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ALGEBRAIZATION OF SOME PARACONSISTENT AND PARACOMPLETE LOGICS

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ALGEBRAIZATION OF SOME PARACONSISTENT AND PARACOMPLETE LOGICS

EDUARDO HIRSH

ABSTRACT. We study the algebraizability of the logics constructed using paraconsistent and paracomplete matrices $\mathcal{M}^3_{2,2}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{1,3}$ and $\mathcal{M}^4$ that were described by Lewin and Mikenberg in [5], proving their algebraizability with the method described by Blok and Pigozzi in [1] and proving that the algebraic reducts of these paraconsistent-paracomplete matrices are the corresponding equivalent algebraic semantics for these logics.

INTRODUCTION.

In [6] the authors present the algebraization of paraconsistent logic P1, which can be characterized as a paraconsistent logic generated by the paraconsistent matrix $\mathcal{M}^3_{2,2}$ as presented in [5]. In this paper we carry out a similar study of the paraconsistent and/or paracomplete logics $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$ as presented in [5], and we also carry out an initial study of the equivalent quasi-variety algebraic semantics of these logics based on the work in [7].

The first three sections contain the facts that are needed for the development of our work. In the first section we review the main results on algebraization. In the second section we present the main results on literal paraconsistent and paracomplete matrices. In the third section we present the definition, axiomatization and completeness theorems for the logics $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$. In the sections four and five we present our results on algebraizability and a study of the equivalent quasi-variety semantics of these logics. In section four we prove the algebraizability of these logics and in the last section we prove that the corresponding quasi-varieties are not varieties and construct the corresponding free algebras over a single generator.

1. Preliminaries.

The present section is a summary of the principal results of Blok and Pigozzi in [1] that we need for the development of our work.

By a propositional language we will understand some set $\mathcal{L}$ of propositional connectives. The $\mathcal{L}$-formulas are built in the usual way from propositional variables $p_0, p_1, p_2, \cdots$ using the connectives of $\mathcal{L}$. The set of all $\mathcal{L}$-formulas is denoted by $Fm_{\mathcal{L}}$.

By an inference rule over $\mathcal{L}$ we mean any pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas and φ is a single formula of $Fm_{\mathcal{L}}$. A deductive system S (over $\mathcal{L}$) is defined by a (possibly infinite) set of inference rules and axioms, it consists of the pair $S = \langle \mathcal{L}, \vdash_S \rangle$ where $\vdash_S$ is the relation between sets of formulas and individual formulas defined by the following condition: $\Delta \vdash_S \varphi$ if and only if φ is contained in the smallest set of formulas that includes Δ together with all substitution instances of the axioms of S, and is closed under direct derivability by the inference rules of S. Let $\mathcal{L}$ be a propositional language and $\mathbf{K}$ any class of $\mathcal{L}$-algebras. Let $\models_{\mathbf{K}}$ be the relation that holds between a set of equations Γ and a single equation $\varphi \approx \psi$, in symbols, $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$, if every interpretation of $\varphi \approx \psi$ in a member of $\mathbf{K}$ holds provided each equation in Γ holds under the same interpretation. Thus, $\Gamma \models_{\mathbf{K}} \varphi \approx \psi$ if and only if for all $\mathbf{A} \in \mathbf{K}$ and every interpretation a of the variables of $\Gamma \cup \{\varphi \approx \psi\}$ as elements of $\mathbf{A}$, for every $\xi \approx \eta \in \Gamma$, $\xi^A(a) = \eta^A(a) \Longrightarrow \varphi^A(a) = \psi^A(a)$. In this case we say that $\varphi \approx \psi$ is a $\mathbf{K}$-consequence of Γ. The relation $\models_{\mathbf{K}}$ is called the (semantical) *equational consequence relation* determined by $\mathbf{K}$.

A matrix is a pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is an arbitrary subset of A; the elements of F are called designated elements of A. Let M be any class of matrices. Let $\models_M$ be the relation that hold between a set Γ of formulas and a single formula φ, in symbols $\Gamma \models_M \varphi$, if every interpretation of φ in a member $\mathcal{A}$ of M holds in $\mathcal{A}$ (i.e., is one of the designated elements) provided each $\psi \in \Gamma$ holds in $\mathcal{A}$ under the same interpretation.

Definition 1.1. *([1], Def. 2.2)* Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a deductive system and $\mathbf{K}$ a class of algebras. $\mathbf{K}$ is called an *algebraic semantics* for S if $\vdash_S$ can be interpreted in $\models_{\mathbf{K}}$ in the following sense: there exists a finite system $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$, of equations with a single variable p such that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and each $j < n$,

$$\Gamma \vdash_S \varphi \Longleftrightarrow \{\delta_i[\psi/p] \approx \epsilon_i[\psi/p] : i < n,\ \psi \in \Gamma\} \models_{\mathbf{K}} \delta_j[\varphi/p] \approx \epsilon_j[\psi/p].$$

The $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$ are called the *defining equations* for S and **K**.

In order to simplify notation we shall use $\delta \approx \epsilon$ as an abbreviation for a system $\delta_i(p) \approx \epsilon_i(p), i < n$.

Definition 1.2. *([1], Def. 2.8)* Let S be a deductive system and **K** an algebraic semantics for S with defining equation $\delta_i \approx \epsilon_i$, for $i < n$, i.e,

i) $\Gamma \vdash_S \varphi \Longleftrightarrow \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathbf{K}} \delta(\varphi) \approx \epsilon(\varphi)$.

K is said to be *equivalent* to **S** if there exists a finite system $\Delta_j(p, q)$, for $j < m$, of composite binary connectives (i.e. formulas with two variables) such that, for every equation $\varphi \approx \psi$, and every $j < m$,

ii) $\varphi \approx \psi \mathrel{=\!\mid\models_{\mathbf{K}}} \delta(\varphi\Delta\psi) \approx \epsilon(\varphi\Delta\psi)$.
The system $\Delta_j, j < m$, of composite binary connectives satisfying (ii) is called a *system of equivalence formulas* for **S** and **K**. As before we will write $\varphi\Delta\psi$ as an abbreviation for the system of formulas $\Delta_j(\varphi, \psi)$, $j < m$.

Corollary 1.3. ([1], Cor. 2.9) *Let* **K** *be an algebraic semantics for* **S** *with defining equations* $\delta \approx \epsilon$*. If* **K** *has equivalence formulas* Δ*, then for each set* Γ *of equations and each equation* $\varphi \approx \psi$*,*

i) $\Gamma \models_{\mathbf{K}} \varphi \approx \psi \Longleftrightarrow \{\xi\Delta\eta : \xi \approx \eta \in \Gamma\} \vdash_S \varphi\Delta\psi$

And for each $\nu \in Fm$*,*

ii) $\nu \dashv\vdash_S \delta(\nu)\Delta\xi(\nu)$.

Conversely, if there exists a system Δ *of formulas satisfying conditions (i) and (ii), then* **K** *is equivalent to* **S** *with equivalence formulas* Δ*. If* **K** *is an equivalent algebraic semantics for* **S***, Definition 1.2(i) guarantees that* $\vdash_S$ *can be interpreted in* $\models_{\mathbf{K}}$*, Corollary 1.3(i) guarantees that* $\models_{\mathbf{K}}$ *can be interpreted in* $\vdash_S$*, and Definition 1.2(ii), Corollary 1.3(ii) guarantees that these interpretations are, essentially, inverse to one another.*

Definition 1.4. *([1], Def. 2.10)* A deductive system **S** is said to be *algebraizable* if it has an equivalent algebraic semantics.

In [1] was proved that. A class of algebras **K** is an equivalent Algebraic semantic for **S** if and only if the quasivariety generated by **K** is an equivalent algebraic. And that two equivalent algebraic semantics of a deductive system **S** generate same quasivariety.

Theorem 1.5. ([1], Thm. 2.17) *Let* **S** *be a deductive system given by a set of axioms Ax and a set of inference rules Ir. Assume* **S** *is algebraizable with equivalence formulas Δ and defining equations $\delta \approx \epsilon$. Then the unique equivalent quasivariety semantics for* **S** *is axiomatized by the identities*

i) $\delta(\varphi) \approx \epsilon(\varphi)$, *for each* $\varphi \in Ax$,
ii) $\delta(p\Delta p) \approx \epsilon(p\Delta p)$,

together with the following quasi-identities

iii) $\delta(\psi_0) \approx \epsilon(\psi_0) \wedge \cdots \wedge \delta(\psi_{n-1}) \approx \epsilon(\psi_{n-1}) \Longrightarrow \delta(\varphi) \approx \epsilon(\varphi)$, *for each* $\langle \{\psi_0, \cdots, \psi_{n-1}\}, \varphi \rangle \in Ir$,

iv) $\delta(p\Delta q) \approx \epsilon(p\Delta q) \Longrightarrow p \approx q$.

