0) = \mathbb{P}_a \left(\int_0^\infty \mathcal{X}_u \rho^{d^{-1}}(du) \le 1 \right) \tag{2.74}$$

On the other hand, using display (2.55) in lemma 3.7 we can prove that

$$\int_0^1 \mathcal{Y}_t \rho^\epsilon(dt) \overset{\epsilon \to 0}{\to} \int_0^1 \mathcal{Y}_t \rho(dt) \qquad \mathbb{P}\text{-almost surely.} \tag{2.75}$$

and

$$\int_0^\infty \mathcal{X}_t \rho^\epsilon(dt) \overset{\epsilon \to 0}{\to} \int_0^\infty \mathcal{X}_t \rho(dt) \qquad \mathbb{P}\text{-almost surely.} \tag{2.76}$$

Thus, there exists ϵ_0 small enough such that, for $m \in \mathbb{N}$ and $t \ge 0$ such that $(m/t)^\alpha \le \epsilon_0$ we have that

$$\mathbb{P}(\tilde{A}_i^0) \ge \frac{1}{2} \mathbb{P}_a \left(\int_0^1 \mathcal{Y}_u \rho(du) \le 1; \mathcal{Y}_1 \le a \right) \tag{2.77}$$

and

$$\mathbb{P}(\tilde{B}^0) \ge \frac{1}{2} \mathbb{P}_a \left(\int_0^\infty \mathcal{X}_u \rho(du) \le 1 \right). \tag{2.78}$$

Hence, we have showed that

LEMMA 3.8. *There exists* ϵ_0 *small enough such that, for* $m \in \mathbb{N}, t \geq 0$ *and* $x = m^{1/(1+\alpha)} t^{\alpha/(1+\alpha)}$ *such that* $(x/t)^\alpha \leq \epsilon_0$ *we have that*

$$\mathbb{P}(G_1^0) \geq \frac{1}{2}\mathbb{P}_a\left(\int_0^\infty \mathcal{X}_s \rho(ds) \leq 1\right)\left(\frac{1}{2}\mathbb{P}_a\left(\int_0^1 \mathcal{Y}_t \rho(dt) \leq 1; \mathcal{Y}_1 \leq a\right)\right)^{m+1}.$$

This lemma states that the probability that X passes trough $-x(m+1)/m$ before time $t(m+2)/m$ is big enough for our purposes. Now we aim to deduce a lower bound for $\mathbb{P}(|X_t| \geq x)$. But we can use displays (2.75) and (2.76) to adapt the proof for the FIN diffusion to the case of the BTM, in the same way we adapted the proof of lemma 3.7 to obtain lemma 3.8.

CHAPTER 4

Randomly trapped random walks

1. Introduction

Here we will introduce a generalization of the one-dimensional Bouchaud trap model. This generalization will consist on allowing the traps to retain the walker by times that are not necessarily exponentially distributed. These models will be called *randomly trapped random walks* (RTRW). First we will need the quenched realization of these models, i.e. when the traps are fixed. We will call these fixed realizations *trapped random walks* (TRW). The RTRW will be obtained from the TRM by considering the traps to be choosen randomly. The scaling limits of RTRW's will be a class of processes called *randomly trapped Brownian motions* (RTBM). The RTBM is a random time change of a Brownian motion which includes as a specific case the FIN diffusion. The quenched realizations of the RTBM's are the *trapped Brownian motions* (TBM). The TBM's are a generalization of speed-measure changed Brownian motion, when, instead of considering a measure for the definition, one uses a *trap measure*. The trap measures are similar to ordinary, one-dimensional measures. But in this case, instead of assigning numbers to sets, we will assign certain stochastic processes to sets. Formally speaking, trap measures are certain kind of ordinary measures over a $\mathbb{R} \times \mathbb{R}_+$. By considering random trap measures to define RTBM's.

Trap measures and random trap measures can also be used to define TRW's and RTRW's respectively. We provide a general method for proving convergence of RTRW's to some scaling limit: As a general theorem, we will show that convergence of trap measures imply convergence of the respective processes. Thus we show that, in order to demonstrate convergence of processes, we will have to show convergence of (ordinary) random measures on a half-plane. Thus, we can profit from the established theory of random measures. Following that line of reasoning we establish conditions under which the scaling limit of the RTRW exists and is a RTBM.

Some interesting and natural examples of RTRW arise in relation to percolation on regular trees. The *incipient infinite cluster* (IIC) in a regular tree is a random graph which is constructed as the limit when $n \to \infty$ of critical percolation clusters on a tree, conditioned on surviving up to level n. As shown by Kesten in [**Kes86b**], this random tree can be viewed as the graph $\mathbb{N}$ (called the *backbone*) in which, from each site $k \in \mathbb{N}$, there emerges a random branch $\mathcal{L}_k$, and $(\mathcal{L}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of critical percolation clusters. Consider a simple, discrete time random walk $(Z_n^{IIC})_{n\in\mathbb{N}}$ on the IIC. Once Z^{IIC}

is on vertex $x \in$ IIC, it jumps to each of its neighbors with probability $deg(x)^{-1}$, where $deg(x)$ is the degree of x in IIC. We can construct a continuous time random walk on $\mathbb{N}$, denoted by $(W_t^{IIC})_{t\geq 0}$, by stating that $W_t^{IIC} = z$ for all $t \geq 0$ such that $Z_{\lfloor t \rfloor}^{IIC} \in \mathcal{L}_z$. That is, W^{IIC} is the projection of Z^{IIC} to the backbone $\mathbb{N}$. For each realization of the IIC, which is to say, each realization of $(\mathcal{L}_k)_{k\in\mathbb{N}}$, W^{IIC} is a random walk with trapping landscape $\boldsymbol{\nu} = (\nu_z)_{z\in\mathbb{N}}$ where, for each $k \in \mathbb{N}$, ν_k is the distribution of the time that Z^{IIC} spends on a single visit to the branch $\mathcal{L}_k$ and for $z \in \mathbb{Z}$ negative, we have that $\nu_z := \delta_0$. As the branches $(\mathcal{L}_z)_{z\in\mathbb{N}}$ are random, we have that W is a RTRW. One can also use, instead of the IIC, the *invasion percolation cluster* (IPC) on a regular tree. As shown in [**AGdHS08**], this random tree can also be viewed as $\mathbb{N}$ adorned with random branches. In this case the sequence of branches will be composed of subcritical percolation clusters (with a varying percolation parameter). As in the case of the IIC, we can construct a randomly trapped random walk W^{IPC} as the projection of a simple random walk on the IPC to the backbone $\mathbb{N}$. It is natural to wonder what kind of scaling limits do the processes W^{IIC}, W^{IPC} have, in case they exist. More generally, it is natural to ask the same questions for a simple random walk on any tree $\mathcal{G}$, random or not, but with a single backbone, like the IPC and the IIC, These questions are particular instances of the broader query: what kind of processes arise as scaling limits of the RTRW?. Here we provide a general theorem exhibiting all possible scaling limits of RTRW's on an i.i.d. environment.

2. Delayed processes and their convergence

In this section we define two large classes of processes which we call *delayed random walks* and *delayed Brownian motions*, and present theorems giving convergence criteria for such processes. Later in the chapter we will explain that all processes mentioned in the Introduction are included in these two classes.

2.1. Delayed random walks and Brownian motion. Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$, $\bar{\mathbb{R}}_+ = [0, \infty]$, and let $\mathbb{H}$ be the half-plane $\mathbb{R} \times \mathbb{R}_+$. For any topological space E, $\mathcal{B}(E)$ stands for the Borel σ-field of E, $C(E)$ for the space of continuous real valued function on E, $C_0(E)$ for the continuous functions with compact support. We write $M(E)$ for the set of positive Radon measures on E, that is for the set of positive Borel measures on E that are finite over compact sets. We will endow $M(E)$ with the topology of vague convergence. $M_1(E)$ stands for the space of probability measures over E endowed with weak convergence. It is know fact [**Kal83**, Lemma 1.4, Lemma 4.1] that the σ-field $\mathcal{B}(M(E))$ coincides with the field generated by the functions $\{\mu \mapsto \mu(A) : A \in \mathcal{B}(E) \text{ bounded } \}$, as well as with the with the σ-field generated by the functions $\{\mu \mapsto \int_E f d\mu : f \in C_0(E)\}$.

For every measure $\nu \in M(\mathbb{R}_+)$, we define its Laplace transform $\hat{\nu} \in C(\mathbb{R}_+)$ as

$$\hat{\nu}(\lambda) := \int_{\mathbb{R}_+} \exp(-\lambda t)\nu(dt). \tag{2.1}$$

We recall that μ is a **random measure on** $\mathbb{H}$ defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ iff $\mu : \tilde{\Omega} \to M(\mathbb{H})$ is a measurable function from the measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to the measurable space $(M(\mathbb{H}), \mathcal{B}(M(\mathbb{H})))$ (see [**Kal83**]). Equivalently, μ is a random measure iff $\mu(A) : \tilde{\Omega} \to \bar{\mathbb{R}}_+$ is a measurable function for every $A \in \mathcal{B}(\mathbb{H})$. The law induced by μ on $M(\mathbb{H})$ will be denoted P_μ,

$$P_\mu = \tilde{\mathbb{P}} \circ \mu^{-1}. \tag{2.2}$$

We call random measure on $\mathbb{H}$ **discrete** when its support is contained in $\mathbb{Z} \times \mathbb{N}$, $\tilde{\mathbb{P}}$-a.s. Finally, for a Borel-measurable function $f : \mathbb{R} \to \mathbb{R}_+$, we define the set

$$U_f := \{(x, y) \in \mathbb{H} : y \le f(x)\}. \tag{2.3}$$

Definition 4.1 (Delayed random walk). Let μ be a *discrete* random measure and $(Z_k)_{k\in\mathbb{N}}$ be a simple random walk on $\mathbb{Z}$ with $Z_1 = 0$. Without loss of generality, we suppose that Z is defined on the same probability space as μ, and that it is independent of it. Let $L(x,t) := \sum_{i=1}^{\lfloor t\rfloor} 1_{\{Z_i = \lfloor x\rfloor\}}$ be the local time of Z. Define $\phi[\mu, Z]_t := \mu(U_{L(\cdot,t)})$ and its right-continuous generalised inverse $\psi[\mu, Z]_t := \min\{s > 0 : \phi[\mu, Z]_s > t\}$. The **$\mu$-delayed random walk** $(Z[\mu]_t)_{t\ge0}$ is the process given by

$$Z[\mu]_t := Z_{\psi[\mu,Z]_t}, \qquad t \ge 0. \tag{2.4}$$

Remark 4.2. (a) If $\mu((x, i)) > 0$ for every $x \in \mathbb{Z}$, $i \in \mathbb{N}$, then the μ-delayed random walk $Z[\mu]$ follows the same trajectory as the simple random walk Z. At i-th visit of site x, it stays there for time $\mu((x, i)) > 0$.

(b) If $\mu(\mathbb{H}) < \infty$, $Z[\mu]$ is not defined for times $t > \mu(\mathbb{H})$ and might not be defined for $t = \mu(\mathbb{H})$.

(c) If $\mu((x, i)) = 0$ for some $(x, i) \in \mathbb{Z} \times \mathbb{N}$, then the trajectory of $Z[\mu]$ is not necessarily nearest-neighbour.

The definition of delayed Brownian motion is very similar.

Definition 4.3 (Delayed Brownian motion). Let μ be a random measure and B be a standard one-dimensional Brownian motion defined on the same probability space as μ, independent of it. Let $\ell(x,t)$ be a bi-continuous version of the local time of B. Define $\phi[\mu, B]_t := \mu(U_{\ell(\cdot,t)})$ and $\psi[\mu, B]_t := \inf\{s > 0 : \phi[\mu, B]_s > t\}$. The **delayed Brownian motion** $(B[\mu]_t)_{t\ge0}$ the process given by

$$B[\mu]_t := B_{\psi[\mu,B]_t}, \qquad t \ge 0. \tag{2.5}$$

Remark 4.4. It is easy to see that the functions $\phi[\mu, Z]$, $\psi[\mu, Z]$, $\phi[\mu, B]$, and $\psi[\mu, B]$ are non-decreasing and right-continuous, $\tilde{\mathbb{P}}$-a.s. Hence, $Z[\mu]$ and $B[\mu]$ have right-continuous trajectories.

2.2. Convergence of delayed random walks. We now present our basic convergence theorems for delayed random walks and Brownian motions. These theorems allow to deduce the convergence of delayed processes from the convergence of the associated random measures. This, in turn, makes possible to use the well developed theory of convergence of random measures, see e.g. [**Kal83**].

First we need few additional definitions. We say that a random measure μ is **dispersed** if

$$\tilde{\mathbb{P}}[\mu(\{(x,y) \in \mathbb{H} : y = f(x)\}) = 0] = 1, \qquad \text{for all } f \in C_0(\mathbb{R}, \mathbb{R}_+). \tag{2.6}$$

We say that a random measure μ is **infinite** if $\mu(\mathbb{H}) = \infty$, $\tilde{\mathbb{P}}$-a.s.

We write $D(\mathbb{R}_+)$, $D(\mathbb{R})$ for the sets of real-valued *càdlàg* functions on $\mathbb{R}_+$, or $\mathbb{R}$, respectively. We endow these sets either with the standard Skorokhod J_1-topology, or with the so called M_1'-topology, and write $D((\mathbb{R}_+), J_1)$, $D((\mathbb{R}_+), M_1')$ when we want to stress the topology used. The main reason to use the M_1'-topology, which is weaker than the J_1-topology, is that the inversion map on $(D(\mathbb{R}_+), M_1')$ is continuous when restricted to monotonous functions. Note also that the convergence in the M_1'-topology implies the convergence in the stronger and slightly more usual M_1-topology, when the limit f satisfies $f(0) = 0$. For definitions and properties of these topologies see [**Whi02**], the proofs of above claims are contained in Section 13.6 of this book.

Let μ be a random measure and $\varepsilon > 0$. We define the scaled random measure $\mathfrak{S}_\varepsilon(\mu)$ by

$$\mathfrak{S}_\varepsilon(\mu)(A) := \mu(\varepsilon^{-1}A), \qquad \text{for each } A \in \mathcal{B}(\mathbb{H}). \tag{2.7}$$

THEOREM 4.5 (convergence of delayed random walks). *Let $(\mu^\varepsilon)_{\varepsilon>0}$ be a family of discrete infinite random measures. Suppose there exists a non-decreasing function $q : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0} q(\varepsilon) = 0$, such that, as $\varepsilon \to 0$, $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$ converges vaguely in distribution to a dispersed infinite random measure μ. Then, as $\varepsilon \to 0$, $(\varepsilon Z[\mu^\varepsilon]_{q(\varepsilon)^{-1}t})_{t\ge 0}$ converges in distribution to $(B[\mu]_t)_{t\ge 0}$ on $(D(\mathbb{R}_+), M_1')$.*

The next theorem, which we will not need later in the thesis, gives a similar criteria for convergence of delayed Brownian motions. We present it as it has intrinsic interest and because its proof is a simplified version of the proof of Theorem 4.5.

THEOREM 4.6. *Let $(\mu^\varepsilon)_{\varepsilon>0}$ be a family of infinite random measures such that μ^ε converges vaguely in distribution to a random measure μ as $\varepsilon \to 0$. Suppose also that μ is both, dispersed and infinite. Then, as $\varepsilon \to 0$, $(B[\mu^\varepsilon]_t)_{t\ge 0}$ converges in distribution to $(B[\mu]_t)_{t\ge 0}$ on $(D(\mathbb{R}_+), M_1')$.*

PROOF OF THEOREM 4.6. As μ^ε converges vaguely in distribution to μ, in virtue of the Skorokhod representation theorem, there exist random measures $(\bar{\mu}^\varepsilon)_{\varepsilon>0}$ and $\bar{\mu}$ on $\mathbb{H}$ with a common reference space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that $\bar{\mu}^\varepsilon$ is distributed as μ^ε, $\bar{\mu}$ is distributed as μ and $\bar{\mu}^\varepsilon$ converges vaguely to $\bar{\mu}$ as $\varepsilon \to 0$, $\tilde{\mathbb{P}}$-a.s. Without loss of generality, we can suppose that on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there is defined a one-dimensional standard Brownian motion $(B_t)_{t\ge 0}$ independent of $(\bar{\mu}^\varepsilon)_{\varepsilon>0}$ and $\bar{\mu}$.

First, we show that $\phi[\bar{\mu}^\varepsilon, B] \to \phi[\bar{\mu}, B]$ in $(D(\mathbb{R}_+), M_1')$, $\tilde{\mathbb{P}}$-a.s. as $\varepsilon \to 0$. Using that $\bar{\mu}$ is a dispersed trap measure,

$$\tilde{\mathbb{P}}[\bar{\mu}(\partial U_{\ell(\cdot,t)}) = 0, \forall 0 \le t \in \mathbb{Q}] = 1, \tag{2.8}$$

where ∂A denotes the boundary of A in $\mathbb{H}$. Since $U_{\ell(\cdot,t)}$ is a bounded set, this implies that for all $0 \le t \in \mathbb{Q}$

$$\phi[\bar{\mu}^\varepsilon, B]_t = \bar{\mu}^\varepsilon(U_{\ell(\cdot,t)}) \xrightarrow{\varepsilon \to 0} \bar{\mu}(U_{\ell(\cdot,t)}) = \phi[\bar{\mu}, B]_t, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{2.9}$$

Since, by [**Whi02**, Theorem 13.6.3], on the set of monotonous functions the convergence on $(D(\mathbb{R}_+), M_1')$ is equivalent to pointwise convergence on a dense subset of $(0,\infty)$, and since $\phi[\bar{\mu}^\varepsilon, B]$ and $\phi[\bar{\mu}, B]$ are non-decreasing in t, we know that $\phi[\bar{\mu}^\varepsilon, B]$ converges to $\phi[\bar{\mu}, B]$ in $(D(\mathbb{R}_+), M_1')$, $\tilde{\mathbb{P}}$-a.s.

Since the random measures $\bar{\mu}^\varepsilon$ and $\bar{\mu}$ are infinite, the functions $\phi[\bar{\mu}^\varepsilon, B]$ and $\phi[\bar{\mu}, B]$ are unbounded. [**Whi02**, Theorem 13.6.3] then implies that

$$(\psi[\bar{\mu}^\varepsilon, B])_{t\ge 0} \xrightarrow{\varepsilon \to 0} (\psi[\bar{\mu}, B]_t)_{t \ge 0}, \qquad \tilde{\mathbb{P}}\text{-a.s. on } (D(\mathbb{R}_+), M_1'). \tag{2.10}$$

In virtue of [**Whi02**, Theorem 13.6.3] once more, we know that $\psi[\bar{\mu}^\varepsilon, B]_t \to \psi[\bar{\mu}, B]_t$ pointwise on a dense subset of $(0,\infty)$. Using the continuity of the Brownian paths we get that $B[\bar{\mu}^\varepsilon]_t \to B[\bar{\mu}, B]_t$ on this dense set. Thus $B[\bar{\mu}^\varepsilon] \to B[\bar{\mu}]$ on $(D(\mathbb{R}_+, M_1'))$, $\tilde{\mathbb{P}}$-a.s. Since $\bar{\mu}^\varepsilon$ and $\bar{\mu}$ are distributed as the μ^ε and μ respectively, the convergence in distribution of $B[\bar{\mu}^\varepsilon]$ and $B[\bar{\mu}]$ follows. $\square$

Proof of Theorem 4.5. As $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$ converges vaguely in distribution to μ, we can, in virtue of the Skorokhod representation theorem, construct random measures $(\bar{\mu}^\varepsilon)_{\varepsilon>0}$ and $\bar{\mu}$ with a common reference space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that $\bar{\mu}^\varepsilon$ is distributed as $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$, $\bar{\mu}$ is distributed as μ, and $\bar{\mu}^\varepsilon$ converges vaguely to $\bar{\mu}$ as $\varepsilon \to 0$, $\tilde{\mathbb{P}}$-a.s. Without loss of generality, we can suppose that on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there is defined a one-dimensional standard Brownian motion $(B_t)_{t\ge 0}$ independent of $(\bar{\mu}^\varepsilon)_{\varepsilon>0}$ and $\bar{\mu}$.

Set $B_t^\varepsilon := \varepsilon^{-1}B_{\varepsilon^2 t}$. For each $\varepsilon > 0$, we define a sequence of stopping times $(\sigma_k^\varepsilon)_{k=0}^\infty$ by $\sigma_0^\varepsilon := 0$,

$$\sigma_k^\varepsilon := \inf\{t > \sigma_{k-1}^\varepsilon : B_t^\varepsilon \in \mathbb{Z} \setminus \{B^\varepsilon_{\sigma^\varepsilon_{k-1}}\}\}. \tag{2.11}$$

Then, the process $(Z_k^\varepsilon)_{k\in\mathbb{N}}$ defined by $Z_k^\varepsilon := B^\varepsilon_{\sigma_k^\varepsilon}$ is a simple, symmetric random walk on $\mathbb{Z}$. We define the local time of Z^ε as $L^\varepsilon(x,s) := \sum_{i=0}^{\lfloor s\rfloor} 1_{\{Z_i^\varepsilon = \lfloor x \rfloor\}}$. Define

$$\bar{\phi}_s^\varepsilon = q(\varepsilon)^{-1}\mathfrak{S}_{\varepsilon^{-1}}(\bar{\mu}^\varepsilon)(U_{L^\varepsilon(\cdot,s)}), \qquad s \ge 0, \varepsilon > 0. \tag{2.12}$$

Note that $q(\varepsilon)^{-1}\mathfrak{S}_{\varepsilon^{-1}}(\bar{\mu}^\varepsilon)$ is distributed as μ^ε. Hence, $(\bar{\phi}_t^\varepsilon)_{t\ge 0}$ is distributed as $(\mu^\varepsilon(U_{L^1(\cdot,t)}))_{t\ge0} = (\phi[\mu^\varepsilon]_t)_{t\ge 0}$. Hence, denoting $\bar{\psi}_t^\varepsilon := \inf\{s > 0 : \bar{\phi}_s^\varepsilon > t\}$, we see that for each $\varepsilon > 0$, the process $(Z^\varepsilon_{\bar{\psi}_t^\varepsilon})_{t\ge 0}$ is distributed as $(Z[\mu^\varepsilon]_t)_{t\ge 0}$.

The proof of Theorem 4.5 relies on the following two lemmas.

Lemma 4.7. *For each $t \ge 0$, there exists a random compact set K_t such that $\cup_{\varepsilon>0} \operatorname{supp} L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)$ is contained in K_t.*

Proof. By the strong Markov property for the Browian motion B, for each $\varepsilon > 0$, $(\sigma_k^\varepsilon - \sigma_{k-1}^\varepsilon)_{k>0}$ is an i.i.d. sequence with $\tilde{\mathbb{E}}[\sigma_i^\varepsilon - \sigma_{i-1}^\varepsilon] = 1$. Thus, by the strong law of large numbers for triangular arrays, $\tilde{\mathbb{P}}$-almost surely, there exists a (random) constant C such that $\varepsilon^2 \sigma^\varepsilon_{\lfloor \varepsilon^{-2} t \rfloor} \leq C$ for all $\varepsilon > 0$. Thus, for each $\varepsilon > 0$, the support of $L^\varepsilon(\varepsilon^{-1} \cdot, \varepsilon^{-2} t)$ is contained in the support of $\ell(\cdot, C)$. Therefore, it is sufficient to choose $K_t = \mathrm{supp}(\ell(\cdot, C))$. $\square$

Lemma 4.8. *$(q(\varepsilon)\bar{\phi}^\varepsilon_{\varepsilon^{-2}t})_{t\geq 0} \xrightarrow{\varepsilon\to 0} (\phi[\bar{\mu}, B]_t)_{t\geq 0}$ $\tilde{\mathbb{P}}$-a.s. on $(D(\mathbb{R}_+), M_1')$.*

Proof. It is easy to see that

$$q(\varepsilon)\bar{\phi}^\varepsilon_{t\varepsilon^{-2}} = \mathfrak{S}_{\varepsilon^{-1}}(\bar{\mu}^\varepsilon)(U_{L^\varepsilon(\cdot, \varepsilon^{-2}t)}) = \bar{\mu}^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)}) \tag{2.13}$$

By [**Bor87**, Theorem 2.1], for each $t \geq 0$, $\tilde{\mathbb{P}}$-a.s., $\varepsilon L^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t) \xrightarrow{\varepsilon\to 0} \ell(x,t)$ uniformly in x. Thus for any $\eta > 0$ there exists ε_η such that, if $\varepsilon < \varepsilon_\eta$ we will have that $\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t) \leq \ell(\cdot, t) + \eta$. Note that $\ell(\cdot, t) + \eta$ is not compactly supported. Let $h_\eta : \mathbb{H} \to \mathbb{R}_+$ be a continuous function which for every $t \geq 0$ coincides with $\ell(\cdot, t) + \eta$ on K_t, $h_\eta(\cdot, t) \leq \eta$ outside K_t, and $h_\eta(\cdot, t)$ is supported on $[\inf K_t - \eta, \sup K_t + \eta]$. Using Lemma 4.7 we find that $\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t) \leq h_\eta(\cdot, t)$. Thus

$$\bar{\mu}^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)}) \leq \bar{\mu}^\varepsilon(U_{h_\eta(\cdot,t)}). \tag{2.14}$$

As $\bar{\mu}$ is a dispersed trap measure, for fixed t, $\bar{\mu}(\partial U_{h_\eta(\cdot,t)}) = \bar{\mu}(\partial U_{\ell(\cdot,t)}) = 0$, $\tilde{\mathbb{P}}$-a.s. For any $\delta > 0$ and all ε small enough (depending on δ), as $\bar{\mu}^\varepsilon$ converges vaguely to $\bar{\mu}$,

$$\bar{\mu}^\varepsilon(U_{h_\eta(\cdot,t)}) \leq \bar{\mu}(U_{h_\eta(\cdot,t)}) + \delta/2. \tag{2.15}$$

For each $\delta > 0$ there exists $\eta > 0$ such that $\bar{\mu}(U_{h_\eta(\cdot,t)}) \leq \bar{\mu}(U_{\ell(\cdot,t)}) + \delta/2$. Combining this with (2.13)–(2.15), we find that

$$\limsup_{\varepsilon\to 0} q(\varepsilon)\bar{\phi}^\varepsilon_{t\varepsilon^{-2}} = \limsup_{\varepsilon\to 0} \bar{\mu}^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)}) \leq \phi[\bar{\mu}, B]_t. \tag{2.16}$$

A lower bound can be obtained in a similar way. Hence, after taking union over $0 \leq t \in \mathbb{Q}$,

$$\tilde{\mathbb{P}}[\lim_{\varepsilon\to 0} q(\varepsilon)\bar{\phi}^\varepsilon_{\varepsilon^{-2}t} = \phi[\bar{\mu}, B]_t, \forall 0 \leq t \in \mathbb{Q}] = 1. \tag{2.17}$$

Since $\bar{\phi}^\varepsilon_t$ and $\phi[\bar{\mu}, B]$ are non-decreasing in t, $(q(\varepsilon)\bar{\phi}^\varepsilon_{\varepsilon^{-2}t})_{t\geq 0}$ converges to $(\phi[\bar{\mu}, B]_t)_{t\geq 0}$, $\tilde{\mathbb{P}}$-a.s. on $(D(\mathbb{R}_+), M_1')$, finishing the proof of the lemma. $\square$

Theorem 4.5 then follows from Lemma 4.8 by repeating the arguments of the last paragraph in the proof of Theorem 4.6. $\square$

3. Trapped random walk and Brownian motion

The class of delayed random walks defined in the last section is very large, and the associated convergence criteria rather general. Applying these criteria, however, requires to check the convergence of underlying random measures, which might be complicated in many situations.

In this section, we introduce several subclasses of delayed random walks/Brownian motions. The underlying random measures of these subclasses will satisfy few additional assumptions which will make checking their convergence easier than in the general case.

3.1. Definition of trapped processes.

DEFINITION 4.9 (Trap measure). A random measure μ on $\mathbb{H}$ with reference space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is called **trap measure** iff for every two disjoint sets $A, B \in \mathcal{B}(\mathbb{H})$ the random variables $\mu(A)$ and $\mu(B)$ are independent under $\tilde{\mathbb{P}}$.

For any random measure μ and $V \in \mathcal{B}(\mathbb{R})$ we define the μ-trapping process $(\mu\langle V\rangle_t)_{t\geq 0}$ by

$$\mu\langle V\rangle_t = \mu(V \times [0,t]). \tag{3.1}$$

Note that, if μ is a trap measure and V and W are disjoint Borel subsets of $\mathbb{R}$, then $\mu\langle V\rangle$ and $\mu\langle W\rangle$ are independent.

DEFINITION 4.10 (Lévy trap measure). A trap measure μ is said to be a **Lévy trap measure** iff $\mu\langle V\rangle$ is a Lévy process for every bounded $V \in \mathcal{B}(\mathbb{R})$.

DEFINITION 4.11 (Discrete trap measure). A trap measure μ is called **discrete trap measure** iff it is discrete and for each $V \in \mathcal{B}(\mathbb{R})$ bounded, $(\mu(V \times \{k\}))_{k\in\mathbb{N}}$ is an i.i.d. sequence.

REMARK 4.12. Observe that discrete trap measure is not simply trap measure that is discrete.

For any discrete trap measure μ and $z \in \mathbb{Z}$, denote by $\pi_z(\mu) \in M_1(\mathbb{R}_+)$ the distribution of $\mu(\{z\} \times \{k\})$, which does not depend on k. We say that $\boldsymbol{\pi}(\mu) := (\pi_z(\mu))_{z\in\mathbb{Z}}$ is the **trapping landscape** associated to μ. Clearly, the distribution of μ is completely determined by its trapping landscape. On the other hand, for every $\pi \in M_1(\mathbb{R}_+)^{\mathbb{Z}}$ there is a discrete trap measure μ such that $\pi = \boldsymbol{\pi}(\mu)$ (see Example 4.16 below).

DEFINITION 4.13 (Trapped random walk/Brownian motion). When μ is a discrete trap measure, we call the μ-delayed random walk $Z[\mu]$ **trapped random walk (TRW)**. Similarly, when μ is a Lévy trap measure, the μ-delayed Brownian motion $B[\mu]$ is called **trapped Brownian motion (TBM)**.

Next, we present some examples of trap measures and corresponding trapped random walks and Brownian motions.

Example 4.14 (Time-change of Brownian motion). *Let $\rho \in M(\mathbb{R})$, Leb_+ be the Lebesgue measure on $\mathbb{R}_+$. Define $\mu(\omega) := \rho \otimes \mathrm{Leb}_+$ for all $\omega \in \tilde{\Omega}$. Then μ is a (deterministic) Lévy trap measure.*

The μ-trapped Brownian motion is simply the process obtained from the Brownian motion B by the time change with the speed measure ρ. Indeed, this time change B^ρ is usually defined as

$$(B^\rho_t)_{t\geq 0} := (B_{\psi_\rho(t)})_{t\geq 0}. \tag{3.2}$$

for

$$\phi_\rho(s) := \int_{\mathbb{R}} \ell(s,x)\rho(dx), \qquad \text{and} \qquad \psi_\rho(t) := \inf\{s > 0 : \phi_\rho(s) > t\}. \tag{3.3}$$

By Fubini's theorem, it is easy to see that

$$\phi_\rho(s) = \int_{\mathbb{R}} \int_0^{\ell(s,x)} dy\rho(dx) = (\rho \otimes \mathrm{Leb}_+)(U_{\ell(\cdot,s)}) = \phi[\rho \otimes \mathrm{Leb}_+, B]_s. \tag{3.4}$$

This implies that B^ρ equals $B[\mu]$.

Example 4.15. *Let $k \in \mathbb{N} \cup \{\infty\}$ and $((S^i_t)_{t\geq 0})_{i<k}$ be a family of independent subordinators defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Let $(x_i)_{i<k}$ be real numbers. Then, it is immediate that $\sum_{i<k} \delta_{x_i} \otimes dS^i$ is a Lévy trap measure (ignoring the issue of the convergence of the sum for the moment).*

Example 4.16 ('Quenched' trap model). *Let $(\pi_z)_{z\in\mathbb{Z}}$ be a sequence of probability distributions over $\mathbb{R}_+$. For each $z \in \mathbb{Z}$, let $(s^z_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random variables defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and distributed according to π_z. We will also impose $(s^z_i)_{i\in\mathbb{N},z\in\mathbb{Z}}$ to be an independent family. Then, it is immediate that $\mu := \sum_{z\in\mathbb{Z},i\in\mathbb{N}} s^z_i \delta_{(z,i)}$ is a discrete trap measure with trapping landscape $(\pi_z)_{z\in\mathbb{Z}}$. The corresponding trapped random walk stays for time s^z_i at i-th visit of z.*

Example 4.17 (Montrol-Weiss continuous-time random walk). *Let $\pi_z = \pi_0$, $z \in \mathbb{Z}$, where π_0 satisfies $\lim_{u\to\infty} u^\gamma \pi_0[u,\infty) = 1$ for some $\gamma \in (0,1)$. Let μ be defined as in the previous example. Then $(s^z_i)_{i,z}$ are i.i.d., and the trapped random walk $Z[\mu]$ needs π_0-distributed time for every step. $Z[\mu]$ is thus an one-dimensional continuous-time random walk à la Montroll-Weiss* [**MW65**].

