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LOGARITHMIC CHERN SLOPES OF ARRANGEMENTS OF RATIONAL SECTIONS IN HIRZEBRUCH SURFACES

por

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1 Introduction

1.1 The Problem of the Behaviour of Chern Numbers

The subject of algebraic surfaces of general type is intriguing and difficult to study, understanding their behaviour is of great concern for algebraic geometers. To make this introduction easier to read, we will only consider complex surfaces of general type, although in the thesis we mention some results dealing with real surfaces and surfaces over fields of positive characteristic.

In order to classify algebraic surfaces of general type two parameters are used, called the *Chern numbers*, denoted c_1^2 and c_2. In the case of complex surfaces, these parameters can be defined as:

- c_1^2 is the self-intersection of the canonical divisor,
- c_2 is the topological Euler characteristic.

In the case of minimal complex surfaces, these numbers are both positive integers. The ratio c_1^2/c_2 is called the *Chern slope* of the surface. It is the behaviour of this slope and the Chern numbers that attracts our attention.

The first natural question to arise is: when does a pair of positive integers (n, m) represent the Chern numbers of a surface X, (c_2, c_1^2)? This is called the "Geography Problem" (for a detailed exposition of this, see [25]). A few restrictions arise quickly as one studies the problem. For example, by Noether's formula the Chern numbers of a surface satisfy: $c_1^2 + c_2 \equiv 0 \mod 12$. Also, if a surface X is blown up at a smooth point, then c_2 increases by 1, while c_1^2 decreses by 1. As [1] explains in section VII.8, the problem of the behaviour of Chern numbers can be precisely stated thus:

Given a pair of integers (n, m) satisfying $n > 0$, $n \leq 3m$, $m \leq 5n + 36$ and $n + m \equiv 0 \mod 12$, is there a minimal surface of general type X such that the Chern numbers of X are $c_1^2 = m$ and $c_2 = n$?

Let us call D the subset of $\mathbb{Z}^2$ of pairs of integers satifying the conditions of the question and let $D_1 = D \cap \{n \leq 2m\}$.

1.2 Results Regarding the Problem

In 1981 Ulf Persson shows in [24] that for any pair $(n, m) \in D_1$, there exists a minimal surface of general type X with $c_1^2 = m$ and $c_2 = n$, except maybe for points on the finitely many lines of the form $m - 3n + 4k = 0$, where $0 \leq k \leq 347$.

This result does not give surfaces satisfying $3c_2 = c_1^2$. A considerably difficult result, known as the Bogomolov-Miyaoka-Yau inequality, implies that for minimal complex surfaces of general type $c_1^2 \leq 3c_2$, with equality if and only if the universal cover of the surface is the unit ball (this last part is due to Miyaoka and Yau). So we know that the Chern slope cannot be bigger than 3. On the other hand, from Noether's inequality it can be deduced that:

$$\frac{1}{5}c_2 - \frac{36}{5} \leq c_1^2.$$

The natural questions now are: Which rational numbers can be obtained as the Chern slope of some surface of general type? Are there special regions in which the Chern slope is dense?

In a paper from 1983 called *Arrangements of Lines and Algebraic Surfaces*, Friedrich Hirzebruch shows a method, which relies mostly on abelian covers, for constructing minimal complex surfaces of general type from line arrangements on the projective plane such that the Chern slope of the resulting surface can be easily calculated from the combinatorial information of the line arrangement. With this method he can prove that the interval $\left[2, \frac{5}{2}\right]$ is densly covered by the Chern slopes of surfaces of general type.

A year later, Andrew J. Sommese published a paper called *On the Density of Ratios of Chern Numbers of Algebraic Surfaces* in which he proved that every rational point in $\left[\frac{1}{5}, 3\right]$ occurs as a limit of Chern slopes of a sequence of minimal surfaces of general type.

Now the next question is: can we impose more conditions on these surfaces to obtain similar results? Specifically: can we make these surfaces be simply connected? Sommese's proof generates algebraic surfaces which are a rather long way away from being simply connected. Hirzebruch's method in [15] however produces such surfaces, but the Chern slopes live in a limited interval. In 2014 Xavier Roulleau and Giancarlo Urzúa published a paper called *Chern slopes of simply connected complex surfaces are dense in* $[2, 3]$ in which they prove, much like the title suggests, that the interval $[2, 3]$ is densly covered by simply connected complex surfaces of general type. The proof is highly technical.

1.3 Our Problems of Interest

In the paper of Sommese we mentioned, he examines Hirzebruch's results and finds that the Chern slopes of the surfaces obtained with Hirzebruch's method lie in the interval $\left[1, \frac{8}{3}\right]$ and that the least limit point is 2. Recall that Hirzebruch had shown density of the Chern slopes in the interval $[2, 2.5]$. Hence the interval $\left]\frac{5}{2}, \frac{8}{3}\right]$ becomes a mysterious interval, because none of these papers give a precise discussion about the behaviour of Chern slopes in it.

On the other hand, Hirzebruch's method is very general, so it can be reproduced on surfaces different from the projective plane. This poses another question: can we use this method on other surfaces to obtain minimal simply connected complex surfaces of general type in an interval different from $[2, 3]$?

The aim of this thesis is to explore these questions, with special emphasis on the last one. Unlike Hirzebruch, I will not be working on just one surface, but on a family of surfaces called Hirzebruch Surfaces. Why a family of surfaces? Because considering a family instead of just one surface introduces a new parameter into the problem which could potentially (and as we will see in part III, it actually does this) give us more freedom to move the Chern slopes to obtain limit points in other intervals. Why this family in particular? Because of a result of Giancarlo Urzúa which provides us with .

1.4 General Structure of the Thesis

The thesis has three parts, each divided into sections and subsections. The first part deals with the general theory. It is a little bit segmented so as to approach different definitions and results needed in the thesis. The main purposes of this first part are: to introduce notation, to introduce the Chern numbers of a surface, to define what we understand by a curve arrangement on a surface, and to show how we plan to use abelian extensions. The results have been taken from a wide variety of sources. The reader who is familiar with these topics may want to skip the first part, as the results which motivate this thesis are all in the other two parts.

The second part is about line arrangements on the projective plane. The purpose is to compile all the results from [15], [16] and [31] and try to extend them. The main question is the one about the behaviour of Chern slopes in the mysterious interval $\left]\frac{5}{2}, \frac{8}{3}\right]$. It is important to study these results because they are not only needed in Part III, but they also describe the way we will address section arrangements. Part II also adresses some questions which I feel were not mentioned or discussed suficiently in the papers aforementioned. There are also some other results regarding the importance of the characteristic of the base field of the projective plane. Hizebruch's and Sommese's work was for the complex projective plane, and I study certain results for projective planes in positive characterisitic. I am able to answer some of these questions, and in the other cases (as is the case of the mysterious interval), I exhibit the specific nature of the problem.