A more useful characterization of algebraizable deductive systems is the following:

Theorem 1.6. ([1], Thm. 4.7) *A deductive system* **S** *is algebraizable if and only if there exists a system Δ of formulas in two variables and a system $\delta \approx \epsilon$ of equations in a single variable such that the following conditions (i)-(v) hold for all $\varphi, \psi, \eta \in Fm_L$:*

i) $\vdash_S \varphi\Delta\varphi$;

ii) $\varphi\Delta\psi \vdash_S \psi\Delta\varphi$;

iii) $\varphi\Delta\psi, \psi\Delta\nu \vdash_S \varphi\Delta\nu$;

For every primitive connective ω and all $\varphi_0, \cdots, \varphi_{n-1}, \psi_1, \cdots, \psi_{n-1} \in Fm_L$ where n is the rank of ω,

iv) $\varphi_0\Delta\psi_0, \cdots, \varphi_{n-1}\Delta\psi_{n-1} \vdash_S \omega\varphi_0 \cdots \varphi_{n-1}\Delta\omega\psi_0 \cdots \psi_{n-1}$.

Finally, for all $\nu \in Fm_L$,

v) $\nu \dashv\vdash_S \delta(\nu)\Delta\epsilon(\nu)$.

In this event Δ *and* $\delta \approx \epsilon$ *are systems of equivalence formulas and defining equations for* **S**.

2. Literal-paraconsistent and Literal-paracomplete matrices.

This section is a summary of the results of Lewin and Mikenberg in [5] that we use for this work.

Let A be a set such that $\{0,1\} \subseteq A$, $F \subseteq A$ such that $1 \in F$ and $0 \notin F$, and $\sim: A \longrightarrow A$ a function such that $\sim 1 = 0$ and $\sim 0 = 1$. We define the *Literal-paraconsistent-paracomplete matrix*, or LPP-matrix, $\langle \mathbf{A}, F, \sim \rangle$ with the following operations

$$a \vee b = \begin{cases} 1 & \text{if } a \in F \text{ or } b \in F, \\ 0 & \text{otherwise.} \end{cases}$$

$$a \wedge b = \begin{cases} 1 & \text{if } a \in F \text{ and } b \in F, \\ 0 & \text{otherwise.} \end{cases}$$

$$a \to b = \begin{cases} 1 & \text{if } a \notin F \text{ or } b \in F, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that the LPP-matrix $\langle \mathbf{A}, F, \sim \rangle$ is just a matrix in the usual sense, where the algebra over the universe A has three binary operations defined as above and one unary operation $\sim$, which is distinguished because of its relevance to the definition.

Observe also that the matrix $\langle \{1,0\}, \{1\} \rangle$, defines classical logic.

2.1. Axiomatization of Literal-Paraconsistent-Paracomplete Matrix Logic.

The sound and complete deductive system for the logic defined by the class of all LPP-matrices $\langle \mathbf{A}, F, \sim \rangle$, with no additional conditions on A, F or $\sim$. This system will be called Literal-Paraconsistent-paracomplete Matrix Logic (**LPPL**).

It is proved in [5] that **LPPL** is axiomatized by the following Axioms with modus ponens as only rule of inference.

AXIOMS

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(A3) $(\alpha \wedge \beta) \to \alpha$
(A4) $(\alpha \wedge \beta) \to \beta$
(A5) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$
(A6) $\alpha \to (\alpha \vee \beta)$
(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Axiom for negation

(A9) $(\neg A \to \neg B) \to (B \to A)$,
where A and B are complex formulas.

Theorem 2.1. ([5], Thm.1) **(Deduction theorem.)** *If* $\Gamma, \varphi \vdash \psi$, *then* $\Gamma \vdash \varphi \to \psi$.

Theorem 2.2. ([5], Thm.2) *Let* $\varphi(x_1, x_2, \ldots, x_n)$ *be a classical tautology. Then*

(1) $\vdash \varphi(A_1, A_2, \ldots, A_n)$, *for complex formulas* $A_1, A_2, \ldots, A_n$.
(2) *If* φ *does not contain negations, then* $\vdash \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n)$, *for any formulas* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

2.2. Negation Structures and Negation–Reduced Matrices.

Definition 2.3. *(*[1], *Def. 1.4)* Let $\mathbf{A}$ be an algebra and $F \subseteq A$. We define the binary relation on A:

$$\mathbf{\Omega}_{\mathbf{A}}(F) = \left\{ \langle a, b\rangle : \begin{array}{r} \varphi^{\mathbf{A}}(a, \bar{c}) \in F \iff \varphi^{\mathbf{A}}(b, \bar{c}) \in F, \text{ for all} \\ \varphi(p, q_1, \ldots, q_n) \in Fm \text{ and all } \bar{c} \in A^n \end{array} \right\}.$$

A congruence Θ of $\mathbf{A}$ is *compatible* with the subset F of A, if for all $a, b \in A$, if $a \in F$ and $\langle a, b\rangle \in \Theta$ then $b \in F$.

Theorem 2.4. ([1], Thm. 1.5) *Given an algebra* $\mathbf{A}$ *and any* $F \subseteq A$, $\mathbf{\Omega}_{\mathbf{A}}(F)$ *is the largest congruence of* $\mathbf{A}$ *compatible with* F.

The congruence $\mathbf{\Omega}_{\mathbf{A}}(F)$ is the *Leibniz relation* on $\mathbf{A}$ *over* F. The operator on the power set of A, denoted $\mathbf{\Omega}_{\mathbf{A}}$, is called the *Leibniz operator* on A. If $\mathbf{A}$ is the formula algebra $\mathbf{Fm}$, the Leibniz operator is simply denoted $\mathbf{\Omega}$.

Let $\mathcal{M} = \langle A, F, \sim\rangle$ be a LPP-matrix. The *negation structure of* $\mathcal{M}$ is a function

$$nstr_{\mathcal{M}} : A \longrightarrow \{0,1\}^{\mathbb{N}}$$

such that

$$nstr_{\mathcal{M}}(a)(k) = 1 \quad \text{if and only if} \quad \sim^k a \in F.$$

The negation type of $a \in A$ is the function $nstr_{\mathcal{M}}(a)$. If $\mathcal{M}$ is finite, then each negation type is eventually periodic, one can think of it as a finite sequence of 0's and 1's.

Lemma 2.5. ([5], Lem.6) *Let v be a valuation. For any $a, b \in A$, define*

$$v(a/b)(p) = \begin{cases} v(p) & \text{if } v(p) \neq a\,, \\ b & \text{if } v(p) = a\,. \end{cases}$$

Then if $nstr_{\mathcal{M}}(a) = nstr_{\mathcal{M}}(b)$ and α is a formula,

$$v(\alpha) \in F \quad \text{if and only if} \quad v(a/b)(\alpha) \in F\,.$$

Theorem 2.6. ([5], Thm.7) $\langle a, b\rangle \in \Omega_{\mathbf{A}}(F)$ *if and only if* $nstr_{\mathcal{M}}(a) = nstr_{\mathcal{M}}(b)$

We say the matrix $\mathcal{M}/_{\Omega_{\mathbf{A}}} = \langle A/_{\Omega_{\mathbf{A}}}, F/_{\Omega_{\mathbf{A}}}, \sim\rangle$ is a *reduced matrix*. Notice that in a reduced matrix there is one element for each negation type present in $\mathcal{M} = \langle A, F, \sim\rangle$.

It is a well known fact that $\mathcal{M}$ and $\mathcal{M}/_{\Omega_{\mathbf{A}}}$ define the same logic, so if the matrices $\mathcal{M}$ and $\mathcal{M}'$ contain elements with the same negation types, then they give rise to the same logic. It is enough then to study reduced matrices.

3. DEDUCTIVE SYSTEMS $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ AND $\mathcal{S}^4$.

In this section we will present the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$ with their axiomatization and the corresponding (weak)completeness theorems.