Example 4.18 (Markovian random walk on $\mathbb{Z}$). *This is a special case of Example 4.16. Let $(m_z)_{z\in\mathbb{Z}}$ a family of positive numbers, and π_z be an exponential distribution with mean m_z. Define μ as in Example 4.16. Then $Z[\mu]$ is a continuous time, symmetric and Markovian random walk with jump rate m_z^{-1} on site z. Clearly, a TRW is Markovian iff its trapping landscape is composed of exponential distributions.*

Example 4.19 (Fractional kinetics process). *Let $\mathcal{P} = (x_i, y_i, z_i)_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{H} \times \mathbb{R}_+$ defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with intensity measure*

$$\varrho = \gamma z^{-1-\gamma} dx\, dy\, dz, \qquad \gamma \in (0,1). \tag{3.5}$$

Define $\mu_{FK} : \tilde{\Omega} \to M_1(\mathbb{H})$ *as*

$$\mu_{FK} = \mu_{FK}^{\gamma} := \sum_i z_i \delta_{(x_i, y_i)}. \tag{3.6}$$

It is easy to see that for every compact set K, $\mu_{FK}(K)$ *has* γ*-stable distribution with the scaling parameter proportional to the Lebesgue measure of* K, *and is thus* $\tilde{\mathbb{P}}$*-a.s. finite. Further,* $\mu_{FK}(K_1)$ *and* $\mu_{FK}(K_2)$ *are independent when* K_1, K_2 *are disjoint, as* $\mathcal{P}$ *is a Poisson point process. In particular,* μ_{FK} *is a trap measure. For the same reasons,* $\mu_{FK}\langle V\rangle$ *is a stable Lévy process, and thus* μ_{FK} *is a Lévy trap measure.*

The trapped Brownian motion corresponding to this measure is the FK process.To see this, it is enough to show that the process $(\phi[\mu, B]_t)_{t\in\mathbb{R}_+}$ *is a* γ*-stable subordinator that is independent of* B.

This can be proved as follows. Fix a realization of the Brownian motion B. *Then its local time is also fixed. As* $\mathrm{Leb}(U_{\ell(\cdot,t)}) = t$ *and* $U_{\ell(\cdot,s)}$, *and* $U_{\ell(\cdot,t)} \setminus U_{\ell(\cdot,s)}$ *are disjoint for every* $s < t$, $\phi[\mu, B]_t$ *has* γ*-stable distribution with scaling parameter proportional to* t, *and* $\phi[\mu, B]_t - \phi[\mu, B]_s$ *is independent of* $\phi[\mu, B]_s$. *Hence, for every realisation of* B, $\phi[\mu, B]$ *is a* γ*-stable subordinator, and thus* $\phi[\mu, B]$ *is a* γ*-stable subordinator independent of* B.

3.2. Convergence of trapped processes. We now specialize Theorems 4.5, 4.6 to trapped processes. We start by presenting a criterion for the vague convergence in distribution of random measures. Let μ be a random measure. Denote

$$\mathcal{T}_\mu := \{A \in \mathcal{B}(\mathbb{H}) : \mu(\partial A) = 0 \ \tilde{\mathbb{P}}\text{-a.s.}\}. \tag{3.7}$$

By a **DC semiring** we shall mean a semiring $\mathcal{U} \subset \mathcal{B}(\mathbb{H})$ with the property that, for any given $B \in \mathcal{B}(\mathbb{H})$ bounded and any $\epsilon > 0$, there exist some finite cover of B by $\mathcal{U}$-sets of diameter less than ϵ. It is a known fact that

Proposition 4.20 (Theorem 4.2 of [**Kal83**]). *Let* μ *be a random measure and suppose that* $\mathcal{A}$ *is a DC semiring contained in* $\mathcal{T}_\mu$. *To prove vague convergence in distribution of random measures* μ^ϵ *to* μ *as* $\epsilon \to 0$, *it suffices to prove convergence in distribution of* $(\mu^\epsilon(A_i))_{i\le k}$ *to* $(\mu(A_i))_{i\le k}$ *as* $\epsilon \to 0$ *for every finite family* $(A_i)_{i\le k}$ *of bounded, pairwise disjoints sets in* $\mathcal{A}$.

As for a trap measure μ and for pairwise disjoint A_i's, $(\mu(A_i))_{i\le k}$ is an independent family of random variables, it is easy to see that the following proposition holds.

Proposition 4.21. *Let* μ^ε, μ *be trap measures and let* $\mathcal{A}$ *be a DC semiring contained in* $\mathcal{T}_\mu$. *If for every* $A \in \mathcal{A}$, $\mu^\epsilon(A)$ *converges in distribution to* $\mu(A)$, *then* μ^ϵ *converges vaguely in distribution to* μ.

This proposition can be further specialised when μ^ε, and μ are discrete, resp. Lévy trap measures.

Proposition 4.22. *(i) Let μ^ε, μ be Lévy trap measures. Then μ^ε converges vaguely in distribution to μ, iff $\mu^\varepsilon(I \times [0,1])$ converges in distribution to $\mu(I \times [0,1])$ for every compact interval $I = [a,b]$ such that $\mu(\{a,b\} \times \mathbb{R}_+) = 0$, $\tilde{\mathbb{P}}$-a.s.*

(ii) The same holds true if $\mu^\varepsilon = \mathfrak{S}_\varepsilon(\nu^\varepsilon)$ for some family of discrete trap measures ν^ε.

Proof. When μ is a Lévy trap measure, the distribution of $\mu([a,b] \times [c,d])$, $a, b \in \mathbb{R}$, $c, d \in \mathbb{R}_+$, is determined by the distribution of $\mu([a,b] \times [0,1])$, since by definition $\mu\langle[a,b]\rangle$ is a Lévy process. In particular, the assumptions of the proposition imply the convergence in distribution of $\mu^\varepsilon(A)$ to $\mu(A)$ for every $A \in \mathcal{A}$ where $\mathcal{A}$ is the set of all rectangles $I \times [c,d]$ with I as in the statement of the proposition and $d \ge c \ge 0$.

As $\mu\langle I\rangle$ is a Lévy process, we have $\mathcal{A} \subset \mathcal{T}_\mu$. Moreover, it is easy to see that $\mathcal{A}$ is a DC semiring. Proposition 4.21 then implies claim (i).

The proof of claim (ii) is analogous. It suffices to observe that the distribution of ν^ε is determined by distributions of $\mu^\varepsilon([a,b] \times [0,1])$, $a, b \in \mathbb{R}$, as well. $\square$

We apply this proposition in few examples.

Example 4.23 (Stone's theorem). *Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of positive, locally finite measures on $\mathbb{R}$. Suppose ρ_ε converges to $\rho \in M_1(\mathbb{R})$ as $\varepsilon \to 0$. Set $\mu_\varepsilon = \rho_\varepsilon \otimes \mathrm{Leb}_+$, $\mu = \rho \otimes \mathrm{Leb}_+$. We have seen in Example 4.14 that μ_ε and μ are Lévy trap measures, and that $B[\mu_\varepsilon]$ and $B[\mu]$ are a time changes of Brownian motion with speed measure ρ_ε and ρ, respectively. Let a, b be such that $\rho(\{a,b\}) = 0$ and thus $\mu(\{a,b\} \times \mathbb{R}_+) = 0$. By vague convergence of ρ_ε to ρ, $\mu_\varepsilon([a,b] \times [0,1]) \to \mu([a,b] \times [0,1])$. Therefore, by Proposition 4.22, μ_ε converges vaguely to μ, and thus, by Theorem 4.6, $B[\mu_\varepsilon]$ converges in distribution to $B[\mu]$ in $(D(0,\infty), M_1)$.*

This result is well known and was originally obtained (with the stronger J_1-topology) by Stone [**Sto63**]*. His result states that convergence of speed measures implies convergence of the corresponding time-changed Brownian motions. Thus, Theorem 4.6 can be viewed a generalization of Stone's result.*

Example 4.24. *μ, $Z[\mu]$ be as in Example 4.17 (a continuous-time random walk à la Montroll-Weiss). Then, using Theorem 4.5 and Proposition 4.22, we can prove that $(\varepsilon Z[\mu]_{\varepsilon^{-2/\gamma}t})_{t\ge0}$ converges in distribution to the FK process. (This result was previously obtained in* [**MS04**]*.)*

Indeed, let K_γ be a positive stable law of index γ. It is easy to see that μ is a discrete trap measure. Example 4.19 implies that FK process is a trapped Brownian motion whose corresponding trap measure μ_{FK} is Lévy. Moreover, from the fact that μ_{FK} is defined via Poisson point process whose intensity has no atoms, we see that for every $a \in \mathbb{R}$, $\mu_{FK}(a \times \mathbb{R}_+) = 0$, $\tilde{\mathbb{P}}$-a.s.

To apply Proposition 4.22 we should check that $\varepsilon^{2/\gamma}\mathfrak{S}_\varepsilon(\mu)([a,b]\times[0,1])$ *converges in distribution to* $(b-a)^{1/\gamma}K_\gamma$*. However,*

$$\varepsilon^{2/\gamma}\mathfrak{S}_\varepsilon(\mu)([a,b]\times[0,1]) = \varepsilon^2 \sum_{i=a\varepsilon^{-1}}^{b\varepsilon^{-1}} \sum_{j=1}^{\varepsilon^{-1}} s(i,j), \tag{3.8}$$

where, by their definition in Example 4.17, the $(s(i,j))_{i\in\mathbb{Z},j\in\mathbb{N}}$ *are i.i.d. random variables variables in the domain of attraction of the* γ*-stable law. The classical result on convergence of i.i.d. random variables (see e.g.* [**GK68**]*) yields that* (3.8) *converges in distribution to* $(b-a)^{1/\gamma}K_\gamma$*. The convergence of processes then follows from Theorem 4.5.*

3.3. Randomly trapped processes. The class of trapped random walks and Brownian motions is too small to include some processes that we want to consider, in particular Bouchaud's trap model, the FIN diffusion and the projections of the random walk on IIC, IPC. More precisely, quenched distributions of these models (given corresponding random environments) are trapped random walks. If we want to consider annealed distributions, we need to introduce a larger classes, *randomly trapped random walks* and *randomly trapped Brownian motion*. Their corresponding random measures will be constructed as random mixtures of trap measures.

First we recall some elementary results about random measures. Let μ be a random measure on $\mathbb{H}$ defined on a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $f:\mathbb{H}\to\mathbb{R}_+$ be a measurable function. We define Laplace transforms

$$L_\mu(f) = \tilde{\mathbb{E}}\Big[\exp\Big\{-\int_{\mathbb{H}} f(t)\mu(dt)\Big\}\Big]. \tag{3.9}$$

The following proposition is well known (see Lemma 1.7 of [**Kal83**]).

Proposition 4.25. *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and let* $(\mu_\omega)_{\omega\in\Omega}$ *be a family of random measures on* $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ *indexed by* $\omega\in\Omega$*. Then there exists a probability measure* $\mathscr{P}$ *on* $M(\mathbb{H})$ *given by (recall* (2.2) *for the notation)*

$$\mathscr{P}(A) = \int_\Omega P_{\mu_\omega}(A)\mathbb{P}(d\omega) \quad \textit{for each } A\in\mathcal{B}(M(\mathbb{H})) \tag{3.10}$$

if and only if the mapping $\omega\mapsto L_{\mu_\omega}(f)$ *is* $\mathcal{F}$*-measurable for each* $f\in C_0(\mathbb{H})$*. The random measure* $\mu:\Omega\times\tilde\Omega\to M(\mathbb{H})$ *given by* $\mu(\omega,\tilde\omega)=\mu_\omega(\tilde\omega)$ *whose distribution is* $\mathscr{P}$ *is called the* ***mixture of*** $(\mu_\omega)_{\omega\in\Omega}$ ***with respect to*** $\mathbb{P}$*.*

Definition 4.26 (Random trap measures). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $(\mu_\omega)_{\omega\in\Omega}$ be a family of trap measures such that $\omega\mapsto L_{\mu_\omega}(f)$ is $\mathcal{F}$-measurable for each $f\in C_0(\mathbb{H})$. Let μ be a random measure whose distribution is the mixture of $(\mu_\omega)_{\omega\in\Omega}$. Then we call μ a **random trap measure driven by the** $(\mu_\omega)_{\omega\in\Omega}$. The law of μ is called the annealed law of μ and the laws of the $(\mu_\omega)_{\omega\in\Omega}$ are referred as the quenched laws.

If for every $\omega \in \Omega$, μ_ω is discrete (resp. Lévy) trap measure, then μ is called **discrete** (resp. **Lévy**) **random trap measure**.

Remark 4.27. Obviously a trap measure is always a random trap measure, the converse might be false.

Definition 4.28 (Traping landscape). If μ is a discrete random trap measure driven by $(\mu_\omega)_{\omega\in\Omega}$, then $\boldsymbol{\pi} := (\pi_z)_{z\in\mathbb{Z}} : \Omega \to M_1(\mathbb{R}_+)^{\mathbb{Z}}$ defined via $\omega \in \Omega \mapsto \boldsymbol{\pi}(\mu_\omega)$ is called **random trapping landscape of** μ. We say that μ is a (discrete) random trap measure with random trapping landscape $\boldsymbol{\pi}$.

Let $\mathbf{P} = \mathbb{P}\circ\boldsymbol{\pi}^{-1}$ be the distribution of $\boldsymbol{\pi}$ on $M_1(\mathbb{R}_+)^{\mathbb{Z}}$. If $\mathbf{P}$ is a product measure, that is $\mathbf{P} = \bigotimes_{z\in\mathbb{Z}} P^z$ for some $P_z \in M_1(M_1(\mathbb{R}_+))$, $z \in \mathbb{Z}$, then the coordinates of the random trapping landscape $(\pi_z)_{z\in\mathbb{Z}}$ are independent. In this case we say that the random trapping landscape is **independent**. If $\mathbf{P} = \bigotimes_{z\in\mathbb{Z}} P$ for some $P \in M_1(M_1(\mathbb{R}_+))$, then the $(\pi_z)_{z\in\mathbb{Z}}$ are i.i.d., and we say the random trapping landscape is **i.i.d.**

The proof of the following proposition is obvious.

Proposition 4.29. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\boldsymbol{\pi} := (\pi_z)_{z\in\mathbb{Z}} : \Omega \to M_1(\mathbb{R}_+)^{\mathbb{Z}}$ be measurable. Then there exists a random discrete trap measure with random trapping landscape $\boldsymbol{\pi}$.*

We now define corresponding processes.

Definition 4.30 (Randomly trapped random walk/Brownian motion). When μ is a random discrete trap measure, we call the μ-delayed random walk $Z[\mu]$ **randomly trapped random walk (RTRW)**. Similarly, when μ is a random Lévy trap measure, the μ-delayed Brownian motion $B[\mu]$ is called **randomly trapped Brownian motion (RTBM)**.

We now present two important examples RTBM's.

Example 4.31 (FIN diffusion). *Let $\mathcal{P} = (x_i, v_i)_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{H}$ with intensity measure $\gamma dx v^{-1-\gamma} dv$, $\gamma \in (0,1)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$, let $\mu_\omega := \sum_{i\in\mathbb{N}} \delta_{x_i(\omega)} \otimes v_i(\omega)\,\mathrm{Leb}_+$. By Proposition 4.25, the mixture of $(\mu_\omega)_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$ exists and thus there exists a random trap measure μ_{FIN} driven by the $(\mu_\omega)_{\omega\in\Omega}$.*

Using Example 4.14, it is easy to see that $B[\mu_\omega]$ is a time change of B with speed measure $\rho(dx) = \sum_i v_i \delta_{x_i}(dx)$. Comparing this with the definition of FIN, we see that the RTBM corresponding to μ_{FIN}, $B[\mu_{FIN}]$, is a FIN diffusion.

Example 4.32 (generalised FIN diffusion). *Let $\mathfrak{F}^*$ be the set of Laplace exponents of subordinators, that is the set of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ that can be expressed as*

$$f(\lambda) = f_{\mathtt{d},\Pi}(\lambda) := \mathtt{d}\lambda + \int_{\mathbb{R}_+} (1 - e^{-\lambda t})\Pi(dt) \tag{3.11}$$

for a $\mathtt{d} \geq 0$ and a measure μ satisfying $\int_{(0,\infty)}(1 \wedge t)\Pi(dt) < \infty$. We endow $\mathfrak{F}^*$ with the topology of pointwise convergence.

Let $\mathbb{F}$ be a σ-finite measure on $\mathfrak{F}^*$, and let $(x_i, f_i)_{i\geq 0}$ be a Poisson point process on $\mathbb{R} \times \mathfrak{F}^*$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with intensity $dx \otimes \mathbb{F}$. Let $(S^i_t)_{t\geq 0}$, $i \geq 0$, be a family of independent subordinators, Laplace exponent of S^i being f_i, defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

For a given realisation of $(x_i, f_i)_{i\geq 0}$, we set similarly as in Example 4.15

$$\mu_{(x_i,f_i)}(dx\, dy) = \sum_{i\geq 0} \delta_{x_i}(dx) dS^i(dy), \tag{3.12}$$

Hence, the measure $\mu_{(x_i,f_i)}$ is a Lévy trap measure on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ (It might be locally infinite, but we ignore this issue for now.)

Using Proposition 4.25, we can show that the mixture of $(\mu_{(x_i(\omega),f_i(\omega))})_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$,

$$\mu_{GFIN}(\omega, \omega') := \mu_{(x_i(\omega),f_i(\omega))}(\tilde{\omega}) \tag{3.13}$$

is a random measure. Since $\mu_{(x_i,f_i)}$'s are Lévy trap measures, μ_{GFIN} is a random Lévy trap measure. The corresponding RTBM is called a ***generalised FIN diffusion*** *(GFIN).*

Remark 4.33. The FIN diffusion presented in Example 4.31 is of course a generalised FIN diffusion. Indeed, it suffices to choose $\mathbb{F}$ to be $\int_0^\infty \gamma v^{-1-\gamma} \delta_{\lambda\mapsto v\lambda}(df) dv$, where $\delta_{\lambda\mapsto v\lambda}(df)$ is a point measure on $\mathfrak{F}^*$ concentrated on the linear function $v\lambda$.

The following processes are important examples of RTRW.

Example 4.34 (Bouchaud trap model). *Symmetric one-dimensional* ***Bouchaud trap model*** *(BTM) is a symmetric continuous time random walk X on $\mathbb{Z}$ with random jump rates. More precisely, to each vertex z of $\mathbb{Z}$ we assign a positive number τ_z where $(\tau_z)_{z\in\mathbb{Z}}$ is an i.i.d. sequence of positive random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\lim_{u\to\infty} u^\gamma \mathbb{P}[\tau_z \geq u] = 1, \tag{3.14}$$

with $\gamma \in (0, 1)$. Each visit of X to $z \in \mathbb{Z}$ lasts an exponentially distributed time with mean τ_z.

It can be seen easily that the BTM is a RTRW. Its random trapping landscape is given by

$$\boldsymbol{\pi}(\omega) = (\nu_{\tau_z(\omega)})_{z\in\mathbb{Z}}, \tag{3.15}$$

where ν_a is the exponential distribution with mean a. As τ_x are i.i.d., the random trapping landscape $\boldsymbol{\pi}$ is i.i.d.

Example 4.35 (Incipient infinite cluster). *The incipient infinite cluster (IIC) in a regular tree is a random graph which is constructed as the limit when $n \to \infty$ of critical percolation clusters on a tree, conditioned on surviving up to level n. As shown by Kesten in* [**Kes86b**], *this random tree can be viewed* *as the graph $\mathbb{N}$ (called the backbone) in which, from each site $z \in \mathbb{N}$, there emerges a random branch $\mathcal{L}_z$, and $(\mathcal{L}_z)_{z\in\mathbb{N}}$ is an i.i.d. sequence of critical percolation clusters. Consider a simple, discrete time random walk $(Z_n^{IIC})_{n\in\mathbb{N}}$ on the IIC. Once Z^{IIC} is on vertex $x \in IIC$, it jumps to each of its neighbors with probability $\deg(x)^{-1}$, where $\deg(x)$ is the degree of x in IIC. We can construct a continuous time random walk on $\mathbb{N}$, denoted by $(W_t^{IIC})_{t\geq 0}$, by stating that $W_t^{IIC} = z$ for all $t \geq 0$ such that $Z_{\lfloor t\rfloor}^{IIC} \in \mathcal{L}_z$. That is, W^{IIC} is the projection of Z^{IIC} to the backbone $\mathbb{N}$. For each realization of the IIC, which is to say, each realization of $(\mathcal{L}_z)_{z\in\mathbb{N}}$, W^{IIC} is a TRW with trapping landscape $\boldsymbol{\nu} = (\nu_z)_{z\in\mathbb{Z}}$ where, for each $z \in \mathbb{N}$, ν_z is the distribution of the time that Z^{IIC} spends on a single visit to the branch $\mathcal{L}_z$, and $\nu_z := \delta_0$ for $z \in \mathbb{Z} \setminus \mathbb{N}$. As the branches $(\mathcal{L}_z)_{z\in\mathbb{N}}$ are random, we have that W^{IIC} is a RTRW. Furthermore, as the sequence of branches $(\mathcal{L}_z)_{z\in\mathbb{N}}$ is i.i.d., we see that the random trapping landscape $\boldsymbol{\nu} = (\nu_z)_{z\in\mathbb{N}}$ is i.i.d. (when restricted to $\mathbb{N}$).*

Example 4.36 (Invasion percolation cluster). *One can also use, instead of the IIC, the invasion percolation cluster (IPC) on a regular tree. As shown in* [**AGdHS08**]*, this random tree can also be viewed as $\mathbb{N}$ adorned with random branches. In this case the sequence of branches will be composed of subcritical percolation clusters (with a varying percolation parameter). As in the case of the IIC, we can construct a RTRW W^{IPC} as the projection of a simple random walk on the IPC to the backbone $\mathbb{N}$. In this case the random trapping landscape is neither independent nor identically distributed. The reason for this is that the sequence of branches are neither independent nor identically distributed (see* [**AGdHS08**]*).*

4. Convergence of RTRW to RTBM

Randomly trapped random walks and Brownian motions are the main subject of the rest of the chapter. In this section we study the convergence of RTRW's with i.i.d. trapping landscape. The first result, Theorem 4.37, gives a complete characterisation of the set of processes that appear as the scaling limit of such RTRW's. We then formulate criteria implying the convergence of RTRW's to several processes in this set. Here, however, our goal is not to characterise completely their domains of attraction. Instead of this we try to state natural criteria which can be easily checked in applications.

4.1. Set of limiting processes. We consider $P \in M_1(M_1(\mathbb{R}_+))$, and on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ an i.i.d. sequence $\boldsymbol{\pi} = (\pi_z)_{z\in\mathbb{Z}}$, $\pi_z \in M_1(\mathbb{R}_+)$ with marginal P. By Definition 4.28, $\boldsymbol{\pi}$ is i.i.d. trapping landscape. For every $\omega \in \Omega$ we define on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ an independent collection $(s_i^z(\omega))_{z\in\mathbb{Z},i\in\mathbb{N}}$ of non-negative random variables such that $\pi_z(\omega)$ is distribution of $s_z^i(\omega)$, and a discrete trap measure $\mu_\omega = \sum_{z,i} s_i^z(\omega)\delta_{z,i}$ as in Example 4.16. Finally, we let μ to stand for random trap measure obtained as a mixture of μ_ω's w.r.t. $\mathbb{P}$. To avoid the trivial situation we assume that P gives a mass to non-zero distributions,

$$P \neq \delta_{\delta_0} \tag{4.1}$$

The following theorem characterises the set of possible scaling limits of μ.

THEOREM 4.37. *Let μ be as above. Assume that there is an non-decreasing function ρ such that $\rho(\varepsilon)\mathfrak{S}_\varepsilon(\mu)$ converges vaguely in distribution on $(\Omega\times\tilde\Omega,\mathcal{F}\otimes\tilde{\mathcal{F}},\mathbb{P}\otimes\tilde{\mathbb{P}})$ as $\varepsilon\to 0$ to a random measure ν which is infinite, locally finite and non-zero. Then ν is random trap measure and one of two following possibilities occurs:*

(1) *$\rho(\varepsilon)=\varepsilon^2L(\varepsilon)$ for a function L slowly varying at 0, and $\nu=c\,\mathrm{Leb}_{\mathbb{H}}$, $c\in(0,\infty)$. In this case the RTRW $\varepsilon Z[\mu]_{\rho(\varepsilon)^{-1}t}$ converges to a Brownian motion.*

(2) *$\rho(\varepsilon)=\varepsilon^\alpha L(\varepsilon)$ for $\alpha>2$ and a function L slowly varying at 0, and ν can be written as*

$$\nu=c_1\mu_{FK}^{2/\alpha}+\mu_{GFIN},\tag{4.2}$$

where $c_1\in[0,\infty)$, $\mu_{FK}^{2/\alpha}$ is the trap measure corresponding to the FK process defined in Example 4.19, and μ_{GFIN} is the random trap measure of GFIN process given in Example 4.32, $\mu_{FK}^{2/\alpha}$ and μ_{GFIN} are mutually independent.

Moreover, the intensity measure $\mathbb{F}$ on the space of Laplace exponents determining μ_{GFIN} satisfies the scaling relation

$$\mathbb{F}(A)=aF(\sigma_\alpha^aA),\qquad \text{for every } A\in\mathcal{B}(\mathfrak{F}^*),a>0,\tag{4.3}$$

where $\sigma_\alpha^a:\mathfrak{F}^*\to\mathfrak{F}^*$ is map which maps $f_{\mathtt{d},\Pi}$ (see (3.11) for the notation) to $f_{\mathtt{d}_a^\alpha,\Pi_a^\alpha}$ with

$$\mathtt{d}_a^\alpha=a^{\alpha-1}\mathtt{d},\qquad \text{and}\qquad \Pi_a^\alpha(dv)=a^{-1}\Pi(dv/a^\alpha).\tag{4.4}$$

In this case the scaled RTRW $\varepsilon Z[\mu]_{\rho(\varepsilon)^{-1}t}$ converges to a process than can be viewed as a 'mixture' of FK and GFIN processes.

PROOF. The proof that $\rho(\varepsilon)$ must be a regularly varying function is standard: For $a>0$, $A\in\mathcal{B}(\mathbb{H})$ bounded, observe that $\mathfrak{S}_{\varepsilon a}(\mu)(aA)=\mathfrak{S}_\varepsilon(\mu)(A)$. Therefore,

$$\nu(A)=\lim_{\varepsilon\to0}\rho(\varepsilon)\mathfrak{S}_\varepsilon(\mu)(A)=\lim_{\varepsilon\to0}\frac{\rho(\varepsilon)}{\rho(a\varepsilon)}\rho(a\varepsilon)\mathfrak{S}_{a\varepsilon}(\mu)(aA)=\nu(aA)\lim_{\varepsilon\to0}\frac{\rho(\varepsilon)}{\rho(a\varepsilon)}.\tag{4.5}$$

As both $\nu(A)$ and $\nu(aA)$ are nontrivial random variables, this implies that the limit $\lim_{\varepsilon\to0}\frac{\rho(\varepsilon)}{\rho(a\varepsilon)}=c_k$ exists and is non-trivial. The theory of regularly varying functions then yields

$$\rho(\varepsilon)=\varepsilon^\alpha L(\varepsilon)\tag{4.6}$$

for $\alpha>0$ and a slowly varying function L. Inserting (4.6) into (4.5) also implies the scaling invariance of ν,

$$a^\alpha\nu(A)\overset{\text{law}}{=}\nu(aA),\qquad A\in\mathcal{B}(\mathbb{H}),a>0.\tag{4.7}$$

We now need to show that ν is as in (1) or (2). To this end we use the theory of 'random measures with symmetries' developed by Kallenberg in [**Kal05**]. We recall from [**Kal05**, Chapter 9.1] that random measure ξ on $\mathbb{H}$ is said separately exchangeable iff for any measure preserving transformations f_1 of $\mathbb{R}$ and f_2 of $\mathbb{R}_+$

$$\xi \circ (f_1 \otimes f_2)^{-1} \overset{\text{law}}{=} \xi. \tag{4.8}$$

Moreover by [**Kal05**, Proposition 9.1], to check separate exchangeability it is sufficient to restrict f_1, f_2 to transpositions of dyadic intervals in $\mathbb{R}$ or $\mathbb{R}_+$, respectively.

We claim that the limiting measure ν is separately exchangeable. Indeed, restricting ε to the sequence $\varepsilon_n = 2^{-n}$, taking $I_1, I_2 \subset \mathbb{R}$ and $J_1, J_2 \subset \mathbb{R}_+$ disjoint dyadic intervals of the same length and defining f_1, f_2 to be transposition of I_1, I_2, respectively J_1, J_2, it is easy to see, using the i.i.d. property of the trapping landscape $\boldsymbol{\pi}$ and independence of s_i^z's, that for all n large enough.

$$\rho(\varepsilon_n)\mathfrak{S}_{\varepsilon_n}(\mu) \circ (f_1 \otimes f_2)^{-1} \overset{\text{law}}{=} \rho(\varepsilon_n)\mathfrak{S}_{\varepsilon_n}(\mu). \tag{4.9}$$

Taking the limit $n \to \infty$ on both sides proves the separate exchangeability of ν.

The set of all separately exchangeable measures on $\mathbb{H}$ is known and given in [**Kal05**, Theorem 9.23] which we recall ([**Kal05**] treats exchangeable measures on the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$, the statement and proof however adapt easily to $\mathbb{H}$).

THEOREM 4.38. *A random measure ξ on $\mathbb{H}$ is separately exchangeable iff a.s.*

$$\begin{aligned} \xi = \gamma \operatorname{Leb}_{\mathbb{H}} &+ \sum_k l(\alpha, \eta_k)\delta_{\rho_k, \rho'_k} + \sum_{i,j} f(\alpha, \theta_i, \theta'_j, \zeta_{ij})\delta_{\tau_i, \tau'_j} \\ &+ \sum_{i,k} g(\alpha, \theta_i, \chi_{ik})\delta(\tau_i, \sigma_{ik}) + \sum_i h(\alpha, \theta_i)(\delta_{\tau_i} \otimes \operatorname{Leb}_+) \\ &+ \sum_{j,k} g'(\alpha, \theta'_j, \chi'_{jk})\delta(\sigma'_{jk}, \tau'_j) + \sum_j h'(\alpha, \theta'_j)(\operatorname{Leb} \otimes \delta_{\tau'_j}), \end{aligned} \tag{4.10}$$

for some measurable functions $f \geq 0$ on $\mathbb{R}_+^4$, $g, g' \geq 0$ on $\mathbb{R}_+^3$, and $h, h', l \geq 0$ on $\mathbb{R}_+^2$, an array of i.i.d. uniform random variables $(\zeta_{i,j})_{i,j\in\mathbb{N}}$, some independent unit rate Poisson processes $(\tau_j, \theta_j)_j$, $(\sigma'_{ij}, \chi'_{ij})_j$, $i \in \mathbb{N}$, on $\mathbb{H}$, $(\tau'_j, \theta'_j)_j$, $(\sigma_{ij}, \chi_{ij})_j$, $i \in \mathbb{N}$ on $\mathbb{R}_+^2$, and $(\rho_j, \rho'_j, \eta_j)_j$ on $\mathbb{H} \times \mathbb{R}_+$, and an independent pair of random variables $\alpha, \gamma \geq 0$.

Ignoring for the moment the issue of convergence of the above sum, let us describe in words various terms in (4.10) to make a link to our result. For this discussion, we ignore the random variable α and omit it from the notations (later we will justify this step).

The term $\sum_k l(\eta_k)\delta_{\rho_k, \rho'_k}$ has the same law as the random measure $\sum_k z_k \delta_{x_k, y_k}$ for a Poisson point process (x_k, y_k, z_k) on $\mathbb{H} \times \mathbb{R}_+$ with intensity $dx\, dy\, \Pi_l(dz)$ where the measure Π_l is given by

$$\Pi_l(A) = \operatorname{Leb}_+(l^{-1}(A)), \qquad A \in \mathcal{B}(\mathbb{R}_+). \tag{4.11}$$

Recalling Example 4.19, this term resembles to the random measure driving the FK process, the z-component of the intensity measure being more general here.

Similarly, terms $\sum_{i,k} g(\theta_i,\xi_{ik})\delta(\tau_i,\sigma_{ik})+\sum_i h(\theta_i)(\delta_{\tau_i}\otimes \mathrm{Leb}_+)$ can be interpreted as the random trap measure μ_{GFIN} defined in Example 4.32: τ_i's correspond to x_i's, and $f_i = f_{h(\theta_i),\Pi_{g(\theta_i,\cdot)}}$ (recall (3.11), (4.11) for the notation). The measure $\mathbb{F}$ is thus determined by functions h and g.

