Non-simple principally polarized abelian varieties

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Pontificia Universidad Católica de Chile, 2014

ABSTRACT

It is a rare phenomenon when an abelian variety contains a non-trivial abelian subvariety. Indeed, the general abelian variety is simple. Here we investigate when an abelian variety is non-simple and we give a characterization of all abelian subvarieties on a fixed abelian variety by means of intersection theory. This is done by canonically associating to an abelian subvariety a numerical class and then classifying all such classes within the Néron-Severi group. We show that studying these classes on a principally polarized abelian variety (A, Θ) is equivalent to studying arithmetic properties of certain homogeneous forms defined on the Néron-Severi group of A.

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A Dissertation
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy
at the
Pontificia Universidad Católica de Chile

2014

APPROVAL PAGE

Doctor of Philosophy Dissertation

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Pontificia Universidad Católica de Chile
2014

ACKNOWLEDGMENTS

First and foremost I want to thank God for everything he's done in my life. I am extremely grateful to my advisor, Rubí Rodríguez, for her guidance, encouragement, generosity and advice these past four years. It has been a privilege to learn such a beautiful subject from such an amazing professor. Next comes my wife, Natalia. I want to thank her for her love and patience with me; I couldn't have finished this thesis without her. My family has been a great encouragement to me as well; thanks Mom, Dad, Sam, Kim, Agustín, Lydia, Megan, and Jon. I'd like to thank Giancarlo Urzúa for his support and encouragement. I am grateful to the other members of my defense committee as well, Anita Rojas and Jennifer Paulhus. Thanks to all my friends who have been with me since we first started learning math: Arié, Bastián, Gabriel, Beto, Gionella, Pablo, Sebastián H., Sebastián T., Danilo and Álvaro. I am indebted to Igor Dolgachev, and in general to the University of Michigan, for inviting me to stay in Ann Arbor for a semester. It was a fantastic experience and I am thankful to Igor for his kindness and insight. An additional thanks to Bruce Glastad, Juan Rivera and Luis Elgueta.

I am grateful to the Chilean government for funding my doctoral studies; I had the scholarship Doctorado Nacional 2010 provided by CONICYT for the duration of my program.

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To Natalia and Rebeca, you mean so much to me.

Introduction

It is well-known that a general abelian variety is simple; that is, a general abelian variety contains no non-trivial abelian subvarieties. However, there are many examples where non-simple abelian varieties appear. For example, the existence of an elliptic curve E in the Jacobian of a smooth projective curve C implies that there exists a finite morphism $C \to E$. Reciprocally, any morphism $C \to E$ automatically gives way to an elliptic curve in the Jacobian of C that is isogenous to E. In other words, the study of abelian subvarieties of dimension 1 of Jacobian varieties amounts to the study of finite covers of elliptic curves.

By Poincaré's Reducibility Theorem, if X is an abelian subvariety of an abelian variety A, then A is isogenous to the product $X \times A/X$ and A/X is isogenous to an abelian subvariety of A. In particular, we can repeat this process and obtain an isogeny decomposition of A into unique (modulo isogeny) simple factors. An interesting question is the following: When is A completely decomposable? In other words, when is A isogenous to the product of elliptic curves? We could also ask that A contain an irreducible polarization, for example when A is the Jacobian of some smooth projective curve. In this case the polarized abelian variety (A, Θ) does not split as a product, but there are many examples when A itself, as a variety, is completely decomposable. This has been studied in [24] and [19].

Ekedahl and Serre [13] found completely decomposable Jacobian varieties for curves of genus up to 1297. A question they ask is: Are there infinite genera g for which there exists a curve of genus g whose Jacobian is completely decomposable? If J_C is the Jacobian of a smooth projective curve defined over a field k, then this question has a negative answer when k is not algebraically closed. Indeed, Serre showed in [37] that over a finite field, there are only finitely many Jacobian varieties isogenous to the product of elliptic curves. Duursma and Enjalbert [12] show that when $k = \mathbb{Z}/2\mathbb{Z}$, the maximum genus a completely decomposable Jacobian can have is 26. If k is algebraically closed and of positive characteristic then the answer is affirmative, but the question remains open when k is algebraically closed of characteristic 0.

Another similar question stems from the following example, discovered by González-Aguilera and Rodríguez [15]. For $\sigma \in \mathbb{H}$, let

$$\tau_\sigma := \sigma \begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{pmatrix};$$

this is a symmetric matrix whose imaginary part is positive definite. This matrix then corresponds to a principally polarized abelian variety $(A_\sigma, \Theta_\sigma)$ of dimension n, and the authors show that the polarization is irreducible. Moreover, it is shown that A_σ is not only isogenous to the product of n elliptic curves, but is actually isomorphic to the product of elliptic curves. The natural question one can ask is: When is a completely decomposable abelian variety *isomorphic* to the product of elliptic curves?

These questions have been studied for abelian surfaces by Humbert ([20]) and Kani ([21],[22]). They describe all principally polarized abelian surfaces that contain an elliptic curve (that is, completely decomposable abelian surfaces). Humbert's description is much more analytic in nature (it is done over $\mathbb{C}$), whereas Kani generalizes Humbert's results to an arbitrary algebraically closed field and uses intersection theory instead of period matrices. Kani [21] used the fact that for an abelian surface A, elliptic curves on A can be seen as Weil divisors, and thus induce numerical classes in the Néron-Severi group of A. He was then able to describe, via a positive definite quadratic form, the set of all numerical classes that come from elliptic curves.

Another point of view for studying non-simple abelian varieties is looking at group actions on A. If G is a subgroup of $\mathrm{Aut}(A, \Theta) := \{\sigma \in \mathrm{Aut}(A) : \sigma^*\Theta \equiv \Theta\}$, then one obtains an algebra homomorphism $\mathbb{Q}[G] \to \mathrm{End}(A)$. Now $\mathbb{Q}[G]$ is a semi-simple $\mathbb{Q}$-algebra and so decomposes as a product

$$\mathbb{Q}[G] = Q_1 \times \cdots \times Q_r$$

where each Q_i is a simple $\mathbb{Q}$-algebra. This decomposition then gives an isogeny decomposition of A into the product of abelian varieties of smaller dimension. This method of decomposing abelian varieties was discovered by Lange-Recillas in [25].

The theory of group actions on abelian varieties has been very successful and has played a central role in the study of non-simple abelian varieties. Groups acting on curves and abelian varieties have been used to find interesting families of Jacobians and principally polarized abelian varieties (such as the family described above) that split isogenously and isomorphically as the product of elliptic curves, and one naturally asks: Does every non-simple polarized abelian variety have non-trivial automorphisms? The author is unaware of any counterexample. Other papers that study group actions on Jacobian varieties and abelian varieties are [7], [8], [34], [35], [36],

[39] and [40].

The motivation, then, for this thesis is to develop an algebraic theory of non-simple abelian varieties that could possibly lead to answers for the questions asked above. The inspiration for our results comes from Kani [21].

Obviously, the problem for obtaining a classification similar to Kani's in higher dimensions lies in the fact that abelian subvarieties of codimension larger than 1 can no longer be seen as divisors. In this thesis we show that in fact, given a principally polarized abelian variety (A, Θ), we can canonically associate to every abelian subvariety $X \leq A$ a numerical class α_X. We can then characterize, via certain homogeneous forms defined on the Néron-Severi group, all numerical classes that come from abelian subvarieties, thus generalizing Kani's results to arbitrary dimension.

If we work over $\mathbb{C}$, these conditions can be written out explicitly in the form of equations on the Siegel semi-space $\mathfrak{H}_n$. Although these equations are long, they could possibly be programmed into a computer in order to find abelian subvarieties. The problem with actually writing a program is the fact that it would involve looking for integer solutions to linear equations with complex coefficients; this problem is not easy. In theory, such a program would receive as input a period matrix and integers d and m, and return all abelian subvarieties of A of exponent d whose divisor classes have norm less than or equal to m. We recall that since the Néron-Severi group is a finitely generated abelian group, we can identify it with a lattice in $\mathbb{R}^\rho$ and thus use the usual norm in $\mathbb{R}^\rho$ for the above calculation.

Chapter 1

Abelian varieties

1.1 Preliminaries

We start off by giving the basic definitions of our objects of study, as well as some of their basic properties. Since this is all well known by anyone who has ventured into the world of abelian varieties, the proofs of the theorems are not included. For the beginner, two good references are [31] (for abelian varieties over arbitrary algebraically closed fields or for group schemes) or [4] (for complex abelian varieties). For learning algebraic geometry in general, [30] and [38] are recommended as introductions, and [18] as the standard reference.

Definition 1.1.1. An *abelian variety* over a field k is a complete connected group variety over k.

We will assume from now on that k is algebraically closed. It can be proven that if A is an abelian variety, then it is actually projective (see [31], Application 1, p. 57) and is an abelian group (see [31], p. 39). Since translation by a fixed element of A is an automorphism of the variety, A is easily seen to be smooth. We will denote by $t_x : A \to A$ translation by $x \in A$.

Proposition 1.1.2. *Let $f : A \to B$ be a morphism between two abelian varieties A and B. Then $t_{-f(0)}f$ is a homomorphism of abelian groups.*

From now on, a *homomorphism* between two abelian varieties will be a morphism that sends 0 to 0.

Definition 1.1.3. An *isogeny* between two abelian varieties A and B is a homomorphism $f : A \to B$ that is surjective and has finite kernel. If such a map exists between A and B, we will write $A \sim B$ and say that they are *isogenous*. We will call $[k(A) : f^*k(B)]$ the *degree* of f, where $k(A)$ is the rational function field of A. Define e_f to be the *exponent of* f, that is, the least common multiple of the orders of all the elements of $\ker f$.

It can be shown that if $f : A \to B$ is an isogeny, then

$$|\ker f| = [k(A) : f^*k(B)]_{\text{sep}};$$

that is, the separable degree of the extension $k(A) \supseteq f^*k(B)$. In particular, in characteristic 0 we have that $\deg f = |\ker f|$. If we consider the category whose objects are abelian varieties and where morphisms between A and B correspond to the elements of the $\mathbb{Q}$-vector space $\mathrm{Hom}_{\mathbb{Q}}(A, B) := \mathrm{Hom}(A, B) \otimes \mathbb{Q}$, then isogenies correspond to the isomorphisms of this category. Indeed, if $f : A \to B$ is an isogeny, then there exists an isogeny $g : B \to A$ such that $gf = e_f$. If $f : A \to B$ is an isogeny, we will denote by f^{-1} the element in $\mathrm{Hom}_{\mathbb{Q}}(B, A)$ such that $ff^{-1} = \mathrm{id}$.

Definition 1.1.4. A *polarization* on A is the choice of an ample divisor D (or equivalently an ample line bundle $\mathcal{O}_A(D)$) on A. The pair (A, D) (or $(A, \mathcal{O}_A(D))$) is called a *polarized abelian variety*. We say that D is a *principal polarization* if $h^0(D) := \dim \Gamma(A, \mathcal{O}_A(D)) = 1$, where $\Gamma(A, \cdot)$ is the global section functor, and in this case, the pair (A, D) will be called a *principally polarized abelian variety* (ppav). If (A, D_1) and (B, D_2) are two polarized abelian varieties and $f : A \to B$ is a homomorphism, we will say that f is a *homomorphism of polarized abelian varieties* if $f^*\mathcal{O}_B(D_2) \simeq t_x^*\mathcal{O}_A(D_1)$ for some $x \in A$ and will often write f as $f : (A, D_1) \to (B, D_2)$.

When the ground field is the complex numbers, one can be much more specific when working with abelian varieties. In this case they take the simple form of a complex torus.

Theorem 1.1.5. *If A is a complex abelian variety of dimension n, then there exists a lattice $\Lambda \subseteq \mathbb{C}^n$ such that A is (analytically) isomorphic to $\mathbb{C}^n/\Lambda$. Moreover, if (A, D) is a ppav, then Λ can be taken to be of the form $(\tau\ I)\mathbb{Z}^n$, where $\tau \in \mathfrak{H}_n := \{M \in M_n(\mathbb{C}) : M = M^t, \Im M > 0\}$ is the Siegel upper half space. Vice versa, a matrix $\tau \in \mathfrak{H}_n$ gives rise to a ppav $\mathbb{C}^n/(\tau\ I)\mathbb{Z}^n$.*

Proof. See [4], Lemma 1.1.1. □

Caution must be taken with the previous theorem; most complex tori are not algebraic, and hence are not abelian varieties! Algebraic tori may be characterized by the following property:

Theorem 1.1.6. *A complex torus $\mathbb{C}^n/\Lambda$ is algebraic if and only if there exists a positive definite hermitian form $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $\Im H(\Lambda \times \Lambda) \subseteq \mathbb{Z}$.*

Proof. See [4], Theorem 4.5.4. □

This mysterious hermitian form can actually be seen as the first Chern class of an ample line bundle $L \in \mathrm{Pic}(A)$, or equivalently as an integral Kähler form. More formally, we have the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_A \to \mathcal{O}_A^\times \to 0$$

which gives rise to the following exact sequence of cohomology groups:

$$H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A^\times) \xrightarrow{c_1} H^2(A, \mathbb{Z}).$$

We will identify $H^1(A, \mathcal{O}_A^\times)$ with $\mathrm{Pic}(A)$. The image of c_1 in $H^2(A, \mathbb{Z})$ is called the *Néron-Severi group* (which we will look at more closely in Chapters 2 and 3) and can be identified with the group of all hermitian forms H on $\mathbb{C}^n$ such that $\Im H(\Lambda\times\Lambda) \subseteq \mathbb{Z}$. We then have that the set of positive definite hermitian forms of this type coincides with the image of c_1 evaluated at all ample line bundles in $\mathrm{Pic}(A)$. Two good references for the general theory of Kähler geometry in algebraic geometry are [16] and [41], and in particular for abelian varieties the previous statements are found in [31]. We will sometimes write that $(A, c_1(\mathcal{O}_A(D)))$ is a polarized abelian variety instead of (A, D); $c_1(\mathcal{O}_A(D))$ will usually be denoted by the letter H.

Proposition 1.1.7. *If $(\mathbb{C}^n/\Lambda, H)$ is a polarized abelian variety, then there exists a basis of Λ such that the matrix of $\Im H$ is of the form*

$$[\Im H] = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is a diagonal matrix diag$(d_1, \ldots, d_n)$ and $d_i \mid d_{i+1}$ for all i. The vector $(d_1, \ldots, d_n)$ is called the type *of H and only depends on H (not the basis chosen).*

Proof. See [4], Section 3.1. □

The numbers d_i are important invariants of the abelian variety $\mathbb{C}^n/\Lambda$.

Theorem 1.1.8. *(Riemann-Roch) Let (A, D) be a complex polarized abelian variety of type $(d_1, \ldots, d_n)$. Then*

$$h^0(D) = \prod_{i=1}^{n} d_i.$$

In particular, D is a principal polarization if and only if $d_i = 1$ for all i.

Proof. See [4], Theorem 3.6.1. □

In Chapter 2 we will give a numerical version of the Riemann-Roch Theorem in terms of intersection theory.

Let $\mathrm{Pic}^0(A)$ consist of all line bundles $L \in \mathrm{Pic}(A)$ such that $t_x^* L \simeq L$ for all $x \in A$. We wish to see $\mathrm{Pic}^0(A)$ as a sort of dual abelian variety to A. Let us consider a pair $(A^\vee, \mathcal{P})$, where $A^\vee$ is a variety over k and $\mathcal{P}$ is a line bundle on $A \times A^\vee$. Moreover, assume that

1. $\mathcal{P}|_{A \times \{x\}} \in \mathrm{Pic}^0(A \times \{x\})$ for all $x \in A^\vee$
2. $\mathcal{P}|_{\{0\} \times A^\vee}$ is trivial.
3. If $(B, \mathcal{Q})$ is another pair that satisfies the previous two conditions on $A \times B$ then there exists a morphism $\alpha : B \to A$ such that $(1 \times \alpha)^* \mathcal{P} \simeq \mathcal{Q}$.

If $(A^\vee, \mathcal{P})$ satisfies the previous three conditions, then we call $A^\vee$ the *dual abelian variety* of A and $\mathcal{P}$ the *Poincaré sheaf* of A.

Theorem 1.1.9. *The dual abelian variety exists, it is an abelian variety isomorphic to $Pic^0(A)$ as a group, and for $L \in Pic(A)$ the map $\phi_L : A \to A^\vee$ where $x \mapsto t_x^* L \otimes L^{-1}$ is a homomorphism. If L is ample, then ϕ_L is an isogeny with $\deg \phi_L = h^0(L)^2$. In particular, if L is a principal polarization, $A \simeq A^\vee$ and in this case if M is a line bundle, then ϕ_M can be thought of as an endomorphism of A.*

Proof. See [29] Section I.8. □

If D is a divisor on A, we will write ϕ_D to denote the morphism $\phi_{\mathcal{O}_A(D)}$. The previous theorem shows that when L is ample, $K(L) := \ker \phi_L$ is finite and thus if ϕ_L is separable, $K(L) \simeq (\bigoplus_{i=1}^n \mathbb{Z}/d_i\mathbb{Z})^2$ for $d_i \in \mathbb{Z}_{>0}$ with $d_i \mid d_{i+1}$. In this way we can define the *type* of L (over any algebraically closed field) to be the tuple $(d_1, \ldots, d_n)$.

Theorem 1.1.10. *Over $\mathbb{C}$, $A^\vee$ is isomorphic to the complex torus $Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C})/\Lambda^\vee$, where $Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C})$ consists of all antilinear maps $\mathbb{C}^n \to \mathbb{C}$ and $\Lambda^\vee$ consists of all antilinear maps h such that $\Im h(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. Moreover, the map ϕ_L from the previous theorem translates into $z + \Lambda \mapsto H(z, \cdot) + \Lambda^\vee$, where $H = c_1(L)$. If H is positive definite (that is, L is ample) and is of type $(d_1, \ldots, d_n)$, then*

$$K(L) \simeq \bigoplus_{i=1}^{n} (\mathbb{Z}/d_i\mathbb{Z})^2$$

and in particular

$$|K(L)| = h^0(L)^2 = (\prod_{i=1}^{n} d_i)^2.$$

Proof. See [11] Proposition V.5.9. □

Let $f : A \to B$ be a homomorphism. By pulling back divisor classes, we get a homomorphism $f^\vee : B^\vee \to A^\vee$. In particular, if $A = B$, for every endomorphism of A we obtain an endomorphism of $A^\vee$. Assume that A has a polarization L and let $f \in \mathrm{End}(A)$. We define an (anti)involution of $\mathrm{End}_{\mathbb{Q}}(A)$, called the *Rosati involution*:

$$\begin{aligned} \dagger : \mathrm{End}(A) &\to \mathrm{End}(A) \\ f &\mapsto f^\dagger := \phi_L^{-1} f^\vee \phi_L \end{aligned} .$$

The Rosati involution will play a crucial role in our characterization of abelian subvarieties of A.

1.2 Abelian subvarieties

An *abelian subvariety* X of an abelian variety A is a closed irreducible subvariety of A that is also an algebraic group by inheriting the group structure of A. Over $\mathbb{C}$, an abelian subvariety is given by a vector subspace $W \leq \mathbb{C}^n$ such that $W \cap \Lambda$ is a lattice in W; the subgroup $X = W/W \cap \Lambda$ is thus an abelian subvariety of $\mathbb{C}^n/\Lambda$. If L is an ample line bundle on A (that is, a polarization), then the restriction $L|_X$ is ample on

B. Thus every abelian subvariety of an abelian variety with a polarization inherits the structure of a polarized abelian variety.

Unlike the real case, most complex tori are *simple*; that is, they don't have abelian subvarieties. This is due to the fact that if $W \leq \mathbb{C}^n$, then $W \cap \Lambda$ is rarely a lattice in W. In fact, if we associate to each matrix $M \in M_{2n}(\mathbb{R})$ the lattice in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ generated over $\mathbb{Z}$ by its columns, then it is easy to show that the set of matrices that induce non-simple tori is the countable union of Zariski-closed subsets of $M_{2n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$.

If $X \leq A$ is an abelian subvariety of a polarized abelian variety (A, L) and $j : X \hookrightarrow A$ is the inclusion, then we define the *norm endomorphism of* A associated to X as the homomorphism

$$N_X := e_{\varphi_{j^*L}} j \phi_{j^*L}^{-1} j^\vee \varphi_L \in \mathrm{End}(A).$$

We note that this is an actual homomorphism, since $e_{\varphi_{j^*L}} \phi_{j^*L}^{-1}$ is a well-defined homomorphism. Note that this endomorphism depends not only on X, but on L as well.

Definition 1.2.1. We define the *exponent* of an abelian subvariety $j : X \hookrightarrow A$ to be $e_X := e_{\varphi_{j^*L}}$.

Theorem 1.2.2. *The function $X \mapsto im(e_X id - N_X)$ gives a bijection between abelian subvarieties of A of dimension r and abelian subvarieties of A of dimension $n - r$.*

Proof. See [4], Theorem 5.3.2 (the arguments presented there work over any algebraically closed field). □

The abelian subvariety $\mathrm{im}(e_X \mathrm{id} - N_X)$ is called the *abelian complement* of X and we will say that $\mathrm{im}(e_X \mathrm{id} - N_X)$ and X are *complementary abelian subvarieties.*

Theorem 1.2.3. *(Poincaré's Reducibility Theorem) Let X be an abelian subvariety of A, and let Y denote the abelian complement of X. Then the addition map $X \times Y \to A$ is an isogeny whose kernel is isomorphic to $X \cap Y$. Moreover, this shows that A is isogenous to a product $X_1^{m_1} \times \cdots \times X_s^{m_s}$, where the X_i are simple abelian varieties.*

Proof. See [4], Theorem 3.5. □

In particular, we can see that an abelian variety contains an elliptic subgroup (that is, an abelian subvariety of dimension 1) if and only if it contains an abelian divisor (an abelian subvariety of codimension 1). This is our first approach to studying elliptic curves on abelian varieties; we associate to an elliptic curve its abelian complement which can be seen as a prime Weil divisor on A.

Norm endomorphisms will be of utmost importance in Chapter 3. We list their most important properties here:

Proposition 1.2.4. *Let $X \leq A$ be an abelian subvariety of A of exponent e_X. Then*

1. $N_X^\dagger = N_X$
2. $N_X^2 = e_X N_X$
3. N_X *is primitive; that is, it is not a multiple of another endomorphism (unless* $X = \{0\}$*)*

Moreover, these conditions characterize norm endomorphisms; that is, if f is an endomorphism that satisfies conditions 1.-3., then $f = N_{im(f)}$.