3.1. **Three Element Matrices.** There are three possible functions $\sim$, namely, either $\sim_1 \frac{1}{2} = \frac{1}{2}$ or $\sim_2 \frac{1}{2} = 1$ or $\sim_3 \frac{1}{2} = 0$. Likewise, there are two possible filters, namely, $F_1 = \{1\}$ and $F_2 = \{1, \frac{1}{2}\}$. Of the six combinations only four of them are reduced.

$$\begin{aligned}
\mathcal{M}^3_{1,1} &= \langle \{0, \frac{1}{2}, 1\}, F_1, \sim_1 \rangle \\
\mathcal{M}^3_{1,3} &= \langle \{0, \frac{1}{2}, 1\}, F_1, \sim_3 \rangle \\
\mathcal{M}^3_{2,1} &= \langle \{0, \frac{1}{2}, 1\}, F_2, \sim_1 \rangle \\
\mathcal{M}^3_{2,2} &= \langle \{0, \frac{1}{2}, 1\}, F_2, \sim_2 \rangle
\end{aligned}$$

3.1.1. $\mathcal{M}^3_{1,1} = \langle \{0, \frac{1}{2}, 1\}, F_1, \sim_1 \rangle$. The following is an axiomatization for the logic $\mathcal{S}_{1,1}$ defined by this matrix. This system appears in [8] under the name I^1_2.

AXIOMS:
We let $\alpha^\bullet = \alpha \vee \neg\alpha$.

$(A_{1,1}.1)$	The axioms of **LPPL**.
$(A_{1,1}.2)$	$\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)))$
$(A_{1,1}.3)$	$\beta \leftrightarrow \neg\neg\beta$

Modus Ponens is the only rule of inference.

Theorem 3.1. ([5], Thm.10) *Let α be an $\mathcal{M}^3_{1,1}$-tautology. Then in $\mathcal{S}_{1,1}$, $\vdash \alpha$.*

3.1.2. $\mathcal{M}^3_{1,3} = \langle\{0, \frac{1}{2}, 1\}, F_1, \sim_3\rangle$. The following is an axiomatization for the logic $\mathcal{S}_{1,3}$ defined by this matrix. This system appears in [8] under the name I^1 and was introduced in [11].

AXIOMS:

($A_{1,3}$.1) The axioms of **LPPL**.
($A_{1,3}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)))$
($A_{1,3}$.3) $(\neg\alpha)^\bullet$
Modus Ponens is the only rule of inference.

Theorem 3.2. ([5], Thm.13) *Let α be an $\mathcal{M}^3_{1,3}$-tautology. Then in $\mathcal{S}_{1,3}$, $\vdash \alpha$.*

3.1.3. $\mathcal{M}^3_{2,1} = \langle\{0, \frac{1}{2}, 1\}, F_2, \sim_1\rangle$. The following is an axiomatization for the logic $\mathcal{S}_{2,1}$ defined by this matrix. This system appears in [2] and [8] under the name P^1_2.

AXIOMS:

We let $\alpha^\circ = \neg(\alpha \wedge \neg\alpha)$.
($A_{2,1}$.1) The axioms of **LPPL**.
($A_{2,1}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)))$
($A_{2,1}$.3) $\alpha \leftrightarrow \neg\neg\, \alpha$
Modus Ponens is the only rule of inference.

Theorem 3.3. ([5], Thm.16) *Let α be an $\mathcal{M}^3_{2,1}$-tautology. Then in $\mathcal{S}_{2,1}$, $\vdash \alpha$.*

3.1.4. $\mathcal{M}^3_{2,2} = \langle\{0, \frac{1}{2}, 1\}, F_2, \sim_2\rangle$. We will give an axiomatization for the system $\mathcal{S}_{2,2}$ that is appropriate in our context. This system is Sette's logic P^1. See [10]. It appears in [2] as system P^1_1 and in [3] as P^1. See also [6] and [9].

AXIOMS:

($A_{2,2}$.1) The axioms of **LPPL**.
($A_{2,2}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)))$
($A_{2,2}$.3) $(\neg\alpha)^\circ$
Modus Ponens is the only rule of inference.

Theorem 3.4. ([5], Thm.19) *Let α be an $\mathcal{M}^3_{2,2}$-tautology. Then in $\mathcal{S}_{2,2}$, $\vdash \alpha$.*

3.2. **A four element matrix.** Consider the matrix $\mathcal{M}^4 = \langle\{0, \bot, \top, 1\}, F, \sim\rangle$, where $\neg\top = \top$ and $\neg\bot = \bot$ and the filter $F = \{1, \top\}$.

We define the system $\mathcal{S}^4$ as follows.

AXIOMS:

(A_4,1) The axioms of **LPPL**.

(A_4,2) $(\alpha^\bullet \wedge \beta^\circ) \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)))$

(A_4,3) $\alpha \leftrightarrow \neg\neg\alpha$

Modus Ponens is the only rule of inference.

Theorem 3.5. ([5], Thm.22) *Let α be an $\mathcal{M}^4$-tautology. Then in $\mathcal{S}^4$, $\vdash \alpha$.*

4. Algebraization of deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$.

In the present section we will study the algebraizability of the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$, proving it in two different ways, the first is syntactical proof using Theorem 1.6 and the second proof is semantical using the definition of algebraizability (Definition 1.4) proving that the algebraic reducts of the matrices $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,3}$, $\mathcal{M}^3_{2,2}$ and $\mathcal{M}^4$ are the correspondent equivalent algebraic semantics for these logics.

Remark 4.1. The logics $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$ and $\mathcal{S}^4$ contain the axiom $\alpha \leftrightarrow \neg\neg\alpha$.

Proposition 4.2. *If S is $\mathcal{S}_{2,2}$ or $\mathcal{S}_{1,3}$ we have*

$$\vdash_S (\neg\alpha \to \neg\beta) \to (\neg\neg\beta \to \neg\neg\alpha)$$

Proof. Since in $\mathcal{S}_{2,2}$ $(\neg\alpha)^\circ$ is an axiom and in $\mathcal{S}_{1,3}$ $(\neg\alpha)^\bullet$ is an axiom, then in both systems we have:

$$\begin{array}{rll} & \vdash (\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \neg\neg\beta) \to \neg\neg\alpha) & \\ (\neg\alpha \to \neg\beta) & \vdash (\neg\alpha \to \neg\neg\beta) \to \neg\neg\alpha & \text{(MP)} \\ (\neg\alpha \to \neg\beta), (\neg\alpha \to \neg\neg\beta) & \vdash \neg\neg\alpha & \text{(MP)} \end{array}$$

But $\neg\neg\beta \vdash \neg\alpha \to \neg\neg\beta$ and hence:

$$\begin{array}{rll} (\neg\alpha \to \neg\beta), \neg\neg\beta & \vdash \neg\neg\alpha & \\ (\neg\alpha \to \neg\beta) & \vdash \neg\neg\beta \to \neg\neg\alpha & \text{Theorem 2.1} \end{array}$$

$$\therefore \vdash_S (\neg\alpha \to \neg\beta) \to (\neg\neg\beta \to \neg\neg\alpha)$$

□

Theorem 4.3. *The deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$ are algebraizable.*

Proof. For φ, ψ formulas, let $\delta(\varphi) = \varphi \wedge \varphi$, $\varepsilon(\varphi) = \varphi \to \varphi$, $\varphi\Delta_1\psi = \varphi \to \psi$, $\varphi\Delta_2\psi = \psi \to \varphi$, $\varphi\Delta_3\psi = \neg\varphi \to \neg\psi$, $\varphi\Delta_4\psi = \neg\psi \to \neg\varphi$. We must show that δ, ε and Δ satisfy conditions of theorem 1.6.

(1) $\vdash_S \varphi\Delta\varphi$ holds by Theorem 2.2.

(2) $\varphi\Delta\psi \vdash_S \psi\Delta\varphi$ is direct from the definition of Δ.

(3) $\varphi\Delta\psi, \psi\Delta\xi \vdash_S \varphi\Delta\xi$ holds by Theorem 2.2.

(4) We must prove:

(a) $\varphi\Delta\psi \vdash_S \neg\varphi\Delta\neg\psi$

(i) If S is $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$ or $\mathcal{S}^4$.