Terms with g', h' can be interpreted analogously, with the role of x-, and y-axis interchanged. Term $\gamma\,\mathrm{Leb}_{\mathbb{H}}$ will correspond to Brownian motion component of ν (recall Example 4.14). Finally, the term containing f can be viewed as a family of atoms placed on the grid $(\tau_i)_i\times(\tau'_j)_j$; we will not need it later.

We now explain why the limiting measure ν appearing in Theorem 4.37 is less general than (4.10). The first reason comes from the fact that the trapping landscape is i.i.d. This implies that ν is not only exchangeable in the x-direction, but also that for every disjoint sets A_1, $A_2\subset\mathbb{R}$ the processes $\nu\langle A_1\rangle$, $\nu\langle A_2\rangle$ are independent. As the consequence of this property, we see that α and γ must be a.s. constant (or f,h,h',g,g',l independent of α). We can thus omit α from the notation.

Further, this independence implies that $h'=g'=f\equiv 0$. Indeed, assume that it is not the case. Then it is easy to see that the processes $\nu\langle A_1\rangle$, $\nu\langle A_2\rangle$ have a non-zero probability to have a jump at the same time, for A_1,A_2 disjoint. On the other hand, for every ω fixed, $\nu\langle A_i\rangle(\omega)$, $i=1,2$, is a Lévy process (it is a limit of an i.i.d. sum), and therefore, for every ω, $\tilde{\mathbb{P}}$-a.s, they do not jump at the same time, contradicting the assumption.

From the previous reasoning implies that $\nu=\nu_1+\nu_2+\nu_3+\nu_4$ where $\nu_1,\dots,\nu_4$ are the Brownian, FK, FIN and GFIN component, respectively:

$$\begin{aligned} \nu_1 &= \gamma\,\mathrm{Leb}_{\mathbb{H}}, & \nu_3 &= \sum_i h(\theta_i)(\delta_{\tau_i}\otimes\mathrm{Leb}_+),\\ \nu_2 &= \sum_k l(\eta_k)\delta_{\rho_k,\rho'_k}, & \nu_4 &= \sum_{i,k} g(\theta_i,\xi_{ik})\delta(\tau_i,\sigma_{ik}). \end{aligned} \tag{4.12}$$

Observe that the functions l, g, and h are not determined uniquely by the law of ν. In particular for any measure preserving transformation f of $\mathbb{R}_+$, l and $l\circ f^{-1}$ give rise to the same law of ν, and similarly for h and $g(\theta,\cdot)$. Hence we may assume that l, h are non-increasing, and g is non-increasing in the second coordinate.

The final restriction of ν comes from its scaling invariance (4.7). To finish the proof, we should thus explore scaling properties of various components of ν.

The Brownian component ν_1 is trivial. It is scale-invariant with $\alpha=2$.

To find the conditions under which the FK component ν_2 is scale-invariant, we set $A=[0,x]\times[0,y]$ and compute the Laplace transform of $\nu_2 A$. To this end we use the formula

$$\mathbb{E}[e^{\pi f}] = \exp\big\{-\lambda(1-e^{-f})\big\}, \tag{4.13}$$

which holds for any Poisson point process π on a measurable space E with intensity measure $\lambda\in M(E)$ and $f:E\to\mathbb{R}$ measurable. Using this formula with $\pi=(\rho_i,\rho'_i,\eta_i)$ and $f(\rho,\rho',\eta)=\mathbf{1}_A(\rho,\rho')\lambda l(\eta)$ we obtain that

$$\mathbb{E}[e^{-\lambda\nu_2 A}] = \exp\Big\{ -xy\int_0^\infty (1-e^{-\lambda l(\eta)})d\eta\Big\}. \tag{4.14}$$

The scaling invariance (4.7) then yields

$$a^2\int_0^\infty (1-e^{-\lambda l(\eta)})d\eta = \int_0^\infty (1-e^{-\lambda a^\alpha l(\eta)})d\eta, \qquad \forall\lambda, a>0, \tag{4.15}$$

implying (together with the fact that l is non-increasing) that $l(\eta) = c\eta^{-\alpha/2}$, for a $c\geq 0$, $\alpha>0$. By [**Kal05**, Theorem 9.25], ν_2 is locally finite iff $\int_0^\infty (1\wedge l(\eta))d\eta < \infty$, yielding $\alpha > 2$. Finally, using the observation from the discussion around (4.11), we see that $\nu_2 = c\mu_{FK}^{2/\alpha}$.

The component ν_3 can be treated analogously. Using formula (4.14) with $\pi = (\tau_i,\theta_i)$ and $f = \lambda y h(\theta)\mathbf{1}_{[0,x]}(\tau)$, we obtain $\mathbb{E}[e^{-\lambda\nu_3 A}] = \exp\{-x\int_0^\infty (1-e^{-\lambda y h(v)})dv\}$. The scaling invariance and the fact that h is non-increasing then yields $h(\theta) = c\theta^{1-\alpha}$, for $c\geq 0$, $\alpha\geq 1$. Using [**?**, Theorem 9.25] again, ν_3 is locally finite iff $\int_0^\infty (1\wedge h(\theta))d\theta<\infty$, implying $\alpha>2$.

The component ν_4 is slightly more difficult as we need to deal with many Poisson point processes. Using formula (4.14) for the processes $(\sigma_{ij})_j$ and $(\chi_{ij})_j$ we get

$$\mathbb{E}[e^{-\lambda\nu_4 A}|(\theta_i),(\tau_i)] = \exp\Big\{-\sum_i \mathbf{1}_{[0,x]}(x_i) y\int_0^\infty (1-e^{-\lambda g(\theta_i,\chi)})d\chi\Big\}. \tag{4.16}$$

Applying (4.14) again, this time for processes (τ_i), (θ_i), then yields

$$\mathbb{E}[e^{-\lambda\nu_4 A}] = \exp\Big\{ -x\int_0^\infty \Big(1-e^{-y\int_0^\infty (1-e^{-\lambda g(\theta,\chi)})d\chi}\Big)d\theta\Big\}. \tag{4.17}$$

Hence, by scaling invariance and trivial substitutions, g should satisfy

$$\int_0^\infty \Big(1-e^{-y\int_0^\infty (1-e^{-\lambda g(\theta,\chi)})d\chi}\Big)d\theta = \int_0^\infty \Big(1-e^{-y\int_0^\infty (1-e^{-\lambda a^{-\alpha} g(\theta/a,\chi/a)})d\chi}\Big)d\theta \tag{4.18}$$

for every $a,y,\lambda>0$.

By [**Kal05**, Theorem 9.25] once more, ν_4 is locally finite iff

$$\int \Big\{1\wedge\int (1\wedge g(\theta,\chi))d\chi\Big\}d\theta < \infty. \tag{4.19}$$

We use this condition to show that for ν_4 the scaling exponent must satisfy $\alpha>2$. As $\alpha\geq 1$ is obvious, we should only exclude $\alpha\in[1,2]$. By (4.18) and the fact that Laplace transform determines measures on $\mathbb{R}_+$,

$$\mathrm{Leb}_+\Big\{\theta : \int (1-e^{-g(\theta,\chi)})d\chi \geq u\Big\} = \mathrm{Leb}_+\Big\{\theta : \int (1-e^{-a^{-\alpha} g(\theta/a,\chi/a)})d\chi\geq u\Big\}. \tag{4.20}$$

For some $c > 1$, $c^{-1}(1 \wedge x) \le 1 - e^{-x} \le 1 \wedge x$, therefore for $u \in (0,1)$

$$\begin{aligned}
K(u) := \mathrm{Leb}_+\{\theta : \int (1 \wedge g(\theta,\chi))d\chi \ge u\} \\
&\ge \mathrm{Leb}_+\{\theta : \int (1 - e^{-g(\theta,\chi)})d\chi \ge u\} \\
&= \mathrm{Leb}_+\{\theta : \int (1 - e^{-a^{-\alpha} g(\theta/a,\chi/a)})d\chi \ge u\} \\
&\ge a\,\mathrm{Leb}_+\{\theta : \int (a^{\alpha} \wedge g(\theta,\chi))d\chi \ge c a^{\alpha-1} u\} \\
&\ge u^{-1/(\alpha-1)}\,\mathrm{Leb}_+\{\theta : \int (1 \wedge g(\theta,\chi))d\chi \ge c\} = u^{-1/(\alpha-1)} K(c),
\end{aligned} \tag{4.21}$$

where for the last inequality we set $a \ge 1$ so that $a^{\alpha-1}u = 1$. Using (4.21), it can be checked easily that the integral over θ in (4.19) is not finite when $\alpha \in [1,2]$, implying $\alpha > 2$.

To complete the proof of Theorem 4.37, it remains to show the scaling relation (4.3). This is easy to be done using the correspondence of $\nu_3 + \nu_4$ and μ_{GFIN}. Indeed, let μ_{GFIN}, $\mu_{(x_i,f_i)}$ be as in Example 4.32. By scaling considerations,

$$a^{-\alpha} \mathfrak{S}_{a^{-1}} \mu_{(x_i,f_i)} \overset{\text{law}}{=} \mu_{(x_i/a, \sigma_a^{\alpha} f_i)}, \tag{4.22}$$

from which (4.3) follows immediately. □

4.2. Convergence to the Brownian motion. We now formulate several conditions implying the convergence of RTRW to the various processes appearing in Theorem 4.37. We recall that we do not want to describe their domains of attraction precisely, instead of this we look for easily applicable conditions.

We start by giving a general criteria for the convergence to a Brownian motion. Let μ be as in the previous section, that is μ is random trap measure with i.i.d. trapping landscape whose marginal is $P \in M_1(M_1(\mathbb{R}_+))$. For any probability measure $\pi \in M_1(\mathbb{R}_+)$ we define $m(\pi)$ to be its mean,

$$m(\pi) = \int_{\mathbb{R}_+} x \pi(dx). \tag{4.23}$$

THEOREM 4.39. *Let μ be as above. Assume that*

$$M := \int m(\pi) P(d\pi) < \infty. \tag{4.24}$$

Then, $\mathbb{P}$-a.s. as $\varepsilon \to 0$, the rescaled RTRW $(\varepsilon Z[\mu]_{M^{-1}\varepsilon^{-2}t})_{t \ge 0}$ converges to a standard Brownian motion in $\tilde{\mathbb{P}}$-distribution on the space $(D(0,\infty), M_1')$.

REMARK 4.40. (a) Observe that Theorem 4.39 is a quenched result: the convergence holds for $\mathbb{P}$-a.e. realisation of the trapping landscape $\boldsymbol{\pi}$.

(b) Since B is continuous and $B(0) = 0$, it is trivial to replace M_1'-topology by the usual J_1-topology.

PROOF. We use of the multidimensional individual ergodic theorem, which we state for the sake of completeness. (see e.g. [**Geo88**, Theorem 14.A5] where the theorem is proved for square domains, its proof works without modifications for rectangles).

THEOREM 4.41 (Multidimensional ergodic theorem). *Let $(X, \mathcal{G}, Q)$ be a probability space and $\Theta = (\theta_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be a group of Q preserving transformations on X such that $\theta_{(i_1,j_1)} \circ \theta_{(i_2,j_2)} = \theta_{(i_1+i_2,j_1+j_2)}$. Let $\mathcal{I}$ be the field of Θ-invariant sets, $a \le 0 < b$ and $c \le 0 < d$ be real numbers, and $\Delta_n = [\lfloor an \rfloor, \lfloor bn \rfloor] \times [\lfloor cn \rfloor, \lfloor dn \rfloor]$. Then, for any Q-measurable f with $Q(|f|) < \infty$*

$$\lim_{n\to\infty} \frac{1}{|\Delta_n|} \sum_{i\in\Delta_n} f \circ \theta_i = Q(f|\mathcal{I}), Q\text{-a.s.} \tag{4.25}$$

We use this theorem for $X = \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}}$, Q the distribution of $(s_z^i)_{z,i\in\mathbb{Z}}$ under $\mathbb{P}\otimes\tilde{\mathbb{P}}$, and $\mathcal{G}$ the cylinder field (here we extend s_i^z to negative i's in the natural way). We define $(\theta_{i,j})_{i,j\in\mathbb{Z}} : \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}}$ via $\theta_{x,j}((s(z,i))_{z,i\in\mathbb{Z}}) = (s(x+z, i+j))_{n,m\in\mathbb{Z}}$. It is clear from the construction that Q is stationary under $\theta_{x,j}$. As the trapping landscape and $(s_i^z)_i$, $z \in \mathbb{Z}$, are i.i.d., Q is ergodic with respect to every $\theta_{x,j}$ with $x \neq 0$. Hence, the invariant field is trivial. The multidimensional ergodic theorem then implies that for any two intervals $I, J \subset \mathbb{R}$

$$\frac{1}{n^2} \sum_{z:\frac{z}{n}\in I} \sum_{i:\frac{i}{n}\in J} s_i^z \xrightarrow{n\to\infty} |I||J|(\mathbb{E}\otimes\tilde{\mathbb{E}})[s_i^z] = |I||J|M, \qquad Q\text{-a.s.} \tag{4.26}$$

Therefore, $\varepsilon^2 \mathfrak{S}_\varepsilon(\mu)(I\times J) \to |I||J|M$, $\mathbb{P}\times\tilde{\mathbb{P}}$-a.s. This together with Theorem 4.5 completes the proof. □

4.3. Convergence to the FK process. We now deal with the convergence to the FK process. Let, as usual, μ be random trap measure with i.i.d. trapping landscape $\boldsymbol{\pi}$ whose marginal is P. We write π for a generic P-distributed probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$, $\hat{\pi}$ for its Laplace transform, and define

$$\Gamma(\varepsilon) := \mathbb{E}[1 - \hat{\pi}(\varepsilon)]. \tag{4.27}$$

Due to assumption (4.1), Γ is strictly increasing on $\mathbb{R}_+$, taking values in $[0, \Gamma_{max})$ for some $0 < \Gamma_{max} < 1$. Therefore, the inverse Γ^{-1} is well defined on this interval. For ε small enough, we can thus introduce the inverse time scale q_{FK} by

$$q_{FK}(\varepsilon) = \Gamma^{-1}(\varepsilon^2). \tag{4.28}$$

THEOREM 4.42. *Let μ be as above and assume that*

$$q_{FK}(\varepsilon) = \varepsilon^\alpha \tilde{L}(\varepsilon) \tag{4.29}$$

for some $\alpha > 2$ and a slowly varying function L. In addition assume that

$$\lim_{\varepsilon\to 0} \varepsilon^{-3} \mathbb{E}\big[\big(1 - \hat{\pi}(q_{FK}(\varepsilon))\big)^2\big] = 0. \tag{4.30}$$

Then, as $\varepsilon \to 0$, the rescaled RTRW $(\varepsilon Z[\mu]_{q_{FK}(\varepsilon)^{-1}t})_{t\geq 0}$ converges to the FK process with parameter $\gamma = 2/\alpha$, in $\tilde{\mathbb{P}}$-distribution on $(D(\mathbb{R}_+), M_1')$, in $\mathbb{P}$-probability.

In addition, if the '3' in (4.30) *is replaced by '4', then the convergence holds in $\tilde{\mathbb{P}}$-distribution, $\mathbb{P}$-a.s.*

REMARK 4.43. (a) To define the convergence in $\tilde{\mathbb{P}}$-distribution, in $\mathbb{P}$ probability, one should equip the space of probability measures on the space $D(\mathbb{R}_+)$ with a metric that is compatible with the M_1'-topology. We do not specify this metric, since to check such convergence we will use the fact that a family ξ_ε of elements of an arbitrary metric space E converges in probability to $\xi \in E$ as $\varepsilon \to 0$ iff for every sequence $\varepsilon_n \to 0$ there is a subsequence ε_{k_n} such that $\xi_{\varepsilon_{k_n}} \to \xi$ a.s. as $n \to \infty$ (see e.g. [**Kal02**, Lemma 4.2]).

(b) Due to (4.28), (4.29) is equivalent to

$$\Gamma(\varepsilon) = \varepsilon^{2/\gamma} \tilde{L}(\varepsilon), \tag{4.31}$$

for some slowly varying function $\tilde{L}$.

(c) As for Theorem 4.39, we can strengthen the M_1'- to J_1-topology, since the FK process has continuous trajectories and $FK(0) = 0$.

PROOF. As explained in the last remark, we fix a sequence $\varepsilon_n \to 0$ and choose subsequence $\varepsilon_{n_k} =: \tilde{\varepsilon}_k$ so that

$$\sum_{k=1}^{\infty} \tilde{\varepsilon}_k^{-3} \mathbb{E}\big[\big(1 - \hat{\pi}(q_{FK}(\tilde{\varepsilon}_k))\big)^2\big] < \infty. \tag{4.32}$$

To show the first claim of the theorem we need to check that $(\tilde{\varepsilon}_k Z[\mu]_{q_{FK}(\tilde{\varepsilon}_k)^{-1}t})_{t\geq 0}$ converges to the FK process with parameter γ, as $k \to \infty$, in $\tilde{\mathbb{P}}$-distribution, $\mathbb{P}$-a.s. By Theorem 4.5, we it is sufficient to show that $\mu_{\tilde{\varepsilon}_k} := q_{FK}(\tilde{\varepsilon}_k)\mathfrak{S}_{\tilde{\varepsilon}_k}(\mu)$ converges vaguely in distribution to μ_{FK}^{γ}, $\mathbb{P}$-a.s., where μ_{FK}^{γ} is the driving measure of the FK process introduced in Example 4.19. For every given $\omega \in \Omega$, $\mu = \mu(\omega, \tilde{\omega})$ is a trap measure (and not random trap measure), and μ_{FK}^{γ} is trap measure by definition. Therefore we can apply Proposition 4.21, and only check that for every rectangle $A = [x_1, x_2] \times [y_1, y_2]$ with rational coordinates, $\mathbb{P}$-a.s, $\mu_{\tilde{\varepsilon}_k}(A) \xrightarrow{k\to\infty} \mu_{FK}^{\gamma}(A)$ (it is easy to see that such rectangles form a DC semiring and are in $\mathcal{T}_{\mu_{FK}^{\gamma}}$). $\mu_{FK}^{\gamma}(A)$ has a γ-stable distribution with scaling parameter proportional to $\mathrm{Leb}_{\mathbb{H}}(A)$ and thus its Laplace exponent is $(x_2 - x_1)(y_2 - y_1)\lambda^{\gamma}$. The Laplace transform of $\mu_\varepsilon(A)$ given ω (and thus given the trapping landscape $(\pi_z)_{z\in\mathbb{Z}}$) is easy to compute. By the independence of s_i^z's,

$$\tilde{\mathbb{E}}[e^{-\lambda\mu_\varepsilon(A)}] = \prod_{x=x_1\varepsilon^{-1}}^{x_2\varepsilon^{-1}} \hat{\pi}_x(\lambda q_{FK}(\varepsilon))^{\varepsilon^{-1}(y_2-y_1)}. \tag{4.33}$$

Hence, taking the $-\log$ to obtain the Laplace exponent, we shall show that $\mathbb{P}$-a.s, for every $x_1 < x_2$, $y_1, y_2 \in \mathbb{Q}$, $0 \leq \lambda \in \mathbb{Q}$,

$$\tilde{\varepsilon}_k^{-1}(y_2 - y_1) \sum_{x=x_1\tilde{\varepsilon}_k^{-1}}^{x_2\tilde{\varepsilon}_k^{-1}} \big(-\log \hat{\pi}_z(\lambda q_{FK}(\tilde{\varepsilon}_k))\big) \xrightarrow{k\to\infty} (y_2 - y_1)(x_2 - x_1)\lambda^{\gamma}. \tag{4.34}$$

As $\mathbb{Q}$ is countable, it is sufficient to show this for fixed x's, y's and λ. This will follow by a standard law-of-large-numbers argument as π_z's are i.i.d. under $\mathbb{P}$. To simplify the notation we set $x_1 = 0$, $x_2 = 1$; y's can be omitted trivially.

We first consider $\lambda \le 1$ and truncate. Using the monotonicity of $\hat\pi$, $\lambda \le 1$, and the Chebyshev inequality

$$\mathbb{P}[\sup_{0\le z\le \tilde\varepsilon_k^{-1}} (1-\hat\pi(q_{FK}(\lambda\tilde\varepsilon_k))) \ge \tilde\varepsilon_k] \le \tilde\varepsilon_k^{-3}\mathbb{E}[(1-\hat\pi(q_{FK}(\tilde\varepsilon_k)))^2]. \tag{4.35}$$

(4.32) then implies that the above supremum is smaller than $\tilde\varepsilon_k$ for all k large enough, $\mathbb{P}$-a.s. Hence, for all k large

$$\tilde\varepsilon_k^{-1}\sum_{x=0}^{\tilde\varepsilon_k^{-1}}\big(-\log\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k))\big) = \tilde\varepsilon_k^{-1}\sum_{x=0}^{\tilde\varepsilon_k^{-1}}\big(-\log\big((1-\tilde\varepsilon_k)\vee\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k))\big)\big). \tag{4.36}$$

For any $\delta > 0$ there is ε small so that $(1-x) \le -\log x \le (1-x) + (\frac12+\delta)(1-x)^2$ on $(1-\varepsilon, 1]$. The expectation of the right-hand side of (4.36) is bounded from above by

$$\begin{aligned}\tilde\varepsilon_k^{-2}&\mathbb{E}[\tilde\varepsilon_k\wedge(1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k)))] + c\tilde\varepsilon_k^{-2}\mathbb{E}\big[\big(\tilde\varepsilon_k\wedge(1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k)))\big)^2\big]\\ &\le \tilde\varepsilon_k^{-2}\mathbb{E}[1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k))] + o(1),\end{aligned} \tag{4.37}$$

as $k\to\infty$, by (4.30). The lower bound for the expectation is then

$$\begin{aligned}&\tilde\varepsilon_k^{-2}\mathbb{E}[\tilde\varepsilon_k\wedge(1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k)))]\\ &\ge \tilde\varepsilon_k^{-2}\mathbb{E}[1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k))] + \tilde\varepsilon_k^{-2}\mathbb{P}[\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k)) \le 1-\tilde\varepsilon_k].\end{aligned} \tag{4.38}$$

The second term is again $o(1)$ as $k\to\infty$ by a similar estimate as in (4.35). Moreover,

$$\tilde\varepsilon_k^{-2}\mathbb{E}[1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k))] = \frac{\Gamma(\lambda q_{FK}(\tilde\varepsilon_k))}{\Gamma(q_{FK}(\tilde\varepsilon_k))} \xrightarrow{k\to\infty} \lambda^\gamma, \tag{4.39}$$

by the fact that Γ is regularly varying. Therefore the expectation of the right-hand side of (4.36) equals λ^γ.

To compute the variance of the right-hand side of (4.36), we observe that the second moment of one term is, for k large, bounded by

$$2\mathbb{E}\big[\big(\tilde\varepsilon_k\wedge(1-\hat\pi_z(\lambda q_{FK}(\tilde\varepsilon_k)))\big)^2\big] \le 2\mathbb{E}\big[\big(1-\hat\pi_z(q_{FK}(\tilde\varepsilon_k))\big)^2\big] = o(\tilde\varepsilon_k^3), \tag{4.40}$$

as $k\to\infty$, by (4.30). Since the first moment of one term is $O(\tilde\varepsilon_k^2)$, by the previous computation, we see that the variance of the right-hand side of (4.36) is bounded by

$$C\tilde\varepsilon_k^{-3}\mathbb{E}\big[\big(1-\hat\pi_z(q_{FK}(\tilde\varepsilon_k))\big)^2\big], \tag{4.41}$$

which is summable over k, by (4.32). This implies the strong law of large numbers for (4.36) and thus (4.34) for $\lambda \le 1$. For $\lambda \ge 1$ (4.34) follows from the analyticity of Laplace transform. This proves (4.34) and thus the first claim of the theorem.

To prove the second claim of the theorem, it is sufficient to repeat the previous argument with $\tilde{\varepsilon}_k = k^{-1+\frac{\delta}{2}}$. From the assumption of the theorem then follows that $\varepsilon^{-4-\delta}\mathbb{E}\big[\big(1-\hat{\pi}(q_{FK}(\varepsilon))\big)^2\big] = o(1)$, and thus

$$\tilde{\varepsilon}_k^{-3}\mathbb{E}\big[\big(1-\hat{\pi}(q_{FK}(\tilde{\varepsilon}_k))\big)^2\big] = o(\tilde{\varepsilon}_k^{1+\delta}) = o\big(k^{(1+\delta)(1-\frac{\delta}{2})}\big), \tag{4.42}$$

and hence (4.32) holds. Therefore $\mathbb{P}$-a.s. holds along $\tilde{\varepsilon}_k$. To pass from the convergence along $\tilde{\varepsilon}_k$ to the convergence as $\varepsilon \to 0$, it is sufficient to observe that, since $\tilde{\varepsilon}_{k+1}^{-1} - \tilde{\varepsilon}_k^{-}1 \xrightarrow{k\to\infty} 0$, for any rectangle A and ε small enough there is k such that $\mathfrak{S}_\varepsilon(\mu)(A) = \mathfrak{S}_{\tilde{\varepsilon}_k}(\mu)(A)$.

$\square$

4.4. Convergence to the GFIN process. This section contains the statement of the main theorem of this chapter, which gives conditions under which the scaling limit of a RTRW is certain RTBM. We start with some notation. Let $\mathbf{P}$ be a probability measure on $M_1(\mathbb{R}_+)^{\mathbb{Z}}$. We will suppose that $\mathbf{P}((\delta_0)_{z\in\mathbb{Z}}) \neq 1$. Let $\boldsymbol{\pi} := (\pi_z)_{z\in\mathbb{Z}}$ be a random trapping landscape distributed according to $\mathbf{P}$. We will be looking for the scaling limit of the RTRW $Z[\mu]$, where μ is a random trap measure with random trapping landscape $\boldsymbol{\pi}$. Next we state the assumptions needed to prove our main theorem.

Assumption (**I**). $\mathbf{P}$ is a product measure, that is $\mathbf{P} = \bigotimes_{z\in\mathbb{Z}} P^z$ where, for each $z \in \mathbb{Z}$, P^z is a probability measure over $M_1(\mathbb{R}_+)$. (This assumption ensures that the random trapping landscape $Z[\mu]$ is independent).

Let $m : M_1(\mathbb{R}_+) \to \mathbb{R}_+ \cup \{\infty\}$ be given by $m(\nu) := \int_{\mathbb{R}_+} t\nu(dt)$, so $m(\nu)$ is the expectation of a random variable having distribution ν. For any $\boldsymbol{\nu} := (\nu_z)_{z\in\mathbb{Z}} \in M_1(\mathbb{R}_+)^{\mathbb{Z}}$ we define the atomic measure $p[\boldsymbol{\nu}]$ as $p[\boldsymbol{\nu}] := \sum_{z\in\mathbb{Z}} m(\nu_z)\delta_z$. We define a random process $(V_x)_{x\in\mathbb{R}}$ as $V_x := p[\boldsymbol{\pi}][0,x]$ if $x \geq 0$ and $V_x := -p[\boldsymbol{\pi}][x,0)$ if $x < 0$. As we have assumed that the $(\pi_z)_{z\in\mathbb{Z}}$ are independent, we will have that $(V_x)_{x\in\mathbb{R}}$ has independent increments. Our second assumption is

Assumption (**PP**). There exists $\gamma \in (0,1)$ and a pure jump, non decreasing stochastic process with independent increments V^0 such that

$$(V_x^\varepsilon)_{x\in\mathbb{R}} := (\varepsilon^{1/\gamma} V_{\varepsilon^{-1}x})_{x\in\mathbb{R}} \overset{\varepsilon\to 0}{\to} (V_x^0)_{x\in\mathbb{R}} \tag{4.43}$$

in distribution over $(D(\mathbb{R}), J_1)$.

We define the random Lebesgue-Stieltjes measure associated with V^0, denoted by ρ^0, as $\rho[a,b) := V_b^0 - V_a^0$ for all a,b real numbers. As V^0 is a pure jump process we will have that ρ^0 is a purely atomic measure, i.e. $\rho^0 := \sum_{i\in\mathbb{N}} v_i\delta_{x_i}$. We can define a random, countable subset of $\mathbb{H}$ as $\mathcal{P}^0 := (x_i, v_i)_{i\in\mathbb{N}}$. The independence of the increments of V^0 implies that $\mathcal{P}^0$ is a Poisson point process. If $(P^z)_{z\in\mathbb{Z}}$ is not only independent, but also identically distributed, assumption (PP) follows from

Assumption (**PP0**). There exists $\gamma \in (0,1)$ such that

$$\lim_{x\to\infty} x^{\gamma} P^0[\pi \in M_1(\mathbb{R}_+) : m(\pi) > x] = 1. \tag{4.44}$$

In this case the process V^0 is a γ-stable subordinator and the measure $\mathcal{P}^0$ is the Poisson point process with intensity measure $\gamma y^{-1-\gamma} dxdy$ (see [**App09**]).

Next, we prepare the statement of the third and final assumption. It is easy to show that, for each $z \in \mathbb{Z}$, there exists a regular conditional probability $\Lambda^z : \mathbb{R}_+ \times \mathcal{B}(M_1(\mathbb{R}_+)) \to \mathbb{R}_+$ of P^z given m. In fact, there might exist more than one version of the regular conditional probability of P^z given m. The probability distributions $\Lambda^z(a,\cdot)$ corresponds to the distribution of π_z conditioned on having expectation a. Let $d_0(\varepsilon) := \varepsilon^{-1/\gamma}$ and $q_0(\varepsilon) := \varepsilon^{(1+\gamma)/\gamma}$. Let $C(\mathbb{R}_+)$ denote the set of continuous functions defined over $\mathbb{R}_+$.We define also $\Psi_\varepsilon : M_1(\mathbb{R}_+) \to C(\mathbb{R}_+)$

$$\Psi_\varepsilon(\nu)(\lambda) := \varepsilon^{-1}(1 - \hat{\nu}(q_0(\varepsilon)\lambda)). \tag{4.45}$$

Our third assumption is

Assumption (**D**). There exist versions $(\Lambda^z)_{z\in\mathbb{Z}}$ of the regular conditional probabilities of P^z given m such that, if π_z^a is a random object having distribution $\Lambda^z(a,\cdot)$, then for each family of integers $(z_\varepsilon)_{\varepsilon\in\mathbb{R}_+}$ we have that

$$\lim_{\varepsilon\to\infty} \Psi_\varepsilon(\pi_{z_\varepsilon}^{d_0(\varepsilon)}) \xrightarrow{d} \mathbb{F}_0 \tag{4.46}$$

where $\mathbb{F}_0$ is any probability distribution over $C(\mathbb{R}_+)$ and $\xrightarrow{d}$ denotes convergence in distribution in $C(\mathbb{R}_+)$ with the topology of pointwise convergence.

Denote by $\mathfrak{F}$ the set of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that allow the expression in display (3.11), where μ is a measure such that $\int_{\mathbb{R}_+} t\mu(dt) = 1$ and v_f, w_f are positive numbers such that $v_f + w_f \leq 1$. Note that $\mathfrak{F}$ is the set of Laplace exponents of subordinators $(S_t)_{t\in\mathbb{R}}$ such that $\mathbb{E}[S_1] \leq 1$. We will endow $\mathfrak{F}$ with the topology of pointwise convergence. It is easy to see that the support of $\mathbb{F}_0$ must be contained in $\mathfrak{F}$.

If $(P^z)_{z\in\mathbb{Z}}$ is not only independent, but also identically distributed, Assumption (D) reduces to

Assumption (**D0**). There exist a version Λ^0 of the regular conditional probability of P^0 given m such that, if π_0^a is a random object having distribution $\Lambda^0(a,\cdot)$, then we have that

$$\lim_{\varepsilon\to\infty} \Psi_\varepsilon(\pi_0^{d_0(\varepsilon)}) \xrightarrow{d} \mathbb{F}_0 \tag{4.47}$$

where $\mathbb{F}_0$ is any probability distribution over $C(\mathbb{R}_+)$ and $\xrightarrow{d}$ denotes convergence in distribution in $C(\mathbb{R}_+)$ with the topology of pointwise convergence.