Proof. See the Norm-endomorphism Criterion in [4], p. 126 (the arguments presented there are valid over any algebraically closed field). □

Proposition 1.2.5. *If X and Y are complementary abelian subvarieties of exponents e_X and e_Y, respectively, then*

$$e_Y N_X + e_X N_Y = e_X e_Y id.$$

If (A, Θ) is a ppav, then $e_X = e_Y =: e$ and so

$$N_X + N_Y = e id.$$

Proof. See [4], p. 127. □

1.3 Jacobian varieties

Let C be a smooth projective curve of genus g defined over k. We will define an abelian variety J_C associated to C such that $J_C \simeq \mathrm{Pic}^0(C)$ as a group. Moreover, we will obtain a closed immersion $C \hookrightarrow J_C$.

To start with, let T be a connected scheme over k, and let $\mathscr{L}$ be an invertible sheaf on $C \times_k T$. Then by [29], $\deg(\mathscr{L}|_{C\times\{t\}})$ is independent of t, and in particular we have that $\deg((\mathrm{pr}_2^*\mathscr{M})|_{C\times\{t\}}) = 0$ for every invertible sheaf $\mathscr{M}$ on T. Define

$$P_C^0(T) := \{\mathscr{L} \in \mathrm{Pic}(C \times T) : \deg(\mathscr{L}|_{C\times\{t\}}) = 0\}/\mathrm{pr}_2^*\mathrm{Pic}(T).$$

We have that P_C^0 is a functor from $\mathrm{Sch}_{/k} \to \mathfrak{Ab}$.

If X and T are two schemes, define $X(T)$ to be the set $\mathrm{Morph}(T, X)$ wich we will call the set of *T-valued points* of X. When $T = \mathrm{Spec}(K)$ for some field K, to give an element of $X(\mathrm{Spec}(K))$ is equivalent to giving a point $x \in X$ and an inclusion map $k(x) \hookrightarrow K$, where $k(x)$ is the residue field of x.

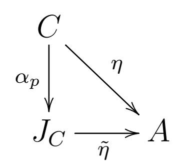
Theorem 1.3.1. *There exists an abelian variety J_C defined over k and a morphism of functors $\iota : P_C^0 \to J_C$ such that $\iota(T) : P_C^0(T) \to J_C(T)$ is an isomorphism whenever $C(T) \neq \varnothing$.*

Proof. See [29] Theorem III.1.6. ☐

We define J_C to be the *Jacobian variety of C*. We can think of J_C as parameterizing all degree 0 line bundles on C modulo linear equivalence.

For every $p \in C$, we have an inclusion $\alpha_p : C \hookrightarrow J_C$ given by $q \mapsto [q - p]$ (we can think of this as a map $C \to P_C^0(k)$ where $q \mapsto \mathcal{O}_{C\times C}(\Delta - C \times \{p\} - \{p\} \times C)$ and Δ is the diagonal). Moreover, we get a divisor $\Theta := \alpha_p(C) + \cdots + \alpha_p(C)$, where the right hand side is a sum of $g - 1$ elements. The divisor Θ is called a *theta divisor.*

Theorem 1.3.2. *(J_C, Θ) is a ppav. Moreover, if $\eta : C \to A$ is a morphism from C to any abelian variety A and $p \in C$, then there is a unique homomorphism $\tilde{\eta} : J_C \to A$ that makes the following diagram commute:*



Proof. See [29] Theorem III.1.7. ☐

We note that Θ is only well defined up to translation; however, we will often abuse this fact and speak of *the* theta divisor.

We have resorted to the *functor of points* of a scheme to construct the Jacobian variety of a curve C. Theorem 1.3.1 essentially says that the functor P_C^0 is *representable* by an abelian variety J_C. Yoneda's Lemma assures us that a scheme is determined by its functor of points and so J_C is uniquely determined.

If $k = \mathbb{C}$, however, much more can be said. If ω_C is the canonical class of C, it is easy to see that $H_1(C, \mathbb{Z})$ injects into $H^0(\omega_C)^\vee$ the following way: if $[\gamma] \in H_1(C, \mathbb{Z})$ is the class of some closed curve γ in C, then we define

$$[\gamma] : \quad H^0(\omega_C) \to \mathbb{C}$$
$$\eta \mapsto \int_\gamma \eta$$

Since $\dim_{\mathbb{C}} H^0(\omega_C) = g$, we can see that $H_1(C,\mathbb{Z})$ actually gives a lattice in $H^0(\omega_C)^\vee$, and so $J := H^0(\omega_C)^\vee / H_1(C,\mathbb{Z})$ is a complex torus. The intersection pairing $E : H_1(C,\mathbb{Z}) \times H_1(C,\mathbb{Z}) \to \mathbb{Z}$ can be shown to extend to all $H^0(\omega_C)^\vee$ as a non-degenerate alternating form that satisfies $E(iv, iw) = E(v, w)$, and so defines a positive definite hermitian form $H(v, w) := E(iv, w) + iE(v, w)$ whose imaginary part takes on integer values on $H_1(C,\mathbb{Z})$.

Theorem 1.3.3. *The Jacobian of C is (analytically) isomorphic to J; moreover H corresponds to the first Chern class of the divisor Θ.*

Proof. See [4], Proposition 11.1.2. □

Let $f : C \to \tilde{C}$ be a finite morphism between two projective non-singular curves. Then f induces two natural morphisms

$$\mathrm{Nm}_f : J_C \to J_{\tilde{C}}$$

$$f^* : J_{\tilde{C}} \to J_C$$

where $\mathrm{Nm}_f(\mathcal{O}_{J_C}(\sum r_i x_i)) = \sum r_i f(x_i)$ and f^* is just pullback of divisors. In particular, if $\tilde{C} = E$ is an elliptic curve, then since $E \simeq J_E$ we get a morphism $E \to J_C$. Therefore, if C maps to an elliptic curve, then J_C contains an elliptic curve.

Vice versa, assume that there is an elliptic subgroup $E \subseteq J_C$ and let Z be its abelian complement. By [31], the quotient abelian variety J_C/Z exists, and we get an exact sequence of abelian groups

$$0 \to Z \to J_C \to \tilde{E} = J_C/Z \to 0,$$

where $\tilde{E}$ is isogenous to E. By composing the embedding α_p with the natural projection, we get a finite morphism $C \to \tilde{E}$, and by composing with an isogeny we get a morphism $C \to E$. Therefore, the study of elliptic curves contained in a fixed Jacobian of a smooth projective curve is equivalent to the study of elliptic coverings by the same curve.

The Torelli Theorem says that the assignation $C \rightsquigarrow (J_C, \Theta)$ gives an injective map $\mathcal{M}_g \to \mathcal{A}_g$ for $g \geq 1$, where $\mathcal{M}_g$ denotes the (coarse) moduli space of genus g curves and $\mathcal{A}_g$ denotes the (coarse) moduli space of dimension g abelian varieties. In other words, the Jacobian of a curve determines the curve. The famous Schottky Problem asks what the locus $\mathcal{M}_g \subseteq \mathcal{A}_g$ is; so far there has only been one complete solution (see [23], [9] and [10]).

Before finishing this section, we explain the terminology *theta divisor*. Over the complex numbers, we already saw that J_C is the quotient of $\mathbb{C}^n$ by some full rank lattice Λ, and we see that $\mathbb{C}^n$ is the universal covering space of J_C. Since (J_C, Θ) is a ppav, $\mathcal{O}_{J_C}(\Theta)$ has one global section (modulo multiplication by a non-zero constant) which can be seen as a *theta function* $\theta : \mathbb{C}^n \to \mathbb{C}$. This is a holomorphic function that satisfies

$$\theta(z + \lambda) = e^{2\pi i(a_\lambda(z)+b_\lambda)}\theta(z)$$

for $\lambda \in \Lambda$, $a_\lambda : \mathbb{C}^n \to \mathbb{C}$ linear and $b_\lambda \in \mathbb{C}$. We can see the divisor Θ, then, as the projection of the zeros of θ to A. See [4] and [11] for more information.

Chapter 2

Intersection theory

Here we briefly introduce some machinery from intersection theory that will allow us to study divisor classes on abelian varieties. The most important result we will need in this section is the characterization of the effective cone on abelian varieties. The result says that for abelian varieties, the effective cone and the nef cone coincide; this is a result of the Nakai-Moishezon Criterion. For more detailed information on intersection theory and the Néron-Severi group see [14], [26] or [6].

2.1 Intersection numbers

Let X be a smooth irreducible projective variety over k of dimension n, let $\mathrm{Div}(X)$ denote the group of (Weil) divisors of X and let $D_1, \ldots, D_n \in \mathrm{Div}(X)$ be effective irreducible divisors. Intuitively, since the intersection of a subvariety of X with a divisor generally lowers the dimension by 1, it is natural to think that if the D_i are general enough, then $D_1 \cap \cdots \cap D_n$ is finite. We would like to define an *intersection number* $(D_1 \cdots D_n)_X = (D_1 \cdots D_n)$ that counts these points and that can be extended to all $\mathrm{Div}(X)^{\oplus n}$; that is, we want to be able to give an intersection number for $D_1, \ldots, D_n$, even if these divisors don't intersect properly.

The following is a list of desired properties we would want an intersection number to possess. Let $D_1, \ldots, D_n \in \mathrm{Div}(X)$ be any divisors.

Desired properties

1. $(D_1 \cdots D_n)$ depends only on the linear equivalence class of the divisors.
2. $(D_1 \cdots D_n)$ is symmetric and multilinear.
3. If the D_i are effective and they meet transversely, then

$$(D_1 \cdots D_n) = \# D_1 \cap \cdots \cap D_n.$$

4. If D_n is effective and meets the other divisors properly, then

$$(D_1 \cdots D_n)_X = (D_1|_{D_n} \cdots D_{n-1}|_{D_n})_{D_n}.$$

It is not hard to show that properties $1. - 3.$ characterize the intersection number (assuming it exists).

Theorem 2.1.1. *Let $D_1, \dots, D_n$ be Cartier divisors on X, let $\mathscr{F}$ be a coherent sheaf on X and let $\chi(L)$ denote the Euler characteristic of a sheaf L. Then*

$$\chi(\mathscr{F} \otimes \mathcal{O}_X(m_1 D_1 + \cdots m_n D_n))$$

is a polynomial in $m_1, \dots, m_n$ of degree less than or equal to the dimension of the support of $\mathscr{F}$.

Proof. See [6], Theorem 1.1. □

Definition 2.1.2. We define $(D_1 \cdots D_n)_X$ as the coefficient of $m_1 \cdots m_n$ in

$$\chi(\mathcal{O}_X(m_1 D_1 + \cdots m_n D_n)).$$

Theorem 2.1.3. *The intersection number of n divisors as defined above satisfies all the desired properties.*

Proof. See [6], Chapter 1. □

In a similar fashion, if $V \subseteq X$ is a subvariety of dimension s and $D_1, \dots, D_s$ are divisors on X, then we can define the intersection number $(D_1 \cdots D_s \cdot V) := (D_1|_V \cdots D_s|_V)$. The most important case is when V is a curve.

Definition 2.1.4. We say that two divisors D_1, D_2 are *numerically equivalent* and we write $D_1 \equiv D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve C on X; this gives us an equivalence relation on divisors on X that divides $\mathrm{Div}(X)$ into *numerical classes*. Since intersection is symmetric, we will sometimes write $(C \cdot D)$ instead of $(D \cdot C)$. We define the group $N^1(X) := \mathrm{Div}(X)/\equiv$. We say that a divisor D is *nef* if $(D \cdot C) \geq 0$ for every curve C.

Definition 2.1.5. Two divisors D_1, D_2 are *algebraically equivalent* if there exists a smooth curve C, a divisor $T \in \mathrm{Div}(C \times X)$ and two points $a, b \in C$ such that $D_1 - D_2 = T|_{\{a\} \times X} - T|_{\{b\} \times X}$. This forms an equivalence relation $\equiv_a$ on $\mathrm{Div}(X)$. It is easy to see that $\equiv_a$ extends to an equivalence relation on $\mathrm{Pic}(X)$. We define the *Néron-Severi group* of X to be $\mathrm{NS}(X) := \mathrm{Div}(X)/\equiv_a$.

We observe that the intersection number of divisors only depends on the numerical class of the divisors. Néron proved that $\mathrm{NS}(X)$ is a finitely generated abelian group (see [33]). The following theorem, due to Matsusaka, shows the relation between numerical and algebraic equivalence.

Theorem 2.1.6. *Let X be a non-singular variety, and let Num(X) (resp. Alg(X)) be the group of divisors that are numerically (resp. algebraically) equivalent to zero. Then Num(X)/Alg(X) is isomorphic to the torsion subgroup of NS(X).*

Proof. See [28], Theorem 4. □

On abelian varieties, algebraic and numerical equivalence coincide for divisor classes. This is a result of the following proposition:

Proposition 2.1.7. *If A is an abelian variety, then NS(A) is torsion-free. In particular, $N^1(A) =$ NS(A).*

Proof. See [31], point (v), pg. 71. □

Numerical classes can give us plenty information about the divisors that represent them. One application is the calculation of cohomology numbers for invertible sheaves; this is the Riemann-Roch Theorem. Since in the following chapters we will only use the Riemann-Roch Theorem for abelian varieties, we will only state it in this context.

Theorem 2.1.8 (Riemann-Roch for abelian varieties). *Let A be an abelian variety of dimension n, let D be a divisor on A and let $\chi(D) := \chi(\mathcal{O}_A(D))$ be its Euler characteristic. Then*

$$\chi(D) = \frac{(D^n)}{n!},$$

$$\chi(D)^2 = \deg \phi_D.$$

Proof. See [31], p. 140. □

Another example of how numerical classes can describe the geometry of a divisor is the astounding fact that ampleness can be characterized numerically; this is the content of the Nakai-Moishezon Criterion:

Theorem 2.1.9 (Nakai-Moishezon Criterion). *Let D be a divisor on a smooth projective variety X. Then D is ample if and only if $(D^s \cdot V) > 0$ for every s-dimensional subvariety V of X.*

Proof. See [6], Theorem 1.22. □

In the case of abelian varieties, we can be much more explicit.

Theorem 2.1.10 (Nakai-Moishezon Criterion for abelian varieties). *Let A be an abelian variety of dimension n and let Θ be an ample divisor. Then a divisor D on A is ample if and only if $(D^i \cdot \Theta^{n-i}) > 0$ for all $i \leq n$.*

Proof. The only interesting part is to prove that if $(D^i \cdot \Theta^{n-i}) > 0$ for all i, then D is ample. We see that if D satisfies this, then in particular $(D^n) > 0$. By Mumford [31], pg. 145, we have that $H^p(A, D) = 0$ for all $p \neq i(D)$ and $H^{i(D)}(A, D) \neq 0$, where $i(D)$ is the number of positive roots of the polynomial $P(t)$ defined by $P(m) := \chi(m\Theta + D)$. By Riemann-Roch,

$$P(m) = \frac{((m\Theta + D)^n)}{n!},$$

and by our assumption on D, all the coefficients of this polynomial are positive. Therefore $i(D) = 0$, and so $H^0(A, D) \neq 0$. This means that D is linearly equivalent to an effective divisor, and by Application 1 on page 57 of Mumford [31], we get that D is ample. □

Note that this particular case of the Nakai-Moishezon Criterion is not true in general. For example, if $\pi : \mathrm{Bl}_p(\mathbb{P}^2) \to \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at a point p, then it is easy to see that the nef cone

$$\{\alpha \in \mathrm{NS}(\mathrm{Bl}_p(\mathbb{P}^2)) \otimes \mathbb{R} : (\alpha \cdot C) \geq 0 \text{ for all irreducible curves } C \subseteq \mathrm{Bl}_p(\mathbb{P}^2)\}$$

is generated over $\mathbb{R}$ by the classes of $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $H - E$, where E is the exceptional divisor of π. In particular, $2H - E$ is ample (it is in the interior of the nef cone, see [26], Theorem 1.4.23). Moreover, $((2H-E)\cdot H) = 2(H^2) - (E \cdot H) = 2$, $(H^2) = 1$, but H is not ample since $(H \cdot E) = 0$.

Proposition 2.1.11. *If A is an abelian variety of dimension n and D is a divisor on A, then D is numerically equivalent to an effective divisor if and only if $(D^i \cdot \Theta^{n-i}) \geq 0$ for $1 \leq i \leq n$.*

Proof. If D is numerically equivalent to an effective divisor, then clearly $(D^i \cdot \Theta^{n-i}) \geq 0$ for $1 \leq i \leq n$, since we can find $x_1, \ldots, x_n \in A$ such that $D_1 + x_1, \ldots, D_i + x_i, \Theta + x_{i+1}, \ldots, \Theta + x_n$ intersect properly, and translation preserves algebraic equivalence. For the other direction, assume that this inequality holds for $1 \leq i \leq n$. Using the Nakai-Moishezon Criterion with $D + m\Theta$ and Θ, we first observe that $D + m\Theta$ is ample for all $m > 0$, and so D is nef.

We will now proceed by induction on n to prove the proposition. We see that for $n = 1$ the proof is trivial. We then assume that $n > 1$. If $(D^n) > 0$, then for the same reasons as in the previous proof we have that $H^0(A, D) \neq 0$, and so D is linearly equivalent to an effective divisor.

If $(D^n) = 0$, then $K(D) = \{x \in A : t_x^*\mathcal{O}_A(D) \simeq \mathcal{O}_A(D)\}$ is not finite and there exists a divisor D' on $A/K(D)_0$ such that $D - \pi^*D'$ is numerically trivial, where $\pi : A \to A/K(D)_0$ is the natural projection and $K(D)_0$ denotes the connected component of 0 in $K(D)$. Since π is proper and surjective and π^*D' is nef, we also have that D' is nef (this can be shown using the projection formula). By our induction hypothesis, we have that D' is numerically equivalent to an effective divisor, and therefore $\pi^*D' \equiv D$ is numerically equivalent to an effective divisor. □

Definition 2.1.12. Let (A, Θ) be a ppav, and let $\dagger$ be the associated Rosati involution. Define

$$\mathrm{End}^s(A) := \{f \in \mathrm{End}(A) : f^\dagger = f\},$$

the group of *symmetric endomorphisms* of A. Notice that this definition depends on the polarization Θ!

An interesting theorem we will use in the next chapter is the following:

Theorem 2.1.13. *Let (A, Θ) be a polarized abelian variety. Then there is an isomorphism*

$$D_\Theta : NS_{\mathbb{Q}}(A) := NS(A) \otimes \mathbb{Q} \to End^s(A) \otimes \mathbb{Q}$$

where $\alpha \mapsto \phi_\Theta^{-1}\phi_\alpha$. In particular, if Θ is a principal polarization then D_Θ induces an isomorphism between $NS(A)$ and $End^s(A)$.

Proof. See [31] p. 176. □

2.2 The Chow ring of a variety

For an irreducible variety X of dimension n defined over k, let $Z^s(X)$ be the free abelian group generated by closed subvarieties of X of codimension s. Elements in $Z^s(X)$ are called *s-cycles*. We define the *group of algebraic cycles on* X as $Z^*(X) := \bigoplus_{s=0}^{n} Z^s(X)$. We notice that any rational function $f \in k(X)$ defines a cycle $\operatorname{div}(f)$ in the usual way, and if $W \subseteq X$ is a subvariety of dimension $s+1$ and $g \in k(W)$, then $\operatorname{div}(g)$ can be seen as an s-cycle on X.

We will say that two s-cycles Z_1 and Z_2 are *rationally equivalent* if there exists a subvariety $W \subseteq X$ of dimension $s+1$ and a rational function $f \in k(W)$ such that $Z_1 - Z_2 = \operatorname{div}(f)$. The group of all s-cycles that are rationally equivalent to 0 form a subgroup $\operatorname{Rat}^s(X)$, and we define the *Chow ring* of X to be

$$\mathrm{CH}^*(X) := \bigoplus_{s=0}^{n} \mathrm{CH}^s(X),$$

where $\mathrm{CH}^s(X) := Z^s(X)/\operatorname{Rat}^s(X)$. Using the intersection pairing described in [14], $\mathrm{CH}^*(X)$ becomes a ring with unit element $[X]$. An s-cycle W is *algebraically equivalent to zero* if there exists a smooth curve C, a cycle $T \in Z^s(C \times X)$ and two points $a, b \in C$ such that $Z = T|_{\{a\}\times X} - T|_{\{b\}\times X}$. This forms an equivalence relation $\equiv_a$ on $Z^*(X)$. It is easy to see that if Z is rationally equivalent to zero, then it is algebraically equivalent to zero, and so algebraic equivalence can be defined on $\mathrm{CH}^*(X)$. We thus define the ring

$$\mathfrak{A}^*(X) := \bigoplus_{s=0}^{n} \mathfrak{A}^s(X),$$

where $\mathfrak{A}^s(X) = \mathrm{CH}^s(X)/\equiv_a$. One of the advantages of passing to algebraic equivalence on an abelian variety is that cycles are algebraically invariant under translation; that is, if W is a subvariety of an abelian variety A, then $t_x^*(W)$ is algebraically equivalent to W.

Chapter 3

Divisor classes associated to abelian subvarieties

As stated in the introduction, our goal is to obtain a characterization of abelian subvarieties of a ppav (A, Θ) in terms of intersection theory. In order to do this, we must somehow assign a divisor class to each abelian subvariety. We will show that this can be done canonically.

If D is a divisor on A, we define its *degree* (with respect to Θ) to be $\deg D := (D \cdot \Theta^{n-1})$. In the same way we define the degree of an algebraic class. The degree of a curve C on A is defined analogously as $\deg C := (C \cdot \Theta)$. We define the *polarized Néron-Severi group* to be $\mathrm{NS}(A, \Theta) := \mathrm{NS}(A)/\mathbb{Z}[\Theta]$.

Notice that if Θ is taken to be a very ample divisor (and not principally polarized), then the degree of a divisor coincides with the degree of $\varphi(D)$ in $\mathbb{P}^{h^0(\Theta)-1}$, where $\varphi : A \to \mathbb{P}^{h^0(\Theta)-1}$ is the embedding associated with Θ.

If G is a finitely generated free $\mathbb{Z}$-module, we say that an element $g \in G$ is *primitive* if g is not a non-trivial multiple of another element in G; that is, $G/\langle g \rangle$ is torsion-free.

3.1 Kani's results in dimension 2

Let (A, Θ) be a principally polarized abelian surface, and let $E \subseteq A$ be an elliptic subgroup (that is, an abelian subvariety of dimension 1). Since E is, in particular, a codimension 1 subvariety of A, we can see it as a divisor on A. Kani's first classification of elliptic curves on A distinguishes them among all divisors.