- $\varphi\Delta\psi \vdash_S \neg\varphi \to \neg\psi$
- $\varphi\Delta\psi \vdash_S \neg\psi \to \neg\varphi$
- $\varphi\Delta\psi \vdash_S \neg\neg\varphi \to \neg\neg\psi$

$\varphi\Delta\psi, \neg\neg\varphi$	$\vdash_S$	φ	remark 4.1 and (MP)
$\varphi\Delta\psi, \neg\neg\varphi$	$\vdash_S$	ψ	$\varphi \to \psi \in \varphi\Delta\psi$ and (MP)
$\varphi\Delta\psi, \neg\neg\varphi$	$\vdash_S$	$\neg\neg\psi$	remark 4.1 and (MP)
$\varphi\Delta\psi$	$\vdash_S$	$\neg\neg\varphi \to \neg\neg\psi$	Theorem 2.1

- $\varphi\Delta\psi \vdash_S \neg\neg\psi \to \neg\neg\varphi$, this case is like the previous one.

(ii) If S is $\mathcal{S}_{2,2}$ or $\mathcal{S}_{1,3}$

- $\varphi\Delta\psi \vdash_S \neg\varphi \to \neg\psi$
- $\varphi\Delta\psi \vdash_S \neg\psi \to \neg\varphi$
- $\varphi\Delta\psi \vdash_S \neg\neg\varphi \to \neg\neg\psi$

$\varphi\Delta\psi$	$\vdash_S$	$(\neg\psi \to \neg\varphi) \to (\neg\neg\varphi \to \neg\neg\psi)$	by proposition 4.2
$\varphi\Delta\psi$	$\vdash_S$	$(\neg\neg\varphi \to \neg\neg\psi)$	$\neg\psi \to \neg\varphi \in \varphi\Delta\psi$ and (MP)

- $\varphi\Delta\psi \vdash_S \neg\neg\psi \to \neg\neg\varphi$, this case is like the previous one.

(b) $\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2 \vdash_S (\varphi_1 * \varphi_2)\Delta(\psi_1 * \psi_2)$ for any binary connective $*$.

(i) $\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2 \vdash_S (\varphi_1 \to \varphi_2)\Delta(\psi_1 \to \psi_2)$

- $\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2 \vdash_S (\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2)$

$\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2, \varphi_1 \to \varphi_2, \psi_1$	$\vdash_S$	ψ_2	(MP)
$\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2$	$\vdash_S$	$(\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2)$	Theorem 2.1

- $\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2 \vdash_S (\psi_1 \to \psi_2) \to (\varphi_1 \to \varphi_2)$
  This case is like the previous one.

- $\varphi_1\Delta\psi_1, \varphi_2\Delta\psi_2 \vdash_S \neg(\varphi_1 \to \varphi_2) \to \neg(\psi_1 \to \psi_2)$
  In this case we have that $((\psi_1 \to \psi_2) \to (\varphi_1 \to \varphi_2)) \to (\neg(\varphi_1 \to \varphi_2) \to \neg(\psi_1 \to \psi_2))$ is a S-tautology by Theorem 2.2), and hence by Theorem 2.1) it is clear that this is true.

- $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S \neg(\psi_1 \to \psi_2) \to \neg(\varphi_1 \to \varphi_2)$
  This case is like the previous one.

(ii) $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\varphi_1 \wedge \varphi_2) \Delta (\psi_1 \wedge \psi_2)$

- $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\varphi_1 \wedge \varphi_2) \to (\psi_1 \wedge \psi_2)$

$$\begin{array}{rcll}
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \varphi_1 & \text{(A3) (MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \varphi_2 & \text{(A4) (MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \psi_1 & \text{(MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \psi_2 & \text{(MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \psi_1 \to \psi_2 & \text{Theorem 2.1} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2, \varphi_1 \wedge \varphi_2 & \vdash_S & \psi_1 \wedge \psi_2 & \text{(A5) (MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 & \vdash_S & (\varphi_1 \wedge \varphi_2) \to (\psi_1 \wedge \psi_2) & \text{Theorem 2.1}
\end{array}$$

- $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\psi_1 \wedge \psi_2) \to (\varphi_1 \wedge \varphi_2)$
  This case is like the previous one.

- The other two cases are analogous to the similar ones for $\to$.

(iii) $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\varphi_1 \vee \varphi_2) \Delta (\psi_1 \vee \psi_2)$

- $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\varphi_1 \vee \varphi_2) \to (\psi_1 \vee \psi_2)$

$$\begin{array}{rcll}
 & \vdash_S & \psi_1 \to (\psi_1 \vee \psi_2) & \text{(A6)} \\
 & \vdash_S & \psi_2 \to (\psi_1 \vee \psi_2) & \text{(A7)} \\
\varphi_1 \to \psi_1 & \vdash_S & \varphi_1 \to (\psi_1 \vee \psi_2) & \text{Theorem 2.1 (MP)} \\
\varphi_2 \to \psi_2 & \vdash_S & \varphi_2 \to (\psi_1 \vee \psi_2) & \text{Theorem 2.1 (MP)} \\
\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 & \vdash_S & (\varphi_1 \vee \varphi_2) \to (\psi_1 \vee \psi_2) & \text{(A8) (MP)}
\end{array}$$

- $\varphi_1 \Delta \psi_1, \varphi_2 \Delta \psi_2 \vdash_S (\psi_1 \vee \psi_2) \to (\varphi_1 \vee \varphi_2)$
  This case is like the previous one.

- The other two cases are analogous to the similar ones for $\to$.

(5) We must prove $v \dashv\vdash_S \delta(v) \Delta \varepsilon(v)$, i.e.

$$v \dashv\vdash_S (v \wedge v) \Delta (v \to v)$$

$(v \wedge v) \Delta (v \to v) \vdash_S v$ is obvious because $(v \wedge v) \to v$ is an axiom of **LPPL**.

We must prove $v \vdash_S (v \wedge v) \Delta (v \to v)$:

- $v \vdash_S (v \wedge v) \to (v \to v)$ is obvious because $\vdash_S v \to ((v \wedge v) \to (v \to v))$ is true by Theorem 2.2.
- $v \vdash_S (v \to v) \to (v \wedge v)$

$$\begin{array}{lll} v \vdash_S & v \wedge v & \text{(A5)(MP)} \\ v \vdash_S & (v \to v) \to (v \wedge v) & \text{(A1)(MP)} \end{array}$$

- $v \vdash_S \neg(v \to v) \to \neg(v \wedge v)$ and $v \vdash_S \neg(v \wedge v) \to \neg(v \to v)$ are directly deductible from the previous two, axiom 9 and Modus Ponens.

□

Remark 4.4. The theorems 3.1, 3.2, 3.3, 3.4 and 3.5 of section 3 are weak-completeness theorems of the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$, but since the defining matrices of these deductive systems are finite, it is easy to see that the weak-completeness implies completeness, in the presence of the deduction theorem.

Remark 4.5. In the next theorem we use the notation $\delta(\Gamma) \approx \varepsilon(\Gamma)$ for the set $\{\delta(\psi) \approx \varepsilon(\psi) : \psi \in \Gamma\}$ as in [4].

Theorem 4.6. *The algebraic reducts of the matrices $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,3}$, $\mathcal{M}^3_{2,2}$ and $\mathcal{M}^4$ with δ, ε and Δ like in Theorem 4.3 are equivalent algebraic semantics for $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}^4$, respectively.*

Proof. Let $\mathcal{M}$ be any of the matrices, $\mathbf{M}$ its algebraic reduct and $\mathcal{S}$ the correspondig deductive system.

(1) $\Gamma \vdash \varphi \Leftrightarrow \delta(\Gamma) \approx \varepsilon(\Gamma) \models_{\mathbf{M}} \delta(\varphi) \approx \varepsilon(\varphi)$.

by Remark 4.4 we have that $\Gamma \models_{\mathcal{M}} \varphi \Leftrightarrow \Gamma \vdash_S \varphi$ and we have that $a \in F$ iff $a \wedge a \in F$, and $a \wedge a \in F$ iff $a \wedge a = 1$, but $a \to a = 1$ for all paraconsistent matrices, and hence $a \in F$ is equivalent to $\delta(a) = \varepsilon(a)$, then $\delta(\Gamma) \approx \varepsilon(\Gamma) \models_{\mathbf{M}} \delta(\varphi) \approx \varepsilon(\varphi)$ if and only if $\Gamma \models_{\mathcal{M}} \varphi$, therefore $\delta(\Gamma) \approx \varepsilon(\Gamma) \models_{\mathbf{M}} \delta(\varphi) \approx \varepsilon(\varphi)$ if and only if $\Gamma \vdash \varphi$.