With the assumptions stated, we now deal with the construction of the RTBM which will appear as scaling limit for the RTRW $Z[\mu]$. All we have to do is to construct the proper random trap measure. Let

$(\Omega, \mathcal{F}, \mathbb{P})$ be the space in which the Poisson point process $\mathcal{P}^0$ is defined. Let $(f_i)_{i\in\mathbb{N}}$, $f_i : \Omega \to \mathfrak{F}$, be an i.i.d. sequence of random objects taking values on $\mathfrak{F}$, distributed according to $\mathbb{F}_0$ and independent of $\mathcal{P}^0$. Define $\tilde{f}_i(\omega)$ as $\tilde{f}_i(\omega)(\lambda) = v_i^{-\gamma}(\omega) f_i(\omega)(v_i(\omega)^{\gamma+1}\lambda)$. For each $\omega \in \Omega$, let $(\Lambda^{(i,\omega)})_{i\in\mathbb{N}}$ be an independent sequence of subordinators, each one having Laplace exponent $\tilde{f}_i(\omega)$. The space where the sequence $(\Lambda^{(i,\omega)})_{i\in\mathbb{N}}$ is defined is of no importance. For each $\omega \in \Omega$ we define the random trap measure

$$\mu^0_\omega := \sum_{i\in\mathbb{Z}} \delta_{x_i(\omega)} \otimes d\Lambda^{(i,\omega)}. \tag{4.48}$$

Let μ be a random trap measure driven by the $(\mu^0_\omega)_{\omega\in\Omega}$. Let B be a Brownian motion defined in the same space that μ^0 and independent of it. Then, the RTBM $B[\mu^0]$ is then process we desire.

Let $\delta_{\mathbf{0}}$ denote the probability distribution over $C(\mathbb{R}_+)$ concentrated on the function $\mathbf{0} \equiv 0$. Now we are in conditions to state the main theorem of this chapter.

THEOREM 4.44. *Under assumptions* $(I), (PP)$ *and* (D)*. Suppose further that* $\mathbb{F}_0 \neq \delta_{\mathbf{0}}$ *we have that* $(\varepsilon Z[\mu]_{q_0(\varepsilon)^{-1}t})_{t>0}$ *converges in distribution to the RTBM* $(B[\mu^0]_t)_{t>0}$ *in* $(D((0,\infty)), M_1)$ *as* $\varepsilon \to \infty$.

REMARK 4.45. In the case where the $(P^z)_{z\in\mathbb{Z}}$ are i.i.d., Theorem 4.44 follows from assumptions (PP0) and (D0).

Here we present the proof for Theorem 4.44, which strongly relies in the following

LEMMA 4.46. *There exists a probability space* $(\Omega^\star, \mathcal{F}^\star, \mathbb{P}^\star)$ *and* $(\bar{\mu}^\varepsilon_{\omega^\star})_{\varepsilon\geq 0, \omega^\star\in\Omega^\star}$*, a family of trap measures, such that*

(1) *for each* $\varepsilon > 0$*, and* $\omega^\star \in \Omega^\star$*,* $\bar{\mu}^\varepsilon_{\omega^\star}$ *is a discrete trap measure*
(2) *Let, for each* $\varepsilon \geq 0$*,* $\bar{\mu}^\varepsilon$ *be a random trapping landscape driven by* $(\bar{\mu}^\varepsilon_{\omega^\star})_{\omega^\star\in\Omega^\star}$*. Then, for each* $\varepsilon > 0$*,* $\bar{\mu}^\varepsilon$ *is distributed as* μ*.*
(3) $\bar{\mu}^0$ *is distributed as* μ^0
(4) $q_0(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}^\varepsilon_{\omega^\star})$ *converges vaguely in distribution to* $\bar{\mu}^0_{\omega^\star}$ *as* $\varepsilon \to 0$*,* $\mathbb{P}^\star$*-almost surely.*

We now prove Theorem 4.44 using Lemma 4.46 : By item 2 in Lemma 4.46, we have that for all $\varepsilon > 0$, $Z[\mu]$ is distributed as $Z[\bar{\mu}^\varepsilon]$ in $D(\mathbb{R}_+)$. As μ^0_ω is $\mathbb{P}$-almost surely a Lévy trap measure, we have that μ^0_ω is a dispersed trap measure. Thus μ^0 is a dispersed trap measure and hence $\bar{\mu}^0$ is also a dispersed trap measure. It is easy to see that the $(\bar{\mu}^\varepsilon)_{\varepsilon>0}$ and $\bar{\mu}^0$ are infinite trap measures. Items 2,3 and 4 of 4.46 imply that $q_0(\varepsilon)\mathfrak{S}_\varepsilon(\mu)$ converges vaguely in distribution to μ^0. Thus we can apply Theorem 4.5 and to deduce convergence in distribution of $(\varepsilon Z[\mu]_{q_0(\varepsilon)^{-1}t})_{t\geq 0}$ to $(B[\mu^0]_t)_{t\geq 0}$.

Hence, if we prove Lemma 4.46 we would have proved Theorem 4.44. We split the proof of the lemma in two parts. In Subsection 4.4.1 we construct the coupling and then, in Subsection 4.4.2 we show $\mathbb{P}^\star$-almost sure convergence.

4.4.1. *Construction of the coupling.* We recall that $p[\boldsymbol{\pi}] := \sum_{z\in\mathbb{Z}} m(\pi_z)\delta_z$ and that $(V_x)_{x\in\mathbb{R}}$ is a random process defined as $V_x := p[\boldsymbol{\pi}][0,x)$ if $t \geq 0$ and $V_x := -p[\boldsymbol{\pi}][x,0)$ if $x < 0$. By assumption (PP), we have that $(\varepsilon^{1/\gamma}V_{\varepsilon^{-1}x})_{x\in\mathbb{R}}$ converges in distribution to $(V_x^0)_{x\in\mathbb{R}}$ in $(D(\mathbb{R}), J_1)$. But $(D(\mathbb{R}, J_1))$ is a separable topological space. Thus, by the Skorohod representation theorem there exists a family of stochastic processes $(\bar{V}^\varepsilon)_{\varepsilon>0}$ and a stochastic process $\bar{V}^0$ defined in a common probability space $(\Omega^\bullet, \mathcal{F}^\bullet, \mathbb{P}^\bullet)$ such that

(1) for each $\varepsilon > 0$, $\bar{V}^\varepsilon$ is distributed as V^ε in $(D(\mathbb{R}), J_1)$
(2) $\bar{V}^0$ is distributed as V^0 in $(D(\mathbb{R}), J_1)$
(3) $\bar{V}^\varepsilon$ converges $\mathbb{P}^\bullet$-almost surely to $\bar{V}^0$ in $(D(\mathbb{R}), J_1)$.

Let $(\bar{\rho}^\varepsilon)_{\varepsilon>0}$ and $\bar{\rho}$ be the random Stieltjes measures associated to $(\bar{V}^\varepsilon)_{\varepsilon>0}$ and $\bar{V}^0$ respectively. As we know, the $(\bar{V}^\varepsilon)_{\varepsilon>0}$ and $\bar{V}^0$ are non-decreasing, pure-jump processes. Thus $(\bar{\rho}^\varepsilon)_{\varepsilon>0}$ and $\bar{\rho}$ are random atomic measures and we can write $\bar{\rho}^\varepsilon = \sum_{z\in\mathbb{Z}} \bar{y}_z^\varepsilon \delta_{\varepsilon z}$ and $\bar{\rho} = \sum_{z\in\mathbb{Z}} \bar{y}_i \delta_{\bar{x}_i}$ with $\bar{x}_i \neq \bar{x}_j$ whenever $i \neq j$, also $\bar{y}_i > 0$ for all $i \in \mathbb{N}$.

Let us fix $\omega^\bullet \in \Omega^\bullet$. By the matching of jumps property of the Skorohod J_1 topology (see, for instance, §3.3 in [**Whi02**]) we have that, for each $i \in \mathbb{N}$, there exists a family $(w_\varepsilon^i)_{\varepsilon>0} \subset \mathbb{R}$ such that $w_\varepsilon^i \overset{\varepsilon\to 0}{\to} \bar{x}_i$ and $\bar{\rho}^\varepsilon(\{w_\varepsilon^i\}) \overset{\varepsilon\to 0}{\to} \bar{y}_i$. We have that, for each $\varepsilon > 0$, the measure ρ^ε is supported on $\varepsilon\mathbb{Z}$. Thus, without loss of generality, we can suppose that the $(w_\varepsilon^i)_{\varepsilon>0}$ is contained in $\varepsilon\mathbb{Z}$. Define $z_\varepsilon^i := \varepsilon^{-1}w_\varepsilon^i$. Using that $w_\varepsilon^1 \overset{\varepsilon\to 0}{\to} \bar{x}_1$, $w_\varepsilon^2 \overset{\varepsilon\to 0}{\to} \bar{x}_2$ and the fact that $\bar{x}_1 \neq \bar{x}_2$, we have that there exist a positive number c_2 small enough such that $(z_\varepsilon^1)_{\varepsilon>0} \bigcap (z_\varepsilon^2)_{0<\varepsilon<c_2} = \emptyset$. The same reasoning can be used to find c_3 small enough such that $(z_\varepsilon^1)_{\varepsilon>0} \bigcap (z_\varepsilon^2)_{0<\varepsilon<c_2} \bigcap (z_\varepsilon^3)_{0<\varepsilon<c_3} = \emptyset$. Iterating this procedure we find a sequence $(c_i)_{i\in\mathbb{N}}$ such that $\bigcap_{i\in\mathbb{N}} (z_\varepsilon^i)_{0<\varepsilon<c_i} = \emptyset$ (define $c_1 = \infty$). Thus, for each $\varepsilon > 0$ and $z \in \mathbb{Z}$, there is at most one i such that $z \in (z_\varepsilon^i)_{0<\varepsilon<c_i}$. Using that fact, for each $\varepsilon > 0$ we can define the function $I^\varepsilon : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ by stating that $I^\varepsilon(z)$ is the index such that $z \in (z_\varepsilon^{I^\varepsilon(z)})_{0<\varepsilon<c_{I^\varepsilon(z)}}$. If there is no such index, we state that $I^\varepsilon(z) = 0$.

J_1 topology (see, for instance, §3.3 in [**Whi02**]) we will also have that $\bar{y}^\varepsilon_{k_\varepsilon(i)} \to \bar{y}_i$.

Let $T_y : \mathfrak{F} \to \mathfrak{F}$ be defined by $T_y(f)(\cdot) := y^{-\gamma} f(y^{1+\gamma}\cdot)$. Let $\mathbb{F}_0^y$ be the probability measure over $\mathfrak{F}$ defined by $\mathbb{F}_0^y(A) := \mathbb{F}_0(T_y^{-1}(A))$ for all $A \in \mathcal{B}(\mathfrak{F})$. We want to prove the following

LEMMA 4.47. *let $(y^\varepsilon)_{\varepsilon>0}$ such that $y^\varepsilon \to y$ as $\varepsilon \to 0$ and $(z^\varepsilon)_{\varepsilon>0} \subset \mathbb{Z}$. Then $\Psi_\varepsilon(\pi_{z^\varepsilon}^{d_0(\varepsilon)y^\varepsilon}) \overset{d}{\to} \mathbb{F}_0^y$ as $\varepsilon \to 0$, where $\overset{d}{\to}$ denotes convergence in distribution in the topology given by pointwise convergence.*

PROOF. Let $t_0(\varepsilon) := (d_0(\varepsilon)y^\varepsilon)^{-\gamma}$, that is $t_0(\varepsilon) := \varepsilon(y^\varepsilon)^{-\gamma}$. Then

$$\begin{aligned}\Psi_\varepsilon(\pi_{z^\varepsilon}^{d_0(\varepsilon)y^\varepsilon}) &= \varepsilon^{-1}(1 - \int_{\mathbb{R}_+} \exp(q_0(\varepsilon)\lambda t)\pi_{z^\varepsilon}^{d_0(\varepsilon)y^\varepsilon}(dt)) \\ &= \varepsilon^{-1}(1 - \int_{\mathbb{R}_+} \exp(\varepsilon^{\frac{1+\gamma}{\gamma}}\lambda t)\pi_{z^\varepsilon}^{t_0(\varepsilon)^{-1/\gamma}}(dt))\end{aligned}$$

$$= (y^\varepsilon)^{-\gamma} t_0(\varepsilon)^{-1}(1 - \int_{\mathbb{R}_+} \exp((y^\varepsilon)^{1+\gamma}) q_0(t_0(\varepsilon))\lambda t) \pi_{z^\varepsilon}^{t_0(\varepsilon)^{-1/\gamma}}(dt))$$
$$= T_{y^\varepsilon}(\Psi_{t_0(\varepsilon)}(\pi_{z^\varepsilon}^{d_0(t_0(\varepsilon))})) = T_{y^\varepsilon y^{-1}}(T_y(\Psi_{t_0(\varepsilon)}(\pi_{z^\varepsilon}^{d_0(t_0(\varepsilon))}))).$$

We can show that $T_{y^\varepsilon y^{-1}}$ converges to the identity as $\varepsilon \to 0$. That plus assumption (D) proves the lemma. $\square$

Using Lemma 4.47 it follows that for each $\omega \in \Omega^\bullet$ and each atom $\bar{y}_i \delta_{\bar{x}_i}$ we have that

$$\Psi_\varepsilon(\pi_{z_\varepsilon^i}^{d_0(\varepsilon)\bar{y}^\varepsilon_{z_\varepsilon^i}}) \xrightarrow{d} \mathbb{F}_0^{\bar{y}_i} \text{ as } \varepsilon \to 0 \tag{4.49}$$

where $\xrightarrow{d}$ denotes convergence in distribution on $\mathfrak{F}^*$ endowed with the pointwise convergence.

The space $C(\mathbb{R}_+)$ endowed with the topology of uniform convergence over compact sets is separable. Thus $\mathfrak{F}^*$ endowed with the topology of uniform convergence in compact sets is also separable. It is a known fact that in the space $\mathfrak{F}^*$, pointwise convergence and uniform convergence over compact sets coincide. We deduce that $\mathfrak{F}^*$ with the topology of pointwise convergence is also separable, and we can apply the Skorohod representation theorem on $\mathfrak{F}^*$. That, plus display (4.49) imply that

LEMMA 4.48. *for each $\omega \in \Omega^\bullet$ and $i \in \mathbb{N}$, there exist a family $(\tilde{\Psi}_\varepsilon^{(i,\omega)})_{0<\varepsilon<c_i}$ and $\tilde{\Psi}^{(i,\omega)}$, random objects taking values in $\mathfrak{F}^*$, defined on a common probability space $(\tilde{\Omega}_{(i,\omega)}, \tilde{\mathcal{F}}_{(i,\omega)}, \tilde{\mathbb{P}}_{(i,\omega)})$ such that*

(1) *for each $\varepsilon > 0$, $\tilde{\Psi}_\varepsilon^{(i,\omega)}$ is distributed as $\Psi_\varepsilon(\pi_{z_\varepsilon^i}^{d_0(\varepsilon)\bar{y}^\varepsilon_{z_\varepsilon^i}})$ in $\mathfrak{F}^*$*
(2) *$\tilde{\Psi}^{(i,\omega)}$ has distribution $\mathbb{F}_0^{\bar{y}_i(\omega)}$*
(3) *$\tilde{\Psi}_\varepsilon^{(i,\omega)} \xrightarrow{u} \tilde{\Psi}^{(i,\omega)}$ as $\varepsilon \to 0$, $\tilde{\mathbb{P}}_{(i,\omega)}$ almost surely, where $\xrightarrow{u}$ denotes uniform convergence.*

For each $\omega^\bullet \in \Omega^\bullet, 0 < \varepsilon < c_i$ and $i \in \mathbb{N}$, we have that $\tilde{\Psi}_\varepsilon^{(i,\omega^\bullet)} \in \mathfrak{F}^*$, $\tilde{\mathbb{P}}_{(i,\omega^\bullet)}$-almost surely. That implies that there exists a random object $\bar{\pi}^{(\omega^\bullet,i,\varepsilon)} : \tilde{\Omega}_{(i,\omega^\bullet)} \to M_1(\mathbb{R}_+)$ such that, for $\tilde{\mathbb{P}}_{(i,\omega^\bullet)}$-almost every $\omega \in \tilde{\Omega}_{(i,\omega^\bullet)}$, we have

$$\tilde{\Psi}_\varepsilon^{(i,\omega^\bullet)}(\omega) = \Psi_\varepsilon(\bar{\pi}^{(\omega^\bullet,i,\varepsilon)}(\omega)) \tag{4.50}$$

Then $\bar{\pi}^{(\omega^\bullet,i,\varepsilon)}$ is distributed as $\pi_{z_\varepsilon^i}^{d_0(\varepsilon)\bar{y}^\varepsilon_{z_\varepsilon^i}}$. Next we define, for each $\omega^\bullet \in \omega^\bullet$, $\varepsilon > 0$ and $i \in \mathbb{N}$ a random element $\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}$ taking values in $M_1(\mathbb{R}_+)$ and defined on the space $(\tilde{\Omega}_{(i,\omega)}, \tilde{\mathcal{F}}_{(i,\omega)}, \tilde{\mathbb{P}}_{(i,\omega)})$. If $I^\varepsilon(z) > 0$, then $\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)} = \bar{\pi}^{(\omega^\bullet,I^\varepsilon(z),\varepsilon)}$. If $I^\varepsilon(z) = 0$ we define $\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}$ as a random object having the same distribution as $\pi_z^{d_0(\varepsilon)\bar{y}^\varepsilon_z}$. Without loss of generality we can suppose that it is defined in $(\tilde{\Omega}_{(i,\omega)}, \tilde{\mathcal{F}}_{(i,\omega)}, \tilde{\mathbb{P}}_{(i,\omega)})$. Furthermore, we will also suppose that $(\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)})_{z\in\mathbb{Z}}$ is an independent sequence of random measures.

Denote by $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) := (\bigotimes_{i\in\mathbb{Z},\omega\in\Omega^\bullet} \tilde{\Omega}_{(i,\omega)}, \bigotimes_{i\in\mathbb{Z},\omega\in\Omega^\bullet} \tilde{\mathcal{F}}_{(i,\omega)}, \bigotimes_{i\in\mathbb{Z},\omega\in\Omega^\bullet} \tilde{\mathbb{P}}_{(i,\omega)})$. We define the probability space $(\Omega^\star, \mathcal{F}^\star, \mathbb{P}^\star) := (\Omega^\bullet \times \bar{\Omega}, \mathcal{F}^\bullet \otimes \bar{\mathcal{F}}, \mathbb{P}^\bullet \otimes \bar{\mathbb{P}})$. For each $\omega^\star = (\omega^\bullet, \omega) \in \Omega^\star$ and $\varepsilon > 0$, we define $\bar{\mu}^\varepsilon_{\omega^\star}$ as a trap measure with trapping landscape $(\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}(\omega))_{z\in\mathbb{Z}}$.

$$\bar{\mu}^\varepsilon_{(\omega^\bullet,\omega)} := \sum_{z\in Q_\varepsilon} \delta_z \otimes d\Lambda^{(\omega^\bullet,j(\varepsilon,z),\varepsilon)}(\omega) \text{ for all } (\omega^\bullet, \omega) \in \Omega^\bullet \times \bar{\Omega}.$$

For each $\omega^\star = (\omega^\bullet, \omega) \in \Omega^\star$, we define $(\bar{\Lambda}^{(\omega^\star,i)})_{i\in\mathbb{N}}$ as an independent sequence of subordinators, each one having Laplace exponent $\tilde{\Psi}^{(i,\omega^\bullet)}(\omega)$. The space in which the subordinators are defined is of no importance. We also define $(\bar{\mu}^0_{\omega^\star})_{\omega^\star\in\Omega^\star}$ as

$$\bar{\mu}^0_{\omega^\star} := \sum_{i\in\mathbb{N}} \delta_{\bar{x}_i(\omega^\bullet)} \otimes d\bar{\Lambda}^{(\omega^\star,i)} \text{ for all } \omega^\star = (\omega^\bullet, \omega) \in \Omega^\bullet \times \bar{\Omega}.$$

For each $\omega^\star \in \Omega^\star$, $\bar{\mu}_{\omega^\star}$ is a discrete trap measure with the trapping landscape $(\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}(\omega))_{z\in\mathbb{Z}}$. Hence, to prove that $\bar{\mu}^\varepsilon$ is distributed as μ, it is enough to show that $(\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}(\omega))_{z\in\mathbb{Z}}$, regarded as a random trapping landscape, is distributed according to P. By assumption (I), we have that P is a product measure, and we have constructed the random trapping landscape $(\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}(\omega))_{z\in\mathbb{Z}}$ in such a way that its coordinates are independent from each other. Thus, it only remains to check that for each $z \in \mathbb{Z}$, $\tilde{\pi}^{(\omega^\bullet,z,\varepsilon)}(\omega)$ its distributed according to P^z, but this is granted by construction.

Now we want to check that $\bar{\mu}^0$ is distributed as μ^0. For each $\omega^\star \in \Omega^\star$, we have that $\bar{\mu}_{\omega^\star} = \sum_{i\in\mathbb{N}} \delta_{\bar{x}_i(\omega^\bullet)} \otimes d\bar{\Lambda}^{(\omega^\star,i)}$, where $(\bar{x}_i, \bar{y}_i)_{i\in\mathbb{N}}$ is distributed as the Poisson point process $\mathcal{P}^0$ and the (random) Laplace exponent of the $(\bar{\Lambda}^{(\omega^\star,i)})_{i\in\mathbb{N}}$ have the same distribution than those of the $(\Lambda^{(\omega,i)})_{i\in\mathbb{N}}$ which appear on the construction of μ^0.

4.4.2. $\mathbb{P}^\star$*-almost sure convergence of the coupled environments.* Here we will prove item (4) in Lemma 4.46. Let μ be a random measure, then denote $\mathcal{A}_\mu := \{[a,b]\times[c,d] \subset \mathbb{H} : \mu(\{a\}\times[c,d]) = \mu(\{b\}\times[c,d]) = 0\}$. As $\bar{\mu}^0_{\omega^\star}$ is $\mathbb{P}$-almost surely a Lévy trap measure,we have that $\bar{\mu}^0_{\omega^\star}$ is $\mathbb{P}$-almost surely a dispersed trap measure. Thus $\mathcal{A}_{\bar{\mu}^0_{\omega_\star}}$ is $\mathbb{P}$ almost surely a DC semiring contained on $\mathcal{T}_{\bar{\mu}^0_{\omega_\star}}$. Using Proposition 4.21 it suffices to show that, $\mathbb{P}^\star$-almost surely, $q_0(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}^\varepsilon_{\omega^\star})([a,b]\times[c,d])$ converges in distribution to $\bar{\mu}^0_{\omega^\star}([a,b]\times[c,d])$ as $\varepsilon \to 0$ for all $[a,b]\times[c,d] \in \mathcal{A}_{\bar{\mu}^0_{\omega^\star}}$. This plus Proposition 4.21 would imply that $\mathbb{P}^\star$-almost surely, $q_0(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}^\varepsilon_{\omega^\star}) \to \bar{\mu}^0_{\omega^\star}$ vaguely in distribution as $\varepsilon \to 0$. Theorem 4.44 would follow.

$$\bar{E}^\varepsilon(\omega^\bullet, \omega) := \sum_{z\in Q_\varepsilon} \delta_{\varepsilon z}^{\Lambda^{(\omega^\bullet,j(\varepsilon,z),\varepsilon)}(\omega)}$$
$$\bar{E}(\omega^\bullet, \omega) := \sum_{i\in\mathbb{Z}} \delta_{\bar{x}_i}^{\Lambda^{(\omega^\bullet,i)}(\omega)}.$$

For each $\omega^\star = (\omega^\bullet, \omega) \in \Omega^\star$, $\varepsilon > 0$ and $z \in \mathbb{Z}$, let

$$\bar{\Lambda}^{(\omega^\star,z,\varepsilon)} := q_0(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}^\varepsilon_{\omega^\star})\langle\{z\}\rangle.$$

That is, $\bar{\Lambda}^{(\omega^\star,z,\varepsilon)}$ is the $q_0(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}^\varepsilon_{\omega^\star})$-trapping process on $\{z\}$. We will prove the following

Lemma 4.49. *For each $i \in \mathbb{N}$, $\bar{\Lambda}^{(\omega^\star,z^i_\varepsilon,\varepsilon)} \overset{\varepsilon\to 0}{\to} \bar{\Lambda}^{(\omega^\star,i)}$ in distribution over $(D(\mathbb{R}_+), J_1)$ to, $\mathbb{P}^\star$-almost surely,*

Also, for each $\omega^\star = (\omega^\bullet, \omega) \in \Omega^\star$, $i \in \mathbb{N}$ and $0 < \varepsilon < c_i$, let $\Theta^{(\omega^\star,i,\varepsilon)}$ be a subordinator having Laplace exponent $\tilde{\Psi}^{(i,\omega^\bullet)}_\varepsilon(\omega)$. Display (4.50) shows that $\Theta^{(\omega^\star,i,\varepsilon)}$ is a compound Poisson process whose intensity of jumps is ε^{-1} and the size of the jumps are distributed according to $\bar{\pi}^{(\omega^\bullet,i,\varepsilon)}(\omega)(q_0(\varepsilon)^{-1}\cdot)$. It is a known fact that convergence of Laplace exponents implies the convergence in distribution over $(D(\mathbb{R}),J_1)$ of the corresponding subordinators . Thus the third item in Lemma 4.48 shows that, $\mathbb{P}^\star$-almost surely, $\Theta^{(\omega^\star,i,\varepsilon)} \overset{\varepsilon\to 0}{\to} \bar{\Lambda}^{(\omega^\star,i)}$ in distribution over $(D(\mathbb{R}_+),J_1)$. It follows easily that, $\mathbb{P}^\star$-almost surely, the family of processes $(\bar{\Lambda}^{(\omega^\star,z^i_\varepsilon,\varepsilon)})_{0<\varepsilon<c_i}$ also converges in distribution over $(D(\mathbb{R}_+),J_1)$ to $\bar{\Lambda}^{(\omega^\star,i)}$ as $\varepsilon\to 0$. This is because, for each $\omega^\star\in\Omega^\bullet$, $i\in\mathbb{N}$ and $0<\varepsilon<c_i$, we have that $\bar{\Lambda}^{(\omega^\star,z^i_\varepsilon,\varepsilon)}$ is just the discrete time analogous of $\Theta^{(\omega^\star,i,\varepsilon)}$. That is, they are discrete time processes in which the step has size ε and each jump is distributed according to $\bar{\pi}^{(\omega^\bullet,i,\varepsilon)}(\omega)(q_0(\varepsilon)^{-1}\cdot)$ (the times of the jumps are de-randomized).

For each $\delta>0$ and $\omega^\star\in\Omega^\star$, let $(\bar{\mu}^{(S,\delta,\varepsilon)}_{\omega^\star})_{\varepsilon>0}$ and $\bar{\mu}^{(S,\delta)}_{\omega^\star}$ be the trap measures defined by

$$\bar{\mu}^{(S,\delta,\varepsilon)}_{\omega^\star} := \sum_{\{z\in\mathbb{Z}:\bar{\rho}^\varepsilon(\{\varepsilon z\})\leq\delta\}} \delta_{\varepsilon z}\otimes d\bar{\Lambda}^{(\omega^\star,z,\varepsilon)}$$

and

$$\bar{\mu}^{(S,\delta)}_{\omega^\star} := \sum_{\{i\in\mathbb{N}:\bar{\rho}(\{\bar{x}_i\})\leq\delta\}} \delta_{\bar{x}_i}\otimes d\bar{\Lambda}^{(\omega^\star,i)}.$$

Those trap measures will take account of the shallow traps of $(\bar{\mu}^\varepsilon_{\omega^\star})_{\varepsilon>0}$ and $\bar{\mu}^0_{\omega^\star}$ respectively. Similarly, we define the trap measures composed of the deep traps, $(\bar{\mu}^{(D,\delta,\varepsilon)}_{\omega^\star})_{\varepsilon>0}$ and $\bar{\mu}^{(D,\delta,\varepsilon)}_{\omega^\star}$ as

$$\bar{\mu}^{(D,\delta,\varepsilon)}_{\omega^\star} := \sum_{\{z\in\mathbb{Z}:\bar{\rho}^\varepsilon(\{\varepsilon z\})>\delta\}} \delta_{\varepsilon z}\otimes d\bar{\Lambda}^{(\omega^\star,z,\varepsilon)}$$

and

$$\bar{\mu}^{(D,\delta)}_{\omega^\star} := \sum_{\{i\in\mathbb{N}:\bar{\rho}(\{\bar{x}_i\})>\delta\}} \delta_{\bar{x}_i}\otimes d\bar{\Lambda}^{(\omega^\star,i)}.$$

Define also the random measures

$$\bar{\rho}^{(S,\delta,\varepsilon)} := \sum_{\{z\in\mathbb{Z}:\bar{\rho}^\varepsilon(\{\varepsilon z\})\leq\delta\}} \bar{y}^\varepsilon_i\delta_z,$$

$$\bar{\rho}^{(S,\delta)} := \sum_{\{i\in\mathbb{N}:\bar{\rho}(\{\bar{x}_i\})\leq\delta\}} \bar{y}_i\delta_{\bar{x}_i},$$

$$\bar{\rho}^{(D,\delta,\varepsilon)} := \sum_{\{z\in\mathbb{Z}:\bar{\rho}^\varepsilon(\{\varepsilon z\})>\delta\}} \bar{y}^\varepsilon_i\delta_z$$

and

$$\bar{\rho}^{(D,\delta)} := \sum_{\{i\in\mathbb{N}:\bar{\rho}(\{x_i\})>\delta\}} \bar{y}_i\delta_{\bar{x}_i}.$$

Without loss of generality, we can suppose that all the trap measures $(\bar{\mu}^\varepsilon_{\omega^\star})_{\varepsilon\geq 0,\omega^\star\in\Omega^\star}$ have a common reference space $(\Omega^\diamond,\mathcal{F}^\diamond,\mathbb{P}^\diamond)$ (just take $(\Omega^\diamond,\mathcal{F}^\diamond,\mathbb{P}^\diamond)$ to be the product of all the reference spaces). First we will neglect the shallow traps. We have that $\bar{\mu}^{(S,\delta,\varepsilon)}_{\omega^\star}([a,b]\times[c,d])\leq\bar{\mu}^{(S,\delta,\varepsilon)}_{\omega^\star}([a,b]\times[0,d])$. Also, for all $\omega^\star=(\omega^\bullet,\omega)\in\Omega^\star$, we have that

$$\mathbb{P}^\diamond[\bar{\mu}^{(S,\delta,\varepsilon)}_{\omega^\star}([a,b]\times[0,d])] = \sum_{\{z\in[a,b]:\bar{\rho}^\varepsilon(\{\varepsilon z\})\leq\delta\}} \mathbb{P}^\diamond[\bar{\Lambda}^{(\omega^\star,z,\varepsilon)}_d]$$

which in turn equals

$$\sum_{\{z\in[a,b]:\bar\rho^\varepsilon(\{\varepsilon z\})\le\delta\}} d\mathbb{P}^\diamond[\bar\Lambda_1^{(\omega^\star,z,\varepsilon)}] = d\bar\rho^{(S,\delta,\varepsilon)}(\omega^\bullet)[a,b].$$

Next, we prove the following

Lemma 4.50. $\lim_{\delta\to0}\limsup_{\varepsilon\to0}\bar\rho^\varepsilon_\delta[a,b]=0$, $\mathbb{P}^\star$*-almost surely*

Proof. For the proof we will suppose that $\omega^\bullet\in\Omega^\bullet$ is fixed. For a fixed $\delta>0$, the set $\mathscr{A}^\delta:=\{j\in\mathbb{N}:\bar x_j\in[a,b],\bar y_j>\delta\}$ is finite. As $\bar V^0$ is a non decreasing, right continuous pure jump process, we have that for any $\eta>0$, there exists δ_0 small enough, such that, for all $\delta<\delta_0$, we have that

$$\sum_{i\in\mathscr{A}^\delta}\bar y_i > \bar V_b^0-\lim_{x\to a^+}\bar V_x^0-\eta/2=\bar V_b^0-\bar V_a^0-\eta/2. \tag{4.51}$$

where the last equality holds because $\bar\rho(\{a\})=0$. We know that for each $i\in\mathscr{A}^\delta$, the sequence $(\varepsilon z^i_\varepsilon)_{0<\varepsilon<c_i}$ converges to $\bar x_i$ as $\varepsilon\to0$. Also the sequence $(\bar\rho^\varepsilon(\{\varepsilon z_i^\varepsilon\}))_{0<\varepsilon<c_i}$ will converge to $\bar y_i$ as $\varepsilon\to0$. Hence, using (4.51), we find that for ε small enough

$$\sum_{i\in\mathscr{A}^\delta}\bar\rho^\varepsilon(\{\varepsilon z^i_\varepsilon\})>\bar V_b^0-\bar V_a^0-\eta$$

On the other hand, from the convergence on the J_1 topology and the fact that $\bar\rho(\{a\})=\bar\rho(\{b\})=0$, we have that $\lim_{\varepsilon\to0}\bar V_a^\varepsilon-\bar V_b^\varepsilon=\bar V_a^0-\bar V_b^0$ and that finishes the proof. □

The previous lemma implies that

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\mathbb{P}^\diamond[\bar\mu_{\omega^\star}^{(S,\delta,\varepsilon)}([a,b]\times[0,d])]=0 \tag{4.52}$$

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\bar\mu_{\omega^\star}^{(S,\delta,\varepsilon)}([a,b]\times[0,d])=0,\mathbb{P}^\diamond-\text{almost surely.}$$

In this way we have neglected the contribution of shallow traps. Having neglected the shallow traps, we just have to take care of a finite number of deep traps. We have that $\bar\mu^0_{\omega^\star}=\bar\mu^{(S,\delta)}_{\omega^\star}+\bar\mu^{(D,\delta)}_{\omega^\star}$, it suffices to show that for any $\delta>0$, $\bar\mu^{(D,\delta,\varepsilon)}_{\omega^\star}([a,b]\times[c,d])\xrightarrow{d}\bar\mu^{(D,\delta)}_{\omega^\star}([a,b]\times[c,d]),\mathbb{P}^\star$-almost surely. With the notation introduced in the proof of Lemma 4.50, we can write

$$\bar\mu^{(D,\delta,\varepsilon)}_{\omega^\star}=\sum_{\{z\in\mathbb{Z}:\bar\rho^\varepsilon(\{\varepsilon z\})>\delta\}}\delta_z\otimes d\bar\Lambda^{(\omega^\star,z,\varepsilon)}$$

$$\bar\mu^{(D,\delta)}_{\omega^\star}([a,b]\times[c,d])=\sum_{i\in\mathscr{A}^\delta}\bar\Lambda_b^{(\omega^\star,i)}-\bar\Lambda_a^{(\omega^\star,i)}$$

We know that for each $i\in\mathscr{A}^\delta$, the sequence $(\varepsilon z^i_\varepsilon)_{0<\varepsilon<c_i}$ converges to $\bar x_i$ as $\varepsilon\to0$. Then, using $\bar\rho(\{a\})=\bar\rho(\{b\})=0$, we find that for ε small enough the set $(\varepsilon z^i_\varepsilon)_{i\in\mathscr{A}^\delta}$ is contained in (a,b). Also the sequence $(\bar\rho^\varepsilon(\{\varepsilon z^i_\varepsilon\}))_{0<\varepsilon<c(i(j))}$ will converge to $\bar y_i$ as $\varepsilon\to0$. Thus, for ε small enough, the trap measures

$(\delta_{\varepsilon z_i^\varepsilon}^{\bar{\Lambda}^{(\omega^\star, z_\varepsilon^i,\varepsilon)}})_{i\in\mathscr{A}^\delta}$ will participate in the sum defining $\bar{\mu}_{\omega^\star}^{(D,\delta,\varepsilon)}$. Furthermore, we will show that, for ε small enough

$$\bar{\mu}_{\omega^\star}^{(D,\delta,\varepsilon)} = \sum_{i\in\mathscr{A}^\delta} \delta_{\varepsilon z_i^\varepsilon} \otimes d\bar{\Lambda}^{(\omega^\star, z_\varepsilon^i,\varepsilon)}. \tag{4.53}$$

We will use our assumption $\bar{\rho}(\{a\}) = \bar{\rho}(\{b\}) = 0$. That assumption Lemma 4.50, together imply that, for ε small, $\bar{\rho}^{(D,\delta,\varepsilon)} = \sum_{i\in\mathscr{A}^\delta} \bar{y}_i \delta_{\bar{x}_i}$.