The third part is what this thesis is all about. Here we study section arrangements on Hirzebruch surfaces. We proceed using abelian extensions, inspired by Hirzebruch. The original results of this thesis have to do with the behaviour of the Chern slope of surfaces of general type obtained from section arrangements. As we will see, we can produce a family of surfaces of general type such that the sequence of their Chern slopes converges to $\frac{4}{3}$. This section also uses some advanced combinatorial results in order to produce certain special examples of section arrangements in positive characteristic.

Part I

Preliminaries

2 Prerequisites

The reader should be familiar with a general background in algebraic surfaces. For the case of complex surfaces: [2], [1] and chapter V of [13] are recommended. The reader should also be familiar with a basic background in algebraic geometry, chapter 2 of [13] is recommended. Now we list some definitions and results we are going to use.

2.1 Miscelaneous

Here are some definitions taken form [13]. The varieties are taken over an algebraically closed field.

1. Let X be a variety and $\zeta \in X$ its generic point. The local ring $\mathcal{O}_{X,\zeta}$ is a field called the *function field* of X, and we denote it by $K(X)$.

2. A variety is *normal* if all of its local rings are integrally closed domains.

3. Let X be a variety. For each open affine subset $U = \text{Spec}A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \text{Spec}\widetilde{A}$. The schemes $\widetilde{U}$ can be glued to produce a normal integral scheme $\widetilde{X}$ called the *normalization* of X. There also exists a morphism $\widetilde{X} \to X$ satisfying the following universal property: for every normal integral scheme Z and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. (See exercise 3.8 of section II.3 of [13]).

4. Let Ω^1_X denote the sheaf of differentials of X over its ground field (see section II.8 of [13]).

2.2 About Surfaces

Surfaces are always smooth and defined over an algebraically closed field.

1. K_X will denote the canonical divisor on X (for the definition see Section 4).

2. We call $q(X) = h^1(X, \mathcal{O}_X)$ the *irregularity* of X, $p_g(X) = h^2(X, \mathcal{O}_X)$ the *geometric genus*, and $P_n(X) = h^0(X, \mathcal{O}_X(nK_X))$ the *plurigenera* of X. By Serre duality we get $p_g(X) = P_1(X)$.

3. Let ϕ_{nK_X} be the rational map from X to the projective space associated with the system $|nK_X|$. The *Kodaira dimension* of X, called $\kappa(X)$, is the maximum dimension of the images of ϕ_{nK_X} for $n \geq 1$. If $|nK_X| = \emptyset$ for all n, then we set $\kappa(X) = -\infty$.

4. X is called *of general type* if $\kappa(X) = 2$.

A great deal can be said about complex surfaces with $\kappa(X) < 2$ (see for example [2] or [1]), that is, they can be classified up to a certain point. The fact that surfaces of general type are not so well understood on the other hand, is what makes them so interesting.

2.3 About Complex Surfaces

Now we focus specifically on surfaces over $\mathbb{C}$. First some definitions.

1. We call $b_i(X) = \dim_{\mathbb{R}} H^i(X, \mathbb{R})$ the *Betti numbers.*

2. We call $e(X) = \sum_i (-1)^i b_i(X)$ the *topological Euler characteristic.*

3. $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$.

Now some results. See [2].

1. The integers $q(X), p_g(X), P_n(X)$ are birational invariants.

2. Poincaré duality yields $b_0(X) = b_4(X) = 1$ and $b_1(X) = b_3(X)$, so $e(X) = 2 - 2b_1(X) + b_2(X)$.

3. $q(X) = h^0(X, \Omega_X) = \frac{1}{2} b_1(X)$.

3 Divisors

3.1 Weil Divisors

This section is taken from [13], section II.6.

Definition 1. We say a variety X is *regular in codimension one* if every local ring $\mathcal{O}_{X,x}$ of X of dimension one is regular.

Definition 2. A scheme X satisfies *condition* $*$ if X is a noetherian integral separated variety which is regular in codimension one.

Definition 3. Let X satisfy condition $*$. A *prime divisor* on X is a closed integral subvariety Y of codimension one. A *Weil divisor* (or simply *divisor*) is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. Divisors are written as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the $n_i \in \mathbb{Z}$, and only finitely many n_i are different from zero. If all the $n_i \geq 0$, the divisor D is called *effective*.

Definition 4. Let X satisfy condition $*$. Let Y be a prime divisor on X, and let η be the generic point of Y. The local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field $K(X)$. The corresponding discrete valuation v_Y is called the *valuation of* Y. By Exercise 4.5 of chapter II of [13] we know that, as X is separated, then Y is uniquely determined by its valuation. Let $f \in K(X)^*$ be any nonzero rational function on X. Then $v_Y(f)$ is an integer. If it is positive, we way that f has a *zero along* Y of that order; if it is negative, we say f has a *pole along* Y, of order $-v_Y(f)$.

Lemma 3.1 (See Lemma 6.1 of [13])**.** *Let X satisfy condition $*$, and let $f \in K(X)^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y on X.*

Definition 5. Let X satisfy condition $*$ and let $f \in K(X)^*$. The *divisor of* f, denoted by (f), is:

$$(f) = \sum v_Y(f) Y,$$

where the sum is taken over all prime divisors of X. By the previous Lemma, this definition is correct as the sum is finite. Any divisor which equals the divisor of a function is called a *principal divisor*.

Remark 1. Let $f, g \in K(X)^*$. Then the properties of discrete valuations give us that $(f/g) = (f) - (g)$. Therefore the function $f \mapsto (f)$ is a group homomorphism between the multiplicative group $K(X)^*$ and the additive group $\mathrm{Div}(X)$. Therefore the principal divisors form a subgroup of $\mathrm{Div}(X)$.

Definition 6. Let X satisfy condition $*$. Two divisors, D and D', are said to be *linearly equivalent*, written as $D \sim D'$, if $D - D'$ is a principal divisor. The quotient group obtained from $\mathrm{Div}(X)$ divided by the subgroup of principal divisors is called the *divisor class group* of X, and is denoted by $\mathrm{Cl}(X)$.

Example 1 (The Class Group of the Projective Plane)**.** Let L be a line in $\mathbb{P}^2$. L is defined a linear form f. Let C be any irreducible curve in $\mathbb{P}^2$. Then C is defined by a form g of some degree, say d. Note that g/f^d is a rational function on $\mathbb{P}^2$, and so the divisor class of C in $\mathrm{Cl}(X)$ is the same as d times the divisor class of L, i.e: $[C] = d[L]$, where $[.]$ denotes divisor class. Therefore the class group of $\mathbb{P}^2$ is the free Abelian group of rank 1: $\mathbb{Z}[L]$.

