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SPECTRAL PROPERTIES OF MAGNETIC QUANTUM SYSTEMS

por

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*"If you want to know me,
look inside your heart."
– Dao De Jing*

Para mi familia

Contents

Resumen

Esta tesis trata sobre la teoría de perturbaciones del hamiltoniano de Landau.

En el transcurso de las últimas tres décadas el hamiltoniano de Landau ha sido objeto de investigación activa en las áreas de la física matemática y la teoría espectral. En particular, dos problemas relacionados con la teoría espectral de perturbaciones de este operador han sido estudiados en una serie de situaciones: la velocidad a la cual el espectro discreto se acumula a los niveles de Landau y la distribución asintótica del espectro discreto dentro de una agrupación de valores propios, a medida que el nivel de energía tiende a infinito.

Investigamos la densidad asintótica de las agrupaciones de valores propios para perturbaciones eléctricas que son asintóticamente homogéneas, en un sentido apropiado, de grado $-\rho$, con $\rho \in (0, 1)$. Obtenemos estimaciones para la talla de las agrupaciones en altas energías, así como una descripción explícita de la distribución asintótica de valores propios en términos de la transformada de valor medio de la función homogénea asociada. Fórmulas explícitas se encuentran en la proposición 2.2.1 y en el teorema 2.2.1 del capítulo 2.

También investigamos el comportamiento del espectro discreto del operador perturbado cerca de los niveles de Landau para perturbaciones métricas con soporte compacto y con decaimiento exponencial o como potencia. Fórmulas asintóticas para la velocidad de acumulación de los valores propios a los niveles de Landau pueden encontrarse en los teoremas 3.2.1, 3.2.2 y 3.2.3 del capítulo 3. El comportamiento del espectro deja de ser semiclásico para perturbaciones con decaimiento rápido.

Adicionalmente, ciertos hechos relacionados con operadores de Berezin–Toeplitz son presentados en el capítulo 1.

Abstract

This thesis is devoted to the perturbation theory of the Landau Hamiltonian.

For the last three decades, the Landau Hamiltonian has been an object of active research in the domains of mathematical physics and spectral theory. In particular, two problems related with the spectral theory of perturbations of this operator have been addressed in a series of situations: the rate of accumulation of discrete spectrum to the Landau levels and the asymptotic distribution of the discrete spectrum within eigenvalue clusters, as the energy level goes to infinity.

We investigate the asymptotic density of eigenvalue clusters for electric perturbations which are asymptotically homogeneous, in an appropriate sense, of order $-\rho$ with $\rho \in (0,1)$. We obtain estimates on the size of the eigenvalue clusters in the high-energy regime, as well as an explicit description of the asymptotic density of the eigenvalues in terms of the mean-value transform of the associated homogeneous function. Formulas can be found in Proposition 2.2.1 and Theorem 2.2.1 on Chapter 2.

We also study the behaviour of the discrete spectrum of the perturbed operator near the Landau levels for metric perturbations of compact support, and of exponential or power-like decay at infinity. Asymptotic formulas for the rate of accumulation of eigenvalues to the Landau levels can be found in Theorems 3.2.1, 3.2.2 and 3.2.3 on Chapter 3. The behavior of the spectrum is seen to cease to be of a semiclassical nature for rapidly decaying perturbations.

Additionally, certain facts regarding Berezin–Toeplitz operators are set forth on Chapter 1.

Chapter 0

Outline and Notations

The contents of this thesis are structured as follows: Chapter 1 concerns the Landau Hamiltonian and its perturbation theory. The purpose here is to state and to contextualize the results contained in the chapters that follow. Also, some facts concerning Berezin–Teoplitz operators are set forth.

The main results of this thesis are presented in Chapters 2 and 3. They are written as scientific articles, as such, they are intended to be self-contained. Some overlapping in the contents of the chapters is therefore to be expected.

Chapter 2 concerns itself with electric perturbations of the Landau Hamiltonian. For asymptotically homogeneous electric potentials of degree $-\rho$, $0 < \rho < 1$, clusters of discrete spectrum appear around the Landau levels. High-energy estimates for the size of these clusters and the asymptotic distribution of eigenvalues within these clusters is studied. These results are published in [48] and [49].

Chapter 3 considers metric perturbations of the Landau Hamiltonian. Asymptotics are given for the rate of convergence of the discrete eigenvalues to the Landau levels for different regimes of asymptotically flat metrics, namely, perturbations that are compactly supported, with exponential decay and with power-like decay. These results can be found in [47].

We next establish some basic notation that will be used throughout this thesis.

Let M be a set, $\mathcal{M}$ a σ-algebra of measurable subsets of M and μ a σ-finite measure defined on $\mathcal{M}$, such that $(M, \mathcal{M}, \mu)$ is a measure space. We denote $L^p(M, d\mu)$, $p \in [1, \infty]$

the Lebesgue space of order p in $(M, \mathcal{M}, \mu)$, and $\|\cdot\|_{L^p(M,d\mu)}$ its corresponding norm.

As usual, for $p \in [1, \infty]$, we denote $L^p(\mathbb{R}^d)$, $d \in \mathbb{N}$, the Lebesgue space of order p in $\mathbb{R}^d$ equipped with the Lebesgue measure. We also define $L^p_w(\mathbb{R}^d)$, the *weak* Lebesgue space of order $p \in [1, \infty)$, as the set of equivalence classes of measurable functions u such that the quasi norm

$$\|u\|_{L^p_w(\mathbb{R}^d)} := \sup_{t>0} t \left|\left\{x \in \mathbb{R}^d \,|\, |u(x)| > t\right\}\right|^{1/p}$$

is finite.

Let X be a Hilbert space, we denote its inner product, linear in the first argument and anti-linear in the second argument, by $\langle \cdot\,, \,\cdot\cdot \rangle_X$. Denote $\mathcal{L}(X)$ the set of bounded operators on X, $S_\infty(X) \subset \mathcal{L}(X)$ the class of compact operators acting on X, and for $p \in [1, \infty)$ we denote $S_p(X)$ the Schatten–von Neumann class. Denote $\{s_j(T)\}_{j=1}^{\operatorname{rank} T}$ the set of non-zero singular numbers of $T \in S_\infty(X)$ enumerated in non-increasing order. Recall that $T \in S_p(X)$ if and only if

$$\|T\|_{S_p(X)} := \left(\sum_{j=1}^{\operatorname{rank} T} s_j(T)^p\right)^{1/p}$$

is finite. The class $S_p(X)$ is a Banach space under the norm $\|\cdot\|_{S_p(X)}$. We will also denote $S_{p,w}(X)$ the *weak* Schatten–von Neumann class of order $p \in [1, \infty)$, i.e. the class of operators $T \in S_\infty(X)$ for which the quasinorm

$$\|T\|_{p,w} := \sup_j j^{1/p} s_j(T)$$

is finite. When X is clear by the context, we will omit it in the notations and simply write $\langle \cdot\,, \,\cdot\cdot \rangle$, S_∞, S_p and $\|\cdot\|_p$.

Let $d \geq 1$ and denote $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_+$ we will use the notations $x^\alpha := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, $\partial_x^\alpha := \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \ldots \frac{\partial^{\alpha_d}}{(\partial x_d)^{\alpha_d}}$, also we will denote $\alpha! := (\alpha_1)! \ldots (\alpha_d)!$.

For $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}_+$, we define the binomial coefficient by the formula

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}.$$

Given a real-valued measurable function f, we denote its positive and negative parts

$$f_+ := \frac{|f| + f}{2}, \quad f_- := \frac{|f| - f}{2},$$

so that $f = f_+ - f_-$, with $f_\pm \geq 0$ and $f_+ f_- = 0$.

We denote $\mathbb{1}_E$ the characteristic function of the subset $E \subset \mathbb{R}^d$ and $\mathbb{1}_R$ the characteristic function of the ball of radius R centered at 0. We will also use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Given a self-adjoint operator A acting in a Hilbert space X, we denote its spectrum $\sigma(A)$, its resolvent set $\rho(A)$, and its essential spectrum $\sigma_{\mathrm{ess}}(A)$. Set

$$\mathcal{N}_{\mathcal{I}}(A) := \operatorname{rank} \mathbb{1}_{\mathcal{I}}(A),$$

where $\mathbb{1}_{\mathcal{I}}(A)$ denotes the spectral projector corresponding to the interval $\mathcal{I} \subset \mathbb{R}$ associated to A. It follows that if $\mathcal{I} \cap \sigma_{\mathrm{ess}}(A) = \emptyset$, then $\mathcal{N}_{\mathcal{I}}(A)$ is equal to the number of eigenvalues of A lying on $\mathcal{I}$, counted with multiplicities. If A is compact, we introduce also the notation

$$n_\pm(\lambda; A) = \mathcal{N}_{(\lambda,\infty)}(\pm A), \quad \lambda > 0.$$

Chapter 1

Introduction

1.1 Magnetic Quantum Hamiltonians

1.1.1 The Schrödinger Equation

According to the quantum mechanical description, to a given physical system we can associate a complex Hilbert space, the one-dimensional subspaces of which are in correspondence with the possible states of the system, and the self-adjoint operators of which are in correspondence with the observable quantities. A "rule of quantization" establishes a correspondence between a classical description of a system and its quantum mechanical description. It associates a Hilbert space to the system and provides a way of mapping observables to self-adjoint operators acting in this Hilbert space. The canonical quantization for a spinless one-particle system establishes the correspondence between such a system and the Hilbert space $L^2(\mathbb{R}^d_x)$, along with the rule of quantization

$$\begin{array}{rcl} q & \mapsto & x \\ p & \mapsto & -i\hbar\nabla \end{array} \tag{1.1.1}$$

assigning self-adjoint operators to the position and momentum observables, which we denote by q and p. Here,

$$\hbar = 1.054\ 571\ 726 \times 10^{-34}\ [J \cdot s]$$

is the reduced Planck constant[1] and the x_k, $k = 1, \dots, d$, denote the operators of multiplication by x_k in $L^2(\mathbb{R}^d)$. In accordance with the spectral theorem, any observable $f(q)$ depending on the position only will be mapped to its corresponding multiplication operator $f(x)$.

The dynamics of the system are governed by a (possibly unbounded) self-adjoint linear operator H acting in this Hilbert space, called the *Hamiltonian* of the system and commonly interpreted as the energy observable. More explicitly, if $\Psi_0 \in \operatorname{Dom} H$ is a vector determining the state of the system at time $t = 0$, the dynamics of the system are described by the equation

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t}\Psi(t) = H\Psi(t) \\ \Psi(0) = \Psi_0, \end{cases} \tag{1.1.2}$$

so that the time evolution of the system is given by

$$\Psi(t) = e^{-itH/\hbar}\Psi_0. \tag{1.1.3}$$

Relation (1.1.2) is called the (time-dependent) Schrödinger equation. This equation plays an important role in modern physics and provides relevant problems and motivation in several mathematical domains, among which spectral theory and microlocal analysis.

For a classical particle in d-dimensional space, $d \in \mathbb{N}$, with mass $m = 1/2$, charge $e = 1$ and momentum p, the energy at position $q \in \mathbb{R}^d$ is given by the expression

$$\mathcal{H}(p, q) = (p - A(q))^2 + V(q), \tag{1.1.4}$$

where A and V are respectively the magnetic vector potential and the electric potential of the electromagnetic field. We are led to a Hamiltonian given by the *Schrödinger operator*

$$H(A, V) = (-i\nabla - A)^2 + V. \tag{1.1.5}$$

Here and in the sequel the choice of units is such that $\hbar = 1$. This thesis is devoted to the spectral properties of Schrödinger operators (1.1.5) with constant magnetic fields.

Under the conditions that $A \in L^4_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, $\operatorname{div} A \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R})$, $V = V_1 - V_2$, with $V_1 \geq -c|x|^2$, $V_1 \in L^2_{\text{loc}}(\mathbb{R}^d)$, $c > 0$, and $V_2 \geq 0$ is Δ-bounded with relative bound smaller than 1, the above operator initially defined on $C_0^\infty(\mathbb{R}^d)$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$ (see [46]). We denote $H(A, V)$ the self-adjoint realization of the formal operator (1.1.5).

[1] Source: 2010 CODATA recommended values, http://physics.nist.gov/cgi-bin/cuu/Value?hbar.

1.1.2 Quantum Hamiltonians in Magnetic Fields

The splitting of degenerate energy levels in a quantum system by the introduction of perturbations is an important phenomenon in quantum mechanics. In many situations, knowledge concerning the way these energy levels split can be used to devise precise measurement techniques and to obtain information about the original system or about the perturbation. For instance, observation of the Zeeman effect (splitting of the atomic spectral lines under the presence of a constant magnetic field) in spectroscopic measurements of the sun allowed for the discovery of strong magnetic fields in sunspots.

Other such effects are, for instance, the Stark effect, in which the atomic degeneracy is removed by the presence of an electric field; nuclear magnetic resonance, a particular case of the Zeeman effect, in which the nuclear spin degeneracy of an atom splits under the presence of a magnetic field; and electronic paramagnetic resonance, an analogous phenomenon occurring in electrons and particles with non-zero internal spin exposed to magnetic fields. The importance of these phenomena is underlined by the fact that the Nobel Prize in physics was awarded to Hendrik A. Lorentz and Pieter Zeeman (1902), to Johannes Stark (1919) and to Isidor Isaac Rabi (1944) for the discovery of these effects.

There are also several other important quantum phenomena related to magnetic fields, not necessarily involving the splitting of degenerate energy levels. A prominent example is the quantum Hall effect: the discretization of the conductance in the Hall effect when it takes place at low temperatures and under strong magnetic fields. Its discovery earned Klaus von Klitzing the Nobel Prize in physics in 1985.

Applications are numerous and important. Other than the direct measurement of intrinsic physical quantities of systems or, inversely, the measurement of the intensity of the perturbation, these effects and their consequences have been used extensively in molecular physics, in chemistry and in medicine. Notably, MRI (magnetic resonance imaging) is based on nuclear magnetic resonance. By varying the magnetic field intensity it is possible to reconstruct images of the body's soft tissues. The trapping property of constant magnetic fields is also used to a great extent in applications.

Beyond its purely mathematical interest, it is conceivable for the results concerning perturbations of the Landau Hamiltonian to have such applications. For example, precise spectral measurement of highly excited particles subject to a constant magnetic field could allow to describe the profile (resp. asymptotic profile) of a short-range (resp. long-range) electric field, in accordance with the results in Chapter 2 and in [55] described below.

Gauge Invariance of the Schrödinger Operator

Let $\mathcal{A} := \sum_{j=1}^{d} A_j dx^j$ be a 1-form in $\mathbb{R}^d$. Then, the 2-form

$$\mathcal{B} := d\mathcal{A} = \frac{1}{2}\sum_{j,k=1}^{d} B_{jk}\, dx^j \wedge dx^k,$$

where

$$B_{jk} := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}, \quad j,k = 1,\dots,d, \tag{1.1.6}$$

can be naturally identified to the anti-symmetric matrix $B = \operatorname{curl} A := \{B_{jk}\}_{j,k=1}^{d}$. Similarly, to a given magnetic vector potential $A = (A_1,\dots,A_d) \in L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, we can associate a *magnetic field* via the expression (1.1.6).

The following theorem reflects the fact that in $\mathbb{R}^d$ (and, more generally, in simply connected domains), given a magnetic field B, the particular choice of $A \in L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\operatorname{curl} A = B$ leaves the spectrum of $H(A,V)$ invariant, and has no physical meaning in the sense that it cannot be distinguished by experiment.

We say that $A^{(1)}$ and $A^{(2)}$, vector potentials in $L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, are gauge equivalent if $\operatorname{curl} A^{(1)} = \operatorname{curl} A^{(2)}$. Equivalently, there exists a real-valued ϕ such that

$$A^{(1)} = A^{(2)} + \nabla\phi. \tag{1.1.7}$$

Theorem 1.1.1. *Let $A^{(j)} \in L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} A^{(j)} \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R})$, $j = 1,2$ be gauge equivalent. Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and suppose V_- is Δ-bounded with relative bound smaller than 1. Then, the operators $H(A^{(j)},V)$, $j = 1,2$, are gauge unitarily equivalent, i.e.*

$$e^{-i\phi} H(A^{(1)},V) e^{i\phi} = H(A^{(2)},V). \tag{1.1.8}$$

Let us comment on the above theorem: (i) Given a vector potential $A^{(1)} \in L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, there exists a gauge equivalent $A^{(2)} \in L^4_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ verifying $\operatorname{div} A^{(2)} = 0$. This can be used to show that the condition $\operatorname{div} A \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R})$ is not essential to define the Schrödinger operator (1.1.5). (ii) In the more general context of Schrödinger operators defined on manifolds M that are not simply connected it is no longer true that

$\operatorname{curl} A^{(1)} = \operatorname{curl} A^{(2)}$ implies the unitary equivalence of the respective Hamiltonians. In such a case, a notion of gauge invariance not completely determined by the magnetic field can be established, so that equivalence classes of vector potentials have a distinguishable physical meaning. For a more detailed discussion on the subject the reader is referred, for example, to [45].

Diamagnetic Inequality

Let $T, S \in \mathcal{L}(L^2(\mathbb{R}^d))$. We say $T \underset{\cdot}{\leq} S$ if

$$|Tf(x)| \leq S|f|(x)$$

for almost every $x \in \mathbb{R}^d$. We have the following theorem, due to Dodds, Fremlin and Pitt, establishing the preservation of compactness under this relation:

Theorem 1.1.2. **[53]** *Let $T \underset{\cdot}{\leq} S$. Then, $S \in S_\infty$ implies $T \in S_\infty$ and $S \in S_{2p}$ implies $T \in S_{2p}$, $p \in \mathbb{N}$.*

Theorem 1.1.3. *Let $A \in L^4_{\mathrm{loc}}(\mathbb{R}^d)$, $\operatorname{div} A \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R})$. Then, for any $t > 0$,*

$$e^{-tH(A,0)} \underset{\cdot}{\leq} e^{t\Delta} \tag{1.1.9}$$

and hence

$$(H(A,0) + E)^{-\gamma} \underset{\cdot}{\leq} (-\Delta + E)^{-\gamma} \tag{1.1.10}$$

for any $E > 0$ and $\gamma > 0$. Further, suppose now V is Δ-bounded with relative bound smaller than 1, then $H(A, V)$ is self-adjoint on $\operatorname{Dom} H(A, 0)$ and

$$e^{-t(H(A,V))} \underset{\cdot}{\leq} e^{-t(-\Delta+V)}. \tag{1.1.11}$$

Let us state some general conditions on V under which the operator $V(-\Delta + E)^{-1}$ is compact and some consequences of this fact. Suppose $V \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, with $p > d/2$, $p \geq 2$, and that $\lim_{|x|\to\infty} V(x) = 0$. For $\epsilon > 0$, let $V = V_{1,\epsilon} + V_{2,\epsilon}$, with $V_{1,\epsilon} \in L^p(\mathbb{R}^d)$ compactly supported and with $\|V_{2,\epsilon}\|_{L^\infty(\mathbb{R}^d)} \leq \epsilon$. We have

$$\|V(-\Delta + E)^{-1} - V_{1,\epsilon}(-\Delta + E)^{-1}\| = \|V_{2,\epsilon}(-\Delta + E)^{-1}\| \xrightarrow{\epsilon \downarrow 0} 0, \tag{1.1.12}$$

so we need only prove that $V_{1,\epsilon}(-\Delta + E)^{-1}$ is compact, which follows from [65, Theorem 4.1].

It follows from (1.1.10) that

$$V(H(A,0)+E)^{-1} \stackrel{\le}{\cdot} |V|(-\Delta+E)^{-1}, \quad E>0.$$

Theorem 1.1.2 then implies that $V(H(A,0)+E)^{-1} \in S_\infty$. In particular, the operator $H(A,V) = H(A,0)+V$ is self-adjoint on $\operatorname{Dom} H(A,0)$ and the resolvent difference

$$(H(A,0)+E)^{-1} - (H(A,V)+E)^{-1} = (H(A,V)+E)^{-1} V (H(A,0)+E)^{-1},$$

with $E>0$, $-E \in \rho(H(A,0)) \cap \rho(H(A,V))$, is compact. As a consequence of this, we have

$$\sigma_{\rm ess}(H(A,0)) = \sigma_{\rm ess}(H(A,V)), \tag{1.1.13}$$

by Weyl's theorem (see [10, Section 9.1, Theorem 4]). For further details on the diamagnetic inequality, the reader may consult [6].

1.2 The Landau Hamiltonian and its Perturbation Theory

1.2.1 Schrödinger Operators with Constant Magnetic Fields

Given a constant magnetic field, the matrix B with entries $B_{jk} := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \in \mathbb{R}$ is anti-symmetric. Let $b_1 \ge b_2 \ge \cdots \ge b_m > 0$ be such that $\pm i b_k$, $k = 1, \ldots, m$, are the non-zero eigenvalues of B and set $n := d - 2m = \dim \operatorname{Ker} B$. Then, there exist cartesian coordinates $(x,y,w) \in \mathbb{R}^d$, with $x, y \in \mathbb{R}^m$, $w \in \mathbb{R}^n$, if $n \ge 1$, or coordinates $(x,y) \in \mathbb{R}^d$, $x, y \in \mathbb{R}^m$, if $n = 0$, such that H defined in (1.1.5) is unitarily equivalent to the operator

$$H := \sum_{k=1}^{m} \left[\left(-i \frac{\partial}{\partial x_k} + b_k \frac{y_k}{2} \right)^2 + \left(-i \frac{\partial}{\partial y_k} - b_k \frac{x_k}{2} \right)^2 \right] - \sum_{j=1}^{n} \frac{\partial^2}{\partial {w_j}^2} + V(x,y,w), \tag{1.2.1}$$

defined initially in $C_0^\infty(\mathbb{R}^d)$ and closed in $L^2(\mathbb{R}^d)$. If $n = 0$, the sum with respect to j and the dependence of V on w should be omitted.

In view of the above, a special role is played by the operator

$$H_0 := \left(-i \frac{\partial}{\partial x_1} + b \frac{x_2}{2} \right)^2 + \left(-i \frac{\partial}{\partial x_2} - b \frac{x_1}{2} \right)^2, \tag{1.2.2}$$

defined in $L^2(\mathbb{R}^2)$, with $b > 0$, for which $m = 1$ and $n = 0$. We will also use the notation $A_0 = b/2(-x_2, x_1)$, so that $H_0 = (-i\nabla - A_0)^2$. This operator is called the Landau Hamiltonian and will be our central object of study.

The spectral properties of operators (1.2.1) differ drastically in the cases $n = 0$ (i.e. magnetic field of full rank) and $n \geq 1$. Let us assume $V = 0$ and compare the spectra of H in these two cases. If $n = 0$, then $\sigma(H)$ coincides with the set of values

$$\left\{ \sum_{k=1}^{m} \Lambda_{q_k}(b_k) \;\middle|\; q_k \in \mathbb{Z}_+,\ k = 1, \dots, m \right\},$$

where $\Lambda_q(b) := b(2q + 1)$. If $n \geq 1$, then

$$\sigma(H) = \left[\sum_{k=1}^{m} b_k, \infty \right).$$

The spectrum is purely essential in both cases, but for $n = 0$ it is purely point spectrum whereas for $n \geq 1$ it is purely absolutely continuous.

Next, we consider in more detail the spectral properties of the Landau Hamiltonian. Set $z := x_1 + ix_2$ and $\overline{z} := x_1 - ix_2$, so that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \right).$$

Define the magnetic annihilation operator

$$a := -2ie^{-b|x|^2/4} \frac{\partial}{\partial \overline{z}} e^{b|x|^2/4} = -2i\left(\frac{\partial}{\partial \overline{z}} + \frac{bz}{4} \right),$$

and the magnetic creation operator

$$a^* := -2ie^{b|x|^2/4} \frac{\partial}{\partial z} e^{-b|x|^2/4} = -2i\left(\frac{\partial}{\partial z} - \frac{b\overline{z}}{4} \right),$$

with common domain $\operatorname{Dom} a = \operatorname{Dom} a^* = \operatorname{Dom} H_0^{1/2}$. They are closed and mutually adjoint in $L^2(\mathbb{R}^2)$ and verify the commutation relation

$$[a, a^*] = 2b \tag{1.2.3}$$

on $\operatorname{Dom} H_0$. Hence, they define a representation of the Heisenberg algebra. This fact together with the relation

$$H_0 = a^* a + b$$

allow us to build a spectral decomposition for the Landau Hamiltonian, similarly to the case of the harmonic oscillator.

The spectrum of H_0 consists of eigenvalues of infinite multiplicity given by

$$\Lambda_q := b(2q+1), \quad q \in \mathbb{Z}_+, \tag{1.2.4}$$

called the Landau levels. We have

$$\begin{aligned} \operatorname{Ker}(H_0 - \Lambda_0) &= \operatorname{Ker} a^* a \\ &= \operatorname{Ker} a = \left\{ u \in L^2(\mathbb{R}^2) \mid u = g e^{-\frac{b}{4}|x|^2},\ g \in C^1(\mathbb{R}^2),\ \partial g / \partial \overline{z} = 0 \right\}, \end{aligned} \tag{1.2.5}$$

an orthonormal eigenbasis being given by the functions

$$\varphi_{0,k}(x) := \sqrt{\frac{b}{2\pi}} \frac{1}{\sqrt{k!}} \left[\sqrt{\frac{b}{2}} (x_1 + i x_2) \right]^k e^{-\frac{b}{4}|x|^2}, \qquad x = (x_1, x_2) \in \mathbb{R}^2,\ k \in \mathbb{Z}_+.$$

Commutation relation (1.2.3) implies

$$\operatorname{Ker}(H_0 - \Lambda_q) = (a^*)^q \operatorname{Ker} a, \quad q \geq 1, \tag{1.2.6}$$

so that

$$\begin{aligned} \varphi_{q,k}(x) &:= \frac{(-i)^q}{\sqrt{(2b)^q\, q!}} (a^*)^q \varphi_{0,k}(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \\ &= (-1)^q \sqrt{\frac{b}{2\pi}} \sqrt{\frac{q!}{k!}} \left[\sqrt{\frac{b}{2}} (x_1 + i x_2) \right]^{k-q} \mathrm{L}_q^{(k-q)}\big(\frac{b}{2}|x|^2\big) e^{-\frac{b}{4}|x|^2}, \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{1.2.7}$$

is an orthonormal basis for $\operatorname{Ker} H_0 - \Lambda_q$. Here

$$\mathrm{L}_q^{(\alpha)}(t) := \frac{1}{q!} t^{-\alpha} e^t \frac{d^q (t^{q+\alpha} e^{-t})}{dt^q} = \sum_{j=0}^{q} \binom{q+\alpha}{q-j} \frac{(-t)^j}{j!}, \quad t \in \mathbb{R},\ q \in \mathbb{Z}_+,\ \alpha \in \mathbb{R},$$

are the generalized Laguerre polynomials. Notice that if $k < q$, $\mathrm{L}_q^{(k-q)}$ has a zero of order $q-k$ at $x=0$, so that $\varphi_{q,k}$ extends continuously to a function in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. These are sometimes referred to as the *angular momentum eigenbases*. Denote by P_q, $q \in \mathbb{Z}_+$, the orthogonal projection onto $\operatorname{Ker} H_0 - \Lambda_q$. A unitary transformation between $P_0 L^2(\mathbb{R}^2)$ and $P_q L^2(\mathbb{R}^2)$ is given by $\varphi_{0,k} \mapsto \varphi_{q,k}$. More precisely, if

$$u = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{0,k} \in P_0 L^2(\mathbb{R}^2), \quad v = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{q,k} \in P_q L^2(\mathbb{R}^2),$$

then the application mapping $u \mapsto v$ is a unitary transformation between the spaces $P_0 L^2(\mathbb{R}^2)$ and $P_q L^2(\mathbb{R}^2)$.

1.2.2 Perturbations of the Landau Hamiltonian

The Landau Hamiltonian is a very good example of the quantization[2] of physical quantities in quantum mechanics, one of its most distinctive features. The energy measured on a system described by this Schrödinger operator cannot take any arbitrary value, it is restricted to the Landau levels. This interesting spectral property has been the starting point for active research since the 1980s, as the natural question arises: what happens to the energy levels when this system is perturbed. Throughout the rest of this section we will denote by H_0 the Landau Hamiltonian defined in (1.2.2) and by H the corresponding perturbed operator.

Let $V \in L^2_{\text{loc}}(\mathbb{R}^2)$ verify $\lim_{|x|\to\infty} V(x) = 0$. Under these hypotheses, we have that V is relatively compact with respect to H_0, so that the essential spectra of $H = H_0 + V$ and H_0 coincide.

In the case of magnetic perturbations, the preservation of the essential spectrum can be a more delicate matter. If the perturbing magnetic field decays slowly at infinity, the perturbation might not be relatively compact with respect to H_0. However, in 1983, A. Iwatsuka proved (see [39]) that if the components of the perturbing magnetic field decay at infinity and belong to $C^\infty(\mathbb{R}^2)$, then the essential spectra of H_0 and the perturbed operator coincide.

Chapter 3 concerns metric perturbations. These are second-order differential operators, so that, in general terms, even if the perturbation decays at infinity, it is not relatively compact with respect to H_0. Nonetheless, the resolvent difference continues to be compact. The reader is referred to Section 3.7 in Chapter 3 for further details concerning this issue.

Having considered this, a first question to be asked regards the nature of the Landau levels as points in the spectrum of the perturbed operator. In principle, a given Landau level Λ_q, being an isolated point of the essential spectrum of H, could be:

- an eigenvalue of H of infinite multiplicity;
- an accumulation point of the discrete spectrum;

[2]Quantization meaning in this context that physical quantities can only take values on a discrete subset of $\mathbb{R}$.

- both.

This issue is addressed by G. Raikov and F. Klopp in [41] in the case of electric perturbations of the Landau Hamiltonian. It turns out that if the perturbation V is sign-definite, and verifies

$$c\mathbb{1}_r(x) \leq |V(x)|, \quad x \in \mathbb{R}^2,$$

for some $c > 0$, $r > 0$, then, under the condition that

$$\|V\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} |V(x)| < 2b,$$

the Landau levels Λ_q are not eigenvalues of H, and hence are accumulation points of the discrete spectrum. On the other hand, given a Landau level Λ_q, there exists a V (with non-constant sign) for which Λ_q is an eigenvalue of infinite multiplicity of H.

1.2.3 Eigenvalue Clusters and High-Energy Asymptotics

This section concerns mainly electric perturbations of the Landau Hamiltonian. Let us examine the operator

$$H := H_0 + V,$$

where the electric potential V is bounded, measurable, and verifies $\lim_{|x|\to\infty} V(x) = 0$. Given the considerations in Subsection 1.2.2, the essential spectra of H and H_0 coincide. Further, we have

$$\sigma(H) \subset \bigcup_{q=0}^{\infty} \left[\Lambda_q + \inf V, \Lambda_q + \sup V\right], \tag{1.2.8}$$

so that if $\|V\|_\infty < b$, the spectrum of the perturbed operator is contained in clusters of width $\|V\|_\infty$ centered around the Landau levels. This fact supports the idea that the Landau levels "split" under perturbations.

Rather than describing the actual distribution of discrete eigenvalues in any such cluster (which amounts to locating precisely the spectrum of H) it is possible to consider the high-energy limit, as $q \to \infty$. A classical result in this direction is A. Weinstein's paper [70], in which such a problem is considered for electric perturbations of the Laplace–Beltrami operator on compact Riemannian manifolds M with periodic geodesics of equal length, i.e. the operator $h := -\Delta_M + V$, $V \in C^\infty(M)$, self-adjoint in $L^2(M, d\mu)$, with μ the measure induced in M by the Riemannian metric.

We illustrate Weinstein's results for $M = \mathbb{S}^2$, the 2-sphere. The spectrum of $h_0 := -\Delta_{\mathbb{S}^2}$ is given by the eigenvalues $\nu_q := q(q+1)$, $q \in \mathbb{Z}_+$, of multiplicity $m_q := 2q+1$ (see [64, Theorem 22.1]). Since the eigenvalues space out as $q \to \infty$, an estimate analogous to (1.2.8) justifies the idea of eigenvalue clusters for the operator h.