That implies that there are no summands other than $(\delta_{\varepsilon z_i^\varepsilon} \otimes d\bar{\Lambda}^{(\omega^\star, z_\varepsilon^i,\varepsilon)})_{i\in\mathscr{A}^\delta}$ in $\bar{\mu}_{\omega^\star}^{(D,\delta,\varepsilon)}$. This shows (4.53) . Then we have that

$$\bar{\mu}_{\omega^\star}^{(D,\delta,\varepsilon)}([a,b]\times[c,d]) = \sum_{i\in\mathscr{A}^\delta} \bar{\Lambda}_b^{(\omega^\star, z_\varepsilon^i,\varepsilon)} - \bar{\Lambda}_a^{(\omega^\star, z_\varepsilon^i,\varepsilon)}$$

Thus, to show that $\bar{\mu}_{\omega^\star}^{(D,\delta,\varepsilon)}([a,b]\times[c,d])$ converges to $\bar{\mu}_{\omega^\star}^{(D,\delta)}([a,b]\times[c,d])$, it is enough to show that $\bar{\Lambda}_b^{(\omega^\star, z_\varepsilon^i,\varepsilon)} - \bar{\Lambda}_a^{(\omega^\star, z_\varepsilon^i,\varepsilon)} \to \bar{\Lambda}_b^{(\omega^\star,i)} - \bar{\Lambda}_a^{(\omega^\star,i)}$ in distribution, for each $i \in \mathscr{A}^\delta$. But that is provided by Lemma 4.49.

5. Applications

In this section we will make use of the results previously obtained to prove convergence results for RTRW which complement Theorem 4.44. In Subsection 4.3 we prove that under some hypothesis, the scaling limit of the RTRW is the fractional kinetic process. In Subsection give conditions under which we find that the scaling limit of the RTRW is a Brownian motion. In Subsection 5.1 we will prove that the model defined in example 3 of RTRW presents four different phases in terms of its scaling limit. In Subsection 5.2 we show different regimes for the comb model in terms of its scaling limit.

5.1. The simplest case of a phase transition.

Example 4.51. *Take α and β in $(0,1)$. Let $(\tau_z)_{z\in\mathbb{Z}}$ be a i.i.d. sequence of positive random variables defined on the space $(\Omega,\mathcal{F},\mathbb{P})$ such that*

$$\lim_{u\to\infty} u^\alpha \mathbb{P}(\tau_0 > u) = 1 \tag{5.1}$$

and $\mathbb{P}(\tau_z > 1) = 1$. For each $z\in\mathbb{Z}$, consider the random probability distribution $\pi_z(\omega) := (1-\tau_z(\omega)^\beta)\delta_0 + \tau_z(\omega)^\beta \delta_{\tau_z(\omega)}$. Then there exists a random trap measure $\mu^{\alpha,\beta}$ driven by $(\sum \delta_z \otimes \pi_z(\omega))_{\omega\in\Omega}$. In this case we have that the random trapping landscape of $\mu^{\alpha,\beta}$ is i.i.d. Consider the RTRW $Z[\mu^{\alpha,\beta}]$. We will see that this RTRW presents a phase transition in terms of its scaling limit as the values of α and β change.

Let $\alpha\in(0,1)$, $\beta\in(0,1)$ and $\mu^{\alpha,\beta}$ as in example 3 of RTRW. By a straightforward computation

$$\lim_{x\to\infty} x^{\frac{\alpha}{1-\beta}} \mathbb{P}(m(\pi_0) \geq x) = 1 \tag{5.2}$$

Hence $\gamma = \frac{\alpha}{1-\beta}$

PROPOSITION 4.52. *The following holds: If $\gamma > 1$ (equivalently $\alpha + \beta > 1$)*

(1) *then define $m := \mathbb{P}(m(\pi_0)) < \infty$. In this case $\varepsilon Z[\mu^{\alpha,\beta}]_{m\varepsilon^{-2}t}$ converges to Brownian motion in the sense of proposition 4.39 if $\gamma < 1$*
(2) *and $\alpha > \beta$, let $q_0(\varepsilon) = \varepsilon^{1+1/\gamma}$. Then $\varepsilon Z[\mu^{\alpha,\beta}]_{q_0(\varepsilon)^{-1}t}$ converges to FIN_γ in the sense of proposition 4.42*
(3) *and $\alpha < \beta$, let $q(\varepsilon) = K_{\alpha,\beta}\varepsilon^{2/\kappa}$, where $\kappa = \alpha + \beta$ and the constant $K_{\alpha,\beta}$ can be computed explicitly. Then $\varepsilon Z[\mu^{\alpha,\beta}]_{q(\varepsilon)^{-1}t}$ converges to FK_κ in the sense of Theorem 4.44*
(4) *and $\alpha = \beta$, let $q_0(\varepsilon) = \varepsilon^{1+1\gamma}$. Then $\varepsilon Z[\mu^{\alpha,\beta}]_{q_0(\varepsilon)^{-1}t}$ converges, in the sense of Theorem 4.44, to a RTRW which is neither FIN nor FK (This RTRW will be referred as a "Poissonian" RTRW).*

We have that $m(\pi_0(\omega)) = \tau(\omega)^{1-\beta}$. Thus, if $\gamma > 1$, we have that $\mathbb{P}(m(\pi_0)) < \infty$, and proposition 4.39 yields (i). Conditioning on $m(\pi_0) = d_0(\varepsilon)$ is equivalent to conditioning on $\tau^{1-\beta} = \varepsilon^{-1/\gamma}$, which, in turn, is equivalent to $\tau = \varepsilon^{-1/\alpha}$. Then, we can choose the $\pi_z^{d_0(\varepsilon)} = (1 - \varepsilon^{\beta/\alpha})\delta_0 + \varepsilon^{\beta/\alpha}\delta_{\varepsilon^{-1/\alpha}}$. Thus $\hat{\pi}^{d_0(\varepsilon)}(\lambda) = 1 - \varepsilon^{\beta/\alpha} + \varepsilon^{\beta/\alpha}\exp(-\lambda\varepsilon^{-1/\alpha})$. Hence we can write

$$\Psi_\varepsilon(\hat{\pi}^{d_0(\varepsilon)})(\lambda) = \varepsilon^{(\beta-\alpha)/\alpha}(1 - \exp(-\lambda\varepsilon^{(\alpha-\beta)/\alpha})). \tag{5.3}$$

Thus, we find that

(1) if $\alpha = \beta$ we have that $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat{\pi}^{d_0(\varepsilon)})(\lambda) = 1 - \exp(-\lambda)$
(2) if $\alpha > \beta$ we have that $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat{\pi}^{d_0(\varepsilon)})(\lambda) = -\lambda$
(3) if $\alpha < \beta$ we have that $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat{\pi}^{d_0(\varepsilon)})(\lambda) = 0$.

If $\alpha > \beta$, item (2) gives assumption (D0). Assumptions (IID) and (PP0) also hold. Thus we can deduce claim (ii) from Theorem 4.44. Similarly, Theorem 4.44 together with item (1) gives claim (iv). In this case, the traps are "Poissonian" in the sense that $\mathbb{F}_0 = \delta_h$, where $h(\lambda) = 1 - \exp(-\lambda)$, which is the Laplace exponent of a Poisson process. Note that if $\alpha < \beta$ we cannot deduce a convergence result from Theorem 4.44. However, we can check the assumptions of proposition 4.42. $\Gamma(\varepsilon) = \mathbb{P}(1 - \hat{\pi}(\varepsilon))$

$$= \int_0^\infty t^{-\beta}(1 - \exp(-\varepsilon t))\tau(dt).$$

Changing variables we obtain

$$= \int_0^\infty t^{-\beta}(1 - \exp(-u))\varepsilon^{\alpha}\varepsilon^{-\alpha}\tau(\varepsilon^{-1}dt).$$

But, by display (5.1), we have that $\varepsilon^{-\alpha}\tau(\varepsilon^{-1}dt)$ converges weakly to $\alpha t^{-1-\alpha}dt$. Thus

$$\begin{aligned}\Gamma(\varepsilon) = \mathbb{P}(1 - \hat{\pi}(\varepsilon)) &= \varepsilon^{\alpha+\beta}\int_0^\infty t^{-1-\alpha-\beta}(1 - \exp(-u))du. \\ &= (-t^{-\beta}(1 - \exp(-\varepsilon t))\mathbb{P}(\tau > t))|_0^\infty \\ &+ \int_0^\infty (-\beta t^{-\beta-1}(1 - \exp(-\varepsilon t)) + t^{-\beta}\varepsilon\exp(-\varepsilon t))\mathbb{P}(\tau > t)dt\end{aligned} \tag{5.4}$$

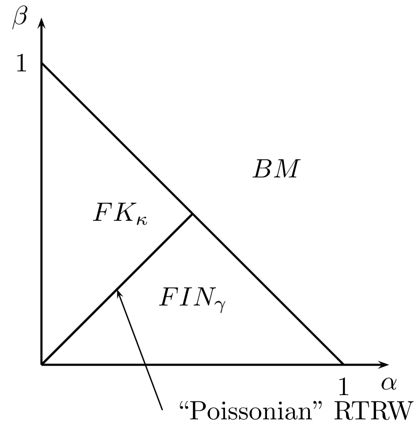

FIGURE 1. The simplest case of a phase transition.

$$= 0 + \int_0^\infty (-\beta t^{-\beta-1}(1-\exp(-\varepsilon t)) + t^{-\beta}\varepsilon\exp(-\varepsilon t))\mathbb{P}(\tau > t)dt.$$

$$\int_0^\infty (-\varepsilon^{\beta+1}\beta u^{-\beta-1}(1-\exp(-u)) + \varepsilon^\beta u^{-\beta}\varepsilon\exp(-u))\mathbb{P}(\tau > \varepsilon^{-1}u)du.$$

$$\varepsilon^\beta \int_0^\infty (-\beta u^{-\beta-1}(1-\exp(-u)) + u^{-\beta}\exp(-u))(\varepsilon^\alpha u^{-\alpha} + o(\varepsilon^\alpha u^{-\alpha}))du.$$

$$\varepsilon^{\beta+\alpha} \int_0^\infty (-\beta u^{-\beta-1}(1-\exp(-u)) + u^{-\beta}\exp(-u))u^{-\alpha}du + o(\varepsilon^{\beta+\alpha}).$$

$$= \alpha\varepsilon^{\alpha+\beta}\int_0^\infty u^{-\beta-1-\alpha}(1-\exp(-u))du.$$

Thus $\kappa = \alpha + \beta$. Similarly, we also find that

$$\mathbb{P}((1-\hat{\nu}(\varepsilon))^2) = \alpha\int_0^\infty t^{-2\beta}(1-\exp(-\varepsilon t))^2\tau(dt) \tag{5.5}$$

$$= \alpha\varepsilon^{2\beta+\alpha}\int_0^\infty u^{-2\beta-1-\alpha}(1-\exp(-u))^2du$$

Leading to

$$\varepsilon^{-3}\mathbb{P}((1-\hat{\nu}(q_{FK}(\varepsilon)))^2) \to 0.$$

Hence, the assumptions of Theorem 4.42 are fulfilled and we deduce (ii).

5.2. The comb model.

EXAMPLE 4.53 (Comb model). *Let $(N_z)_{z\in\mathbb{Z}}$ be an i.i.d. collection of heavy tailed random variables satisfying*

$$\mathbb{P}(N_0 = n) = \zeta(1+\alpha)^{-1}n^{-1-\alpha}, \alpha > 0. \tag{5.6}$$

We use G_z to denote a line segment of length N_z with nearest-neighbor edges. Let G_{comb} be the tree-like graph with leaves $(G_z)_{z\in\mathbb{Z}}$. We refer to the geometric trap model on G_{comb} as the ***comb model****. We refer to the leaves of G_{comb} as teeth. The comb graph is a great simplification compared to IIC or IPC since the distribution of the time spent in the teeth is easy to compute. As in the previous examples, we can project the simple random walk X^{comb} on G_{comb} to $\mathbb{Z}$ to obtain a RTRW $Z[\mu]$. As we will see in Section 5.2, the scaling limit of $Z[\mu]$ is either Brownian motion or fractional kinetics depending on the value of α. When $\alpha > 1$, the teeth are "short" and the mean time spent in traps has finite expectation. Thus $Z[\mu]$ is diffusive. If, on the other hand, $\alpha < 1$ the teeth are "long" and the expectation of the mean time spent in traps is infinite. However, $Z[\mu]$ does not explore deep traps and, therefore, does not "remember" the environment. Hence fractional kinetics is the limit. The comb model can be enriched further. In order to do so, we write each vertex of G_{comb} as a pair (n, z) with $z \in \mathbb{Z}$ and $n \in [0, N_z]$ in a straightforward fashion so that the points on the backbone have vanishing second coordinate. We (re-)define X^{comb} as follows. Whenever X^{comb} is not on the backbone, it performs a drifted random walk on $(z, 0) \cup G_z$ with a drift $g(N_z) \geq 0$ pointing away from the backbone and reflecting wall at the end of the tooth, i.e. for any $0 < n < N_z$*

$$\mathbb{P}(X^{comb}(k+1) = (z, n+1)|X^{comb}(k) = (z, n)) = (1 + g(N_z))/2, \tag{5.7}$$

$$\mathbb{P}(X^{comb}(k+1) = (z, n-1)|X^{comb}(k) = (z, n)) = (1 - g(N_z))/2, \tag{5.8}$$

$$\mathbb{P}(X^{comb}(k+1) = (z, N_z - 1)|X^{comb}(k) = (z, N_z)) = 1. \tag{5.9}$$

Otherwise, if $X^{comb} = (z, 0)$, it jumps to one of the three vertices $(z-1, 0), (z+1, 0)$ and $(z, 1)$ with equal probability. We will reason in Subsection 5.2 that the appropriate choice of g is

$$g(N) = \beta N^{-1} \log(N) \tag{5.10}$$

for some $\beta \geq 0$. The presence of the drift might force X^{comb} to explore even the deepest traps. The statements of the convergence results with their proofs are given in Subsection 5.2.

In this section we investigate the scaling limits of the model described in example 6 of RTRW. There, we assumed $(N_z)_{z\in\mathbb{Z}}$ to be an i.i.d. collection of random variables with

$$\mathbb{E}(N_0 = n) = \zeta(1+\alpha)^{-1} n^{-1-\alpha}, \alpha > 0.$$

Let V be a random walk on $[0, N] \cap \mathbb{Z}$ reflected at the boundary N: $V_0 = 1$,

$$V_n = \begin{cases} V_{n-1} + \xi_n, & V_{n-1} < N; \\ V_{n-1} - 1, & V_n = N. \end{cases} \tag{5.11}$$

With i.i.d. Bernoulli random variables $(\xi_n)_{n\in\mathbb{N}}$

$$\mathbb{P}(\xi_1 = 1) = p = 1 - \mathbb{P}(\xi_1 = -1), p \geq 1/2.$$

Let $\tau(N) := \inf\{n \geq 0 : V_n = 0\}$ be the hitting time of 0. We use ϑ_N to denote the law of $\tau(N)$ if we choose $p = (1 + g(N))/2$. The function g represents the drift along the teeth.

Set $\nu'_z = \vartheta_{N_z}$ and notice that, conditioned on N_z, ν'_z coincides with the distribution of the time which is required by X^{comb} in order to go from $(z, 1)$ to $(z, 0)$. Let ν_z denote the distribution of a geometric, with parameter $1/3$, sum of independent random variables drawn from ν'_z, i.e.

$$\hat{\nu}_0(\lambda) = \frac{2}{3 - \hat{\nu'}_0(\lambda)}. \tag{5.12}$$

It is clear that the comb model is a RTRW on the environment $(\nu_z, z \in \mathbb{Z})$. The following observation will be used throughout our calculations.

REMARK 4.54.

$$1 - \hat{\nu}_0(\lambda) \sim \frac{1 - \hat{\nu'}_0(\lambda)}{2} \tag{5.13}$$

as $\lambda \to 0$.

5.2.1. *Random walk with reflection.* This section is devoted to studying the properties of the distribution ϑ. We start by computing the moment-generating function $\hat{\vartheta}_N(-\log(s))$.

LEMMA 4.55. *Let $p = (1 + g(N))/2$ and $\xi(p) = (1 - p)/p$. We use $\chi(s)$ to denote*

$$\frac{1 + \sqrt{1 - 4s^2 p(1-p)}}{2sp}. \tag{5.14}$$

Then, the generating function of $\tau(N)$ is given by

$$\hat{\vartheta}_N(-\log(s)) = \frac{\xi(p)\chi(s)^{2N-2}(\chi(s) - s) + \xi(p)^{N-1}\chi(s)(s\chi(s) - \xi(p))}{\chi(s)^{2N-1}(\chi(s) - s) + \xi(p)^{N-1}(s\chi(s) - \xi(p))}. \tag{5.15}$$

and, if $p = q$, then $\hat{\vartheta}_N(-\log(1)) = 1$

PROOF. We need to compute the moment generating function of $\tau(N)$. Let $q_z(n) = \mathbb{P}(V_n = 0)$, for all $n \geq 0$ and $0 \leq z \leq N$. By the Markov property

$$q_z(n+1) = pq_{z+1}(n) + (1-p)q_{z-1}(n), 1 \leq z \leq N-1 \tag{5.16}$$

$$q_N(n+1) = q_{N-1}(n), \text{ for all } n \geq 0. \tag{5.17}$$

We also define the boundary conditions as

$$q_0(0) = 1; q_0(n) = 0, n \geq 1; q_z(0) = 0, z \geq 1. \tag{5.18}$$

For the moment-generating function $f_z(s) = \sum_{n=0}^{\infty} q_z(n)s^n$ we have

$$f_z(s) = spf_{z+1}(s) + s(1-p)f_{z-1}(s), \text{ for } 1 \leq z \leq N-1, \tag{5.19}$$

with the boundary conditions $f_0(s) = 1$, and $f_N(s) = sf_{N-1}(s)$.

Let s be fixed. Then, it is natural to seek solutions for (5.90) of the form $\lambda(s)^z$. Replacing in 5.90, we are led to seek for solutions of

$$\lambda(s) = sp\lambda(s)^2 - s(1-p). \tag{5.20}$$

Let $\lambda_\pm$ be the solutions of λ. Thus, we can write the general solution of 5.90 as

$$f_z(s) = A_+(s)\lambda_+(s)^z + A_-(s)\lambda_-(s)^z. \tag{5.21}$$

With the boundary conditions transforming to

$$A_+(s) + A_-(s) = 1 \tag{5.22}$$

$$A_+(s)\lambda_+(s)^N + A_-(s)\lambda_-(s)^N = s(A_+(s)\lambda_+(s)^{N-1} + A_-(s)\lambda_-(s)^{N-1}). \tag{5.23}$$

Solving for $A_\pm$ we obtain

$$A_+(s) = -\frac{\lambda_-(s)^{N-1}(\lambda_-(s)-s)}{\lambda_+(s)^{N-1}(\lambda_+(s)-s) - \lambda_-(s)^{N-1}(\lambda_-(s)-s)}, \tag{5.24}$$

$$A_-(s) = \frac{\lambda_+(s)^{N-1}(\lambda_+(s)-s)}{\lambda_+(s)^{N-1}(\lambda_+(s)-s) - \lambda_-(s)^{N-1}(\lambda_-(s)-s)}. \tag{5.25}$$

Finally,

$$f_z(s) = \frac{\lambda_+(s)^{N-1}\lambda_-(s)^z(\lambda_+(s)-s) - \lambda_-(s)^{N-1}\lambda_+(s)^z(\lambda_-(s)-s)}{\lambda_+(s)^{N-1}(\lambda_+(s)-s) - \lambda_-(s)^{N-1}(\lambda_-(s)-s)}. \tag{5.26}$$

A simple rearrangement yields the claim. □

In the next lemma we compute the first and second moments of $\tau(N)$.

LEMMA 4.56. *We assume $g(N) > 0$ and set $p = (1+g(N))/2$ as before. Then*

$$\int_{\mathbb{R}_+} t\vartheta_N(dt) = \frac{2(1-p)\xi(p)^{-N} - 1}{2p-1} \tag{5.27}$$

and

$$\begin{aligned}\int t^2\vartheta_N(dt) = {} & (1+\xi(p)^{-N})\frac{4p(1-p)(2(1-p)\xi(p)^{-N}-1)}{(2p-1)} \\ & - 2p\frac{4Np-1}{(1-p)(2p-1)^2} + \frac{2(1-p)\xi(p)-1}{2p-1}.\end{aligned} \tag{5.28}$$

PROOF. For $p > 1/2$ and any small $\varepsilon > 0$ we have the following asymptotic expansion

$$\chi(1-\varepsilon) = 1 + \frac{\varepsilon}{2p-1} + \frac{(2p-1)^2 - (1-p)}{(2p-1)^3}\varepsilon^2 + o(\varepsilon^2). \tag{5.29}$$

Using this expansion and 5.15 we write $\hat{\nu}'(-\log(1-\varepsilon))$, keeping terms up to second order, as

$$\frac{1+N_1\varepsilon+N_2\varepsilon^2}{1+D_1\varepsilon+D_2\varepsilon^2} = 1 + (N_1-D_1)\varepsilon + (D_1^2 - D_2 + N_2 - D_1N_1)\varepsilon^2 \tag{5.30}$$

with

$$N_1 = \frac{2(1-p)^2}{(2p-1)^2}\xi(p)^{-N} - 2p^2 - 4p + 1(2p-1)^2$$

$$N_2 = \frac{p^2(4p-3)}{(2p-1)^4} + \xi(p)^{-N}\left(\frac{4(1-p)^2}{(2p-1)^3}N - \frac{(1-p)^2(4p-1)}{(2p-1)^4}\right),$$

$$D_1 = (1 - \xi(p)^{-N})\frac{2p(1-p)}{(2p-1)^2}$$
$$D_2 = -\frac{p(1-p)}{(2p-1)^4} + \xi(p)^{-N}(\frac{4p(1-p)}{(2p-1)^3}N - \frac{p(1-p)}{(2p-1)^4}).$$

$\square$

From display (5.30) we obtain

$$\int t\vartheta_N(dt) = D_1 - N_1 \tag{5.31}$$

$$\frac{1}{2}\int t(t-1)\vartheta_N(dt) = D_1^2 - D_2 + N_2 - D_1N_1. \tag{5.32}$$

Subsequently,

$$\int t^2\vartheta_N(dt) = 2(D_1^2 - D_2 + N_2 - D_1N_1) - \int t\vartheta_N(dt). \tag{5.33}$$

Now it is easy to conclude

$$\int t\vartheta_N(dt) = \frac{2(1-p)\xi(p)^{-N} - 1}{2p-1} \tag{5.34}$$

and

$$\int t^2\vartheta_N(dt) = (1 + \xi(p)^{-N})\frac{4p(1-p)}{(2p-1)^2}\int t\vartheta_N(dt) - 2p\frac{(4Np-1)}{(1-p)(2p-1)^2} + \int t\vartheta_N(dt). \tag{5.35}$$

This completes the proof.

LEMMA 4.57. *If* $g \equiv 0$ *then*

$$1 - \hat{\vartheta}_N(\varepsilon) \sim \sqrt{2\varepsilon}\tanh(N\sqrt{2\varepsilon}).$$

If $\lim_{N\to\infty}\frac{N}{\log N}g(N) = \beta > 0$ *we define*

$$\frac{d(\varepsilon)^{1+\alpha}}{\log(d(\varepsilon))} = \varepsilon^{-2}, q(\varepsilon) = \frac{\log^2(d(\varepsilon))}{d(\varepsilon^{2+2\beta})}.$$

Then

$$1 - \hat{\vartheta}_N(q(\varepsilon)) \leq \begin{cases} q(\varepsilon)m_{\vartheta_N}, & N < d(\varepsilon)^{1/2} \\ c(q(\varepsilon) + g(N)), & N > d(\varepsilon)^{1+\beta/2} \end{cases}$$

Moreover,

$$1 - \hat{\vartheta}_N(q(\varepsilon)) \sim \frac{2\beta N^{1+2\beta}\log(N)}{N^{2+2\beta} + 2\beta^2 q(\varepsilon)^{-1}\log^2(N)} \tag{5.36}$$

for $d(\varepsilon)^{1/2} \leq N \leq d(\varepsilon)^{1+\beta/2}$.

OBSERVATION 4.58. $\varepsilon^{2/(1+\alpha)}d(\varepsilon)$ and $\varepsilon^{-4(1-\beta)/(1+\alpha)}q(\varepsilon)$ are slowly varying functions.

PROOF. Let $g \equiv 0$. This corresponds to $\xi = 1$ and

$$\chi(s) = \frac{1 + \sqrt{1 - s^2}}{s}. \tag{5.37}$$

Therefore

$$|\chi(\exp(-\varepsilon)) - 1 - \sqrt{2\varepsilon}| \leq c\varepsilon, \tag{5.38}$$

and we obtain from (5.15)

$$1-\hat{\vartheta}_N(\varepsilon) \sim \sqrt{2\varepsilon}\tanh(N\sqrt{2\varepsilon}). \tag{5.39}$$

Let us assume

$$\lim_{N\to\infty}\frac{N}{\log(N)}g(N) = \beta > 0. \tag{5.40}$$

Then $1-\hat{\vartheta}_N(q(\varepsilon)) \leq q(\varepsilon)m_{\vartheta_N}$, for any N.

Next, from (5.15) we obtain

$$1-\hat{\vartheta}_N(q(\varepsilon)) \leq \frac{\chi(s)-\xi(p)}{\chi(s)} \leq \chi_N(s)-\xi_N, \tag{5.41}$$

where $\chi_N = \frac{1-g(N)}{1+g(N)}$,

$$\chi_N(s) = \frac{1+\sqrt{1-s^2+s^2g(N)^2}}{s(1+g(N))}. \tag{5.42}$$

Therefore,

$$1-\hat{\vartheta}_N(q(\varepsilon)) \leq c(q(\varepsilon)+g(N)). \tag{5.43}$$

Finally, let $N \in [d(\varepsilon)^{1/2}, d(\varepsilon)^{1-\beta/2}]$. Then

$$q(\varepsilon) = \frac{\log^2(d(\varepsilon))}{d(\varepsilon)^{2+2\beta}} \ll \frac{\log^2(d(\varepsilon))}{d(\varepsilon)^{2+\beta}} \leq cg(N)^2. \tag{5.44}$$

This implies

$$|\chi(\exp(-q(\varepsilon)))-1-\frac{q(\varepsilon)}{g(N)}| \leq c\frac{q(\varepsilon)}{g(N)}(d(\varepsilon^{-\beta})+\frac{\log^2(d(\varepsilon))}{d(\varepsilon)}). \tag{5.45}$$

Using the fact that $\xi_N^N \sim N^{-2\beta}$ combined with (5.15), we obtain

$$1-\hat{\vartheta}_N(q(\varepsilon)) \sim \frac{2\beta N^{1+2\beta}\log(N)}{N^{2\beta+2}+2\beta^2 q(\varepsilon)^{-1}\log^2(N)}. \tag{5.46}$$

The proof is complete. □

5.2.2. *Results:* We first discuss feasible choices of the drift g. If

$$\limsup_{N\to\infty} g(N) > 0 \tag{5.47}$$

we deduce from (5.27) that m_0 grows exponentially fast in N_0 along some subsequence. This, combined with (5.6) violates the crucial assumption (PP0). Therefore,

$$\lim_{N\to\infty} g(N) = 0 \tag{5.48}$$

must hold. Provided (5.48) holds, we can write

$$m_0 \sim \begin{cases} 2N_0, & g(N_0)=0 \\ g(N_0)^{-1}\exp(2g(N_0)N_0), & g(N_0)\neq 0. \end{cases}$$

If

$$\lim_{N\to\infty}\frac{N}{\log(N)}g(N) = \infty, \tag{5.49}$$

we obtain

$$m_0 \geq cN_0^{2g(N_0)N_0/\log(N_0)} \tag{5.50}$$

violating assumption (PP0) again. Thus we have to analyze the case

$$\lim_{N\to\infty} \frac{N}{\log(N)} g(N) < \infty. \tag{5.51}$$

Due to a technical difficulty, we will not treat the case of a very weak non-vanishing drift, that is

$$\lim_{N\to\infty} \frac{N}{\log(N)} g(N) = 0, \text{ while } g \not\equiv 0. \tag{5.52}$$

Rather we discuss the situation where either

$$\beta := \lim_{n\to\infty} \frac{N}{\log(N)} g(N) \in (0, \infty) \tag{5.53}$$

or $g \equiv 0$. We set $\beta = 0$ whenever $g \equiv 0$.

PROPOSITION 4.59. *If $\alpha \geq 1$ and*

(i)$1 + 2\beta < \alpha$, then $\mathbb{E}(m_0) < \infty$. In this case, $\varepsilon Z[\mu]_{m\varepsilon^{-2}t}$ converges to Brownian motion in the sense of proposition 4.39

(ii)$1+2\beta > \alpha$, then, set $\gamma = \alpha/(1+2\beta)$ and $q_0(\varepsilon) = \varepsilon^{1+1/\gamma}$. In this case, $\varepsilon Z[\mu]_{q_0(\varepsilon)^{-1}}$ converges to FIN_γ in the sense of Theorem 4.44.

If $\alpha < 1$

(iii) set $\kappa = (1 + \alpha)/2$. Moreover, let $q_{FK} = \varepsilon^{2/\kappa}$ if $g(N) \sim 0$ or, if $\beta > 0$, let

$$\frac{d_{FK}(\varepsilon)^{2\kappa}}{\log(d_{FK}(\varepsilon))} = \varepsilon^{-1}, q_{FK}(\varepsilon) = \frac{\log^2(d_{FK}(\varepsilon))}{d_{FK}(\varepsilon)^{2+2\beta}}.$$

Then $\varepsilon Z[\mu]_{q_{FK}(\varepsilon)^{-1}}$ converges to the fractional kinetic process in the sense of proposition 4.42.