3.2 Cartier Divisors

This section is taken from section II.6 of [13], except when otherwise stated.

Definition 7. Let X be a variety, $U \subseteq X$ open. Let $S(U)$ denote the set of elements of $\mathcal{O}_X(U)$ which are not zero divisors in each local ring $\mathcal{O}_{X,x}$ for $x \in U$. Then $S(U)$ is a multiplicative system, so one can take the localization $S(U)^{-1}\mathcal{O}_X(U)$. The assignment $U \mapsto S(U)^{-1}\mathcal{O}_X(U)$ defines a presheaf. The induced sheaf $\mathscr{K}$ is called the *sheaf of total quotient rings* of $\mathcal{O}_X$.

Definition 8. Given a sheaf of rings $\mathscr{F}$ on a scheme X, let $\mathscr{F}^*$ denote the sheaf of abelian groups such that $\mathscr{F}^*(U)$ is the group of invertible elements in $\mathscr{F}(U)$, for all $U \subseteq X$ open.

Definition 9. A *Cartier divisor* on a variety X is an element of $\Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$. Alternatively, a Cartier divisor can be described by a family of the form $\{(U_i, f_i)\}$, where $\{U_i\}$ forms an open covering of X, and $f_i \in \Gamma(U_i, \mathscr{K}^*)$, such that for each i, j, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. Note that there is a natural homomorphism $\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$ given by the projection. The elements in the image of this homomorphism are called *principal Cartier divisors*. Two Carties divisors are said to be *linearly dependent* if their ratio is principal.

On some situations, like in the case of surfaces, Weil divisors and Cartier divisors "coincide" in the sense of the following proposition.

Proposition 3.1 (See Proposition 6.11 of [13])**.** *Let X satisfy condition $*$ and be locally factorial. Then the group* $\mathrm{Div}(X)$ *of Weil divisors on X is isomorphic to the group of Cartier divisors* $\Gamma(X, \mathscr{K}^*/\mathcal{O}_X^*)$*, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Because of this isomorphism, one tends to use the language of additive groups when speaking about Cartier divisors, even though they are constructed from multiplicative groups.

The next definitions are taken from Remark 3.8.1 of section V.3 of [13].

Definition 10. Let X be a surface. Two divisors D and D' on X are said to *intersect transversally* if, for every point p in the intersection, their local equations f_1 and f_2 around p are linearly independent $\mod \mathfrak{m}_p^2$.

Definition 11. Let X satisfy condition $*$. A divisor $D = \sum D_j$ on X is said to have *simple normal crossings* if the D_j are non-singular components intersecting each other transversally.

3.3 Invertible Sheaves

This section follows section II.6 of [13].

Definition 12. Let X be a scheme. An *invertible sheaf* on X is a locally free $\mathcal{O}_X$-module of rank one. For an invertible sheaf $\mathscr{L}$ on X, define the *dual sheaf of* $\mathscr{L}$ as $\mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$. We denote it by $\mathscr{L}^{-1}$, because of the following proposition.

Proposition 3.2 (See Proposition 6.12 of [13]). *If $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves on a scheme X, then so is $\mathscr{L} \otimes \mathscr{M}$. Also, $\mathscr{L}^{-1}$ is an invertible sheaf such that $\mathscr{L} \otimes \mathscr{L}^{-1} = \mathcal{O}_X$.*

Given an integer n, we will use the notation $\mathscr{L}^n$ to represent $\mathscr{L}^{\otimes n}$, and $\mathscr{L}^{-n}$ denotes the inverse of $\mathscr{L}^n$ under $\otimes$.

Definition 13. By the previous proposition, for a scheme X, the set of isomorphism classes of invertible sheaves forms a group under the operation $\otimes$. This group is called the *Picard group* of X: $\mathrm{Pic}(X)$.

Definition 14. Let $D = \{(U_i, f_i)\}$ be a Cartier divisor on a variety X. Let $\mathscr{K}$ be the sheaf of total quotient rings on X. Let $\mathcal{O}_X(D)$ be the subsheaf of $\mathscr{K}$ generated by f_i^{-1} on U_i. This is well-defined as the definition of Cartier divisor requires f_i/f_j to be invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. $\mathcal{O}_X(D)$ is called the *sheaf associated to D*.

Proposition 3.3 (See Proposition 6.13 in chapter II of [13]). *Let X be a variety. Then for any Cartier divisor D on X, $\mathcal{O}_X(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{O}_X(D)$ is a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathscr{K}$.*

Corollary 3.1 (See Corollary 6.16 in chapter II of [13]). *If X is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism* $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

Due to this Corollary, when we want to calculate the Picard group of a surface, we will do it via calculating the class group.

4 The Chow Ring of a Surface

This section is taken from [13], appendix A.

Definition 15. Let X a smooth surface. A *cycle of codimension r* on X is an element of the free abelian group generated by the closed irreducible subvarieties of X of codimension r.

For example, any curve C in X is a cycle of codimension 1, and any point p in X is a cycle of codimension 2.

Definition 16. Given a morphism of smooth surfaces $f : X \to X'$, we associate the *push forward* functor f_* defined in the following way. Given Y a subvariety of X, if $\dim f(Y) < \dim Y$, then set $f_*(Y) = 0$. If instead $\dim f(Y) = \dim Y$, then the function field $K(Y)$ is a finite field extension of $K(f(Y))$, so we can set $f_*(Y) = [K(Y) : K(f(Y))]\overline{f(Y)}$. Extending these definitions by linearity, f_* becomes a homomorphism from the group of cycles of a given codimension on X to the group of cycles of the same codimension on X'.

Remark 2. Let V be a subvariety of X, $f : \widetilde{V} \to V$ the normalization of V. Then $\widetilde{V}$ satisfies condition $*$. This means we can define Weil divisors and the notion of linear equivalence on $\widetilde{V}$.

This remark allows us to make the following definition:

Definition 17. Take X, V and $\widetilde{V}$ as in the previous remark. Let D and D' be linearly equivalent Weil divisors on $\widetilde{V}$. Then $f_*(D)$ and $f_*(D')$ are called *rationally equivalent* cycles on X. Then we define *rational equivalence* of cycles on X by dividing out by the group generated by all such $f_*D \sim f_*D'$ for all subvarieties V, and all linearly equivalent Weil divisors D, D' on $\widetilde{V}$.

Remark 3. Rational equivalence is certainly an equivalence relation.

Remark 4. If X is normal, then the rational equivalence of cycles of codimension 1 is the same as the linear equivalence.