For $[\alpha, \beta] \subset \mathbb{R}$ and $q \in \mathbb{Z}_+$ introduce eigenvalue counting measures by setting

$$\mu_q([\alpha, \beta]) := \operatorname{rank} \mathbb{1}_{[\alpha,\beta]}(h - \nu_q),$$

where $\mathbb{1}_{\mathcal{I}}(h - \nu_q)$ is the spectral projector onto the interval $\nu_q + \mathcal{I}$ corresponding to the self-adjoint operator h. Weinstein's results read as follows: given $u \in C(\mathbb{S}^2)$, introduce the function

$$\bar{u}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} u(\gamma(t; \omega))\, dt, \quad \omega \in \mathbb{S}^2,$$

where $\gamma(\,\cdot\,; \omega)$ is the oriented great circle in $\mathbb{S}^2$, orthogonal to ω. Note that the great circles are exactly the geodesic curves in $\mathbb{S}^2$ parametrized by the arc length t. Then,

$$\lim_{q\to\infty} m_q^{-1} \int_{\mathbb{R}} \varphi(\lambda)\, d\mu_q(\lambda) = \int_{\mathbb{S}^2} \varphi(\bar{V}(\omega))\, d\sigma(\omega), \quad \varphi \in C^\infty(\mathbb{R}), \tag{1.2.9}$$

where σ is the normalized Lebesgue measure on $\mathbb{S}^2$. For $[\alpha, \beta] \subset \mathbb{R}$ define the measure

$$\mu_\infty([\alpha, \beta]) := \sigma\left(\bar{V}^{-1}([\alpha, \beta])\right).$$

Equation (1.2.9) states that μ_∞ is the weak limit of the measures $m_q^{-1}\mu_q$.

Weinstein's results are valid in a more abstract setting. Namely, let A be a positive, self-adjoint, first-order elliptic pseudodifferential operator on an n-dimensional compact manifold such that $e^{2\pi i A} = cI$, for some $c \in \mathbb{R}$, and such that its bicharacteristic flow is simply periodic with period 2π. Let also B be a self-adjoint pseudodifferential operator of order zero. Then, the eigenvalues of $A^2 + B$ (which cluster around the eigenvalues of A^2, in a sense analogous to (1.2.8)) can be well approximated, as the cluster number q goes to infinity, by those of $A^2 + \langle B \rangle$, where

$$\langle B \rangle := \frac{1}{2\pi} \int_0^{2\pi} e^{-itA} B e^{itA}\, dt$$

is the Weinstein average of B. The operators A^2 and $\langle B \rangle$ can be jointly diagonalized, which simplifies the asymptotic analysis of the eigenvalue counting measures, yielding a limiting distribution as $q \to \infty$.

The structure of eigenvalue clusters for electric perturbations of the Landau Hamiltonian was studied for the first time in its high-energy limit by E. Korotyaev and A. Pushnitski in [42]. A more thorough description including the asymptotic distribution of eigenvalues within clusters was later given by A. Pushnitski, G. Raikov and C. Villegas–Blas in [55]. Chapter 2 extends these results to a supplementary class of potentials.

The following theorem establishes the notion of eigenvalue clusters for potentials V verifying

$$|V(x)| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^2, \tag{1.2.10}$$

for some $\rho > 0$, $C > 0$. Moreover, it provides estimates for the size of these clusters as $q \to \infty$.

Theorem 1.2.1. *(i) Assume that V satisfies* (1.2.10) *with $\rho > 1$. Then*

$$\sigma(H) \subset \bigcup_{q\in\mathbb{Z}_+} \left(\Lambda_q - C_1\Lambda_q^{-1/2}, \Lambda_q + C_1\Lambda_q^{-1/2}\right) \tag{1.2.11}$$

with a constant $C_1 > 0$ independent of q.

(ii) Assume that V satisfies (1.2.10) *with $\rho \in (0,1)$. Then*

$$\sigma(H) \subset \bigcup_{q\in\mathbb{Z}_+} \left(\Lambda_q - C_2\Lambda_q^{-\rho/2}, \Lambda_q + C_2\Lambda_q^{-\rho/2}\right) \tag{1.2.12}$$

with a constant $C_2 > 0$ independent of q.

Notice the fundamental difference between the cases $\rho > 1$ and $\rho \in (0,1)$, in which an explicit dependence on ρ appears in the cluster size estimates. The critical case $\rho = 1$, which is not covered by either of these results, establishes the splitting point between two different regimes. Namely, potentials verifying (1.2.10) with $\rho > 1$, described as being *short-range*, are considered in [55], whereas Chapter 2 concerns *long-range* potentials which decay at infinity as $|x|^{-\rho}$ with $\rho \in (0,1)$.

Let us now describe the aforementioned limiting eigenvalue distribution for short-range potentials. For $[\alpha,\beta] \subset \mathbb{R}\setminus\{0\}$ and $q \in \mathbb{Z}_+$ define eigenvalue counting measures by setting

$$\mu_q^{\text{short}}([\alpha,\beta]) := \sum_{\Lambda_q+\alpha\Lambda_q^{-1/2}\leq\lambda\leq\Lambda_q+\beta\Lambda_q^{-1/2}} \dim\operatorname{Ker}(H-\lambda),$$

so that the above quantity is simply the number of eigenvalues in the interval $[\Lambda_q + \alpha\Lambda_q^{-1/2}, \Lambda_q + \beta\Lambda_q^{-1/2}]$ counted with multiplicities. Notice that the interval $[\alpha, \beta]$ is rescaled in accordance with (1.2.11). For large values of q, we have that

$$[\Lambda_q + \alpha\Lambda_q^{-1/2},\ \Lambda_q + \beta\Lambda_q^{-1/2}] \cap \sigma_{\text{ess}}(H) = \emptyset,$$

so the above quantity is finite. Note that if $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, then

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu_q^{\text{short}}(\lambda) = \text{Tr } \varphi(\Lambda_q^{1/2}(H - \Lambda_q)). \tag{1.2.13}$$

This description in terms of traces of operators is the object of study in [55], for which asymptotics are calculated to obtain a limiting measure as $q \to \infty$.

In order to state their results, given V satisfying (1.2.10) with $\rho > 1$, define its *Radon transform*

$$\tilde{V}(\omega, s) := \frac{1}{2\pi} \int_{\mathbb{R}} V(s\omega + t\omega^\perp) dt, \quad \omega = (\omega_1, \omega_2) \in \mathbb{S}^1,\ s \in \mathbb{R},$$

where $\omega^\perp = (-\omega_2, \omega_1) \in \mathbb{S}^1$.

Theorem 1.2.2. *Let V satisfy* (1.2.10) *for some $\rho > 1$. Then*

$$\lim_{q\to\infty} \Lambda_q^{-1/2} \operatorname{Tr} \varphi(\Lambda_q^{1/2}(H - \Lambda_q)) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \varphi(b\tilde{V}(\omega, s))\, ds\, d\omega \tag{1.2.14}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Condition (1.2.10) with $\rho > 1$ entails the following decay property:

$$|\tilde{V}(\omega, s)| \leq C(1 + |s|)^{1-\rho}, \quad \omega \in \mathbb{S}^1,\ s \in \mathbb{R}, \tag{1.2.15}$$

so that the r.h.s of (1.2.14) is a finite quantity for all $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

For $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ define the measure

$$\mu_\infty^{\text{short}}([\alpha, \beta]) := \frac{1}{2\pi} \left|\tilde{V}^{-1}([b^{-1}\alpha, b^{-1}\beta])\right|_{\mathbb{S}^1 \times \mathbb{R}},$$

$|\cdot|_{\mathbb{S}^1\times\mathbb{R}}$ being the Lebesgue measure on $\mathbb{S}^1 \times \mathbb{R}$. In terms of convergence of measures, Theorem 1.2.2 is equivalent to

$$\lim_{q\to\infty} \Lambda_q^{-1/2} \mu_q^{\text{short}}([\alpha, \beta]) = \mu_\infty^{\text{short}}([\alpha, \beta]), \tag{1.2.16}$$

for any α, β such that $\alpha\beta > 0$ and $\mu_\infty^{\text{short}}(\{\alpha\}) = \mu_\infty^{\text{short}}(\{\beta\}) = 0$. However, the above condition does not generically hold. Indeed, it is possible for V to be such that its Radon transform is constant and non-vanishing over a subset of $\mathbb{S}^1 \times \mathbb{R}$ of positive Lebesgue measure, which equivalently means that $\mu_\infty^{\text{short}}$ has an atom.

Again, the Berezin–Toeplitz operators play a significant role in the proof. The analysis of the trace of the operator $\varphi(\Lambda_q^{1/2}(H - \Lambda_q))$ is reduced using Stone–Weierstrass and the Cauchy integral formula to that of operators of the form $(P_q V P_q)^\ell$ for ℓ large enough. The unitary equivalence of Brerezin–Toeplitz operators to certain pseudodifferential operators is employed to obtain an approximate integral representation of these traces which is then evaluated using the stationary phase method to obtain (1.2.14).

Chapter 2 concerns itself in detail with the long-range regime, let us briefly overview the results. We assume that the potential V is asymptotically homogeneous of order $-\rho$, $\rho > 0$, in the sense that $V \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ and that there exists $\mathbb{V} \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, homogeneous of order $-\rho$, such that

$$|V(x) - \mathbb{V}(x)| \le C|x|^{-\rho-\varepsilon}, \quad |x| > 1, \tag{1.2.17}$$

with given constants C and $\varepsilon > 0$. Similarly to the short-range case, we define for $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{Z}_+$ the corresponding eigenvalue counting measures

$$\mu_q^{\text{long}}([\alpha, \beta]) := \sum_{\Lambda_q + \alpha\Lambda_q^{-\rho/2} \le \lambda \le \Lambda_q + \beta\Lambda_q^{-\rho/2}} \dim \operatorname{Ker}(H - \lambda),$$

differing from the above in the scaling factor $\Lambda_q^{-\rho/2}$ applied to the interval. An expression analogous to (1.2.13) in terms of traces of operators is readily obtained.

Define the *mean-value transform* of $\mathbb{V}$ via the expression

$$\mathring{\mathbb{V}}(x) := \frac{1}{2\pi} \int_{\mathbb{S}^1} \mathbb{V}(x - \omega)\, d\omega, \quad x \in \mathbb{R}^2 \setminus \mathbb{S}^1.$$

Theorem 1.2.3. *Let V satisfy* (1.2.17). *Then*

$$\lim_{q \to \infty} \Lambda_q^{-1} \operatorname{Tr} \varphi(\Lambda_q^{\rho/2}(H - \Lambda_q)) = \frac{1}{2\pi b} \int_{\mathbb{R}^2} \varphi(b^\rho \mathring{\mathbb{V}}(x))\, dx \tag{1.2.18}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Notice that the mean-value transform of $\mathbb{V}$ extends continuously to all of $\mathbb{R}^2$, is bounded, and decays at infinity. In particular, the r.h.s of (1.2.18) is finite. Also, $\mathring{\mathbb{V}}(x) = 0$ for each $x \in \mathbb{R}^2$ if and only if $\mathbb{V}(x) = 0$ for each $x \in \mathbb{R}^2 \setminus \{0\}$.

Again, defining the measure

$$\mu_\infty^{\mathrm{long}}([\alpha,\beta]) := \frac{1}{2\pi b}\left|\mathring{\mathbb{V}}^{-1}([b^{-\rho}\alpha, b^{-\rho}\beta])\right|_{\mathbb{R}^2}, \quad [\alpha,\beta]\subset\mathbb{R}\setminus\{0\},$$

with $|\cdot|_{\mathbb{R}^2}$ the Lebesgue measure on $\mathbb{R}^2$, we find that (1.2.18) is equivalent to

$$\lim_{q\to\infty}\Lambda_q^{-1}\mu_q^{\mathrm{long}}([\alpha,\beta]) = \mu_\infty^{\mathrm{long}}([\alpha,\beta]), \tag{1.2.19}$$

for any α,β such that $\alpha\beta>0$ and such that α and β are not atoms of the measure $\mu_\infty^{\mathrm{long}}$.

Notice that $\operatorname{Tr}\varphi(\Lambda_q^{\rho/2}(H-\Lambda_q))$ is of order Λ_q, while the corresponding short-range expression is of order $\Lambda_q^{1/2}$, so that a discontinuity takes place when passing from one case to the other. The limiting case $\rho=1$ differs from both the short-range and long-range situations. In fact, for, say, V asymptotically homogeneous of order -1, the Radon transform $\tilde{V}$ is not well defined. Further, since $\mathring{\mathbb{V}}$ can generically have a logarithmic singularity at $\mathbb{S}^1$ if $\rho=1$, the support of the limiting measure $\mu_\infty^{\mathrm{long}}$ need not be compact, which would mean that (1.2.12) fails to hold in this case for each $C_2>0$.

As in Weinstein's paper [70], the averaged operator $\langle H\rangle := H_0+\langle V\rangle$, where

$$\langle V\rangle := \frac{b}{\pi}\int_0^{\pi/b} e^{-itH_0}Ve^{itH_0}dt = \sum_{s\in\mathbb{Z}_+} P_sVP_s, \tag{1.2.20}$$

provides a good approximation for H and has the additional advantage that H_0 and $\langle V\rangle$ admit a joint spectral decomposition. In particular, passing to the Weinstein average allows to reduce the problem to the analysis of $\operatorname{Tr}\varphi(\Lambda_q^{\rho/2}P_qVP_q)$, which results in Theorem 1.2.3 by means of considerations which develop on the pseudodifferential representation of Berezin–Toeplitz operators, presented in the following section.

Both of these results admit a semiclassical interpretation. Both transforms appearing on the r.h.s. of Theorems 1.2.2 and 1.2.3 can be obtained by averaging the perturbation V over the orbits of the free dynamics and making the energy of these orbits tend to infinity. Namely, set

$$\mathrm{Av}(V)(x,E) := \frac{1}{2\pi}\int_{\mathcal{S}^1} V\left(x+E^{1/2}b^{-1}\omega'\right)\,d\omega', \quad x\in\mathbb{R}^2, E>0,$$

the average of V over the orbit of energy E centered at x. If V satisfies (1.2.10) with $\rho>1$, we have

$$\lim_{E\to\infty} E^{1/2}\,\mathrm{Av}(V)((s+E^{1/2}b^{-1})\omega,E) = b\tilde{V}(\omega,s), \quad \omega\in\mathbb{S}^1,\ s\in\mathbb{R}.$$

Similarly, for V verifying (1.2.17), we have

$$\lim_{E\to\infty} E^{\rho/2}\,\mathrm{Av}(V)(E^{1/2}b^{-1}x,E) = b^{\rho}\mathring{\mathbb{V}}(x), \quad x\in\mathbb{R}^2.$$

Therefore, the limiting measures can be represented by

$$\begin{aligned}\int_{\mathbb{R}}\varphi(\lambda)\,d\mu_{\infty}^{\mathrm{short}}(\lambda) &= \frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{E^{1/2}}\int_{\mathbb{R}^2}\varphi(E^{1/2}\,\mathrm{Av}(V)(x,E))\,b\,dx,\\ \int_{\mathbb{R}}\varphi(\lambda)\,d\mu_{\infty}^{\mathrm{long}}(\lambda) &= \frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{E}\int_{\mathbb{R}^2}\varphi(E^{\rho/2}\,\mathrm{Av}(V)(x,E))\,b\,dx.\end{aligned} \tag{1.2.21}$$

1.2.4 Local Spectral Asymptotics

Under perturbations preserving the essential spectrum, among the first questions to be asked is the one concerning the local behavior of the resulting discrete spectrum. Particularly, the rate of convergence of the discrete eigenvalues to the Landau levels has been the subject of many of the works present in the literature around the Landau Hamiltonian today. The first such result was published by G. Raikov in [58], it concerns electric perturbations with power-like decay at infinity.

Recall $\langle x\rangle := (1+|x|^2)^{1/2}$, $x\in\mathbb{R}^2$. For $\rho\in\mathbb{R}$ define the Hörmander class

$$\mathcal{S}^{-\rho}(\mathbb{R}^2) := \{u\in C^{\infty}(\mathbb{R}^2)\mid \sup_{x\in\mathbb{R}^2}|\partial_x^{\alpha}u(x)|\langle x\rangle^{\rho+|\alpha|}<\infty,\ \alpha\in\mathbb{Z}_+^2\}.$$

Given $V\in\mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho>0$, let $H := H_0+V$. The results in Subsection 1.2.2 entail

$$\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = \{\Lambda_q\}_{q\in\mathbb{Z}_+}.$$

Denote by $\lambda_{q,k}^{+}$, $k\in\mathbb{Z}_+$, the eigenvalues of H in the interval (Λ_q,Λ_q+b) enumerated in decreasing order and by $\lambda_{q,k}^{-}$, $k\in\mathbb{Z}_+$, the eigenvalues of H in the interval (Λ_q-b,Λ_q) enumerated in increasing order. Also, set

$$\begin{aligned}\mathcal{N}_q^{+}(\lambda) &:= \operatorname{rank}\mathbb{1}_{(\Lambda_q+\lambda,\,\Lambda_q+b)}(H),\\ \mathcal{N}_q^{-}(\lambda) &:= \operatorname{rank}\mathbb{1}_{(\Lambda_q-b,\,\Lambda_q-\lambda)}(H),\end{aligned} \qquad \lambda\in(0,b), \tag{1.2.22}$$

where $\mathbb{1}_{\mathcal{I}}(H)$ is the spectral projector of the self-adjoint operator H onto the interval $\mathcal{I}$. The problem of local spectral asymptotics involves studying asymptotics for the quantities $\pm\left(\lambda_{q,k}^{\pm}-\Lambda_q\right)$, which also relate to those of the eigenvalue counting functions $\mathcal{N}_q^{\pm}$.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\lim_{|x|\to\infty} \psi(x) = 0$ and define the volume functions

$$\Phi_\psi^\pm(\lambda) := \left|\left\{x \in \mathbb{R}^2 \mid \pm\psi(x) > \lambda\right\}\right|, \quad \lambda > 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$.

Theorem 1.2.4. [58, Theorem 2.6] *Let $V \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 0$. Assume that there exists $\lambda_0 > 0$ such that the function $\Phi_V^\pm$ is differentiable on $(0, \lambda_0)$, and*

$$-\lambda \Phi_V^{\pm'}(\lambda) \asymp \Phi_V^\pm(\lambda),$$
$$\Phi_V^\pm(\lambda) \geq C\lambda^{-2/\rho}, \quad \lambda \in (0, \lambda_0),$$

for some constant $C > 0$. Then we have

$$\mathcal{N}_q^\pm(\lambda) = \frac{b}{2\pi}\Phi_V^\pm(\lambda)(1 + o(1)) \asymp \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \quad q \in \mathbb{Z}_+. \tag{1.2.23}$$

A more explicit version of Theorem 1.2.4 can be stated when the electric potential V is asymptotically homogeneous, in the sense that there exist $0 \neq v_\pm \in C^\infty(\mathbb{S}^1)$ such that

$$\lim_{|x|\to\infty} \left(|x|^\rho V_\pm(x) - v_\pm(x/|x|)\right) = 0,$$

with $V_\pm$ the positive and negative parts of V respectively. Then,

$$\mathcal{N}_q^\pm(\lambda) = \mathcal{C}_\pm \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \quad q \in \mathbb{Z}_+ \tag{1.2.24}$$

where

$$\mathcal{C}_\pm := \frac{b}{4\pi}\int_0^{2\pi} v_\pm(\cos\theta, \sin\theta)^{2/\rho}.$$

Equivalently, we have the following asymptotics for the rate of convergence of the discrete eigenvalues:

$$\pm\left(\lambda_{q,k}^\pm - \Lambda_q\right) = \mathcal{C}_\pm^{\rho/2} k^{-\rho/2}(1 + o(1)), \quad k \to \infty. \tag{1.2.25}$$

This result is semiclassical in nature insofar the counting functions $\mathcal{N}_q^\pm$ are expressed in terms of a Weyl formula involving the volume function of the perturbing potential V. In this paper the problem is reduced, for the first time, to the spectral analysis of *Berezin–Toeplitz operators*, $P_q V P_q$, which are also seen to be unitarily equivalent to certain pseudodifferential operators with Weyl symbols. Techniques stemming from micro-local analysis are then employed to obtain the result.

Later, V. Ivrii was able to extend the results in [58] to perturbations of more general type. In [38], combined electric, magnetic and metric perturbations were considered. These results were obtained making use of micro-local machinery.

Consider an electric potential $V \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ and a perturbing magnetic field $B \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 0$, with an associated vector potential $A = (A_1, A_2)$ such that $A + A_0 \in L^4_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R})^2$ and $\mathrm{div}(A + A_0) \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R})$. Consider also real-valued functions $m_{jk} \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $j, k = 1, 2$, such that the matrix g with entries $g_{jk}(x) := \delta_{jk} + m_{jk}(x)$ is positive definite for each $x \in \mathbb{R}^2$. Under these conditions, the operator in $L^2(\mathbb{R}^2)$

$$H := \sum_{j,k=1}^{2} \left(-i\frac{\partial}{\partial x_j} - A_{0,j} - A_j \right) g_{jk}(x) \left(-i\frac{\partial}{\partial x_k} - A_{0,k} - A_k \right) + V, \tag{1.2.26}$$

initially defined on $C_0^\infty(\mathbb{R}^2)$ and closed in $L^2(\mathbb{R}^2)$, is self-adjoint and verifies $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$.

Set $\gamma := (\det g)^{-1/2}$ and $F(x) := \gamma^{-1}(B(x) + b)$, $x \in \mathbb{R}^2$. Fix $q \in \mathbb{Z}_+$ and define $\mathcal{V}_q(x) := V(x) + (2q + 1)(F(x) - b)$, $x \in \mathbb{R}^2$. Under some additional hypotheses the asymptotics [38, Theorem 11.3.17]

$$\mathcal{N}_q^{\pm}(\lambda) = \frac{1}{2\pi} \int_{\{\pm \mathcal{V}_q > \lambda\}} F(x)\gamma(x)\, dx + O(\ln \lambda), \quad \lambda \downarrow 0, \tag{1.2.27}$$

hold true.

Results so far were restricted to perturbations with power-like decay and known methods were unable to deal with such cases as compact support, in which the successive derivatives do not decay faster at infinity than the perturbation itself, as was required in some sense by the semiclassical formalism.

In [59], G. Raikov and S. Warzel were able to find the first asymptotic term for $\mathcal{N}_q$ in the cases of electric perturbations with exponential decay and compact support, albeit with a constant sign. They made use of an explicit description, which involves the angular momentum basis, of the eigenvalues of Berezin–Toeplitz operators for radially symmetric electric perturbations.

Consider the perturbed operator given by $H := H_0 + V$, with $V \geq 0$ bounded and verifying

$$\ln V(x) = -\gamma |x|^{2\beta}(1 + o(1)), \quad |x| \to \infty, \tag{1.2.28}$$

for some $\gamma > 0$, $\beta > 0$. Set $\mu := \gamma(2/b)^\beta$. Then, preserving the previous notations, the asymptotics

$$\ln\left(\lambda_{k,q}^+ - \Lambda_q\right) = \begin{cases} -\mu k^\beta(1+o(1)) & \text{if } \beta \in (0,1), \\ -\left(\ln\left(1+\mu\right)\right) k \left(1+o(1)\right) & \text{if } \beta = 1, \\ -\frac{\beta-1}{\beta} k \ln k \left(1+o(1)\right) & \text{if } \beta \in (1,\infty), \end{cases} \qquad k \to \infty. \tag{1.2.29}$$

hold true. Equivalently, in terms of the eigenvalue counting functions, the results in [59] read as follows:

$$\mathcal{N}_q^+(\lambda) = \begin{cases} \mu^{-1/\beta}|\ln\lambda|^{1/\beta}(1+o(1)) & \text{if } \beta \in (0,1), \\ \frac{1}{\ln\left(1+\mu\right)}|\ln\lambda|(1+o(1)) & \text{if } \beta = 1, \\ \frac{\beta}{\beta-1}\frac{|\ln\lambda|}{\ln|\ln\lambda|}(1+o(1)) & \text{if } \beta \in (1,\infty), \end{cases} \qquad \lambda \downarrow 0. \tag{1.2.30}$$

If $V \geq 0$ is now bounded with compact support and such that $V \geq c > 0$ on some open non-empty subset of $\mathbb{R}^2$, then the asymptotic relation

$$\ln\left(\lambda_{k,q}^+ - \Lambda_q\right) = -k\ln k\left(1+o(1)\right), \quad k \to \infty, \tag{1.2.31}$$

and its equivalent formulation

$$\mathcal{N}_q^+(\lambda) = \frac{|\ln\lambda|}{\ln|\ln\lambda|}(1+o(1)), \quad \lambda \downarrow 0, \tag{1.2.32}$$

describe the local spectral asymptotics.

The $\varphi_{q,k}$ being eigenfunctions for the angular momentum operator $-i(x\partial/\partial y - y\partial/\partial x)$ with eigenvalue $k-q$, it can be seen that if $V(x) = v(|x|)$ is radially symmetric, then they are also eigenfunctions for the Berezin–Toeplitz operator P_qVP_q. An explicit description of the eigenvalues is therefore given by the expression

$$\langle V\varphi_{q,k},\, \varphi_{q,k}\rangle = \frac{q!}{k!}\int_0^\infty v\left((2t/b)^{1/2}\right) e^{-t}t^{k-q}\,\mathrm{L}_q^{(k-q)}(t)^2\,dt, \quad k \in \mathbb{Z}_+. \tag{1.2.33}$$

Asymptotics as $k \to \infty$ for the above expression are calculated for certain model potentials, and the minimax principle then allows to extend the results to more general potentials verifying the stated decay properties.

It is important to notice that these results do not adjust to the semiclassical intuition anymore in some cases. In fact, $\mathcal{N}_q$ and the volume function Φ_V are asymptotically equivalent for $0 < \beta < 1$, but differ when $1 \leq \beta < \infty$ and for the compact support case, having the same order with a different constant if $\beta = 1$ and having different orders altogether if $\beta > 1$. In particular, we see that the volume function of a compactly supported V remains bounded as $\lambda \downarrow 0$, which would imply no accumulation of eigenvalues to the Landau levels, whereas these results show that accumulation does take place at a superexponential rate.

These results were later extended by M. Melgaard and G. Rozenblum in [51] to Schrödinger operators in higher dimensions with full-rank constant magnetic fields.

The description

$$P_0 L^2(\mathbb{R}^2) = \operatorname{Ker} a = \left\{ u \in L^2(\mathbb{R}^2) \mid u = g e^{-\frac{b}{4}|x|^2},\ g \in C^1(\mathbb{R}^2),\ \partial g / \partial \overline{z} = 0 \right\} \qquad (1.2.34)$$

establishes a unitary transformation between the eigenspace $\operatorname{Ker} H_0 - \Lambda_0$ and the space F^2 of entire functions g verifying

$$\int_{\mathbb{C}} |g(z)|^2 e^{-\frac{b}{2}|z|^2} \, dx \, dy < \infty,$$

commonly called Fock or Segal–Bargmann space.

Based on variational arguments and this description applied to Berezin–Toeplitz operators, the complex analysis carried out in [29] by N. Filonov and A. Pushnitski conveyed a second asymptotic term in expression (1.2.31) for electric perturbations, provided that the support of the perturbation is compact with a Lipschitz boundary and that $V(x) \geq c > 0$ for all x in its support. Under these hypotheses, the asymptotics involve more specific properties of the perturbation, namely, they depend on the logarithmic capacity $\mathcal{C}$ of its support. We recall that the logarithmic capacity of a compact non-empty set $K \subset \mathbb{R}^2$ is defined as $\mathcal{C}(K) := e^{-W(K)}$, where $W(K)$ is the minimum of the functional

$$I(\mu) := \int_{K \times K} \ln \frac{1}{|x - y|} d\mu(x) \, d\mu(y)$$

over the set of Baire measures μ supported in K and such that $\mu(K) = 1$ (see [44, Section 4, Chapter II] for further details). Then we have

$$\ln \left(\lambda_{k,q}^{+} - \Lambda_q \right) = -k \ln k + \left(1 + \ln \left(b\mathcal{C}^2 / 2 \right) \right) k + o(k), \quad k \to \infty. \qquad (1.2.35)$$

The question of local spectral asymptotics turns out to be more complicated when considering magnetic perturbations, since such perturbations are never of a definite sign.

This poses rather serious difficulties, the reader is referred to [56] for a discussion on the complications encountered and for some preliminary results.

Despite this fact, in [62], G. Rozenblum and G. Tashchiyan were able to obtain results for compactly supported magnetic perturbations, given certain technical assumptions. They consider the operator

$$H := (-i\nabla - A_0 - A)^2 + V,$$

where the perturbative terms are given by the electric potential V, and the magnetic field B, both compactly supported. Recall that even though B, generated by the magnetic potential A, is compactly supported, it is generally not the case that A itself has a compact support.

The authors managed to reduce the problem to the spectral analysis of a Berezin–Toeplitz operator with an effective potential $W_\pm$ given in terms of some differential operators acting on B and V, depending on both the constant magnetic field intensity b and on the Landau level. Under the condition that the effective potentials verify $W_\pm \geq 0$ and further, $W_\pm \geq c > 0$ on some open subset of $\mathbb{R}^2$, asymptotics for the accumulation of the discrete eigenvalues to the Landau levels both from above and below are obtained:

$$\mathcal{N}_q^\pm(\lambda) = \frac{1}{2}\frac{|\ln\lambda|}{\ln|\ln\lambda|}\left(1 + o(1)\right), \quad \lambda \downarrow 0. \tag{1.2.36}$$

The results that will be presented in Chapter 3 concern metric perturbations of the Landau Hamiltonian. More precisely, let $m(x) = \{m_{jk}(x)\}_{j,k=1,2}$ be a Hermitian positive semi-definite matrix for all $x \in \mathbb{R}^2$. Suppose that the matrix entries $m_{jk} \in C_{\rm b}^\infty(\mathbb{R}^2)$, i.e they are of class C^∞, bounded as well as all their derivatives. Under these conditions, define on $\operatorname{Dom} H_0$ the perturbed operators $H_\pm$, self-adjoint in $L^2(\mathbb{R}^2)$ and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, given by

$$\begin{aligned} H_\pm &:= \sum_{j,k=1}^{2}\left(-i\frac{\partial}{\partial x_j} - A_{0,j}\right)(\delta_{j,k} \pm m_{j,k}(x))\left(-i\frac{\partial}{\partial x_k} - A_{0,k}\right) \\ &= H_0 \pm W, \end{aligned}$$

where $W = \sum_{j,k=1}^{2}(-i\partial/\partial x_j - A_{0,j})\, m_{j,k}(x)\,(-i\partial/\partial x_k - A_{0,k}) \geq 0$ is the perturbative term. In the case of H_-, it is additionally supposed that $\sup_{x\in\mathbb{R}^2}\|m(x)\| < 1$, so that $g^\pm(x) := \{\delta_{jk} \pm m_{jk}(x)\}_{j,k=1,2}$ is positive definite for all $x \in \mathbb{R}^2$.

As mentioned in Subsection 1.2.2 above, the perturbation, being a second-order differential operator, is no longer relatively compact with respect to H_0. However, the resolvent difference $H_\pm^{-1} - H_0^{-1}$ is still a compact operator, so the essential spectra of the operators coincide.

Denote by $m_<(x)$ and $m_>(x)$, $m_<(x) \le m_>(x)$, the two eigenvalues of the matrix $m(x)$, $x \in \mathbb{R}^2$. Since the perturbation has a definite sign, the discrete spectrum can accumulate only from above in the case of H_+ and from below in the case of H_-, so that we consider $\lambda_{k,q}^+$ for H_+ only and $\lambda_{k,q}^-$ for H_- only. The results in Chapter 3 read as follows:

Theorem 1.2.5. *Assume that the support of the matrix m is compact, and its smaller eigenvalue $m_<$ does not vanish identically. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\ln\left(\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right)\right) = -k\ln k + O(k), \quad k \to \infty. \tag{1.2.37}$$

Under certain technical conditions, these asymptotics could be improved by applying the results in [29], yielding the better error estimates

$$\left(1+\ln\left(b\mathcal{C}_<^2/2\right)\right)k + o(k) \le \ln\left(\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right)\right) + k\ln k \le \left(1+\ln\left(b\mathcal{C}_>^2/2\right)\right)k + o(k),$$

with $\mathcal{C}_\gtrless$ the logarithmic capacities of the supports of $m_>$ and $m_<$ respectively.