PROOF. If

$$\lim \frac{N}{\log(N)} g(N) = 0 \tag{5.54}$$

there exists $\delta > 0$ so that

$$\mathbb{P}(m_0 > x) = O(x^{-1-\delta}). \tag{5.55}$$

Subsequently, $\mathbb{E}(m_0) < \infty$. If, on the other hand,

$$\lim_{N\to\infty} \frac{N}{\log(N)} g(N) = \beta > 0 \tag{5.56}$$

it can be easily shown that

$$x^\gamma \mathbb{P}(m_0 > x), \gamma = \alpha/(1 + 2\beta) \tag{5.57}$$

varies slowly.

(i): From the discussion above, $\mathbb{E}(m_0) < \infty$. We obtain the statement from proposition 4.39.

(ii): In this case, $\gamma < 1$ and it follows from (5.27) and (5.28) that

$$m_0 \asymp \frac{N_0^{1+2\beta}}{\log(N_0)} \text{ and } \sigma_0^2 \asymp \frac{N_0^{3+4\beta}}{\log^3(N_0)}. \tag{5.58}$$

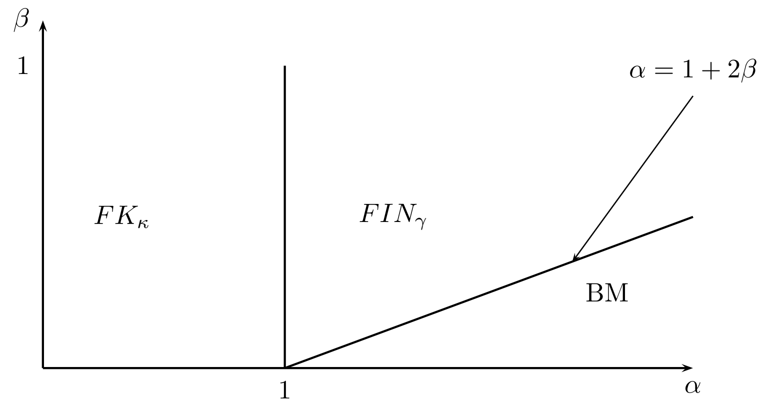

FIGURE 2. Phase diagram for the comb model.

Thus $\sigma_0^2 \ll m_0^{2+\gamma}$, Therefore, with some simple calculations we can show that $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat{\pi}^{d_0(\varepsilon)})(\lambda) = -\lambda$ and Theorem 4.44 yields the result.

(iii): The strategy of the proof is simple . We have to verify assumptions (S) and (V). We first discuss the easier case $g(N) \equiv 0$. By Lemma 4.57,

$$1 - \hat{\vartheta}_N(\exp(-\varepsilon)) \sim \sqrt{2\varepsilon}\tanh(N\sqrt{2\varepsilon}). \tag{5.59}$$

Thus, $\Gamma(\varepsilon) = \mathbb{E}(1 - \hat{\nu}(\varepsilon))$ can be approximated by

$$\frac{1}{2}\sqrt{2\varepsilon}\zeta(1+\alpha)^{-1}\sum_{N=1}^{\infty} N^{-1-\alpha}\tanh(N\sqrt{2\varepsilon}). \tag{5.60}$$

Since $x^{-1-\alpha}\tanh(cx)$ is decreasing in x on $[0,\infty)$ for any $a > 0$, we can substitute the sum trough the integral to obtain

$$\Gamma(\varepsilon) \sim \sqrt{\frac{\varepsilon}{2}}\zeta(1+\alpha)^{-1}\int_{N=1}^{\infty} N^{-1-\alpha}\tanh(N\sqrt{2\varepsilon})dN. \tag{5.61}$$

This leads to

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\Gamma(q_{FK}(\varepsilon)) = \frac{1}{\sqrt{2}}\zeta(1+\alpha)^{-1}\int_0^{\infty} y^{-1-\alpha}\tanh(\sqrt{2}y)dy, \tag{5.62}$$

justifying (up to a multiplicative constant) the choice of q_{FK}. Repeating the same line of reasoning, we obtain

$$\mathbb{E}((1-\hat{\nu}(q_{FK}(\varepsilon)))^2) \sim \frac{q_{FK}(\varepsilon)}{2\zeta(1+\alpha)}\sum_{N=1}^{\infty} N^{-1-\alpha}\tanh^2(N\sqrt{2q(\varepsilon)}) \sim c\varepsilon^{(2\kappa+1)/\kappa}. \tag{5.63}$$

It follows

$$\varepsilon^{-3}\mathbb{E}((1-\hat{\nu}(q_{FK}(\varepsilon)))^2) \sim c\varepsilon^{(1-\kappa)/\kappa}. \tag{5.64}$$

We follow the same steps in case $\beta > 0$. The main tool is Lemma 4.57 . Using this lemma we can bound

$$\varepsilon^{-2}\mathbb{E}(1-\hat{\nu}_N(q_{FK}(\varepsilon)); N < d_{FK}(\varepsilon)^{1/2}) \tag{5.65}$$

by

$$\leq c\varepsilon^{-2} q_{FK}(\varepsilon) \sum_{N=1}^{d_{FK}(\varepsilon)^{1/2}} N^{-1-\alpha} m_{\vartheta_N} \tag{5.66}$$

$$\leq c\varepsilon^{-2} q_{FK}(\varepsilon) \sum_{N=1}^{d_{FK}(\varepsilon)^{1/2}} N^{2\beta-\alpha} \tag{5.67}$$

$$\leq c\frac{\log d_{FK}(\varepsilon)}{d_{FK}(\varepsilon)^{(1+2\beta-\alpha)/2}} = o(1). \tag{5.68}$$

In the second estimate we used display (5.58). Similarly,

$$\varepsilon^{-2}\mathbb{E}(1-\hat{\nu}_N(q_{FK}(\varepsilon)); N > d_{FK}(\varepsilon)^{1+2\beta}) \tag{5.69}$$

is bounded by

$$c\varepsilon^{-2} \sum_{N=d_{FK}(\varepsilon)^{1+2\beta}}^{\infty} N^{-1-\alpha}(q_{FK}(\varepsilon)+g(N)) \tag{5.70}$$

$$\leq c\varepsilon^{-2} q_{FK}(\varepsilon) \sum_{N=d_{FK}(\varepsilon)^{1+2\beta}}^{\infty} N^{-1-\alpha} + c\varepsilon^{-2} + \sum_{N=d_{FK}(\varepsilon)^{1+2\beta}}^{\infty} N^{-2-\alpha}\log(N) \tag{5.71}$$

$$\leq c d_{FK}(\varepsilon)^{-1} + c d_{FK}(\varepsilon)^{1+\alpha} \sum_{N=d_{FK}(\varepsilon)^{1+2\beta}}^{\infty} N^{-2-\alpha}\log(N) = o(1). \tag{5.72}$$

To check assumption (S) it remains to compute

$$\mathbb{E}(1-\hat{\nu}_N(q_{FK}(\varepsilon)); d_{FK}(\varepsilon)^{1/2} \leq N \leq d_{FK}(\varepsilon)^{1+\beta/2}). \tag{5.73}$$

It can be approximated by

$$\frac{1}{2\zeta(1+\alpha)} \sum_{N=d_{FK}(\varepsilon)^{1/2}}^{d_{FK}(\varepsilon)^{1+\beta/2}} \frac{2\beta N^{2\beta-\alpha}\log N}{N^{2+2\beta}+2\beta^2 q_{FK}(\varepsilon)^{-1}\log^2(N)} \tag{5.74}$$

$$\sim \frac{\beta\varepsilon^2}{\zeta(1+\alpha)} \int_{d_{FK}(\varepsilon)^{-1/2}}^{d_{FK}(\varepsilon)^{\beta/2}} \frac{y^{2\beta-\alpha}}{y^{2+2\beta}+2\beta^2} dy. \tag{5.75}$$

By dominated convergence theorem and taking into account previous estimates, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\Gamma(q_{FK}(\varepsilon)) = \frac{\beta}{\zeta(1+\alpha)} \int_0^{\infty} \frac{y^{2\beta-\alpha}}{y^{2+2\beta}+2\beta^2} dy. \tag{5.76}$$

This justifies (up to a multiplicative constant) the choice of q_{FK}. Similarly, it can be shown that

$$\mathbb{E}((1-\hat{\nu}(q_{FK}(\varepsilon)))^2) \leq c d_{FK}(\varepsilon)^{-1-2\kappa}\log^2(d_{FK}(\varepsilon)). \tag{5.77}$$

Hence

$$\varepsilon^{-3}\mathbb{E}((1-\hat{\nu}(q_{FK}(\varepsilon)))^2) \leq c d_{FK}(\varepsilon)^{-(1-\kappa)/2}. \tag{5.78}$$

This finishes the proof.

As present an alternative proof:

If $\beta > 0$, and N is big, then the expectation of ϑ_N is asymptotically equivalent to $\frac{N^{2\beta+1}}{\beta\log(N)}$. Thus, we see that the introduction of the drift $g(N)$ on the comb model increases drastically the expectation ϑ_N. As we will explain, this increase comes from particles that are trapped on the bottom of the teeth. On one hand, the probability of hitting the bottom of a deep trap is asymptotically equivalent to $\frac{2\beta\log(N)}{N}$. That is a situation relatively similar to the behavior on the comb model without drift, where that probability is of order N^{-1}. On the other hand, the expectation of the time that it takes for the walker to go from the bottom of the teeth to the surface is asymptotically equivalent to $\frac{N^{2\beta+2}}{2\beta^2\log^2(N)}$. Here is where the drifted comb model radically differs from the symmetric one, where this expectation is much smaller.

In fact, we can heuristically say that, in order to reach the bottom on a trap of depth N, the comb model must give $N^{\alpha+1}$ steps. It will take $N^{2\beta+2}$ time.

We aim to prove that $\Gamma(\epsilon)$ is regularly varying at 0. We will first do it neglecting the contribution of the excursions that do not reach the bottom of their teeth. We will also neglect the time that it takes to reach the bottom on a excursion which reach that bottom. Let us denote by p_N^b the probability of hitting the bottom before the backbone, starting from 1.

$$\epsilon m_N^b p_N^b - \epsilon^2/2 p_N^b \sigma_N^{2,b} \leq 1 - \hat{\vartheta}_N(\epsilon) \leq \epsilon m_N^b p_N^b. \tag{5.79}$$

We will show that the relevant traps are the ones of order $d(\epsilon)$ where

$$\frac{d(\epsilon)^{2\beta+2}}{2\beta^2\log^2(d(\epsilon))} := \epsilon^{-1} \tag{5.80}$$

That is, traps in which the expected exit time starting from the bottom is of order ϵ^{-1}.

Some long computations yields that

$$\sigma_N^{2,b} := \int_{\mathbb{R}} t^2 \vartheta_N(dt) \sim \frac{N^{4\beta+4}}{2\beta^4\log^4(N)}. \tag{5.81}$$

Note that, by the previous formula, we have that

$$\sigma_N^{2,b} \sim \frac{(m_N^b)^2}{2}. \tag{5.82}$$

Thus for $N \leq d(2\epsilon)$, we have that

$$\epsilon/2 m_N^b p_N^b \leq 1 - \hat{\vartheta}_N(\epsilon) \leq \epsilon m_N^b p_N^b \tag{5.83}$$

Neglecting deep traps (In fact, the bound that we find is not negligible, but of the same index at 0), we see that

$$\zeta(1+\alpha)^{-1}\sum_{0}^{d(2\epsilon)}\epsilon/2m_N^b p_N^b N^{-1-\alpha}\leq\Gamma(\epsilon)\leq\zeta(1+\alpha)^{-1}\sum_{0}^{d(\delta\epsilon)}\epsilon m_N^b p_N^b N^{-1-\alpha}\tag{5.84}$$

for δ small enough. Now we write the contribution of the excursions which reach the bottom of a very deep trap.

$$\sum_{d(\epsilon)}(1-\hat{\vartheta}_N(\epsilon))p_N^b N^{-1-\alpha}\leq\sum_{d(\epsilon)}p_N^b N^{-1-\alpha}.\tag{5.85}$$

It is easy to see that the right side of the above inequality is a regularly varying function at $\epsilon=0$ of index $(\alpha+1)/(2\beta+2)$. Denote $F(\epsilon):=\zeta(1+\alpha)^{-1}\sum_0^{d(\epsilon)}\epsilon/2m_N^b p_N^b N^{-1-\alpha}$. Then we have that

$$1/2F(2\epsilon)\leq\Gamma(\epsilon)\leq F(\delta\epsilon).\tag{5.86}$$

On the other side, it is not hard to deduce that F is a regularly varying function at zero of index $\frac{\alpha+1}{2\beta+2}$. Thus, we can deduce that $\Gamma(\epsilon)$ is also regularly varying at zero of index $\frac{\alpha+1}{2\beta+2}$.

It remains to show the variance assumption.

To neglect the time that the walk spends on excursion which do not reach the bottom, we will compute the moment-generating function of the time of a walk on a teeth with periodic boundary conditions Let $q_z(n)=\mathbb{P}(V_n=0)$, for all $n\geq 0$ and $0\leq z\leq N$. By the Markov property

$$q_z(n+1)=pq_{z+1}(n)+(1-p)q_{z-1}(n),1\leq z\leq N-1\tag{5.87}$$

We also define the boundary conditions as

$$q_0(0)=1;q_0(n)=0,n\geq 1;q_z(0)=0,z\geq 1\tag{5.88}$$

and

$$q_N(0)=1;q_N(n)=0,\text{ for all }n\geq 0.\tag{5.89}$$

For the moment-generating function $f_z(s)=\sum_{n=0}^{\infty}q_z(n)s^n$ we have

$$f_z(s)=spf_{z+1}(s)+s(1-p)f_{z-1}(s),\text{ for }1\leq z\leq N-1,\tag{5.90}$$

with the boundary conditions $f_0(s)=1$, and $f_N(1)=1$. We compute the moment generating function and we find that the expectation of that time is asymptotically equivalent to $\frac{N^2}{2\beta\log(N)}$. I think that it is not correct. For the symmetric case is N. In this case it will also be N!!! (I corrected the computations). The excursion which do not reach the origin will be neglected by choosing a scale of very large traps $r(\epsilon):=\epsilon^{-r}$ where r is to be chosen.

$$\sum_{r(\epsilon)}(1-\hat{\vartheta^s}_N(\epsilon))N^{-1-\alpha}\leq\sum_{r(\epsilon)}N^{-1-\alpha}.\tag{5.91}$$

The right hand side of the above inequality is a regularly varying function at $\epsilon = 0$ of index αr. Thus, choosing $r := \frac{1+\alpha}{\alpha(2+2\beta)}$, we have that the mentioned expression is of index $\frac{1+\alpha}{1+\beta}$. Now we compute the contribution of the short excursion on short teeth.

$$\sum^{r(\epsilon)^{-1}} (1 - \hat{\vartheta}^s{}_N(\epsilon))N^{-1-\alpha} \leq \epsilon \sum^{r(\epsilon)} m_N^s N^{-1-\alpha} \tag{5.92}$$

$$\sim \epsilon \sum^{r(\epsilon)} N^{-\alpha} \tag{5.93}$$

which is a regularly varying function of index $1 - \frac{(1-\alpha)(1+\alpha)}{\alpha(2+2\beta)}$. But, using that $\alpha < 1$, we see that such index is greater than $\frac{1+\alpha}{2+2\beta}$.

To check assumption (V) we bound $\mathbb{E}((1 - \hat{\nu}(q(\epsilon)))^2)$ by

$$\mathbb{E}((1 - \hat{\nu}(q(\epsilon)))^2) = \zeta(1+\alpha)^{-1} \sum_{N \in \mathbb{N}} (1 - \hat{\vartheta}_N(q(\epsilon)))^2 N^{-1-\alpha} \tag{5.94}$$

$$= \zeta(1+\alpha)^{-1} \sum^{d(\epsilon)} (1 - \hat{\vartheta}_N(q(\epsilon)))^2 N^{-1-\alpha} + \zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (1 - \hat{\vartheta}_N(q(\epsilon)))^2 N^{-1-\alpha} \tag{5.95}$$

We can bound

$$\zeta(1+\alpha)^{-1} \sum^{d(\epsilon)} (1 - \hat{\vartheta}_N(q(\epsilon)))^2 N^{-1-\alpha} \leq \zeta(1+\alpha)^{-1} \sum^{d(\epsilon)} (q(\epsilon) m_N)^2 N^{-1-\alpha} \tag{5.96}$$

where the right hand side of the above inequality is a regularly varying function of index $\frac{4+2\alpha}{\alpha+1}$ at $\epsilon = 0$. And

$$\zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (1 - \hat{\vartheta}_N(q(\epsilon)))N^{-1-\alpha} = \zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (p_N^b (1 - \hat{\vartheta}_N^b(q(\epsilon))))^2 N^{-1-\alpha} \tag{5.97}$$

$$+ \zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (1 - p_N^b)((1 - \hat{\vartheta}_N^s(q(\epsilon)))^2) N^{-1-\alpha}. \tag{5.98}$$

But

$$\zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (p_N^b (1 - \hat{\vartheta}_N^b(q(\epsilon))))^2 N^{-1-\alpha} \leq \zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (p_N^b)^2 N^{-1-\alpha} \tag{5.99}$$

where the right hand side of the above inequality is a slowly varying function of index $\frac{4+2\alpha}{1+\alpha}$ at $\epsilon = 0$. And

$$\zeta(1+\alpha)^{-1} \sum_{d(\epsilon)} (1 - p_N^b)((1 - \hat{\vartheta}_N^s(q(\epsilon)))^2) N^{-1-\alpha} \leq \zeta(1+\alpha)^{-1} q(\epsilon)^2 \sum_{d(\epsilon)} (m_N^s)^2 N^{-1-\alpha} \tag{5.100}$$

$\square$

CHAPTER 5

Geometric trap models on the IIC and the IPC

1. Introduction

Let $d \geq 2$ and T_d be a rooted, regular random tree with forward degree d. That is, T_d is a connected tree with a distinguished vertex ρ called the root, in which each vertex has degree $d+1$ except the root which has degree d.

Suppose we perform bond-percolation on T_d of parameter p. It is trivial to notice that the cluster of the root corresponds to the genealogical tree of a branching process whose offspring distribution is Binomial of parameters d and p. From this correspondence, it follows that the critical percolation parameter p_c equals $1/d$. Let $\mathcal{T}^n$ be a critical percolation cluster (of the root) conditioned on intersecting the boundary of a ball of radius n (with the graph-distance on T_d). With the branching-process terminology, $\mathcal{T}^n$ is a critical, genealogical-tree conditioned on surviving up to generation n. The *incipient infinite cluster* (IIC) on T_d is defined as the limit as $n \to \infty$ of $\mathcal{T}^n$. We will denote the IIC as $\mathcal{C}$. For details of the definition, we refer to [**Kes86b**].

It is a known fact (see [**Kes86b**]) that $\mathcal{C}$ possesses a single path to infinity, called the backbone. This backbone, denoted BB, is isomorphic (as a graph) to $\mathbb{N}$. Thus, $\mathcal{C}$ can be seen as $\mathbb{N}$ adorned with random branches. More precisely, $\mathcal{C}$ is a graph which is obtained by attaching a random, finite tree to each vertex of the backbone. We denote by $\mathcal{B}_k$ the branch attached to the k-th vertex of the backbone. We will not distinguish between the backbone and $\mathbb{N}$. In [**Kes86b**] it is showed that $(\mathcal{B}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of critical percolation clusters on $\tilde{T}_d$, where $\tilde{T}_d$ is a rooted graph in which each vertex has degree $d+1$, except for the root which has degree $d-2$.

Consider a simple random walk Z^{IIC} on $\mathcal{C}$. That is, for each fixed realization of $\mathcal{C}$, Z^{IIC} is a discrete-time, symmetric random walk on $\mathcal{C}$ starting from the root. We can project Z^{IIC} to the backbone, to obtain a symmetric random walk Z^{IIC} on $\mathbb{N}$ with random jump times. The jumping times at a site $k \in \mathbb{N}$ will correspond to the times that Z^{IIC} spends on each visit to $\mathcal{B}_k$.

Our first goal is to identify the scaling limit for W^{IIC}. We will briefly discuss a related process to have a feeling about what kind of process should arise as the scaling limit. The one-dimensional *Bouchaud trap model* (BTM) is a random walk on a random media taking values on $\mathbb{Z}$: Let $(\tau_z)_{z\in\mathbb{Z}}$ be an i.i.d.

sequence of positive random variables satisfying

$$\lim_{x\to\infty} x^{-\alpha}\mathbb{P}(\tau_0 \geq x) = 1 \tag{1.1}$$

for some $\alpha \in (0,1)$. This indicates that the distribution of τ_0 has heavy tails.
Once the media $\boldsymbol{\tau} := (\tau_z)_{z\in\mathbb{Z}}$ is fixed, we define a Markov process $X^{\boldsymbol{\tau}}$ on $\mathbb{Z}$, starting from 0, stating that each time that $X^{\boldsymbol{\tau}}$ visits a site $z \in \mathbb{Z}$, it stays there an exponentially distributed time with mean τ_z, then it jumps to $z+1$ with probability $1/2$ and to $z-1$ with probability $1/2$. The Bouchaud trap model is obtained from $X^{\boldsymbol{\tau}}$ by averaging with respect to the media distribution. Thus, the BTM is a random walk moving on a random landscape composed of traps. The analogy between the one-dimensional BTM and W^{IIC} is evident: W^{IIC} is a one-dimensional, symmetric random walk moving on a random landscape composed by the random trees $(\mathcal{B}_k)_{k\in\mathbb{N}}$. These random trees play the role of geometric traps which hold the walker by some time.

The scaling limit of the one-dimensional BTM has been identified on [**FIN02**] as a quasi-diffusion moving on a random media. More precisely, let V_α be a two-sided α-stable subordinator. Is well know that V_α is a pure-jump process, thus, its corresponding Lebesgue-Stieltjes measure ρ is purely-atomic. We can write

$$\rho := \sum_{k\in\mathbb{N}} y_i \delta_{x_i}. \tag{1.2}$$

Let B be a standard, one-dimensional Brownian motion independent from V_α and $l(x,t)$ be its corresponding local time. Define

$$\phi[\rho]_t := \int_{\mathbb{R}} l(x,t)\rho(dx) = \sum_{i\in\mathbb{N}} l(x_i,t) y_i \tag{1.3}$$

and its right-continuous generalized inverse

$$\psi[\rho]_t := \inf\{s \geq 0 : \phi[\rho]_s \geq t\}. \tag{1.4}$$

We define the *Fontes Isopi Newman singular diffusion* (FIN), denoted by Z as

$$Z_t := B_{\psi[\rho]_t}. \tag{1.5}$$

Thus, the FIN diffusion is a speed measure changed Brownian motion, with ρ as its random speed measure. In [**FIN02**] it was showed that the FIN diffusion is the scaling limit of the one-dimensional BTM.

The FIN diffusion is a sub-diffusive process. The sub-diffusivity is caused by the fact that the media is highly inhomogeneous (because τ_0 is heavy-tailed). It is not hard to show that if the media is tame (if τ_0 has finite expectation), then the scaling limit of the BTM would be simply a Brownian motion. The media of W^{IIC} can be regarded as a heavy-tailed one, because the random branches $(\mathcal{B}_k)_{k\in\mathbb{N}}$ are critical percolation cluster, and it is a known fact that thus critical percolation clusters will create a highly inhomogeneous landscape. For instance, it is not hard to show that

$$\lim_{x\to\infty} x^{1/2}\mathbb{P}(|\mathcal{L}_0|\geq x)=1. \tag{1.6}$$

where $|\cdot|$ denotes cardinality. This suggests that the scaling limit of W^{IIC} should be a process similar to the FIN diffusion, if not the FIN diffusion itself.

The walk projected to the backbone W^{IIC} belongs to a general class of processes called *Randomly trapped random walks* (RTRW). These RTRW's are one-dimensional, continuous-time, symmetric random walks that move on a random landscape of traps. These processes where studied on chapter 4. There are given conditions under which the scaling limit of a RTRW is a *Randomly trapped Brownian motion* (RTBM). Randomly trapped Brownian motions constitute a general class of symmetric, one-dimensional processes which are obtained as certain random-time changes of a Brownian motion. As particular cases of RTBM we can mention the FIN diffusion, the *Fractional kinetics process* (FK), speed-measure changed Brownian motion and Brownian motion itself. Furthermore the RTBM family is much broader than that, in particular it contains a class of processes called the *Generalized FIN diffusions* (GFIN). The GFIN family is a class of processes which are an enrichment of the FIN diffusion: The FIN diffusion is a speed-measure changed Brownian motion through a random, purely atomic speed-measure ρ. Each atom $y_i\delta_{x_i}$ represents a trap of depth y_i located at x_i. Thus, in the FIN diffusion each trap is characterized by its location and a single parameter, namely, its depth. The GFIN diffusions are also Brownian motions moving among random traps, but in this case the traps will be much more complex, they will be characterized by their location and a whole stochastic process (instead of just their depth).

In this chapter we will show that the scaling limit of W^{IIC} is a generalized FIN diffusion. Thus we are lead to answer the following question: what kind of (complex) traps appear on the scaling limit?. To identify the processes appearing on our traps we will have to understand the behavior of the times that Z^{IIC} spend on the visits to a large percolation cluster. We will see that the answer to this question is strongly related to the *Continuum random tree*(CRT) of Aldous. The CRT appears as a scaling limit for a some families of large random trees. In particular, the CRT is the scaling limit of the conditioned percolation trees $\mathcal{T}^n$ as $n\to\infty$ (see [**Ald93**]). But, as we have said, we will be concerned not only with $\mathcal{T}^n$, but with a simple random walk on $\mathcal{T}^n$. The scaling limit (as $n\to\infty$) of such walks has been identified by Croydon on [**Cro08**] as the *Brownian motion on the Continuum random tree*. Using his ideas we will show that the local time at the root of the simple random walks on $\mathcal{T}^n$ converges to the local time at the root of the continuum random tree. It is fairly easy to see that the inverse of the local time at the root of the simple random walks on $\mathcal{T}^n$ gives the successive excursion times away from the root of that random walk, which is the object we want to control. Thus we will see that the processes appearing on the traps of the scaling limit of W^{IIC} are the inverse local times at the root of the Brownian motion on the CRT. More specifically, the construction of this environment can be performed as follows: First, we randomly choose a measure ρ in exactly the same way as we did in the definition of the FIN diffusion. That is, a random measure associated to a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $\alpha y^{-1-\alpha} dx dy$. Now, for each atom $y_i \delta_{x_i}$ of ρ, we randomly choose a realization of the CRT conditioned on having "size" y_i. Then, the trap located at x_i will be characterized by the inverse local time process (at the root) of the Brownian motion on that realization of the CRT. Then, the scaling limit of W^{IIC} will can be seen as a Brownian motion moving among these complex traps.

In order to state precisely our results, we need to define the Continuum random tree Let $\mathcal{W} := \{w : [0,\infty) \to [0,\infty) : w \text{ is continuous}; w(t) = 0 \text{ if and only if } t \in (0,1)\}$ be the space of (positive) excursions away from 0 of duration 1. Given $w \in \mathcal{W}$, we define a pseudometric d_w over $[0,1]$ by

$$d_w(s,t) := w(s) + w(t) - 2\inf\{w(r) : r \in [s \wedge t, s \vee t]\}. \tag{1.7}$$

Define the equivalence relation $\sim$ on $[0,1]$ by stating that $s \sim t$ if and only if $d_w(s,t) = 0$. Then define the topological space $\mathcal{T}_w := [0,1]/\sim$. We denote by $[r]$ the equivalence class of $r \in [0,1]$. We can endow $\mathcal{T}_w$ with a metric $d_{\mathcal{T}_w}([s],[t]) := d_w(s,t)$. The space $\mathcal{T}_w$ is arc-connected and contains no subspace homeomorphic to the circle, that is, $\mathcal{T}_w$ is a *dendrite*. Moreover $d_{\mathcal{T}_w}$ is a *shortest-path* metric, that is, $d_{\mathcal{T}_w}$ is additive along the non-self intersecting paths of $\mathcal{T}_d$. The Lebesgue measure λ on $[0,1]$ induces a probability measure μ_w over $\mathcal{T}_w$ by

$$\mu_w(A) := \lambda(\{t \in [0,1] : [t] \in A\}) \tag{1.8}$$

for any Borelian $A \subset \mathcal{T}_w$.

Now, let $W = (W_t)_{t\in[0,1]}$ be a random process defined on a probability space $(\mathcal{X}, \mathcal{G}, P)$ having the law of a normalized Brownian excursion. Clearly, W can be viewed as a random object taking values in $\mathcal{W}$. Thus, starting from the Brownian excursion W, we can construct a random dendrite $\mathcal{T}_W$, equipped with a shortest path metric d_{T_W} and a measure μ_W.

DEFINITION 5.1. *The triple $(\mathcal{T}_W, d_{T_W}, \mu_W)$ is the* ***Continuum Random Tree*** .

Having defined the CRT, we prepare the definition of the Brownian motion on the CRT. We need to recall some facts about processes on abstract trees. Let $\mathcal{K}$ be a compact dendrite equipped with a shortest-path metric $d_{\mathcal{K}}$ and a σ-finite Borel measure ν. We will suppose that $\nu(A) > 0$ for any non-empty open set $A \subset \mathcal{K}$. As $\mathcal{K}$ is a dendrite, it is easy to show that, for all $x, y \in \mathcal{K}$, there exists a unique (non-self intersecting) path from x to y. We denote such a path by $[[x,y]]$ Fox all $x, y, z \in \mathcal{K}$ we define the **branching point between** x, y **and** z as the unique point $b^{\mathcal{K}}(x,y,z)$ that satisfies

$$\{b^{\mathcal{K}}(x,y,z)\} := [[x,y]] \cap [[x,z]] \cap [[y,z]]. \tag{1.9}$$

Proposition 2.2 in [**Cro08**] ensures the existence of a reversible Markov process $((X_t^{\mathcal{K},\nu})_{t\geq 0}, (P_z^{\mathcal{K},\nu})_{z\in\mathcal{K}})$ taking values in $\mathcal{K}$ with the following properties

(1) Continuous sample paths.

(2) Strong Markov.

(3) Reversible with respect to its invariant measure ν.

(4) For $x, y \in \mathcal{K}, x \neq y$ we have

$$P_z^{\mathcal{K},\nu}(\sigma_x < \sigma_y) = \frac{d_{\mathcal{K}}(b^{\mathcal{K}}(z,x,y),y)}{d_{\mathcal{K}}(x,y)}, \text{ for all } z \in \mathcal{K}$$

where $\sigma_x := \inf\{t > 0 : X_t^{\mathcal{K},\nu} = x\}$ is the hitting time of x.

(5) For $x, y \in \mathcal{K}$, the mean occupation measure for the process started at x and killed on hitting y has density

$$2d_{\mathcal{K}}(b^{\mathcal{K}}(z,x,y),y)\nu(dz) \text{ for all } z \in \mathcal{K}.$$

Moreover, in section 5.2 of [**Ald91b**] is shown that such a process must be unique (in law).

Definition 5.2. *The process $X^{\mathcal{K},\nu}$ is the* ***Brownian motion on*** $(\mathcal{K}, d_{\mathcal{K}}, \nu)$.