Definition 18. Given $r \in \mathbb{N}$, let $A^r(X)$ be the group of cycles of codimension r on X modulo rational equivalence.

Remark 5. Note that $A^0(X) = \mathbb{Z}$ and $A^r(X) = 0$ for all $r > 2$.

Theorem 1.1 of appendix A of [13] assures the existence of a unique intersection theory that satisfies certain given properties. These properties imply that, given a smooth surface X, there exists a pairing $A^r(X) \times A^s(X) \to A^{r+s}(X)$, for all r and s, that turns $A(X) = \bigoplus_{r=0}^{2} A^r(X)$ into a commutative graded ring with identity.

Definition 19. The ring $A(X)$ is called the *Chow ring* of X. Given two cycles, D and D', we will denote the intersection cycle class by $D.D'$.

Definition 20. Given X, X' smooth surfaces and a morphism $f : X \to X'$, we define the *pull back* $f^* : A(X') \to A(X)$ as follows. For a subvariety Y' of X' set $f^*(Y') = p_{1*}\left(\Gamma_f.p_2^{-1}(Y')\right)$, where p_1 and p_2 are the projections of $X \times X'$ to X and X' respectively, and Γ_f is the graph of f considered as a cycle on $X \times X'$.

Proposition 4.1 (See A2 of [13]). *For any morphsim $f : X \to X'$ of smooth surfaces, $f^* : A(X') \to A(X)$ is a ring homomorphism.*

Proposition 4.2 (See A3 of [13]). *For any proper morphism $f : X \to X'$ of smooth surfaces, $f_* : A(X) \to A(X')$ is a homomorphism of graded groups (which shifts degrees).*

This intersection theory can be done in a much more general way (like it is done in appendix A of [13]), but we are not interested in describing it so abstractly. The following Theorem says, in the more natural language of divisors, what the intersection theory on a surface looks like.

Theorem 4.1 (See Theorem 1.1 of section V.1 of [13]). *There is a unique pairing* $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ *such that for any three divisors C, D, E we have:*

(a) If C and D are smooth curves meeting transversally, then $C.D$ equals the number of points in $C \cap D$ (counting multiplicities).

(b) $C.D = D.C$

(c) $(C + D).E = C.E + D.E$

(d) If $C \sim E$, then $C.D = E.D$.

In the case of plane curves, a more direct approach can be taken to define the intersection theory (see section 3.3 of [9]).

Remark 6. For any divisor D on a smooth surface X we can define the *self-intersection* $D.D$, which is usually denoted D^2. This can be done by finding a divisor E such that $E \sim D$ and E meets D transversally. Then $D^2 = E.D$. (See Example 1.4.1 of section V.1 of [13]).

Definition 21. For a smooth surface X, define $\omega_X = \bigwedge^2 \Omega^1_X$ to be the *canonical sheaf* of X. This is an invertible sheaf. Any divisor K linearly equivalent to ω_X is called a *canonical divisor* and it is denoted by K_X.

Example 2 (The canonical divisor of the projective plane)**.** Let $[x_0 : x_1 : x_2]$ be affine coordinates in $\mathbb{P}^2$. To obtain the canonical divisor, we look at the divisor of any rational 2-form, for example the one given by $s = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ in the affine chart $\mathbb{A}^2_0 = (x_0 \neq 0)$. The canonical divisor will be given by the divisor class of s, so we need to calculate the zeros and poles of s, which will be located in the line at "infinity" of this chart $(x_0 = 0)$. Note that the coordinate change from $\mathbb{A}^2_0$ to $\mathbb{A}^2_1 = (x_1 \neq 0)$ can be described by $y_1 = 1/x_0$, $y_2 = x_2/x_0$. The Jacobian of this coordinate change is x_0^{-3}, which means that s has a pole of order 3 at the line at "infinity". If $[L]$ is the divisor class of a line (and from Example 1 we know all lines are linearly equivalent), then a canonical divisor of $\mathbb{P}^2$ is $-3[L]$.

A very useful technique in algebraic geometry is that of blowing up smooth points in a surface. Blow ups have interesting effects on divisors, which we look at now.

Definition 22. Let p a point on a smooth surface X. Let $\pi : Y \to X$ be the blow up at p. The inverse image of p is a curve E_p called an *exceptional divisor*. We will call a divisor E on a surface Y *exceptional* if exactly the preceding situation occurs.

Proposition 4.3 (See Proposition 3.1 and Theorem 5.7 of chapter V of [13])**.** *A divisor E on a surface Y is exceptional if and only if $E^2 = -1$.*

For this reason, exceptional divisors are also called (-1)-*curves*.

Proposition 4.4 (See Propositions 3.2 and 3.3 of sectoin V.3 of [13])**.** *Let $\pi : Y \to X$ be a blow-up of a surface X at a point p. Let E_p be the exceptional divisor. Then:*

(a) If C, D are divisors on X, then $(\pi^*C).(\pi^*D) = C.D$.

(b) If C is a divisor on X, then $(\pi^*C).E_p = 0$.

(c) The canonical divisor of Y is $K_Y = \pi^*K_X + E_p$. Therefore $K_Y^2 = K_X^2 - 1$.

5 Chern Classes

This section is taken from [13], appendix A, unless otherwise stated.

Proposition 5.1 (See A11 of [13]. For a proof, see [5])**.** *Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth surface X. Let $\mathbb{P}(\mathscr{E})$ be the associated projective space bundle (for the definition see section II.7 of [13]). Let $\zeta \in A^1(\mathbb{P}(\mathscr{E}))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the projection. Then π^* makes $A(\mathbb{P}(\mathscr{E}))$ into a free $A(X)$-module of rank 1 if $r = 1$, and of rank 2 if $r = 2$ (in which case it is generated by $\{1, \zeta\}$).*

Definition 23. Here we follow the notation of Proposition 5.1. Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth surface X. For each $i = 0, \dots, r$ define the i-th *Chern class* $c_i(\mathscr{E}) \in A^i(X)$ as elements satisfying the requirements: $c_0(\mathscr{E}) = 1$ and:

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathscr{E}).\zeta^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathscr{E}))$.