The proofs in this section are all found in [21]; we will provide proofs for the general case in the following sections.

Proposition 3.1.1. *Let D be a divisor on A. Then $D \equiv mE$ for some elliptic curve E on A and $m \in \mathbb{Z}$ if and only if $(D^2) = 0$.*

Every elliptic curve defines a numerical class $[E] \in \mathrm{NS}(A)$, and Kani characterized all classes that come from elliptic curves:

Theorem 3.1.2. *The map $E \mapsto [E]$ induces a one-to-one correspondence between the set of elliptic subgroups of A and the set of primitive classes $\alpha \in NS(A)$ such that $(\alpha^2) = 0$ and $\deg \alpha > 0$.*

Now consider the following quadratic form on $\mathrm{NS}(A)$:

$$q(\alpha) := (\alpha \cdot \Theta)^2 - 2(\alpha^2).$$

We notice in particular that if E is an elliptic subgroup on A, then $q([E]) = (\deg E)^2$. Another observation is that $q(\alpha + [\Theta]) = q(\alpha)$ for all α, and so q actually descends to a positive definite quadratic from on $\mathrm{NS}(A, \Theta)$.

Another way of writing q that will serve as inspiration for further generalizations is

$$q(\alpha) = -\frac{1}{2}(\alpha^\natural \cdot \Theta),$$

where $\alpha^\natural = 2\alpha - (\deg \alpha)[\Theta]$.

Theorem 3.1.3. *For any positive integer $d \in \mathbb{N}$, the map $E \mapsto [E]$ induces a bijection between*

1. *the set of elliptic subgroups $E \leq A$ with $\deg E = d$, and*
2. *the set of primitive classes $\alpha \in NS(A, \Theta)$ with $q(\alpha) = k^2$.*

This theorem not only gives us a characterization of elliptic curves on A, but lets us distinguish elliptic curves by their degrees. This will come in handy when we do calculations on the moduli space $\mathcal{A}_n$.

3.2 Elliptic subgroups in arbitrary dimension

Let (A, Θ) be a polarized abelian variety of dimension n, and let $E \leq A$ be an elliptic subgroup of A. We wish to generalize Kani's results to higher dimension, but we already face the problem of dimension: an elliptic curve can no longer be seen as a divisor. However, Theorem 1.2.2 says that we can canonically associate to E an abelian subvariety Z of codimension 1 called its abelian complement. The obvious choice of a divisor class associated to E then is the class of its abelian complement. This section will be concerned with the classification of abelian subvarieties of codimension 1.

Definition 3.2.1. An *abelian divisor* is an abelian subvariety of codimension 1 on A, seen as a prime Weil divisor.

We would like to characterize abelian divisors by their numerical classes. We first characterize them among all prime divisors.

Proposition 3.2.2. *Let X be a prime divisor on an abelian variety A of dimension n. Then X is the translation of an abelian divisor if and only if $[X]^2 = 0$ in $\mathfrak{A}^*(A)$.*

Proof. If X is the translation of an abelian divisor and $x \in X$, let $z \notin X - x := t_{-x}(X)$. Then $X \cap (X + z) = \varnothing$, and so in particular $[X]^2 = 0$. Conversely, assume that X is a prime divisor and $(X^2 \cdot \Theta^{n-2}) = 0$. By translating, we can assume that X contains 0. We see that if $x \in X$, then $0 \in X \cap (X - x)$, and therefore, if $X \cap (X - x) \neq X$, this would be a subvariety of A of codimension 2 and hence $(X^2 \cdot \Theta^{n-2}) > 0$, a contradiction. This implies that $X = X - x$. Similarly, we see that for $x, y \in X$, $X - (x + y) = (X - x) - y = X - y = X$, and we therefore conclude that X is a group. Since we can see X as an irreducible subvariety of A, we obtain that it is an abelian subvariety of codimension 1. □

An easy corollary to the proposition is the following:

Corollary 3.2.3. *An abelian variety A is isogenous to the product of elliptic curves if and only if there exists a strictly increasing chain of irreducible subvarieties $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{n-1}$ such that $\dim X_i = i$ and*

$$(X_i^2 \cdot X_{i+1} \cdot \Theta^{i-1}) = 0.$$

We see that abelian divisors correspond to certain elements $\alpha \in \mathrm{NS}(A)$ such that $\alpha^2 = 0$. The question we would like to answer is: How can we characterize abelian divisors among all such elements?

Proposition 3.2.4. *If X is an effective divisor, then $X \equiv mY$ for some abelian divisor Y and some $m \in \mathbb{Z}$ if and only if $[X]^2 = 0$ in $\mathfrak{A}^*(A)$ (and this occurs if and only if $(X^2 \cdot \Theta^{n-2}) = 0$).*

Proof. Assume that $[X]^2 = 0$ and $X \neq 0$. Write $X = \sum m_i F_i$, where the F_i are irreducible codimension 1 subvarieties of A and $m_i > 0$. We have that

$$0 = (X^2 \cdot \Theta^{n-2}) = \sum_{i,j} m_i m_j (F_i \cdot F_j \cdot \Theta^{n-2}) \geq 0.$$

In particular, $(F_i^2 \cdot \Theta^{n-2}) = 0$, and so by Corollary 3.2.3 we have that each F_i is the translate of an abelian divisor. Since we can move all the F_i inside their numerical equivalence classes, assume that all are abelian subvarieties. If F_i and F_j are different, for instance, we have that $(F_i \cap F_j)_0$ is an abelian subvariety of codimension 2. However, this contradicts the fact that $(F_i \cdot F_j \cdot \Theta^{n-2}) = 0$. Therefore we must have that all the F_i are the same, and the result follows. □

Notice that Proposition 2.1.11 guarantees that if X is a divisor that satisfies $[X]^2 = 0$ in $\mathfrak{A}^*(A)$, then either X or $-X$ is effective.

Lemma 3.2.5. *The class of an abelian divisor is primitive.*

Proof. Let X be an abelian divisor, and assume that $X \equiv mD$ for some divisor D such that $[D]$ is primitive. Suppose that $m > 0$; if not, then we replace D by $-D$. Since X is effective and $m > 0$, by the previous proposition we can assume that D is effective. Now using the same proposition, we see that $D \equiv Y$ for some abelian divisor Y, and so $X \equiv mY$. However, we then see that $[X \cdot Y] = 0$ in $\mathfrak{A}^*(A)$, and using the argument used in the proof of Proposition 3.2.4, we get that $Y \equiv X$. Therefore $(m-1)X \equiv 0$, and so $m = 1$. □

Remark 3.2.6. Another way of proving this lemma in characteristic 0 is using the following criterion: The class of a divisor D is primitive if and only if $A[m] \nsubseteq K(D)$ for some $m \in \mathbb{Z}$ (where $A[m]$ denotes the group of m-torsion points of A). If D is an abelian divisor, then by cardinality $A[m] \nsubseteq K(D) = D$ for all m.

Using this lemma and Proposition 3.2.4, we get:

Corollary 3.2.7. *A class $\alpha \in NS(A)$ comes from an abelian divisor if and only if it is effective, primitive and $\alpha^2 = 0$ in $\mathfrak{A}^*(A)$.*

With all we have said so far, we can prove our first main result:

Theorem 3.2.8. *Let A be an abelian variety of dimension n. Then the map $X \mapsto [X]$ induces a bijective correspondence between abelian divisors on A and primitive elements $\alpha \in NS(A)$ that satisfy $\alpha^2 = 0$ in $\mathfrak{A}^*(A)$ and $\deg \alpha > 0$.*

Proof. Injectivity was already proven. To show surjectivity, let α be a primitive class that satisfies $(\alpha \cdot \Theta^{n-1}) > 0$ and $\alpha^2 = 0$. Proposition 2.1.11 says that α is effective, and so Corollary 3.2.7 assures us that α comes from an abelian divisor. $\square$

3.2.1 Homogeneous forms on NS(A)

We now wish to mimic Kani's characterization of elliptic curves on abelian surfaces using homogeneous forms on $\mathrm{NS}(A, \Theta)$.

Let $\natural : \mathrm{NS}(A) \to \mathrm{NS}(A)$ denote the endomorphism

$$\alpha \mapsto \alpha^\natural = (\Theta^n)\alpha - (\deg \alpha)[\Theta].$$

We define the homogeneous polynomials

$$q_r(\alpha) := -\frac{1}{(r-1)(\Theta^n)}((\alpha^\natural)^r \cdot \Theta^{n-r})$$

for $2 \le r \le n$; for $n = 2$ we have that q_2 is precisely the quadratic form introduced by Kani.

By expanding the right hand side, we have that

$$q_r(\alpha) = (-1)^r(\deg \alpha)^r + \frac{(\Theta^n)}{r-1}\sum_{m=2}^{r}\binom{r}{m}(\Theta^n)^{m-2}(-1)^{r-m+1}(\deg \alpha)^{r-m}(\alpha^m \cdot \Theta^{n-m}).$$

Lemma 3.2.9. *If $[\Theta]$ is primitive in NS(A) and $\alpha \in NS(A)$, then $q_r(\alpha) \le 0$ for all $r = 2, \ldots, n$ if and only if $\alpha \in \mathbb{Z}[\Theta]$.*

Proof. It is clearly seen that $q_r([\Theta]) = 0$. Conversely, if $q_r(\alpha) \le 0$, then we would have that $((\alpha^\natural)^r \cdot \Theta^{n-r}) \ge 0$ for all r (and by the definition of $\alpha^\natural$, $\deg \alpha^\natural = 0$), so by Proposition 2.1.11 we have that $\alpha^\natural = [D]$, where D is an effective divisor on A. But if $D \not\equiv 0$, then $\deg D > 0$, a contradiction. Therefore $\alpha^\natural \equiv 0$, and so $(\Theta^n)\alpha \in \mathbb{Z}[\Theta]$. Since $[\Theta]$ is primitive, we obtain that $\alpha \in \mathbb{Z}[\Theta]$. $\square$

Remark 3.2.10. We observe that if $\alpha \in \mathrm{NS}(A)$ satisfies $\alpha^2 = 0$ in $\mathfrak{A}^*(A)$, then

$$q_r(\alpha) = (-1)^r(\deg \alpha)^r$$

for all r.

Remark 3.2.11. It is easy to see that the forms q_r descend to forms on $\mathrm{NS}(A, \Theta)$. In dimension 3, the previous lemma shows that q_2 is positive definite on $\mathrm{NS}(A, \Theta)$. Indeed, if $q_2(\alpha) \leq 0$ and $q_3(\alpha) \leq 0$, then Lemma 3.2.9 says that $\alpha \in \mathbb{Z}[\Theta]$. If $q_2(\alpha) \leq 0$ and $q_3(\alpha) \geq 0$, then $q_2(-\alpha) \leq 0$ and $q_3(-\alpha) \leq 0$, and we have the same situation.

Lemma 3.2.12. *If X is an abelian divisor on A, then* $\deg X = (n-1)!(E \cdot \Theta)$, *where E is the abelian complement of X in A.*

Proof. Set $d := (\Theta \cdot E)$; in other words, Θ restricted to E is a divisor of degree d. Using Riemann-Roch and the fact that $K(\Theta|_X) \simeq X \cap E \simeq K(\Theta|_E)$ and $\chi(\Theta|_X)^2 = |K(\Theta|_X)|$, we have that

$$\deg X = (X \cdot \Theta^{n-1}) = ((\Theta|_X)^{n-1}) = (n-1)!\chi(\Theta|_X) = (n-1)!\chi(\Theta|_E) = (n-1)!d.$$

□

The next three lemmas are technical in nature and will be used in the proof of our main theorem. The first of the three is elementary and well-known.

Lemma 3.2.13. *If $m \in \mathbb{Z}\backslash\{\pm 1\}$ and $n \in \mathbb{Z}_{>0}$, then $m^{n-1} \mid n!$ if and only if $m = \pm 2$ and n is a power of* 2.

Proof. We will prove this for a prime number p that divides m. If p is a prime number such that $p^{n-1} \mid n!$, then Legendre's formula for the highest power of a prime appearing in $n!$ says that

$$n - 1 \leq \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^l} \right\rfloor$$

for $l = \lfloor \log_p(n) \rfloor$. If S denotes the right hand side of the inequality, we have that

$$S \leq n\left(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^l}\right) = n\left(\frac{1 - \frac{1}{p^l}}{p-1}\right).$$

This is obviously less than or equal to $\frac{n}{p-1}$, and since $n-1 \leq S$, we get that p is necessarily 2. Replacing $p=2$ above and clearing the equations, we arrive at $n \leq 2^l = 2^{\lfloor \log_2(n) \rfloor}$. If n is not a multiple of 2, then this is impossible. Therefore we conclude that $n = 2^k$ for some k and $p=2$. □

Lemma 3.2.14. *Let (A, Θ) be a ppav. Then the class of an abelian divisor in $NS(A, \Theta)$ is primitive.*

Proof. Let X be an abelian divisor on A, and assume that $X \equiv mD + s\Theta$ for some divisor D and $m, s \in \mathbb{Z}$. We can also assume that $s \neq 0$ and actually $(m, s) = 1$, since X is primitive in $\mathrm{NS}(A)$. Moreover, after changing D with $-D$ if necessary, we can assume that $m > 0$. We get the following formula:

$$m^r(D^r \cdot \Theta^{n-r}) = ((X - s\Theta)^r \cdot \Theta^{n-r}) = (-s)^{r-1}(r \deg X - s(\Theta^n))$$

for $1 \leq r \leq n$. Assume that $(D^n) \neq 0$ and $(D^{n-1} \cdot \Theta) \neq 0$. We see that

$$m^{n-1}(D^{n-1} \cdot \Theta) = (-s)^{n-2}((n-1) \deg X - s(\Theta^n))$$

and

$$m^n(D^n) = (-s)^{n-2}(n \deg X - s(\Theta^n)).$$

This means that $m^{n-1} \mid (n-1)\deg X - s(\Theta^n)$ and $m^n \mid n \deg X - s(\Theta^n)$, and so $m^{n-1} \mid \deg X$. But then $m^{n-1} \mid s(\Theta^n)$ and so $m^{n-1} \mid n!$, since $(\Theta^n) = n!$ in this case. By Lemma 3.2.13 we conclude that $n = 2^k$ for some k and $m = 2$. In this case, we have that $2^n \mid n!(d-s)$, where $\deg X = d(n-1)!$. If d is even, then $d-s$ is odd and so $2^n \mid n!$. Based on the previous lemma, it is easy to see that this is impossible. If d is odd, then

$$2^{n-1} \mid (n-1)!((n-1)d - sn) = (2^k-1)!((2^k-1)d - s2^k),$$

and since $(2^k-1)d - s2^k$ is odd, we have that $2^{n-1} \mid (n-1)!$, a contradiction. Therefore we must have that $m = 1$.

If $(D^n) = 0$, then $\deg X = (n-1)!s$ and $(D^{n-1} \cdot \Theta) \neq 0$. Therefore, $m^{n-1} \mid (n-1)(n-1)!s - sn!$, and so $m^{n-1} \mid (n-1)!$. Thus $m = 1$. If $(D^{n-1} \cdot \Theta) = 0$, then $(n-1)\deg X = sn!$. Let $\deg X = d(n-1)!$. Then $m^n \mid n!(d-s)$, and since $d = \frac{sn}{n-1}$, we get that

$$m^n \mid n!\left(\frac{sn}{n-1} - s\right) = s(n^2(n-2)! - n!) = s(n-2)!n.$$

Therefore, if p is a prime that divides m, we have that either $p^n \mid (n-2)!$, which is impossible based on what we have said above, or $p^n \mid n$ which is even more ridiculous. We conclude that $m = 1$, and so $[X]$ is primitive in $\mathrm{NS}(A, \Theta)$. □

We observe that this is no longer true when Θ is not a principal polarization. For instance, take $A = E_1 \times E_2$ and $\Theta = \{0\} \times E_2 + 2(E_1 \times \{0\})$ where E_1 and E_2 are elliptic curves. Putting $X = \{0\} \times E_2$, we get that $X = -2(E_1 \times \{0\}) + \Theta$, and so is not primitive in $\mathrm{NS}(A, \Theta)$. This gives a counterexample to Theorem 3.2 of Kani [21].

Lemma 3.2.15. *Take $\alpha \in NS(A, \Theta)$ and let $d, k \in \mathbb{Z}$ be such that k is positive, $\deg \alpha = d$ and $q_r(\alpha) = (-1)^r k^r$ for $r = 2, \dots, n$. Then $d - k \equiv 0$ (mod $n!$).*

Proof. Let $x_r := n!^{r-1}(\alpha^r \cdot \Theta^{n-r})$. It is easy to see, using the definition of q_r, that

$$x_r = (r-1)(-1)^r(d^r - k^r) + \sum_{m=2}^{r-1} \binom{r}{m} (-1)^{r-m+1} d^{r-m} x_m.$$

By using induction, it is first easy to prove that

$$x_r = d^r - \sum_{l=2}^{r} \binom{r}{l} (-1)^l (l-1) d^{r-l} k^l.$$

Using this expression, again by induction we arrive at the following expression for the general term:

$$x_r = (d-k)^{r-1}(d + (r-1)k).$$

We can replace α by $\alpha + m\Theta$ for $m \in \mathbb{Z}$ and assume that $d \geq 0$; we further assume that $d \neq k$. This shows in particular that $(\alpha^r \cdot \Theta^{n-r}) \neq 0$ for all r.

Assume that $n > 2$ (the case $n = 2$ is trivial). Let p be a prime such that $p^s \mid n!$ with $s \in \mathbb{Z}_{>0}$ maximal and let t be the largest integer such that $p^t \mid d - k$. We wish to prove that $t \geq s$. Assume the contrary; that is, assume that $t < s$. We then have that $p^{(s-t)(r-1)} \mid d + (r-1)k$ for every r. Then $p^{(s-t)(n-2)} \mid k$, and therefore $p^{(s-t)(n-2)} \mid d$. In particular, $p^{(s-t)(n-2)} \mid d - k$, and so $(s-t)(n-2) \leq t < s$. But then $s \geq n-1$, and so by Lemma 3.2.13 necessarily $p = 2$ and n is a power of 2. This means that for every odd prime that divides $n!$, the same prime divides $d - k$ with the same or greater power. We have now reduced the proof to showing that the same is also true when $p = 2$.

With $p = 2$, we have that $s = n - 1$, and so $(n-1-t)(n-2) < n-1$. After rearranging, we have that $t > n - 1 - \frac{n-1}{n-2}$, and so $t \geq n - 1 = s$. □

We can now state and prove our second main theorem.

Theorem 3.2.16. *Let (A, Θ) be a ppav of dimension n and let $d > 0$. Then the map $X \mapsto [X]$ induces a bijection between*

1. *abelian divisors of degree d*

2. *primitive numerical classes $[\alpha] \in NS(A, \Theta)$ that satisfy $q_r(\alpha) = (-1)^r d^r$ for all $r = 2, \ldots, n$.*

Proof. First we will prove injectivity. If $[X] = [Y]$ in $\mathrm{NS}(A, \Theta)$, then $X \equiv Y + m\Theta$ for some $m \in \mathbb{Z}$. By squaring and then intersecting with Θ^{n-2}, we get that

$$\begin{aligned} m(2 \deg Y + mn!) &= 0 \\ m(-2 \deg X + mn!) &= 0 \end{aligned}$$

But this is only possible if $m = 0$, since $\deg X$ and $\deg Y$ are positive.

For surjectivity, first let $[\alpha] \in \mathrm{NS}(A, \Theta)$ be a primitive class such that $q_r(\alpha) = (-1)^r d^r$. By Lemma 3.2.15, we get that $\deg \alpha - d \equiv 0 \pmod{n!}$, and we define

$$\beta := \alpha - \frac{\deg \alpha - d}{n!}[\Theta].$$

Since α is primitive in $\mathrm{NS}(A, \Theta)$, it is trivial to see that β is primitive in $\mathrm{NS}(A)$. Moreover, $q_r(\beta) = q_r(\alpha)$ and $\deg \beta = d$. This means that $(\beta^r \cdot \Theta^{n-r}) = 0$ for all $2 \leq r \leq n$, and so by Theorem 3.2.8, β, and thus α, comes from an abelian divisor. □

Corollary 3.2.17. *There is a bijection between*

1. *elliptic curves of degree d on A*

2. *primitive numerical classes $[\alpha] \in NS(A, \Theta)$ that satisfy $q_r(\alpha) = (-1)^r (n-1)!^r d^r$ for all $r = 2, \ldots, n$.*

It may seem that the forms q_r for $r \geq 3$ are extraneous, especially since Kani's characterization of elliptic curves on an abelian surface is by means of a single quadratic form. However, all the forms q_r are needed for this characterization. For example, if $A = E_1 \times E_2 \times E_3$ for elliptic curves E_i and Θ is the product polarization, let

$$\begin{aligned} D_1 &:= \{0\} \times E_2 \times E_3 \\ D_2 &:= E_1 \times \{0\} \times E_3 \\ D_3 &:= E_1 \times E_2 \times \{0\}. \end{aligned}$$

For $k \in \mathbb{Z}_{>0}$, let $\alpha_k := -kD_1 + k(k+1)D_2 + (k+1)D_3$. We have that $\deg \alpha_k = (k+1)^2 - k$, $(\alpha_k^2 \cdot \Theta) = 0$ and $(\alpha_k^3) = -k^2(k+1)^2$. We see that α_k is primitive and $q_2(\alpha_k)$ is a square, but α_k does not come from an abelian divisor (since $(\alpha_k^3) \neq 0$).

This shows that the form q_3 is indispensable here. Similar examples can be found in higher dimension.

In the case that Θ is not a principal polarization we cannot be as explicit as in Theorem 3.2.16, but something can be said. One of the main obstructions to obtaining a similar theorem in the non-principally polarized case is the fact that abelian divisors are not necessarily primitive in $\mathrm{NS}(A,\Theta)$. Nonetheless, we can still use the forms q_r to find abelian divisors.