(2) $\varphi \approx \psi \models_{\mathbf{M}} \delta(\varphi\Delta\psi) \approx \varepsilon(\varphi\Delta\psi)$ is obvious because for any $p \in M$, $p \to p \approx 1$ and $1 \wedge 1 \approx 1$.

(3) $\delta(\varphi\Delta\psi) \approx \varepsilon(\varphi\Delta\psi) \models_{\mathbf{M}} \varphi \approx \psi$.

Since all the matrices are reduced it is enough to prove that for any interpretations of φ and ψ into $\mathbf{M}$ they have the same negation type, i.e., $(\neg^k\varphi)^{\mathbf{M}} \in F$ if and only if $(\neg^k\psi)^{\mathbf{M}} \in F$, where F is the set of designated elements of the matrix.

(a) $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$ and $\mathcal{S}^4$.

In these logics $\alpha \leftrightarrow \neg\neg\alpha$ is an axiom, so it is enough to prove that $\varphi^{\mathbf{M}} \in F$ if and only if $\psi^{\mathbf{M}} \in F$ and that $(\neg\varphi)^{\mathbf{M}} \in F$ if and only if $(\neg\psi)^{\mathbf{M}} \in F$, but by definition of $\rightarrow$ in the respective algebras and the definition of Δ it is clear that this is true.

(b) $\mathcal{S}_{1,3}$ and $\mathcal{S}_{2,2}$.

In the corresponding algebras of these logics we have that for any $a \in M \sim a$ is a classical value, i.e., $\sim a = 0$ or $\sim a = 1$ so it is enough to prove that $\varphi^{\mathbf{M}} \in F$ if and only if $\psi^{\mathbf{M}} \in F$ and that $(\neg\varphi)^{\mathbf{M}} \in F$ if and only if $(\neg\psi)^{\mathbf{M}} \in F$, but by the same argument used in the previous case, this is true.

$\square$

5. ABOUT QUASI-VARIETIES

In this section we study the proprieties of the equivalent quasi-variety semantics of the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$ and $\mathcal{S}_4$, this are the quasi-varieties generated by the algebraic reducts of the matrices $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,3}$, $\mathcal{M}^3_{2,2}$ and $\mathcal{M}^4$.

From now on we will use $\mathbf{M}^3_{1,1}$, $\mathbf{M}^3_{2,1}$, $\mathbf{M}^3_{1,3}$, $\mathbf{M}^3_{2,2}$ and $\mathbf{M}^4$ for the algebraic reducts of the matrices $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,3}$, $\mathcal{M}^3_{2,2}$ and $\mathcal{M}^4$, respectively.

We will define the algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ as.

$$\mathbf{A}_i = \langle \{b, c\}, \wedge, \vee, \rightarrow, \sim_i \rangle,$$

where.

$*$	b	c	$\sim_1$	$\sim_2$
b	b	b	b	b
c	b	b	b	c

with $*$ one of the binary connectives $\wedge$, $\vee$ and $\rightarrow$.

Theorem 5.1. *The quasi-varieties generated by* $\mathbf{M}^3_{1,1}$, $\mathbf{M}^3_{2,1}$, $\mathbf{M}^3_{1,3}$, $\mathbf{M}^3_{2,2}$ *and* $\mathbf{M}^4$ *are not varieties.*

Proof. (1) If $\mathbf{M}$ is $\mathbf{M}^3_{1,3}$ or $\mathbf{M}^3_{2,2}$.
We define $f : \mathbf{M} \longrightarrow \mathbf{A}_1$ as.

$$f(x) = \begin{cases} b & \text{if } x = 1 \text{ or } x = 0. \\ c & \text{in other case.} \end{cases}$$

It is easy to see that f is an homomorphism.

(2) If $\mathbf{M}$ is $\mathbf{M}^3_{1,1}$, $\mathbf{M}^3_{2,1}$ or $\mathbf{M}^4$. We define $f : \mathbf{M} \longrightarrow \mathbf{A}_2$ as.

$$f(x) = \begin{cases} b & \text{if } x = 1 \text{ or } x = 0. \\ c & \text{in other case.} \end{cases}$$

It is easy to see that f is an homomorphism.

Now it is easy to check that $\mathbf{A}_1$ and $\mathbf{A}_2$ do not belong to any of the these quasi-varieties, since.

$$\delta^{\mathbf{A}_i}(b\Delta^{\mathbf{A}_i}c) = \varepsilon^{\mathbf{A}_i}(b\Delta^{\mathbf{A}_i}c)$$

But $b \neq c$.

Therefore the quasi-varieties generated by $\mathbf{M}^3_{1,1}$, $\mathbf{M}^3_{2,1}$, $\mathbf{M}^3_{1,3}$, $\mathbf{M}^3_{2,2}$ and $\mathbf{M}^4$ are not varieties.

□

Now we will call $\mathfrak{M}^3_{1,1}$, $\mathfrak{M}^3_{2,1}$, $\mathfrak{M}^3_{1,3}$, $\mathfrak{M}^3_{2,2}$ and $\mathfrak{M}^4$ the quasi-varieties generated by $\mathbf{M}^3_{1,1}$,$\mathbf{M}^3_{2,1}$, $\mathbf{M}^3_{1,3}$, $\mathbf{M}^3_{2,2}$ and $\mathbf{M}^4$ respectively. Since they are generated by a single finite algebra we have that $ISPP_u\{\mathbf{M}\} = ISP\{\mathbf{M}\}$, where $\mathbf{M}$ is $\mathbf{M}^3_{1,1}$,$\mathbf{M}^3_{2,1}$, $\mathbf{M}^3_{1,3}$, $\mathbf{M}^3_{2,2}$ or $\mathbf{M}^4$, because any ultrapower of a single finite algebra is isomorphic to that algebra.

Now we will construct the corresponding free algebras over one generator for these quasi-varieties.

Remark 5.2. Let $\mathbf{A}$ be $\mathfrak{M}$-algebra (where $\mathfrak{M}$ can be $\mathfrak{M}^3_{1,1}$, $\mathfrak{M}^3_{2,1}$, $\mathfrak{M}^3_{1,3}$, $\mathfrak{M}^3_{2,2}$ or $\mathfrak{M}^4$), since $\mathbf{A}$ is isomorphic to a subalgebra of a power of the corresponding $\mathbf{M}$, in the next sections we will call *Boolean element* the elements $a \in A$ such that $\forall i(a(i) = 0 \vee a(i) = 1)$ and we will denote $\mathbf{Bl(A)}$ the set of all Boolean elements of the algebra $\mathbf{A}$. It is easy to see that $x \in \mathbf{Bl(A)}$ if and only if $x \wedge x = x$ and that for every $x \in A$, $x \wedge x \in \mathbf{Bl(A)}$.

Lemma 5.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathfrak{M}^3_{1,1}$, $\mathfrak{M}^3_{2,1}$, $\mathfrak{M}^3_{1,3}$, $\mathfrak{M}^3_{2,2}$ *or* $\mathfrak{M}^4$*-algebras and* $f : A \longrightarrow B$. *Then* f *is an homomorphism if and only if* f *verifies the next conditions:*

(1) f *restricted to* $\mathbf{Bl(A)}$ *is a homomorphism between* $\mathbf{Bl(A)}$ *and* $\mathbf{Bl(B)}$.
(2) $a \notin \mathbf{Bl(A)}$ *implies* $f(\sim a) = \sim f(a)$
(3) $a \notin \mathbf{Bl(A)}$ *implies* $f(a \wedge a) = f(a) \wedge f(a)$

Proof. $\Rightarrow$) if f is an homomorphism it is easy to see that f verifies the second and third items of this lemma and for the first one we have that if $x \in \mathbf{Bl(A)}$ then $x \wedge x = x$, hence we have $f(x) = f(x \wedge x) = f(x) \wedge f(x)$ and therefore $f(x) \in \mathbf{Bl(B)}$

$\Leftarrow$) for $a, b \in A$ we have to prove that $f(a * b) = f(a) * f(b)$, where $*$ is a binary connective (because $f(\sim a) = \sim f(a)$ is true by hypothesis).

$$
\begin{array}{rcll}
a * b & = & (a \wedge a) * (b \wedge b) & \\
f(a * b) & = & f((a \wedge a) * (b \wedge b)) & \\
f(a * b) & = & f(a \wedge a) * f(b \wedge b) & \text{since } x \wedge x \in \mathbf{Bl(A)} \\
f(a * b) & = & (f(a) \wedge f(a)) * (f(b) \wedge f(b)) & \text{by hypothesis} \\
f(a * b) & = & f(a) * f(b) &
\end{array}
$$