In the case of exponential decay, asymptotics as $k \to \infty$ are obtained up to order $\ln k$, an improvement with respect to the remainder estimates obtained in [59]. Again, the constant $\beta = 1$ acts as a critical value separating different asymptotic regimes, semiclassical for $\beta < 1$ and purely quantum for $\beta \ge 1$.

Theorem 1.2.6. *Fix $q \in \mathbb{Z}_+$. Let $m_\gtrless$ satisfy*

$$\ln m_\gtrless(x) = -\gamma|x|^{2\beta} + O(\ln|x|), \quad |x| \to \infty. \tag{1.2.38}$$

(i) If $\beta \in (0,1)$, then there exist constants $f_j = f_j(\beta,\mu)$, $j \in \mathbb{N}$, with $f_1 = \mu$, such that

$$\ln\left(\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right)\right) = -\sum_{1\le j<\frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{1.2.39}$$

(ii) If $\beta = 1$, then

$$\ln\left(\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right)\right) = -\left(\ln\left(1+\mu\right)\right)k + O(\ln k), \quad k \to \infty. \tag{1.2.40}$$

(iii) If $\beta \in (1,\infty)$, then there exist constants $g_j = g_j(\beta,\mu)$, $j \in \mathbb{N}$, such that

$$\ln\left(\pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right)\right) = \tag{1.2.41}$$
$$-\frac{\beta-1}{\beta} k \ln k + \left(\frac{\beta - 1 - \ln(\mu\beta)}{\beta}\right) k - \sum_{1 \le j < \frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + O(\ln k), \quad k \to \infty.$$

The constants f_j and g_j correspond to the coefficients of the Taylor series of certain functions depending on the parameters β and μ. A more precise description of these is given in Chapter 3. Notice that condition (1.2.38) yielding these results, is less general than (1.2.28), considered in [59].

Finally, the results for power-like decay are stated as follows:

Theorem 1.2.7. *Fix $q \in \mathbb{Z}_+$, and introduce the function*

$$\mathcal{T}_q(x) := \frac{1}{2}\left(\Lambda_q \operatorname{Tr} m(x) - 2b \operatorname{Im} m_{12}(x)\right) \ge 0, \quad x \in \mathbb{R}^2.$$

Let $m_{jk} \in \mathcal{S}^{-\rho}(\mathbb{R}^d)$, $j,k = 1,2$, with $\rho > 0$. Assume that there exists a function $0 < \tau_q \in C^\infty(\mathbb{S}^1)$, such that

$$\lim_{|x|\to\infty} |x|^\rho \mathcal{T}_q(x) = \tau_q(x/|x|).$$

Then we have

$$\mathcal{N}_q^{\pm}(\lambda) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)(1+o(1)) \asymp \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \tag{1.2.42}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho}\mathcal{N}_q^{\pm}(\lambda) = \mathcal{C}_q := \frac{b}{4\pi}\int_0^{2\pi} \tau_q(\cos\theta,\sin\theta)^{2/\rho} d\theta, \tag{1.2.43}$$

or to

$$\pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right) = \mathcal{C}_q^{\rho/2} k^{-\rho/2}(1+o(1)), \quad k \to \infty. \tag{1.2.44}$$

Statements analogous to (1.2.23) for $\mathcal{T}_q$ for which there exists $\lambda_0 > 0$ such that the function $\Phi_{\mathcal{T}_q}$ is differentiable on $(0,\lambda_0)$, and

$$-\lambda\Phi_{\mathcal{T}_q}(\lambda) \asymp \Phi_{\mathcal{T}_q}(\lambda),$$
$$\Phi_{\mathcal{T}_q}(\lambda) \ge C\lambda^{-2/\rho}, \quad \lambda \in (0,\lambda_0),$$

for some constant $C > 0$, remain valid in this context.

These results are readily extensible to operators of the form $H_\pm \pm V$, provided $V \geq 0$ decays at a rate comparable to that of the matrix m, so that the combined perturbation remains sign-definite.

In Chapter 3, a proof of the power-like decay case is given which uses methods that differ significantly from those present in the previous results by V. Ivrii, appearing in [38]. Also, the cases of exponential decay and compact support are considered, based on and further developing the methods used by G. Raikov and S. Warzel in [59]. As far as the authors are aware of, no results involving rapidly decaying metric perturbations (exponential decay, compact support) have been previously introduced in the literature around the Landau Hamiltonian.

Despite the fact that the perturbation is no longer relatively compact with respect to H_0, the Birman–Schwinger principle can still be applied, not to the operator difference, but to that of the resolvents, in order to reduce the problem to the spectral asymptotics of the operator $P_q W P_q$. The structure of the Heisenberg algebra is then used to derive a level-reduction formula establishing a unitary equivalence between the operator $P_q W P_q$ and the operator $P_0\, \omega_q\, P_0$, where ω_q is a multiplication operator by a function given in terms of some differential operators depending on b and q acting on the entries of the matrix m. This allows for the use of relation (1.2.33) at the level $q = 0$, for which the Laguerre polynomials are trivially simple. This, in turn, reduces technicalities so that more precise asymptotics that those previously obtained are possible in certain cases.

1.3 Berezin–Toeplitz Operators

The operators $P_q V P_q$, $q \in \mathbb{Z}_+$, are central elements in the methods exposed in this thesis, as well as in many of the mentioned publications. A general strategy when studying the behavior of the spectrum of the perturbed operator near a Landau level Λ_q is to express appropriately the spectral quantity to be investigated in terms of the operator $P_q Q P_q$, where Q denotes the generic perturbation of H_0 under consideration. Then, this operator is seen to be approximated in some suitable sense by Berezin–Toeplitz operators $P_{q'} w P_{q'}$, possibly with q' possibly different from q and with w an effective multiplication operator depending on the parameters of the problem. The problem is consequently reduced to the analysis of spectral properties of Berezin–Toeplitz operators.

Berezin–Toeplitz operators can be defined in a more general context. These are formally defined by the expression PVP, where the symbol V is a multiplication operator and P is the orthogonal projection onto a generalized holomorphic function space (see e.g. (1.2.5) and (1.2.6)). Berezin–Toeplitz operators are a relevant class of concrete operators, appearing in a range of different domains, both in pure and applied mathematics. Their study engages several different branches of mathematics.

Given a non-empty open subset $U \subset \mathbb{C}$ and a continuous strictly positive function α defined on U, the holomorphic function space $\mathcal{H}L^2(U, \alpha\, dx)$ is the space of holomorphic functions g which verify

$$\int_U |g(z)|^2 \alpha(z)\, dx_1\, dx_2 < \infty.$$

Being closed subspaces of $L^2(U, \alpha\, dx)$, they are in fact Hilbert spaces, and they have the remarkable property that pointwise evaluation is a continuous functional (see [34], for example).

As was previously mentioned in (1.2.34), there is a unitary transformation identifying the space

$$P_0 L^2(\mathbb{R}^2) = \operatorname{Ker} a = \left\{ u \in L^2(\mathbb{R}^2) \mid u = g e^{-\frac{b}{4}|x|^2},\ g \in C^1(\mathbb{R}^2),\ \partial g / \partial \overline{z} = 0 \right\}$$

and the holomorphic space $F^2 = \mathcal{H}L^2\big(\mathbb{C}, e^{-\frac{b}{2}|z|^2}\, dx\big)$, commonly called Fock or Segal–Bargmann space, given explicitly by the mapping $u \mapsto g$. It follows that the operator $P_0 V P_0$ can be seen as a Berezin–Toeplitz operator, unitarily equivalent to the operator generated in F^2 by the quadratic form

$$\int_{\mathbb{C}} V(z) e^{-\frac{b}{2}|z|^2} |g(z)|^2\, dx\, dy, \quad g \in F^2.$$

This section contains results concerning operators of the form $P_q V P_q$. Norm estimates in Schatten–von Neumann classes are given both in terms of the functional parameter V and for large values of q. Several unitary equivalences for these operators are stated as well: a level-reduction formula relating Berezin–Toeplitz operators $P_q V P_q$, $q \geq 1$, to operators of the form $P_0 U P_0$ is established, and a unitary equivalence to certain pseudodifferential operators is formulated. In Subsection 1.3.3 we consider generalized Berezin–Toeplitz operators.

1.3.1 Norm Estimates for Berezin–Toeplitz Operators

We begin by giving estimates for operators of Berezin–Toeplitz type in terms of the functional parameter V (see Proposition 2.3.6 or [27, Lemma 3.1]).

Proposition 1.3.1. *Let $q \in \mathbb{Z}_+$.*

(i) Suppose $V \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$. Then, $P_q V P_q \in S_p$ and

$$\|P_q V P_q\|_p^p \leq \frac{b}{2\pi} \|V\|_{L^p(\mathbb{R}^2)}^p . \tag{1.3.1}$$

Moreover, if $V \in L^1(\mathbb{R}^2)$, we have

$$\operatorname{Tr} P_q V P_q = \frac{b}{2\pi} \int_{\mathbb{R}^2} V(x)\, dx. \tag{1.3.2}$$

(ii) Suppose $V \in L_w^p(\mathbb{R}^2)$, $1 < p < \infty$. Then, $P_q V P_q \in S_{p,w}$ and

$$\|P_q V P_q\|_{p,w}^p \leq \frac{b}{2\pi} \|V\|_{L_w^p(\mathbb{R}^2)}^p . \tag{1.3.3}$$

Next, we present norm estimates for Berezin–Toeplitz operators for large values of the parameter q, appearing in Proposition 2.3.7 and [49, Proposition 3.5].

Proposition 1.3.2. *Assume that V verifies* (1.2.10) *with $\rho \in (0, \infty)$. Then, there exists a constant c_∞ independent of q such that*

$$\|P_q V P_q\| \leq c_\infty \begin{cases} \Lambda_q^{-\rho/2} & \text{if } \rho \in (0,1), \\ \Lambda_q^{-1/2} |\ln \Lambda_q| & \text{if } \rho = 1, \\ \Lambda_q^{-1/2} & \text{if } \rho > 1, \end{cases} \qquad q \in \mathbb{Z}_+. \tag{1.3.4}$$

Estimates (1.3.4) are sharp for $\rho \neq 1$. Effectively,

$$\liminf_{q \to \infty} \Lambda_q^{1/2} \langle \mathbb{1}_R\, \varphi_{q,q}, \varphi_{q,q} \rangle > 0, \tag{1.3.5}$$

which implies the sharpness of estimate (1.3.4) for $\rho > 1$. Similarly, if $\rho \in (0,1)$ we can show that

$$\liminf_{q \to \infty} \Lambda_q^{\rho/2} \langle \langle \cdot \rangle^{-\rho}\, \varphi_{q,0}, \varphi_{q,0} \rangle > 0, \tag{1.3.6}$$

which entails the sharpness for $\rho \in (0,1)$ (see [42] and the remark after 2.3.7 for further details). We do not know whether the estimate for $\rho = 1$ is sharp.

Corollary 1.3.1. *(i) Assume that V satisfies* (1.2.10) *with $\rho > 1$. Then, for each $\ell > 1/(\rho-1)$ we have $P_q V P_q \in S_\ell$. Moreover, there exists a constant c_ℓ such that*

$$\|P_q V P_q\|_\ell \le c_\ell \Lambda_q^{\frac{1}{\ell}-\frac{1}{2}}, \quad q \in \mathbb{Z}_+. \tag{1.3.7}$$

(ii) Assume that V satisfies (1.2.10) *with $\rho \in (0,1)$. Then, for each $\ell > 2/\rho$ we have $P_q V P_q \in S_\ell$. Moreover, there exists a constant c_ℓ such that*

$$\|P_q V P_q\|_\ell \le c_\ell \Lambda_q^{\frac{1}{\ell}-\frac{\rho}{2}} \left(1 + |\ln \Lambda_q|\right)^{1/\ell}, \quad q \in \mathbb{Z}_+. \tag{1.3.8}$$

Equation (1.3.8) should be considered an *a priori* estimate, we do not know wether it is sharp.

1.3.2 Unitary Equivalences for Berezin–Toeplitz Operators

Metaplectic Mapping of Berezin–Toeplitz Operators

The purpose of this subsection is to establish some unitary equivalences for the operator $P_q V P_q$. We begin by relating Berezin–Toeplitz operators to certain pseudodifferential operators, which we introduce next.

Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class in $\mathbb{R}^d$, and by $\mathcal{S}'(\mathbb{R}^d)$ its dual class. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R}^d, \tag{1.3.9}$$

is the Fourier transform of f. Whenever necessary, we extend by duality the Fourier transform to $\mathcal{S}'(\mathbb{R}^d)$.

Assume that $s \in \mathcal{S}(\mathbb{R}^{2d})$. Then, the operator $\mathrm{Op}^{\mathrm{w}}(s)$ defined by

$$\left(\mathrm{Op}^{\mathrm{w}}(s)u\right)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} s\left(\frac{x+x'}{2}, \xi\right) e^{i(x-x')\cdot\xi} u(x') dx' d\xi, \quad x \in \mathbb{R}^d, \tag{1.3.10}$$

is a mapping from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. It is called a *pseudodifferential operator* with Weyl symbol s. The application $s \mapsto \mathrm{Op}^{\mathrm{w}}(s)$ extends continuously to $s \in \mathcal{S}'(\mathbb{R}^{2d})$, taking values in the space of linear continuous mappings from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

Let $\Gamma(\mathbb{R}^{2d})$ denote the completion of the set $C_{\mathrm{b}}^\infty(\mathbb{R}^{2d})$ of smooth functions $s \in C^\infty(\mathbb{R}^{2d})$, bounded with all their derivatives, under the norm

$$\|s\|_{\Gamma(\mathbb{R}^{2d})} := \sup_{\{\alpha,\beta \in \mathbb{Z}_+^d \,|\, |\alpha|,|\beta| \le [\frac{d}{2}]+1\}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta s(x,\xi)|.$$

Suppose $s \in \Gamma(\mathbb{R}^{2d})$. Then, $\mathrm{Op}^{\mathrm{w}}(s)$ extends uniquely to an operator bounded in $L^2(\mathbb{R}^d)$, and there exists a constant c_0 independent of s such that

$$\|\mathrm{Op}^{\mathrm{w}}(s)\| \leq c_0 \|s\|_{\Gamma(\mathbb{R}^{2d})}, \tag{1.3.11}$$

(see e.g. [12, Corollary 2.5(i)]). Estimates of this type are known as *Calderón–Vaillancourt* estimates.

Proposition 1.3.3. *Let $\kappa : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d}$ be a linear symplectic transformation, $s_1 \in \Gamma(\mathbb{R}^{2d})$, and $s_2 := s_1 \circ \kappa$. Then, there exists a unitary operator $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ depending only on κ, such that*

$$\mathrm{Op}^{\mathrm{w}}(s_2) = \mathcal{U}^* \mathrm{Op}^{\mathrm{w}}(s_1) \mathcal{U}, \quad s_1 \in \Gamma(\mathbb{R}^{2d}).$$

The operator $\mathcal{U}$ is called the *metaplectic operator* corresponding to the linear symplectic transformation κ. There is a one-to-one correspondence between metaplectic operators and linear symplectic transformations, except for a factor of modulus one. Proposition 1.3.3 extends to a large class of not necessarily bounded operators (see [25, Ch. 7, Th. A.2] and [37, Th. 18.5.9] for further details).

For $(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4$, set

$$\varkappa_b(x_1, x_2, \xi_1, \xi_2) := \left(\frac{1}{\sqrt{b}}(x_1 - \xi_2), \frac{1}{\sqrt{b}}(\xi_1 - x_2), \frac{\sqrt{b}}{2}(\xi_1 + x_2), -\frac{\sqrt{b}}{2}(\xi_2 + x_1) \right).$$

The transformation $\varkappa_b$ is linear and symplectic, denote by $\mathcal{U}_b$ its corresponding metaplectic operator in $L^2(\mathbb{R}^2)$, an explicit expression for it is given by

$$(\mathcal{U}_b u)(x) := \frac{\sqrt{b}}{2\pi} \int_{\mathbb{R}^2} e^{i\phi_b(x,x')} u(x')\, dx', \quad x := (x_1, x_2) \in \mathbb{R}^2, \; x' := (x'_1, x'_2) \in \mathbb{R}^2,$$

defined initially in $\mathcal{S}(\mathbb{R}^2)$ and extended by continuity to $L^2(\mathbb{R}^2)$, where

$$\phi_b(x, x') := b\frac{x_1 x_2}{2} + b^{1/2}(x_1 x'_2 - x_2 x'_1) - x'_1 x'_2, \quad x, x' \in \mathbb{R}^2.$$

Note that

$$\mathcal{H} \circ \varkappa_b(x_1, x_2, \xi_1, \xi_2) = b\left(\xi_1^2 + x_1^2\right), \quad (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4,$$

where

$$\mathcal{H}(x, \xi) := \left(\xi_1 + \frac{1}{2}bx_2\right)^2 + \left(\xi_2 - \frac{1}{2}bx_1\right)^2$$

is the Weyl symbol of H_0. Applying Proposition 1.3.3 we obtain the following unitary equivalence of operators:

Corollary 1.3.2. **[55, Corollary 2.9]**

(i) We have

$$\begin{aligned} \mathcal{U}_b^* \, H_0 \, \mathcal{U}_b &= b\,(h \otimes I_{x_2}), \\ \mathcal{U}_b^* \, P_q \, \mathcal{U}_b &= (p_q \otimes I_{x_2}), \qquad q \in \mathbb{Z}_+, \end{aligned} \tag{1.3.12}$$

where h is the harmonic oscillator, defined below.

(ii) If $V \in \Gamma(\mathbb{R}^2)$, then

$$\mathcal{U}_b^* \, V \, \mathcal{U}_b = \mathrm{Op}^{\mathrm{w}}(\mathbf{V}_b), \tag{1.3.13}$$

where

$$\mathbf{V}_b(x_1, x_2, \xi_1, \xi_2) := V(b^{-1/2}(x_1 - \xi_2), b^{-1/2}(\xi_1 - x_2)), \quad (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4.$$

Introduce the harmonic oscillator

$$h := -\frac{d^2}{dx^2} + x^2,$$

self-adjoint in $L^2(\mathbb{R})$. It is well known that the spectrum of h is purely discrete and simple, and consists of the eigenvalues $2q + 1$, $q \in \mathbb{Z}_+$. Denote by p_q the orthogonal projection onto $\mathrm{Ker}\,(h - (2q + 1))$, $q \in \mathbb{Z}_+$. Set

$$\Psi_q(x, \xi) := \frac{(-1)^q}{\pi} \, \mathrm{L}_q(2(x^2 + \xi^2))e^{-(x^2+\xi^2)}, \quad (x, \xi) \in \mathbb{R}^2, \; q \in \mathbb{Z}_+.$$

Then $2\pi\Psi_q$ is the Wigner function associated to the rank one orthogonal projector p_q (i.e. it is the Weyl symbol of the operator p_q, see for instance [55, Lemma 2.6]).

The following result describes the image of Berezin–Toeplitz operators under the metaplectic transformation:

Theorem 1.3.1. **[55, Theorem 2.11]** *Let $q \in \mathbb{Z}_+$. For any $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ we have*

$$\mathcal{U}_b^* \, P_q V P_q \, \mathcal{U}_b = \big(p_q \otimes \mathrm{Op}^{\mathrm{w}}(V_b * \Psi_q)\big), \tag{1.3.14}$$

where

$$V_b(x_1, x_2) := V(-b^{-1/2}x_2, -b^{-1/2}x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{1.3.15}$$

Notice that for $q = 0$ we get the anti-Wick quantization $\mathrm{Op}^{\mathrm{aw}}(V_b) := \mathrm{Op}^{\mathrm{w}}(V_b * \Psi_0)$.

Applications and further developments on these results can be found in the chapters that follow.

Level-Reduction Formula

Finally, we state a level-reduction formula which gives a unitary equivalence between the operator $P_q V P_q$ defined on $P_q L^2(\mathbb{R}^2)$ and the operator $P_0\,(\mathcal{D}_q V) P_0$ defined on $P_0 L^2(\mathbb{R}^2)$, for some differential operator $\mathcal{D}_q$ acting on V and depending on b and q.

Theorem 1.3.2. [16, Corollary 9.3] *Set $q \in \mathbb{Z}_+$ and let $V \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$. The operator $P_q V P_q$, defined in $P_q L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator $P_0\,(\mathcal{D}_q V) P_0$ defined in $P_0 L^2(\mathbb{R}^2)$, where*

$$(\mathcal{D}_q V)\,(x) := \mathrm{L}_q \left(-\frac{\Delta}{2b}\right) V(x) = \sum_{j=0}^{q} \binom{q}{j} \frac{(\Delta/2b)^j}{j!} V(x). \tag{1.3.16}$$

1.3.3 Generalized Berezin–Toeplitz Operators

In this subsection we consider generalizations of Berezin–Toeplitz operators of the form $P_q\, a\, V a^* P_q$, $P_q\, a^* V a\, P_q$, $P_q\, a^* V a^* P_q$, and $P_q\, a\, V a\, P_q$, appearing, for example, in the analysis of metric perturbations of the Landau Hamiltonian. We state a level-reduction formula which generalizes Theorem 1.3.2.

Define the operator $\mathbb{A} : \mathrm{Dom}\, H_0^{1/2} \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ by

$$\mathbb{A}u := \begin{pmatrix} a^* u \\ a u \end{pmatrix}, \quad u \in \mathrm{Dom}\, H_0^{1/2},$$

with adjoint given by

$$\mathbb{A}^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = a v_1 + a^* v_2.$$

Introduce the matrix-valued function

$$\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix},$$

with $\omega_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$. Fix $q \in \mathbb{Z}_+$ and define the operator

$$P_q \mathbb{A}^* \Omega \mathbb{A} P_q = \Lambda_q P_q H_0^{-1/2} \mathbb{A}^* \Omega \mathbb{A} H_0^{-1/2} P_q,$$

bounded in $P_q L^2(\mathbb{R}^2)$.

Theorem 1.3.3. *Fix $q \in \mathbb{Z}_+$. Then, the operator $P_q \mathbb{A}^* \Omega \mathbb{A} P_q$ with domain $P_q L^2(\mathbb{R}^2)$, is unitarily equivalent to the operator $P_0 w_q P_0$ with domain $P_0 L^2(\mathbb{R}^2)$ where*

$$\begin{aligned} w_q = w_q(\Omega) := 2b(q+1)\,\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right)\omega_{11} + 2bq\,\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\omega_{22} \\ - 4\,\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 \omega_{12}}{\partial \overline{z}^2} - 4\,\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 \omega_{21}}{\partial z^2} \end{aligned} \tag{1.3.17}$$

if $q \geq 1$ and

$$w_q = w_q(\Omega) := 2b\,\mathrm{L}_1\left(-\frac{\Delta}{2b}\right)\omega_{11} \tag{1.3.18}$$

if $q = 0$.

This theorem follows from Proposition 3.4.1 in Chapter 3, which states the result for Hermitian matrix-valued Ω, the decomposition $\Omega = \Omega_1 + i\Omega_2$, with $\Omega_1 = \Omega_1^*$, $\Omega_2 = \Omega_2^*$, and from the fact that the unitary operator defined in the proof of Proposition 3.4.1 does not depend on Ω, but only on b and q.

Combining Theorem 1.3.1 and Theorem 1.3.3 we obtain

Corollary 1.3.3. *The operator $P_q \mathbb{A}^* \Omega \mathbb{A} P_q$ with domain $P_q L^2(\mathbb{R}^2)$ is unitarily equivalent to $p_0 \otimes \mathrm{Op}^{\mathrm{w}}(w_{q,b} * \Psi_0)$ with domain $(p_0 \otimes I_{x_2}) L^2(\mathbb{R}^2)$, where*

$$w_{q,b}(x_1, x_2) := w_q(-b^{-1/2} x_2, -b^{-1/2} x_1), \quad (x_1, x_2) \in \mathbb{R}^2.$$

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Chapter 2

A Trace Formula for Long-Range Perturbations of the Landau Hamiltonian

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Abstract. We consider the Landau Hamiltonian perturbed by a long-range electric potential V. The spectrum of the perturbed operator consists of eigenvalue clusters which accumulate to the Landau levels. First, we estimate the rate of the shrinking of these clusters to the Landau levels as the number of the cluster tends to infinity. Further, we assume that there exists an appropriate $\mathbb{V}$, homogeneous of order $-\rho$ with $\rho \in (0,1)$, such that $V(x) = \mathbb{V}(x) + O(|x|^{-\rho-\varepsilon})$, $\varepsilon > 0$, as $|x| \to \infty$, and investigate the asymptotic distribution of the eigenvalues within the qth cluster as $q \to \infty$. We obtain an explicit description of the asymptotic density of the eigenvalues in terms of the mean-value transform of $\mathbb{V}$.

Keywords: Landau Hamiltonian, long-range perturbations, asymptotic density for eigenvalue clusters, mean-value transform

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2.1 Introduction

Our unperturbed operator is the Landau Hamiltonian

$$H_0 := (-i\nabla - A)^2,$$

self-adjoint in $L^2(\mathbb{R}^2)$. Here, $A := \left(-\frac{Bx_2}{2}, \frac{Bx_1}{2}\right)$ is the magnetic potential, and $B > 0$ is the generated constant magnetic field. It is well known that the spectrum $\sigma(H_0)$ of H_0 consists of infinitely degenerate eigenvalues $\lambda_q := B(2q+1)$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, called *Landau levels*.

The perturbation of H_0 is an electric potential $V : \mathbb{R}^2 \to \mathbb{R}$ which is supposed to be bounded and continuous. Set $H := H_0 + V$. Evidently,

$$\sigma(H) \subset \bigcup_{q=0}^{\infty} \left[\lambda_q + \inf V, \lambda_q + \sup V\right].$$

Moreover, if V decays at infinity, and, hence, is relatively compact with respect to H_0, then $\sigma(H) \setminus \sigma(H_0)$ consists of discrete eigenvalues which could accumulate only to the Landau levels. Recently, in [55] it was shown that if V satisfies

$$|V(x)| \leq c\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^2, \tag{2.1.1}$$

with $\rho > 1$, then $\sigma(H)$ is contained in the union of intervals centered at the Landau levels λ_q, of size $O(\lambda_q^{-1/2})$ as $q \to \infty$. Moreover, in [55] the asymptotic density of the eigenvalue clusters was studied. To this end, the asymptotic behaviour of the trace $\operatorname{Tr} \varphi(\lambda_q^{1/2}(H - \lambda_q))$ with $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ was investigated, and it was found that $\operatorname{Tr} \varphi(\lambda_q^{1/2}(H - \lambda_q))$ is of order $\lambda_q^{1/2}$ as $q \to \infty$, and its first asymptotic term could be written explicitly using the Radon transform of V.

In the present article we assume that V is long-range, i.e. in contrast to [55], it satisfies (2.1.1) with $\rho \in (0, 1)$. First, we show that the eigenvalue clusters of H shrink to the Landau levels at rate $\lambda_q^{-\rho/2}$ as $q \to \infty$ (see Proposition 2.2.1). Further, we suppose that there exists an appropriate $\mathbb{V}$, homogeneous of order $-\rho$, which is asymptotically equivalent to V, and study the asymptotic behaviour of the trace $\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(H - \lambda_q))$. We show that $\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(H - \lambda_q))$ is of order λ_q as $q \to \infty$, and its main asymptotic term could be written explicitly using the mean-value transform of $\mathbb{V}$ (see Theorem 2.2.1).

The article is organized as follows. In the next section we formulate our main results, and briefly comment on them. Section 2.3 contains auxiliary facts concerning the properties of Weyl pseudodifferential operators and Berezin–Toeplitz operators which are the main tools in the proof of Theorem 2.2.1. The proof itself could be found in Section 2.4, and is divided into several steps, contained in separate subsections.

2.2 Main Results

Our first result concerns the shrinking of the eigenvalue clusters of H in the case of long-range potentials V.

Proposition 2.2.1. *Assume that V satisfies (2.1.1) with $\rho \in (0,1)$. Then there exists a constant $C > 0$ such that*

$$\sigma(H) \subset \bigcup_{q=0}^{\infty} \left(\lambda_q - C\lambda_q^{-\rho/2}, \lambda_q + C\lambda_q^{-\rho/2}\right). \tag{2.2.1}$$

The proof of Proposition 2.2.1 could be found in Subsection 2.3.4.
Remarks: (i) Simple considerations (see the remark after Proposition 2.3.7) show that the estimate $O(\lambda_q^{-\rho/2})$ of the size of the eigenvalue clusters is sharp. This will follow also from Theorem 2.2.1.
(ii) In [55, Proposition 1.1] it was shown that if V satisfies (2.1.1) with $\rho > 1$, then there exists a constant $C > 0$ such that

$$\sigma(H) \subset \bigcup_{q=0}^{\infty} \left(\lambda_q - C\lambda_q^{-1/2}, \lambda_q + C\lambda_q^{-1/2}\right).$$

In the case of compactly supported V, such a result was already obtained in [42].
In order to formulate our main result we need the following notations. For $d \geq 1$ put

$$C_{\mathrm{b}}^{\infty}(\mathbb{R}^d) = \left\{ u \in C^{\infty}(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}} |D^{\alpha}u(x)| \leq c_{\alpha},\ \alpha \in \mathbb{Z}_+^d \right\}.$$

Following [68, Section 8, Chapter 3], we write $u \in \mathcal{H}^{\sharp}_{-\rho}(\mathbb{R}^d)$ if $u \in C^{\infty}(\mathbb{R}^d \setminus \{0\})$, $\rho \in (0,\infty)$, is a homogeneous function of order $-\rho$. Moreover, for $\rho \in [0,\infty)$ we set

$$\mathcal{S}_1^{-\rho}(\mathbb{R}^d) := \left\{ u \in C^{\infty}(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}^d} \langle x\rangle^{\rho+|\alpha|} |D^{\alpha}u(x)| \leq c_{\alpha},\ \alpha \in \mathbb{Z}_+^d \right\}.$$

Assume $u \in C(\mathbb{R}^2 \setminus \{0\})$, and define the *mean-value transform*

$$\mathring{u}(x) := \frac{1}{2\pi} \int_{\mathbb{S}^1} u(x-\omega)d\omega, \quad x \in \mathbb{R}^2 \setminus \mathbb{S}^1.$$

Our mean-value transform coincides with the 2D mean-value operator M^1 defined in [36, Eq. (15), Chapter I] with $n = 2$, and is quite closely related to the so-called planar circular Radon transform defined, for instance, in [4].

Next, we describe some elementary but yet useful properties of the mean-value transforms of functions from appropriate classes. The proofs are quite simple, so that we omit the details. If $u \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$, $\rho \in (0,\infty)$, then the mean-value transform $\mathring{u}$ extends to a function $\mathring{u} \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$. If $u \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^2)$, $\rho \in (0,\infty)$, then $\eta\mathring{u} \in \mathcal{S}_1^{-\rho}$ provided that $\eta \in \mathcal{S}_1^0(\mathbb{R}^2)$ and $\operatorname{supp}\eta \cap \mathbb{S}^1 = \emptyset$. Moreover, if $\rho \in (0,1)$, then the mean-value transform of $u \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^2)$, $\rho \in (0,1)$, extends to a function $\mathring{u} \in C(\mathbb{R}^2)$. Finally, if $u \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^2)$, $\rho \in (0,1)$, and $\mathring{u}(x) = 0$ for each $x \in \mathbb{R}^2$, then $u(x) = 0$ for each $x \in \mathbb{R}^2 \setminus \{0\}$.