Proposition 3.2.18. *Let (A,Θ) be a polarized abelian variety of dimension n. If $[\alpha] \in NS(A,\Theta)$ and $d \in \mathbb{Z}_{>0}$ such that $\deg\alpha \equiv d \pmod{(\Theta^n)}$ and $q_r(\alpha) = (-1)^r d^r$ for $r \leq n$, then $\alpha = m[X]$ for some abelian divisor X on A and some $m \in \mathbb{Z}$.*

Proof. Take $\beta = \alpha - \frac{\deg\alpha - d}{(\Theta^n)}[\Theta] \in \mathrm{NS}(A)$. We see that $\deg\beta = d$ and $q_r(\beta) = (-1)^r d^r$ for $r \leq n$, and so $(\beta^r \cdot \Theta^{n-r}) = 0$ for $r \geq 2$. Since $\deg\beta > 0$, Proposition 2.1.11 says that β is effective and Proposition 3.2.4 says that β is algebraically equivalent to a multiple of an abelian divisor. □

We say that a polarized abelian variety (A,Θ) *represents* the vector $(d_2,\ldots,d_n) \in \mathbb{Z}^{n-1}$ if there exists a class $\alpha \in \mathrm{NS}(A)$ (or equivalently in $\mathrm{NS}(A,\Theta)$) such that $q_r(\alpha) = d_r$. We say that it *primitively represents* the same vector if there is a primitive class that satisfies the same equation.

Proposition 3.2.19. *Let (A,Θ) be a ppav of dimension n. Then (A,Θ) is isomorphic to a product abelian variety $(E\times Y, pr_1^*(0)+pr_2^*\Theta_2)$ for E an elliptic curve and (Y,Θ_2) an $n-1$ dimensional polarized abelian variety if and only if (A,Θ) represents the vector $((-1)^r(n-1)!^r)_{r=2}^n$. This is equivalent to the existence of an elliptic curve E in A with $(\Theta\cdot E)=1$.*

Proof. If A splits in the way that is stated, then $(\Theta\cdot(E\times\{0\})) = 1$, and so Lemma 3.2.12 says that the abelian complement of $E\times\{0\}$ has degree $(n-1)!$.

For the other direction, we will first show that if $q_r(\alpha)$ satisfies the equation above, then α must be primitive. If $\alpha = m\beta$ for some primitive β (we can assume m positive), then $q_2(\beta) = ((n-1)!/m)^2$, and so $m \mid (n-1)!$. But then $q_r(\beta) = (-1)^r((n-1)!/m)^r$, and so by Theorem 3.2.16 there exists an abelian divisor Y on A with $\deg Y = (n-1)!/m$. If $m > 1$, this contradicts Lemma 3.2.12.

Now assume that $q_r(\alpha) = ((-1)^r((n-1)!)^r)_{r=2}^n$. Since α is primitive, there exists an abelian divisor X of degree $(n-1)!$ such that $[X] = [\alpha]$ in $\mathrm{NS}(A,\Theta)$. By Lemma 3.2.12, the abelian complement of X is an elliptic curve E with $(\Theta\cdot E)=1$. Now

$$((\Theta - X)^r \cdot \Theta^{n-r}) = (n-r)(n-1)! \geq 0,$$

and by Proposition 2.1.11 $\Theta - X \equiv D$ for some effective divisor D. This implies that $((\Theta - X) \cdot E) \geq 0$, and so $1 - (X \cdot E) \geq 0$. But then E intersects X in only one point, and so the addition map $E \times X \to A$ is an isomorphism of varieties.

Since $(D \cdot E) = 0$, if we see D as a divisor on $E \times X$, this means that $\mathcal{O}_{E\times X}(D)|_{E\times\{z\}}$ is trivial for every $z \in X$. By the Seesaw Theorem, we get that $\mathcal{O}_{E\times X}(D) \simeq \mathrm{pr}_2^*\mathcal{O}_X(\Theta_2)$ for some divisor Θ_2 on X. Summing everything up, we get that Θ, seen as a divisor on the product, is numerically equivalent to $\mathrm{pr}_1^*(0) + \mathrm{pr}_2^*(\Theta_2)$. □

As a corollary, we obtain a nice geometric result in dimension 3.

Corollary 3.2.20. *A ppav $(A, \Theta) \in \mathcal{A}_3$ is not the Jacobian of a curve if and only if (q_2, q_3) represents $(4, -8)$.*

Proof. It is known that a principally polarized abelian 3-fold (A, Θ) is the Jacobian of some curve if and only if it is indecomposable. Therefore, by Proposition 3.2.19, (A, Θ) is not the Jacobian of a curve if and only if it represents $(4, -8)$. □

Assume now that (J_C, Θ_C) is the Jacobian of a curve C. A *minimal elliptic cover* is a finite morphism $f : C \to E$ to an elliptic curve E that does not factor through any other elliptic curve non-trivially. Two covers $f : C \to E$ and $f' : C \to E'$ are *isomorphic* if there is an isomorphism $\phi : E \to E'$ such that $\phi \circ f = f'$. Kani [21] gives the following classification:

Proposition 3.2.21. *The map $f \mapsto f^*E$ gives a 1-1 correspondence between the set of isomorphism classes of minimal elliptic covers $f : C \to E$ of degree k and elliptic subgroups $E \leq J_C$ with $(E \cdot \Theta_C) = k$.*

Translating this to our language, we get:

Proposition 3.2.22. *Let C be a curve of genus g, and let J_C be its Jacobian. Then there is a bijective correspondence between the following sets:*

1. *Isomorphism classes of minimal elliptic covers $C \to E$ of degree k.*
2. *Elliptic subgroups $E \leq J_C$ such that $(E \cdot \Theta_C) = k$.*
3. *Primitive elements $\alpha \in NS(J_C, \Theta_C)$ such that $q_r(\alpha) = (-1)^r (g-1)!^r k^r$.*

Corollary 3.2.23. *A 3-dimensional ppav (A, Θ) is the Jacobian of a genus 3 curve and splits isogenously as the product of elliptic curves if and only if (q_2, q_3) does not represent $(4, -8)$ but there exist two distinct primitive elements in $NS(A, \Theta)$ that represent vectors of the form $(d^2, -d^3)$ for $d > 2$.*

3.3 The general case

The results of the previous section are substantial, but one is left unsatisfied. The natural question is the following: Does a similar characterization exist for *all* abelian subvarieties of A? The obvious problem is that the abelian complement of a u-dimensional abelian subvariety X is no longer a divisor if $u \geq 2$, so using the above technique is out of the question. We do have a useful tool at our disposal, however: norm endomorphisms.

Let (A, Θ) be a ppav. Recall that to each abelian subvariety X of A we can associate a symmetric endomorphism N_X that satisfies the properties of Proposition 1.2.4.

Now comes the crucial observation that permits a generalization of the previous section:

Key Fact: Since N_X is a symmetric endomorphism, by Theorem 2.1.13 there exists a numerical class $\alpha_X \in \mathrm{NS}(A)$ such that $N_X = \phi_{\alpha_X}$.

This is our first beacon of hope in the quest to characterize abelian subvarieties by numerical classes.

Theorem 3.3.1. *If $f \in End(A)$, then there exists a unique monic polynomial $P_f(t) \in \mathbb{Z}[t]$ of degree $2n$ such that for every $m \in \mathbb{Z}$, $P_f(m) = \deg(f - m)$.*

Proof. See [29], Theorem 10.9. □

We note that when $k = \mathbb{C}$, $P_f(t)$ is the characteristic polynomial of the rational representation of f. For a general field, if $p \neq \mathrm{char}(k)$, then $P_f(t)$ is the characteristic polynomial of the action of f on

$$V_pA = (T_pA) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p := (\lim_{\leftarrow} A[p^l]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $A[p^l]$ denotes the group of p^l-torsion points of A, $\mathbb{Z}_p$ denotes the ring of p-adic integers and $\mathbb{Q}_p$ is the field of p-adic numbers (see [29]). Moreover, V_p is a vector space of dimension $2n$ over $\mathbb{Q}_p$.

By Proposition 1.2.4, if $X \leq A$ is a non-trivial abelian subvariety of exponent e_X, then $N_X^2 = e_X N_X$, and so the only eigenvalues N_X can have on $(T_pA) \otimes \mathbb{Q}_p$ are 0 and e_X. This means that

$$P_X(t) := P_{N_X}(t) = t^{2n-r}(t - e_X)^r$$

for some positive integer r.

Proposition 3.3.2. *If $X \leq A$ is an abelian subvariety of dimension u and exponent e_X, then*

$$P_X(t) = t^{2n-2u}(t - e_X)^{2u}.$$

Proof. Let Y be the abelian complement of X in A (defined by Θ). We have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{(N_X,N_Y)} & X \times Y \\ {\scriptstyle N_X}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{(N_X,N_Y)} & X \times Y \end{array}$$

where $g = \begin{pmatrix} e_X \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix}$. Now (N_X, N_Y) is an isogeny, and

$$N_X = (N_X, N_Y)^{-1} \begin{pmatrix} e_X \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix} (N_X, N_Y)$$

lies in $\mathrm{End}_{\mathbb{Q}}(A)$. In particular,

$$\mathrm{tr}(N_X) = \mathrm{tr} \begin{pmatrix} e_X \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix},$$

where *tr* denotes the trace function on $\mathrm{End}(A) \hookrightarrow \mathrm{End}(V_p)$. Let p be a prime such that $p \neq \mathrm{char}(k)$ and such that $A[p] \cap X \cap Y = \{0\}$ (p exists since $X \cap Y$ is finite). In this case, $A[p] \simeq X[p] \oplus Y[p]$, and so

$$V_pA \simeq (T_pZ \oplus T_pY) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq V_pZ \oplus V_pY.$$

It is then obvious that g acts on V_pA as $\begin{pmatrix} e_X \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix}$, and so

$$\mathrm{tr}(N_X) = \mathrm{tr} \begin{pmatrix} e_X \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix} = 2ue_X$$

(since $\dim_{\mathbb{Q}_p}(T_pZ) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 2u$). Note that this argument was taken from [4] Corollary 4.3.10.

From our previous discussion, we have that

$$P_X(t) = t^{2n-r}(t - e_X)^r = \sum_{m=0}^{r} \binom{r}{m} (-e_X)^m t^{2n-m}.$$

Moreover, we have that $\mathrm{tr}(N_X)$ is precisely -1 times the coefficient of t^{2n-1}. Putting everything together, we get

$$2ue_X = \operatorname{tr}(N_X) = e_X\binom{r}{1} = e_X r$$

and so $r = 2u$. □

Corollary 3.3.3. *For all $m \in \mathbb{Z}$, we have that* $\deg(N_X - m) = m^{2n-2u}(m - e_X)^{2u}$.

Let α_X be the divisor class associated to X and let D_X be any divisor in the class of α_X. We see that

$$\begin{aligned}\chi(m\Theta - D_X)^2 &= \deg(m\phi_\Theta - \phi_{D_X}) = \deg(m - \phi_\Theta^{-1}\phi_{D_X}) = \deg(m - N_X)\\ &= \deg(N_X - m) = m^{2n-2u}(m - e_X)^{2u}.\end{aligned}$$

Since $m\Theta - D_X$ is ample for $m \gg 0$, we get that

$$\chi(m\Theta - D_X) = m^{n-u}(m - e_X)^u$$

for $m \gg 0$. The Riemman-Roch Theorem (Theorem 2.1.8) then says that

$$m^{n-u}(m - e_X)^u = \frac{1}{n!}((m\Theta - D_X)^n)$$

for $m \gg 0$, and since both sides are polynomials, we get this equality for all $m \in \mathbb{Z}$. After expanding both sides, we get that

$$\sum_{s=0}^{u}\binom{u}{s}(-e_X)^s m^{n-s} = \sum_{r=0}^{n}(-1)^r\frac{(D_X^r \cdot \Theta^{n-r})}{(n-r)!r!}m^{n-r}.$$

Theorem 3.3.4. *Let $X \leq A$ be an abelian subvariety of dimension u and exponent d, and let α_X be the unique numerical class such that $N_X = \phi_{\alpha_X}$. Then*

$$(\alpha_X^r \cdot \Theta^{n-r}) = \begin{cases} (n-r)!r!\binom{u}{r}d^r & 1 \leq r \leq u \\ 0 & u+1 \leq r \leq n \end{cases}$$

Reciprocally, if α is a primitive class that has these same intersection numbers for certain positive integers d and u, then there exists an abelian subvariety $X \leq A$ of dimension u and exponent d such that $\alpha = \alpha_X$. Moreover, if Y is the abelian complement of X in A, in the Chow ring modulo algebraic equivalence $\mathfrak{A}^*(A)$ we have the equality

$$[\alpha_X]^u = \frac{u!d^u}{\sqrt{|X \cap Y|}}[Y].$$

Before proving the theorem we prove a lemma.

Lemma 3.3.5. *Let $\phi \in End^s(A)$ be a symmetric endomorphism with characteristic polynomial of the form $\chi_\phi(t) = t^{2n-2u}(t-d)^{2u}$. Then the minimal polynomial of ϕ is $M_\phi(t) = t(t-d)$.*

Proof. Let $\mathbb{Q}[\phi]$ be the (commutative) subalgebra of $\mathrm{End}_\mathbb{Q}(A)$ generated by 1 and ϕ, and let $T_\phi \in \mathrm{End}(\mathbb{Q}[\phi])$ be multiplication by ϕ. We know that $H : (f,g) \mapsto \mathrm{Tr}(fg^\dagger)$ is a positive definite symmetric bilinear form, and we see that for all $f, g \in \mathbb{Q}[\phi]$

$$H(T_\phi(f), g) = \mathrm{Tr}(\phi f g^\dagger) = \mathrm{Tr}(f g^\dagger \phi^\dagger) = \mathrm{Tr}(f(\phi g)^\dagger) = H(f, T_\phi(g)),$$

and so T_ϕ is self-adjoint with respect to H. In particular, T_ϕ is diagonalizable on $\mathbb{R}[\phi] = \mathbb{Q}[\phi] \otimes \mathbb{R}$, and so its minimum polynomial splits as the product of distinct linear factors:

$$M_{T_\phi}(t) = \prod_{i=1}^{r}(t - \lambda_i)$$

where $r = \dim_\mathbb{Q} \mathbb{Q}[\phi]$ and $\lambda_i \in \mathbb{R}$. Evaluating in $T_\phi(1)$, we get that

$$0 = \prod_{i=1}^{r}(\phi - \lambda_i \mathrm{id}).$$

Therefore the minimal polynomial of ϕ divides M_{T_ϕ}, and by our hypothesis on the characteristic polynomial of ϕ, we have that $M_\phi(t)$ must be $t(t-d)$. ☐

Proof of Theorem 3.3.4. From what we have already said, we see that any α_X coming from an abelian subvariety of dimension u and exponent d has these intersection numbers. Now let α be a primitive class that has the intersection numbers above. By our previous analysis, we have that $P_{\phi_\alpha}(t) = t^{2n-2u}(t-d)^{2u}$. Since ϕ_α is a symmetric endomorphism, by the previous lemma we get that $\phi_\alpha^2 = d\phi_\alpha$. Moreover ϕ_α is primitive since α is, and so by Proposition 1.2.4 we get that $\phi_\alpha = N_{\mathrm{Im}(\phi_\alpha)}$.

Since $(\ker \phi_{\alpha_X})_0 = Y$ (the connected component of $\ker \phi_{\alpha_X}$ containing 0), we have that there exists a line bundle $\tilde{L}_X$ on A/Y such that the numerical class of $L_X := p_Y^* \tilde{L}_X$ is α_X, where $p_Y : A \to A/Y$ is the natural projection. Moreover,

$$h^0(A, L_X) = h^0(A/Y, \tilde{L}_X) \geq 0$$

(since L_X is effective). We see that if we intersect the numerical class of $\tilde{L}_X$ u times (as algebraic cycles), then we obtain a finite number of points. Since any two points are algebraically equivalent on an abelian variety, we have that $\alpha_X^u = m_Y p_Y^*[\{0\}]$ for some integer $m_Y \geq 0$ in the Chow ring of A. Now, $p_Y^*[\{0\}] = [Y]$, and so we get that

$$\alpha_X^u = m_Y[Y]$$

for some integer Y.

We get that

$$(n-u)!^2 u!^2 d^{2u} = (\alpha_X^u \cdot \Theta^{n-u})^2 = m_Y^2 (\Theta|_Y^{n-u})^2 = m_Y^2 (n-u)!^2 |X \cap Y|,$$

and so $m_Y = u! \frac{d^u}{\sqrt{|X \cap Y|}}$. □

In particular, the previous proposition implies that the classes α_X are nilpotent in $\mathfrak{A}^*(A)$. It would be interesting to characterize all classes that come from abelian subvarieties among all nilpotent elements of $\mathfrak{A}^*(A)$ that lie in NS(A).

Corollary 3.3.6. *If A and B are two abelian varieties of dimension n such that there exists an isomorphism between NS(A) and NS(B) that preserves the intersection pairing, then there is a bijection between abelian subvarieties of A and abelian subvarieties of B that preserves dimension and exponents.*

Example 3.3.7. We have that $\alpha_{\{0\}} = 0$ and $\alpha_A = [\Theta]$.

Example 3.3.8. Let E be an elliptic subgroup of A. Then $(\alpha_E^r \cdot \Theta^{n-r}) = 0$ for all $r \geq 2$ and so by Theorem 3.2.8 $\alpha_E = [Z]$ for some abelian divisor $Z \leq A$. By comparing the kernels of N_E and ϕ_Z, we obtain that Z is the abelian complement of E in A.

In general if X and Y are complementary abelian subvarieties then by Proposition 1.2.5 they have the same exponent d, and $N_X + N_Y = d\text{Id}$. Translating this to divisor classes, we get:

Proposition 3.3.9. *Let X and Y be complementary abelian subvarieties of A. Then*

$$\alpha_X = d[\Theta] - \alpha_Y.$$

In particular, $[\alpha_X] = -[\alpha_Y]$ in NS(A, Θ) and so the natural function between abelian subvarieties of dimension u and abelian subvarieties of codimension u can be seen as the inverse homomorphism.

3.3.1 Characterization using q_r

Let u and r be integers such that $1 \leq u \leq n$ and $2 \leq r \leq n$. We define the number

$$f(u,r) \quad := \quad \frac{1}{r-1} \sum_{m=0}^{\min\{r,u\}} \frac{\binom{r}{m}\binom{u}{m}}{\binom{n}{m}} n!^m (-1)^{r-m+1} ((n-1)!u)^{r-m}.$$

Lemma 3.3.10. *Let $X \leq A$ be an abelian subvariety of exponent d, and let α_X be its corresponding numerical class. Then*

$$q_r(\alpha_X) = f(u,r)d^r,$$

where q_r is the homogeneous form defined in Section 3.2.1.

Proof. By expanding q_r, we see that if $\alpha \in \mathrm{NS}(A)$, then

$$q_r(\alpha) = (-1)^r(\deg\alpha)^r + \frac{n!}{r-1}\sum_{m=2}^{r}\binom{r}{m} n!^{m-2}(-1)^{r-m+1}(\deg\alpha)^{r-m}(\alpha^m \cdot \Theta^{n-m}).$$

If we evaluate α_X above and use the intersection numbers from Theorem 3.3.4, we get the expression we seek. □

Corollary 3.3.11. *$f(u,r) = (-1)^r f(n-u,r)$.*

Proof. This follows immediately from the previous lemma and from Proposition 3.3.9. □

Lemma 3.3.12. *Let α be a primitive numerical class in NS(A) such that*

$$\deg\alpha = (n-1)!ud$$

for $u, d \in \mathbb{Z}_{>0}$, and assume that $q_r(\alpha) = f(u,r)d^r$ for $2 \leq r \leq n$. Then α comes from an abelian subvariety of A of dimension u and exponent d.

Proof. We will proceed by induction on r to show that under the hypotheses of the lemma, we obtain the same intersection numbers as Theorem 3.3.4 for α.

For $r = 2$, we get that

$$q_2(\alpha) = (n-1)!^2u^2d^2 - n!(\alpha^2 \cdot \Theta^{n-2})$$

and

$$f(u,2)d^2 = (n-1)!^2u^2d^2 - 2n!\binom{u}{2}(n-2)!d^2.$$

This implies that $(\alpha^2 \cdot \Theta^{n-2}) = 2(n-2)!\binom{u}{2}d^2$, the same as in Theorem 3.3.4. For $2 < r \leq u$, we assume that the induction hypothesis is true for all $m < r$. Therefore we have

$$\begin{aligned} q_r(\alpha) &= (-1)^r(n-1)!^r u^r d^r \\ &\quad + \frac{d^r n!}{r-1}\sum_{m=2}^{r-1}\binom{r}{m}\binom{u}{m} n!^{m-2}(n-1)!^{r-m}(n-m)!m!u^{r-m}(-1)^{r-m+1} \\ &\quad - \frac{n!^{r-1}}{r-1}(\alpha^r \cdot \Theta^{n-r}) \\ &= f(u,r)d^r + \frac{d^r n!^{r-1}}{r-1}\binom{u}{r}(n-r)!r! - \frac{n!^{r-1}}{r-1}(\alpha^r \cdot \Theta^{n-r}) \end{aligned}$$

Since we are assuming that $q_r(\alpha) = f(u,r)d^r$, we get that

$$(\alpha^r \cdot \Theta^{n-r}) = (n-r)!r!\binom{u}{r}d^r.$$

When $r \geq u+1$, a similar argument shows that $(\alpha^r \cdot \Theta^{n-r}) = 0$, and so by Theorem 3.3.4, α comes from an abelian subvariety of dimension u and exponent d. □

Lemma 3.3.13. *The map that takes an abelian subvariety X to its class $[\alpha_X] \in NS(A, \Theta)$ is injective.*

Proof. We already saw that the map that takes X to $\alpha_X \in \mathrm{NS}(A)$ is injective. Assume then that there exists $m \in \mathbb{Z}$ such that $\alpha_X = \alpha_Y + m[\Theta]$ for some abelian subvariety Y of exponent e in A. Moreover, by switching X and Y if needed, we may assume that m is negative. We then have the equality

$$N_X = N_Y + m\mathrm{Id}.$$

In particular, since $N_X^2 = dN_X$ where d is the exponent of X, we get that

$$N_Y^2 + 2mN_Y + m^2\mathrm{Id} = dN_X = dN_Y + dm\mathrm{Id}.$$

Therefore $N_Y^2 + (2m-d)N_Y + (m^2 - dm)\mathrm{Id} = 0$. This implies that the minimal polynomial of N_Y, which is $M_Y(t) = t^2 - et$ divides $t^2 + (2m-d)t + m^2 - dm$. Since m is negative, $m \neq d$ and so m must be 0. Therefore $X = Y$. □

Lemma 3.3.14. *If X is an abelian subvariety of A, then $[\alpha_X]$ is primitive in $NS(A, \Theta)$.*

Proof. If X is a trivial subvariety, then the result is trivial (we will think of 0 as a primitive element). Assume then that X is non-trivial and assume that there exists a class $\beta \in \mathrm{NS}(A)$ and integers $m, s \in \mathbb{Z}$ such that $\alpha_X = s\beta + m\Theta$, where $s \neq \pm 1$. Therefore we have $N_X = s\phi_\beta + m\mathrm{Id}$. We can also assume that $\gcd(s, m) = 1$, since a norm endomorphism is primitive.