□

5.1. **Free $\mathfrak{M}^3_{1,1}$-algebra over a single generator.** Let x be an element of an $\mathfrak{M}^3_{1,1}$-algebra $\mathbf{B}$, since $\mathbf{B}$ is isomorphic to a subalgebra of a power of $\mathbf{M}^3_{1,1}$, we can say that $x \in {\mathbf{M}^3_{1,1}}^I$ for some set I and

${\mathbf{M}^3_{1,1}}^I \cong {\mathbf{M}^3_{1,1}}^J \times {\mathbf{M}^3_{1,1}}^K \times {\mathbf{M}^3_{1,1}}^L$, where $J = \{i \in I : x_i = 0\}$, $K = \{i \in I : x_i = 1\}$ and $L = \{i \in I : x_i = \frac{1}{2}\}$. We take $x \in {M^3_{1,1}}^I$, $I \geq 3$, such that $J \neq \emptyset$, $K \neq \emptyset$ and $L \neq \emptyset$, x can be regarded as the tuple $(0, 1, \frac{1}{2})$. Let $\mathcal{F}$ be the $\mathfrak{M}^3_{1,1}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are:

$$\begin{aligned}
x &= (0,1,\frac{1}{2}) \\
\sim x &= (1,0,\frac{1}{2}) \\
x \wedge x &= (0,1,0) \\
\sim x \wedge \sim x &= (1,0,0) \\
x \wedge \sim x &= (0,0,0) \\
x \vee \sim x &= (1,1,0) \\
x \rightarrow \sim x &= (1,0,1) \\
\sim x \rightarrow x &= (0,1,1) \\
x \rightarrow x &= (1,1,1) \\
(x \rightarrow \sim x) \wedge (\sim x \rightarrow x) &= (0,0,1)
\end{aligned}$$

And its diagram is in Figure 5.1

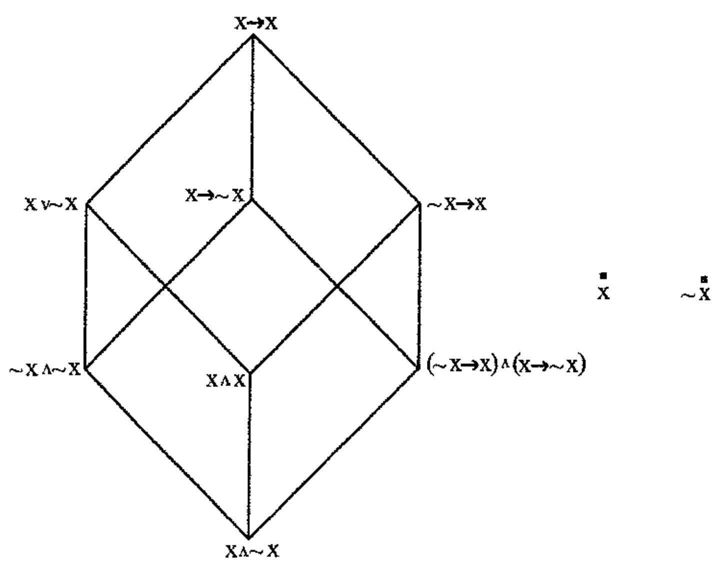
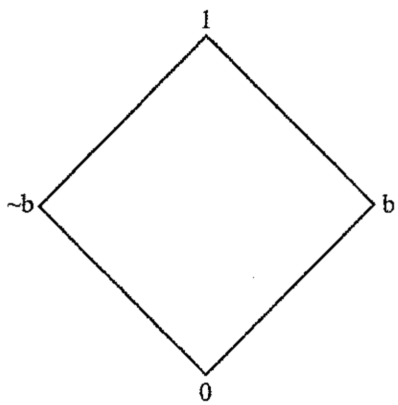


FIGURE 5.1. Free $\mathfrak{M}^3_{1,1}$-algebra generated by x

Theorem 5.4. *$\mathcal{F}$ is the free $\mathfrak{M}^3_{1,1}$-algebra over a single generator x.*

Proof. Let b be an element of a $\mathfrak{M}^3_{1,1}$-algebra B, with the same procedure of the last paragraph we could say that $b \in \mathbf{M}^3_{1,1}{}^J \times \mathbf{M}^3_{1,1}{}^K \times \mathbf{M}^3_{1,1}{}^L$ we have five cases for this element. In all these cases we present the $\mathfrak{M}^3_{1,1}$-algebra generated by b and the unique homomorphism between $\mathcal{F}$ and the corresponding $\mathfrak{M}^3_{1,1}$-algebra generated by b, such that $f(x) = b$.

(1) $L = \emptyset$



In this case we have.

$$\begin{aligned} b &= (0,1) \\ \sim b &= (1,0) \\ b \wedge \sim b &= (0,0) \\ b \to b &= (1,1) \end{aligned}$$

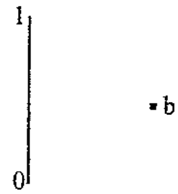
And the homomorphism f will be.

$$\begin{array}{rcccccc} f(x) &=& f(x \wedge x) &=& f(\sim x \to x) &=& b \\ f(\sim x) &=& f(\sim x \wedge \sim x) &=& f(x \to \sim x) &=& \sim b \\ f(x \wedge \sim x) &=& f((x \to \sim x) \wedge (\sim x \to x)) &=& 0 && \\ f(x \to x) &=& f(x \vee \sim x) &=& 1 && \end{array}$$

(2) $J = K = \emptyset$ and $L \neq \emptyset$

In this case we have.

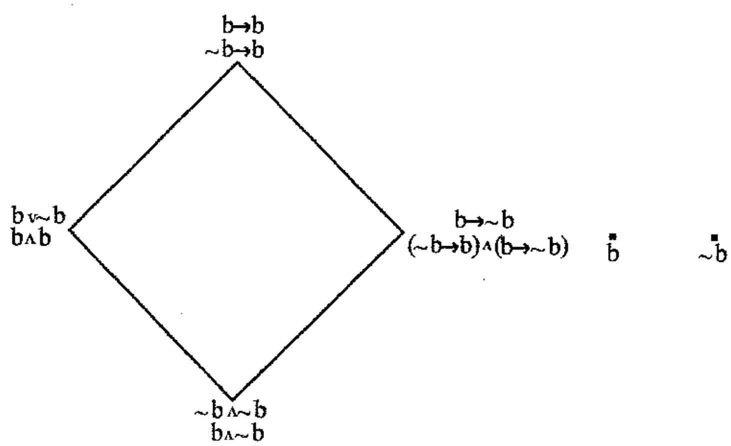
$$\begin{aligned} b &= \frac{1}{2} \\ b \to b &= 1 \\ b \wedge \sim b &= 0 \end{aligned}$$



And the homomorphism f will be.

$$\begin{array}{rcccccccc} f(x) & = & f(\sim x) & = & b & & & & \\ f(x\to x) & = & f(\sim x\to x) & = & f(x\to\sim x) & = & f((x\to\sim x)\wedge(\sim x\to x)) & = & 1 \\ f(x\wedge\sim x) & = & f(x\wedge x) & = & f(x\vee\sim x) & = & f(\sim x\wedge\sim x) & = & 0 \end{array}$$

(3) $J=\emptyset$



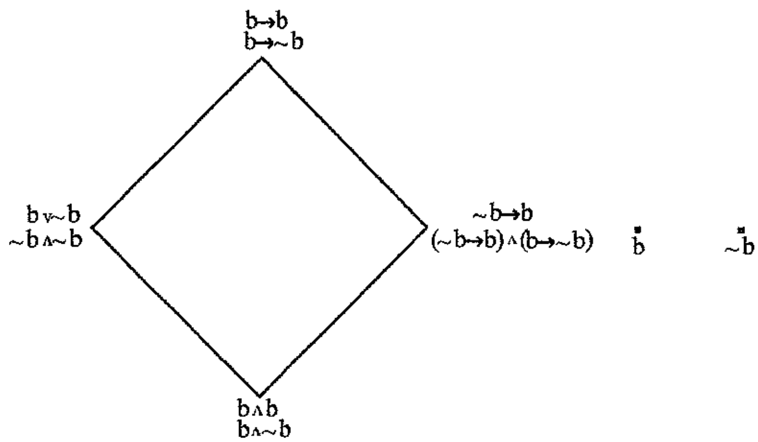
In this case we have

$$\begin{aligned} b &= (1,\frac{1}{2}) \\ \sim b &= (0,\frac{1}{2}) \\ b\wedge b &= (1,0) \\ b\to\sim b &= (0,1) \\ b\wedge\sim b &= (0,0) \\ b\to b &= (1,1) \end{aligned}$$