Theorem 2.2.1. *Let $\rho \in (0,1)$. Assume that $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ and there exists $\mathbb{V} \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^2)$ such that*

$$|V(x) - \mathbb{V}(x)| \leq C|x|^{-\rho-\varepsilon}, \quad x \in \mathbb{R}^2, \quad |x| > 1, \tag{2.2.2}$$

with some constant C, and $\varepsilon > 0$. Then we have

$$\lim_{q\to\infty} \lambda_q^{-1} \operatorname{Tr} \varphi(\lambda_q^{\rho/2}(H - \lambda_q)) = \frac{1}{2\pi B} \int_{\mathbb{R}^2} \varphi(B^\rho \mathring{\mathbb{V}}(x))dx \tag{2.2.3}$$

for each $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Let us comment briefly on Theorem 2.2.1.

- Let $[\alpha,\beta] \subset \mathbb{R} \setminus \{0\}$ be a bounded interval with $\alpha < \beta$. For $q \in \mathbb{Z}_+$ set

  $$\mu_q([\alpha,\beta]) := \sum_{\lambda_q + \alpha\lambda_q^{-\rho/2} \leq \lambda \leq \lambda_q + \beta\lambda_q^{-\rho/2}} \dim \operatorname{Ker}(H - \lambda).$$

  Evidently, $\mu_q([\alpha,\beta]) < \infty$ if $q \in \mathbb{Z}_+$ is large enough. Put

  $$\mu([\alpha,\beta]) := \frac{1}{2\pi B} \left| \left\{ x \in \mathbb{R}^2 \,|\, \alpha B^{-\rho} \leq \mathring{\mathbb{V}}(x) \leq \beta B^{-\rho} \right\} \right|$$

  where $|\cdot|$ denotes the Lebesgue measure, and $\mathring{\mathbb{V}}$ is the mean-value transform of $\mathbb{V} \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^2)$, $\rho \in (0,1)$, the homogeneous function introduced in the statement of Theorem 2.2.1. Evidently, $0 \notin [\alpha, \beta]$ implies $\mu([\alpha, \beta]) < \infty$. We extend μ to a σ-finite measure defined on the Borel sets $\mathcal{O} \subset \mathbb{R} \setminus \{0\}$, and supported on $\left[B^\rho \inf_{x\in\mathbb{R}^2} \mathring{\mathbb{V}}(x), B^\rho \sup_{x\in\mathbb{R}^2} \mathring{\mathbb{V}}(x)\right] \setminus \{0\}$; the compactness of the support of the limiting measure μ agrees with the fact that, in accordance, with (2.2.1), we have $\operatorname{supp} \mu_q \subset [-C, C] \setminus \{0\}$ for sufficiently large q. Then the validity of (2.2.3) for any $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is equivalent to the validity of

$$\lim_{q\to\infty} \lambda_q^{-1} \mu_q([\alpha, \beta]) = \mu([\alpha, \beta])$$

for any bounded $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ such that $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$. Note that if, for instance, the function $\mathbb{V}$ is radially symmetric, then $\mu(\{\alpha\}) = 0$ for any $\alpha \in \mathbb{R}\setminus\{0\}$.

- As already mentioned, in [55] it was supposed that V satisfies (2.1.1) with $\rho > 1$. Then the Radon transform

$$\tilde{V}(s, \omega) := \frac{1}{2\pi} \int_{\mathbb{R}} V(s\omega + t\omega^\perp)dt, \quad s \in \mathbb{R}, \quad \omega = (\omega_1, \omega_2) \in \mathbb{S}^1, \quad \omega^\perp := (-\omega_2, \omega_1),$$

is well defined, continuous, and decays as $|s| \to \infty$ uniformly with respect to $\omega \in \mathbb{S}^1$. Then, instead of (2.2.3), we have

$$\lim_{q\to\infty} \lambda_q^{-1/2} \operatorname{Tr} \varphi(\lambda_q^{1/2}(H - \lambda_q)) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{S}^1} \varphi(B\tilde{V}(s, \omega))d\omega ds \tag{2.2.4}$$

(see [55, Theorem 1.3]). Note, in particular, that if V satisfies (2.1.1) with $\rho > 1$, then (2.2.4) implies that $\operatorname{Tr} \varphi(\lambda_q^{1/2}(H - \lambda_q))$ is of order $\lambda_q^{1/2}$, while it follows from (2.2.3) that under the hypotheses of Theorem 2.2.1 $\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(H - \lambda_q))$ is of order λ_q as $q \to \infty$.

- Theorem 2.2.1 admits a similar semiclassical interpretation as [55, Theorem 1.3]. Namely, consider the classical Hamiltonian function

$$\mathcal{H}(\xi, x) = (\xi_1 + Bx_2/2)^2 + (\xi_2 - Bx_1/2)^2, \tag{2.2.5}$$

with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$. The projections onto the configuration space of the orbits of the Hamiltonian flow of $\mathcal{H}$ are circles of radius $\sqrt{E}/B$, where $E > 0$ is the energy corresponding to the orbit. The classical particles move around these circles with period $T_B = \pi/B$. These orbits are parameterized by the energy

$E > 0$ and the center $\mathbf{c} \in \mathbb{R}^2$ of the circle. Denote the path in the configuration space corresponding to such an orbit by $\gamma(\mathbf{c}, E, t)$, $t \in [0, T_B)$, and set

$$\mathrm{Av}(V)(\mathbf{c}, E) = \frac{1}{T_B} \int_0^{T_B} V(\gamma(\mathbf{c}, E, t))dt, \quad T_B = \pi/B.$$

It is easy to see that the r.h.s. of (2.2.3) can be rewritten as

$$\frac{1}{2\pi B} \int_{\mathbb{R}^2} \varphi(B^\rho \mathring{\mathbb{V}}(x))dx = \frac{1}{2\pi} \lim_{E\to\infty} \frac{1}{E} \int_{\mathbb{R}^2} \varphi(E^{\rho/2}\, \mathrm{Av}(V)(\mathbf{c}, E))\, B\, d\mathbf{c}. \tag{2.2.6}$$

Given (2.2.6), we can rewrite (2.2.3) as

$$\lim_{q\to\infty} \frac{1}{\lambda_q} \mathrm{Tr}\, \varphi(\lambda_q^{\rho/2}(H - \lambda_q)) = \frac{1}{2\pi} \lim_{E\to\infty} \frac{1}{E} \int_{\mathbb{R}^2} \varphi(E^{\rho/2}\, \mathrm{Av}(V)(\mathbf{c}, E))\, B\, d\mathbf{c}. \tag{2.2.7}$$

Formula (2.2.7) agrees with the so-called "averaging principle" for systems close to integrable ones, according to which a good approximation is obtained if one replaces the original perturbation by its average along the orbits of the free dynamics (see e.g. [5, Section 52]).

- Neither Theorem 2.2.1, nor [55, Theorem 1.3], treat the border-line case $\rho = 1$, i.e. the case where V is, say, asymptotically homogeneous of order -1. In this case the Radon transform of V is not well defined, while the mean-value transform $\mathring{\mathbb{V}}$ of $\mathbb{V} \in \mathcal{H}^\sharp_{-1}(\mathbb{R}^2)$ generically is not bounded since it may have a logarithmic singularity at $\mathbb{S}^1$. Therefore, in the border-line case the asymptotic density of the eigenvalue clusters of H should be different from both the short-range case $\rho > 1$ and the long-range case $\rho \in (0, 1)$. Hopefully, we will consider in detail the border-line case in a future work.

The proof of Theorem 2.2.1 is contained on Section 2.4.

2.3 Auxiliary Results

2.3.1 Weyl Pseudodifferential Operators

Let $d \geq 1$. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class, and by $\mathcal{S}'(\mathbb{R}^d)$ its dual class. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d,$$

is the Fourier transform of f. Whenever necessary, we extend by duality the Fourier transform to $\mathcal{S}'(\mathbb{R}^d)$.

Let $\Gamma(\mathbb{R}^{2d})$, $d \geq 1$, denote the closure of $C_{\rm b}^\infty(\mathbb{R}^{2d})$ with respect to the norm

$$\|s\|_{\Gamma(\mathbb{R}^{2d})} := \sup_{\{\alpha,\beta\in\mathbb{Z}_+^d \mid |\alpha|,|\beta|\leq[\frac{d}{2}]+1\}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta s(x,\xi)| < \infty.$$

Proposition 2.3.1. ([17, 21]) *Assume that* $s \in \Gamma(\mathbb{R}^{2d})$. *Then the operator* $\mathrm{Op}^{\rm w}(s)$ *defined initially as a mapping from* $\mathcal{S}(\mathbb{R}^d)$ *into* $\mathcal{S}'(\mathbb{R}^d)$ *by*

$$\left(\mathrm{Op}^{\rm w}(s)u\right)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} s\left(\frac{x+x'}{2},\xi\right) e^{i(x-x')\cdot\xi} u(x')dx'd\xi, \quad x\in\mathbb{R}^d, \qquad (2.3.1)$$

extends uniquely to an operator bounded in $L^2(\mathbb{R}^d)$, *and there exists a constant* c_0 *independent of* s *such that*

$$\|\mathrm{Op}^{\rm w}(s)\| \leq c_0 \|s\|_{\Gamma(\mathbb{R}^{2d})}.$$

The operator $\mathrm{Op}^{\rm w}(s)$ is called *a pseudodifferential operator* (ΨDO) with Weyl symbol s. Assume that $s \in \mathcal{S}'(\mathbb{R}^{2d})$, $\hat{s} \in L^1(\mathbb{R}^{2d})$. Then the operator defined in (2.3.1) extends to an operator bounded in $L^2(\mathbb{R}^d)$, and we have

$$\|\mathrm{Op}^{\rm w}(s)\| \leq (2\pi)^{-d} \|\hat{s}\|_{L^1(\mathbb{R}^{2d})} \qquad (2.3.2)$$

(see [37, Lemma 18.6.1]). Assume now $s \in L^2(\mathbb{R}^{2d})$. Then, evidently, the operator defined in (2.3.1) extends to a Hilbert–Schmidt operator in $L^2(\mathbb{R}^d)$, and we have

$$\|\mathrm{Op}^{\rm w}(s)\|_2^2 = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} |s(x,\xi)|^2\, dx\, d\xi = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} |\hat{s}(x,\xi)|^2\, dx\, d\xi. \qquad (2.3.3)$$

Let X be a separable Hilbert space. Then $S_\infty(X)$ denotes the class of compact linear operators acting in X. If $T \in S_\infty(X)$, then $\{s_j(T)\}_{j=1}^{\mathrm{rank}\, T}$ denotes the set of the non-zero singular numbers of T enumerated in non-increasing order, and $S_\ell(X)$ is the Schatten–von Neumann class of order $\ell \in [1,\infty)$, i.e. the class of operators $T \in S_\infty(X)$ for which the norm $\|T\|_\ell := \left(\sum_{j=1}^{\mathrm{rank}\, T} s_j(T)^\ell\right)$ is finite. Thus, $S_1(X)$ is the trace class, and $S_2(X)$ is the Hilbert–Schmidt class. Similarly, $S_{\ell,w}(X)$ denotes the *weak* Schatten–von Neumann class of order $\ell \in [1,\infty)$, i.e. the class of operators $T \in S_\infty(X)$ for which the quasinorm $\|T\|_{\ell,w} := \sup_j j^{1/\ell} s_j(T)$ is finite. Whenever appropriate, we omit X in the notations $S_\ell(X)$ and $S_{\ell,w}(X)$.

Next, we recall that $u \in L^p_w(\mathbb{R}^d)$, $d \geq 1$, the weak Lebesgue space of order $p \in [1,\infty)$, if the quasinorm $\|u\|_{L^p_w(\mathbb{R}^d)} := \sup_{t>0} t \left|\{x \in \mathbb{R}^d \,|\, |u(x)| > t\}\right|^{1/p}$ is finite. Evidently, $u \in \mathcal{H}^\sharp_{-\rho}(\mathbb{R}^d)$, $\rho \in (0,d)$, implies $u \in L^{d/\rho}_w(\mathbb{R}^d)$.

Interpolating between (2.3.2) and (2.3.3) (see [9, Theorem 3.1]), we obtain the following

Proposition 2.3.2. *Let $m \in (2,\infty)$, $m' := m/(m-1)$.*
(i) *Assume that $s \in \mathcal{S}'(\mathbb{R}^{2d})$, $\hat{s} \in L^{m'}(\mathbb{R}^{2d})$. Then $\mathrm{Op}^{\mathrm{w}}(s) \in S_m(L^2(\mathbb{R}^d))$, and*

$$\|\mathrm{Op}^{\mathrm{w}}(s)\|_m \leq (2\pi)^{-d\left(1-\frac{1}{m}\right)} \|\hat{s}\|_{L^{m'}(\mathbb{R}^{2d})}.$$

(ii) *Assume that $s \in \mathcal{S}'(\mathbb{R}^{2d})$, $\hat{s} \in L^{m'}_w(\mathbb{R}^{2d})$. Then $\mathrm{Op}^{\mathrm{w}}(s) \in S_{m,w}(L^2(\mathbb{R}^d))$, and*

$$\|\mathrm{Op}^{\mathrm{w}}(s)\|_{m,w} \leq (2\pi)^{-d\left(1-\frac{1}{m}\right)} \|\hat{s}\|_{L^{m'}_w(\mathbb{R}^{2d})}.$$

2.3.2 Operators $\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q)$ and $\mathrm{Op}^{\mathrm{w}}(V_B * \delta_k)$

Introduce the harmonic oscillator

$$h := -\frac{d^2}{dx^2} + x^2,$$

self-adjoint in $L^2(\mathbb{R})$. It is well known that the spectrum of h is purely discrete and simple, and consists of the eigenvalues $2q+1$, $q \in \mathbb{Z}_+$. Denote by p_q the orthogonal projection onto $\mathrm{Ker}\,(h-(2q+1))$, $q \in \mathbb{Z}_+$. Set

$$\Psi_q(x,\xi) = \frac{(-1)^q}{\pi} \mathrm{L}_q(2(x^2+\xi^2))e^{-(x^2+\xi^2)}, \quad (x,\xi) \in \mathbb{R}^2, \quad q \in \mathbb{Z}_+, \tag{2.3.4}$$

where

$$\mathrm{L}_q(t) := \frac{1}{q!} e^t \frac{d^q(t^q e^{-t})}{dt^q} = \sum_{k=0}^{q} \binom{q}{k} \frac{(-t)^k}{k!}, \quad t \in \mathbb{R}, \tag{2.3.5}$$

are the Laguerre polynomials. Then $2\pi\Psi_q$ is the Weyl symbol of the operator p_q.

Denote by P_q, $q \in \mathbb{Z}_+$, the orthogonal projection onto $\mathrm{Ker}\,(H_0 - \lambda_q)$. Set

$$V_B(x) = V(-B^{-1/2}x_2, -B^{-1/2}x_1), \quad x = (x_1,x_2) \in \mathbb{R}^2. \tag{2.3.6}$$

Proposition 2.3.3. ([55, Corollary 2.13]) *There exists a unitary operator $\mathcal{U}_B : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ such that for each $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and each $q \in \mathbb{Z}_+$ we have*

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = p_q \otimes \mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q). \tag{2.3.7}$$

For $k > 0$, define the distribution $\delta_k \in \mathcal{S}'(\mathbb{R}^2)$ by

$$\delta_k(\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \varphi(k\cos\theta, k\sin\theta)d\theta, \quad \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Proposition 2.3.4. *Assume that $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ with $\rho \in (0,\infty)$. Then the operator $\mathrm{Op}^{\mathrm{w}}(V_B * \delta_k)$, $k > 0$, is bounded and there exists a constant c_1 such that*

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_k)\| \leq c_1 \begin{cases} k^{-\rho} & \text{if} \quad \rho \in (0,1), \\ k^{-1}\ln k & \text{if} \quad \rho = 1, \\ k^{-1} & \text{if} \quad \rho \in (1,\infty), \end{cases} \qquad k \in [2,\infty). \tag{2.3.8}$$

Proof. Proposition 2.3.4 is an extension of [55, Lemma 3.2] which concerned only the case $\rho > 1$. By Proposition 2.3.1,

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_k)\| \leq c_0 \max_{\alpha\in\mathbb{Z}_+^2 \ | \ 0\leq|\alpha|\leq 2} \sup_{z\in\mathbb{R}^2} |(D^\alpha V * \delta_k)(z)|. \tag{2.3.9}$$

By $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$, we have

$$|D^\alpha V(x)| \leq c_{1,\alpha}\langle x\rangle^{-|\alpha|-\rho} \leq c_{1,\alpha}\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{Z}_+^2, \tag{2.3.10}$$

with constants $c_{1,\alpha}$ which may depend on B but are independent of x. Now (2.3.9) and (2.3.10) imply

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_k)\| \leq c_1' \sup_{z\in\mathbb{R}^2} \langle\cdot\rangle^{-\rho} * \delta_k(z). \tag{2.3.11}$$

Note that the function $\langle\cdot\rangle^{-\rho} * \delta_k$ is radially symmetric. Arguing as in the proof of [55, Lemma 3.2], we get

$$(\langle\cdot\rangle^{-\rho} * \delta_k)(z) = \frac{1}{2\pi}\int_0^{2\pi} ((k\cos\theta - |z|)^2 + k^2\sin^2\theta + 1)^{-\rho/2}d\theta \leq \tag{2.3.12}$$

$$\frac{1}{2\pi}\int_0^{2\pi} (k^2\sin^2\theta + 1)^{-\rho/2}d\theta = \frac{2}{\pi}\int_0^{\pi/2} (k^2\sin^2\theta + 1)^{-\rho/2}d\theta \leq \int_0^1 (k^2t^2+1)^{-\rho/2}dt =: I_\rho(k).$$

Elementary calculations yield

$$I_\rho(k) = \begin{cases} O(k^{-\rho}) & \text{if} \quad \rho \in (0,1), \\ O(k^{-1}\ln k) & \text{if} \quad \rho = 1, \\ O(k^{-1}) & \text{if} \quad \rho \in (1,\infty), \end{cases} \qquad k \in [2,\infty). \tag{2.3.13}$$

Putting together (2.3.11) – (2.3.13), we obtain (2.3.8). □

Proposition 2.3.5. *Assume that* $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$ *with* $\rho \in (0,\infty)$. *Then* $\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}}) \in S_2$, *and there exists a constant* c_2 *independent of* q, *such that*

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\|_2 \leq c_2 \lambda_q^{-3/4}, \quad q \in \mathbb{Z}_+. \tag{2.3.14}$$

Proof. Proposition 2.3.5 is an extension of the second part of [55, Lemma 3.1] which concerned the case $V \in C_0^\infty(\mathbb{R}^2)$. By (2.3.3) we have

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} |(V_B * \Psi_q)(z) - (V_B * \delta_{\sqrt{2q+1}})(z)|^2 dz =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |(\widehat{V_B * \Psi_q})(\zeta) - (\widehat{V_B * \delta_{\sqrt{2q+1}}})(\zeta)|^2 d\zeta. \tag{2.3.15}$$

An explicit calculation (see [55, Eq. (3.9)]) yields

$$(\widehat{V_B * \Psi_q})(\zeta) - (\widehat{V_B * \delta_{\sqrt{2q+1}}})(\zeta) = \left(\mathrm{L}_q(|\zeta|^2/2)e^{-|\zeta|^2/4} - J_0(\sqrt{2q+1}|\zeta|)\right)\widehat{V_B}(\zeta), \tag{2.3.16}$$

for $\zeta \in \mathbb{R}^2$, where L_q is the Laguerre polynomial defined in (2.3.5), and J_0 is the Bessel function of zeroth order. Moreover, there exists a constant $\tilde{c}_2$ such that, for $q \in \mathbb{Z}_+$,

$$\left|\mathrm{L}_q(r)e^{-r/2} - J_0(\sqrt{(4q+2)r})\right| \leq \tilde{c}_2\left((q+1)^{-3/4} r^{5/4} + (q+1)^{-1} r^3\right), \quad r > 0, \tag{2.3.17}$$

(see [55, Eq. (3.10)] for the generic case $q \in \mathbb{N}$; if $q = 0$, then (2.3.17) follows from $|e^{-r/2} - J_0(\sqrt{r})| = O(r^2)|$, $r \in (0,1)$, and $|e^{-r/2} - J_0(\sqrt{r})| = O(1)$, $r \geq 1$). Further,

$$|\widehat{V_B}(\zeta)| = \begin{cases} O(|\zeta|^{-2+\rho}) & \text{if} \quad \rho \in (0,2), \\ O(|\ln|\zeta||) & \text{if} \quad \rho = 2, \\ O(1) & \text{if} \quad \rho > 2, \end{cases} \qquad |\zeta| \leq 1/2,$$

and

$$|\widehat{V_B}(\zeta)| = O(|\zeta|^{-N}), \quad |\zeta| > 1/2, \quad N > 0,$$

(see [67, Lemma 3.1, Chapter XII]). In particular, the functions $|\zeta|^m \widehat{V_B}(\zeta)$, $\zeta \in \mathbb{R}^2$, with $m > 1-\rho$ if $\rho \in (0,2)$ or with $m > -1$ if $\rho \geq 2$, are in $L^2(\mathbb{R}^2)$. Combining (2.3.15), (2.3.16), and (2.3.17), we get

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\|_2^2 \leq \frac{\tilde{c}_2^2}{\pi}\int_{\mathbb{R}^2}\left((q+1)^{-3/2}|\zeta|^5 + (q+1)^{-2}|\zeta|^{12}\right)|\widehat{V_B}(\zeta)|^2 d\zeta,$$

which yields (2.3.14). □

Remark: Estimate (2.3.14) could be interpreted as a manifestation of the equipartition of the eigenfunctions of the harmonic oscillator h, i.e. the appropriate weak convergence as $q \to \infty$ of the Wigner function $2\pi\Psi_q$ associated with the qth normalized eigenfunction of H, to the measure invariant with respect to the classical flow (see e.g. [13, 18, 71] for related results concerning various ergodic quantum systems).

2.3.3 Norm Estimates for Berezin–Toeplitz Operators

Proposition 2.3.6. (i) *Let $V \in L^p(\mathbb{R}^2)$, $p \in [1,\infty)$. Then for each $q \in \mathbb{Z}_+$ we have $P_qVP_q \in S_p$ and*

$$\|P_qVP_q\|_p^p \leq \frac{B}{2\pi}\|V\|^p_{L^p(\mathbb{R}^2)}. \tag{2.3.18}$$

(ii) *Let $V \in L^p_w(\mathbb{R}^2)$, $p \in (1,\infty)$. Then for each $q \in \mathbb{Z}_+$ we have $P_qVP_q \in S_{p,w}$ and*

$$\|P_qVP_q\|_{p,w}^p \leq \frac{B}{2\pi}\|V\|^p_{L^p_w(\mathbb{R}^2)}. \tag{2.3.19}$$

Proof. Using the explicit expression for the integral kernel of P_q (see e.g. [27, Eq. (3.2)]), we easily obtain

$$\|P_qVP_q\|_1 \leq \frac{B}{2\pi}\|V\|_{L^1(\mathbb{R}^2)} \tag{2.3.20}$$

with an equality if $V = \overline{V} \geq 0$. Moreover, evidently,

$$\|P_qVP_q\| \leq \|V\|_{L^\infty(\mathbb{R}^2)}. \tag{2.3.21}$$

Interpolating between (2.3.20) and (2.3.21) (see [9, Theorem 3.1]), we obtain (2.3.18) and (2.3.19). □

Remark: The first part of Proposition 2.3.6 has been known since long ago (see [27, Lemma 3.1], [58, Lemma 5.1]).

Corollary 2.3.1. *Let $V \in L^p(\mathbb{R}^2)$, $p \in [2,\infty)$. Then for each $q \in \mathbb{Z}_+$ we have $P_qV = (VP_q)^* \in S_p$, and*

$$\|P_qV\|_p^p = \|VP_q\|_p^p \leq \frac{B}{2\pi}\|V\|^p_{L^p(\mathbb{R}^2)}. \tag{2.3.22}$$

Proof. Estimates (2.3.22) follow immediately from (2.3.18) since we have

$$\|P_qV\|_p^p = \|P_q|V|^2P_q\|_{p/2}^{p/2} \leq \frac{B}{2\pi}\|V^2\|^{p/2}_{L^{p/2}(\mathbb{R}^2)} = \frac{B}{2\pi}\|V\|^p_{L^p(\mathbb{R}^2)}.$$

□

Proposition 2.3.7. *Assume that* V *satisfies* (2.1.1) *with* $\rho \in (0,\infty)$. *Then there exists a constant* c_∞ *such that*

$$\|P_q V P_q\| \leq c_\infty \begin{cases} \lambda_q^{-\rho/2} & \text{if} \quad \rho \in (0,1), \\ \lambda_q^{-1/2} |\ln \lambda_q| & \text{if} \quad \rho = 1, \\ \lambda_q^{-1/2} & \text{if} \quad \rho \in (1,\infty), \end{cases} \quad q \in \mathbb{Z}_+. \tag{2.3.23}$$

Proof. An elementary variational argument implies that we may assume without loss of generality that $V(x) = \langle x \rangle^{-\rho}$, $x \in \mathbb{R}^2$; then, $V \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$. By Propositions 2.3.3 and 2.3.5 we have

$$\|P_q V P_q\| = \|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q)\| \leq$$

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\| + \|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\| \leq$$

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\| + \|\mathrm{Op}^{\mathrm{w}}(V_B * \Psi_q) - \mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\|_2 \leq$$

$$\|\mathrm{Op}^{\mathrm{w}}(V_B * \delta_{\sqrt{2q+1}})\| + c_2 \lambda_q^{-3/4}. \tag{2.3.24}$$

Now, (2.3.24) and (2.3.8) yield immediately (2.3.23). □

Remark: Estimates (2.3.23) with $\rho \neq 1$ are sharp. For $\rho > 1$ this follows from the argument of [42] where the estimate $\|P_q V P_q\| \leq c_\infty \lambda_q^{-1/2}$ was obtained for compactly supported V. Namely, if $\{\phi_{k,q}\}_{k=-q}^{\infty}$ is the so called *angular-momentum* orthonormal basis of the Hilbert space $P_q L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$ (see e.g. [59]), and $\mathbb{1}_R$ is the characteristic function of a disk of finite radius $R > 0$, centered at the origin, then

$$\liminf_{q \to \infty} \lambda_q^{1/2} \langle \mathbb{1}_R \phi_{0,q}, \phi_{0,q} \rangle_{L^2(\mathbb{R}^2)} > 0,$$

which implies the sharpness of estimates (2.3.23) with $\rho > 1$. Similarly, if $\rho \in (0,1)$, we can show that

$$\liminf_{q \to \infty} \lambda_q^{\rho/2} \langle \langle \cdot \rangle^{-\rho} \phi_{-q,q}, \phi_{-q,q} \rangle_{L^2(\mathbb{R}^2)} > 0,$$

which entails the sharpness of estimates (2.3.23) with $\rho \in (0,1)$. We do not know whether estimate (2.3.23) with $\rho = 1$ is sharp, but it is sufficient for our purposes.

Corollary 2.3.2. *Assume that* V *satisfies* (2.1.1) *with* $\rho \in (0,1)$. *Then for each* $\ell > 2/\rho$ *there exists a constant* c_ℓ *such that*

$$\|P_q V P_q\|_\ell \leq c_\ell \lambda_q^{\frac{1}{\ell} - \frac{\rho}{2}} (1 + |\ln \lambda_q|)^{1/\ell}, \quad q \in \mathbb{Z}_+. \tag{2.3.25}$$

Proof. Similarly to the proof of Proposition 2.3.7 above, an elementary variational argument shows that we may assume without loss of generality that $V(x) = \langle x \rangle^{-\rho}$, $x \in \mathbb{R}^2$. Note also that in the proof of estimate (2.3.25) we may assume that q is large enough since for any fixed q it follows from (2.3.18).

For brevity, set $T_q := P_q V P_q$, $q \in \mathbb{Z}_+$; by [59], $\operatorname{rank} T_q = \infty$. By (2.3.19) with $p = 2/\rho$, there exists a constant $\mathcal{C}$ such that

$$s_j(T_q) \le \mathcal{C} j^{-\rho/2}, \quad j \in \mathbb{N}, \quad q \in \mathbb{Z}_+. \tag{2.3.26}$$

On the other hand, (2.3.23) implies

$$s_1(T_q) \le c_\infty \lambda_q^{-\rho/2}, \quad q \in \mathbb{Z}_+. \tag{2.3.27}$$

Fix $\ell > 2/\rho$. By (2.3.26) – (2.3.27), for any $N \in \mathbb{N}$, we have

$$\|T_q\|_\ell^\ell = \sum_{j=1}^{\infty} s_j(T_q)^\ell = \sum_{j=1}^{N} s_j(T_q)^\ell + \sum_{j=N+1}^{\infty} s_j(T_q)^\ell \le$$

$$s_1(T_q)^{\ell-\frac{2}{\rho}} \sum_{j=1}^{N} s_j(T_q)^{\frac{2}{\rho}} + \mathcal{C}^\ell \sum_{j=N+1}^{\infty} j^{-\frac{\ell\rho}{2}} \le c_\infty^{\ell-\frac{2}{\rho}} \mathcal{C}^{2/\rho} \lambda_q^{1-\frac{\ell\rho}{2}} \sum_{j=1}^{N} j^{-1} + \mathcal{C}^\ell \sum_{j=N+1}^{\infty} j^{-\frac{\ell\rho}{2}} \le$$

$$\text{const.} \left(\lambda_q^{1-\frac{\ell\rho}{2}} (1 + \ln N) + N^{1-\frac{\ell\rho}{2}} \right)$$

with a constant independent of N and q. Assuming that q is large enough, and choosing N equal to the integer part of λ_q, we obtain (2.3.25). □

Remark: Estimate (2.3.25) should be regarded as an *a priori* estimate which is sufficient for our purposes.

2.3.4 Proof of Proposition 2.2.1

Given estimate (2.3.23) with $\rho \in (0,1)$, the proof of Proposition 2.2.1 is analogous to the one of [55, Proposition 1.1]; we include it just for the convenience of the reader.

In order to prove (2.2.1) it suffices to show that there exist $\tilde{C} > 0$ and $s_0 \in \mathbb{N}$ such that $s \ge s_0$ implies

$$\sigma(H) \cap [\lambda_s - B, \lambda_s + B] \subset \left(\lambda_s - \tilde{C} \lambda_s^{-\rho/2}, \lambda_s + \tilde{C} \lambda_s^{-\rho/2} \right). \tag{2.3.28}$$

Set $R_0(z) = (H_0 - z)^{-1}$, $z \in \mathbb{C} \setminus \sigma(H_0)$. By the Birman–Schwinger principle, $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ is an eigenvalue of H if and only if -1 is an eigenvalue of $|V|^{1/2} R_0(\lambda) V^{1/2}$ where

$$V^{1/2}(x) := \begin{cases} |V(x)|^{1/2} \operatorname{sign} V(x) & \text{if} \quad V(x) \neq 0, \\ 0 \quad \text{if} \quad V(x) = 0. \end{cases}$$

Hence, in order to prove (2.3.28), it suffices to show that for some $\tilde{C} > 0$ and $s_0 \in \mathbb{N}$, the inequalities $s \geq s_0$ and

$$\tilde{C} \lambda_s^{-\rho/2} < |\lambda_s - \lambda| \leq B \tag{2.3.29}$$

imply

$$\| |V|^{1/2} R_0(\lambda) |V|^{1/2} \| < 1. \tag{2.3.30}$$

Pick $m \in \mathbb{N}$ such that $\|V\|_{L^\infty(\mathbb{R}^2)} \leq \lambda_m / 2$. For $s \geq m$ write

$$R_0(\lambda) = \sum_{k=s-m}^{s+m} (\lambda_k - \lambda)^{-1} P_k + \tilde{R}_0(\lambda; s, m).$$

Then

$$\| |V|^{1/2} R_0(\lambda) V^{1/2} \| \leq \sum_{k=s-m}^{s+m} |\lambda_k - \lambda|^{-1} \| P_k |V| P_k \| + \| |V|^{1/2} \tilde{R}_0(\lambda; s, m) |V|^{1/2} \|. \tag{2.3.31}$$

By the choice of m, we have

$$\| |V|^{1/2} \tilde{R}_0(\lambda; s, m) |V|^{1/2} \| < \frac{1}{2}. \tag{2.3.32}$$

On the other hand, by (2.3.23) with $\rho \in (0, 1)$, we have

$$\sum_{k=s-m}^{s+m} |\lambda_k - \lambda|^{-1} \| P_k |V| P_k \| \leq c_\infty \lambda_{s-m}^{-\rho/2} (2m+1) |\lambda_s - \lambda|^{-1}$$

which implies

$$\sum_{k=s-m}^{s+m} |\lambda_k - \lambda|^{-1} \| P_k |V| P_k \| < \frac{1}{2}, \tag{2.3.33}$$

provided that the first inequality in (2.3.29) holds with appropriate $\tilde{C}$. Now, (2.3.30) follows from (2.3.31), (2.3.32), and (2.3.33).