Let U be the connected component of $\ker N_X$. This is an abelian subvariety of positive dimension, and the restriction of $s\phi_\beta$ to U is equal to $-m\mathrm{Id}$. Therefore if $x \in U[s]$ (the s-torsion points of U), we get that

$$0 = \phi_\beta(sx) = s\phi_\beta(x) = -mx,$$

and so $U[s] \subseteq U[m]$. This means that $\gcd(s, m) \neq 1$, a contradiction. □

We finally come to our main theorem which is a vast generalization of Theorem 3.2.16.

Theorem 3.3.15. *Let (A, Θ) be a ppav of dimension n. Then the map $X \mapsto [\alpha_X]$ induces a bijection between*

1. *abelian subvarieties of dimension u and exponent d*
2. *primitive numerical classes $[\alpha] \in NS(A, \Theta)$ that satisfy $\deg \alpha \equiv (n-1)!ud \pmod{n!}$ and $q_r(\alpha) = f(u, r)d^r$ for all $r = 2, \ldots, n$.*

For $u = 1$, the condition $\deg \alpha \equiv (n-1)!ud \pmod{n!}$ may be dropped.

Proof. We already saw that abelian subvarieties induce such classes, and that this map is injective. Assume now that $[\alpha] \in \mathrm{NS}(A, \Theta)$ is a class that satisfies the above, and let

$$\beta := \alpha + \frac{(n-1)!ud - \deg(\alpha)}{n!}[\Theta] \in \mathrm{NS}(A).$$

Then $\deg \beta = (n-1)!ud$ and $q_r(\beta) = q_r(\alpha) = f(u, r)d^r$. By Lemma 3.3.12, we get that β comes from an abelian subvariety of dimension u and exponent d. □

Corollary 3.3.16. *We have a decomposition $(A, \Theta) \simeq (X, \Theta_X) \times (Y, \Theta_Y)$ for (X, Θ_X) and (Y, Θ_Y) ppavs if and only if there is a primitive class $[\alpha] \in NS(A, \Theta)$ such that $\deg \alpha \equiv (n-1)!u \pmod{n!}$ and $q_r(\alpha) = f(u, r)$ for $2 \leq r \leq n$ and a certain $u \leq n$.*

Proof. If $(A, \Theta) \simeq (X, \Theta_X) \times (Y, \Theta_Y)$ for certain ppavs, then X and Y can be seen as abelian subvarieties of A of exponent 1, and so α_X (resp. α_Y) give classes α_X (resp. α_Y) such that $q_r(\alpha_X) = f(\dim X, r)$ (resp. $q_r(\alpha_Y) = f(\dim Y, r)$).

Reciprocally, assume that α is a primitive class such that $\deg \alpha \equiv (n-1)!ud \pmod{n!}$ and $q_r(\alpha) = f(u, r)$. This implies that it comes from an abelian subvariety X of exponent 1 and dimension u; this can only mean that (A, Θ) splits as a product $(X, \Theta_X) \times (Y, \Theta_Y)$ for some abelian subvariety Y of exponent 1. □

Unfortunately, we have been unable to prove that the condition $\deg \alpha \equiv (n-1)!ud \pmod{n!}$ is redundant for $u \geq 2$. It seems plausible that it can be dropped, especially after seeing the case $u = 1$. We therefore leave this as a conjecture.

Conjecture 3.3.17. *There is a bijection between abelian subvarieties of dimension u on A with exponent d and primitive numerical classes $[\alpha] \in NS(A, \Theta)$ that satisfy*

$$q_r(\alpha) = f(u,r)d^r$$

for $2 \leq r \leq n$, given by $X \mapsto [\alpha_X]$.

This would follow from the following (seemingly elementary) statement:

Conjecture 3.3.18. *Let $d, l, u, n \in \mathbb{N}$ such that $1 \leq u \leq n$ and*

$$d^r - \sum_{m=2}^{r} \binom{r}{m} f(u,m) d^{r-m} l^m \equiv 0 \pmod{n!^{r-1}}$$

for all $2 \leq r \leq n$. Then $d \equiv (n-1)!ul \pmod{n!}$.

The proof that Conjecture 3.3.17 follows from this statement is very similar to the proof of Lemma 3.2.15. This conjecture is easy to prove by hand for $n \leq 4$, and so Conjecture 3.3.17 is true for $n \leq 4$.

3.4 Projective varieties associated to (A, Θ)

Since the forms q_r are homogeneous on $\mathrm{NS}(A, \Theta)$, one is tempted to look at the projective varieties they define. Because of our failure to prove Conjecture 3.3.17 for $u \geq 2$ and our success at proving it for $u = 1$, we will concentrate on the elliptic curve case. Let ρ be the rank of $\mathrm{NS}(A)$ and fix a basis of $\mathrm{NS}(A, \Theta)$; we define the *elliptic variety associated to* (A, Θ):

$$\mathbb{E}_\Theta = \mathbb{E} := \bigcap_{r=2}^{n} \{[\boldsymbol{x} : x_{\rho-1}] \in \mathbb{P}^{\rho-1}_{\mathbb{Q}} : q_r(\boldsymbol{x}) = (-1)^r x_{\rho-1}^r\}.$$

Theorem 3.4.1. *There is a bijection between elliptic subgroups on A and rational points of $\mathbb{E}$. Moreover, if $x \in \mathbb{E}(\mathbb{Q})$ and $H : \mathbb{P}^{\rho-1}_{\overline{\mathbb{Q}}} \to \mathbb{R}$ is the usual height function, then $(E \cdot \Theta) \leq \frac{1}{(n-1)!}H(x)$.*

Proof. By Theorem 3.2.16, there is a bijection between elliptic subgroups on A of degree d and primitive elements $\alpha \in \mathrm{NS}(A, \Theta)$ such that $q_r(\alpha) = (-1)^r((n-1)!d)^r$. Such an α clearly defines a rational point on $\mathbb{E}$.

If $x \in \mathbb{E}(\mathbb{Q})$, then we can write $x = [\beta : d]$ where $d \in \mathbb{Z}_{>0}$ and $\beta \in \mathrm{NS}(A, \Theta)$ is such that $q_r(\beta) = (-1)^r d^r$ for all r. Assume moreover that the greatest common divisor of all the coordinates is 1 (we can obviously assume this). Write $\beta = m\alpha$ for α a primitive element and $m \in \mathbb{Z}_{>0}$. We then have that $q_r(m\alpha) = m^r q_r(\alpha) = (-1)^r d^r$, which implies that

$$q_r(\alpha) = (-1)^r \left(\frac{d}{m}\right)^r \in \mathbb{Z}.$$

But this means that $m \mid d$, and so we must have that $m = 1$. Thus $\alpha = \beta$, and this corresponds to an abelian divisor of degree d on A. To get the bound on the height, we see that for an integral point $x = [x_1 : \cdots : x_\rho] \in \mathbb{P}^{\rho-1}_{\overline{\mathbb{Q}}}$ with relatively prime coordinates, $H(x)$ is defined as $\max\{|x_1|, \ldots, |x_\rho|\}$. Since d is the degree of the abelian complement Z of the elliptic curve E associated to x, we have that $H(x) \geq d = (Z \cdot \Theta^{n-1}) = (n-1)!(E \cdot \Theta)$. □

Corollary 3.4.2. *A contains an elliptic curve if and only if there exists a $\mathbb{Q}$-divisor D such that*

$$(D^r \cdot \Theta^{n-r}) = (1-r)n!$$

for all $1 \leq r \leq n$.

Proof. If D is a $\mathbb{Q}$-divisor that satisfies the above equation then $[-[D] : n!]$ is a rational point of $\mathbb{E}$. Viceversa, we can see $\mathrm{NS}_{\mathbb{Q}}(A, \Theta) \simeq \{\alpha \in \mathrm{NS}_{\mathbb{Q}}(A) : \deg \alpha = 0\}$ using the isomorphism $\alpha \mapsto \alpha^{\natural}$. Therefore a rational point of $\mathbb{E}$ can be represented by a point $[-[D] : n!]$ with $\deg D = 0$; such a D satisfies the equation above. □

Corollary 3.4.3. *The number of elliptic curves on A of bounded degree is finite.*

Proof. This follows from the fact that there are only finitely many points $x \in \mathbb{P}^{\rho-1}_{\overline{\mathbb{Q}}}$ of bounded height. □

Of course this last corollary is well known.

It seems natural to believe that if Conjecture 3.3.17 is true, then a similar theorem could be proven for analogously defined varieties $\mathbb{E}(u)$.

We will show that these varieties give us some geometric insight. We analyze dimensions 2 and 3.

3.4.1 Dimension 2

Here we will analyze the variety $\mathbb{E}$ when A is an abelian surface. Since $q = q_2$ is a positive definite quadratic form on $\mathrm{NS}(A, \Theta)$, by using Cholesky factorization we know there exists a basis of this space (over $\mathbb{Q}$) such that

$$q(x_1, \ldots, x_{\rho-1}) = a_1 x_1^2 + \cdots + a_{\rho-1} x_{\rho-1}^2,$$

for $a_i \in \mathbb{Q}_{>0}$. We can assume that the a_i are either 1 or are square-free.

It is known that over $\mathbb{C}$, $1 \leq \rho \leq 4$. We will analyze these cases, except for the case $\rho = 1$ which is not interesting in our context.

$\boxed{\rho = 2}$

Here $q(x_1) = a_1 x_1^2$, and so

$$\mathbb{E} = \{[x_1 : x_2] \in \mathbb{P}^2_{\overline{\mathbb{Q}}} : a_1 x_1^2 - x_2^2 = 0\} = \{[1 : \sqrt{a_1}], [-1 : \sqrt{a_1}]\}.$$

We see that these points are rational if and only if $a_1 = 1$, and therefore the points of $\mathbb{E}$ are rational if and only if q represents a square. We see that if $a_1 = 1$ and $[1 : 1]$ corresponds to an elliptic curve E, then $[-1 : 1]$ corresponds to its abelian complement. Of course in this case we didn't need to go to $\mathbb{E}$ to reach this conclusion, this is evident from directly looking at q on $\mathrm{NS}(A, \Theta) \simeq \mathbb{Z}$.

For $\rho = 2$, then, A either contains zero or two elliptic subgroups.

$\boxed{\rho = 3}$

Here

$$\mathbb{E} = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2_{\overline{\mathbb{Q}}} : a_1 x_1^2 + a_2 x_2^2 - x_3^2 = 0\};$$

it is a non-singular quadric in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$. It is well-known that such a conic either has no rational points or is isomorphic over $\mathbb{Q}$ to $\mathbb{P}^1$ via the Veronese embedding. Therefore in this case A either contains no elliptic curves or contains infinitely many elliptic subgroups. In the case that A does contain an elliptic curve, then $\mathbb{E}(\mathbb{Q})$ is dense in $\mathbb{E}$.

$\boxed{\rho = 4}$

We have the non-singular conic

$$\mathbb{E} = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^3_{\overline{\mathbb{Q}}} : a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 - x_4^2 = 0\}.$$

It is known over $\mathbb{C}$ (see [4]) that $\rho = n^2$ if and only if A is isogenous to the nth power of an elliptic curve with complex multiplication. In this case, then, a_1, a_2 and a_3 must be such that $\mathbb{E}$ contains infinite rational points. Actually, using stereographic projection, $\mathbb{E}$ is birational to $\mathbb{P}^2_{\mathbb{Q}}$ over $\mathbb{Q}$, and so the rational points on $\mathbb{E}$ are dense.

The above analysis gives a more complete description of the Bolza-Poincaré Theorem:

Theorem 3.4.4 (Bolza-Poincaré). *If A is an abelian surface that contains more than two elliptic subgroups, then it contains infinitely many.*

Our slight refinement (in which we assume that $k = \mathbb{C}$) then says:

Corollary 3.4.5. *Let A be an abelian surface that contains an elliptic curve. Then A has finitely many elliptic subgroups if and only if $\rho = 2$, and in this case it contains exactly two elliptic subgroups.*

3.4.2 Dimension 3

By Remark 3.2.11, we see that q_2 is positive definite. Just as in the previous section, we can find a basis of $\mathrm{NS}_{\mathbb{Q}}(A, \Theta)$ such that

$$q_2(x_1, \dots, x_{\rho-1}) = a_1 x_1^2 + \cdots + a_{\rho-1} x_{\rho-1}^2,$$

with $a_i \in \mathbb{Q}_{>0}$ square-free.

Proposition 3.4.6. *If $n = 3$, $\rho \geq 3$ and $\mathbb{E}(\mathbb{Q}) \neq \varnothing$, then $\mathbb{E}$ is a complete intersection in $\mathbb{P}^{\rho-1}_{\overline{\mathbb{Q}}}$.*

Proof. If not, then $q_2 - x_\rho^2$ and $q_3 + x_\rho^3$ share a component. We first prove that $q_2 - x_\rho^2$ is irreducible over $\overline{\mathbb{Q}}$. If not then q_2 would be a square in $\overline{\mathbb{Q}}$. Writing $q_2 = L^2$, we see that since q_2 is positive definite over $\mathbb{Q}$, then the line defined by L in $\mathbb{P}^{\rho-1}_{\overline{\mathbb{Q}}}$ has no rational points. If the coefficient of x_i in L is a_i; we see that $a_i a_j \in \mathbb{Q}$ for all i, j, and so $a_i / a_j \in \mathbb{Q}$ for all i, j (since $a_i^2/(a_i a_j) = a_i/a_j$). But then $a_i = b_i a_1$ for $b_i \in \mathbb{Q}$ and after factoring we can assume that $q_2 = a_1^2 \tilde{L}^2$, where $\tilde{L}$ is of degree 1 and defined over $\mathbb{Q}$. However, this implies that $\tilde{L} = 0$ has a non-trivial solution in $\mathbb{Q}$; this contradicts the fact that q_2 is positive definite.

Since $q_2 - x_\rho^2$ is irreducible, if $q_2 - x_\rho^2$ and $q_3 + x_\rho^3$ share a common factor, we must have that

$$q_3 + x_\rho^3 = (q_2 - x_\rho^2) L$$

for some homogeneous L of degree 1. Since $\mathbb{E}(\mathbb{Q}) \neq \varnothing$, and in this case $\mathbb{E} = \{q_2 - x_\rho^2 = 0\}$, we get that $\#\mathbb{E}(\mathbb{Q}) = \infty$ and so A contains infinite elliptic curves. In particular, $A \sim E_1 \times E_2 \times E_3$ for three elliptic curves E_1, E_2 and E_3. Let $\phi : A \to E_1 \times E_2 \times E_3$ be an isogeny, write

$$D_1 := \{0\} \times E_2 \times E_3$$
$$D_2 := E_1 \times \{0\} \times E_3$$
$$D_3 := E_1 \times E_2 \times \{0\},$$

and let $d_i = \deg \phi^* D_i$. Let $F(m) := \phi^* D_1 + m\phi^* D_2 \in \mathrm{Div}_{\mathbb{Q}}(A)[m]$.

Let $m_0 \in \overline{\mathbb{Q}}$ be such that $q_2([F(m_0)]) = 0$; we will show that $q_3([F(m_0)]) \neq 0$. We see that $((\phi^* D_i)^2 \cdot \Theta) = \frac{1}{(\deg \phi)^2}((\phi^* D_i)^2 \cdot \phi^* \psi^* \Theta) = \frac{1}{\deg \phi}(D_i^2 \cdot \psi^* \Theta) = 0$, where ψ is the isogeny $E_1 \times E_2 \times E_3 \to A$ such that $\psi\phi = \deg \phi$. A direct application of the degree formula also shows that $((\phi^* D_i)^3) = 0$. Using this, we have that m_0 satisfies the equation

$$d_2^2 m_0^2 + (2d_1 d_2 - 12(D_1 \cdot D_2 \cdot \Theta))m_0 + d_1^2 = 0.$$

Now

$$\begin{aligned} q_3([F(m)]) &= -d_2^3 m^3 + (18d_2(\phi^* D_1 \cdot \phi^* D_2 \cdot \Theta) - 3d_2^2 d_1)m^2 + \\ &\quad (18d_1(\phi^* D_1 \cdot \phi^* D_2 \cdot \Theta) - 3d_1^2 d_2)m - d_1^3. \end{aligned}$$

A simple calculation shows that m_0 is a zero of $q_3([F(m)])$ if and only if the polynomial $q_2([F(m)])$ divides $q_3([F(m)])$, and that this happens if and only if $(\phi^* D_1 \cdot \phi^* D_2 \cdot \Theta) = 0$. This is not the case, and we arrive at a contradiction. Therefore q_2 and q_3 share no common components, and $\mathbb{E}$ is a complete intersection. ☐

Corollary 3.4.7. *If $n = 3$, $\rho \geq 3$ and q_2 represents a square (over $\mathbb{Q}$), then $\mathbb{E}$ is a complete intersection.*

Proof. If not, then $\mathbb{E}$ contains a rational point. This contradicts the previous proposition. ☐

Unfortunately, we can't say as much as in the case of dimension 2.

$\boxed{\rho = 2}$

We have

$$\mathbb{E} = \{[x_1 : x_2] \in \mathbb{P}^1_{\overline{\mathbb{Q}}} : a_1 x_1^2 - x_2^2 = 0, q_3(x_1) + x_2^3 = 0\}.$$

We see that $\{a_1 x_1^2 - x_2^2 = 0\} = \{[1 : \sqrt{a_1}], [-1 : \sqrt{a_1}]\}$, and these two points will not in general satisfy the cubic relation above. However, if E is an elliptic curve and Z

is an abelian surface with rank(NS(Z)) = 1, then

$$\mathrm{NS}(E \times Z) \simeq \mathrm{NS}(E) \times \mathrm{NS}(Z) \times \mathrm{Hom}(E, Z) = \mathrm{NS}(E) \times \mathrm{NS}(Z) \simeq \mathbb{Z}^2,$$

and so rank(NS($E \times Z$)) = 2. This shows that the intersection of the quadric and the cubic may be non-empty.

Since q_3 is defined over $\mathbb{Q}$, we see that at most one of $[1 : \sqrt{a_1}]$ or $[-1 : \sqrt{a_2}]$ could satisfy the cubic relation in $\mathbb{E}$, and for this it would be necessary to have $a_1 = 1$. Therefore, we conclude that if $\rho = 2$ and A contains an elliptic subgroup, then said subgroup is unique.

$\boxed{\rho = 3}$

Here we have a quadric and a cubic intersection in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$, which consists of at most 6 points. Now, it is easy to show that if an abelian 3-fold contains more than 3 elliptic subgroups, it contains infinitely many. Therefore, if $\rho = 3$, we necessarily have that there are at most 3 elliptic subgroups on A.

$\boxed{4 \leq \rho \leq 9}$

By Proposition 5.5.7 in [4], in this case A cannot be simple. Therefore $\mathbb{E}(\mathbb{Q}) \neq \varnothing$ and so is a complete intersection. For $\rho = 4$ we get a curve in $\mathbb{P}^3$ of arithmetic genus 4. Not much more can be said in these cases, however, given the tools we have developed so far.

The arithmetic genus p_a of $\mathbb{E}$ is easy to calculate given the results of [1]. We have the following table:

ρ	p_a
3	5
4	4
5	1
6	0
7	0
8	0
9	0

It would be interesting to see if geometric properties of $\mathbb{E}$ could give us geometric information about A. For example, what do the singular points of $\mathbb{E}$ tell us? Is $\mathbb{E}$ a complete intersection for all n? What does the density of $\mathbb{E}(\mathbb{Q})$ in $\mathbb{E}$ tell us about the distribution of abelian subvarieties on A?

3.5 Analytic calculations on $\mathcal{A}_n$

Throughout this section, (A, Θ) will be a ppav over $\mathbb{C}$. Since we are working over the field of complex numbers, we can assume that $A = \mathbb{C}^n/\Lambda$, where Λ is a rank $2n$ lattice in $\mathbb{C}^n$. Furthermore, we may assume that $\Lambda = (\tau\ I)\mathbb{Z}^g$, where $\tau \in \mathfrak{H}_n := \{N \in M_n(\mathbb{C}) : N = N^t, \Im N > 0\}$. We have a map

$$c_1 : \mathrm{Pic}(A) \simeq H^1(A, \mathcal{O}_A^\times) \to H^2(A, \mathbb{Z}) \simeq \mathbb{Z}^{n(2n-1)}$$

that assigns to each $L \in \mathrm{Pic}(A)$ its first Chern class $c_1(L)$. The first Chern class can be seen as a hermitian form on $\mathbb{C}^n$ whose imaginary part takes on integer values on Λ. Actually, the image of c_1 is isomorphic to $\mathrm{NS}(A)$, and so in this section we will write numerical classes as integral differential forms. It can be shown that $\mathrm{NS}(A) = H^2(A, \mathbb{Z}) \cap H^{1,1}(A, \mathbb{Z})$, and this means that an integral cohomology class ω is in $\mathrm{NS}(A)$ if and only if $\omega \wedge dz_1 \wedge \cdots \wedge dz_n = 0$. Intersection of line bundles can be shown to be equal to

$$(L_1 \cdots L_n) = \int_A c_1(L_1) \wedge \cdots \wedge c_1(L_n).$$

Choose a symplectic basis $\{\lambda_1, \ldots, \lambda_{2n}\}$ for Λ, and take x_i to be the real coordinate function of λ_i. By Lemma 3.6.4 of [4], we have that

$$c_1(\Theta) = -\sum_{i=1}^{n} dx_i \wedge dx_{i+n}.$$

Given an integral form $\omega \in \mathrm{NS}(A)$, we wish to find the intersection number $(\omega^r \cdot c_1(\Theta)^{n-r})$. Using the explicit basis we just wrote, it is easy to see that

$$c_1(\Theta)^{\wedge s} = (-1)^s s! \sum_{1 \le i_1 < \cdots < i_s \le n} dx_{i_1} \wedge dx_{i_1+n} \wedge \cdots \wedge dx_{i_s} \wedge dx_{i_s+n}.$$

In particular, for $s = n$, we get that

$$n! = (c_1(\Theta)^n) = \int_A c_1(\Theta)^{\wedge n} = (-1)^n n! \int_A dx_1 \wedge dx_{n+1} \wedge \cdots \wedge dx_n \wedge dx_{2n}.$$

If we put $\eta := dx_1 \wedge dx_{n+1} \wedge \cdots \wedge dx_n \wedge dx_{2n}$, this implies that when calculating intersection numbers, we have

$$(L_1 \cdots L_n) = (-1)^n \cdot \text{coefficient of } \eta \text{ in } c_1(L_1) \wedge \cdots \wedge c_1(L_n).$$

Using the binomial theorem, if $\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j$, then

$$\omega^{\wedge r} = r! \sum_{I} \bigwedge_{(i,j)\in I} (a_{ij} dx_i \wedge dx_j)$$

where I runs over all subsets of $\{1, \ldots, 2n\}^2$ with r elements, each coordinate of each element of I does not appear in any other element of I, and for all $(i, j) \in I$, we have that $i < j$.