Definition 24. Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth surface X. We define the *Chern polynomial* of $\mathscr{E}$ as:

$$c_t(\mathscr{E}) = c_0(\mathscr{E}) + c_1(\mathscr{E})t + \cdots + c_r(\mathscr{E})t^r.$$

Proposition 5.2 (See C1 of [13])**.** *If $\mathscr{E} = \mathcal{O}(D)$ for a divisor D (that is, if $\mathscr{E}$ is an invertible sheaf), then $c_t = 1 + Dt$.*

For the following propositions we need a result of algebraic topology called the *splitting principle*. This principle, in terms of the objects we are dealing with, states that, given a locally free sheaf $\mathscr{E}$ of rank r on a smooth surface X, there exists a surface X' and a morphism $f : X' \to X$ such that $f^* : A(X) \to A(X')$ is injective, and $\mathscr{E}' = f^*\mathscr{E}$ *splits*, that is, it has a filtration:

$$0 = \mathscr{E}'_r =\subseteq \cdots \subseteq \mathscr{E}'_1 \subseteq \mathscr{E}'_0 = \mathscr{E}',$$

such that the succesive quotients are invertible sheaves. For a general statement of the principle and its proof see Proposition 3.3 of Chapter 3 of [14]. This principle allows us to express the Chern polynomials in a more explicit way, as the next proposition shows.

Proposition 5.3 (See C5 of [13]). *Consider a locally free sheaf $\mathscr{E}$ of rank r on a smooth surface X. If $\mathscr{E}$ splits, and the filtration has invertible sheaves $\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_r)$ as quotients, then:*

$$c_t(\mathscr{E}) = \prod_{i=1}^{r} c_t(\mathcal{O}_X(D_i)).$$

Note that thanks to Proposition 5.2, we know what $c_t(\mathcal{O}_X(D_i))$ is.

Proposition 5.4 (See C5 of [13]). *Let $\mathscr{E}$ and $\mathscr{F}$ be locally free sheaves of rank r and s respectively on a smooth surface X. Using the splitting principle we can write:*

$$c_t(\mathscr{E}) = \prod_{i=1}^{r}(1 + a_i t), \qquad c_t(\mathscr{F}) = \prod_{i=1}^{s}(1 + b_i t).$$

Then:

(a) $c_t(\mathscr{E} \otimes \mathscr{F}) = \prod_{i,j}(1 + (a_i + b_j)t)$

(b) $c_t(\bigwedge^p \mathscr{E}) = \prod_{1 \leq i_1 < \ldots < i_p \leq r}(1 + (a_{i_1} + \cdots + a_{i_p})t)$

(c) $c_t(\mathscr{E}^\vee) = c_{-t}(\mathscr{E})$.

Definition 25. Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth projective variety X. Let $c_t = \prod_{i=1}^{r}(1 + a_i t)$. We define the *exponential Chern character* as:

$$\mathrm{ch}(\mathscr{E}) = \sum_{i=1}^{r} e^{a_i},$$

where:

$$e^x = \sum_{k\geq 0} \frac{x^k}{k!}.$$

We also define the *Todd class* of $\mathscr{E}$ as:

$$\mathrm{td}(\mathscr{E}) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}},$$

where:

$$\frac{x}{1-e^{-x}} = x\sum_{k\geq 0} e^{-kx}.$$

So far we have made a series of abstract definitions. Our interest is to be able to effectively compute the Chern classes of certain locally free shaves on surfaces, which is not something immediate from the definitions. The following Lemma will be used to make all explicit calculations.

Lemma 5.1 (See section 4 of appendix A of [13])**.** *Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth surface X. Let $c_i = c_i(\mathscr{E})$, $c_i = 0$ if $i > r$. Then:*

$$\begin{aligned} \mathrm{ch}(\mathscr{E}) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 2c_2^2) + \cdots \\ \mathrm{td}(\mathscr{E}) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2) + \cdots \end{aligned}$$

Proof. Long tedious calculations. □

Theorem 5.1 (Hirzebruch-Riemann-Roch, see [3] for proof)**.** *For a locally free sheaf $\mathscr{E}$ of rank r on a smooth surface X*

$$\chi(\mathscr{E}) = \deg(\mathrm{ch}(\mathscr{E}).\mathrm{td}(\mathscr{T}_X))_2,$$

where $(\,)_2$ denotes the component of degree 2 in $A(X)\otimes\mathbb{Q}$.

Definition 26. Let X be a smooth surface over an algebraically closed field k. Let c_1 and c_2 be the Chern classes of the tangent sheaf $\mathscr{T}_X$. As these numbers depend only on X, we can call them the *Chern classes of* X, and we will write $c_1(X) = c_1$, $c_2(X) = c_2$.

Example 3 (Chern classes of a surface)**.** Let X be a smooth surface over an algebraically closed field k. Let K_X be a canonical divisor on X. Let $\mathscr{E} = \mathcal{O}(D)$ be an invertible sheaf. Then $\mathrm{ch}(\mathscr{E}) = 1 + D + \frac{1}{2}D^2$.

Now notice the following: $\mathscr{T}_X$ is the dual of the sheaf of differentials Ω^1_X, $c_1(\Omega^1_X) = c_1(\bigwedge^2 \Omega^1_X)$ by Proposition 5.4, and $\bigwedge^2 \Omega^1_X$ is just $\mathcal{O}(K_X)$. Therefore we get that $c_1(\mathscr{T}_X) = -K_X$. Using this we can write:

$$\mathrm{td}(\mathscr{T}_X) = 1 - \frac{1}{2}K_X + \frac{1}{12}\left(K_X^2 + c_2(X)\right).$$

We multiply this by $\mathrm{ch}(\mathscr{E})$ and then take the component of degree 2. In this way, we can write Theorem 5.1 as (here we abuse notation and use D^2 for both the class in $A^2(X)$ and its degree):

$$\chi(\mathcal{O}(D)) = \frac{1}{2}D.\left(D - K_X\right) + \frac{1}{12}\left(K_X^2 + c_2(X)\right).$$

In particular, for $D = 0$, we have Noether's formula:

$$\chi(\mathcal{O}_X) = \frac{1}{12}\left(K_X^2 + c_2(X)\right). \tag{1}$$

Proposition 5.5. *Let X be a surface over $\mathbb{C}$. Then $c_2(X) = e(X)$.*

Proof. Ω^1_X is a locally free sheaf of rank 2. Let $c_i = c_i\left(\Omega^1_X\right)$. Then, by Lemma 5.1 we have that:

$$\mathrm{ch}\left(\Omega^1_X\right) = 2 + c_1 + \frac{1}{2}(c_1^2 - 2c_2)$$

By Proposition 5.4 we have that $c_1\left(\Omega^1_X\right) = c_1\left(\bigwedge^2 \Omega^1_X\right) = K_X$. Using that $\mathscr{T}_X$ is the dual of Ω^1_X, part (c) of Proposition 5.4 gives us $c_2\left(\Omega^1_X\right) = c_2(X)$. As $\mathscr{T}_X$ is the dual of Ω^1_X, then $c_1\left(\mathscr{T}_X\right) = -K_X$. Also, in the previous example we saw that:

$$\mathrm{td}(\mathscr{T}_X) = 1 - \frac{1}{2}K_X + \frac{1}{12}\left(K_X^2 + c_2(X)\right).$$