In the proof of Theorem 2.2.1, we will need also the following

Proposition 2.3.8. *Assume that V satisfies* (2.1.1) *with $\rho \in (0,1)$. Then there exists a constant $C' > 0$ such that for each $q \in \mathbb{Z}_+$ we have*

$$\sigma\left((I-P_q)H(I-P_q)_{|(I-P_q)\mathrm{Dom}(H_0)}\right) \subset \bigcup_{s\in\mathbb{Z}_+\setminus\{q\}} \left(\lambda_s - C'\lambda_s^{-\rho/2}, \lambda_s + C'\lambda_s^{-\rho/2}\right). \quad (2.3.34)$$

The proof of Proposition 2.3.8 is quite the same as that of Proposition 2.2.1, so that we omit the details.

2.4 Proof of Theorem 2.2.1

2.4.1 Passing from V to its Weinstein Average $\langle V\rangle$

Assume that $V \in L^\infty(\mathbb{R}^2)$ and set

$$\langle V\rangle := \sum_{s\in\mathbb{Z}_+} P_s V P_s \quad (2.4.1)$$

where, a priori, the series converges strongly; this is the case if, for instance, $V = 1$ identically. If

$$\lim_{q\to\infty} \|P_q V P_q\| = 0 \quad (2.4.2)$$

which, by Proposition 2.3.7, is the case if V satisfies (2.1.1) with $\rho > 0$, then the series in (2.4.1) converges in norm. In order to check this, it suffices to show that $\{\langle V\rangle_q\}_{q\in\mathbb{Z}_+}$ with $\langle V\rangle_q := \sum_{s=0}^q P_s V P_s$, $q \in \mathbb{Z}_+$, is a Cauchy sequence in the uniform operator topology. By $P_j P_s = 0$ for $j \neq s$, we have

$$\|\langle V\rangle_{q+m} - \langle V\rangle_q\| \leq \sup_{j\geq q+1} \|P_j V P_j\|, \quad q \in \mathbb{Z}_+, \quad m \in \mathbb{N},$$

which combined with (2.4.2), implies the required property of the sequence $\{\langle V\rangle_q\}_{q\in\mathbb{Z}_+}$. Since

$$\langle V\rangle = \frac{B}{\pi}\int_0^{\pi/B} e^{-itH_0} V e^{itH_0}\, dt,$$

we call $\langle V\rangle$ the Weinstein average of V (see [70]). Set $\langle H\rangle := H_0 + \langle V\rangle$.

Proposition 2.4.1. *Under the hypotheses of Theorem 2.2.1 we have*

$$\mathrm{Tr}\,\varphi(\lambda_q^{\rho/2}(H-\lambda_q)) = \mathrm{Tr}\,\varphi(\lambda_q^{\rho/2}(\langle H\rangle - \lambda_q)) + o(\lambda_q), \quad q\to\infty, \quad (2.4.3)$$

for each $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$.

Proof. First, let us write the difference of the traces in (2.4.3) according to the Helffer–Sjöstrand formula (see the original works [26, 35], or the monographs [23, Section 2.2], [25, Chapter 8]). Let $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, and let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^2)$ be an almost analytic continuation of φ which satisfies

$$\operatorname{supp} \tilde{\varphi} \subset ((a_-, b_-) \cup (a_+, b_+)) \times (-c, c) \tag{2.4.4}$$

with $-\infty < a_- < b_- < 0 < a_+ < b_+ < \infty$, and $0 < c < \infty$, as well as

$$|\psi(x,y)| \leq C_N |y|^N, \quad (x,y) \in \mathbb{R}^2, \quad N > 0, \tag{2.4.5}$$

where $\psi := \frac{1}{2}\left(\frac{\partial \tilde{\varphi}}{\partial x} + i \frac{\partial \tilde{\varphi}}{\partial y}\right)$. For $(x,y) \in \mathbb{R}^2$ set $z = x + iy$ and

$$\psi_q(x,y) := \lambda_q^{\rho/2} \psi(\lambda_q^{\rho/2}(x - \lambda_q), \lambda_q^{\rho/2} y), \quad q \in \mathbb{Z}_+. \tag{2.4.6}$$

Then the Helffer–Sjöstrand formula yields

$$\varphi(\lambda_q^{\rho/2}(H - \lambda_q)) = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x,y)(\lambda^{\rho/2}(H - \lambda_q) - z)^{-1} dxdy = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi_q(x,y)(H - z)^{-1} dxdy.$$

Similarly,

$$\varphi(\lambda_q^{\rho/2}(\langle H \rangle - \lambda_q)) = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi_q(x,y)(\langle H \rangle - z)^{-1} dxdy.$$

Further, let ℓ_+ be the smallest integer (strictly) greater than $2/\rho$. Write the iterated resolvent identity

$$(H - z)^{-1} = \sum_{s=0}^{\ell_+ - 1} (-1)^s ((H_0 - z)^{-1} V)^s (H_0 - z)^{-1} + (-1)^{\ell_+} ((H_0 - z)^{-1} V)^{\ell_+} (H - z)^{-1}. \tag{2.4.7}$$

In the sequel, assume that $q \in \mathbb{Z}_+$ is so large that $-2B < a_- \lambda_q^{-\rho/2}$ and $b_+ \lambda_q^{-\rho/2} < 2B$ (see (2.4.4) and (2.4.6)). Then the sum on the r.h.s of (2.4.7) is holomorphic on the support of ψ_q. Therefore,

$$\varphi(\lambda_q^{\rho/2}(H - \lambda_q)) = \frac{(-1)^{\ell_+}}{\pi} \int_{\mathbb{R}^2} \psi_q(x,y)((H_0 - z)^{-1} V)^{\ell_+} (H - z)^{-1} dxdy.$$

Similarly,

$$\varphi(\lambda_q^{\rho/2}(\langle H \rangle - \lambda_q)) = \frac{(-1)^{\ell_+}}{\pi} \int_{\mathbb{R}^2} \psi_q(x,y)((H_0 - z)^{-1} \langle V \rangle)^{\ell_+} (\langle H \rangle - z)^{-1} dxdy.$$

Thus we get

$$(-1)^{\ell_+}\pi \operatorname{Tr}\left(\varphi(\lambda_q^{\rho/2}(H-\lambda_q)) - \varphi(\lambda_q^{\rho/2}(\langle H\rangle - \lambda_q))\right) =$$
$$\int_{\mathbb{R}^2} \psi_q(x,y)\left(((H_0-z)^{-1}V)^{\ell_+}(H-z)^{-1} - ((H_0-z)^{-1}\langle V\rangle)^{\ell_+}(\langle H\rangle - z)^{-1}\right) dxdy =$$
$$\sum_{j=1,2,3} T_j(q) \tag{2.4.8}$$

where

$$T_1(q) := \operatorname{Tr} \int_{\mathbb{R}^2} \psi_q(x,y)\left(((H_0-z)^{-1}V)^{\ell_+} - (\lambda_q - z)^{-\ell_+}(P_qV)^{\ell_+}\right)(H-z)^{-1}dxdy,$$

$$T_2(q) := \operatorname{Tr} \int_{\mathbb{R}^2} \psi_q(x,y)\left((\lambda_q - z)^{-\ell_+}(P_q\langle V\rangle)^{\ell_+} - ((H_0-z)^{-1}\langle V\rangle)^{\ell_+}\right)(\langle H\rangle - z)^{-1}dxdy,$$

$$T_3(q) := \operatorname{Tr} \int_{\mathbb{R}^2} \psi_q(x,y)(\lambda_q - z)^{-\ell_+}(P_qV)^{\ell_+}\left((H-z)^{-1} - P_q(\langle H\rangle - z)^{-1}\right) dxdy.$$

Writing $T_3(q)$, we have taken into account that $(P_q\langle V\rangle)^{\ell_+} = (P_qVP_q)^{\ell_+} = (P_qV)^{\ell_+}P_q$. Hence, in order to prove (2.4.3) it suffices to show that

$$T_j(q) = o(\lambda_q), \quad j = 1,2,3, \quad q \to \infty.$$

To this end we need some preliminary estimates. Namely, we will show that if $V \in L^p(\mathbb{R}^2)$, $p \in [2,\infty)$, $(x,y) \in \operatorname{supp}\psi_q$, and $q \in \mathbb{Z}_+$, then $(I-P_q)(H_0-z)^{-1}V \in S_p$, and there exists a constant c_p such that

$$\sup_{(x,y)\in \operatorname{supp}\psi_q} \|(I-P_q)(H_0-z)^{-1}V\|_p \le c_p\|V\|_{L^p(\mathbb{R}^2)} \tag{2.4.9}$$

for sufficiently large q. Assume at first $V \in L^\infty(\mathbb{R}^2)$. Then, evidently,

$$\|(I-P_q)(H_0-z)^{-1}V\| \le \sup_{s\in\mathbb{Z}_+\setminus\{q\}} |\lambda_s - z|^{-1}\|V\|_{L^\infty(\mathbb{R}^2)}. \tag{2.4.10}$$

Note that if $(x,y) \in \operatorname{supp}\psi_q$, then

$$|x-\lambda_q| = O(\lambda_q^{-\rho/2}), \quad q \to \infty, \tag{2.4.11}$$

(see (2.4.4) and (2.4.6)). Hence, for q large enough we have

$$\sup_{s\in\mathbb{Z}_+\setminus\{q\}} |\lambda_s - z|^{-1} \le B^{-1}, \quad (x,y) \in \operatorname{supp}\psi_q. \tag{2.4.12}$$

Assume now that $V \in L^2(\mathbb{R}^2)$. Then we have

$$\|(I-P_q)(H_0-z)^{-1}V\|_2^2 = \sum_{s\in\mathbb{Z}_+\setminus\{q\}} |\lambda_s - z|^{-2}\mathrm{Tr}P_s|V|^2P_s = \frac{B}{2\pi}\sum_{s\in\mathbb{Z}_+\setminus\{q\}} |\lambda_s - z|^{-2}\|V\|^2_{L^2(\mathbb{R}^2)} \tag{2.4.13}$$

(see (2.3.20)). Taking into account again (2.4.11), we find that for q large enough we have

$$\sum_{s\in\mathbb{Z}_+\setminus\{q\}} |\lambda_s - z|^{-2} \leq \frac{\pi^2}{3B^2}, \quad (x,y)\in \mathrm{supp}\,\psi_q. \tag{2.4.14}$$

Interpolating between (2.4.10) and (2.4.13) (see [9, Theorem 3.1]), and bearing in mind (2.4.12) and (2.4.14), we obtain (2.4.9). Further, since

$$(H_0-z)^{-1}V = (\lambda_q - z)^{-1}P_qV + (I-P_q)(H_0-z)^{-1}V,$$

elementary combinatorial arguments yield the estimate

$$|T_1(q)| \leq \sum_{m=0}^{\ell_+-1}\binom{\ell_+}{m}T_{1,m}(q) \tag{2.4.15}$$

where

$$T_{1,m}(q) := \int_{\mathbb{R}^2}|\psi_q(x,y)|\|(I-P_q)(H_0-z)^{-1}V\|_{\ell_+}^{\ell_+-m}|\lambda_q-z|^{-m}\|P_qV\|^m_{\ell_+}\|(H-z)^{-1}\|dxdy. \tag{2.4.16}$$

Our next goal is to show that

$$T_{1,m}(q) = o(\lambda_q), \quad q\to\infty, \quad m = 0,\dots,\ell_+-1. \tag{2.4.17}$$

To this end we apply:

- estimate (2.4.5) with $N=2$ in order to get
$$\sup_{(x,y)\in\mathrm{supp}\,\psi_q}|\psi_q(x,y)| \leq C_2\lambda_q^{3\rho/2}y^2;$$
- estimate (2.4.9) in order to handle $\|(I-P_q)(H_0-z)^{-1}V\|_{\ell_+}$;
- the fact that, due to (2.4.4) and (2.4.6), we have
$$\sup_{(x,y)\in\mathrm{supp}\,\psi_q}|\lambda_q - z|^{-1} = O(\lambda_q^{\rho/2}); \tag{2.4.18}$$

- estimate (2.3.22) in order to handle $\|P_q V\|_{\ell_+}$;
- the standard resolvent estimate $\|(H-z)^{-1}\| \leq |y|^{-1}$;
- the elementary estimate

$$\int_{\mathrm{supp}\,\psi_q} |y| dx dy = O(\lambda_q^{-3\rho/2}). \tag{2.4.19}$$

As a result, we obtain

$$T_{1,m}(q) \leq \text{const.}\, \lambda_q^{m\rho/2}, \quad m = 0, \ldots, \ell_+ - 1, \tag{2.4.20}$$

with a constant independent of q. We have $\lambda_q^{m\rho/2} = o(\lambda_q)$ as $q \to \infty$ in all the cases except the one where $2/\rho$ is an integer, and $m = \ell_+ - 1 = 2/\rho$. In this exceptional case however we have $m \geq 3$ and in all the terms of

$$\left((H_0 - z)^{-1} V\right)^{\ell_+} - (\lambda_q - z)^{-\ell_+} (P_q V)^{\ell_+}$$

which contain $m = \ell_+ - 1$ factors of the type $(\lambda_q - z)^{-1} P_q V$, at least two of these factors are neighbours. Therefore, in this exceptional case we can replace $\|P_q V\|_{\ell_+}^m$ by $\|P_q V\|_{\ell_+}^{m-1} \|P_q V P_q\|_{\ell_+}$ in (2.4.16), apply (2.3.25) with $\ell = \ell_+ = m+1$, and obtain

$$T_{1,m}(q) = O\left(\lambda_q^{1+\frac{1}{m+1}-\frac{1}{m}} (\ln \lambda_q)^{\frac{1}{m+1}}\right) = o(\lambda_q), \quad q \to \infty. \tag{2.4.21}$$

Now, (2.4.20) and (2.4.21) entail (2.4.17), which combined with (2.4.15) implies

$$|T_1(q)| = o(\lambda_q), \quad q \to \infty. \tag{2.4.22}$$

Similarly, we get

$$|T_2(q)| = o(\lambda_q), \quad q \to \infty. \tag{2.4.23}$$

Let us now turn to $T_3(q)$. First, note that due to the cyclicity of the trace, we have

$$T_3(q) =$$

$$\mathrm{Tr} \int_{\mathbb{R}^2} \psi_q(x,y) (\lambda_q - z)^{-\ell_+} (P_q V)^{\ell_+} \left((H-z)^{-1} P_q - P_q (P_q V P_q + \lambda_q - z)^{-1} P_q\right) dx dy.$$

Next, we need the Schur–Feshbach formula (see the original works [28, 63], or a contemporary exposition available, for instance, in [8, Appendix]). According to this formula,

$$(H-z)^{-1} =$$

$$P_qR_{\|}(z)P_q - P_qR_{\|}(z)V(I-P_q)R_{\perp}(z) - (I-P_q)R_{\perp}(z)VP_qR_{\|}(z)+$$
$$(I-P_q)\left(R_{\perp}(z) + R_{\perp}(z)VP_qR_{\|}(z)V(I-P_q)R_{\perp}(z)\right) \tag{2.4.24}$$

where $R_{\perp}(z)$ is the inverse of the operator $(I-P_q)(H-z)(I-P_q)$ defined on $(I-P_q)\mathrm{Dom}\,H_0$, and considered as an operator in the Hilbert space $(I-P_q)L^2(\mathbb{R}^2)$, while $R_{\|}(z)$ is the inverse of the operator

$$P_qVP_q - P_qV(I-P_q)R_{\perp}(z)VP_q + \lambda_q - z$$

considered as an operator in the Hilbert space $P_qL^2(\mathbb{R}^2)$. Applying (2.4.24) and the resolvent identity, we obtain

$$(H-z)^{-1}P_q - P_q(\lambda_q + P_qVP_q - z)^{-1}P_q =$$
$$R_{\|}P_qV(I-P_q)R_{\perp}(z)VP_q(P_qVP_q+\lambda_q-z)^{-1} - (I-P_q)R_{\perp}(z)VR_{\|}(z)P_q$$

Thus,

$$T_3(q) = T_{3,1}(q) + T_{3,2}(q) \tag{2.4.25}$$

where

$$T_{3,1}(q) :=$$
$$\mathrm{Tr}\int_{\mathbb{R}^2}\frac{\psi_q(x,y)}{(\lambda_q-z)^{\ell_+}}(P_qVP_q)^{\ell_+}R_{\|}(z)P_qV(I-P_q)R_{\perp}(z)VP_q(P_qVP_q+\lambda_q-z)^{-1}dxdy,$$

and

$$T_{3,2}(q) := -\mathrm{Tr}\int_{\mathbb{R}^2}\psi_q(x,y)(\lambda_q-z)^{-\ell_+}(P_qVP_q)^{\ell_+-1}P_qV(I-P_q)R_{\perp}(z)VP_qR_{\|}(z)P_qdxdy.$$

We have

$$|T_{3,1}(q)| \leq \tag{2.4.26}$$
$$\int_{\mathbb{R}^2}\frac{|\psi_q(x,y)|}{|\lambda_q-z|^{\ell_+}}\|P_qVP_q\|_{\ell_+}^{\ell_+}\|R_{\|}(z)\|\|P_qV\|\|VP_q\|\|R_{\perp}(z)\|\|(P_qVP_q+\lambda_q-z)^{-1}\|dxdy.$$

In order to show that

$$|T_{3,1}(q)| = o(\lambda_q), \quad q\to\infty, \tag{2.4.27}$$

we apply:

- estimate (2.4.5) with $N=3$ in order to get

$$\sup_{(x,y)\in\mathrm{supp}\,\psi_q}|\psi_q(x,y)| \leq C_3\lambda_q^{2\rho}|y|^3;$$

- estimate (2.4.18) in order to handle $|\lambda_q - z|^{-\ell_+}$;
- estimate (2.3.25) with $\ell = \ell_+$ in order to handle $\|P_q V P_q\|_{\ell_+}$;
- the standard resolvent estimates

$$\|R_{\|}(z)\| \leq |y|^{-1}, \quad \|(P_q V P_q + \lambda_q - z)^{-1}\| \leq |y|^{-1}; \tag{2.4.28}$$

- estimate (2.3.23) in order to conclude that

$$\|P_q V\| \|V P_q\| = \|P_q V^2 P_q\| \leq c_\infty \begin{cases} \lambda_q^{-\rho} & \text{if} \quad \rho \in \left(0, \frac{1}{2}\right), \\ \lambda_q^{-1/2} |\ln \lambda_q| & \text{if} \quad \rho = \frac{1}{2}, \\ \lambda_q^{-1/2} & \text{if} \quad \rho \in \left(\frac{1}{2}, 1\right), \end{cases} \quad q \in \mathbb{Z}_+; \tag{2.4.29}$$

- the elementary estimate (2.4.19);
- Proposition 2.3.8 in order to deduce the estimate

$$\sup_{(x,y) \in \operatorname{supp} \psi_q} \|R_\perp(z)\| = O(1), \quad q \to \infty; \tag{2.4.30}$$

As a result, we obtain

$$|T_{3,1}(q)| = O\left(\Phi_{1,\rho}(\lambda_q)\right), \quad q \to \infty, \tag{2.4.31}$$

where

$$\Phi_{1,\rho}(t) := \begin{cases} t^{1-\rho/2} |\ln t| & \text{if} \quad \rho \in \left(0, \frac{1}{2}\right), \\ t^{3/4} (\ln t)^2 & \text{if} \quad \rho = \frac{1}{2}, \\ t^{(1+\rho)/2} |\ln t| & \text{if} \quad \rho \in \left(\frac{1}{2}, 1\right), \end{cases} \quad t > 0.$$

Now, (2.4.31) implies (2.4.27). Finally, we have

$$|T_{3,2}(q)| \leq \int_{\mathbb{R}^2} |\psi_q(x,y)| |\lambda_q - z|^{-\ell_+} \|P_q V P_q\|_{\ell_+}^{\ell_+ - 1} \|P_q V\|_{\ell_+} \|R_\perp(z)\| \|P_q V\| \|R_{\|}(z)\| dx dy$$

In order to show that

$$|T_{3,2}(q)| = o(\lambda_q), \quad q \to \infty, \tag{2.4.32}$$

we apply (2.4.5) with $N = 2$, (2.4.18), (2.3.25), (2.3.22), the first estimate in (2.4.28), (2.4.30), (2.4.19), and (2.4.29). Thus we obtain

$$|T_{3,2}(q)| = O\left(\lambda_q^{\frac{(\ell_+ - 1)}{\ell_+} + \frac{\rho}{2}} (\ln \lambda_q)^{\frac{(\ell_+ - 1)}{\ell_+}} \Phi_{2,\rho}(\lambda_q)\right), \quad q \to \infty, \tag{2.4.33}$$

where

$$\Phi_{2,\rho}(t) := \begin{cases} t^{-\rho/2} \quad \text{if} \quad \rho \in \left(0, \frac{1}{2}\right), \\ t^{-1/4}|\ln t|^{1/2} \quad \text{if} \quad \rho = \frac{1}{2}, \qquad t > 0, \\ t^{-1/4} \quad \text{if} \quad \rho \in \left(\frac{1}{2}, 1\right), \end{cases}$$

and (2.4.32) follows from (2.4.33).

Now the combination of (2.4.8), (2.4.22), (2.4.23), (2.4.25), (2.4.27), and (2.4.32) yields (2.4.3). □

2.4.2 Passing to Individual Berezin–Toeplitz Operators

Proposition 2.4.2. *Assume the hypotheses of Theorem 2.2.1. Then for each* $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ *there exists* $q_0 \in \mathbb{Z}_+$ *such that*

$$\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(\langle H \rangle - \lambda_q)) = \operatorname{Tr} \varphi(\lambda_q^{\rho/2} P_q V P_q) \tag{2.4.34}$$

for $q \geq q_0$.

Proof. We have

$$\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(\langle H \rangle - \lambda_q)) = \sum_{s \in \mathbb{Z}_+} \operatorname{Tr} \varphi(\lambda_q^{\rho/2}(\lambda_s - \lambda_q + P_s V P_s)). \tag{2.4.35}$$

Due to the presence of the factor $\lambda_q^{\rho/2}$ in the traces $\operatorname{Tr} \varphi(\lambda_q^{\rho/2}(\lambda_s - \lambda_q + P_s V P_s))$, $s \in \mathbb{Z}_+$, it suffices to show that there exists q_0 such that for $q \geq q_0$ the operators

$$\lambda_s - \lambda_q + P_s V P_s = 2B(s-q) + P_s V P_s, \quad s \neq q, \tag{2.4.36}$$

are invertible, and

$$\sup_{q \geq q_0} \sup_{s \in \mathbb{Z}_+ \setminus \{q\}} \|(\lambda_s - \lambda_q + P_s V P_s)^{-1}\| < \infty. \tag{2.4.37}$$

Since $\|P_s V P_s\| \leq \|V\|_{L^\infty(\mathbb{R}^2)}$, $s \in \mathbb{Z}_+$, there exists $m \in \mathbb{N}$ such that the operators in (2.4.36) with $|s-q| > m$ are invertible, and

$$\sup_{q \in \mathbb{Z}_+} \sup_{s \in \mathbb{Z}_+ : |s-q| > m} \|(\lambda_s - \lambda_q + P_s V P_s)^{-1}\| < \infty. \tag{2.4.38}$$

On the other hand, Proposition 2.3.7 implies that for any fixed $j \in \mathbb{Z}$ we have

$$\lim_{q \to \infty} \|P_{q+j} V P_{q+j}\| = 0.$$

Therefore, there exists $q_0 \in \mathbb{Z}_+$ such that the operators in (2.4.36) with $|s-q| \le m$ are invertible for $q \ge q_0$, and

$$\sup_{q \ge q_0} \max_{s \in \mathbb{Z}_+ \setminus \{q\} : |s-q| \le m} \|(\lambda_s - \lambda_q + P_s V P_s)^{-1}\| < \infty. \tag{2.4.39}$$

Putting together (2.4.38) and (2.4.39), we obtain (2.4.37), and hence (2.4.34). □

2.4.3 Passing from $P_q V P_q$ to $\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}})$

Introduce the operator $\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}})$. We have

$$\widehat{\mathbb{V}_B * \delta_{\sqrt{2q+1}}}(\zeta) = \widehat{\mathbb{V}_B}(\zeta) J_0(\sqrt{2q+1}|\zeta|), \quad \zeta \in \mathbb{R}^2.$$

Note that J_0 is an entire function, and $|J_0(r)| \le 1$, $r \in \mathbb{R}$. On the other hand, $\widehat{\mathbb{V}_B} \in \mathcal{H}^{\sharp}_{-2+\rho}(\mathbb{R}^2)$. Therefore, $\widehat{\mathbb{V}_B * \delta_{\sqrt{2q+1}}} \in L_w^{2/(2-\rho)}(\mathbb{R}^2)$, and by Proposition 2.3.2 we have $\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}}) \in S_{2/\rho;w}$. The main result of this subsection is

Proposition 2.4.3. *Under the hypotheses of Theorem 2.2.1 we have*

$$\mathrm{Tr}\, \varphi(\lambda_q^{\rho/2} P_q V P_q) = \mathrm{Tr}\, \varphi(\lambda_q^{\rho/2} \mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}})) + o(\lambda_q), \quad q \to \infty. \tag{2.4.40}$$

In the proof of Proposition 2.4.3, as well as in the next subsection, we will use systematically the following auxiliary result.

Lemma 2.4.1. *Let $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, and let $T = T^*$ and $Q = Q^*$ be compact operators.*
(i) *Assume $Q \in S_m$, $m \in [2, \infty)$, $T \in S_\ell$, $\ell \in [m, \infty)$. Then*

$$\|\varphi(T+Q) - \varphi(T)\|_1 \le c_{\varphi,m,\ell} \|Q\|_m \left(\|T\|_\ell^{\ell/m'} + \|Q\|_\ell^{\ell/m'} \right) \tag{2.4.41}$$

with $m' = m/(m-1)$ and a constant $c_{\varphi,m,\ell}$ independent of T and Q.
(ii) *Assume $Q \in S_2$, $T \in S_\ell$, $\ell \in [2, \infty)$. Then*

$$\|\varphi(T+Q) - \varphi(T)\|_1 \le c_{\varphi,\ell} \|Q\|_2 \left(\|T\|_\ell^{\ell/2} + \|Q\|_2 \right) \tag{2.4.42}$$

with a constant $c_{\varphi,\ell}$ independent of T and Q.
(iii) *Assume $Q \in S_2$, $T \in S_{\ell;w}$, $\ell \in [2, \infty)$. Then*

$$\|\varphi(T+Q) - \varphi(T)\|_1 \le c_{\varphi,\ell;w} \|Q\|_2 \left(\|T\|_{\ell,w}^{\ell/2} + \|Q\|_2 \right) \tag{2.4.43}$$

with a constant $c_{\varphi,\ell;w}$ independent of T and Q.

Proof. By [54], for each $p \in (1,\infty)$ and each Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_{\rm Lip} := \sup_{\lambda\in\mathbb{R},\, \mu\in\mathbb{R},\, \lambda\neq\mu} \frac{|f(\lambda)-f(\mu)|}{|\lambda-\mu|} < \infty,$$

there exists a constant c_p such that

$$\|f(M+N)-f(M)\|_p \leq c_p \|f\|_{\rm Lip}\|N\|_p \tag{2.4.44}$$

for each self-adjoint M, and each $N = N^* \in S_p$. Further, assume $f \in C_0^\infty(\mathbb{R}\setminus\{0\})$, and set $\delta := {\rm dist}\,(0, {\rm supp}\, f)$. Then for any $M = M^* \in S_\infty$ we have

$$\|f(M)\|_s \leq \max_{\lambda\in\mathbb{R}} |f(\lambda)| \left(\sum_{j: s_j(M)>\delta} 1\right)^{1/s}. \tag{2.4.45}$$

Let $M = M^* \in S_p$, $p \in [1,\infty)$. Then

$$\sum_{j: s_j(M)>\delta} 1 \leq \delta^{-p}\|M\|_p^p. \tag{2.4.46}$$

Finally, let $M = M^* \in S_{p,w}$, $p \in (1,\infty)$. Then

$$\sum_{j: s_j(M)>\delta} 1 \leq \sum_{j: \|M\|_{p,w} j^{-1/p}>\delta} 1 \leq \delta^{-p}\|M\|_{p,w}^p. \tag{2.4.47}$$

Next, pick a real function $\nu \in C_0^\infty(\mathbb{R}\setminus\{0\})$ such that $\nu = 1$ on the support of φ. Then $\varphi(T) = \varphi(T)\nu(T)$, $\varphi(T+Q) = \varphi(T+Q)\nu(T+Q)$. Assume now $Q \in S_m$, $m \in [2,\infty)$, $T \in S_\ell$, $\ell \in [m,\infty)$. Then

$$\|\varphi(T+Q)-\varphi(T)\|_1 \leq \|\nu(T+Q)-\nu(T)\|_m\|\varphi(T)\|_{m'} + \|\varphi(T+Q)-\varphi(T)\|_m\|\nu(T+Q)\|_{m'}, \tag{2.4.48}$$

and (2.4.41) follows from (2.4.48), (2.4.44) with $M = T$, $N = Q$, and $p = m$, (2.4.45) with $M = T$ or $M = T+Q$ and $s = m'$, (2.4.46) with $M = T$ or $M = T+Q$ and $p = \ell$, and the convexity of the function $t \mapsto t^{\ell/m'}$, $t \geq 0$. Further, assume $Q \in S_2$. Then, instead of (2.4.48), we can write

$$\|\varphi(T+Q)-\varphi(T)\|_1 \leq \tag{2.4.49}$$

$$\|\nu(T+Q)-\nu(T)\|_2\|\varphi(T)\|_2 + \|\varphi(T+Q)-\varphi(T)\|_2\left(\|\nu(T)\|_2 + \|\nu(T+Q)-\nu(T)\|_2\right).$$

If we assume now $T \in S_\ell$ (resp., $T \in S_{\ell,w}$) with $\ell \in [2,\infty)$, then (2.4.42) (resp., (2.4.43)) follows from (2.4.49), (2.4.44) with $M = T$, $N = Q$, and $p = 2$, (2.4.45) with $M = T$ and $s = 2$, and (2.4.46) (resp., (2.4.47)) with $M = T$ and $p = \ell$. □

Proof of Proposition 2.4.3. Pick a real radially symmetric $\eta \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^2$, $\eta(x) = 1$ for $|x| \leq 1/2$, $\eta(x) = 0$ for $|x| > 1$. Our first goal is to show that

$$\operatorname{Tr}\varphi(\lambda_q^{\rho/2}P_qVP_q) = \operatorname{Tr}\varphi(\lambda_q^{\rho/2}P_q(1-\eta)\mathbb{V}P_q) + o(\lambda_q), \quad q \to \infty. \tag{2.4.50}$$

Evidently,

$$|\operatorname{Tr}\varphi(\lambda_q^{\rho/2}P_qVP_q) - \operatorname{Tr}\varphi(\lambda_q^{\rho/2}P_q(1-\eta)\mathbb{V}P_q)| \leq \tag{2.4.51}$$

$$\|\varphi(\lambda_q^{\rho/2}P_qVP_q)-\varphi(\lambda_q^{\rho/2}P_q(1-\eta)VP_q)\|_1+\|\varphi(\lambda_q^{\rho/2}P_q(1-\eta)VP_q)-\varphi(\lambda_q^{\rho/2}P_q(1-\eta)\mathbb{V}P_q)\|_1.$$