Definition 5.3. *The CRT is always compact, thus there is no problem in defining the* ***Brownian motion on the Continuum Random Tree*** *as $X^{\mathcal{T}_W,\mu_W}$.*

Now we focus in the construction of the scaling limit. Let $V^{1/2}$ be a two-sided 1/2-stable subordinator. Let ϱ be its corresponding Lebesgue-Stieltjes random measure. As we have said, ϱ will be a purely-atomic measure. Thus we can write

$$\varrho = \sum_{i\in\mathbb{Z}} \bar{y}_i \delta_{\bar{x}_i}. \tag{1.10}$$

Let $(\mathcal{D}_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random dendrites having the distribution of the CRT and independent of ϱ. For any $i \in \mathbb{N}$ and any fixed realization of $\mathcal{D}_i$, we let $(B_t^i)_{t\geq 0}$ be the Brownian motion on $\mathcal{D}_i$ and $(L^i(\rho)_t)_{t\geq 0}$ be its local time at the root. Let $(L_t^{i,\leftarrow})_{t\geq 0}$, be the corresponding right-continuous generalized inverse. Finally, let $(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion independent of the processes just defined and let $l(x,t)$ be its local time. Define

$$\phi_t^{IIC} := \sum_{i\in\mathbb{N}} \bar{y}_i^{3/2} L^{i,\leftarrow}_{\bar{y}_i^{-1/2} l(\bar{x}_i,t)} \tag{1.11}$$

and its right-continuous generalized inverse

$$\psi_t^{IIC} := \inf\{s \geq 0 : \phi_s^{IIC} \geq t\}. \tag{1.12}$$

The scaling limit of W^{IIC} will be $(B_{\psi_t^{IIC}})_{t\geq 0}$

Having defined the scaling limit, we proceed to state the define the topology we will use. We write $D(\mathbb{R}_+)$, $D(\mathbb{R})$ for the sets of real-valued *càdlàg* functions on $\mathbb{R}_+$, or $\mathbb{R}$, respectively. We endow these sets either with the standard Skorokhod J_1-topology, or with the so called M_1'-topology, and write $D((\mathbb{R}_+), J_1)$, $D((\mathbb{R}_+), M_1')$ when we want to stress the topology used. The main reason to use the M_1'-topology, which is weaker than the J_1-topology, is that the inversion map on $(D(\mathbb{R}_+), M_1')$ is continuous when restricted to monotonous functions. Note also that the convergence in the M_1'-topology implies the convergence in the stronger and slightly more usual M_1-topology, when the limit f satisfies $f(0) = 0$. For definitions and properties of these topologies see [**Whi02**], the proofs of above claims are contained in Section 13.6 of this book. We are ready to state the theorem:

THEOREM 5.4. *$(\epsilon W^{IIC}_{\epsilon^{-3}t})_{t\geq 0}$ converges in distribution to $(B_{\psi_t^{IIC}})_{t\geq 0}$ on $(D(\mathbb{R}_+), M_1')$, as $\epsilon \to 0$.*

Now we will refer to the *Invasion percolation cluster* (IPC) on a regular tree. The IPC is a random graph obtained by a stochastic growth process and it was introduced by Wilkinson and Willemsen on [**WW83**]. We pass to recall its construction. Suppose we have an infinite, connected graph G with a distinguished vertex o. We randomly assign, to each edge e, a weight w_e. We assume the family $(w_e)_{e \text{ vertex of } G}$ to be i.i.d. and uniformly distributed over $[0, 1]$. Then define C_0 as o. C_1 will be obtained from C_o by adding the neighbor x_1 of o whose corresponding edge has smaller weight. That is $w_{x_1} = \min\{w_y : x \sim o\}$. Generally, C_n is constructed from C_{n-1} by attaching the vertex on the outer boundary of C_{n-1} with smaller weight. The invasion percolation cluster is $C_\infty := \cup_{n\in\mathbb{N}} C_n$. We will be concerned with the IPC on the regular tree, this is, when $G = T_d$. We denote the IPC on T_d by $\mathcal{C}_\infty$.

Like in the IIC case, it can be shown that $\mathcal{C}_\infty$ is composed by a single infinite backbone, denoted BB, which is a graph isomorphic to $\mathbb{N}$, and, from every vertex of the backbone, there emerges a branch (see [**AGdHS08**]). We denote $\mathcal{L}_k$ the branch emerging from the k-th vertex of backbone. In fact, in that article there is given an structure theorem for the IPC on a regular tree: Let B_l be the weight of the l-th vertex of the backbone and $B^k := \sup_{l>k} B_l$. Conditioned on a fixed realization of $(B^k)_{k\in\mathbb{N}}$, the sequence of branches $(\mathcal{B}_k)_{k\in\mathbb{N}}$ is an independent sequence of trees where each $\mathcal{B}_k$ is distributed as a supercritical percolation cluster on $\tilde{T}_d$ with parameter B^k conditioned to stay finite (see [**AGdHS08**, Proposition 2.1]). The percolation parameter B^k corresponding to the cluster attached at $k \in \mathbb{N}$ converges to $p_c = 1/d$ as k goes to ∞. In fact, it can be shown [**AGdHS08**, Proposition 3.3] that for any $\epsilon > 0$

$$(k[dB^{\lceil kt\rceil} - 1])_{t>\epsilon} \Longrightarrow^* (L_t)_{t>\epsilon} \tag{1.13}$$

where $\Longrightarrow^*$ denotes convergence in distribution in the space of cadlag paths endowed with the Skorohod topology and L_t is the lower envelope of a homogeneous Poisson point process in $\mathbb{R}^+ \times \mathbb{R}^+$. More specifically, let $\mathcal{P}$ be a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}^+$ with intensity 1, then

$$L_t := min\{y : (x, y) \in \mathcal{P} \text{ for some } x \leq t\}. \tag{1.14}$$

One can use duality of the percolation to see that a supercritical cluster, with parameter p, conditioned to stay finite is distributed as a subcritical cluster with dual parameter $\tilde{p}$ which satisfies (see [**AGdHS08**, Lemma 2.2])

$$p - p_c \sim p_c - \tilde{p} \text{ as } p \downarrow p_c. \tag{1.15}$$

Here $\sim$ denotes asymptotic equivalence. Thus, $\mathcal{C}_\infty$ can be viewed as an infinite backbone adorned with subcritical clusters having parameter $(\tilde{B}^k)_{k\in\mathbb{N}}$, and (1.15) implies that (1.13) also holds with B replaced by $\tilde{B}$.

We want to study a simple symmetric random walk on $\mathcal{C}_\infty$ projected on the backbone. Let $(Z_n^{IPC})_{n\in\mathbb{N}}$ be a simple symmetric random walk on $\mathcal{C}_\infty$. Define

$$\sigma_0 := 0$$

$$\sigma_k := \min\{n > \sigma_k : Y_n \in BB - \{Y_{\sigma_{k-1}}\}\}.$$

Then we define $(W_t^{IPC})_{t\in\mathbb{R}_+}$ as

$$W_t^{IPC} := Z_j^{IPC}$$

where $\sigma_j \le t < \sigma_{j+1}$. It is clear that $(W_t^{IPC})_{t\in\mathbb{R}_+}$ is a time-change of a symmetric random walk on the backbone. We want to prove that the scaling limit of W^{IPC} is a specific randomly trapped Brownian motion which is neither FIN nor FK, but a GFIN process. We will also show that this process is different to the scaling limit of W^{IIC}.

The difference between the scaling limits of W^{IIC} and W^{IPC} stems on the random measure used to construct the traps. In the IIC case, we used a random atomic measure ϱ to set the locations and "sizes" of the traps. The measure ρ was constructed by means of a 1/2-stable subordinator. In the IPC, this measure will not be related to an α-stable subordinator, but to an *inverse Gaussian subordinator* with changing parameters. Thus, the way we select the "sizes" of our traps is different. The rest of the construction of the environment is the same for both cases. This differences comes from the fact that the percolation parameter of the attached percolation clusters to the backbone is always critical in the IIC. Whereas, in the IPC, this percolation parameter is subcritical and varies depending on the vertex of the backbone that it corresponds.

Now we state the theorem that gives the scaling limit for W^{IPC}. Let L_t as in (1.14). For each realization of L_t, let $(b_i)_{i\in\mathbb{R}_+}$ be a enumeration of the points of discontinuity of L_t. For each b_i, let $a_i := \max\{b_j : b_j < a_i\}$. For $t \in \mathbb{R}_+$ let $a^* := \min\{a_i > t\}$. Then define

$$E_t := \sum_{i:a_i<t} V^i_{b_i - a_i} + V^*_{t-a^{ast}}$$

where (V^i) is an independent family of inverse Gaussian subordinators, each one with parameters $\delta = \sqrt{\frac{d-1}{d}}$ and $\gamma = 2L_{a_i}\sqrt{\frac{d-1}{d}}$ and V^* is an inverse Gaussian subordinator with parameters $\delta = \sqrt{\frac{d-1}{d}}$ and $\gamma = 2L_{a^*}\sqrt{\frac{d-1}{d}}$. The process E is increasing and purely-atomic, thus, its associated Lebesgue-Stiletjes random measure ι is purely atomic. Thus we can write

$$\iota = \sum_{i \in \mathbb{Z}} \tilde{y}_i \delta_{\tilde{x}_i}. \tag{1.16}$$

Let $(\mathcal{D}_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random dendrites having the distribution of the CRT and independent of ϱ. For any $i \in \mathbb{N}$ and any fixed realization of $\mathcal{D}_i$, we let $(B^i_t)_{t\geq 0}$ be the Brownian motion on $\mathcal{D}_i$ and $(L^i(\rho)_t)_{t\geq 0}$ be its local time at the root and $L^{i,\leftarrow}$ be the corresponding right-continuous generalized inverse. Finally, let $(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion independent of the processes just defined and let $l(x,t)$ be its local time. Define

$$\phi^{IPC}_t := \sum_{i\in\mathbb{N}} \tilde{y}_i^{3/2} L^i(\rho)_{\tilde{y}_i^{-1/2} l(\tilde{x_i},t)} \tag{1.17}$$

and its right-continuous generalized inverse

$$\psi^{IPC}_t := \inf\{s \geq 0 : \phi^{IIC}_s \geq t\}. \tag{1.18}$$

The scaling limit of W^{IPC} will be $(B_{\psi^{IPC}_t})_{t\geq 0}$, more precisely

THEOREM 5.5. *$(\epsilon W^{IPC}_{\epsilon^{-3}t})_{t\geq 0}$ converges in distribution to $(B_{\psi^{IPC}_t})_{t\geq 0}$ in the Skorohod M_1 topology, as $\epsilon \to 0$.*

One can make a comparison between the inverse Gaussian subordinator and the 1/2-stable subordinator. It is well known that T_t has a distribution with density f_{T_t} such that

$$f_{T_t}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \gamma} s^{-3/2} \exp(-1/2(t^2\delta^2 s^{-1} + \gamma^2 s)).$$

Also T has Levy measure with density

$$2\sqrt{2}\Gamma(1/2)^{-1}\delta t s^{-3/2} \exp(-1/2\gamma s).$$

An 1/2- stable subordinator can be viewed as the passage times of the Brownian motion. If we define

$$\tilde{V}(t) := \inf\left\{ s : B_s = \frac{t}{\sqrt{2}} \right\}$$

then $\tilde{V}$ is an 1/2-stable subordinator. In order to obtain the inverse Gaussian subordinator, we just have to consider the passage times of a Brownian motion with drift. If we define

$$V(t) := \inf\left\{ s : B(s) + \gamma s = \delta t \right\}$$

then V is an Inverse Gaussian subordinator with parameters δ and γ. Thus, it become intuitively clear that the small jumps of the two processes look similar, because the Brownian motion with drift is locally like Brownian motion, but the large jumps of T are smaller than those of $\tilde{V}$.

2. Preliminaries

2.1. The Continuum random tree. For each $n \in \mathbb{N}$, let $\mathcal{T}^n$ be a random tree having the law of the connected component of the root under critical percolation on $\mathcal{T}_d$, conditioned on having n vertices. We will regard $\mathcal{T}^n$ as an ordered, rooted tree. The order being the one induced from $\mathcal{T}_d$ and the root ρ being the root of $\mathcal{T}_d$. The random tree $\mathcal{T}^n$ increases its size as n goes to infinity. One might ask if it is possible to find a scaling limit for the sequence $(\mathcal{T}^n)_{n\in\mathbb{N}}$ as $n \to \infty$. Aldous settled this question in [**Ald93**] where he showed that the scaling limit of $\mathcal{T}^n$ is the Contimuun Random Tree.

Now we will briefly recall a way of describing trees through excursions. For the details of this description, we will refer to [**Ald93**]. Let T^n be an ordered, rooted tree having n vertices. Without loss of generality, we can suppose that the root of T^n is ρ. Let $\tilde{w}_n : \{1, 2, .., 2n-1\} \to T^n$ be the *depth-first search around* T^n. For the definition of $\tilde{w}_n$ we refer to [**Ald93**]. Here we content to give a heuristical description of $\tilde{w}_n$. Suppose T^n is embedded in the plane in a way that sons are "above" their progenitor and siblings are ordered from left to right according to they order on T^n. Then $\tilde{w}_n$ moves along the vertices of T^n "clockwise" (according to the embedding in the plane), starting from the root and ending on the root. Note that $\tilde{w}_n$ respects the order of T_n, in the sense that, for any pair of vertices $x_1, x_2 \in T_n$ with $x_1 \preceq x_2$, there exists $i_1, i_2 \in \{1, .., 2n-1\}$ with $i_1 \le i_2$ such that $\tilde{w}_n(i_1) = x_1$ and $\tilde{w}_n(i_2) = x_2$. Define the *search-depth process* ω_n by

$$\omega_n(i/2n) := d_{T^n}(\rho, \tilde{\omega}_n(i)), \qquad 1 \le i \le 2n-1 \tag{2.1}$$

where d_{T^n} is the graph distance on T^n. We also set $w_n(0) = w_n(1) = 0$ and extend w_n to the whole interval $[0, 1]$ by linear interpolation.

The critical percolation cluster on the regular tree T_d can also be viewed as a critical branching process, whose offspring distribution ξ is Binomial of parameters $n = d$ and $p = 1/d$. Let $\sigma^2 := var(\xi) = (d-1)/d$. Let w_n denote the search depth process associated to the trees $\mathcal{T}_n$. Theorem 23 on [**Ald93**] states that

$$(\sigma n^{-1/2} w_n(t))_{t\in[0,1]} \xrightarrow{d} (2W_t)_{t\in[0,1]} \qquad \text{as } n \to \infty \tag{2.2}$$

on $C[0, 1]$ endowed with the uniform topology. Moreover, Theorem 20 on [**Ald93**] states that the convergence on display (2.2) is equivalent to the convergence of (a rescaling of) the sequence of random trees $(\mathcal{T}^n)_{n\in\mathbb{N}}$ to the CRT $\mathcal{T}_W$ on a sense that is specified there.

We now turn our attention to the large scale behavior of a sequence of random walks on $\mathcal{T}^n$. Let $\mathcal{X}^n = (\mathcal{X}^n_m)_{m\in\mathbb{N}}$ be a simple random walk on $\mathcal{T}^n$ starting from the root. In [**Cro08**], it is shown that the scaling limit for the sequence of processes $\mathcal{X}^n$ is the Brownian Motion on the Continuum Random Tree.

Next, we prepare the definitions to state the mentioned convergence result Let K be a compact metric space, ν a Borel probability measure on K and Q be a probability measure over $C([0, R], K)$, where $R > 0$.

We say that $(\tilde{K}, \tilde{\nu}, \tilde{Q})$ is an **isometric embedding of** (K, ν, Q) **into** l^1 (the Banach space of summable sequences) if there exists a distance-preserving map $\Psi : K \to l^1$ such that $\Psi(K) = \tilde{K}$, $\nu \circ \Psi^{-1} = \tilde{\nu}$ and $Q \circ \Psi^{-1} = \tilde{Q}$. Let $\tilde{K}(l^1)$ be the space of compact subsets of l^1 endowed with the usual Hausdorff topology for compact subsets of l^1. We will also denote by $\mathcal{M}_1(l^1)$ and $\mathcal{M}_1(C([0,R],l^1))$ the space of probability measures over l^1 and $C([0,R],l^1)$ respectively. Those two spaces are endowed with their respective weak topologies. We define the rescaling operators $\Theta_n : \tilde{K}(l^1) \times \mathcal{M}_1(l^1) \times \mathcal{M}_1(C([0,R],l^1)) \to \tilde{K}(l^1) \times \mathcal{M}_1(l^1) \times \mathcal{M}_1(C([0,1],l^1))$ by

$$\Theta_n(\tilde{K}, \tilde{\nu}, \tilde{Q}) := (n^{-1/2}\tilde{K}, \tilde{\nu}(n^{1/2}\cdot), \tilde{Q}(\{f \in C([0,R],l^1) : (n^{-1/2}f(tn^{3/2}))_{t\in[0,1]} \in \cdot\})).$$

We are ready to state main result of [**Cro08**].

THEOREM 5.6. *(Croydon) There exists a set $\mathfrak{W}^* \subset C([0,1],[0,\infty))$ with $P(W \in \mathfrak{W}^*) = 1$ such that if $(T^n)_{n\in\mathbb{N}}$ is a sequence of ordered graph trees whose search-depth functions $(w_n)_{n\in\mathbb{N}}$ satisfy*

$$n^{-1/2}w_n \to w$$

in $C([0,1],[0,\infty))$ with the uniform topology for some $w \in \mathfrak{W}^*$, then, for each $n \in \mathbb{N}$, there exists an isometric embedding $(\tilde{T}^n, \tilde{\mu}_n, P^{\tilde{T}_n})$ of (T^n, μ_n, P^{T_n}) into l^1 such that

$$\Theta_n(\tilde{T}^n, \tilde{\mu}_n, P^{\tilde{T}_n}) \to (\tilde{\mathcal{T}}, \tilde{\nu}, P^{\tilde{\mathcal{T}}})$$

in the space

$$\mathcal{K}(l^1) \times \mathcal{M}_1(l^1) \times \mathcal{M}_1(C([0,1],l^1))$$

where $(\tilde{\mathcal{T}}, \tilde{\nu}, P^{\tilde{\mathcal{T}}})$ is an isometric embedding of the triple $(\mathcal{T}_w, \mu_w, P^{\mathcal{T}_w,\mu_w})$ into l^1.

Note that the theorem above is *quenched*, because the convergence holds for all $w \in \mathfrak{W}^*$ and $P(W \in \mathfrak{W}^*) = 1$. Theorem 1.2 in [**Cro08**] gives the *annealed* version of Theorem 5.6.

3. Proof of Assumption (PPO) and Assumption (PP) for W^{IIC} and W^{IPC} respectively

We will first focus on W^{IIC}. We aim to prove Assumption PP0 for the RTRW W^{IIC}. For any realization of $(\mathcal{B}_k)_{k\in\mathbb{N}}$, we define σ_k as

$$\sigma_k := \min\{n > 0 : Y_n^k \in BB - \{k\}\}$$

where $(Y_n^k)_{n\in\mathbb{N}}$ is a symmetric random walk on $\mathcal{C}$ started from the k-th vertex of the backbone. Let m_k be the mean of the time spent in a single visit to $\mathcal{B}_k$ conditioned on the branch $\mathcal{B}_k$

$$m_k := \mathbb{E}(\sigma_k|\mathcal{B}_k).$$

We aim to prove

LEMMA 5.7. *Assumption PP0 holds for the RTRW W^{IIC} with $\gamma = 1/2$. That is*

$$\lim_{x\to\infty} x^{1/2}\mathbb{P}[\pi \in M_1(\mathbb{R}_+) : m_0 > x] = 1. \tag{3.1}$$

PROOF. It turns out that the distribution of m_0 depends only on the size $|\mathcal{B}_0|$ of $\mathcal{B}_0$ and not on his structure. Moreover, they are equal, (see [**Kes86b**, Lemma 2.28]). Using that, it is easy to compute the Laplace transform $\hat{m_0}$ of m_0 using that m_0 is the size of a subcritical percolation cluster of parameter p_c. That can be done by conditioning on the status of the edges emerging from the root. The condition follows easily. $\square$

Now we deal with W^{IIC}. Let $(I_x)_{x\in\mathbb{R}_+}$ be defined as

$$I_x := \iota(0, x] \tag{3.2}$$

where ι is the function defined in (1.16). We aim to show that for the IPC case, Assumption (PP) holds with $(I_x)_{x\in\mathbb{R}}$ as the limiting process.

LEMMA 5.8. *Let $M_x^{IPC} := \sum_0^{\lfloor x\rfloor} \bar{m}_i$, where $\bar{m}_i$ is the mean of the duration of each visit to W^{IPC} to $\mathcal{L}_i$. Then*

$$(\epsilon^2 M^{IPC}_{\epsilon^{-1}x})_{x\in\mathbb{R}} \overset{\epsilon\to 0}{\to} (I_x)_{x\in\mathbb{R}} \tag{3.3}$$

in distribution with the Skorohod J_1-topology.

As $L(t)$ (the limiting process of $(\tilde{B}^k)_{k\in\mathbb{N}}$) is piecewise constant, it will be helpful to consider a related, simpler, sequence of processes. Let $\mathcal{C}_\epsilon$ be the random tree constructed by attaching to each vertex $k \in \mathbb{N}$ in the backbone, i.i.d. branches $\mathcal{B}_k^{(\epsilon)}$ which are distributed as subcritical percolation clusters on $\tilde{T}_d$ with parameter p_ϵ such that $1 - dp_\epsilon \sim a\epsilon$ where a is a constant positive number. For any k and any realization of $(\mathcal{B}_k^{(\epsilon)})_{k\in\mathbb{N}}$, we define $\tilde{\sigma}_k^{(\epsilon)}$ as

$$\tilde{\sigma}_k^{(\epsilon)} := \min\{n > \tilde{\sigma}_k^{(\epsilon)} : Y_n^{(\epsilon,k)} \in BB - \{Y^{(\epsilon,k)}\}\}$$

where $(Y_n^{(\epsilon,k)})_{n\in\mathbb{N}}$ is a symmetric random walk on $\mathcal{C}_\epsilon$ started from the k-th vertex of the backbone. Let $m_k^{(\epsilon)}$ be the mean of the time spent by a simple random walk on a single visit to the branch $\mathcal{B}_k^{(\epsilon)}$, i.e.

$$m_k^{(\epsilon)} := \mathbb{E}(\tilde{\sigma}_k^{(\epsilon)} | \mathcal{B}_k^{(\epsilon)}).$$

We denote

$$M_x^{(\epsilon)} := \sum_{i=0}^{\lfloor x\rfloor} m_i^{(\epsilon)}$$

The process $(M_x^{(\epsilon)} : x \in \mathbb{R}^+)$ describes the environment generated by the branches $(\mathcal{B}_k^{(\epsilon)})_{k\in\mathbb{N}}$, but only keeping track of the expectation of $\tilde{\sigma}_k^{(\epsilon)}$.

LEMMA 5.9. *$(\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}x})_{x\geq 0}$ converges to an inverse Gaussian subordinator $(T_x)_{x\geq 0}$ with parameters $\delta = \sqrt{\frac{d-1}{d}}$ and $\gamma = 2a\sqrt{\frac{d-1}{d}}$, which is a subordinator characterized by*

$$\mathbb{E}(\exp(-\lambda T_x)) = \exp(-x\delta(\sqrt{2\lambda+\gamma^2}-\gamma)).$$

We will first establish convergence of single-time distributions.

LEMMA 5.10.

$$\mathbb{E}(\exp(-\lambda(\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}x}))) = \exp(-x\delta(\sqrt{2\lambda+\gamma^2}-\gamma)).$$

PROOF. As we have said, $m_k^{(\epsilon)}$ is equal to the size $N_k^{(\epsilon)}$ of $\mathcal{B}_k^{(\epsilon)}$. It is easy to compute the Laplace transform $\hat{N}^{(\epsilon)}$ of $N_k^{(\epsilon)}$ using that $N_k^{(\epsilon)}$ is the size of a subcritical percolation cluster of parameter p_ϵ. That can be done by conditioning on the status of the edges emerging from the root. It is easier to first compute the Laplace transform of $N_*^{(\epsilon)}$ which is the size of a percolation cluster on $\mathcal{T}_d$ with parameter p_ϵ. We will first treat the case $d = 2$. In this case we have

$$\hat{N}_*^{(\epsilon)}(\lambda) = \exp(-\lambda)[(1-p_\epsilon) + p_\epsilon \hat{N}_*^{(\epsilon)}]^2$$

Thus

$$\hat{N}_*^{(\epsilon)}(\lambda) = \frac{1 - 2p_\epsilon(1-p_\epsilon)\exp(-\lambda) - \sqrt{1-4p_\epsilon(1-p_\epsilon)\exp(-\lambda)}}{2p_\epsilon^2\exp(-\lambda)}.$$

Now, it is easy to see that

$$\hat{N}^{(\epsilon)} = \exp(-\lambda)(p_\epsilon \hat{N}_*^{(\epsilon)} + 1 - p_\epsilon).$$

So

$$\hat{N}^{(\epsilon)} = \frac{1 - \sqrt{1-4p_\epsilon(1-p_\epsilon)\exp(-\lambda)}}{2p_\epsilon}.$$

We are interested in the behavior of $\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}x}$, we can compute its Laplace transform

$$\mathbb{E}(\exp(-\lambda\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}x}))$$

as

$$\hat{N}(\epsilon^2\lambda)^{\lfloor x\epsilon^{-1}\rfloor}$$

which, in turn can be approximated by

$$\exp(x\epsilon^{-1}(1-\hat{N}^{(\epsilon)}(\epsilon^2\lambda))).$$

Using that

$$\exp(-\epsilon^2\lambda) \sim 1 - \epsilon^2\lambda \text{ as } \epsilon \to 0$$

we can see that

$$\lim_{\epsilon\to 0} x\epsilon^{-1}(1-\hat{N}^{(\epsilon)}(\epsilon^2\lambda)) = x(a - \sqrt{a^2+\lambda}).$$

That implies that

$$\lim_{\epsilon \to 0} \mathbb{E}(\exp\{-\lambda\epsilon^2 V^{(\epsilon)}(\epsilon^{-1}x)\}) = \exp(x(a - \sqrt{a^2+\lambda})).$$

Now we analyze the general case ($d \geq 2$). We have that

$$\hat{N}_*^{(\epsilon)}(\lambda) = \exp(-\lambda)(1 - p_\epsilon + p_\epsilon \hat{N}_*^{(\epsilon)}(\lambda))^d$$

So $\hat{N}(\lambda)$ is a root of F with

$$F(x) = \exp(-\lambda)(1 - p_\epsilon + p_\epsilon x)^d - x$$

As $\hat{N}(\lambda)$ is close to 1 if λ is close to zero (which is the regime that we are interested of), we will expand this function around 1 up to the second order. Let us call this expansion $\bar{F}$, then

$$\bar{F}(x) = \exp(-\lambda) - 1 + (\exp(-\lambda)dp_\epsilon - 1)(x-1) + \exp(-\lambda)d/2(d-1)p_\epsilon^2(x-1)^2$$

we can easily find the roots of the equation $\bar{F}(x) = 0$, let us call $r^{(\epsilon,\lambda)}$ the root which lies below one (which is the root that concern us), then

$$r^{(\epsilon,\lambda)} - 1 = \frac{1 - p_\epsilon d\exp(-\lambda) - \sqrt{(\exp(-\lambda)dp_\epsilon - 1)^2 - 2(\exp(-\lambda)-1)d(d-1)p^2\exp(-\lambda)}}{\exp(-\lambda)d(d-1)p_\epsilon^2}$$

If we choose a sequence $\lambda_\epsilon = \epsilon^2\lambda$ we will have that

$$r^{(\epsilon,\lambda\epsilon^2)} \sim \epsilon d(d-1)^{-1}(a - \sqrt{a^2 + 2\lambda d^{-1}(d-1)}).$$

Define

$$\bar{r}^{(\epsilon,\lambda)} := \epsilon d(d-1)^{-1}(a - \sqrt{a^2+2\lambda d^{-1}(d-1)})$$

then we can see that

$$\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}) = \exp(-\lambda\epsilon^2) - 1 + (\exp(-\lambda\epsilon^2)(1-a\epsilon) - 1)\epsilon d(d-1)^{-1}(a - \sqrt{a^2+2\lambda d^{-1}(d-1)}) \tag{3.4}$$

$$+ \exp(-\lambda\epsilon^2)d^{-1}/2(d-1)(1-a\epsilon)^2(\epsilon d(d-1)^{-1}(a - \sqrt{a^2+2\lambda d^{-1}(d-1)}))^2 \tag{3.5}$$

So, using $\exp(-\lambda\epsilon^2) - 1 \sim -\lambda\epsilon^2$ and $\exp(-\lambda\epsilon^2)(1-a\epsilon) - 1 \sim -a\epsilon$ we see that the terms cancel, so $\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)})$ is of order ϵ^3.

Let $\delta > 0$, we evaluate $\bar{F}^{(\epsilon)}$ at $\bar{r}^{(\epsilon,\lambda)} \pm \epsilon\delta$

$$\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)} \pm \epsilon\delta) = \bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}) + \epsilon\delta(\exp(-\lambda\epsilon^2)(1-a\epsilon) - 1) \tag{3.6}$$

$$\pm\epsilon\delta\exp(-\lambda\epsilon^2)d^{-1}(d-1)(1-a\epsilon)^2(\bar{r}^{(\epsilon,\lambda)} - 1) + \delta^2\epsilon^2\exp(-\delta\epsilon^2)d^{-1}/2(d-1)(1-a\epsilon)^2. \tag{3.7}$$

Then

$$\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)} \pm \epsilon\delta) = \epsilon^2(d^{-1}/2(d-1)\delta^2 \mp \delta\sqrt{a^2+2\lambda d^{-1}(d-a)}) + O(\epsilon^3).$$

For δ small enough we have that

$$|d^{-1}/2(d-1)\delta^2| < |\delta\sqrt{a^2+2\lambda d^{-1}(d-a)}|.$$

So,$\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}+\epsilon\delta)>0$, $\bar{F}^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}-\epsilon\delta)<0$ and both terms are of order ϵ^2. Using the Taylor theorem we see that $F^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}+\epsilon\delta)>0$, $F^{(\epsilon)}(\bar{r}^{(\epsilon,\lambda)}-\epsilon\delta)<0$ so, for each $\delta>0$ there exists ϵ small enough such that

$$\hat{N}^{(\epsilon)}(\lambda\epsilon^2)\in(\bar{r}^{(\epsilon,\lambda)}-\delta\epsilon,\bar{r}^{(\epsilon,\lambda)}+\delta\epsilon).$$

That implies that $1-\hat{N}^{(\epsilon)}(\lambda\epsilon^2)\sim 1-\bar{r}^{(\epsilon,\lambda)}$.

We can see that

$$\hat{N}^{(\epsilon)}(\lambda)=\exp(-\lambda)(p_\epsilon\hat{N}_*^{(\epsilon)}+1-p_\epsilon)^{d-1}$$

From which it follows that

$$\hat{N}^{(\epsilon)}(\lambda)=(\exp(-\lambda/(d-1))\hat{N}_*^{(\epsilon)})(\lambda)^{\frac{d-1}{d}}$$

As we have convergence of $\hat{N}_*^{(\epsilon)}(\epsilon^2\lambda)^{\lfloor x\epsilon^{-1}\rfloor}$ to $\exp(xd(d-1)^{-1}(a-\sqrt{a^2+2\lambda d^{-1}(d-1)}))$ we will have convergence of $\hat{N}^{(\epsilon)}(\epsilon^2\lambda)^{\lfloor x\epsilon^{-1}\rfloor}$ to $\exp(x(a-\sqrt{a^2+2\lambda d^{-1}(d-1)}))$ This corresponds to the Laplace transform of an Inverse Gaussian distribution with parameters $\delta=x\sqrt{\frac{d-1}{d}}$ and $\gamma=2a\sqrt{\frac{d}{d-1}}$ (see [**App09**, page 51]).