Definition 3.5.1. Let $S \subseteq \{1, \ldots, 2n\}$ be a set with $2r$ elements, such that $\#S \cap \{1, \ldots, n\} = r$ and such that $i + n \in S$ for all $i \in S \cap \{1, \ldots, n\}$; such a set will be called a *balanced set* of weight r. A *matching* of S is a subset $P_S \subset S \times S$, maximal with respect to the property that if $(i, j) \in P_S$, then $i < j$, and neither i nor j appear in any other elements of P_S.

Let S and P_S be as above, and define $\varepsilon(P_S) \in \{0, 1\}$ to be such that

$$\left(\bigwedge_{\substack{1\le k\le n \\ k\notin S}} dx_i \wedge dx_{n+i} \right) \wedge \bigwedge_{(i,j)\in P_S} dx_i \wedge dx_j = (-1)^{\varepsilon(P_S)} \eta.$$

Using this number, we can calculate intersection numbers of forms.

Lemma 3.5.2. *For $\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j$, we have that*

$$(\omega^r \cdot c_1(\Theta)^{n-r}) = (-1)^r r!(n-r)! \left(\sum_{S} \sum_{P_S} (-1)^{\varepsilon(P_S)} \prod_{(i,j)\in P_S} a_{ij} \right),$$

where S ranges over all balanced sets of weight r, and P_S ranges over all matchings of S.

Proof. This follows from calculating with a lot of patience. The idea is that upon taking the wedge product of $\omega^{\wedge r}$ with $c_1(\Theta)^{\wedge(n-r)}$, we are faced with a sum whose general term is of the form

$$\left(\prod_{(i,j)\in I} a_{ij} \right) \left(\bigwedge_{k=1}^{n-r} dx_{i_k} \wedge dx_{i_k+n} \right) \wedge \bigwedge_{(i,j)\in I} dx_i \wedge dx_j,$$

where I is as before. We see that this is non-zero exactly when I is a pairing for some balanced set of weight r.

Corollary 3.5.3. *The degree of* ω *is* $-(n-1)!(a_{1,n+1}+a_{2,n+2}+\cdots+a_{n,2n})$.

All these calculations can be done much faster using a computer program that works with differential forms, such as Sage. In this thesis, we used the above method for calculating the equations by hand for dimensions 2 and 3, and for dimension 4 we resorted to a computer program.

3.5.1 Dimension 2

By writing all these formulas out for a 2-dimensional ppav and passing to the quotient $H^2(A,\mathbb{Z})/\mathbb{Z}c_1(\Theta) \simeq \mathbb{Z}^5$, we get that we can write the form $q = q_2$ as

$$q(b_1,b_2,b_3,b_4,b_5) = b_2^2 - 4(b_1b_3 - b_4b_5).$$

Moreover, a vector $(b_1,b_2,b_3,b_4,b_5) \in \mathbb{Z}^5$ is in $\mathrm{NS}(A,\Theta)$, where A is defined by the period matrix $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$, if and only if

$$b_1\tau_1 + b_2\tau_2 + b_3\tau_3 + b_4 \begin{vmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{vmatrix} + b_5 = 0.$$

Therefore we get the following characterization:

Proposition 3.5.4. *The ppav associated to the matrix* $\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ *contains an elliptic curve of degree* d *if and only if there exists a primitive vector* $(b_1,b_2,b_3,b_4,b_5) \in \mathbb{Z}^5$ *that satisfies*

$$b_2^2 - 4(b_1b_3 - b_4b_5) = d^2$$

and

$$b_1\tau_1 + b_2\tau_2 + b_3\tau_3 + b_4 \begin{vmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{vmatrix} + b_5 = 0.$$

This is the condition obtained by Humbert [20].

3.5.2 Dimension 3

For $n = 3$, we can write the forms q_r on $H^2(A, \mathbb{Z}) \simeq \mathbb{Z}^{15}$ as follows:

$$\begin{aligned} q_2(\omega) &= 12a_{12}a_{45} + 12a_{13}a_{46} + 4a_{14}^2 - 4a_{14}a_{25} - 4a_{14}a_{36} + 12a_{15}a_{24} + \\ &\quad 12a_{16}a_{34} + 12a_{23}a_{56} + 4a_{25}^2 - 4a_{25}a_{36} + 12a_{26}a_{35} + 4a_{36}^2 \end{aligned}$$

$$\begin{aligned} q_3(\omega) &= 36a_{12}a_{14}a_{45} + 36a_{12}a_{25}a_{45} - 108a_{12}a_{34}a_{56} + 108a_{12}a_{35}a_{46} - \\ &\quad 72a_{12}a_{36}a_{45} + 36a_{13}a_{14}a_{46} + 108a_{13}a_{24}a_{56} - 72a_{13}a_{25}a_{46} + \\ &\quad 108a_{13}a_{26}a_{45} + 36a_{13}a_{36}a_{46} + 8a_{14}^3 - 12a_{14}^2a_{25} - \\ &\quad 12a_{14}^2a_{36} + 36a_{14}a_{15}a_{24} + 36a_{14}a_{16}a_{34} - 72a_{14}a_{24}a_{56} - \\ &\quad 12a_{14}a_{25}^2 + 48a_{14}a_{25}a_{36} - 72a_{14}a_{26}a_{35} - 12a_{14}a_{36}^2 + \\ &\quad 108a_{15}a_{23}a_{46} + 36a_{15}a_{24}a_{25} - 72a_{15}a_{24}a_{36} + 108a_{15}a_{26}a_{34} - \\ &\quad 108a_{16}a_{23}a_{45} + 108a_{16}a_{24}a_{35} - 72a_{16}a_{25}a_{34} + 36a_{16}a_{34}a_{36} + \\ &\quad 36a_{23}a_{25}a_{56} + 36a_{23}a_{36}a_{56} + 8a_{25}^3 - 12a_{25}^2a_{36} + 36a_{25}a_{26}a_{35} - \\ &\quad 12a_{25}a_{36}^2 + 36a_{26}a_{35}a_{36} + 8a_{36}^3. \end{aligned}$$

Since

$$c_1(\Theta) = -(dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6),$$

we can take the projection

$$\mathbb{Z}^{15} \to H^2(A, \Theta) := H^2(A, \mathbb{Z})/\mathbb{Z}c_1(\Theta) \simeq \mathbb{Z}^{14}.$$

We choose the projection

$$(a_{ij})_{1 \le i < j \le 6} \mapsto (a'_{ij})_{1 \le i < j \le 6, (i,j) \ne (3,6)},$$

where $a'_{ij} = a_{ij}$ if $(i, j) \ne (1, 4), (2, 5)$, and $a'_{14} = a_{14} - a_{36}$ and $a'_{25} = a_{25} - a_{36}$. Using the change of coordinates from real coordinates to complex coordinates given by the matrix $(\tau\ I)$, we get the following proposition:

Proposition 3.5.5. *Let (A, Θ) be a ppav of dimension 3, corresponding to a matrix $\tau = (\tau_{ij}) \in \mathfrak{H}_3$, and let $\beta = (b_1, \ldots, b_{14}) \in \mathbb{Z}^{14}$. Then $\beta \in NS(A, \Theta)$ if and only if β satisfies the following six linear equations:*

$$
\begin{aligned}
0 &= b_6 - \tau_{13}b_7 - \tau_{23}b_8 - \tau_{33}b_9 + \tau_{12}b_{10} + \tau_{22}b_{11} + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} b_{12} + \\
& \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_{13} + \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_{14} \\
0 &= b_2 - \tau_{13}b_3 - \tau_{23}b_4 - \tau_{33}b_5 + \tau_{11}b_{10} + \tau_{12}b_{11} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{13} & \tau_{23} \end{vmatrix} b_{12} + \\
& \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{13} & \tau_{33} \end{vmatrix} b_{13} + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_{14} \\
0 &= b_1 - \tau_{12}b_3 - \tau_{22}b_4 - \tau_{23}b_5 + \tau_{11}b_7 + \tau_{12}b_8 + \tau_{13}b_9 + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{vmatrix} b_{12} + \\
& \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{13} & \tau_{23} \end{vmatrix} b_{13} + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} b_{14} \\
0 &= \tau_{13}b_1 - \tau_{12}b_2 + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} b_4 + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_5 + \tau_{11}b_6 - \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{13} & \tau_{23} \end{vmatrix} b_8 - \\
& \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{13} & \tau_{33} \end{vmatrix} b_9 + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{vmatrix} b_{11} + (\det \tau) b_{14} \\
0 &= -\tau_{23}b_1 + \tau_{22}b_2 + \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} b_3 - \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_5 - \tau_{12}b_6 - \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{13} & \tau_{23} \end{vmatrix} b_7 + \\
& \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_9 + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{vmatrix} b_{10} + (\det \tau) b_{13} \\
0 &= \tau_{33}b_1 - \tau_{23}b_2 - \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_3 - \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_4 + \tau_{13}b_6 + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{13} & \tau_{33} \end{vmatrix} b_7 + \\
& \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{33} \end{vmatrix} b_8 - \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{13} & \tau_{23} \end{vmatrix} b_{10} - \begin{vmatrix} \tau_{12} & \tau_{13} \\ \tau_{22} & \tau_{23} \end{vmatrix} b_{11} + (\det \tau) b_{12}
\end{aligned}
$$

After rewriting the equations for q_r in $\mathbb{Z}^{14}$ and using Theorem 3.2.16, we get the following corollary:

Corollary 3.5.6. *The ppav of dimension 3 corresponding to a matrix $\tau = (\tau_{ij}) \in \mathfrak{H}_3$ contains an elliptic curve whose abelian complement has degree d if and only if there exists a non-zero primitive vector $\beta = (b_1, \ldots, b_{14}) \in \mathbb{Z}^{14}$ that satisfies the equations* *of Proposition 3.5.5, and such that*

$$\begin{aligned} d^2 &= 12b_1b_{12} + 12b_2b_{13} + 4b_8^2 - 4b_3b_8 + 4b_3^2 + 12b_4b_7 + 12b_5b_{10} + 12b_6b_{14} + 12b_9b_{11} \\ -d^3 &= 108b_4b_9b_{10} + 36b_1b_8b_{12} - 72b_5b_8b_{10} + 36b_3b_5b_{10} - 108b_5b_6b_{12} + \\ &\quad 108b_1b_{11}b_{13} - 72b_3b_7b_{14} + 36b_9b_{11}b_8 + 108b_2b_7b_{14} - 72b_2b_8b_{13} - \\ &\quad 72b_3b_9b_{11} + 108b_4b_6b_{13} + 36b_3b_4b_7 + 36b_4b_7b_8 + 36b_2b_3b_{13} + \\ &\quad 108b_5b_7b_{11} - 108b_1b_{10}b_{14} - 12b_3b_8^2 + 108b_2b_9b_{12} + 36b_6b_8b_{14} + \\ &\quad 36b_1b_3b_{12} + 8b_3^3 + 8b_8^3. \end{aligned}$$

Example 3.5.7. González-Aguilera and Rodríguez [15] found, for every $n \geq 3$, a family of indecomposable principally polarized abelian varieties each of whose underlying abelian variety is isomorphic to the product of elliptic curves. More specifically, they give the family

$$\mathscr{F}_n := \{\sigma\tau_0 : \sigma \in \mathfrak{H}_1\} \subseteq \mathfrak{H}_n,$$

where

$$\tau_0 = \begin{pmatrix} n & -1 & \cdots & -1 \\ -1 & n & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{pmatrix}.$$

If $(A_\sigma, \Theta_\sigma)$ denotes the ppav associated to $\sigma\tau_0$, then

$$A_\sigma \simeq E_{(n+1)\sigma}^{n-1} \times E_\sigma,$$

where $E_\sigma := \mathbb{C}/\langle 1, \sigma\rangle$. For $n = 3$, the equations of Proposition 3.5.5 become

$$\begin{aligned} 0 &= b_6 + \sigma b_7 + \sigma b_8 - 3\sigma b_9 - \sigma b_{10} + 3\sigma b_{11} + 4\sigma^2 b_{12} - 4\sigma^2 b_{13} + 8\sigma^2 b_{14} \\ 0 &= b_2 + \sigma b_3 + \sigma b_4 - 3\sigma b_5 + 3\sigma b_{10} - \sigma b_{11} - 4\sigma^2 b_{12} + 8\sigma^2 b_{13} - 4\sigma^2 b_{14} \\ 0 &= b_1 + \sigma b_3 - 3\sigma b_4 + \sigma b_5 + 3\sigma b_7 - \sigma b_8 - \sigma b_9 + 8\sigma^2 b_{12} - 4\sigma^2 b_{13} + 4\sigma^2 b_{14} \\ 0 &= -\sigma b_1 + \sigma b_2 + 4\sigma^2 b_4 - 4\sigma^2 b_5 + 3\sigma b_6 + 4\sigma^2 b_8 - 8\sigma^2 b_9 + 8\sigma^2 b_{11} + 16\sigma^3 b_{14} \\ 0 &= \sigma b_1 + 3\sigma b_2 + 4\sigma^2 b_3 - 8\sigma^2 b_5 + \sigma b_6 + 4\sigma^2 b_7 - 4\sigma^2 b_9 + 8\sigma^2 b_{10} + 16\sigma^3 b_{13} \\ 0 &= 3\sigma b_1 + \sigma b_2 + 4\sigma^2 b_3 - 8\sigma^2 b_4 - \sigma b_6 + 8\sigma^2 b_7 - 4\sigma^2 b_8 + 4\sigma^2 b_{10} - 4\sigma^2 b_{11} + 16\sigma^3 b_{12} \end{aligned}$$

If $[\mathbb{Q}(\sigma) : \mathbb{Q}] > 2$, then if we have an integral solution to the equations above and

$x_1, \dots, x_{11}$ are the coefficients of the generators, we get the following relations:

$$\begin{aligned}
0 &= -2x_1 - x_6 + x_{11} \\
0 &= x_1 + x_6 - 2x_{11} \\
0 &= -x_1 - 2x_6 + x_{11} \\
0 &= -x_3 + 3x_4 - x_5 - 3x_8 + x_9 + x_{10} \\
0 &= x_2 + 3x_3 - x_4 - x_5 - 3x_7 \\
0 &= -3x_2 + x_7 - x_8 - x_9 + 3x_{10}
\end{aligned}$$

This necessarily leads to $x_1 = x_6 = x_{11} = 0$. The general integral solution to this system of equations is

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = \begin{pmatrix} a+c-2d+e \\ 3a+b+c+d \\ a+c+d \\ e \\ 3a+b+c \\ c \\ b \\ c-2d+e \end{pmatrix}$$

for $a, b, c, d, e \in \mathbb{Z}$. Therefore, a vector in $\mathbb{Z}^{14}$ is a solution for the Néron-Severi equations if and only if it is of the form

$$\begin{pmatrix} 0 \\ 0 \\ e \\ a+c+d \\ 3a+b+c+d \\ 0 \\ c \\ b \\ c-2d+e \\ 3a+b+c \\ a+c-2d+e \\ 0 \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + e\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using this basis, we will write an element of $\mathrm{NS}(A_\sigma, \Theta_\sigma)$ as a 5-tuple (a, b, c, d, e).

The forms can then be written as

$$\begin{aligned} q_2(a,b,c,d,e) &= 108a^2+16b^2+36c^2+48d^2+16e^2+72ab+96ac+12ad+\\ &\quad 12ae+24bc+12bd-4be-24cd+24ce-48de\\ q_3(a,b,c,d,e) &= 576ead-504bcd+936cea+24e^2b+24eb^2-504bad-\\ &\quad 1080cad+576ced+360bae+360bce-144bca-64e^3-64b^3+\\ &\quad 216bc^2-72bd^2+216c^2e-144ce^2-288ed^2+288de^2-72ae^2+\\ &\quad 216c^3+648ea^2+648ca^2+864c^2a-216c^2d-432cd^2-\\ &\quad 648a^2d-648d^2a-432ab^2-144b^2c-72db^2-648ba^2 \end{aligned}$$

Using a simple Java program, we can find a cornucopia of primitive elements α such that $q_2(\alpha)$ is a square d^2 and such that $q_3(\alpha) = -d^3$. For example, the following table shows all abelian divisors in $\mathrm{NS}(A_\sigma, \Theta_\sigma)$ whose coordinates lie between -3 and 3 and whose degree is less than or equal to 6:

Divisor class α_E	$(\alpha_E \cdot \Theta_\sigma^2)$	$(E \cdot \Theta_\sigma)$
(0,0,0,-1,-2)	4	2
(1,-1,-1,-1,-1)	4	2
(0,-1,1,0,-1)	4	2
(1,-2,-1,0,0)	4	2
(0,1,0,0,0)	4	2
(-1,1,1,0,0)	4	2
(0,0,0,0,1)	4	2
(0,0,0,1,1)	4	2
(-1,2,0,1,2)	4	2
(0,0,1,-1,-3)	6	3
(0,0,-1,0,0)	6	3
(1,-3,0,0,0)	6	3
(-1,3,0,1,3)	6	3

The fact that there are exactly 13 divisors above is strikingly similar to what Guerra obtained in [17] for the self product of an elliptic curve. We also get the following table:

$\mathbb{E}_{A_\sigma,\Theta_\sigma}(\mathbb{Q})$	
Dimension	3
Dimension of singular locus	2
Irreducible	yes
Singular locus irreducible	yes

Example 3.5.8. Let $(A_\lambda, \Theta_\lambda)$ be the ppav associated to the matrix

$$\begin{pmatrix} i & -1 & \frac{-1-i}{2} \\ -1 & i & \frac{-1+i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} & \lambda \end{pmatrix},$$

where $\lambda \in \mathfrak{H}_1$ and such that no degree 1 polynomial in two variables over $\mathbb{Q}$ vanishes at $(\Re\lambda, \Im\lambda)$. Then a basis for $\mathrm{NS}(A_\lambda, \Theta_\lambda)$ is

$$\begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \\ -2 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We notice that since the Picard number of A_λ is 5, there must appear two elliptic curves in A_λ that are not isogenous. Using coordinates (a, b, c, d) for this basis, we have that

$$\begin{aligned} q_2(a,b,c,d) &= -8ab + 8db + 24c^2 + 24ca - 72cd + 16a^2 - 32da + 16b^2 + 64d^2 \\ q_3(a,b,c,d) &= 64b^3 + 96bc^2 + 336bd^2 + 48bca - 336bcd - 96bda + 12c^3 - 48ab^2 + \\ &\quad 48b^2c + 48db^2 - 96c^2a + 48ba^2 + 480cda - 96ca^2 + 96c^2d - 384cd^2 + \\ &\quad 192da^2 - 480d^2a - 64a^3 + 352d^3 \end{aligned}$$

Again using Magma (see [5]) and Java, we obtain the following tables:

Divisor class α_E	$(\alpha_E \cdot \Theta_\sigma^2)$	$(E \cdot \Theta_\sigma)$
(0,0,-2,-1)	4	2
(1,1,-2,-1)	4	2
(0,-1,0,0)	4	2
(1,0,0,0)	4	2

$\mathbb{E}_{A_\lambda,\Theta_\lambda}(\mathbb{Q})$	
Dimension	2
Dimension of singular locus	1
Irreducible	yes
Singular locus irreducible	yes
Singular locus non-singular	no
Geometric genus	1
Irregularity	0
Arithmetic genus of singular locus	-8

Example 3.5.9. Take the product of three elliptic curves E_τ^3 where $E_\tau = \mathbb{C}/\langle 1, \tau\rangle$ and $[\mathbb{Q}(\tau) : \mathbb{Q}] > 2$. A basis for the polarized Néron-Severi group is

$$\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\1\\0\\0\\0\end{pmatrix}.$$

Using coordinates (a, b, c, d, e), we have that

$$\begin{aligned} q_2(a,b,c,d,e) &= 4d^2 - 4ad + 4a^2 + 12b^2 + 12c^2 + 12e^2 \\ q_3(a,b,c,d,e) &= -24a^2d + 72ab^2 + 72ac^2 - 24ad^2 + 72b^2d + 16a^3 + 16d^3 \\ &\quad -144c^2d + 72de^2 - 144ae^2 + 432bec. \end{aligned}$$

Since the elliptic curves that lie on E_τ^3 are easily found, we only show the following table:

$\mathbb{E}_{E_\tau^3,\theta^{\boxtimes 3}}(\mathbb{Q})$	
Dimension	3
Dimension of singular locus	1
Irreducible	yes
Singular locus irreducible	yes
Singular locus non-singular	yes
Genus of singular locus	0

3.5.3 Dimension 4

In dimension 4, the equations are very long. We have included them, as well as the expressions for q_2, q_3 and q_4, in the appendix. Though these equations may seem daunting, they are not difficult to handle when working with a computer. We include an example to show that calculations can be made.