Applying (2.4.42) with $\ell > 2/\rho$, $T = \lambda_q^{\rho/2}P_qVP_q$ and $Q = -\lambda_q^{\rho/2}P_q\eta VP_q$, (2.3.18) with $p = 2$, and (2.3.25), we obtain the estimate

$$\|\varphi(\lambda_q^{\rho/2}P_qVP_q) - \varphi(\lambda_q^{\rho/2}P_q(1-\eta)VP_q)\|_1 = O\left(\lambda_q^{(1+\rho)/2}(\ln\lambda_q)^{1/2}\right) = o(\lambda_q), \quad q \to \infty. \tag{2.4.52}$$

Similarly, assuming without loss of generality that $\varepsilon \in (0, 1-\rho)$ in (2.2.2), and then applying (2.4.41), with $\ell = m > 2/\rho$, $T = \lambda_q^{\rho/2}P_q(1-\eta)VP_q$, $Q = -\lambda_q^{\rho/2}P_q(1-\eta)(V - \mathbb{V})P_q$, as well as (2.3.25), we obtain

$$\|\varphi(\lambda_q^{\rho/2}P_q(1-\eta)VP_q) - \varphi(\lambda_q^{\rho/2}P_q(1-\eta)\mathbb{V}P_q)\|_1 = O\left(\lambda_q^{1-\frac{\varepsilon}{2}}(\ln\lambda_q)\right) = o(\lambda_q), \quad q \to \infty. \tag{2.4.53}$$

Now, (2.4.51), (2.4.52), and (2.4.53) imply (2.4.50). Further, by Proposition 2.3.3 we have

$$\operatorname{Tr}\varphi(\lambda_q^{\rho/2}P_q(1-\eta)\mathbb{V}P_q) = \operatorname{Tr}\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B * \Psi_q)). \tag{2.4.54}$$

Our next goal is to show that

$$\operatorname{Tr}\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B * \Psi_q)) = \operatorname{Tr}\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}})) + o(\lambda_q), \quad q \to \infty. \tag{2.4.55}$$

Similarly to (2.4.51), we have

$$|\operatorname{Tr}\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B * \Psi_q)) - \operatorname{Tr}\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}}))| \leq$$

$$\|\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\Psi_q))-\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}))\|_1+$$

$$\|\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}))-\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B*\delta_{\sqrt{2q+1}}))\|_1. \qquad (2.4.56)$$

Applying (2.4.42) with $\ell > 2/\rho$,

$$T = \lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\Psi_q),$$

$$Q = \lambda_q^{\rho/2}\left(\mathrm{Op}^{\mathrm{w}}((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}) - \mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\Psi_q)\right),$$

as well as Proposition 2.3.3, (2.3.25), and Proposition 2.3.5, we get

$$\|\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\Psi_q))-\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}))\|_1 =$$

$$O\left(\lambda_q^{\frac{1+\rho}{2}-\frac{3}{4}}(\ln\lambda_q)^{1/2}\right) = o(\lambda_q), \quad q\to\infty. \qquad (2.4.57)$$

In order to estimate the second factor at the r.h.s. of (2.4.56), we need an estimate of the Hilbert–Schmidt norm of the operator $\mathrm{Op}^{\mathrm{w}}((\eta\mathbb{V})_B*\delta_{\sqrt{2q+1}}))$. By (2.3.3) and the generalized Young inequality (see e.g. [60, Section IX.4]) we have

$$\|\mathrm{Op}^{\mathrm{w}}((\eta\mathbb{V})_B*\delta_{\sqrt{2q+1}}))\|_2^2 = \frac{1}{2\pi}\|\widehat{(\eta\mathbb{V})_B*\delta_{\sqrt{2q+1}}}\|^2_{L^2(\mathbb{R}^2)} =$$

$$= \frac{B^2}{(2\pi)^3}\int_{\mathbb{R}^2}|(\hat\eta*\hat{\mathbb{V}})(B^{1/2}\zeta)|^2 J_0(\sqrt{2q+1}|\zeta|)^2 d\zeta \le$$

$$\frac{B}{(2\pi)^3}\|\hat\eta*\hat{\mathbb{V}}\|^2_{L^2(\mathbb{R}^2)} \le \frac{cB}{(2\pi)^3}\|\hat\eta\|^2_{L^{2/(1+\rho)}(\mathbb{R}^2)}\|\hat{\mathbb{V}}\|^2_{L_w^{2/(2-\rho)}(\mathbb{R}^2)} \qquad (2.4.58)$$

with a constant c which depends only on ρ. Applying (2.4.42) with $\ell > 2/\rho$ and $T = \lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}))$, $Q = \lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}((\eta\mathbb{V})_B*\delta_{\sqrt{2q+1}}))$, as well as Propositions 2.3.3 and 2.3.5, Corollary 2.3.2, and (2.4.58), we get

$$\|\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(((1-\eta)\mathbb{V})_B*\delta_{\sqrt{2q+1}}))-\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B*\delta_{\sqrt{2q+1}}))\|_1 =$$

$$= O\left(\lambda_q^{(1+\rho)/2}(\ln\lambda_q)^{1/2}\right) = o(\lambda_q), \quad q\to\infty. \qquad (2.4.59)$$

Now, (2.4.56), (2.4.57), and (2.4.59) imply (2.4.55), while (2.4.50), (2.4.54), and (2.4.55) imply (2.4.40). □

2.4.4 Semiclassical Analysis of $\mathrm{Tr}\,\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}}))$

Proposition 2.4.4. *Under the hypotheses of Theorem 2.2.1 we have*

$$\lim_{q\to\infty} \lambda_q^{-1}\mathrm{Tr}\,\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}})) = \frac{1}{2\pi B}\int_{\mathbb{R}^2} \varphi(B^\rho \mathring{\mathbb{V}}(x))dx. \tag{2.4.60}$$

Proof. Let $s : \mathbb{R}^{2d} \to \mathbb{C}$ be an appropriate Weyl symbol. For $\hbar > 0$ set

$$s_\hbar(x,\xi) := s(x,\hbar\xi), \quad (x,\xi) \in \mathbb{R}^{2d},$$

and define the $\hbar$-ΨDO $\mathrm{Op}^{\mathrm{w}}_\hbar(s) := \mathrm{Op}^{\mathrm{w}}(s_\hbar)$. Set

$$\tilde{s}_\hbar(x,\xi) := s(\sqrt{\hbar}x, \sqrt{\hbar}\xi), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

A simple rescaling argument shows that the operators $\mathrm{Op}^{\mathrm{w}}_\hbar(s)$ and $\mathrm{Op}^{\mathrm{w}}(\tilde{s}_\hbar)$ are unitarily equivalent (see e.g. [64, Section A2.1]). Set

$$\mathbf{s}(z) := B^\rho\, \mathring{\mathbb{V}}_1(z), \quad z \in \mathbb{R}^2.$$

Due to the homogeneity of $\mathbb{V}$ we have

$$\lambda_q^{\rho/2}\, \mathbb{V}_B * \delta_{\sqrt{2q+1}}(z) = \mathbf{s}((2q+1)^{-1/2}z), \quad z \in \mathbb{R}^2.$$

Therefore, the operator $\varphi(\lambda_q^{\rho/2}\mathrm{Op}^{\mathrm{w}}(\mathbb{V}_B * \delta_{\sqrt{2q+1}}))$ is unitarily equivalent to $\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}))$ with $\hbar := (2q+1)^{-1}$. Now, in order to prove (2.4.60), it suffices to show that

$$\lim_{\hbar\to 0} \hbar\,\mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \varphi(\mathbf{s}(x))dx, \tag{2.4.61}$$

since $\int_{\mathbb{R}^2} \varphi(\mathbf{s}(x))dx = \int_{\mathbb{R}^2} \varphi(B^\rho\mathring{\mathbb{V}}(x))dx$. If the symbol $\mathbf{s}$ were regular, then (2.4.61) would follow from standard semiclassical results (see e.g. [25, Theorem 9.6]). Due to the singularity of $\mathbf{s}$ at $\mathbb{S}^1$, we need some additional final estimates. Pick the function η defined at the beginning of the proof of Proposition 2.4.3, and for $r > 0$ set $\eta_r(x) := \eta(r^{-1}x)$, $x \in \mathbb{R}^2$. Define the symbols

$$\mathbf{s}_{1,r}(x) := B^\rho\,((1-\eta_r)\mathbb{V}_1)(x), \mathbf{s}_{2,r}(x) := B^\rho\,(\eta_r\mathbb{V}_1)(x), \quad x \in \mathbb{R}^2,$$

so that $\mathbf{s} = \mathbf{s}_{1,r} + \mathbf{s}_{2,r}$. Evidently, $\mathbf{s}_{1,r} \in \mathcal{S}_1^{-\rho}(\mathbb{R}^2)$. By [25, Theorem 9.6],

$$\lim_{\hbar\to 0} \hbar\,\mathrm{Tr}\,(\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{1,r})) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \varphi(\mathbf{s}_{1,r}(x))dx. \tag{2.4.62}$$

On the other hand, estimate (2.4.43) with $\ell = 2/\rho$, $T = \mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})$ and $Q = -\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{2,r})$ implies

$$|\mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})) - \mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{1,r}))| \leq$$
$$c_{\varphi,\ell;w}\|\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{2,r})\|_2\left(\|\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})\|^{1/\rho}_{2/\rho;w} + \|\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{2,r})\|_2\right). \tag{2.4.63}$$

By Proposition 2.3.2,

$$\|\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})\|_{2/\rho;w} \leq \hbar^{-\rho/2}(2\pi)^{-(1-\frac{2}{\rho})}\|\hat{\mathbf{s}}\|_{L^{2/(2-\rho)}_w(\mathbb{R}^2)} \tag{2.4.64}$$

and, similarly to (2.4.58),

$$\|\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{2,r})\|_2 = \hbar^{-1/2}(2\pi)^{-1/2}\|\widehat{\mathbf{s}_{2,r}}\|_{L^2(\mathbb{R}^2)} \leq \hbar^{-1/2}c\|\hat{\mathbb{V}}\|_{L^{2/(2-\rho)}_w(\mathbb{R}^2)}\|\widehat{\eta_r}\|_{L^{2/(1+\rho)}(\mathbb{R}^2)}. \tag{2.4.65}$$

Finally,

$$\|\widehat{\eta_r}\|_{L^{2/(1+\rho)}(\mathbb{R}^2)} = r^{1-\rho}\|\widehat{\eta}\|_{L^{2/(1+\rho)}(\mathbb{R}^2)}. \tag{2.4.66}$$

As a result, we find that (2.4.63) – (2.4.66) imply the existence of a constant C such that the estimate

$$|\mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})) - \mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s}_{1,r}))| \leq C\hbar^{-1}r^{1-\rho} \tag{2.4.67}$$

is valid for each $\hbar > 0$ and $r \in (0,1)$. Now, (2.4.62) and (2.4.67) yield

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\varphi(\mathbf{s}_{1,r}(x))dx - Cr^{1-\rho} \leq$$
$$\liminf_{\hbar\downarrow 0}\hbar\mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})) \leq \limsup_{\hbar\downarrow 0}\hbar\mathrm{Tr}\,\varphi(\mathrm{Op}^{\mathrm{w}}_\hbar(\mathbf{s})) \leq$$
$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\varphi(\mathbf{s}_{1,r}(x))dx + Cr^{1-\rho}.$$

Letting $r \downarrow 0$, and taking into account that

$$\lim_{r\downarrow 0}\int_{\mathbb{R}^2}\varphi(\mathbf{s}_{1,r}(x))dx = \int_{\mathbb{R}^2}\varphi(\mathbf{s}(x))dx,$$

we obtain (2.4.61), and hence (2.4.60). □

Putting together (2.4.3), (2.4.34), (2.4.40), and (2.4.60), we arrive at (2.2.3) which completes the proof of Theorem 2.2.1.

Chapter 3

Local Spectral Asymptotics for Metric Perturbations of the Landau Hamiltonian

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Abstract. We consider metric perturbations of the Landau Hamiltonian. We investigate the asymptotic behaviour of the discrete spectrum of the perturbed operator near the Landau levels, for perturbations of compact support, and of exponential or power-like decay at infinity.

Keywords: Landau Hamiltonian, metric perturbations, position-dependent mass, spectral asymptotics

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3.1 Introduction

Let

$$H_0 := (-i\nabla - A_0)^2,$$

with $A_0 = (A_{0,1}, A_{0,2}) := \frac{b}{2}(-x_2, x_1)$, be the Landau Hamiltonian, self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. In other words, H_0 is the 2D Schrödinger operator with constant scalar magnetic field $b > 0$, i.e. the Hamiltonian of a 2D spinless non relativistic quantum particle subject to a constant magnetic field. As is well known, the spectrum $\sigma(H_0)$ consists of infinitely degenerate eigenvalues $\Lambda_q := b(2q+1)$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, called *Landau levels* (see e.g. [30, 43]).

In the present article we consider metric perturbations of H_0. Namely, let

$$m(x) = \{m_{jk}(x)\}_{j,k=1,2}, \quad x \in \mathbb{R}^2,$$

be a Hermitian 2×2 matrix such that $m(x) \geq 0$ for all $x \in \mathbb{R}^2$. Throughout the article we assume that $m_{jk} \in C_{\rm b}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$, i.e. $m_{jk} \in C^\infty(\mathbb{R}^2)$, and m_{jk} together with all its derivatives are bounded on $\mathbb{R}^2$. Set

$$\Pi_j := -i\frac{\partial}{\partial x_j} - A_{0,j}, \quad j = 1, 2, \tag{3.1.1}$$

so that $H_0 = \Pi_1^2 + \Pi_2^2$. On $\operatorname{Dom} H_0$ define the operators

$$H_\pm := \sum_{j,k=1,2} \Pi_j(\delta_{jk} \pm m_{jk})\Pi_k = H_0 \pm W$$

where $W := \sum_{j,k=1,2} \Pi_j m_{jk} \Pi_k$; in the case of H_-, we suppose additionally that

$$\sup_{x \in \mathbb{R}^2} |m(x)| < 1.$$

Thus the matrices $g_\pm(x) = \left\{g_{jk}^\pm(x)\right\}_{j,k=1,2}$ with $g_{jk}^\pm := \delta_{jk} \pm m_{jk}$ are positive definite for each $x \in \mathbb{R}^2$. Under these assumptions, the operators $H_\pm$ are self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see Section 3.7).

From mathematical physics point of view, the operators $H_\pm$ are special cases of Schrödinger operators with *position-dependent mass* which have been investigated since long ago (see e.g. [7, 69]), but the interest towards which increased essentially during the last decade (see e.g. [31, 40, 52]). Here we would like to mention especially the article [24] where the

model considered is quite close to the operators $H_\pm$ discussed in the present paper.
The operators $H_\pm$ admit also a geometric interpretation since they are related to the Bochner Laplacians corresponding to connections with constant non-vanishing curvature (see e.g. [19, 61]); we discuss this relation in more detail at the end of Section 3.2. Further, assume that

$$\lim_{|x|\to\infty} m_{jk}(x) = 0, \quad j,k = 1,2. \tag{3.1.2}$$

Thus m models a localized perturbation with respect to a reference medium. Under condition (3.1.2) the resolvent difference $H_\pm^{-1} - H_0^{-1}$ is a compact operator (see Section 3.7), and therefore the essential spectra of $H_\pm$ and H_0 coincide, i.e.

$$\sigma_{\rm ess}(H_\pm) = \sigma_{\rm ess}(H_0) = \sigma(H_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}\,.$$

The spectrum $\sigma(H_\pm)$ on $\mathbb{R} \setminus \bigcup_{q=0}^{\infty} \{\Lambda_q\}$ may consist of discrete eigenvalues whose only possible accumulation points are the Landau levels. Moreover, taking into account that $W \geq 0$, and applying [10, Th. 7, Section 9.4], we find that the eigenvalues of H_+ (resp., H_-) may accumulate to a given Landau level Λ_q only from above (resp., from below). Fix $q \in \mathbb{Z}_+$. Let $\left\{\lambda_{k,q}^-\right\}$ be the eigenvalues of H_- lying on the interval $(\Lambda_{q-1}, \Lambda_q)$ with $\Lambda_{-1} := -\infty$, counted with the multiplicities, and enumerated in increasing order. Similarly, let $\left\{\lambda_{k,q}^+\right\}$ be the eigenvalues of H_+ lying on the interval $(\Lambda_q, \Lambda_{q+1})$, counted with the multiplicities, and enumerated in decreasing order.
The aim of the article is to investigate the rate of convergence of $\lambda_{k,q}^\pm - \Lambda_q$ as $k \to \infty$, $q \in \mathbb{Z}_+$ being fixed, for perturbations m of compact support, of exponential decay, or of power-like decay at infinity.
The properties of the discrete spectrum generated by perturbative second-order differential operators with decaying coefficients have been considered also in [2, 14, 15, 57].
The article is organized as follows. In Section 3.2 we formulate our main results, and briefly comment on them. In Section 3.3 we reduce our analysis to the study of operators of Berezin–Toeplitz type, and in Section 3.4 we establish several useful unitary equivalences for these operators. Section 3.5 contains the proofs of our results in the case of rapid decay, i.e. of compact support or exponential decay, while the proofs for slow, i.e. power-like decay, could be found in Section 3.6. Finally, in Section 3.7 we address some standard issues concerning the domain of the operators $H_\pm$, and the compactness of the resolvent difference $H_0^{-1} - H_\pm^{-1}$.

3.2 Main Results

First, we formulate our results concerning perturbations m of compact support. Denote by $m_<(x)$ and $m_>(x)$ with $m_<(x) \le m_>(x)$, the two eigenvalues of the matrix $m(x)$, $x \in \mathbb{R}^2$.

Theorem 3.2.1. *Assume that the support of the matrix m is compact, and its smaller eigenvalue $m_<$ does not vanish identically. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\ln\left(\pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right)\right) = -k\ln k + O(k), \quad k \to \infty. \tag{3.2.1}$$

Remarks: (i) Under additional technical hypotheses on $m_{\gtrless}$, we could make asymptotic relation (3.2.1) more precise. Namely, assume that there exists a non increasing sequence $\{s_j\}_{j\in\mathbb{N}}$, such that $s_j > 0$, $j \in \mathbb{N}$, $\lim_{j\to\infty} s_j = 0$, and the level lines

$$\left\{x \in \mathbb{R}^2 \,|\, m_<(x) = s_j\right\}, \quad j \in \mathbb{N},$$

are bounded Lipschitz curves. In particular, the existence of such sequence follows from the Sard lemma (see e.g. [66, Th. 3.1, Chapter II]) if we assume that $m_< \in C^2(\mathbb{R}^2)$. Further, denote by $\mathcal{C}_{\gtrless}$ the *logarithmic capacities* (see e.g. [44, Section 4, Chapter II]) of $\operatorname{supp} m_{\gtrless}$. Then we have

$$\left(1 + \ln\left(b\mathcal{C}_<^2/2\right)\right)k + o(k) \le \ln\left(\pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right)\right) + k\ln k \le \left(1 + \ln\left(b\mathcal{C}_>^2/2\right)\right)k + o(k) \tag{3.2.2}$$

as $k \to \infty$. We omit the details of the proof of (3.2.2), inspired by [29].
(ii) For $q \in \mathbb{Z}_+$ and $\lambda > 0$, set

$$\mathcal{N}_q^{\pm}(\lambda) := \#\left\{k \in \mathbb{Z}_+ \,|\, \pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right) > \lambda\right\}. \tag{3.2.3}$$

Then a less precise version of (3.2.1), namely

$$\ln\left(\pm\left(\lambda_{k,q}^{\pm} - \Lambda_q\right)\right) = -k\ln k\,(1 + o(1)), \quad k \to \infty,$$

is equivalent to

$$\mathcal{N}_q^{\pm}(\lambda) = \frac{|\ln\lambda|}{\ln|\ln\lambda|}(1 + o(1)), \quad \lambda \downarrow 0. \tag{3.2.4}$$

Further, we state our results concerning perturbations of exponential decay. Assume that there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\ln m_{\gtrless}(x) = -\gamma |x|^{2\beta} + O(\ln |x|), \quad |x| \to \infty. \tag{3.2.5}$$

Remark: In (3.2.5), we suppose that the values of γ and β are the same for $m_<$ and $m_>$. Of course, the remainder $O(\ln |x|)$ could be different for $m_<$ and $m_>$.
Given $\beta > 0$ and $\gamma > 0$, set $\mu := \gamma(2/b)^\beta$, $b > 0$ being the constant magnetic field.

Theorem 3.2.2. *Let $m_{\gtrless}$ satisfy* (3.2.5)*. Fix $q \in \mathbb{Z}_+$.*
(i) *If $\beta \in (0,1)$, then there exist constants $f_j = f_j(\beta, \mu)$, $j \in \mathbb{N}$, with $f_1 = \mu$, such that*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q} - \Lambda_q\right)\right) = -\sum_{1 \le j < \frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{3.2.6}$$

(ii) *If $\beta = 1$, then*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q} - \Lambda_q\right)\right) = -\left(\ln\left(1+\mu\right)\right) k + O(\ln k), \quad k \to \infty. \tag{3.2.7}$$

(iii) *If $\beta \in (1,\infty)$, then there exist constants $g_j = g_j(\beta, \mu)$, $j \in \mathbb{N}$, such that*

$$\ln\left(\pm\left(\lambda^{\pm}_{k,q} - \Lambda_q\right)\right) =$$

$$-\frac{\beta-1}{\beta} k \ln k + \left(\frac{\beta - 1 - \ln(\mu\beta)}{\beta}\right) k - \sum_{1 \le j < \frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + O(\ln k), \quad k \to \infty. \tag{3.2.8}$$

Remarks: (i) Let us describe explicitly the coefficients f_j and g_j, $j \in \mathbb{N}$, appearing in (3.2.6) and (3.2.8) respectively. Assume at first $\beta \in (0,1)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| << 1$, introduce the function

$$F(s;\epsilon) := s - \ln s + \epsilon \mu s^\beta. \tag{3.2.9}$$

Denote by $s_<(\epsilon)$ the unique positive solution of the equation $s = 1 - \epsilon\beta\mu s^\beta$, so that $\frac{\partial F}{\partial s}(s_<(\epsilon);\epsilon) = 0$. Set

$$f(\epsilon) := F(s_<(\epsilon);\epsilon). \tag{3.2.10}$$

Note that f is a real analytic function for small $|\epsilon|$. Then $f_j := \frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0)$, $j \in \mathbb{N}$.
Let now $\beta \in (1,\infty)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| << 1$, introduce the function

$$G(s;\epsilon) := \mu s^\beta - \ln s + \epsilon s. \tag{3.2.11}$$

Denote by $s_>(\epsilon)$ the unique positive solution of the equation $\beta\mu s^\beta = 1-\epsilon s$ so that $\frac{\partial G}{\partial s}(s_>(\epsilon);\epsilon) = 0$. Define

$$g(\epsilon) := G(s_>(\epsilon);\epsilon), \tag{3.2.12}$$

which is a real analytic function for small $|\epsilon|$. Then $g_j := \frac{1}{j!}\frac{d^j g}{d\epsilon^j}(0)$, $j \in \mathbb{N}$.
(ii) If, instead of (3.2.5), we assume that

$$\ln m_{\gtrless}(x) = -\gamma|x|^{2\beta}(1+o(1)), \quad |x| \to \infty, \tag{3.2.13}$$

then we can prove less precise versions of (3.2.6), (3.2.7), and (3.2.8), namely

$$\ln\left(\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right)\right) = \begin{cases} -\mu k^\beta(1+o(1)) & \text{if} \quad \beta \in (0,1), \\ -\left(\ln\left(1+\mu\right)\right) k(1+o(1)) & \text{if} \quad \beta = 1, \\ -\frac{\beta-1}{\beta} k \ln k\,(1+o(1)) & \text{if} \quad \beta \in (1,\infty), \end{cases} \qquad k \to \infty,$$

which are equivalent to

$$\mathcal{N}_q^\pm(\lambda) = \begin{cases} \mu^{-1/\beta}|\ln\lambda|^{1/\beta}(1+o(1)) & \text{if} \quad \beta \in (0,1), \\ \frac{1}{\ln(1+\mu)}|\ln\lambda|(1+o(1)) & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta-1}\frac{|\ln\lambda|}{\ln|\ln\lambda|}(1+o(1)) & \text{if} \quad \beta \in (1,\infty), \end{cases} \qquad \lambda \downarrow 0. \tag{3.2.14}$$

Note that in (3.2.13), similarly to (3.2.5), we assume that the values of γ and β are the same for $m_<$ and $m_>$. However, since the coefficient in (3.2.14) with $\beta > 1$ does not depend on γ, in this case we could assume different values of $\gamma > 0$ for $m_<$ and $m_>$.
Finally, we consider perturbations m which admit a power-like decay at infinity. For $\rho > 0$ recall the definition of the Hörmander class

$$\mathcal{S}^{-\rho}(\mathbb{R}^2) := \left\{\psi \in C^\infty(\mathbb{R}^2) \,|\, |\partial_x^\alpha \psi(x)| \le c_\alpha \langle x\rangle^{-\rho-|\alpha|},\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{Z}_+^2\right\},$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\lim_{|x|\to\infty}\psi(x) = 0$. Set

$$\Phi_\psi(\lambda) := \left|\left\{x \in \mathbb{R}^2 \,|\, \psi(x) > \lambda\right\}\right|, \quad \lambda > 0, \tag{3.2.15}$$

where $|\cdot|$ denotes the Lebesgue measure. Fix $q \in \mathbb{Z}_+$, and introduce the function

$$\mathcal{T}_q(x) := \frac{1}{2}\left(\Lambda_q \mathrm{Tr}\, m(x) - 2b\,\mathrm{Im}\, m_{12}(x)\right), \quad x \in \mathbb{R}^2. \tag{3.2.16}$$

Note that $\mathcal{T}_q(x) \ge 0$ for any $x \in \mathbb{R}^2$ and $q \in \mathbb{Z}_+$.

Theorem 3.2.3. *Let $m_{jk} \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $j,k = 1,2$, with $\rho > 0$. Fix $q \in \mathbb{Z}_+$. Suppose that there exists a function $0 < \tau_q \in C^\infty(\mathbb{S}^1)$, such that*

$$\lim_{|x|\to\infty} |x|^\rho \mathcal{T}_q(x) = \tau_q(x/|x|).$$

Then we have

$$\mathcal{N}_q^\pm(\lambda) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)(1+o(1)) \asymp \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \tag{3.2.17}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho}\mathcal{N}_q^\pm(\lambda) = \mathcal{C}_q := \frac{b}{4\pi}\int_0^{2\pi} \tau_q(\cos\theta, \sin\theta)^{2/\rho} d\theta, \tag{3.2.18}$$

or to

$$\pm\left(\lambda_{k,q}^\pm - \Lambda_q\right) = \mathcal{C}_q^{\rho/2} k^{-\rho/2}(1+o(1)), \quad k \to \infty. \tag{3.2.19}$$

Remarks: (i) Relation (3.2.17) could be regarded as a semiclassical one, although here the semiclassical interpretation is somewhat implicit. In Propositions 3.4.1 and 3.4.3 below, we show that the effective Hamiltonian which governs the asymptotics of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ is a pseudodifferential operator (ΨDO) with anti-Wick symbol $w_{q,b} := w_q \circ \mathcal{R}_b$, defined by (3.4.8) and (3.4.31). Under the assumptions of Theorem 3.2.3, $\mathcal{T}_{q,b} := \mathcal{T}_q \circ \mathcal{R}_b$ (see (3.2.16) and (3.4.31)) can be considered as the principal part of the symbol $w_{q,b}$, while the difference between the anti-Wick and the Weyl quantization is negligible. Then $\frac{1}{2\pi}\Phi_{\mathcal{T}_{q,b}}(\lambda) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)$ is just the main semiclassical asymptotic term for the eigenvalue counting function for a compact ΨDO with Weyl symbol $\mathcal{T}_{q,b}$.

(ii) There exists an extensive family of alternative sets of assumptions for Theorem 3.2.3 (see e.g. [22, 38]). We have chosen here hypotheses which, for certain, are not the most general ones, but are quite explicit and, hopefully, easy to absorb.

Let us comment briefly on our results. Nowadays, there exists a relatively wide literature on the local spectral asymptotics for various magnetic quantum Hamiltonians. Let us concentrate here on three types of perturbations of H_0 which are considered to be of a particular interest (see e.g. [38, 50]):

- Electric perturbations $H_0 + Q$ where $Q : \mathbb{R}^2 \to \mathbb{R}$ plays the role of the perturbative *electric potential*;

- Magnetic perturbations $(-i\nabla - A_0 - A)^2$ where $A = (A_1, A_2)$, and $B := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$ is the perturbative *magnetic field*;

- Metric perturbations $\sum_{j,k=1,2} \Pi_j (\delta_{jk} + m_{jk}) \Pi_k$ where $m = \{m_{jk}\}_{j,k=1,2}$ is an appropriate perturbative matrix-valued function.

Typically, the perturbations Q, B, or m are supposed to decay in a suitable sense at infinity. Slowly decaying Q, e.g. $Q \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ with $\rho > 0$ were considered in [58], and the main asymptotic terms of the corresponding counting functions $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found, utilizing, in particular, anti-Wick ΨDOs . In [38, Th. 11.3.17], the case of *combined* electric, magnetic, and metric slowly decaying perturbations was investigated, the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$, as well as certain remainder estimates were obtained. The semiclassical microlocal analysis applied in [38] imposed restrictions on the symbols involved which, in some sense or another, had to decay at infinity less rapidly than their derivatives. These restrictions did not allow to handle some rapidly decaying perturbations, e.g. those of compact support, or of exponential decay with $\beta \geq 1/2$ (see (3.2.5)).

In [59] the authors used a different approach based on the spectral analysis of Berezin–Toeplitz operators and obtained the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ in the case of potential perturbations Q of exponential decay or of compact support. In particular, in [59] formulas of type (3.2.4) or (3.2.14) appeared for the first time. In the present article, we essentially improve the methods developed in [59]. These improvements lead also to more precise results for certain rapidly decaying electric perturbations. Namely, assume that $Q \geq 0$ admits a decay at infinity which is compatible in a suitable sense with the decay of m. Then the results of the article extend quite easily to operators of the form

$$H_{\pm} \pm Q, \tag{3.2.20}$$

so that $H_{\pm} \pm Q$ are perturbations of H_0 having a definite sign. We do not include these generalizations just in order to avoid an unreasonable increase of the size of the article due to results which do not require any really new arguments.

Combined perturbations of H_0 by compactly supported B and Q were considered in [62] where the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found. Note that the magnetic perturbations of H_0 are never of fixed sign which creates specific difficulties, successfully overcome in [62].

To authors' best knowledge, no results on the spectral asymptotics for rapidly decaying *metric* perturbations of H_0 appeared before in the literature. We also included in the article our result on slowly-decaying metric perturbations (see Theorem 3.2.3) since it is coherent with the unified approach of the article, and is proved by methods quite different from those in [38].

Finally, let us discuss briefly the relation of $H_\pm$ to the Bochner Laplacians. Assume that the elements of m are real. In $\mathbb{R}^2$ introduce a Riemannian metric generated by the inverse of $g^\pm$, and the connection 1-form $\sum_{j=1,2} A_{0,j} dx_j$. Set $\gamma_\pm := (\det g^\pm)^{-1/2}$. Then the standard Bochner Laplacian, self-adjoint in $L^2(\mathbb{R}^2; \gamma_\pm dx)$, is written in local coordinates as

$$\mathcal{L}_\pm := -\gamma_\pm^{-1} \sum_{j,k=1,2} \Pi_j g_{jk}^\pm \gamma_\pm \Pi_k.$$

Let $U_\pm : L^2(\mathbb{R}^2; \gamma_\pm dx) \to L^2(\mathbb{R}^2; dx)$ be the unitary operator defined by $U_\pm f := \gamma_\pm^{-1/2} f$. Then we have

$$U_\pm \mathcal{L}_\pm U_\pm^* = H_\pm + Q_\pm \tag{3.2.21}$$

where

$$Q_\pm := \frac{1}{4} \sum_{j,k=1,2} \left(g_{jk}^\pm \frac{\partial \ln \gamma_\pm}{\partial x_k} \frac{\partial \ln \gamma_\pm}{\partial x_j} - 2 \frac{\partial}{\partial x_j} \left(g_{jk}^\pm \frac{\partial \ln \gamma_\pm}{\partial x_k} \right) \right).$$

Generally speaking, the functions $Q_\pm$ do not have a definite sign coinciding with the sign of the operators $H_\pm - H_0$; hence, the operators on the r.h.s of (3.2.21) are not exactly of the form of (3.2.20). The fact that the symbol of a Toeplitz operator does not have a definite sign may cause considerable difficulties in the study of the spectral asymptotics of this operator if the symbol decays rapidly and, in particular, when its support is compact (see e.g. [56]). Hopefully, we will overcome these difficulties in a future work where we would consider the local spectral asymptotics of $\mathcal{L}_\pm$.