□

We continue with the proof of lemma 5.9. We have established convergence of fixed-time distributions. Using independence, it is easy to deduce convergence of finite-dimensional distributions. Thus, it remains to show tightness. It is a known fact that tightness in the Skorohod J_1-topology is implied by the two following conditions:

(1) for every $\eta>0$ and $T>0$ there exists K such that

$$\sup_{\epsilon>0}\mathbb{P}\left(\sup_{x\le T}|\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}x}|>K\right)\le\eta$$

(2) for every $\eta>0$ and $T>0$ there exists $\delta>0$ such that

$$\sup_{\epsilon>0}\mathbb{P}\left(w(\epsilon^2 M^{(\epsilon)}_{\epsilon^{-1}\cdot},T,\delta)\ge\eta\right)\le\eta$$

where

$$\begin{aligned} w(f(\cdot),T,\delta): \;=\; & \inf\{\max_{1\le i\le n}\{\sup_{s,t\in[t_{i-1},t_i)}|f(s)-f(t)|\}: && (3.8)\\ & n\in\mathbb{N},0=t_0<t_1...<T<t_n,\min_{1\le i\le n}(t_i-t_{i-1})\ge\delta\}. && (3.9)\end{aligned}$$

We know that condition (1) is implied by the weak convergence of $\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}T}$, which was already showed. To prove condition (2) we can replace $w(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},T,\delta)$ by $\tilde{w}(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},T,\delta)+v(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},\delta,0)+v(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},\delta,T)$ where

$$\tilde{w}(f(\cdot),T,\delta):=\sup\{\min(|f(t)-f(t_1)|,|f(t_2)-f(t)|):t_1\le t\le t_2\le T,t_2-t_1\le\delta\}$$

and

$$v(f(\cdot),t,\delta):=\sup\{|f(t_1)-f(t_2)|,t_1,t_2\in[0,\infty]\cap(t-\delta,t+\delta)\}.$$

It is easy to check that

$$w(f(\cdot),T,\delta)\leq \tilde{w}(f(\cdot),T,\delta)+v(f(\cdot),0,\delta)+v(f(\cdot),t,\delta)$$

It is easy to see that, for any $\eta>0$, $T>0$ we can find $\delta>0$ such that

$$\sup_{\epsilon>0}\mathbb{P}(v(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},0,\delta)+v(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},T,\delta)\geq\eta/2)\leq\eta/2$$

On the other side, $\tilde{w}(\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}\cdot},T,\delta)$ can only be large if there are three times $t_1<t<t_2$ with $t_2-t_1<\delta$ with $\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}t}-\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}t_1}$ and $\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}t_2}-\epsilon^2M^{(\epsilon)}_{\epsilon^{-1}t}$ are large. This can only happen if there are two big jumps on a time interval of size δ, because, as we will show, the small jumps can be neglected. Let

$$M^{(\epsilon,\xi)}_x:=\sum_{i=0}^{\lfloor x\rfloor}m^{(\epsilon)}_i1_{\{m^{(\epsilon)}_i<\xi\epsilon^{-2}\}}$$

be the contribution of the traps with depth smaller than ξ.
We can write

$$\mathbb{E}[m^{(\epsilon)}_i1_{\{m^{(\epsilon)}_i<\xi\epsilon^{-2}\}}]=\int_0^{\xi\epsilon^2}xm^{(\epsilon)}_i(dx).$$

Using that $\mathbb{P}[m_i\geq x]\sim cx^{-2}$, and that $\mathbb{P}[m^{(\epsilon)}_i\geq x]\leq\mathbb{P}[m_i\geq x]$ and integration by parts we deduce that

$$\mathbb{E}[m^{(\epsilon)}_i1_{\{m^{(\epsilon)}_i<\xi\epsilon^{-2}\}}]<C(\xi\epsilon^{-2})^{\frac{1}{2}}.$$

So

$$\mathbb{E}(\epsilon^2M^{(\epsilon,\xi,s)}_{\epsilon^{-1}T})\leq\epsilon^{-1}T\epsilon^2C(\xi\epsilon^{-2})^{\frac{1}{2}}\leq\tilde{C}\xi^{1/2}.$$

Using Chevychev's inequality, it is possible to choose ξ such that

$$\sup_{\epsilon\geq0}\mathbb{P}(\epsilon^2M^{(\epsilon,\xi,s)}_{\epsilon^{-1}T}\geq\eta/4)\leq\eta/4.$$

Let $W^{(\epsilon,\xi,d)}:=\sum_{i=0}^{\lfloor x\rfloor}m_i1_{\{m_i>\xi\epsilon^{-2}\}}$ and $M^{(\epsilon,\xi,d)}:=\sum_{i=0}^{\lfloor x\rfloor}m^{(\epsilon)}_i1_{\{m^{(\epsilon)}_i>\xi\epsilon^{-2}\}}$. The number of jumps of $W^{(\epsilon,\xi,d)}$ (before time $\epsilon^{-1}T$) converges to Poisson distribution with mean $T\xi^{-1/2}$ and is bigger than the number of jumps of $M^{(\epsilon,\xi,d)}$ (before time $\epsilon^{-1}T$). That implies that

$$\sup_{\epsilon}\mathbb{P}[\exists t_1,t_2\in[0,T]:0<t_2-t_1<\delta:\Delta\epsilon^2M^{(\epsilon,\xi,d)}_{\epsilon^{-1}t_i}\geq\xi\wedge(\eta/4),i=1,2]\leq CT\delta\xi^{-1/2}\leq\eta/4$$

for δ small enough. Then we deduce that $\tilde{w}(V^{(\epsilon),T,\delta})\leq\frac{\eta}{2}$ on an event of probability at least $1-\frac{\eta}{2}$. This tightness result and lemma 5.10 together yields lemma 5.9.

Display (11) states that $k[d\tilde{B}^{\lceil tk\rceil}-1]$ converges in distribution (in the Skorohod J_1-topology) to L_t away from zero. Thus, we can show that

LEMMA 5.11. *$(k^{-1}(d\tilde{B}^{\lceil tk\rceil}-1)^{-1})_{t\geq0}$ converge is distribution to $(L_t^{-1})_{t\geq0}$ in the Skorohod J_1 topology as $k\to\infty$.*

PROOF: For all $n \in \mathbb{N}$ and $0 < t_1 \geq t_2.. \geq t_n$, convergence of $(k^{-1}(d\tilde{B}^{\lceil t_i k \rceil} - 1)^{-1})_{1 \leq i \leq n}$ to $(L_{t_i}^{-1})_{1 \leq i \leq n}$ follows from display (1.13) (choosing $\epsilon \leq t_1$) and continuity of the inversion map. Convergence for $t = 0$ follows from display (1.13) and by noting that the processes involved are increasing. Tightness away from 0 is immediate. Tightness near 0 follows because L_t^{-1} is small near zero.

□

Using the previous lemma and the Skorohod representation theorem, we can find, a family of processes $(\bar{B}_t^k)_{t\geq 0}$, $k \in \mathbb{N}$, and a process $(\bar{L}_t)_{t\geq 0}$ defined on a common probability space such that

- $k^{-1}(d\bar{B}^k_{\lceil tk \rceil} - 1)^{-1}$ converges almost surely to $(\bar{L}_t^{-1})_{\geq 0}$ in the Skorohod J_1-topology as $k \to \infty$.
- for each $k \in \mathbb{N}$, $(\bar{B}^k)_{t\geq 0}$ is distributed as $(\tilde{B}^k)_{t\geq 0}$
- $(\bar{L}_t)_{t>0}$ is distributed as $(L_t)_{t>0}$.

Using the coupling above, we can use lemma 5.9 to prove lemma 5.8. More precisely, for each fixed realization of $\bar{L}_t$ and $(\bar{B}^k)_{t\geq 0}, k \in \mathbb{N}$, we define $\bar{M}_x^{(k)} := \sum_{i=0}^{\lfloor x \rfloor} m_i^{(k)}$, where $m_i^{(k)}$ is a random variable distributed as the mean duration time of a visit of a simple random walk on a random tree distributed a percolation cluster on $\tilde{T}_d$ of parameter $\bar{B}_i^k$. Hence, for all $k \in \mathbb{N}$, $(\bar{M}_x^{(k)})_{x\geq 0}$ is distributed as $(M_x^{IIC})_{x\geq 0}$. Applying lemma 5.9 on each interval on which $(\bar{L}_t)_{t\geq 0}$, we get lemma 5.8.

4. Scaling limit for the local time of the random walk on random trees

In this section we will prove that **Assumption D0** and **Assumption D** hold for W^{IIC} and W^{IPC} respectively. This task will be achieved with the aid of the ideas developed in [**Cro08**]. As we have said, there are given conditions for a sequence of random graph trees $(\mathcal{J}_n)_{n\in\mathbb{N}}$ under which the scaling limit of the simple random walk on $\mathcal{J}_n$ converges to the Brownian motion on the Continuum Random Tree.

Sometimes it is useful to consider, along with the CRT, some collection of sub-trees of it. Let $\mathcal{K}$ be a dendrite and A be a subset of $\mathcal{K}$. We will suppose that $\mathcal{K}$ has a distinguished point ρ which we will regard as the root. We define the subspace $r(\mathcal{K}, A)$ as

$$r(\mathcal{K}, A) := \bigcup_{x\in A} [[\rho, x]], \tag{4.1}$$

where we recall that $[[\rho, x]]$ denotes the unique non-sel intersecting path between ρ and x. This subspace is clearly a dendrite. Moreover, if A is finite, $r(\mathcal{K}, A)$ is closed and is called the **reduced sub-tree**.

Given a tree T (continuous or discrete), and a sub-tree of it T', we can define the projection $\phi_{T,T'}$ of T onto T' by simply stating that, for each $x \in T$, $\phi_{T,T'}(x)$ is the point on T' which is closest to x. The uniqueness of the projection follows easily from the tree structure of T and T'.

Now, suppose we have a pair of random variables (W, U) (defined over a probability space $(\mathcal{X}, \mathcal{G}, P)$), where $U = (U_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables which are uniformly distributed over $[0, 1]$ and $W = (W_t)_{t\in[0,1]}$ has the law of a normalized Brownian excursion. Let $T_n, n \in \mathbb{N}$ be a sequence of (deterministic) ordered, rooted trees in which $|T_n| = n$. Let $w_n, n \in \mathbb{N}$, be the search depth processes associated to the trees $T_n, n \in \mathbb{N}$, under consideration. For each $n \in \mathbb{N}$, we consider a sequence $u^n = (u_i^n)_{i\in\mathbb{N}} \subset [0,1]$ that we will use to span subtrees of T_n. We will state the assumption that we will make for the sequence of trees T_n and its corresponding family of sub-trees.

Lemma 2.3 in [**Cro08**] guarantees the existence of a set $\Gamma \subset C([0,1],\mathbb{R}_+) \times [0,1]^{\mathbb{N}}$ with "good properties" and $P((W,U) \in \Gamma) = 1$. For the definition of Γ we refer to [**Cro08**].

Assumption 1: For each n, the sequence $(u_k^n)_{k\geq 1}$ is dense in $[0,1]$, and also

$$(n^{-1/2}w_n, u^n) \xrightarrow{d} (w,u) \tag{4.2}$$

in $C([0,1],\mathbb{R}_+) \times [0,1]^{\mathbb{N}}$, for some $(w,u) \in \Gamma$.

For $(w,u) = ((w(t))_{t\in[0,1]}, (u_i)_{i\in\mathbb{N}}) \in \Gamma$ and $k \in \mathbb{N}$, we define the dendrite $\mathcal{T}_w(k)$ as

$$\mathcal{T}_w(k) := r(\mathcal{T}_w, \{[u_i] : i \leq k\}). \tag{4.3}$$

As $\mathcal{T}_w(k)$ is composed of a finite number of line segments, we can define the Lebesgue measure $\lambda_{w,u}^{(k)}$ over $\mathcal{T}_w$. Moreover we will suppose that $\lambda_{w,u}^{(k)}$ is normalized to become a probability measure. Similarly, for n fixed, we would like to use u^n to choose subtrees of T_n. Let

$$\gamma_n(t) := \begin{cases} \lfloor 2nt \rfloor/2n & \text{if } w_n(\lfloor 2nt \rfloor/2n) \geq w_n(\lceil 2nt \rceil/2n), \\ \lceil 2nt \rceil/2n & \text{otherwise.} \end{cases} \tag{4.4}$$

This function is constructed so that, if U is uniformly distributed over $[0,1]$, then $\tilde{w}_n(\gamma_n(U))$ is uniformly distributed over the vertices of T_n (recall that $\tilde{w}_n$ is the depth first search around T_n). We define the discrete reduced sub-trees as

$$T_n(k) := r(T_n, \{[\tilde{w}_n(\gamma_n(u_i^n))]; i \leq n\}). \tag{4.5}$$

Let X^n be a simple random walk on T_n started from the root and $X_m^{n,k} := \phi_{T_n,T_n(k)}(X_m^n)$. Define

$$A_m^{n,k} := \min\left\{ l \geq A_{m-1}^{n,k} : X_l^n \in T_n(k) - \{X^n_{A_{m-1}^{n,k}}\}\right\}, \tag{4.6}$$

$$\tau^{n,k}(m) := \max\{l : A_l^{n,k} \leq m\} \tag{4.7}$$

and

$$J_m^{n,k} := X^n_{A_m^{n,k}}. \tag{4.8}$$

Then we will have that $J^{n,k}$ is a simple random walk on $T_n(k)$ started from the root.Also we have that

$$X_m^{n,k} := J^{n,k}_{\tau^{n,k}(m)}. \tag{4.9}$$

Next we will present a method introduced in [**Cro08**] to couple the jump processes $J^{n,k}$ $n,k \in \mathbb{N}$ with the Brownian motion on $\mathcal{T}_w$. Lemma 2.5 in [**Cro08**] states the existence of $P^{\mathcal{T}_w,\mu_w}$-a.s. jointly continuous local times $(L(x,t))_{t\geq 0, x\in\mathcal{T}_w}$ for $X^{\mathcal{T}_w,\mu_\omega}$ (here $X^{\mathcal{T}_w,\mu_\omega}$ is the Brownian motion on $\mathcal{T}_w,\mu_\omega$ according to definition 5.2). Define

$$A_t^{(k)} := \int_{\mathcal{T}_w(k)} L(x,t)\lambda_{w,u}^{(k)}(dx), \tag{4.10}$$

and its right continuous generalized inverse

$$\tau^{(k)}(t) := \inf\{s : A_s^{(k)} > t\}. \tag{4.11}$$

Define the process

$$B_t^{(k)} := X_{\tau^{(k)}(t)}. \tag{4.12}$$

Lemma 2.6 in [**Cro08**] states that $B^{(k)}$ is the Brownian motion on $(\mathcal{T}_w(k),\lambda^{(k)})$ (according to definition 5.2). Moreover lemma 3.3 [**Cro08**] implies that $B^{(k)}$ has jointly continuous local times $(L^{(k)}(x,t))_{t\geq 0, x\in\mathcal{T}_w(k)}$ and lemma 3.4 in [**Cro08**] states that

$$L^{(k)}(x,t) := L(x,\tau^{(k)}(t)). \tag{4.13}$$

We will consider elements of the form $T = (T^\star, |e_1|,..,|e_l|)$ for some l. Here $T^\star$ is an ordered graph tree having $l+1$ vertices and $|e_1|,..,|e_l|$ are the corresponding edge lengths. By considering line segments along edges, naturally associated with T there is a dendrite T^* with its corresponding shortest path distance d_{T^*}. Moreover, we can consider T^* to be ordered in the sense that the set composed of its branching points and its end points are ordered according to the order on $T^\star$. We can consider a distance d_1 between such objects by stating that $d_1(T_1,T_2) = \infty$ if their corresponding graph trees $T_1^\star$ and $T_2^\star$. Otherwise we set

$$d_1(T_1,T_2) := \sup_{i=1..l} ||e_i^1| - |e_i^2||, \tag{4.14}$$

where the $|e_i^1|, |e_i^2|, i \leq l$ are the corresponding edge lengths. Moreover, if $T_1 = T'$, we can define a homeomorphism $\Upsilon_{T_1^*,T_2^*}$ between their associated dendrites T_1^*, T_2^* under which the point $x\in T_1^*$ which is at distance α along the edge e_i (measured from the vertex at the end of e_i which is closer to the root) goes to the point which is at distance $|e_i^2|\alpha/|e_i^1|$ along e_i^2. DEFINE LEAVES Now, we return to our sequence of trees satisfying Assumption 1. To each ordered trees $T_n(k)$, we associate the graph tree $T_n^\star(k)$ given by the root and leaves of $T_n(k)$ along with their corresponding branching points. Let $l+1$ denote the number of vertices of $T_n^\star(k)$. We consider the edge lengths $|e_1^{(n,k)}|,..,|e_l^{(n,k)}|$ to be given by the graph-distance on $T_n(k)$. Let $\bar{T}_n(k) := (T_n^\star(k), |e_1^{(n,k)}|,..,|e_l^{(n,k)}|)$. The corresponding ordered dendrite (obtained by adding line segments) is denoted $T_n^*(k)$. A similar procedure can be done to obtain $\bar{\mathcal{T}}_w(k) := (\mathcal{T}_w^\star(k), |e^{(w,k)}|_1,..,|e^{w,k}|_l)$ from $\mathcal{T}_w(k)$. By lemma 4.1 in [**Cro08**] it is proved that, for each fixed $k\in\mathbb{N}$, we have that

$$\bar{T}_n(k) \xrightarrow{n} \bar{\mathcal{T}}_w(k) \tag{4.15}$$

in the d_1 distance.

Let k be fixed, display (4.15) implies that, for n big enough, we have that $T_n^\star(k) = \mathcal{T}_w^\star(k)$. Thus, we can define the homeomorphism $\Upsilon_{\mathcal{T}_w(k),T_n^*(k)}$ between $\mathcal{T}_w(k)$ and $T_n^*(k)$. Using this homeomorphism, we define the process $B_t^{n,k} := \Upsilon_{\mathcal{T}_w(k),T_n^*(k)}(B_t^{(k)})$ taking values in $T_n^*(k)$. We also define the set of vertices of $T_n^*(k)$ as

$$V(T_n^*(k)) := \left\{x \in T_n^*(k) : d_{T_n^*(k)}(\rho, x) = m \in \mathbb{N} \cup \{0\}\right\}. \tag{4.16}$$

Define

$$h^{n,k}(m) := \inf\left\{t \geq h^n(m-1) : B_t^{n,k} \in V(T_n^*(k)) - \{B^{n,k}_{h^{n,k}(m-1)}\}\right\}. \tag{4.17}$$

Define $\bar{J}_m^{n,k} := B^{n,k}_{h^{n,k}(m)}$. For k and n fixed, the process $\bar{J}^{n,k}$ naturally induces a process $\tilde{J}^{n,k}$ taking values in $T_n(k)$. $\tilde{J}^{n,k}$ will be a simple random walk on $T_n(k)$. Thus $\tilde{J}^{n,k}$ is distributed as $J^{n,k}$. The family of processes $\bar{J}^{n,k} : n, k \in \mathbb{N}$ is coupled, because all the processes where defined using the same Brownian motion on the CRT.

We would like to have a coupled family of simple random walks on T_n. Let k and n be fixed. We can add excursions to the processes $\tilde{J}^{n,k}$ in order to obtain $(\tilde{X}_l^{n,k})_{l\geq 0}$ a simple random walk on T_n. This is, we add random excursions to the trajectory of $\tilde{J}^{n,k}$ to obtain a simple random walk on $\tilde{T}^{n,k}$. Define

$$\tilde{A}_m^{n,k} := \min\left\{l \geq \tilde{A}_{m-1}^{n,k} : \tilde{X}_l^{n,k} \in T_n(k) - \{\tilde{X}^n_{A^{n,k}_{m-1}}\}\right\}, \tag{4.18}$$

$$\tilde{\tau}^{n,k}(m) := \max\{l : \tilde{A}_l^{n,k} \leq m\}. \tag{4.19}$$

Then we will have that

$$\tilde{J}_m^{n,k} := \tilde{X}^{n,k}_{\tilde{A}_m^{n,k}}. \tag{4.20}$$

Next, we focus our attention in the local times of X^n. Let

$$l^n(x, m) := \sum_{l=0}^{m} 1_x(X_l^n). \tag{4.21}$$

Analogously, define

$$l^{n,k}(x, m) := \sum_{l=0}^{m} 1_x(J_l^{n,k}). \tag{4.22}$$

For all $n \in \mathbb{N}$, we extend the functions $l^n(x, \cdot)$, $l^{n,k}(x, \cdot), k \in \mathbb{N}$, to $\mathbb{R}_+$ by linear interpolation. Let

$$L^n(x, t) := \frac{2l^n(x, t)}{\deg_n(x)} \tag{4.23}$$

and

$$L^{n,k}(x, t) := \frac{2l^{n,k}(x, t)}{\deg_{n,k}(x)} \tag{4.24}$$

where $\deg_n(x)$ ($\deg_{n,k}(x)$) denote the degree of the vertex x on the graph T_n ($T_n(k)$). Similarly, we define the corresponding quantities using the processes $\tilde{J}^{n,k}$ and $\tilde{X}^{n,k}$

$$\tilde{l}^{(n,k)}(x, m) := \sum_{l=0}^{m} 1_x(\tilde{X}_l^{n,k}). \tag{4.25}$$

Analogously, define

$$\tilde{l}^{n,k}(x,m) := \sum_{l=0}^{m} 1_x(\tilde{J}_l^{n,k}). \tag{4.26}$$

Let

$$\tilde{L}^{(n,k)}(x,t) := \frac{2\tilde{l}^{(n,k)}(x,t)}{\deg_n(x)} \tag{4.27}$$

and

$$\tilde{L}^{n,k}(x,t) := \frac{2\tilde{l}^{n,k}(x,t)}{\deg_{n,k}(x)}. \tag{4.28}$$

Now, we will suppose that the trees under consideration have all their vertices with degree at most 2. Supposing so, it is elementary to notice that, for each $n \in \mathbb{N}$.

$$n^{-1/2}\tilde{l}^{(n,k)}(\rho, tn^{3/2}) = n^{-1/2}\tilde{l}^{n,k}(\rho, \tilde{\tau}^{n,k}(tn^{3/2})). \tag{4.29}$$

Using the fact that $\deg_n(\rho) = \deg_{n,k}(\rho) = 1$ we have that

$$n^{-1/2}\tilde{L}^{(n,k)}(\rho, tn^{3/2}) = n^{-1/2}\tilde{L}^{n,k}(\rho, \tilde{\tau}^{n,k}(tn^{3/2})). \tag{4.30}$$

Fix $k \in \mathbb{N}$ and $R > 0$. Lemma 4.8 in [**Cro08**] states the convergence of the local times of $\tilde{X}^{n,k}$ to those of the Brownian motion on $\mathcal{T}_w(k)$. Furthermore, that theorem states

PROPOSITION 5.12 (Croydon).

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{x\in T(k)} \sup_{t\in[0,R]} |L^{(k)}(\rho,t) - n^{-1/2}\tilde{L}^{n,k}(\rho, tn\Lambda_n^{(k)})| \geq \epsilon\right) = 0. \tag{4.31}$$

OBSERVATION 5.13. *In fact, the theorem is stronger, because it states convergence for local times considered on the whole tree, not only on the root. Nevertheless we will only use the result for the root. We decide to state the theorem in this weaker form for simplicity.*

The key lemma to verify **Assumption (D0)** for W^{IIC} and **Assumption (D)** for W^{IPC} is the convergence in probability of $n^{-1/2}\tilde{L}^{(n,k)}(\rho, tn^{3/2})$ to $L(\rho,t)$.

LEMMA 5.14. *Under Assumption (1), we have that, for each $M > 0$ and $\epsilon > 0$*

$$\lim_{n\to\infty} \limsup_{k} P_\rho^{\mathcal{T}_w,\mu_w}\left(\sup_{t\leq M} |n^{-1/2}\tilde{L}^{(n,k)}(\rho, tn^{3/2}) - L(\rho,t)| \geq \epsilon\right) = 0. \tag{4.32}$$

We recall that the local time $(L(x,t))_{x\in\mathcal{T}_w, t\geq 0}$ is P^ω-a.s. jointly continuous in x and t.

PROOF: For all $n \in \mathbb{N}, k \in \mathbb{N}$ and $\epsilon > 0$ we define

$$A(n,k,\epsilon) := \left\{\sup_{t\leq R} |n^{-3/2}\tilde{A}^{n,k}_{tn\Lambda_n^{(k)}} - t| > \epsilon\right\}, \tag{4.33}$$

$$B(n,k,\epsilon) := \left\{\sup_{t\leq R} |L^{(k)}(\rho,t) - n^{-1/2}\tilde{L}^{n,k}(\rho, tn\Lambda_n^{(k)})| > \epsilon\right\} \tag{4.34}$$

$$C(k,\epsilon) := \left\{ \sup_{t\leq R} |\tau^{(k)}(t) - t| > \epsilon \right\}. \tag{4.35}$$

and

$$D(\epsilon,\eta) := \left\{ \sup_{t\leq M} \sup_{|\delta|<\eta} |L(\rho, t+\delta) - L(\rho,t)| \geq \epsilon/2 \right\} \tag{4.36}$$

Fix $\epsilon > 0$ and $M > 0$, we aim to prove that

$$P^{\omega}\left(\sup_{t\leq M} |n^{-1/2} L^n(\rho, tn^{3/2}) - L(\rho,t)| \geq \epsilon \right) \leq \epsilon \tag{4.37}$$

for n big enough. By the uniform continuity on time of the local time $(L(\rho,t))_{t\geq 0}$ on $[0, M+1]$, we will have that there exists $\eta_0 \in (0,1)$ small enough such that

$$P^{\omega}\left(D(\epsilon,\eta_0)\right) \leq \epsilon/4. \tag{4.38}$$

Define $\epsilon_1 := \min\{\epsilon/4, \eta_0/2\}$. Corollary 5.3 in [**Cro08**] states that, for each $R > 0$, we have that

$$\lim_{k\to\infty} \limsup_{n\to\infty} P^{w}\left(A(n,k,\epsilon_1)\right) = 0. \tag{4.39}$$

On the other side, on $A(n,k,\epsilon_1)^c$ we have that

$$\sup_{t\leq M} |(n\Lambda_n^{(k)})^{-1} \tilde{\tau}^{n,k}_{tn^{3/2}} - t| \leq \epsilon_1. \tag{4.40}$$

which, in turn, implies that

$$\sup_{t\leq M} |(\tilde{\tau}^{n,k}(tn^{3/2}) - n\Lambda_n^{(k)} t| \leq \epsilon_1 n \Lambda^{(k)} \tag{4.41}$$

By display (4.39) we can choose k_0 big enough such that

$$\limsup_{n\to\infty} P^{\omega}\left(A(n,k_0,\epsilon_1)\right) \leq \epsilon_1/2. \tag{4.42}$$

Thus, there exists n_0 such that, for all $n \geq n_0$, we have that

$$P^{\omega}\left(A(n,k_0,\epsilon_1)\right) \leq \epsilon_1. \tag{4.43}$$

Furthermore, by Proposition 3.5 in [**Cro08**], k_0 can be chosen large enough to also satisfy

$$P^{\omega}\left(C(k_0,\epsilon_1)\right) \leq \epsilon_1. \tag{4.44}$$

Proposition 5.12 implies that there exists n_1 such that, if $n \geq n_1$ we have that

$$P(B(n,k_0,\epsilon_1)) \leq \epsilon_1. \tag{4.45}$$

Using displays (4.30) and (4.41), it can be deduced that, on $A(n,k_0,\epsilon_1)^c \cap B(n,k_0,\epsilon_1)^c$ we have that

$$n^{-1/2} \tilde{L}^{(n,k)}(\rho, tn^{3/2}) \in [L^{(k)}(\rho, t-\epsilon_1) - \epsilon_1, L^{(k)}(\rho, t+\epsilon_1) + \epsilon_1] \tag{4.46}$$

for all $t \leq M$.

On $A(n,k_0,\epsilon_1)^c \cap B(n,k_0,\epsilon_1)^c \cap C(k_0,\epsilon_1)^c$ we have that

$$n^{-1/2} \tilde{L}^{(n,k)}(\rho, tn^{3/2}) \in [L(\rho, t-2\epsilon_1) - \epsilon_1, L(\rho, t+2\epsilon_1) + \epsilon_1] \tag{4.47}$$

for all $t \leq M$.

And, by definition $2\epsilon_1 \leq \eta_0$, so, on we have that on $A(n, k_0, \epsilon_1)^c \cap B(n, k_0, \epsilon_1)^c \cap C(k_0, \epsilon_1)^c \cap D(\epsilon, \eta_0)^c$

$$n^{-1/2}\tilde{L}^{(n,k)}(\rho, tn^{3/2}) \in [L(\rho, t) - 2\epsilon_1, L(\rho, t) + 2\epsilon_1] \tag{4.48}$$

for all $t \leq M$.

As $\epsilon_1 \leq \epsilon/2$ we have that on $A(n, k_0, \epsilon_1)^c \cap B(n, k_0, \epsilon_1)^c \cap C(k_0, \epsilon_1)^c \cap D(\epsilon, \eta_0)^c$

$$n^{-1/2}\tilde{L}^{(n,k)}(\rho, tn^{3/2}) \in [L(\rho, t) - 2\epsilon, L(\rho, t) + 2\epsilon] \tag{4.49}$$

for all $t \leq M$.

If $n \geq n_0 \vee n_1$, $P(A(n, k_0, \epsilon_1)^c \cap B(n, k_0, \epsilon_1)^c \cap C(k_0, \epsilon_1)^c \cap D(\epsilon, \eta_0)^c) \geq 1 - \epsilon$

□

By construction, for each $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the coupled local time $(\tilde{L}^{(n,k)}(\rho, t))_{t\geq 0}$ is distributed as the local time $(L^n(\rho, t))_{t\geq 0}$ of a simple random walk on T_n, thus, from lemma 5.14, we can deduce that

PROPOSITION 5.15. *Under Assumption (1), the processes $(n^{-1/2}L^n(\rho, tn^{3/2}))_{t\geq 0}$ converges in distribution to $(L(\rho, t))_{t\geq 0}$ as $n \to \infty$ with the uniform topology on $D[0, T]$, for all $T \geq 0$.*

As we know that the inversion map on $(D(\mathbb{R}_+), M_1')$ is continuous when restricted to monotonous functions, we can obtain convergence in the Skorohod M_1'-topology for the corresponding inverted processes.

PROPOSITION 5.16. *Suppose Assumption (1) holds. Let $(L_t^{n,\leftarrow})_{t\geq 0}$ and $(L_t^{\leftarrow})_{t\geq 0}$ denote the right-continuous generalized inverses of $(n^{-1/2}L^n(\rho, tn^{3/2}))_{t\geq 0}$ and $(L(\rho, t))_{t\geq 0}$ respectively, Then we have that, for all $T \geq 0$, $(L_t^{n,\leftarrow})_{t\geq 0}$ converges in distribution to $(L_t^{\leftarrow})_{t\geq 0}$ on $D[0, T]$ with the Skorohod M_1'-topology.*

We proceed to state the main proposition of this section

PROPOSITION 5.17. *Let $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ be a sequence of random, rooted, ordered, graph-trees such that $\mathcal{Y}_n$ has n vertices and, for all $n \in \mathbb{N}$, $\mathcal{Y}_n$ has all their vertices with degree at most 2. Let $(\omega_n(t))_{t\in[0,1]}$ denote its corresponding search depth process. Let $(W_t)_{t\geq 1}$ be a normalized Brownian excursion. For each $n \in \mathbb{N}$, let $(\mathcal{J}_l^n)_{l\in\mathbb{N}}$ be a simple random walk on $\mathcal{Y}_n$ started from the root and $(\mathscr{L}_t^n)_{t\geq 0}$ be its corresponding local time at the root (extended to all $\mathbb{R}_+$ by linear interpolation). Let $(\mathscr{L}_t^{n,\leftarrow})_{t\geq 0}$ denote its corresponding right-continuous generalized inverse. Suppose that.*

$$(n^{-1/2}\omega_n(t))_{t\in[0,1]} \xrightarrow{d} (W_t)_{t\in[0,1]} \qquad \text{as } n \to \infty \tag{4.50}$$

on $C[0, 1]$ endowed with the uniform topology. Then we have that, for each $T \geq 0$

$$(\mathscr{L}_t^{n,\leftarrow})_{t\geq 0} \text{converges in distribution to} (\mathscr{L}_t^{\leftarrow})_{t\geq 0} \tag{4.51}$$

on $D[0, T]$ with the Skorohod M_1'-topology, where $(\mathscr{L}_t)_{t\geq 0}$ is the local time at the root of the Brownian motion on the CRT and $(\mathscr{L}_t^{\leftarrow})_{t\geq 0}$ denote its corresponding right-continuous generalized inverse.

PROOF. The Skorohod representation theorem guarantee the existence of a sequence of random functions $(\omega^{n,*})_{n\in\mathbb{N}} \in \mathcal{W}$ and $W^* \in \mathcal{W}$ defined on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ such that

(1) for each $n \in \mathbb{N}$, $\omega^{n,*}$ is distributed as w_n
(2) W^* is distributed as W
(3) $\sup_{t\in[0,1]} |\sigma n^{-1/2}\omega^{n,*}(t) - 2W_t^*| \overset{n\to\infty}{\to} 0$, $\mathbb{P}^*$-a.s.

A quick analysis allows us to deduce that one can reconstruct a graph-tree starting from its search depth process. Thus, using the functions $\omega^{n,*}$, we can construct a sequence of ordered, rooted graph-trees $\mathcal{T}_n^*$. For each $n \in \mathbb{N}$, $\mathcal{T}_n^*$ is distributed as $\mathscr{T}_n$. Analogously, we can construct a CRT $\mathcal{T}_{W^*}$ using the normalized Brownian excursion W^*. Then we can apply proposition 5.16. $\square$

Using that proposition we can easily deduce Assumption (D0) for W^{IIC}.

PROPOSITION 5.18. *Assumption (D0) holds for W^{IIC}.*

PROOF: The proof is based on the mentioned fact that a critical percolation cluster is the genealogical tree of a critical Galton-Watson process. Thus the proposition follows easily from display (2.2) and proposition 5.17.

$\square$

The corresponding proposition for W^{IPC} needs to extend the theorem on display (2.2) to trees which are non-critical

LEMMA 5.19. *Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ a sequence of trees which are percolation clusters with parameter $p_n \sim d^{-1}(1-a/n)$ conditioned on having n vertices converge to the Continuum random tree in the sense of display (2.2).*

PROOF: Let t be a tree on n vertices. There are $(d-1)n+1$ closed vertices in the boundary of t and $n-1$ open vertices. Then

$$\mathbb{P}_p[\mathcal{T} = t] = p^{n-1}(1-p)^{(d-1)(n-1)+d}$$

where $\mathbb{P}_p$ denotes percolation probability under the parameter p. This can be proved by induction on the number of vertices. This implies that the probability conditioned on $|\mathcal{T}| = n$ is the uniform tree on n vertices, for all p. So the convergence in distribution of the conditioned critical search depth functions to a Brownian excursion implies the convergence for any p, even variable p.

$\square$

PROPOSITION 5.20. *Assumption (D) holds for W^{IPC}.*

PROOF: Using lemma 5.19, the proof is the same as in proposition 5.18.

$\square$

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