And the homomorphism f will be.

$$\begin{array}{rcccl} f(x) & = & b & & \\ f(\sim x) & = & \sim b & & \\ f(x\wedge x) & = & f(x\vee\sim x) & = & b\wedge b \\ f(x\to\sim x) & = & f((x\to\sim x)\wedge(\sim x\to x)) & = & b\to\sim b \\ f(x\wedge\sim x) & = & f(\sim x\wedge\sim x) & = & b\wedge\sim b \\ f(x\to x) & = & f(\sim x\to x) & = & b\to b \end{array}$$

(4) $K=\emptyset$



In this case we have

$$\begin{array}{rcl} b & = & (0,\frac{1}{2}) \\ \sim b & = & (1,\frac{1}{2}) \\ \sim b\wedge\sim b & = & (1,0) \\ \sim b\to b & = & (0,1) \\ b\wedge\sim b & = & (0,0) \\ b\to b & = & (1,1) \end{array}$$

And the homomorphism f will be.

$$\begin{array}{rcccl} f(x) & = & b & & \\ f(\sim x) & = & \sim b & & \\ f(\sim x\wedge\sim x) & = & f(x\vee\sim x) & = & b\wedge b \\ f(\sim x\to x) & = & f((x\to\sim x)\wedge(\sim x\to x)) & = & b\to\sim b \\ f(x\wedge\sim x) & = & f(x\wedge x) & = & b\wedge\sim b \\ f(x\to x) & = & f(x\to\sim x) & = & b\to b \end{array}$$

(5) $J \neq \emptyset$, $K \neq \emptyset$ and $L \neq \emptyset$.
This case is obvious.

In all cases is easy to check using Remark 5.2 and Lemma 5.3 that f is an homomorphism, and it is obvious unique. □

5.2. **Free $\mathfrak{M}^3_{2,1}$-algebra over a single generator.** Let x be an element of an $\mathfrak{M}^3_{2,1}$-algebra $\mathbf{B}$, since $\mathbf{B}$ is isomorphic to a subalgebra of a power of $\mathbf{M}^3_{2,1}$, we can say that $x \in {\mathbf{M}^3_{2,1}}^I$ for some set I and ${\mathbf{M}^3_{2,1}}^I \cong {\mathbf{M}^3_{2,1}}^J \times {\mathbf{M}^3_{2,1}}^K \times {\mathbf{M}^3_{2,1}}^L$, where $J = \{i \in I : x_i = 0\}$, $K = \{i \in I : x_i = 1\}$ and $L = \left\{i \in I : x_i = \frac{1}{2}\right\}$. We take $x \in {M^3_{2,1}}^I$, $I \geq 3$, such that $J \neq \emptyset$, $K \neq \emptyset$ and $L \neq \emptyset$, x can be regarded as the tuple $(0,1,\frac{1}{2})$. Let $\mathcal{F}$ be the $\mathfrak{M}^3_{2,1}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are:

$$\begin{aligned}
x &= (0,1,\frac{1}{2}) \\
\sim x &= (1,0,\frac{1}{2}) \\
x \wedge x &= (0,1,1) \\
\sim x \wedge \sim x &= (1,0,1) \\
x \wedge \sim x &= (0,0,1) \\
x \vee \sim x &= (1,1,1) \\
\sim (x \wedge x) &= (1,0,0) \\
\sim (\sim x \wedge \sim x) &= (0,1,0) \\
\sim (x \wedge \sim x) &= (1,1,0) \\
\sim (x \vee \sim x) &= (0,0,0)
\end{aligned}$$

And its diagram is in Figure 5.2

Theorem 5.5. *$\mathcal{F}$ is the free $\mathfrak{M}^3_{2,1}$-algebra over a single generator x.*

Proof. It is analogous to the proof of Theorem 5.4. □

5.3. **Free $\mathfrak{M}^3_{1,3}$-algebra over a single generator.** Let x be an element of an $\mathfrak{M}^3_{1,3}$-algebra $\mathbf{B}$, since $\mathbf{B}$ is isomorphic to a subalgebra of a power of $\mathbf{M}^3_{1,3}$, we can say that $x \in {\mathbf{M}^3_{1,3}}^I$ for some set I and ${\mathbf{M}^3_{1,3}}^I \cong {\mathbf{M}^3_{1,1}}^J \times {\mathbf{M}^3_{1,3}}^K \times {\mathbf{M}^3_{1,3}}^L$, where $J = \{i \in I : x_i = 0\}$, $K = \{i \in I : x_i = 1\}$ and $L = \left\{i \in I : x_i = \frac{1}{2}\right\}$. We take $x \in {M^3_{1,3}}^I$, $I \geq 3$,

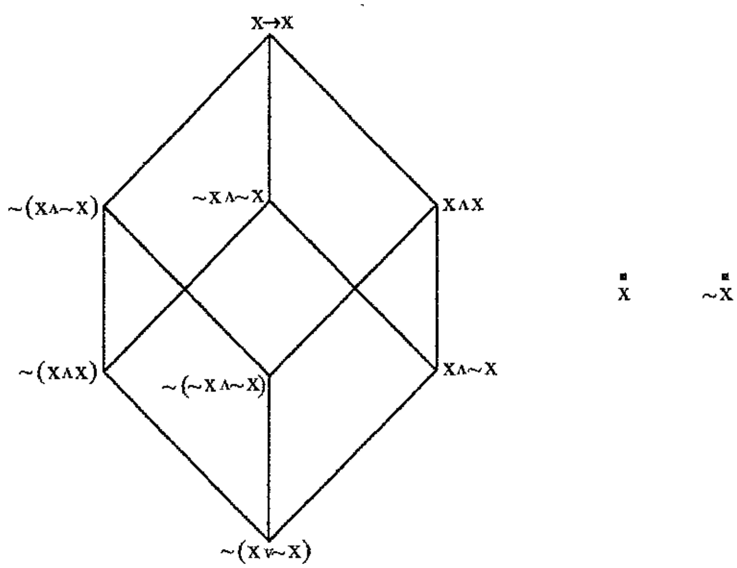


FIGURE 5.2. Free $\mathfrak{M}^3_{2,1}$-algebra generated by x

suhch that $J \neq \emptyset$, $K \neq \emptyset$ and $L \neq \emptyset$, x can be regarded as the tuple $(0, 1, \frac{1}{2})$. Let $\mathcal{F}$ be the $\mathfrak{M}^3_{1,3}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are:

$$\begin{aligned} x &= (0,1,\frac{1}{2}) \\ \sim x &= (1,0,0) \\ \sim\sim x &= (0,1,1) \\ x \wedge x &= (0,1,0) \\ x \wedge \sim x &= (0,0,0) \\ x \vee \sim x &= (1,1,0) \\ x \to\sim x &= (1,0,1) \\ x \to x &= (1,1,1) \\ (x \to\sim x) \wedge (\sim\sim x) &= (0,0,1) \end{aligned}$$

And its diagram is in Figure 5.3

Theorem 5.6. *$\mathcal{F}$ is the free $\mathfrak{M}^3_{1,3}$-algebra over a single generator x.*