Using Noether's formula we get:

$$\begin{aligned}\left(\mathrm{ch}\left(\Omega^1_X\right).\mathrm{td}\left(\mathscr{T}_X\right)\right)_2 &= \frac{1}{2}(c_1^2(X) - 2c_2(X)) - \frac{1}{2}c_1(X)K_X + \frac{1}{6}\left(K_X^2 + c_2(X)\right) \\ &= 2\chi\left(\mathcal{O}_X\right) - c_2(X).\end{aligned}$$

So, using Theorem 5.1 we get that $\chi(\Omega^1_X) = 2\chi(\mathcal{O}_X) - c_2(X)$. On the other hand $\chi(\Omega^1_X) = 2q(X) - h^1(\Omega^1_X)$ and $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$. Hodge theory (see [37]) says that the Betti numbers can be written as: $b_1 = 2q(X)$ and $b_2 = h^1(\Omega_X) + 2p_g(X)$. With all this, we get $c_2(X) = 2 - 2b_1 + b_2 = e(X)$. □

Proposition 5.6. *Let X be a minimal surface of general type over $\mathbb{C}$. Then $c_1^2(X)$ and $c_2(X)$ are positive.*

Proof. For $c_1(X) > 0$ see Proposition X.1 of [2]. For $c_2(X) > 0$ see Theorem X.4 of [2] (the minimality condition is not needed in this part). □

Proposition 5.7 (Noether's inequality)**.** *Let X be a smooth minimal surface of general type over an algebraically closed field. Then:*

$$p_g \leq \frac{1}{2}c_1^2 + 2.$$

Definition 27. Let X be a smooth surface. A divisor D on X is said to be *numerically effective* (*nef* for short) if for every curve C on X, $c_1(\mathcal{O}_X(D)).C \geq 0$

5.1 The Chern Slope

Definition 28. Let X be a smooth projective surface such that $c_2(\mathscr{T}_X) \neq 0$. Then we define the *Chern slope* of X to be:

$$S(X) = \frac{c_1^2(\mathscr{T}_X)}{c_2(\mathscr{T}_X)}.$$

Remark 7. For a smooth minimal surface of general type, we can combine Noether's formula (1) with Noether's inequality (Proposition 5.7) to obtain:

$$0 \leq 12q \leq 5c_1^2 - c_2 + 36,$$

from which we get:

$$\frac{1}{5} - \frac{36}{5c_2} \leq S(X). \tag{2}$$

Therefore, if we have a family of minimal smooth surfaces such that the sequence of c_2's tends to infinity, then the limit (if it exists) of the sequence of Chern slopes is bounded below by $\frac{1}{5}$.

Theorem 5.2 (Bogomolov-Miyaoka-Yau inequality, see [15], [20], [38])**.** *Let X be an projective smooth complex surface that is not birationally equivalent to a ruled surface with base curve of genus $g \geq 2$. Then: $c_1^2(X) \leq 3c_2(X)$.*

Therefore, with X as in the theorem, we have that $S(X) \leq 3$. This statement of the theorem is a bit more general than the way it is usually stated (in which X is asked to be of general type and minimal), but Hirzebruch notes in [15] that it can be stated in this fashion. It is important to have this version of the theorem so as to use the results of that paper.

Remark 8. If X is a minimal complex surface of general type, then Miyaoka and Yau also proved that the equality $c_1^2(X) = 3c_2(X)$ holds if and only if the universal cover of X is the ball $B = \{z \in \mathbb{C}^2 \mid |z| < 1\}$.

5.2 Log Surfaces

This section has been taken from section 2 of [8].

Definition 29. Let X be a smooth surface and D a simple normal crossings divisor on X. Let $U = X \setminus D$, and write $\tau : U \to X$. Define $\Omega_X^1(*D)$ as $\tau_*(\Omega_U^1)$. We define the *sheaf of differential one-forms with logarithmic poles* along D, denoted by $\Omega_X^1(\log D)$, as: If $V \subseteq X$ is open, then $\Gamma(V, \Omega_X^1(\log D))$ is the set of elements $\alpha \in \Gamma(V, \Omega_X^1(*D))$ such that α and $\mathrm{d}\alpha$ have simple poles along D.

In the following sections, D will be taken to be a finite set of curves with simple normal corssings.

Proposition 5.8 (See 2.2 Properties in section 2 of [8])**.** *Let X be a smooth surface, D a simple normal crossings divisor on X. Then $\Omega_X^1(\log D)$ is the $\mathcal{O}_X$-submodule of $\Omega_X^1 \otimes \mathcal{O}_X(D)$ satisfying:*

(a) $\Omega_X^1(\log D)|_{X\setminus D} = \Omega_{X\setminus D}^1$

(b) If $p \in D$ is a closed point, then:

$$\omega_p \in \Omega^1_X(\log D)_p \iff \omega_p = \sum_{i=1}^{s} a_i \frac{dz_i}{z_i} + \sum_{j=s+1}^{2} b_j dz_j,$$

where (z_1, z_2) is a local coordinate system at p for X such that $\{z_1 z_2 = 0\}$ defines D at p.

Remark 9. The proposition implies that $\Omega^1_X(\log D)$ is a locally free sheaf of rank 2 on X. Moreover, since we have the following identities:

$$\begin{aligned} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} &= \frac{1}{z_1 z_2} dz_1 \wedge dz_2 \\ \frac{dz_1}{z_1} \wedge dz_2 &= \frac{1}{z_1} d_{z_1} \wedge dz_2, \end{aligned}$$

then $\bigwedge^2 \Omega^1_X(\log D) \cong \mathcal{O}_X(K_X + D)$.

Definition 30. Let X be a smooth surface, D a simple normal crossings divisor on X. We call the pair (X, D) a *log surface.* The *logarithmic Chern classes* of (X, D) are defined as: $\overline{c}_i = c_i(\Omega^1_X(\log D)^\vee)$ for $i = 1, 2$.

The results of the thesis will revolve around log Chern numbers rather than Chern numbers. The reasons and motivation for this will be explained in section 6.1.

Proposition 5.9. *Let X be a smooth complex surface and D a simple normal crossings divisor on X. Then:*

(a) $\overline{c_1}^2(X, D) = (K_X + D)^2$

(b) $\overline{c_2}(X, D) = e(X) - e(D)$.

Proof. We know from remark 9 that $\bigwedge^2 \Omega^1_X \cong \mathcal{O}_X(K_X + D)$. The result is now obtained by repeating the procedure of 3 that we used to calculate the Chern numbers of a surface. □

6 Curve Arrangements on Surfaces

We have arrived to the main object of study of this thesis. Besides all the algebraic technology that we have described so far, we will see in this section that we are also interested in the combinatorial view of curves on surfaces. Certainly both views will be combined in Parts II and III.