Example 3.5.10. Let $(A_\sigma, \Theta_\sigma)$ be a ppav in the family $\mathscr{F}_4$, from Example 3.5.7. This family is interesting for the fact that only one Jacobian appears in $\mathscr{F}_4$, and it corresponds precisely to the element $(A_\sigma, \Theta_\sigma)$ with $j(\sigma) = -\frac{25}{2}$ and $j(5\sigma) = -\frac{5\cdot 29^3}{2^5}$. After using Maple, we get that the polarized Néron-Severi group of $(A_\sigma, \Theta_\sigma)$ (for $[\mathbb{Q}(\sigma) : \mathbb{Q}] > 2$) is generated by the vectors

$$
\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\1\\0\\1\\1\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\-1\\0\\-3\\0\\0\\0\\4\\0\\1\\0\\0\\0\\0\\0\\0\\4\\0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\10\\0\\3\\0\\0\\0\\-3\\0\\0\\1\\0\\0\\0\\0\\-2\\-4\\0\\-3\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\-4\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\10\\0\\2\\0\\0\\0\\-3\\0\\0\\0\\0\\0\\1\\0\\1\\-3\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\-3\\0\\-1\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\1\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\-6\\0\\-2\\0\\0\\0\\2\\0\\0\\0\\0\\0\\0\\0\\2\\3\\1\\3\\0\\0\\0\\0\\0\\0\end{pmatrix}.
$$

We identify $\mathrm{NS}(A_\sigma, \Theta_\sigma)$ with $\mathbb{Z}^9$ using this basis. We can then use a Java program to find elliptic curves and abelian surfaces on A_σ. For example, we get the following chart that shows all non-trivial abelian subvarieties of A_σ of exponent 2 whose coordinates are bounded by 1:

α	dimension
(0,0,0,0,0,1,0,-1,0)	1
(0,0,-1,1,0,0,1,-1,0)	1
(1,0,-1,0,0,0,0,0,0)	1
(-1,-1,0,0,0,0,0,0,0)	1
(0,0,1,0,0,0,0,0,0)	1
(0,1,1,0,0,0,0,1,0)	1
(0,0,0,0,1,0,0,1,1)	1
(1,1,0,0,-1,0,0,-1,-1)	2
(0,0,0,0,-1,-1,0,0,-1)	2
(0,0,1,1,0,0,1,1,-1)	2
(0,0,0,0,-1,1,1,1,-1)	2
(0,1,1,0,0,-1,-1,-1,0)	2
(0,-1,1,1,1,0,0,-1,0)	2
(-1,-1,-1,1,0,0,1,-1,0)	2
(0,-1,0,-1,0,0,-1,0,0)	2
(0,0,-1,-1,-1,0,0,0,0)	2
(1,1,-1,0,0,0,0,0,0)	2
(-1,0,0,0,0,0,0,0,0)	2
(1,0,0,0,0,0,0,0,0)	2
(-1,-1,1,0,0,0,0,0,0)	2
(0,0,1,1,1,0,0,0,0)	2
(0,1,0,1,0,0,1,0,0)	2
(1,1,1,-1,0,0,-1,1,0)	2
(0,1,-1,-1,-1,0,0,1,0)	2
(0,-1,-1,0,0,1,1,1,0)	2
(0,0,0,0,1,-1,-1,-1,1)	2
(0,0,-1,-1,0,0,-1,-1,1)	2
(0,0,0,0,1,1,0,0,1)	2
(-1,-1,0,0,1,0,0,1,1)	2
(0,0,0,0,-1,0,0,-1,-1)	3
(0,-1,-1,0,0,0,0,-1,0)	3
(0,0,-1,0,0,0,0,0,0)	3
(1,1,0,0,0,0,0,0,0)	3
(-1,0,1,0,0,0,0,0,0)	3
(0,0,1,-1,0,0,-1,1,0)	3
(0,0,0,0,0,-1,0,1,0)	3

Notice that the elements that appear above of dimension 3 correspond to the additive inverses of the elements of dimension 1. Moreover, since the abelian complement of an abelian surface in this case is a surface, the above elements of dimension 2 are closed under multiplication by -1.

Chapter 4

Concluding remarks

This thesis was concerned with characterizing abelian subvarieties on a fixed ppav, and at the end we found equations on the moduli spaces $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ that described the locus of ppavs with an abelian subvariety of exponent d. An obvious direction for future research would be to describe these loci explicitly. In [22], Kani used the quadratic form $q = q_2$ to describe the locus of Jacobian surfaces that contain an elliptic curve of degree d. He found that each irreducible component of this locus is isomorphic to either a modular curve or a degree 2 quotient of a modular curve. It seems plausible that the forms q_r could be used to acquire similar results in higher dimension.

Before pursuing this, however, Conjecture 3.3.17 must be proven. We must gain a better understanding of $f(u, r)$, and see if there is a way of simplifying the calculations done in this thesis.

Question 1: Do there exist homogeneous forms $\tilde{q}_2, \ldots, \tilde{q}_n$ on $\mathrm{NS}(A, \Theta)$ that simplify the calculations of Chapter 3?

Question 2: What does the moduli space of abelian varieties (or Jacobians) that contain an abelian subvariety of exponent d look like?

If we restrict out attention once again to a specific abelian variety, there are still many questions that need to be answered.

Question 3: If $X, Y \leq A$ are abelian subvarieties (not complementary), how can we write $\alpha_{X\cap Y}$ and α_{X+Y} in terms of α_X and α_Y?

Question 4: If α_X is the divisor class of an abelian subvariety $X \leq A$, can we determine the type of $\Theta|_X$ by α_X? In theory the answer should be yes, since α_X determines X, but is there a formula for calculating it?

Question 5: How can we characterize all nilpotent divisor classes that come from abelian subvarieties in $\mathfrak{A}^*(A)$?

Question 6: Let $(d_1, \ldots, d_u)$ be the type of $\Theta|_X$ as in the previous question, and assume that A is the Jacobian of a smooth projective curve. What conditions can we impose on α_X so that $d_1 = d_u$? That is, how can we distinguish Prym-Tyurin varieties among divisor classes?

Question 7: If $f : C \to C'$ is a finite morphism between smooth projective curves, then we get a morphism $f^* : J_{C'} \to J_C$ which is an isogeny with its image. In particular, f induces a numerical class $\alpha_f := \alpha_{f^*(J_{C'})}$ in $\mathrm{NS}(J_C)$. How can we distinguish which classes come from morphisms between curves?

Question 8: Can we relate this theory to the theory of decomposition via automorphisms? Can we find families of ppavs that decompose but do not have automorphisms (as polarized abelian varieties)? What can the action of the automorphism group tell us about the Néron-Severi group?

Another area for future research would be to focus on computational aspects of the results stated here. Identify $\mathrm{NS}(A,\Theta) \otimes \mathbb{R}$ with $\mathbb{R}^{\rho-1}$, and let $|\cdot| : \mathbb{R}^{\rho-1} \to \mathbb{R}_{\geq 0}$ be the usual norm on $\mathbb{R}^{\rho-1}$. Moreover, assume that Conjecture 3.3.17 is true. Ideally, we would like to write a computer program that takes a period matrix along with positive integers d, u and B, and returns the set of all abelian subvarieties of the abelian variety associated to the period matrix of dimension u, exponent d, and with norm less than or equal to B. This would give us a function

$$(\tau, u, d, B) \rightsquigarrow \bigcap_{r=2}^{n} \{\beta \in \mathrm{NS}(A,\Theta) : q_r(\beta) = f(u,r)d^r, |\beta| \leq B\}.$$

One problem with writing such a program is the fact that the Néron-Severi equations (the equations that tell us when a vector in $\mathbb{Z}^{n(2n-1)-1}$ is in $\mathrm{NS}(A,\Theta)$) ask for *integer solutions* to certain linear equations with complex coefficients. In practice (such as in the examples for $n = 3$) this is not hard to do by hand, but a general method may not exist.

Appendix

Proposition Let $\tau = (\tau_{ij}) \in \mathfrak{H}_4$ correspond to a ppav A_τ, and let $\beta = (b_i) \in \mathbb{Z}^{27}$. Then $\beta \in \mathrm{NS}(A, \Theta)$ if and only if it satisfies the following 28 equations:

$$0 = -(\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_{22} - (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{23} - (\tau_{14}^2 - \tau_{11}\tau_{44})b_{24} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_{25} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{26} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_{27} + b_{19}\tau_{11} + b_{20}\tau_{12} + b_{21}\tau_{13} - b_4\tau_{14} - b_5\tau_{24} - b_6\tau_{34} - b_7\tau_{44} + b_3$$

$$0 = -(\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_{22} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{23} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{24} - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_{25} - (\tau_{24}^2 - \tau_{22}\tau_{44})b_{26} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_{27} + b_{19}\tau_{12} - b_{10}\tau_{14} + b_{20}\tau_{22} + b_{21}\tau_{23} - b_{11}\tau_{24} - b_{12}\tau_{34} - b_{13}\tau_{44} + b_9$$

$$0 = -(\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_{22} - (\tau_{13}^2 - \tau_{11}\tau_{33})b_{23} - (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{24} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_{25} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{26} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_{27} + b_{15}\tau_{11} + b_{16}\tau_{12} - b_4\tau_{13} + b_{17}\tau_{13} + b_{18}\tau_{14} - b_5\tau_{23} - b_6\tau_{33} - b_7\tau_{34} + b_2$$

$$0 = -(\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_{11} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_{13} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_{16} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{17} + (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{18} - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{20} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_{21} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{13} + (\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{14})b_{25} - ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{13} + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{14})b_{26} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{13} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{14})b_{27} + b_{14}\tau_{12} - b_9\tau_{13} + b_8\tau_{14}$$

$$0 = -(\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_4 - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_5 - (\tau_{34}^2 - \tau_{33}\tau_{44})b_6 + (\tau_{14}^2 - \tau_{11}\tau_{44})b_{15} + (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{16} + (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_{17} - (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{19} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{20} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_{21} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{13} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{14})b_{22} + ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{11} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{13} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{14})b_{23} + ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})b_{25} - b_{14}\tau_{14} + b_3\tau_{34} - b_2\tau_{44}$$

$$0 = -(\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_1 + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_2 - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_3 + ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{13} + (\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{14})b_6 + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{13} + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{14})b_7 - (\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_8 + (\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_9 - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{13} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{14})b_{12} - ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{13} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{14})b_{13} - (\tau_{12}^2 - \tau_{11}\tau_{22})b_{14} + ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{14})b_{17} + ((\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{14})b_{18} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})b_{21} - ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{12} - (((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{22} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{23} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{24})\tau_{11} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})\tau_{12} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})\tau_{13} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{27}$$

$$0 = (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_4 + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_5 - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_7 - (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{10} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_{11} + (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_{13} + (\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_{19} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{20} - ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{14})b_{22} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{12} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{14})b_{24} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})b_{26} + b_9\tau_{13} - b_3\tau_{23} + b_1\tau_{34}$$

$$0 = -(\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_5 - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_6 - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_7 + (\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_{16} + (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{17} + (\tau_{14}^2 - \tau_{11}\tau_{44})b_{18} - (\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_{20} - (\tau_{13}^2 - \tau_{11}\tau_{33})b_{21} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{13} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{14})b_{25} - ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{13} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{14})b_{26} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{11} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{13} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{14})b_{27} + b_{14}\tau_{11} - b_3\tau_{13} + b_2\tau_{14}$$

$$0 = -(\tau_{24}\tau_{33} - \tau_{23}\tau_{34})b_1 + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_2 - (\tau_{23}^2 - \tau_{22}\tau_{33})b_3 + ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{13} + (\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{14})b_4 + ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{22} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{23} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{24})b_7 - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_8 + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_9 - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{13} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{14})b_{10} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})b_{13} - (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{14} + ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{14})b_{15} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})b_{18} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})b_{19} - (((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{22} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{23} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{24})\tau_{11} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})\tau_{12} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})\tau_{13} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{24}$$

$$0 = -(\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_{22} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_{23} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})b_{24} - (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})b_{25} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_{26} - (\tau_{34}^2 - \tau_{33}\tau_{44})b_{27} + b_{19}\tau_{13} - b_{15}\tau_{14} + b_{20}\tau_{23} - b_{16}\tau_{24} + b_{21}\tau_{33} - b_{17}\tau_{34} - b_{18}\tau_{44} + b_{14}$$

$$0 = -(\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{22} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_{23} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_{24} - (\tau_{23}^2 - \tau_{22}\tau_{33})b_{25} - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_{26} - (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})b_{27} + b_{15}\tau_{12} - b_{10}\tau_{13} + b_{16}\tau_{22} - b_{11}\tau_{23} + b_{17}\tau_{23} + b_{18}\tau_{24} - b_{12}\tau_{33} - b_{13}\tau_{34} + b_8$$

$$0 = -(\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_4 + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})b_6 + (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_7 + (\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_{15} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_{17} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{18} - (\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_{19} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_{21} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{13} + (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{14})b_{23} - ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{13} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{14})b_{24} + ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})b_{27} - b_{14}\tau_{12} + b_3\tau_{23} - b_2\tau_{24}$$

$$0 = (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})b_1 - (\tau_{24}^2 - \tau_{22}\tau_{44})b_2 + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_3 - ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{13} + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{14})b_4 + ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{22} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{23} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{24})b_6 + (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_8 - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_9 + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{13} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{14})b_{10} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})b_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_{14} - ((\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{14})b_{15} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})b_{17} + ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{13})b_{19} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})b_{21} - (((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{22} - (\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{23} + (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{24})\tau_{11} - ((\tau_{34}^2 - \tau_{33}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{23} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{24})\tau_{12} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})\tau_{13} - ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{23}$$

$$0 = -(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})b_1+(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})b_2-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})b_3-((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{24}^2-\tau_{22}\tau_{44})\tau_{13}+(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{14})b_5-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{13}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{14})b_6-(\tau_{14}^2-\tau_{11}\tau_{44})b_8+(\tau_{13}\tau_{14}-\tau_{11}\tau_{34})b_9+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})\tau_{13}+(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})\tau_{14})b_{11}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{13}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{14})b_{12}-(\tau_{12}\tau_{14}-\tau_{11}\tau_{24})b_{14}-((\tau_{24}^2-\tau_{22}\tau_{44})\tau_{11}-(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})\tau_{12}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})\tau_{14})b_{16}-((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{14})b_{17}+((\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{11}-(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})\tau_{12}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})\tau_{13})b_{20}+((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{13})b_{21}-(((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})\tau_{11}-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{23}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{24})\tau_{12}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{24})\tau_{13}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{25}$$

$$0 = -(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})b_5-(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})b_6-(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})b_7+(\tau_{12}\tau_{14}-\tau_{11}\tau_{24})b_{11}+(\tau_{13}\tau_{14}-\tau_{11}\tau_{34})b_{12}+(\tau_{14}^2-\tau_{11}\tau_{44})b_{13}-(\tau_{12}^2-\tau_{11}\tau_{22})b_{20}-(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{21}-((\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{11}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})\tau_{12}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})\tau_{14})b_{25}-((\tau_{24}^2-\tau_{22}\tau_{44})\tau_{11}-(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})\tau_{12}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})\tau_{14})b_{26}-((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{14})b_{27}+b_9\tau_{11}-b_3\tau_{12}+b_1\tau_{14}$$

$$0 = -(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})b_{10}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_{12}+(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_{13}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})b_{15}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})b_{17}-(\tau_{24}^2-\tau_{22}\tau_{44})b_{18}-(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})b_{19}+(\tau_{23}^2-\tau_{22}\tau_{33})b_{21}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{13}+(\tau_{23}^2-\tau_{22}\tau_{33})\tau_{14})b_{23}-((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{24}^2-\tau_{22}\tau_{44})\tau_{13}+(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{14})b_{24}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})b_{27}-b_{14}\tau_{22}+b_9\tau_{23}-b_8\tau_{24}$$

$$0 = -(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})b_{10}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_{11}-(\tau_{34}^2-\tau_{33}\tau_{44})b_{12}+(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})b_{15}+(\tau_{24}^2-\tau_{22}\tau_{44})b_{16}+(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_{17}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})b_{19}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})b_{20}-(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_{21}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{24}^2-\tau_{22}\tau_{44})\tau_{13}+(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{14})b_{22}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{13}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{14})b_{23}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})b_{25}-b_{14}\tau_{24}+b_9\tau_{34}-b_8\tau_{44}$$

$$0 = -(\tau_{34}^2-\tau_{33}\tau_{44})b_1+(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_2-(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_3+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{13}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{14})b_4+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})b_5-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})b_8+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})b_9-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{13}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{14})b_{10}-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{23}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{24})b_{11}-(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})b_{14}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{14})b_{15}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{24})b_{16}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{13})b_{19}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{23})b_{20}-(((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})\tau_{11}-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{23}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{24})\tau_{12}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{24})\tau_{13}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{22}$$

$$0 = (\tau_{14}\tau_{33}-\tau_{13}\tau_{34})b_1-(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})b_2+(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_3+((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{13}+(\tau_{23}^2-\tau_{22}\tau_{33})\tau_{14})b_5-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{13}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{14})b_7+(\tau_{13}\tau_{14}-\tau_{11}\tau_{34})b_8-(\tau_{13}^2-\tau_{11}\tau_{33})b_9-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})\tau_{13}+(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})\tau_{14})b_{11}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{13}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{14})b_{13}+(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{14}+((\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{11}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})\tau_{12}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})\tau_{14})b_{16}-((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{14})b_{18}-((\tau_{23}^2-\tau_{22}\tau_{33})\tau_{11}-(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})\tau_{12}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})\tau_{13})b_{20}-(((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})\tau_{11}-((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{23}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{24})\tau_{12}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{24})\tau_{13}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{23})\tau_{14})b_{26}$$

$$0 = (\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_4+(\tau_{23}^2-\tau_{22}\tau_{33})b_5-(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_7-(\tau_{13}^2-\tau_{11}\tau_{33})b_{10}-(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_{11}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})b_{13}+(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{15}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})b_{16}-(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})b_{18}-((\tau_{23}^2-\tau_{22}\tau_{33})\tau_{11}-(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})\tau_{12}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})\tau_{13})b_{22}+((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{13})b_{24}+((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{23})b_{26}+b_8\tau_{13}-b_2\tau_{23}+b_1\tau_{33}$$

$$0 = -(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})b_5-(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_6-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})b_7+(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{11}+(\tau_{13}^2-\tau_{11}\tau_{33})b_{12}+(\tau_{13}\tau_{14}-\tau_{11}\tau_{34})b_{13}-(\tau_{12}^2-\tau_{11}\tau_{22})b_{16}-(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{17}-(\tau_{12}\tau_{14}-\tau_{11}\tau_{24})b_{18}-((\tau_{23}^2-\tau_{22}\tau_{33})\tau_{11}-(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})\tau_{12}+(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})\tau_{13})b_{25}-((\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{11}-(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})\tau_{12}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})\tau_{13})b_{26}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{13})b_{27}+b_8\tau_{11}-b_2\tau_{12}+b_1\tau_{13}$$

$$0 = -(\tau_{12}^2-\tau_{11}\tau_{22})b_{22}-(\tau_{12}\tau_{13}-\tau_{11}\tau_{23})b_{23}-(\tau_{12}\tau_{14}-\tau_{11}\tau_{24})b_{24}-(\tau_{13}\tau_{22}-\tau_{12}\tau_{23})b_{25}-(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})b_{26}-(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})b_{27}+b_{10}\tau_{11}-b_4\tau_{12}+b_{11}\tau_{12}+b_{12}\tau_{13}+b_{13}\tau_{14}-b_5\tau_{22}-b_6\tau_{23}-b_7\tau_{24}+b_1$$

$$0 = (\tau_{14}\tau_{33}-\tau_{13}\tau_{34})b_{10}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_{11}-(\tau_{34}^2-\tau_{33}\tau_{44})b_{13}-(\tau_{14}\tau_{23}-\tau_{12}\tau_{34})b_{15}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})b_{16}+(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_{18}+(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_{19}+(\tau_{23}^2-\tau_{22}\tau_{33})b_{20}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{12}-(\tau_{23}\tau_{24}-\tau_{22}\tau_{34})\tau_{13}+(\tau_{23}^2-\tau_{22}\tau_{33})\tau_{14})b_{22}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{13}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{14})b_{24}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{22}-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{23}+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{24})b_{26}+b_{14}\tau_{23}-b_9\tau_{33}+b_8\tau_{34}$$

$$0 = -(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})b_4-(\tau_{24}^2-\tau_{22}\tau_{44})b_5-(\tau_{24}\tau_{34}-\tau_{23}\tau_{44})b_6+(\tau_{14}^2-\tau_{11}\tau_{44})b_{10}+(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})b_{11}+(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})b_{12}-(\tau_{12}\tau_{14}-\tau_{11}\tau_{24})b_{19}-(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})b_{20}-(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})b_{21}+((\tau_{24}^2-\tau_{22}\tau_{44})\tau_{11}-(\tau_{14}\tau_{24}-\tau_{12}\tau_{44})\tau_{12}+(\tau_{14}\tau_{22}-\tau_{12}\tau_{24})\tau_{14})b_{22}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{12}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{14})b_{23}+((\tau_{24}\tau_{34}-\tau_{23}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{22}+(\tau_{14}\tau_{23}-\tau_{13}\tau_{24})\tau_{24})b_{25}-b_9\tau_{14}+b_3\tau_{24}-b_1\tau_{44}$$

$$0 = (\tau_{14}\tau_{33}-\tau_{13}\tau_{34})b_4+(\tau_{24}\tau_{33}-\tau_{23}\tau_{34})b_5-(\tau_{34}^2-\tau_{33}\tau_{44})b_7-(\tau_{13}\tau_{14}-\tau_{11}\tau_{34})b_{15}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})b_{16}+(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})b_{18}+(\tau_{13}^2-\tau_{11}\tau_{33})b_{19}+(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})b_{20}-((\tau_{24}\tau_{33}-\tau_{23}\tau_{34})\tau_{11}-(\tau_{13}\tau_{24}-\tau_{12}\tau_{34})\tau_{13}+(\tau_{13}\tau_{23}-\tau_{12}\tau_{33})\tau_{14})b_{22}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{11}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{13}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{14})b_{24}+((\tau_{34}^2-\tau_{33}\tau_{44})\tau_{12}-(\tau_{14}\tau_{34}-\tau_{13}\tau_{44})\tau_{23}+(\tau_{14}\tau_{33}-\tau_{13}\tau_{34})\tau_{24})b_{26}+b_{14}\tau_{13}-b_3\tau_{33}+b_2\tau_{34}$$

$$0 = -(\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_4 + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_6 + (\tau_{24}^2 - \tau_{22}\tau_{44})b_7 + (\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_{10} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_{12} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})b_{13} - (\tau_{12}^2 - \tau_{11}\tau_{22})b_{19} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{21} - ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{13}\tau_{24} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{14})b_{23} - ((\tau_{24}^2 - \tau_{22}\tau_{44})\tau_{11} - (\tau_{14}\tau_{24} - \tau_{12}\tau_{44})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{14})b_{24} + ((\tau_{24}\tau_{34} - \tau_{23}\tau_{44})\tau_{12} - (\tau_{14}\tau_{34} - \tau_{13}\tau_{44})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{24})b_{27} - b_9\tau_{12} + b_3\tau_{22} - b_1\tau_{24}$$

$$0 = -(\tau_{13}\tau_{24} - \tau_{12}\tau_{34})b_4 - (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_5 - (\tau_{24}\tau_{33} - \tau_{23}\tau_{34})b_6 + (\tau_{13}\tau_{14} - \tau_{11}\tau_{34})b_{10} + (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{11} + (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})b_{12} - (\tau_{12}\tau_{14} - \tau_{11}\tau_{24})b_{15} - (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_{16} - (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})b_{17} + ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{13})b_{22} + ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{11} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{12} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{13})b_{23} + ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})b_{25} - b_8\tau_{14} + b_2\tau_{24} - b_1\tau_{34}$$

$$0 = -(\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_4 + (\tau_{23}^2 - \tau_{22}\tau_{33})b_6 + (\tau_{23}\tau_{24} - \tau_{22}\tau_{34})b_7 + (\tau_{12}\tau_{13} - \tau_{11}\tau_{23})b_{10} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})b_{12} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})b_{13} - (\tau_{12}^2 - \tau_{11}\tau_{22})b_{15} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})b_{17} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})b_{18} - ((\tau_{23}^2 - \tau_{22}\tau_{33})\tau_{11} - (\tau_{13}\tau_{23} - \tau_{12}\tau_{33})\tau_{12} + (\tau_{13}\tau_{22} - \tau_{12}\tau_{23})\tau_{13})b_{23} - ((\tau_{23}\tau_{24} - \tau_{22}\tau_{34})\tau_{11} - (\tau_{14}\tau_{23} - \tau_{12}\tau_{34})\tau_{12} + (\tau_{14}\tau_{22} - \tau_{12}\tau_{24})\tau_{13})b_{24} + ((\tau_{24}\tau_{33} - \tau_{23}\tau_{34})\tau_{12} - (\tau_{14}\tau_{33} - \tau_{13}\tau_{34})\tau_{22} + (\tau_{14}\tau_{23} - \tau_{13}\tau_{24})\tau_{23})b_{27} - b_8\tau_{12} + b_2\tau_{22} - b_1\tau_{23}$$