3.3 Reduction to Berezin–Toeplitz Operators

In this section we reduce the analysis of the functions $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the spectral asymptotics for certain compact operators of Berezin–Toeplitz type. To this end, we will need some more notations, and several auxiliary results from the abstract theory of compact operators in Hilbert space.

In what follows, we denote by $\mathbb{1}_M$ the characteristic function of the set M. Let T be a self-adjoint operator in a Hilbert space[1], and $\mathcal{I} \subset \mathbb{R}$ be an interval. Set

$$N_{\mathcal{I}}(T) := \operatorname{rank} \mathbb{1}_{\mathcal{I}}(T)$$

where, in accordance with our general notations, $\mathbb{1}_{\mathcal{I}}(T)$ is the spectral projection of T corresponding to $\mathcal{I}$. Thus, if $\mathcal{I} \cap \sigma_{\rm ess}(T) = \emptyset$, then $N_{\mathcal{I}}(T)$ is just the number of the eigenvalues of T, lying on $\mathcal{I}$, and counted with their multiplicities. In particular,

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_{q-1},\Lambda_q-\lambda)}(H_-), \quad q \in \mathbb{Z}_+, \quad \lambda \in (0, 2b), \tag{3.3.1}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q+\lambda,\Lambda_{q+1})}(H_+), \quad q \in \mathbb{Z}_+, \quad \lambda \in (0, 2b), \tag{3.3.2}$$

the functions $\mathcal{N}_q^{\pm}$ being defined in (3.2.3). Let $T = T^*$ be a linear compact operator in a Hilbert space. For $s > 0$ set

$$n_{\pm}(s; T) := N_{(s,\infty)}(\pm T);$$

thus, $n_+(s;T)$ (resp., $n_-(s;T)$) is just the number of the eigenvalues of the operator T larger than s (resp., smaller than $-s$), counted with their multiplicities. If $T_j = T_j^*$, $j = 1, 2$, are two linear compact operators, acting in a given Hilbert space, then the Weyl inequalities

$$n_{\pm}(s_1 + s_2; T_1 + T_2) \leq n_{\pm}(s_1; T_1) + n_{\pm}(s_2; T_2) \tag{3.3.3}$$

hold for $s_j > 0$ (see e.g. [10, Th. 9, Section 9.2]).

Fix $q \in \mathbb{Z}_+$ and denote by P_q the orthogonal projection onto $\operatorname{Ker}(H_0 - \Lambda_q)$. Since the operator $H_0^{-1} W H_0^{-1}$ is compact, the operator $P_q W P_q = \Lambda_q^2 P_q H_0^{-1} W H_0^{-1} P_q$ is compact as well. Similarly, the operators $H_0^{-1} W H_{\pm}^{-1/2}$ are compact, and hence the operators

$$P_q W H_{\pm}^{-1} W P_q = \Lambda_q^2 P_q (H_0^{-1} W H_{\pm}^{-1/2})(H_{\pm}^{-1/2} W H_0^{-1}) P_q$$

are compact as well.

Proposition 3.3.1. *Under the general assumptions of the article we have*

$$\begin{aligned} n_+((1+\varepsilon)\lambda; P_q W P_q \mp P_q W H_{\pm}^{-1} W P_q) + O(1) \leq \\ \mathcal{N}_q^{\pm}(\lambda) \leq \\ n_+((1-\varepsilon)\lambda; P_q W P_q \mp P_q W H_{\pm}^{-1} W P_q) + O(1), \quad \lambda \downarrow 0, \end{aligned} \tag{3.3.4}$$

for each $\varepsilon \in (0, 1)$.

[1]All the Hilbert spaces considered in the article are supposed to be separable.

Proof. The argument is close in spirit to the proof of [59, Proposition 4.1], and is based again on the (generalized) Birman–Schwinger principle. However, since the operator $H_0^{-1/2} W H_0^{-1/2}$ is only bounded but not compact, we cannot apply the Birman–Schwinger principle to the operator pair $(H_0, H_\pm)$, and apply it instead to the resolvent pair $(H_0^{-1}, H_\pm^{-1})$. First of all, note that there exist Λ_- and Λ_+ with $\Lambda_- \in (0, \Lambda_0)$ if $q = 0$, $\Lambda_- \in (\Lambda_{q-1}, \Lambda_q)$ if $q \in \mathbb{N}$, and $\Lambda_+ \in (\Lambda_q, \Lambda_{q+1})$ if $q \in \mathbb{Z}_+$, such that

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-), \quad \lambda \in (0, \Lambda_q - \Lambda_-), \tag{3.3.5}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q + \lambda, \Lambda_+)}(H_+), \quad \lambda \in (0, \Lambda_+ - \Lambda_q). \tag{3.3.6}$$

Further, evidently,

$$N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-) = N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_-^{-1}) = N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-), \tag{3.3.7}$$

$$N_{(\Lambda_q + \lambda, \Lambda_+)}(H_+) = N_{(\Lambda_+^{-1}, (\Lambda_q + \lambda)^{-1})}(H_+^{-1}) = N_{(\Lambda_+^{-1}, (\Lambda_q + \lambda)^{-1})}(H_0^{-1} - T_+), \tag{3.3.8}$$

with $T_- := H_-^{-1} - H_0^{-1}$ and $T_+ := H_0^{-1} - H_+^{-1}$. Note that the operators $T_\pm$ are non negative and compact. By the generalized Birman–Schwinger principle (see e.g. [3, Theorem 1.3]) we have

$$\begin{aligned} N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-) &= n_+(1; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1} T_-^{1/2}) \\ &\quad - n_+(1; T_-^{1/2}(\Lambda_-^{-1} - H_0^{-1})^{-1} T_-^{1/2}) \\ &\quad - \dim \operatorname{Ker}(H_- - \Lambda_-), \end{aligned} \tag{3.3.9}$$

$$\begin{aligned} N_{(\Lambda_+^{-1}, (\Lambda_q + \lambda)^{-1})}(H_0^{-1} - T_+) &= n_+(1; T_+^{1/2}(H_0^{-1} - (\Lambda_q + \lambda)^{-1})^{-1} T_+^{1/2}) \\ &\quad - n_+(1; T_+^{1/2}(H_0^{-1} - \Lambda_+^{-1})^{-1} T_+^{1/2}) \\ &\quad - \dim \operatorname{Ker}(H_+ - \Lambda_+). \end{aligned} \tag{3.3.10}$$

Since the operators $T_\pm$ are compact, and $\Lambda_\pm \notin \sigma(H_0)$, we find that the two last terms on the r.h.s. of (3.3.9) and (3.3.10) which are independent of λ, are finite. Next, the Weyl inequalities (3.3.3) imply

$$\begin{aligned} & n_+(1+\varepsilon; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1} P_q T_-^{1/2}) - n_-(\varepsilon; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1}(I - P_q) T_-^{1/2}) \leq \\ & \qquad n_+(1; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1} T_-^{1/2}) \leq \\ & n_+(1-\varepsilon; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1} P_q T_-^{1/2}) + n_+(\varepsilon; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1}(I - P_q) T_-^{1/2}) \end{aligned} \tag{3.3.11}$$

for any $\varepsilon \in (0,1)$. The operator $T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}$ tends in norm as $\lambda \downarrow 0$ to the compact operator

$$T_-^{1/2}\left(\sum_{j\in\mathbb{Z}_+\setminus\{q\}}(\Lambda_q^{-1}-\Lambda_j^{-1})^{-1}P_j\right)T_-^{1/2}.$$

Therefore,

$$n_\pm(\varepsilon;T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2})=O(1),\quad \lambda\downarrow 0, \tag{3.3.12}$$

for any $\varepsilon>0$. Next, for any $s>0$ we have

$$n_+(s;T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2})=$$

$$n_+(s;((\Lambda_q-\lambda)^{-1}-\Lambda_q^{-1})^{-1}T_-^{1/2}P_qT_-^{1/2})=n_+(s\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};P_qT_-P_q). \tag{3.3.13}$$

Hence, (3.3.9) and (3.3) - (3.3.13) yield

$$n_+((1+\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};P_qT_-P_q)+O(1)\leq$$

$$N_{((\Lambda_q-\lambda)^{-1},\Lambda_-^{-1})}(H_0^{-1}+T_-)\leq$$

$$n_+((1-\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};P_qT_-P_q)+O(1),\quad \lambda\downarrow 0, \tag{3.3.14}$$

for any $\varepsilon\in(0,1)$. Similarly, (3.3.10) and the analogues of (3.3) - (3.3.13) for positive perturbations, imply

$$n_+((1+\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};P_qT_+P_q)+O(1)\leq$$

$$N_{(\Lambda_+^{-1},(\Lambda_q+\lambda)^{-1})}(H_0^{-1}-T_+)\leq$$

$$n_+((1-\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};P_qT_+P_q)+O(1),\quad \lambda\downarrow 0, \tag{3.3.15}$$

By the resolvent identity, we have $T_\pm=H_0^{-1}WH_0^{-1}\mp H_0^{-1}WH_\pm^{-1}WH_0^{-1}$, so that

$$P_qT_\pm P_q=\Lambda_q^{-2}(P_qWP_q\mp P_qWH_\pm^{-1}WP_q).$$

Thus,

$$n_+(s;P_qT_\pm P_q)=n_+(s\Lambda_q^2;P_qWP_q\mp P_qWH_\pm^{-1}WP_q),\quad s>0. \tag{3.3.16}$$

Putting together (3.3.5) – (3.3.8) and (3.3.14) – (3.3.16), we easily obtain (3.3.4). □

3.4 Unitary Equivalence for Berezin–Toeplitz Operators

Our first goal in this section is to show that under certain regularity conditions on the matrix m, the operator P_qWP_q, $q \in \mathbb{Z}_+$, with domain $P_qL^2(\mathbb{R}^2)$, is unitarily equivalent to $P_0w_qP_0$ with domain $P_0L^2(\mathbb{R}^2)$, where w_q is the multiplier by a suitable function $w_q : \mathbb{R}^2 \to \mathbb{C}$. In fact, we will need a slightly more general result, and that is why we introduce at first the appropriate notations.
As usual, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $z := x_1 + ix_2$ and $\overline{z} := x_1 - ix_2$ so that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Introduce the magnetic annihilation operator

$$a := -2ie^{-b|x|^2/4}\frac{\partial}{\partial \overline{z}}e^{b|x|^2/4} = -2i\left(\frac{\partial}{\partial \overline{z}} + \frac{bz}{4}\right),$$

and the magnetic creation operator

$$a^* := -2ie^{b|x|^2/4}\frac{\partial}{\partial z}e^{-b|x|^2/4} = -2i\left(\frac{\partial}{\partial z} - \frac{b\overline{z}}{4}\right),$$

with common domain $\mathrm{Dom}\, a = \mathrm{Dom}\, a^* = \mathrm{Dom}\, H_0^{1/2}$. The operators a and a^* are closed and mutually adjoint in $L^2(\mathbb{R}^2)$. On $\mathrm{Dom}\, H_0$ we have $[a, a^*] = 2b$ and

$$H_0 = a^*a + b = aa^* - b = \frac{1}{2}(aa^* + a^*a). \tag{3.4.1}$$

Moreover, on $\mathrm{Dom}\, H_0^{1/2}$ we have

$$\Pi_1 = \frac{1}{2}(a + a^*), \quad \Pi_2 = \frac{1}{2i}(a - a^*), \tag{3.4.2}$$

the operators Π_j, $j = 1, 2$, being introduced in (3.1.1). Next, define the operator $\mathbb{A} : \mathrm{Dom}\, H_0^{1/2} \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ by

$$\mathbb{A}u := \begin{pmatrix} a^*u \\ au \end{pmatrix}, \quad u \in \mathrm{Dom}\, H_0^{1/2}.$$

Then, (3.4.1) implies that $H_0 = \frac{1}{2}\mathbb{A}^*\mathbb{A}$. Further, introduce the Hermitian matrix-valued function

$$\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix},$$

with $\omega_{jk} \in L^\infty(\mathbb{R}^2)$, $j,k = 1,2$. Fix $q \in \mathbb{Z}_+$ and define the operator

$$P_q\mathbb{A}^*\Omega\mathbb{A}P_q = \Lambda_q P_q H_0^{-1/2}\mathbb{A}^*\Omega\mathbb{A}H_0^{-1/2}P_q \tag{3.4.3}$$

bounded and self-adjoint in $P_qL^2(\mathbb{R}^2)$. Utilizing (3.4.2), we easily find that

$$P_qWP_q = \frac{1}{2}P_q\mathbb{A}^*U\mathbb{A}P_q \tag{3.4.4}$$

where

$$U := \mathcal{O}^*m\mathcal{O}, \quad \mathcal{O} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{3.4.5}$$

so that $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, with

$$u_{11} := \frac{1}{2}\left(\mathrm{Tr}\, m - 2\mathrm{Im}\, m_{12}\right), \quad u_{22} := \frac{1}{2}\left(\mathrm{Tr}\, m + 2\mathrm{Im}\, m_{12}\right),$$

$$u_{12} = \overline{u_{21}} := \frac{1}{2}\left(m_{11} - m_{22} - 2i\mathrm{Re}\, m_{12}\right).$$

Introduce the Laguerre polynomials

$$\mathrm{L}_q^{(m)} := \sum_{j=0}^{q}\binom{q+m}{q-j}\frac{(-t)^j}{j!}, \quad t \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \quad m \in \mathbb{Z}_+; \tag{3.4.6}$$

as usual, we write $\mathrm{L}_q^{(0)} = \mathrm{L}_q$, and for notational convenience we set $q\mathrm{L}_{q-1} = 0$ for $q = 0$. By [33, Eq. 8.974.3] we have

$$\sum_{j=0}^{q}\mathrm{L}_j^{(m)}(t) = \mathrm{L}_q^{(m+1)}(t), \quad t \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \quad m \in \mathbb{Z}_+. \tag{3.4.7}$$

Proposition 3.4.1. *Let Ω be a Hermitian 2×2 matrix-valued function with entries $\omega_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j,k = 1,2$. Fix $q \in \mathbb{Z}_+$. Then the operator $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ with domain $P_qL^2(\mathbb{R}^2)$, is unitarily equivalent to the operator $P_0w_qP_0$ with domain $P_0L^2(\mathbb{R}^2)$ where*

$$w_q = w_q(\Omega) := \tag{3.4.8}$$

$$\begin{cases} 2b(q+1)\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right)\omega_{11} + 2bq\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\omega_{22} - 8\mathrm{Re}\,\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2\omega_{12}}{\partial\bar{z}^2} & \text{if} \quad q \geq 1, \\ 2b\mathrm{L}_1\left(-\frac{\Delta}{2b}\right)\omega_{11} \quad \text{if} \quad q = 0, \end{cases}$$

Δ is the standard Laplacian in $\mathbb{R}^2$ so that, in accordance to (3.4.6), $\mathrm{L}_s^{(m)}\left(-\frac{\Delta}{2b}\right)$ with $s \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$, is just the differential operation $\sum_{j=0}^{s}\binom{s+m}{s-j}\frac{\Delta^j}{j!(2b)^j}$ of order $2s$ with constant coefficients.

Proof. Set

$$\varphi_{0,k}(x) := \sqrt{\frac{b}{2\pi k!}}\left(\frac{b}{2}\right)^{k/2} z^k e^{-b|x|^2/4}, \quad x \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+,$$

$$\varphi_{q,k}(x) := \sqrt{\frac{1}{(2b)^q q!}}(a^*)^q \varphi_{0,k}(x), \quad x \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+, \quad q \in \mathbb{N}.$$

Then $\{\varphi_{q,k}\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis of $P_q L^2(\mathbb{R}^2)$ called sometimes *the angular momentum basis* (see e.g. [59] or [16, Subsection 9.1]). Evidently, for $k \in \mathbb{Z}_+$ we have

$$a^*\varphi_{q,k} = \sqrt{2b(q+1)}\varphi_{q+1,k}, \quad q \in \mathbb{Z}_+, \quad a\varphi_{q,k} = \begin{cases} \sqrt{2bq}\varphi_{q-1,k}, & q \geq 1, \\ 0, & q = 0. \end{cases} \tag{3.4.9}$$

Define the unitary operator $\mathcal{W} : P_q L^2(\mathbb{R}^2) \to P_0 L^2(\mathbb{R}^2)$ by $\mathcal{W} : u \mapsto v$ where

$$u = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{q,k}, \quad v = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{0,k}, \quad \{c_k\}_{k\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+). \tag{3.4.10}$$

We will show that

$$P_q \mathbb{A}^*\Omega\mathbb{A}P_q = \mathcal{W}^* P_0 w_q P_0 \mathcal{W}. \tag{3.4.11}$$

For $V \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $m, s \in \mathbb{Z}_+$, and $k, \ell \in \mathbb{Z}_+$, set

$$\Xi_{m,s}(V; k, \ell) := \langle V\varphi_{m,k}, \varphi_{s,\ell}\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$. Taking into account (3.4.9) and (3.4.10), we easily find that

$$\langle P_q\mathbb{A}^*\Omega\mathbb{A}P_q u, u\rangle =$$

$$2b\sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+}\left((q+1)\Xi_{q+1,q+1}(\omega_{11}; k,\ell) + q\Xi_{q-1,q-1}(\omega_{22}; k,\ell)\right)c_k\overline{c_\ell} +$$

$$2b\sqrt{q(q+1)}2\mathrm{Re}\sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+}\Xi_{q+1,q-1}(\omega_{21}; k,\ell)c_k\overline{c_\ell}, \tag{3.4.12}$$

if $q \geq 1$, and

$$\langle P_0 \mathbb{A}^* \Omega \mathbb{A} P_0 u, u\rangle = 2b \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} \Xi_{1,1}(\omega_{11}; k, \ell) c_k \overline{c_\ell}. \tag{3.4.13}$$

Moreover,

$$\langle P_0 w_q P_0 v, v\rangle = \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} \Xi_{0,0}(w_q; k, \ell) c_k \overline{c_\ell}, \quad q \in \mathbb{Z}_+. \tag{3.4.14}$$

In [16, Lemma 9.2] (see also the remark after Eq.(2.2) in [11]), it was shown that

$$\Xi_{m,m}(V; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_m\left(-\frac{\Delta}{2b}\right) V; k, \ell\right), \quad m \in \mathbb{Z}_+. \tag{3.4.15}$$

Now (3.4.13), (3.4.15) with $m = 1$ and $V = \omega_{11}$, and (3.4.14) with $q = 0$, imply (3.4.11) in the case $q = 0$. Assume $q \geq 1$. By (3.4.15), we have

$$\Xi_{q+1,q+1}(\omega_{11}; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right) \omega_{11}; k, \ell\right), \tag{3.4.16}$$

$$\Xi_{q-1,q-1}(\omega_{22}; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right) \omega_{22}; k, \ell\right). \tag{3.4.17}$$

Let us now consider the quantity $\Xi_{q+1,q-1}(V; k, \ell)$. Using (3.4.9), we easily find that for $q \geq 2$ we have

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{\sqrt{2b(q+1)}} \Xi_{q,q-1}([V, a^*]; k, \ell) + \sqrt{\frac{q-1}{q+1}} \Xi_{q,q-2}(V; k, \ell), \tag{3.4.18}$$

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \Xi_{q-1,q-1}([[V, a^*], a^*]; k, \ell) + \sqrt{\frac{q-1}{q}} \Xi_{q-1,q-2}([V, a^*]; k, \ell). \tag{3.4.19}$$

Moreover, $[V, a^*] = 2i\frac{\partial V}{\partial z}$, and

$$[[V, a^*], a^*] = -4\frac{\partial^2 V}{\partial z^2}. \tag{3.4.20}$$

Using (3.4.19), it is not difficult to prove by induction that

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \sum_{j=0}^{q-1} \Xi_{j,j}([[V, a^*], a^*]; k, \ell), \quad q \geq 1. \tag{3.4.21}$$

Now (3.4.15), (3.4.20), and (3.4.7) imply

$$\sum_{j=0}^{q-1} \Xi_{j,j}([[V, a^*], a^*]; k, \ell) = \sum_{j=0}^{q-1} \Xi_{0,0}\left(-4\mathrm{L}_j\left(-\frac{\Delta}{2b}\right) \frac{\partial^2 V}{\partial z^2}; k, \ell\right) =$$

$$\Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 V}{\partial z^2}; k, \ell\right). \tag{3.4.22}$$

Setting

$$\mathcal{D}_q := -4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}, \quad q \in \mathbb{N}, \tag{3.4.23}$$

we find that (3.4.21) and (3.4.22) imply

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}}\Xi_{0,0}\left(\mathcal{D}_q V; k, \ell\right). \tag{3.4.24}$$

Bearing in mind (3.4.18), (3.4.15), and (3.4.24), it is not difficult to prove by induction that

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{2b\sqrt{q(q+1)}}\sum_{s=1}^{q}\Xi_{0,0}\left(\mathcal{D}_s V; k, \ell\right). \tag{3.4.25}$$

Note that (3.4.7) and (3.4.25) imply

$$\sum_{s=1}^{q}\mathcal{D}_s = -4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}. \tag{3.4.26}$$

Now, (3.4.25) and (3.4.26) entail

$$2b\sqrt{q(q+1)}\Xi_{q+1,q-1}(\omega_{21}; k, \ell) = \Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 \omega_{21}}{\partial z^2}; k, \ell\right). \tag{3.4.27}$$

Finally, (3.4.12) and (3.4.14) combined with (3.4.16), (3.4.17), and (3.4.27), yield (3.4.11) with $q \geq 1$. □

In the rest of the section we establish two other suitable representations for the operators $P_q V P_q$, $q \in \mathbb{Z}_+$, with $V : \mathbb{R}^2 \to \mathbb{C}$.

Proposition 3.4.2. (i) [11, Subsection 2.3], [27, Lemma 3.1] *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ satisfy $\lim_{|x|\to\infty} V(x) = 0$. Then for each $q \in \mathbb{Z}_+$ the operator $P_q V P_q$ is compact.*
(ii) [59, Lemma 3.3] *Assume in addition that V is radially symmetric, i.e. there exists $v : [0,\infty) \to \mathbb{C}$ such that $V(x) = v(|x|)$, $x \in \mathbb{R}^2$. Then the eigenvalues of the operator $P_q V P_q$ with domain $P_q L^2(\mathbb{R}^2)$, counted with the multiplicities, coincide with the set*

$$\left\{\langle V\varphi_{q,k}, \varphi_{q,k}\rangle\right\}_{k\in\mathbb{Z}_+}. \tag{3.4.28}$$

In particular, the eigenvalues of $P_0 V P_0$ coincide with

$$\frac{1}{k!}\int_0^\infty v((2t/b)^{1/2})e^{-t}t^k dt, \quad k \in \mathbb{Z}_+. \tag{3.4.29}$$

Remarks: (i) Let us recall that if f is, say, a bounded function of exponential decay, then

$$(\mathcal{M}f)(z) := \int_0^\infty f(t)t^{z-1}dt, \quad z \in \mathbb{C}, \quad \operatorname{Re} z > 0,$$

is called sometimes *the Mellin transform* of f. Some of the asymptotic properties as $k \to \infty$ of the integrals (3.4.29) which we will later obtain and use in the proofs of Theorem 3.2.1 and 3.2.2, could possibly be deduced from the general theory of the Mellin transform.

(ii) Combining Propositions 3.4.1 and 3.4.2, we find that if the matrix-valued function Ω is radially symmetric and diagonal, then the operator $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ acting in $P_qL^2(\mathbb{R}^2)$ is unitarily equivalent to a *diagonal* operator in $\ell^2(\mathbb{Z}_+)$. If Ω is just radially symmetric, then $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ is unitarily equivalent to a *tridiagonal* operator acting in $\ell^2(\mathbb{Z}_+)$.

The last proposition in this section concerns the unitary equivalence between the Berezin–Toeplitz operator P_0WP_0 and a certain Weyl pseudodifferential operator (ΨDO). Let us recall the definition of Weyl ΨDOs acting in $L^2(\mathbb{R})$. Denote by $\Gamma(\mathbb{R}^2)$ the set of functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|\psi\|_{\Gamma(\mathbb{R}^2)} := \sup_{(y,\eta)\in\mathbb{R}^2} \sup_{\ell,m=0,1} \left| \frac{\partial^{\ell+m}\psi(y,\eta)}{\partial y^\ell \partial \eta^m} \right| < \infty.$$

Then the operator $\mathrm{Op}^{\mathrm{w}}(\psi)$ defined initially as a mapping between the Schwartz class $\mathcal{S}(\mathbb{R})$ and its dual class $\mathcal{S}'(\mathbb{R})$ by

$$\left(\mathrm{Op}^{\mathrm{w}}(\psi)u\right)(y) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(\frac{y+y'}{2}, \eta\right) e^{i(y-y')\eta} u(y')dy'd\eta, \quad y \in \mathbb{R},$$

extends uniquely to an operator bounded in $L^2(\mathbb{R})$. Moreover, there exists a constant c such that

$$\|\mathrm{Op}^{\mathrm{w}}(\psi)\| \leq c\|\psi\|_{\Gamma(\mathbb{R}^2)} \tag{3.4.30}$$

(see e.g. [12, Corollary 2.5(i)]).

Remark: Inequalities of type (3.4.30) are known as *Calderón–Vaillancourt* estimates.

Put

$$\mathcal{R}_b := -b^{-1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.4.31}$$

and for $V : \mathbb{R}^2 \to \mathbb{C}$, define

$$V_b(x) := V\left(\mathcal{R}_b x\right), \quad x \in \mathbb{R}^2, \quad b > 0.$$

Moreover, set $\mathcal{G}(x) := \frac{e^{-|x|^2}}{\pi}$, $x \in \mathbb{R}^2$.

Proposition 3.4.3. [55, Theorem 2.11, Corollary 2.8] *Let* $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. *Then the operator* $P_0 V P_0$ *with domain* $P_0 L^2(\mathbb{R}^2)$ *is unitarily equivalent to the operator* $\mathrm{Op}^{\mathrm{w}}(V_b * \mathcal{G})$.

Remark: The operator $\mathrm{Op}^{\mathrm{aw}}(\psi) : = \mathrm{Op}^{\mathrm{w}}(\psi * \mathcal{G})$ is called ΨDO with *anti-Wick symbol* ψ (see e.g. [64, Section 24]).

3.5 Proofs of Theorem 3.2.1 and Theorem 3.2.2

In this section we complete the proofs of Theorem 3.2.1 and Theorem 3.2.2, concerning perturbations of compact support, and of exponential decay.
Let $T = T^*$ be a compact operator in a Hilbert space, such that $\operatorname{rank} \mathbb{1}_{(0,\infty)}(T) = \infty$. Denote by $\{\nu_k(T)\}_{k=0}^\infty$ the non-increasing sequence of the positive eigenvalues of T, counted with the multiplicities.
Recall that $m_<(x) \le m_>(x)$ are the eigenvalues of the matrix $m(x)$, $x \in \mathbb{R}^2$. Since the matrix U (see (3.4.5)) is unitarily equivalent to m, $m_\gtrless$ are also the eigenvalues of U. Next, we check that Proposition 3.3.1 implies the following

Corollary 3.5.1. *Under the general assumptions of the article, there exist constants* $0 < c_<^\pm \le c_>^\pm < \infty$ *and* $k_0 \in \mathbb{Z}_+$ *such that*

$$c_<^\pm \nu_{k+k_0}(P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le \pm(\lambda_{k,q}^\pm - \Lambda_q) \le c_>^\pm \nu_{k-k_0}(P_q \mathbb{A}^* m_> \mathbb{A} P_q) \tag{3.5.1}$$

for sufficiently large $k \in \mathbb{N}$.