Definition 31. A set of curves on a surface Z is said to have *simple crossings* if every pair of curves of the set intersects transversally.

Definition 32. An *arrangement of curves* on a surface Z is a set of curves $\Lambda = \{C_1, \ldots, C_d\}$, $d \geq 4$, with simple crossings on Z such that $\bigcap_{i=1}^{d} C_i = \emptyset$. We also define some notation:

(a) For a point $p \in Z$ let r_p be the number of elements of Λ that pass through p. We will call r_p the *multiplicity* of p in Λ. Note that r_p depends on Λ, and when necessary we will write $r_p(\Lambda)$ to emphasize the curve arrangement being considered. A point $p \in Z$ such that $r_p \geq 2$ is called a *multiple point.*

(b) For a positive integer $k \geq 2$, let t_k be the number of points $p \in Z$ such that $r_p = k$. Then, for example, the condition $\bigcap_{i=1}^{d} C_i = \emptyset$ means $t_d = 0$.

(c) $f_0 = \sum_{k \geq 2} t_k$, that is, f_0 is the number of multiple points.

(d) $f_1 = \sum_{k \geq 2} k t_k$.

Definition 33. Given an arrangement of curves Λ on a surface Z, the tuple $(t_2, t_3, \cdots, t_{d-1})$ is called the *multiplicity information* of Λ.

On certain cases we will have a combinatorial situation which could potentially describe a curve arrangement. For instance we may have a tuple $(a_1, \cdots, a_{d-2})$ of non-negative integers, or we may have more information regarding the way the curves should intersect. In all these cases we say that the combinatorial situation is *realizable* on Z if there exists a curve arrangement Λ whose combinatorial situation equals the given one.

Definition 34. We will say that two curve arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ on Z are *combinatorially isomorphic* if there exists a bijection $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that for every $C \in \mathcal{A}_1$ there exists a bijection

$$g_C : \{p \in C \mid r_p \geq 2\} \to \{q \in f(C) \mid r_q \geq 2\},$$

such that $r_p = r_{g_C(p)}$, for all $p \in C$ satisfying $r_p \geq 2$.

6.1 Log Chern Slope

As the following proposition shows, calculating log Chern numbers is easy if one knows certain combinatorial information about the divisor used to define the log surface.

Proposition 6.1 (See Proposition 2.4 of [35]). *Let Z be a smooth projective surface, $\Lambda = \{C_1, \ldots, C_d\}$, $d \geq 3$, be a curve arrangement with simple crossings on Z. Let $\sigma : Y \to Z$ be the blow up at all points p of Λ with $r_p \geq 3$, and let Λ' be the reduced total transform of Λ. Then the logarithmic Chern numbers for (Y, Λ') are:*

$$\begin{aligned}\overline{c_1}^2(Y, \Lambda') &= c_1^2(Z) - \sum_{i=1}^{d} C_i^2 + \sum_{k \geq 2}(3k-4)t_k + 4\sum_{i=1}^{d}\left(g\left(C_i\right) - 1\right)\\ \overline{c_2}(Y, \Lambda') &= c_2(Z) + \sum_{k \geq 2}(k-1)t_k + 2\sum_{i=1}^{d}\left(g\left(C_i\right) - 1\right)\end{aligned}$$

Definition 35. Let Λ be a curve arrangement on a smooth projective surface X. Λ need not have simple normal crossings, so that (X, Λ) is not necessarily a log surface. But using Proposition 6.1, we will define the *log Chern slope* of (X, Λ) as:

$$\overline{S}(X, D) = \frac{\overline{c_1}^2(Y, D')}{\overline{c_2}(Y, D')}.$$

In this thesis we will be working with the log Chern slope instead of the Chern slope. This is motivated by the following theorems:

Theorem 6.1 (See Theorem V.2 in [33]). *Let Λ be a divisor on $\mathbb{P}^2$ with given by an arrangement of lines (so Λ has simple crossings). Let (Y, Λ') be the corresponding associated pair (as defined in Proposition 6.1) and assume $e(Y) \neq e(\Lambda')$ (so that $\overline{c_2}(Y, \Lambda') \neq 0$). Then, there are smooth projective surfaces X having $S(X)$ arbitrarily close to $\overline{S}(Y, \Lambda')$.*

The definitions for the following theorem can be found in Part III.

Theorem 6.2 (Corollary 8.3 of [36]). *Let Λ be a section arrangement on a Hirzebruch surface. Then, there exist nonsingular projective surfaces X of general type such that $2 < S(X) < 3$ and having $S(X)$ arbitrarily close to $\overline{S}(\Sigma_e, \Lambda)$.*

6.2 Regular Curve Arrangements

The curve arrangements that will be examined in Parts II and III of this thesis all satisfy a combinatorial regularity condition which we define next. The terminology chosen is not standard, it is only used to help the exposition of the thesis.

Definition 36. Let Λ be a curve arrangement on a surface Z for which there exists a positive integer n such that for any pair of curves C_1, C_2 (not necessarily distinct) of the arrangement, we have that $C_1.C_2 = n$. Then Λ is said to be *regular of index n*.

For example, a finite set of lines in a projective plane is a regular curve arrangement of index 1.

Lemma 6.1 (See Remark 7.4 of [36]). *Let Λ be a regular curve arrangement of index n on a surface Z. Then $f_0 \geq d$.*

Proof. Let $r = f_0$. Enumerate the multiple points of Z as $p_1, p_2, \ldots, p_r$. Enumerate the curves of Λ as $C_1, C_2, \ldots, C_d$. Consider the vector space $\mathbb{Q}^r$ with the usual inner product (which we will denote as $a \cdot b$, for $a, b \in \mathbb{Q}^r$). Given $C_\ell \in \Lambda$, associate to it the vector v_ℓ which has as its i-th entry 1, if C_ℓ passes through p_i, and 0 otherwise.

Notice that if $i \neq j$, then $v_i \cdot v_j$ is precisely the number of points common to C_i and C_j. As our assumption is that $C_i.C_j = n$, then $v_i \cdot v_j = n$. On the other hand, $v_i \cdot v_i$ is the number of multiple points that are contained in C_i. Recall that our curve arrangements have to satisfy $t_d = 0$. Therefore $v_i \cdot v_i > n$.