Using Sage, we can calculate the forms q_r as well; let $\beta \in \mathbb{Z}^{27} \simeq \mathrm{NS}(A, \Theta)$. Then

$$q_2(\beta) = 36b_{11}^2 - 24b_{11}b_{17} - 24b_{11}b_4 + 36b_{17}^2 - 24b_{17}b_4 + 36b_4^2 + 96b_{12}b_{16} + 96b_{13}b_{20} + 96b_{18}b_{21} + 96b_1b_{22} + 96b_2b_{23} + 96b_{14}b_{27} + 96b_{24}b_3 + 96b_{10}b_5 + 96b_{15}b_6 + 96b_{19}b_7 + 96b_{25}b_8 + 96b_{26}b_9$$

$$q_3(\beta) = 1728b_{23}b_{18}b_3 - 864b_{26}b_9b_4 + 1728b_{19}b_{18}b_6 - 1728b_{19}b_2b_{27} + 432b_{11}b_{17}b_4 + 1728b_{10}b_{26}b_3 + 1728b_{15}b_{21}b_7 + 864b_2b_{23}b_4 + 1728b_{21}b_{26}b_8 + 864b_{13}b_{20}b_{11} + 864b_1b_{22}b_{11} - 864b_1b_{17}b_{22} + 1728b_{22}b_{12}b_2 - 1728b_{22}b_7b_9 - 864b_{14}b_{27}b_{11} + 1728b_{10}b_{20}b_7 - 864b_{10}b_{17}b_5 + 1728b_{10}b_{16}b_6 - 864b_{18}b_{21}b_4 + 864b_{18}b_{21}b_{17} - 864b_{12}b_{16}b_4 - 864b_{13}b_{20}b_{17} + 1728b_{13}b_{19}b_5 - 1728b_1b_{19}b_{26} + 1728b_1b_{16}b_{23} + 864b_2b_{23}b_{17} + 1728b_{21}b_2b_{24} + 1728b_1b_{20}b_{24} + 1728b_{10}b_2b_{25} + 864b_{24}b_3b_4 - 864b_{18}b_{21}b_{11} + 1728b_{14}b_{12}b_{26} + 1728b_{15}b_{27}b_3 - 1728b_{14}b_{23}b_7 - 1728b_{22}b_6b_8 - 1728b_{20}b_{27}b_8 + 1728b_{15}b_{12}b_5 - 1728b_1b_{15}b_{25} + 1728b_{20}b_{12}b_{18} - 864b_2b_{23}b_{11} + 864b_{12}b_{16}b_{11} + 864b_1b_{22}b_4 - 864b_{13}b_{20}b_4 + 1728b_{13}b_{16}b_{21} - 1728b_{13}b_{14}b_{25} + 864b_{12}b_{16}b_{17} + 864b_{15}b_6b_4 + 1728b_{14}b_{24}b_6 + 1728b_{16}b_{27}b_9 + 1728b_{18}b_{25}b_9 - 864b_{15}b_6b_{11} + 864b_{10}b_5b_4 + 864b_{10}b_5b_{11} - 864b_{14}b_{27}b_4 + 1728b_{13}b_{22}b_3 - 864b_{24}b_3b_{17} + 864b_{14}b_{27}b_{17} - 864b_{24}b_3b_{11} + 864b_{25}b_8b_{17} + 864b_{25}b_8b_{11} + 864b_{26}b_9b_{11} - 864b_{25}b_8b_4 + 864b_{15}b_6b_{17} - 864b_{19}b_7b_{11} - 864b_{19}b_7b_{17} + 864b_{19}b_7b_4 - 864b_{26}b_9b_{17} + 1728b_{23}b_5b_8 + 1728b_{24}b_5b_9 - 216b_{11}^2b_4 - 216b_{11}b_{17}^2 - 216b_{11}b_4^2 - 216b_{17}^2b_4 - 216b_{17}b_4^2 + 216b_{11}^3 + 216b_{17}^3 + 216b_4^3 - 216b_{11}^2b_{17}$$

$$
\begin{aligned}
q_4(\beta) = {} & 8640 b_{11} b_{17} b_4^2 - 20736 b_{24} b_3 b_{17}^2 + 6912 b_{15} b_6 b_4^2 + 6912 b_1 b_{22} b_4^2 + 6912 b_1 b_{22} b_{11}^2 - 20736 b_1 b_{22} b_{17}^2 + \\
& 6912 b_2 b_{23} b_4^2 + 6912 b_{14} b_{27} b_{17}^2 - 20736 b_{15} b_6 b_{11}^2 + 6912 b_{26} b_9 b_{11}^2 - 20736 b_{26} b_9 b_4^2 - \\
& 20736 b_{14} b_{27} b_{11}^2 + 6912 b_{10} b_5 b_4^2 - 20736 b_{10} b_5 b_{17}^2 + 8640 b_{11} b_{17}^2 b_4 + 6912 b_2 b_{23} b_{17}^2 + \\
& 6912 b_{19} b_7 b_4^2 - 20736 b_{19} b_7 b_{11}^2 - 20736 b_{26} b_9 b_{17}^2 + 6912 b_{12} b_{16} b_{17}^2 + 6912 b_{25} b_8 b_{11}^2 - \\
& 20736 b_2 b_{23} b_{11}^2 + 6912 b_{12} b_{16} b_{11}^2 - 20736 b_{25} b_8 b_4^2 - 20736 b_{14} b_{27} b_4^2 + 110592 b_{10} b_{14} b_{26} b_6 + \\
& 110592 b_{10} b_2 b_{21} b_{26} + 110592 b_{10} b_{18} b_{25} b_3 - 110592 b_1 b_{15} b_{21} b_{26} + 110592 b_1 b_{18} b_{20} b_{23} - \\
& 110592 b_1 b_{18} b_{21} b_{22} - 110592 b_1 b_{14} b_{22} b_{27} + 110592 b_1 b_{15} b_{20} b_{27} - 110592 b_1 b_{16} b_{19} b_{27} - \\
& 82944 b_1 b_{17} b_{20} b_{24} - 110592 b_1 b_{18} b_{19} b_{25} - 110592 b_1 b_{14} b_{24} b_{25} + 110592 b_1 b_{14} b_{23} b_{26} + \\
& 110592 b_1 b_{16} b_{21} b_{24} + 82944 b_1 b_{17} b_{19} b_{26} - 82944 b_{23} b_{18} b_3 b_{11} + 27648 b_{23} b_{18} b_3 b_{17} + \\
& 27648 b_{23} b_{18} b_3 b_4 - 82944 b_{19} b_{18} b_6 b_{11} + 27648 b_{19} b_{18} b_6 b_{17} + 27648 b_{19} b_{18} b_6 b_4 + \\
& 82944 b_{19} b_2 b_{27} b_{11} - 27648 b_{19} b_2 b_{27} b_{17} - 27648 b_{19} b_2 b_{27} b_4 + 27648 b_{10} b_{26} b_3 b_{11} + \\
& 27648 b_{10} b_{26} b_3 b_4 - 82944 b_{15} b_{21} b_7 b_{11} + 27648 b_{15} b_{21} b_7 b_{17} + 27648 b_{15} b_{21} b_7 b_4 + \\
& 27648 b_{21} b_{26} b_8 b_{11} + 27648 b_{21} b_{26} b_8 b_{17} - 82944 b_{21} b_{26} b_8 b_4 + 27648 b_{22} b_{12} b_2 b_{11} + \\
& 27648 b_{22} b_{12} b_2 b_{17} + 27648 b_{22} b_{12} b_2 b_4 - 27648 b_{22} b_7 b_9 b_{11} + 82944 b_{22} b_7 b_9 b_{17} - \\
& 27648 b_{22} b_7 b_9 b_4 + 27648 b_{10} b_{20} b_7 b_{11} + 27648 b_{10} b_{20} b_7 b_4 + 27648 b_{10} b_{16} b_6 b_{11} + \\
& 27648 b_{10} b_{16} b_6 b_{17} + 27648 b_{10} b_{16} b_6 b_4 + 27648 b_{13} b_{19} b_5 b_{11} - 82944 b_{13} b_{19} b_5 b_{17} + \\
& 27648 b_{13} b_{19} b_5 b_4 - 27648 b_1 b_{19} b_{26} b_{11} - 27648 b_1 b_{19} b_{26} b_4 + 27648 b_1 b_{16} b_{23} b_{11} + \\
& 27648 b_1 b_{16} b_{23} b_{17} + 27648 b_1 b_{16} b_{23} b_4 - 82944 b_{21} b_2 b_{24} b_{11} + 27648 b_{21} b_2 b_{24} b_{17} + \\
& 27648 b_{21} b_2 b_{24} b_4 + 27648 b_1 b_{20} b_{24} b_{11} + 27648 b_1 b_{20} b_{24} b_4 - 20736 b_{12} b_{16} b_4^2 + 6912 b_{24} b_3 b_4^2 + \\
& 13824 b_{12} b_{16} b_{11} b_{17} - 110592 b_{10} b_{14} b_{27} b_5 - 82944 b_{10} b_{17} b_{26} b_3 - 110592 b_{10} b_{14} b_{25} b_7 + \\
& 110592 b_{10} b_{16} b_{21} b_7 + 110592 b_{10} b_{16} b_{27} b_3 + 69120 b_{26} b_9 b_{17} b_4 - 110592 b_{10} b_2 b_{20} b_{27} - \\
& 13824 b_{26} b_9 b_{11} b_4 + 110592 b_{10} b_{18} b_{20} b_6 - 82944 b_{10} b_{17} b_{20} b_7 - 110592 b_{10} b_{18} b_{21} b_5 + \\
& 69120 b_{14} b_{27} b_{11} b_4 - 13824 b_{14} b_{27} b_{11} b_{17} - 13824 b_{14} b_{27} b_{17} b_4 - 13824 b_2 b_{23} b_{11} b_{17} + \\
& 13824 b_2 b_{23} b_{17} b_4 - 13824 b_2 b_{23} b_{11} b_4 - 13824 b_{10} b_5 b_{17} b_4 + 13824 b_{10} b_5 b_{11} b_4 - \\
& 13824 b_{15} b_6 b_{11} b_4 - 13824 b_{15} b_6 b_{11} b_{17} - 13824 b_{24} b_3 b_{11} b_4 + 69120 b_{24} b_3 b_{11} b_{17} - \\
& 13824 b_{10} b_5 b_{11} b_{17} - 13824 b_{24} b_3 b_{17} b_4 - 13824 b_{13} b_{20} b_{11} b_4 - 13824 b_{13} b_{20} b_{11} b_{17} + \\
& 69120 b_{13} b_{20} b_{17} b_4 - 13824 b_{12} b_{16} b_{11} b_4 - 13824 b_{12} b_{16} b_{17} b_4 - 13824 b_1 b_{22} b_{11} b_{17} - \\
& 13824 b_{18} b_{21} b_{17} b_4 - 13824 b_1 b_{22} b_{17} b_4 + 13824 b_1 b_{22} b_{11} b_4 - 13824 b_{18} b_{21} b_{11} b_{17} + \\
& 69120 b_{18} b_{21} b_{11} b_4 + 13824 b_{25} b_8 b_{11} b_{17} - 13824 b_{19} b_7 b_{17} b_4 - 13824 b_{25} b_8 b_{17} b_4 - \\
& 13824 b_{25} b_8 b_{11} b_4 + 69120 b_{19} b_7 b_{11} b_{17} + 13824 b_{15} b_6 b_{17} b_4 - 13824 b_{19} b_7 b_{11} b_4 - \\
& 13824 b_{26} b_9 b_{11} b_{17} - 1728 b_{11} b_4^3 - 1728 b_{17}^3 b_4 - 1728 b_{17} b_4^3 - 6048 b_{17}^2 b_4^2 - 1728 b_{11} b_{17}^3 - \\
& 20736 b_{18} b_{21} b_{11}^2 - 20736 b_{13} b_{20} b_4^2 + 6912 b_{18} b_{21} b_{17}^2 + 6912 b_{13} b_{20} b_{11}^2 - 20736 b_{13} b_{20} b_{17}^2 + \\
& 6912 b_{25} b_8 b_{17}^2 - 1728 b_{11}^3 b_{17} - 1728 b_{11}^3 b_4 + 8640 b_{11}^2 b_{17} b_4 + 6912 b_{15} b_6 b_{17}^2 - 20736 b_{19} b_7 b_{17}^2 + \\
& 6912 b_{10} b_5 b_{11}^2 - 20736 b_{24} b_3 b_{11}^2 + 1296 b_{11}^4 + 1296 b_4^4 - 20736 b_{18} b_{21} b_4^2 - 6048 b_{11}^2 b_{17}^2 - \\
& 6048 b_{11}^2 b_4^2 + 1296 b_{17}^4 + 27648 b_{10} b_2 b_{25} b_{11} + 27648 b_{10} b_2 b_{25} b_{17} + 27648 b_{10} b_2 b_{25} b_4 + \\
& 27648 b_{14} b_{12} b_{26} b_{11} + 27648 b_{14} b_{12} b_{26} b_{17} - 82944 b_{14} b_{12} b_{26} b_4 - 82944 b_{15} b_{27} b_3 b_{11} + \\
& 27648 b_{15} b_{27} b_3 b_{17} + 27648 b_{15} b_{27} b_3 b_4 + 82944 b_{14} b_{23} b_7 b_{11} - 27648 b_{14} b_{23} b_7 b_{17} - \\
& 27648 b_{14} b_{23} b_7 b_4 - 27648 b_{22} b_6 b_8 b_{11} - 27648 b_{22} b_6 b_8 b_{17} - 27648 b_{22} b_6 b_8 b_4 - \\
& 27648 b_{20} b_{27} b_8 b_{11} - 27648 b_{20} b_{27} b_8 b_{17} + 82944 b_{20} b_{27} b_8 b_4 + 27648 b_{15} b_{12} b_5 b_{11} + \\
& 27648 b_{15} b_{12} b_5 b_{17} + 27648 b_{15} b_{12} b_5 b_4 - 27648 b_1 b_{15} b_{25} b_{11} - 27648 b_1 b_{15} b_{25} b_{17} - \\
& 27648 b_1 b_{15} b_{25} b_4 + 27648 b_{20} b_{12} b_{18} b_{11} + 27648 b_{20} b_{12} b_{18} b_{17} - 82944 b_{20} b_{12} b_{18} b_4 + \\
& 27648 b_{13} b_{16} b_{21} b_{11} + 27648 b_{13} b_{16} b_{21} b_{17} - 82944 b_{13} b_{16} b_{21} b_4 - 27648 b_{13} b_{14} b_{25} b_{11} - \\
& 27648 b_{13} b_{14} b_{25} b_{17} + 82944 b_{13} b_{14} b_{25} b_4 - 82944 b_{14} b_{24} b_6 b_{11} + 27648 b_{14} b_{24} b_6 b_{17} +
\end{aligned}
$$

$27648 b_{14} b_{24} b_6 b_4 + 27648 b_{16} b_{27} b_9 b_{11} + 27648 b_{16} b_{27} b_9 b_{17} - 82944 b_{16} b_{27} b_9 b_4 + 27648 b_{18} b_{25} b_9 b_{11} + 27648 b_{18} b_{25} b_9 b_{17} - 82944 b_{18} b_{25} b_9 b_4 + 27648 b_{13} b_{22} b_3 b_{11} - 82944 b_{13} b_{22} b_3 b_{17} + 27648 b_{13} b_{22} b_3 b_4 + 27648 b_{23} b_5 b_8 b_{11} + 27648 b_{23} b_5 b_8 b_{17} + 27648 b_{23} b_5 b_8 b_4 + 27648 b_{24} b_5 b_9 b_{11} - 82944 b_{24} b_5 b_9 b_{17} + 27648 b_{24} b_5 b_9 b_4 - 110592 b_{12} b_{14} b_{22} b_7 + 110592 b_{12} b_{14} b_{24} b_5 + 110592 b_{12} b_{15} b_{26} b_3 - 110592 b_{12} b_{16} b_{24} b_3 + 110592 b_{12} b_2 b_{20} b_{24} + 110592 b_{12} b_{18} b_{22} b_3 + 110592 b_{12} b_{18} b_{19} b_5 - 110592 b_{12} b_{16} b_{19} b_7 - 110592 b_{12} b_{19} b_2 b_{26} + 110592 b_{12} b_{15} b_{20} b_7 + 110592 b_{13} b_{19} b_2 b_{25} + 110592 b_{13} b_{14} b_{22} b_6 - 110592 b_{13} b_2 b_{20} b_{23} + 110592 b_{13} b_{15} b_{21} b_5 - 110592 b_{13} b_{14} b_{23} b_5 - 110592 b_{13} b_{15} b_{20} b_6 + 110592 b_{13} b_2 b_{21} b_{22} - 110592 b_{13} b_{15} b_{25} b_3 + 110592 b_{13} b_{16} b_{23} b_3 + 110592 b_{15} b_{27} b_5 b_9 + 110592 b_{15} b_{25} b_7 b_9 - 110592 b_{15} b_{26} b_6 b_9 - 110592 b_{16} b_{23} b_7 b_9 + 110592 b_{16} b_{24} b_6 b_9 + 110592 b_{13} b_{16} b_{19} b_6 - 110592 b_{19} b_{25} b_7 b_8 + 110592 b_{19} b_{26} b_6 b_8 - 110592 b_{19} b_{27} b_5 b_8 + 110592 b_2 b_{22} b_{27} b_9 - 110592 b_2 b_{23} b_{26} b_9 + 110592 b_2 b_{24} b_{25} b_9 + 110592 b_{20} b_{23} b_7 b_8 - 110592 b_{20} b_{24} b_6 b_8 - 110592 b_{18} b_{22} b_6 b_9 + 110592 b_{18} b_{23} b_5 b_9 - 110592 b_{21} b_{22} b_7 b_8 + 110592 b_{21} b_{24} b_5 b_8 - 110592 b_{22} b_{27} b_3 b_8 + 110592 b_{23} b_{26} b_3 b_8 - 110592 b_{25} b_{24} b_3 b_8.$

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Nomenclature

t_x	translation by a point x
$\sim$	isogeny relation
$\deg f$	degree of a homomorphism f
e_f	exponent of a homomorphism f
$\mathrm{Hom}(A, B)$	morphisms between A and B that send 0 to 0
$\mathrm{Hom}_{\mathbb{Q}}(A, B)$	$= \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$
f^{-1}	inverse of an isogeny f in $\mathrm{Hom}_{\mathbb{Q}}(A, B)$
$\mathfrak{H}_n$	Siegel upper half space
ppav	principally polarized abelian variety
Λ	full rank lattice in $\mathbb{C}^n$
$A^\vee$	dual abelian variety of A
$\mathrm{Pic}^0(A)$	group of line bundles algebraically equivalent to 0 on A modulo isomorphism
ϕ_L	morphism from A to $A^\vee$ induced by L
$f^\vee$	dual homomorphism of f
$\dagger$	Rosati involution
N_X	norm endomorphism of the abelian subvariety X
e_X	exponent of the abelian subvariety X

C	smooth projective curve
J_C	Jacobian variety of C
$\mathrm{Pic}^0(C)$	degree 0 line bundles on C, modulo isomorphism
α_p	closed immersion $C \hookrightarrow J_C$ defined by $p \in C$
Θ	principal polarization (divisor)
ω_C	canonical divisor of C
$\mathcal{M}_g$	(coarse) moduli space of genus g curves
$\mathcal{A}_g$	(coarse) moduli space of g-dimensional ppavs
$\mathrm{Div}(X)$	group of divisors of the variety X
$(D_1 \cdots D_n)$	intersection number of the divisors $D_1, \ldots, D_n$
$(D \cdot C)$	intersection number of a divisor D with a curve C
$\equiv_a$	algebraic equivalence
$\equiv$	numerical equivalence
$\mathrm{NS}(A)$	Néron-Severi group of A
$\mathrm{End}^s(A)$	symmetric endomorphisms of A
$\deg D$	$= (D \cdot \Theta^{n-1})$
$\mathfrak{A}^*(X)$	Chow ring modulo algebraic equivalence of the variety X
$\deg C$	$= (C \cdot \Theta)$
$\mathrm{NS}(A, \Theta)$	$= \mathrm{NS}(A)/\mathbb{Z}[\Theta]$, the polarized Néron-Severi group of A
$A[m]$	m-torsion points of A
$K(D)$	$= \ker \phi_D$
$P_f(t)$	characteristic polynomial of f
$T_p A$	$= \lim_{\leftarrow} A[p^l]$
$\mathbb{Z}_p$	ring of p-adic integers

$\mathbb{Q}_p$	field of p-adic numbers
V_pA	$= (T_pA) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$
$P_X(t)$	characteristic polynomial of N_X
α_X	numerical class associated to the abelian subvariety X
$\mathbb{E}$	elliptic variety associated to (A, Θ)
$c_1(L)$	first Chern class of a line bundle L
E_τ	elliptic curve associated to $\tau \in \mathfrak{H}_1$

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