Proof. It is easy to see that

$$0 \le P_q W H_\pm^{-1} W P_q \le c_\pm P_q W P_q \tag{3.5.2}$$

with

$$c_\pm := \|H_\pm^{-1/2} W H_\pm^{-1/2}\| \le \sup_{x \in \mathbb{R}^2} |m(x)(I \pm m(x))^{-1}|.$$

Note that $0 \le c_- < \infty$ and $0 \le c_+ < 1$. Moreover, by (3.4.4) and the mini-max principle,

$$n_+(2s; P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le n_+(s; P_q W P_q) \le n_+(2s; P_q \mathbb{A}^* m_> \mathbb{A} P_q), \quad s > 0. \tag{3.5.3}$$

Now, (3.3.4), (3.5.2), and (3.5.3), imply that for any $\varepsilon \in (0,1)$ we have

$$n_+(2\lambda(1+\varepsilon); P_q \mathbb{A}^* m_< \mathbb{A} P_q) + O(1) \le$$

$$\mathcal{N}_q^-(\lambda) \leq$$

$$n_+(2\lambda(1-\varepsilon);(1+c_-)P_q\mathbb{A}^*m_>\mathbb{A}P_q) + O(1), \tag{3.5.4}$$

$$n_+(2\lambda(1+\varepsilon);(1-c_+)P_q\mathbb{A}^*m_<\mathbb{A}P_q) + O(1) \leq$$

$$\mathcal{N}_q^+(\lambda) \leq$$

$$n_+(2\lambda(1-\varepsilon);P_q\mathbb{A}^*m_>\mathbb{A}P_q) + O(1), \tag{3.5.5}$$

as $\lambda \downarrow 0$, and estimates (3.5.4) - (3.5.5) yield (3.5.1) with

$$c_<^- = \frac{1}{2(1+\varepsilon)}, \quad c_>^- = \frac{1+c_-}{2(1-\varepsilon)}, \quad c_<^+ = \frac{1-c_+}{2(1+\varepsilon)}, \quad c_>^+ = \frac{1}{2(1-\varepsilon)},$$

and sufficiently large $k_0 \in \mathbb{N}$. □

Let us now complete the proof of Theorem 3.2.1. Let $\zeta_1 \in C_0^\infty(\mathbb{R}^2)$, $\zeta_1 \geq 0$, $\zeta_1 = 1$ on $\operatorname{supp} m_>$. Set $\zeta_2(x) := (\max_{y\in\mathbb{R}^2} m_>(y))\,\zeta_1(x)$, $x \in \mathbb{R}^2$. Evidently, $m_> \leq \zeta_2$ on $\mathbb{R}^2$, so that

$$\nu_k(P_q\mathbb{A}^*m_>\mathbb{A}P_q) \leq \nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q), \quad k \in \mathbb{Z}_+. \tag{3.5.6}$$

Further, by Proposition 3.4.1, the operator $P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q$ is unitarily equivalent to the operator $P_0\zeta_3P_0$ where

$$\zeta_3 := 2b\left((q+1)\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right) + q\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\right)\zeta_2.$$

Therefore,

$$\nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q) = \nu_k(P_0\zeta_3P_0), \quad k \in \mathbb{Z}_+. \tag{3.5.7}$$

Let $R_> > 0$ be so large that the disk $B_{R_>}(0)$ of radius $R_>$, centered at the origin contains the support of ζ_3. Then,

$$\nu_k(P_0\zeta_3P_0) \leq \max_{x\in\mathbb{R}^2}|\zeta_3(x)|\,\nu_k(P_0\mathbb{1}_{B_{R_>}(0)}P_0), \quad k \in \mathbb{Z}_+. \tag{3.5.8}$$

Putting together (3.5.6), (3.5.7), and (3.5.8), we find that there exists a constant $K_> < \infty$ such that

$$\nu_k(P_q\mathbb{A}^*m_>\mathbb{A}P_q) \leq K_>\nu_k(P_0\mathbb{1}_{B_{R_>}(0)}P_0), \quad k \in \mathbb{Z}_+. \tag{3.5.9}$$

On the other hand,

$$\nu_k(P_q\mathbb{A}^*m_<\mathbb{A}P_q) \geq \nu_k(P_q a\, m_<\, a^*P_q). \tag{3.5.10}$$

Applying (3.4.9), we easily find that the operators $P_q a\, m_< a^*\, P_q$ and $2b(q+1)P_{q+1}\, m_<\, P_{q+1}$ are unitarily equivalent. Hence,

$$\nu_k(P_q a m_< a^* P_q) = 2b(q+1)\nu_k(P_{q+1} m_< P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{3.5.11}$$

Further, since $m_<$ is non-negative, continuous, and does not vanish identically, there exist $c_0 > 0$, $R_< \in (0,\infty)$, and $x_0 \in \mathbb{R}^2$, such that $m_<(x) \geq c_0 \mathbb{1}_{B_{R_<}(x_0)}(x)$, $x \in \mathbb{R}^2$. Therefore,

$$\nu_k(P_{q+1} m_< P_{q+1}) \geq c_0 \nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(x_0)} P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{3.5.12}$$

The operators $P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}$ and $P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}$ are unitarily equivalent under the magnetic translation which maps x_0 into 0 (see e.g. [59, Eq. (4.21)]). Therefore,

$$\nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}) = \nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{3.5.13}$$

Combining (3.5.10) - (3.5.13), we find that there exists a constant $K_<$ such that

$$K_< \, \nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}) \leq \nu_k(P_q \mathbb{A}^* m_< \mathbb{A} P_q), \quad k \in \mathbb{Z}_+. \tag{3.5.14}$$

By (3.5.9) and (3.5.14), it remains to study the asymptotic behaviour as $k \to \infty$ of the operator $\nu_k(P_m \mathbb{1}_{B_R(0)} P_m)$, $m \in \mathbb{Z}_+$ and $R \in (0,\infty)$ being fixed. This asymptotic analysis relies on the representation (3.4.28), and results sufficient for our purposes, are available in the literature. Namely, we have

Lemma 3.5.1. [20, Corollary 2, Section 4] *Let $m \in \mathbb{Z}_+$, $R \in (0,\infty)$, $b \in (0,\infty)$. Set $\varrho := bR^2/2$. Then*

$$\nu_k(P_m \mathbb{1}_{B_R(0)} P_m) = \frac{e^{-\varrho}\varrho^{-m+1} k^{2m-1} \varrho^k}{m!\, k!}(1 + o(1)), \quad k \to \infty. \tag{3.5.15}$$

Now, asymptotic relation (3.2.1) follows from (3.5.1), (3.5.9), (3.5.14), (3.5.15), and the elementary fact that $\ln k! = k \ln k + O(k)$ as $k \to \infty$.
In the remaining part of this section we prove Theorem 3.2.2 concerning perturbations m of exponential decay. Assume that m satisfies (3.2.5). Then there exist $\delta_\gtrless \in \mathbb{R}$, $\delta_< \leq \delta_>$, and $r > 1$, such that

$$|x|^{\delta_<} e^{-\gamma |x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) \leq m_<(x) \leq$$
$$m_>(x) \leq |x|^{\delta_>} e^{-\gamma |x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) + \max_{y \in \mathbb{R}^2} m_>(y) \mathbb{1}_{B_r(0)}(x), \quad x \in \mathbb{R}^2. \tag{3.5.16}$$

Let $\eta_{\gtrless,0} \in C^\infty(\mathbb{R}^2; [0,1])$ be two radially symmetric functions such that $\eta_{<,0} = 1$ on $\mathbb{R}^2 \setminus B_{r+1}(0)$, $\eta_{<,0} = 0$ on $B_r(0)$, and $\eta_{>,0} = 1$ on $\mathbb{R}^2 \setminus B_r(0)$, $\eta_{>,0} = 0$ on $B_{r-1}(0)$. For $x \in \mathbb{R}^2$ set

$$\eta_{<,1}(x) := |x|^{\delta_<} e^{-\gamma|x|^{2\beta}} \eta_{<,0}(x),$$

$$\eta_{>,1}(x) := |x|^{\delta_>} e^{-\gamma|x|^{2\beta}} \eta_{>,0}(x) + \max_{y\in\mathbb{R}^2} m_>(y)(1-\eta_{<,0}(x)).$$

Evidently, $\eta_{\gtrless,1} \in C_{\rm b}^\infty(\mathbb{R}^2)$, and by (3.5.16),

$$\eta_{<,1}(x) \le m_<(x), \quad m_>(x) \le \eta_{>,1}(x), \quad x \in \mathbb{R}^2.$$

Therefore, for $k \in \mathbb{Z}_+$, we have

$$\nu_k(P_q\mathbb{A}^* m_< \mathbb{A}P_q) \ge \nu_k(P_q\mathbb{A}^*\eta_{<,1}\mathbb{A}P_q), \quad \nu_k(P_q\mathbb{A}^* m_> \mathbb{A}P_q) \le \nu_k(P_q\mathbb{A}^*\eta_{>,1}\mathbb{A}P_q). \quad (3.5.17)$$

Further, set

$$\eta_{\gtrless,2} := 2b\left((q+1)\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right) + q\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\right)\eta_{\gtrless,1}.$$

According to Proposition 3.4.1, the operators $P_q\mathbb{A}^*\eta_{\gtrless,1}\mathbb{A}P_q$, $q \in \mathbb{Z}_+$, and $P_0\eta_{\gtrless,2}P_0$ are unitarily equivalent. Therefore,

$$\nu_k(P_q\mathbb{A}^*\eta_{\gtrless,1}\mathbb{A}P_q) = \nu_k(P_0\eta_{\gtrless,2}P_0), \quad k \in \mathbb{Z}_+. \quad (3.5.18)$$

Next, a tedious but straightforward calculation shows that

$$\eta_{\gtrless,2}(x) = \eta_{\gtrless,3}(x)(1+o(1)), \quad |x| \to \infty, \quad (3.5.19)$$

where

$$\eta_{\gtrless,3}(x) := C_{q,\beta}|x|^{\delta_\gtrless} e^{-\gamma|x|^{2\beta}} \begin{cases} 1 \quad \text{if} \quad \beta \in (0,1/2], \\ |x|^{2(q+1)(2\beta-1)} \quad \text{if} \quad \beta \in (1/2,\infty), \end{cases} \quad x \in \mathbb{R}^2 \setminus \{0\},$$

and $C_{q,\beta} > 0$ are some constants. Even though the exact values of $C_{q,\beta}$ will not play any role in the sequel, we indicate here these values for the sake of the completeness of the exposition:

$$C_{q,\beta} = \begin{cases} 2\Lambda_q \quad \text{if} \quad \beta \in (0,1/2), \\ 2b\left((q+1)\mathrm{L}_{q+1}\left(-\frac{(2\beta\gamma)^2}{2b}\right) + q\mathrm{L}_{q-1}\left(-\frac{(2\beta\gamma)^2}{2b}\right)\right) \quad \text{if} \quad \beta = 1/2, \\ \frac{(2\beta\gamma)^{2(q+1)}}{(2b)^q q!} \quad \text{if} \quad \beta \in (1/2,\infty). \end{cases}$$

Hence, by (3.5.19), there exists $R \in (0,\infty)$ such that for $x \in \mathbb{R}^2$ we have

$$\eta_{<,2} \geq \frac{1}{2}\eta_{<,3}\mathbb{1}_{\mathbb{R}^2\setminus B_R(0)} - c_< \mathbb{1}_{B_R(0)} =: \eta_{<,4}(x), \tag{3.5.20}$$

$$\eta_{>,2} \leq \frac{3}{2}\eta_{>,3}\mathbb{1}_{\mathbb{R}^2\setminus B_R(0)} + c_> \mathbb{1}_{B_R(0)} =: \eta_{>,4}(x), \tag{3.5.21}$$

with $c_{\gtrless} := \max_{y\in\mathbb{R}^2} |\eta_{\gtrless,2}(y)|$. Thus, for any admissible $k \in \mathbb{Z}_+$ we have

$$\nu_k(P_0\eta_{<,2}P_0) \geq \nu_k(P_0\eta_{<,4}P_0), \quad \nu_k(P_0\eta_{>,2}P_0) \leq \nu_k(P_0\eta_{>,4}P_0). \tag{3.5.22}$$

In order to complete the proof of Theorem 3.2.2, we need a couple of auxiliary results. For $\beta > 0$, $\mu > 0$, and $\varrho > 0$, set

$$\mathcal{J}_{\beta,\mu}(k) := \int_0^\infty e^{-\mu t^\beta - t} t^k dt, \quad \mathcal{E}_\varrho(k) := \int_0^\varrho e^{-t} t^k dt, \quad k > -1, \tag{3.5.23}$$

and for $\delta \in \mathbb{R}$, $c_0 > 0$ and $c_1 \in \mathbb{R}$, put

$$\mathcal{L}(k) = \mathcal{L}_{\beta,\mu,\varrho,\delta}(k; c_0, c_1) := \frac{c_0\mathcal{J}_{\beta,\mu}(k+\delta) + c_1\mathcal{E}_\varrho(k-\delta_-)}{\Gamma(k+1)}, \quad k > \max\{-1, -\delta-1\},$$

where $\delta_- := \max\{0, -\delta\}$.

Lemma 3.5.2. *Let $\beta > 0$, $\mu > 0$, $\varrho > 0$, $c_0 > 0$, and $\delta \in \mathbb{R}$, $c_1 \in \mathbb{R}$.*
(i) *The asymptotic relations*

$$\ln \mathcal{L}(k) = \tag{3.5.24}$$

$$\begin{cases} -\sum_{1\leq j<\frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + O(\ln k) \quad \text{if} \quad \beta \in (0,1), \\ -(\ln(1+\mu))\, k + O(\ln k) \quad \text{if} \quad \beta = 1, \\ -\frac{\beta-1}{\beta} k \ln k + k\left(\frac{\beta-1-\ln(\mu\beta)}{\beta}\right) - \sum_{1\leq j<\frac{\beta}{\beta-1}} g_j k^{(\frac{1}{\beta}-1)j+1} + O(\ln k) \quad \text{if} \quad \beta \in (1,\infty), \end{cases}$$

hold true as $k \to \infty$, the coefficients f_j and g_j being introduced in the statement of Theorem 3.2.2.
(ii) *We have $\mathcal{L}'(k) < 0$ for sufficiently large k.*

Proof. Let at first $\delta = 0$. Assume $\beta \in (0,1)$, $k > 0$, and change the variable $t \mapsto ks$ in the first integral in (3.5.23). Thus we find that

$$\mathcal{J}_{\beta,\mu}(k) = k^{k+1} \int_0^\infty e^{-kF(s;k^{\beta-1})} ds. \tag{3.5.25}$$

The function $F(s;k^{\beta-1})$ defined in (3.2.9), attains its unique minimum at $s_<(k^{\beta-1})$, and we have $\frac{\partial^2 F}{\partial s^2}(s_<(k^{\beta-1});k^{\beta-1}) = 1+o(1)$, $k \to \infty$. Therefore, applying a standard argument close to the usual Laplace method for asymptotic evaluation of integrals depending on a large parameter, we easily find that

$$\int_0^\infty e^{-kF(s;k^{\beta-1})}ds = (2\pi)^{1/2} e^{-kF(s_<(k^{\beta-1});k^{\beta-1})} k^{-1/2}(1+o(1)), \quad k \to \infty. \tag{3.5.26}$$

Bearing in mind that $F(s_<(k^{\beta-1});k^{\beta-1}) = f(k^{\beta-1})$ (see (3.2.10)), $f(0)=1$, and

$$\ln \Gamma(k+1) = k \ln k - k + \frac{1}{2}\ln k + O(1), \quad k \to \infty, \tag{3.5.27}$$

(see e.g. [1, Eq. 6.1.40]), we find that (3.5.25) – (3.5.26) imply

$$\begin{aligned}
\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= k - kf(k^{\beta-1}) + O(\ln k) \\
&= k - k \sum_{0 \le j < \frac{1}{1-\beta}} \frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0) k^{(\beta-1)j} + O(\ln k) \\
&= - \sum_{1 \le j < \frac{1}{1-\beta}} \frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0) k^{(\beta-1)j+1} + O(\ln k) \\
&= - \sum_{1 \le j < \frac{1}{1-\beta}} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k \to \infty.
\end{aligned} \tag{3.5.28}$$

In the case $\beta = 1$, we simply have

$$\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)} = \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-(\mu+1)t} t^k dt = (\mu+1)^{-k-1},$$

i.e

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = -(\ln(1+\mu))k + O(1), \quad k \to \infty. \tag{3.5.29}$$

Let now $\beta \in (1,\infty)$. Changing the variable $t \mapsto k^{1/\beta}s$ with $k>0$ in (3.5.23), we find

$$\mathcal{J}_{\beta,\mu}(k) := k^{(k+1)/\beta} \int_0^\infty e^{-kG(s;k^{(\frac{1}{\beta}-1)})} ds. \tag{3.5.30}$$

The function $G(s;k^{\frac{1}{\beta}-1})$ defined in (3.2.11), attains its unique minimum at $s_>(k^{\frac{1}{\beta}-1})$, and we have

$$\frac{\partial^2 G}{\partial s^2}(s_>(k^{\frac{1}{\beta}-1}), k^{\frac{1}{\beta}-1}) = \beta(\mu\beta)^{2/\beta}(1+o(1)), \quad k \to \infty.$$

Arguing as in the derivation of (3.5.26), we obtain

$$\int_0^\infty e^{-kG(s;k^{\frac{1}{\beta}-1})}ds = \sqrt{2\pi\beta}\,(\mu\beta)^{-1/\beta}e^{-kG(s_>(k^{\frac{1}{\beta}-1});k^{\frac{1}{\beta}-1})}k^{-1/2}(1+o(1)), \quad k\to\infty. \tag{3.5.31}$$

Bearing in mind that $G(s_>(k^{\frac{1}{\beta}-1});k^{\frac{1}{\beta}-1}) = g(k^{\frac{1}{\beta}-1})$ (see (3.2.12)), and $g(0) = \frac{1+\ln(\mu\beta)}{\beta}$, we find that (3.5.30), (3.5.31), and (3.5.27), imply

$$\begin{aligned}
\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= -\frac{\beta-1}{\beta}k\ln k + k - kg(k^{\frac{1}{\beta}-1}) + O(\ln k) \qquad (3.5.32)\\
&= -\frac{\beta-1}{\beta}k\ln k + k - k\sum_{0\le j<\frac{\beta}{\beta-1}}\frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(\frac{1}{\beta}-1)j} + O(\ln k)\\
&= -\frac{\beta-1}{\beta}k\ln k + k(1-g(0)) - \sum_{1\le j<\frac{\beta}{\beta-1}}\frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(\frac{1}{\beta}-1)j+1} + O(\ln k)\\
&= -\frac{\beta-1}{\beta}k\ln k + k\left(\frac{\beta-1-\ln(\mu\beta)}{\beta}\right) - \sum_{1\le j<\frac{\beta}{\beta-1}} g_jk^{(\frac{1}{\beta}-1)j+1} + O(\ln k),
\end{aligned}$$

as $k\to\infty$. Let us now consider general $\delta\in\mathbb{R}$. By (3.5.27),

$$\ln\left(\frac{\Gamma(k+\delta+1)}{\Gamma(k+1)}\right) = \delta\ln k + O(1), \quad k\to\infty. \tag{3.5.33}$$

Putting together (3.5.28), (3.5.29), (3.5.32), and (3.5.33), we find that

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right) - \ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = O(\ln k), \quad k\to\infty. \tag{3.5.34}$$

Finally, by (3.5.15), we easily find that for each $\delta\in\mathbb{R}$ fixed, we have

$$\frac{\mathcal{E}_\varrho(k-\delta_-)}{\Gamma(k+1)} = o\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right), \quad k\to\infty. \tag{3.5.35}$$

The combination of (3.5.28), (3.5.29), (3.5.32), (3.5.34), and (3.5.35) implies (3.5.24).
(ii) We have

$$\begin{aligned}
&\mathcal{L}'(k) =\\
&c_0\left(\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{J}_{\beta,\mu}(k+\delta)\right) +\\
&c_1\left(\frac{\mathcal{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{E}_\varrho(k-\delta_-)\right),
\end{aligned} \tag{3.5.36}$$

$$\mathcal{J}'_{\beta,\mu}(k) = \int_0^\infty e^{-\mu t^\beta - t} t^k \ln t \, dt, \quad \mathcal{E}'_\varrho(k) = \int_0^\varrho e^{-t} t^k \ln t \, dt,$$

and

$$\frac{\Gamma'(k+1)}{\Gamma(k+1)} = \ln k + \frac{1}{2k} + O(k^{-2}), \quad k \to \infty,$$

(see e.g. [1, Eq. 6.3.18]). Performing an asymptotic analysis similar to the one in the proof of the first part of the lemma, we find that there exists a function $\Psi = \Psi_{\beta,\mu,\delta}$ such that $\Psi(k) < 0$ for k large enough, and

$$\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2} \mathcal{J}_{\beta,\mu}(k+\delta) = \Psi(k)(1+o(1)), \tag{3.5.37}$$

$$\frac{\mathcal{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2} \mathcal{E}_\varrho(k-\delta_-) = o(\Psi(k)), \tag{3.5.38}$$

as $k \to \infty$. Putting together (3.5.36), (3.5.37), and (3.5.38), we conclude that $\mathcal{L}'(k) < 0$ for sufficiently large k. □

Taking into account the definition of the functions $\eta_{\gtrless,4}$ in (3.5.20) - (3.5.21), the mini-max principle, representation (3.4.29), as well as Lemma 3.5.2 (ii), we find that there exist constants $c_{j,\gtrless} > 0$, $j = 0, 1$, $\tilde{\delta}_\gtrless \in \mathbb{R}$, and $k_0 \in \mathbb{Z}_+$, such that

$$\nu_k(P_0 \eta_{<,4} P_0) \geq \mathcal{L}_{\beta,\mu,\varrho,\tilde{\delta}_<}(k + k_0; c_{0,<}, -c_{1,<}), \quad \nu_k(P_0 \eta_{>,4} P_0) \leq \mathcal{L}_{\beta,\mu,\varrho,\tilde{\delta}_>}(k; c_{0,>}, c_{1,>}), \tag{3.5.39}$$

for $\mu = \gamma(2/b)^\beta$, $\varrho = bR^2/2$, and sufficiently large $k \in \mathbb{Z}_+$.
Putting together (3.5.1), (3.5.17), (3.5.18), (3.5.22), (3.5.39), and (3.5.24), we obtain (3.2.6) – (3.2.8).

3.6 Proof of Theorem 3.2.3

Estimates (3.3.4) combined with the Weyl inequalities (3.3.3) and the mini-max principle, entail

$$n_+(\lambda(1+\varepsilon); P_q W P_q) + O(1) \leq$$

$$\mathcal{N}_q^-(\lambda) \leq$$

$$n_+(\lambda(1-\varepsilon)^2; P_q W P_q) + n_+(\lambda\varepsilon(1-\varepsilon); P_q W H_-^{-1} W P_q) + O(1), \tag{3.6.1}$$

$$n_+(\lambda(1+\varepsilon)^2; P_q W P_q) - n_+(\lambda\varepsilon(1+\varepsilon); P_q W H_+^{-1} W P_q) + O(1) \leq$$

$$\mathcal{N}_q^+(\lambda) \leq$$
$$n_+(\lambda(1-\varepsilon); P_q W P_q) + O(1), \tag{3.6.2}$$

as $\lambda \downarrow 0$. It is easy to check that we have

$$P_q W H_\pm^{-1} W P_q \leq C_{1,\pm} P_q \mathbb{A}^* \langle\cdot\rangle^{-2\rho} \mathbb{A} P_q$$

with

$$C_{1,\pm} := \|H_0^{1/2} H_\pm^{-1/2}\|^2 \left(\sup_{x\in\mathbb{R}^2} \langle x\rangle^\rho m_>(x) \right)^2.$$

Therefore, for any $s > 0$,

$$n_+(s; P_q W H_\pm^{-1} W P_q) \leq n_+(s; C_{1,\pm} P_q \mathbb{A}^* \langle\cdot\rangle^{-2\rho} \mathbb{A} P_q). \tag{3.6.3}$$

Further, by Proposition 3.4.1, the operator $P_q W P_q$ (resp., $P_q \mathbb{A}^* \langle\cdot\rangle^{-2\rho} \mathbb{A} P_q$) is unitarily equivalent to $\frac{1}{2} P_0 w_q(U) P_0$ (resp., to $P_0 w_q(\langle\cdot\rangle^{-2\rho} I) P_0$). Hence, for any $s > 0$,

$$n_+(s; P_q W P_q) = n_+(2s; P_0 w_q(U) P_0), \tag{3.6.4}$$

$$n_+(s; P_q \mathbb{A}^* \langle\cdot\rangle^{-2\rho} \mathbb{A} P_q) = n_+(s; P_0 w_q\left(\langle\cdot\rangle^{-2\rho} I\right) P_0) \leq n_+(s; C_2 P_0 \langle\cdot\rangle^{-2\rho} P_0) \tag{3.6.5}$$

with $C_2 := \sup_{x\in\mathbb{R}^2} \langle x\rangle^{2\rho} |w_q(\langle x\rangle^{-2\rho} I)|$. Now, write

$$\frac{1}{2} w_q(U) = \mathcal{T}_q + \tilde{\mathcal{T}}_q,$$

the symbol $\mathcal{T}_q$ being defined in (3.2.16), and note the crucial circumstance that $\tilde{\mathcal{T}}_q \in \mathcal{S}^{-\rho-2}(\mathbb{R}^2)$. Then the Weyl inequalities (3.3.3) entail

$$n_+(s(1+\varepsilon); P_0 \mathcal{T}_q P_0) - n_-(s\varepsilon; P_0 \tilde{\mathcal{T}}_q P_0) \leq$$
$$n_+(2s; P_0 w_q(U) P_0) \leq$$
$$n_+(s(1-\varepsilon); P_0 \mathcal{T}_q P_0) + n_+(s\varepsilon; P_0 \tilde{\mathcal{T}}_q P_0), \tag{3.6.6}$$

for any $s > 0$ and $\varepsilon \in (0,1)$. Evidently,

$$n_\pm(s; P_0 \tilde{\mathcal{T}}_q P_0) \leq n_+(s; C_3 P_0 \langle\cdot\rangle^{-\rho-2} P_0), \quad s > 0, \tag{3.6.7}$$

with $C_3 := \sup_{x\in\mathbb{R}^2} \langle x\rangle^{\rho+2} |\tilde{\mathcal{T}}_q(x)|$. Recalling Proposition 3.4.3, we find that we have reduced the asymptotic analysis of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the eigenvalue asymptotics for a ΨDO with elliptic anti-Wick symbol of negative order. The spectral asymptotics for operators of this type has been extensively studied in the literature since the 1970s. In particular, we have the following

Proposition 3.6.1. *Let* $0 \leq \psi \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 0$. *Assume that there exists* $0 < \psi_0 \in C^\infty(\mathbb{S}^1)$ *such that* $\lim_{|x|\to\infty} |x|^\rho \psi(x) = \psi_0(x/|x|)$. *Then we have*

$$n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\psi)) = (2\pi)^{-1}\Phi_\psi(\lambda)(1 + o(1)), \quad \lambda \downarrow 0, \tag{3.6.8}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\psi)) = \mathcal{C}(\psi_0) := \frac{1}{4\pi}\int_0^{2\pi} \psi_0(\cos\theta, \sin\theta)^{2/\rho} d\theta.$$

Proof. Evidently, for each $\varepsilon \in (0, 1)$ there exist real functions $\psi_{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2)$ such that

$$\psi_{-,\varepsilon}(x) \leq \psi(x) \leq \psi_{+,\varepsilon}(x), \quad x \in \mathbb{R}^2,$$

$$\psi_{\pm,\varepsilon}(x) = (1 \mp \varepsilon)^{-1}|x|^{-\rho}\psi_0(x/|x|), \quad x \in \mathbb{R}^2, \quad |x| \geq R,$$

for some $R \in (0, \infty)$. Applying the monotonicity of the anti-Wick quantization with respect to the symbol (see e.g. [64, Proposition 24.1]), the mini-max principle, and the Weyl inequalities, we obtain

$$n_+((1+\varepsilon)\lambda; \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon})) - n_-(\varepsilon\lambda; (\mathrm{Op}^{\mathrm{aw}}(\psi_{-,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon}))) \leq$$

$$n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\psi)) \leq$$

$$n_+((1-\varepsilon)\lambda; \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon})) + n_+(\varepsilon\lambda; (\mathrm{Op}^{\mathrm{aw}}(\psi_{+,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon}))). \tag{3.6.9}$$

By [22], we have the following semiclassical result

$$n_+(\lambda; \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})) = (2\pi)^{-1}\Phi_{\psi_{\pm,\varepsilon}}(\lambda)(1 + o(1)), \quad \lambda \downarrow 0. \tag{3.6.10}$$

Further, by [64, Theorem 24.1] the differences $\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$ are ΨDOs of lower order than $\mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$, so that we easily obtain

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho} n_\pm(\varepsilon\lambda; (\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})) = 0, \quad \varepsilon > 0. \tag{3.6.11}$$

Now, (3.6.9) – (3.6.11) imply

$$(1+\varepsilon)^{-4/\rho}\mathcal{C}(\psi_0) \leq \liminf_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\psi)) \leq$$

$$\limsup_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\psi)) \leq (1-\varepsilon)^{-4/\rho}\mathcal{C}(\psi_0),$$

for $\varepsilon \in (0, 1)$. Letting $\varepsilon \downarrow 0$, we obtain (3.6.8).

□

By Propositions 3.4.3 and 3.6.1, we have

$$n_+(\lambda; P_0 \mathcal{T}_q P_0) = n_+(\lambda; \mathrm{Op}^{\mathrm{aw}}(\mathcal{T}_{q,b})) =$$

$$\frac{1}{2\pi}\Phi_{\mathcal{T}_{q,b}}(\lambda)(1+o(1)) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)(1+o(1)), \quad \lambda \downarrow 0, \tag{3.6.12}$$

with $\mathcal{T}_{q,b} = \mathcal{T}_q \circ \mathcal{R}_b$, $\mathcal{R}_b$ being defined in (3.4.31). Finally, for $\rho_0 > \rho$, we have

$$n_+(\lambda; P_0 \langle \cdot \rangle^{-\rho_0} P_0) = O(\lambda^{-2/\rho_0}) = o(\Phi_{\mathcal{T}_q}(\lambda)), \quad \lambda \downarrow 0. \tag{3.6.13}$$

Now, (3.2.17) easily follows from (3.6.1) – (3.6.8), (3.6.12), and (3.6.13). The equivalence of (3.2.18) and (3.2.19) can be checked by arguing as in the proof of [64, Proposition 13.1].

3.7 Compactness of the Resolvent Differences

A priori, the operators H_0 and $H_\pm$, self-adjoint in $L^2(\mathbb{R}^2)$, could be defined as the Friedrichs extensions of the operators $\sum_{j=1,2}\Pi_j^2$ and $\sum_{j,k=1,2}\Pi_j g_{jk}^{\pm}\Pi_k$ defined on $C_0^\infty(\mathbb{R}^2)$. Such a definition implies immediately that

$$\mathrm{Dom}\, H_0^{1/2} = \mathrm{Dom}\, H_\pm^{1/2} = \left\{u \in L^2(\mathbb{R}^2) \,|\, \Pi_j u \in L^2(\mathbb{R}^2),\ j=1,2\right\},$$

and that the operators $H_\pm^{1/2} H_0^{-1/2}$ and $H_0^{1/2} H_\pm^{-1/2}$ are bounded. By [32, Proposition A.2], the operators H_0 and $H_\pm$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, and have a common domain

$$\mathrm{Dom}\, H_0 = \mathrm{Dom}\, H_\pm = \left\{u \in L^2(\mathbb{R}^2) \,|\, \Pi_j \Pi_k u \in L^2(\mathbb{R}^2),\ j,k=1,2\right\}.$$

Let us now prove the compactness of the operator $H_0^{-1} - H_\pm^{-1}$ in $L^2(\mathbb{R}^2)$. Since we have

$$H_0^{-1} - H_\pm^{-1} = \pm H_0^{-1} W H_\pm^{-1} = \pm H_0^{-1} W H_0^{-1} H_0 H_\pm^{-1},$$

it suffices to prove the compactness of $H_0^{-1} W H_0^{-1}$. The operators

$$H_0^{-1} W H_0^{-1} = \frac{1}{2} H_0^{-1} \mathbb{A}^* U \mathbb{A} H_0^{-1}$$

and $\frac{1}{2} H_0^{-1} \mathbb{A}^* m_> \mathbb{A} H_0^{-1}$ are bounded, self-adjoint, and positive. Moreover,

$$H_0^{-1} \mathbb{A}^* U \mathbb{A} H_0^{-1} \le H_0^{-1} \mathbb{A}^* m_> \mathbb{A} H_0^{-1}. \tag{3.7.1}$$

On the other hand,

$$H_0^{-1}\mathbb{A}^* m_> \mathbb{A} H_0^{-1} = H_0^{-1} a^* m_> a H_0^{-1} + H_0^{-1} a m_> a^* H_0^{-1}. \tag{3.7.2}$$

By (3.7.1) and (3.7.2), it suffices to prove the compactness of the operator $m_>^{1/2} a^* H_0^{-1}$. We have

$$m_>^{1/2} a^* H_0^{-1} = m_>^{1/2} H_0^{-1/2} \left(H_0^{-1/2} a^* + 2b H_0^{-1/2} a^* H_0^{-1} \right).$$

The operator $H_0^{-1/2} a^* + 2b H_0^{-1/2} a^* H_0^{-1}$ is bounded, so that it suffices to prove the compactness of $m_>^{1/2} H_0^{-1/2}$ which follows from $m_> \in L^\infty(\mathbb{R}^2)$, $\lim_{|x|\to\infty} m_>(x) = 0$, and the diamagnetic inequality (see e.g.[6, Theorem 2.5]).

Index of Notations

a, a^*	magnetic annihilation and creation operators (p. 10)
A	magnetic vector potential (p. 21)
$\mathbb{A}$	magnetic annihilation/creation vector operator (p. 75)
$A_0 := b/2(-x_2, x_1)$	symmetric gauge vector potential of a constant magnetic field (p. 9)
$b > 0$ (B in Chapter 2)	constant scalar magnetic field (p. 9)
B	magnetic field (p. 21)
$C_{\mathrm{b}}^{\infty}(\mathbb{R}^2)$	the set of functions of class C^{∞} with bounded derivatives (p. 24)
h	harmonic oscillator (p. 32)
H, $H_{\pm}$	perturbed Hamiltonian (p. 21, 64)
H_0	Landau Hamiltonian (p. 9)
$\mathrm{L}_q^{(\alpha)}$	generalized Laguerre polynomials (p. 11)
$m = \{m_{jk}\}_{j,k=1,2}$	metric perturbation (p. 64)
$\mathcal{N}_q^{\pm}$	eigenvalue counting functions (p. 19)
$\mathrm{Op}^{\mathrm{w}}(s)$	Weyl pseudodifferential operator with symbol s (p. 30)
P_q	orthogonal projector onto $\mathrm{Ker}\, H_0 - \Lambda_q$ (p. 11)
$\mathcal{S}^{-\rho}(\mathbb{R}^2)$	Hörmander class of order $-\rho$ (p. 19)
$\mathring{u}$	mean-value transform of u (p. 38)
V	electric potential (p. 21)
$\mathbb{V}$	homogeneous function associated to the potential V (p. 38)
W	metric perturbation (p. 64)
$x := (x_1, x_2)$	coordinates in $\mathbb{R}^2$
$(x, \xi) := (x_1, x_2, \xi_1, \xi_2)$	coordinates in phase space $T^*\mathbb{R}^2$
δ_k	uniformly distributed measure on the circle of radius k (p. 43)
Λ_q (λ_q in Chapter 2)	Landau levels (p. 11)
$\lambda_{q,k}^{\pm}$	ordered sequences of discrete eigenvalues of H (p. 19)

$\Pi_j,\ j = 1, 2$	magnetic derivatives (p. 64)
$\varphi_{q,k}$	angular momentum eigenbasis (p. 11)

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