Proof. It is analogous to the proof of Theorem 5.4. □

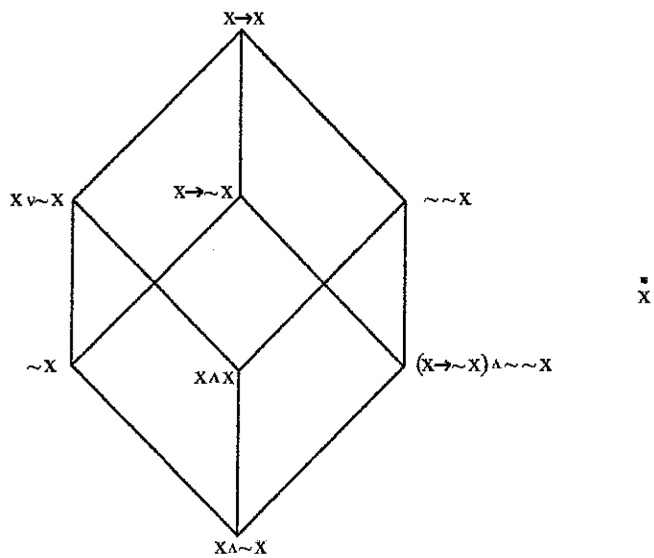


FIGURE 5.3. Free $\mathfrak{M}^3_{1,3}$-algebra generated by x

5.4. **Free $\mathfrak{M}^3_{2,2}$-algebra over a single generator.** Let x be an element of an $\mathfrak{M}^3_{2,2}$-algebra **B**, since **B** is isomorphic to a subalgebra of a power of $\mathbf{M}^3_{2,2}$, we can say that $x \in {\mathbf{M}^3_{2,2}}^I$ for some set I and ${\mathbf{M}^3_{2,2}}^I \cong {\mathbf{M}^3_{2,2}}^J \times {\mathbf{M}^3_{2,2}}^K \times {\mathbf{M}^3_{2,2}}^L$, where $J = \{i \in I : x_i = 0\}$, $K = \{i \in I : x_i = 1\}$ and $L = \{i \in I : x_i = \frac{1}{2}\}$. We take $x \in {M^3_{2,2}}^I$, $I \geq 3$, such that $J \neq \emptyset$, $K \neq \emptyset$ and $L \neq \emptyset$, x can be regarded as the tuple $(0, 1, \frac{1}{2})$. Let $\mathcal{F}$ be the $\mathfrak{M}^3_{2,2}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are:

$$\begin{aligned}
x &= (0, 1, \frac{1}{2}) \\
\sim x &= (1, 0, 1) \\
\sim\sim x &= (0, 1, 0) \\
x \wedge x &= (0, 1, 1) \\
x \wedge \sim x &= (0, 0, 1) \\
\sim (x \wedge x) &= (1, 0, 0) \\
\sim (x \vee \sim x) &= (0, 0, 0) \\
\sim (x \wedge \sim x) &= (1, 1, 0) \\
x \to x &= (1, 1, 1)
\end{aligned}$$

And its diagram is in Figure 5.4

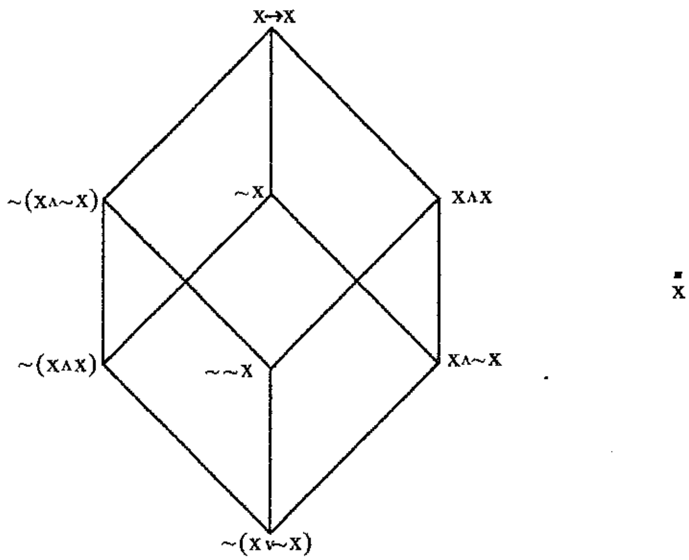


FIGURE 5.4. Free $\mathfrak{M}^3_{2,2}$-algebra generated by x

Theorem 5.7. *$\mathcal{F}$ is the free $\mathfrak{M}^3_{2,2}$-algebra over a single generator x.*

Proof. It is analogous to the proof of Theorem 5.4. □

5.5. **Free $\mathfrak{M}^4$-algebra over a single generator.** Let x be an element of an $\mathfrak{M}^4$-algebra $\mathbf{B}$, since $\mathbf{B}$ is isomorphic to a subalgebra of a power of $\mathbf{M}^4$, we can say that $x \in \mathbf{M}^{4^I}$ for some set I and $\mathbf{M}^{4^I} \cong \mathbf{M}^{4^J} \times \mathbf{M}^{4^K} \times \mathbf{M}^{4^L} \times \mathbf{M}^{4^M}$, where $J = \{i \in I : x_i = 0\}$, $K = \{i \in I : x_i = 1\}$, $L = \{i \in I : x_i = \top\}$ and $M = \{i \in I : x_i = \bot\}$. We take $x \in M^{4^I}$, $I \geq 4$, such that $J \neq \emptyset$, $K \neq \emptyset$, $L \neq \emptyset$ and $M \neq \emptyset$, x can be regarded as the tuple $(0, 1, \top, \bot)$. Let $\mathcal{F}$ be the $\mathfrak{M}^4$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are:

$$\begin{aligned}
x &= (0,1,\top,\bot)\\
\sim x &= (1,0,\top,\bot)\\
x \wedge x &= (0,1,1,0)\\
\sim x \wedge \sim x &= (1,0,1,0)\\
x \wedge \sim x &= (0,0,1,0)\\
x \vee \sim x &= (1,1,1,0)\\
x \rightarrow \sim x &= (1,0,1,1)\\
\sim x \rightarrow x &= (0,1,1,1)\\
x \rightarrow x &= (1,1,1,1)\\
(x \rightarrow \sim x) \wedge (\sim x \rightarrow x) &= (0,0,1,1)
\end{aligned}$$

and its negations.

And its diagram is in Figure 5.5

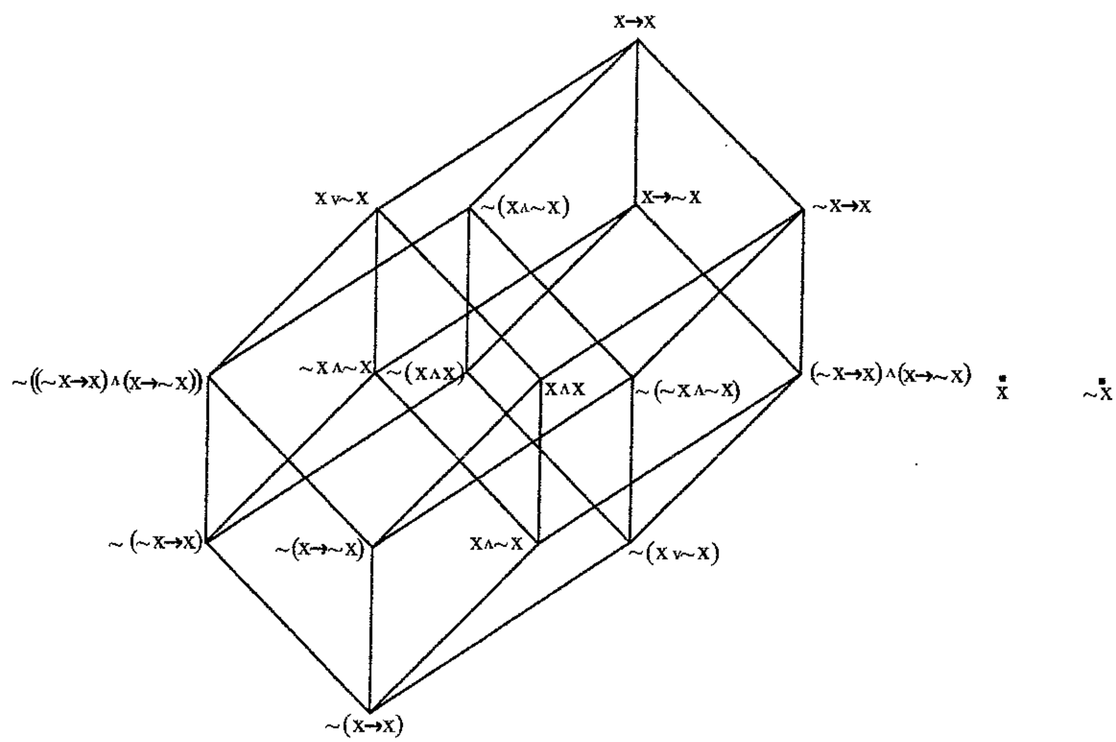


FIGURE 5.5. Free $\mathfrak{M}^4$-algebra generated by x

Theorem 5.8. *$\mathcal{F}$ is the free $\mathfrak{M}^4$-algebra over a single generator x.*

Proof. It is analogous to the proof of Theorem 5.4. □

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