To prove the lemma, we can prove that the set $\{v_1, \ldots, v_d\}$ is linearly independent. We will proceed by contradiction. Suppose the set is not linearly independent, so that we can write:

$$v_1 = \sum_{j=2}^{d} a_j v_j.$$

Now consider $v_1 \cdot (v_1 - v_h)$, where $h \geq 2$. We are interested in finding a_h.

$$\begin{aligned} v_1 \cdot v_1 - n &= v_1 \cdot (v_1 - v_h) \\ &= \sum_{i=2}^{d} a_i v_i \cdot (v_1 - v_h) \\ &= \sum_{i=2}^{d} a_i (n - v_i \cdot v_h) \\ &= a_h (n - v_h \cdot v_h). \end{aligned}$$

Therefore $a_h = \frac{v_1 \cdot v_1 - n}{n - v_h \cdot v_h}$, which is a negative number. This means that v_1 is written a linear combination in which all coefficients are negative numbers. However, the entries of v_i, for any $i = 1, \ldots, d$, are just 0 or 1, a contradiction. □

Theorem 6.3 (Ryser-Woodall Theorem, see Theorem 1.3.5 of [17])**.** *Let Λ be a regular curve arrangement of index n with d curves on a surface Z. Suppose that the number of multiple points equals d. Then either all multiple points have the same multiplicity, or there exist two integers, r and r', greater than 1, such that $r + r' = d + 1$ and for every multiple point p, either $r_p = r$ or $r_p = r'$.*

Lemma 6.2. *Let Λ be a regular curve arrangement of index n on a surface Z. Then:*

(a) For every $C \in \Lambda$ we have that:

$$\sum_{p \in C} (r_p - 1) = n(d - 1).$$

(b)

$$\sum_{k \geq 2} k(k-1) t_k = nd(d-1).$$

Proof. (a) Given a multiple point $p \in C$, $r_p - 1$ is the number of curves of the arrangement passing through p different from C. As every curve meets every other curve in n distinct points, the expression $\sum_{p\in C}(r_p-1)$ counts all curves of the arrangement different from C, n times each.

(b) Use the result from (a) and take the sum over all curves:

$$\sum_{C\in\Lambda}\sum_{p\in C}(r_p-1)=\sum_{k\geq 2}k(k-1)t_k.$$

□

Example 4. When the curves of a regular curve arrangement of index n on Z are in general position, then the only mutliple points that appear are double points. So $f_0=t_2$. Using Lemma 6.2 we can calculate $t_2=nd(d-1)/2$. Note that, for a fixed d, when the arrangements are regular, then the arrangements in general position are all isomorphic.

7 Abelian Extensions

This section follows [10]. For a more comprehensive study of abelian and cyclic extensions which shows that this procedure can be adapted to work over algebraically closed fields of any characteristic, see section 3 of [8]. Another canonical reference, although it approaches subject in a slightly different mannner than we will do, is [27]. The purpose of this section is to describe a technical procedure which is central to the most important results that arise later. With this method one can use curve arrangements on certain surfaces to obtain surfaces of general type. Let X be a smooth variety throughout.

Definition 37. A finite cover of smooth varieties $\pi:Y\to X$ is said to be an *abelian cover* if the function field of Y is an abelian extension of the function field of X. If the Galois group of this field extension is G, then we say that G is the group associated to the cover $\pi:Y\to X$.

7.1 The Defining Equations of an Abelian Cover

See section 2 of [10]. Suppose we are given the following data:

1. A smooth variety X
2. $D_1, \dots, D_k$ prime divisors on X
3. An abelian group of the form $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$.
4. Divisors $L_1, \dots, L_k$ on X such that they satisfy the following linear equivalence relations:

$$D_i \sim n_i L_i, \quad i = 1, \dots, k. \tag{3}$$

With this information, we want to construct a variety Y and a morphism $\pi : Y \to X$ such that π is an abelian cover with associated group G. We will use this process in Part III of this thesis.

Let $\mathscr{L}_i = \mathcal{O}_X(L_i)$. Now we choose nonzero global sections $f_i \in \Gamma(X, \mathscr{L}_i^{\otimes n_i})$ such that: $D_i = \operatorname{div}(f_i)$. Let $V(\mathscr{L}_i) = \operatorname{Spec} S(\mathscr{L}_i)$ (for the definition of this, see exercise 5.8 of section II.5 of [13]) be the associated line bundle of $\mathscr{L}_i$, where $S(\mathscr{L}_i)$ is the symmetric $\mathcal{O}_X$-algebra of $\mathscr{L}_i$. Let $p_i : V(\mathscr{L}_i) \to X$ be the bundle projection, and let z_i be a global section of $p_i^*(\mathscr{L}_i)$ (for instance, one can take z_i to be the fiber coordinate of $V(\mathscr{L}_i)$). Thus one obtains a polynomial section $p_i(z_i) = z_i^{n_i} - f_i$ of $p^*\mathscr{L}_i^{n_i}$. Let $\Sigma \subseteq V = V(\mathscr{L}_1) \oplus \cdots \oplus V(\mathscr{L}_k)$ be defined by the equations:

$$z_i^{n_i} = f_i, \quad i = 1, \dots, k. \tag{4}$$

Set Y as the normalization of Σ, and set $\pi : Y \to X$ as the composition of $Y \to \Sigma$ and the restriction to Σ of the bundle projection. The following theorem says that every abelian cover can be obtained in this manner. Readers familiar with [27] will note that we have chosen a different approach to our construction of Y.

Theorem 7.1 (See Theorem 2.1 of [10])**.** *For any finite abelian cover $\pi : Y \to X$ with group $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$, there exist effective divisors $D_1, \dots, D_k$ (maybe zero) on X, and some other divisors $L_1, \dots, L_k$ satisfying the equations (3), such that $\pi : Y \to X$ is defined by the equations (4).*

7.2 The Invariants of Abelian Covers

See section 3 of [10]. From now on we assume that the abelian cover $\pi : \Sigma \to X$ is defined by (4), that G is the group of the cover and that Y is the normalization of Σ. Notice that in order to compute the normalization, we can do it locally. Take $X = \mathrm{Spec}(A)$ and $\Sigma = \mathrm{Spec}(B)$. Then:

$$B = A[z_1, \ldots, z_k]/(z_1^{n_1} - f_1, \ldots, z_k^{n_k} - f_k).$$

We can write $B = A[\alpha_1, \ldots, \alpha_k]$, where $\alpha_i^{n_i} - f_i = 0$. As X is nonsingular, the local rings are always UFD, so we can assume that A is a UFD. If X is defined over an algebraically closed field $\mathbb{K}$, then we can assume that A contains $\mathbb{K}$. Let F and K denote the fraction fields of A and B respectively. Note that $K = F(\alpha_1, \ldots, \alpha_k)$.

Write $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$. Choose representatives for each $\mathbb{Z}/n_i\mathbb{Z}$ so that if $g = (g_1, \ldots, g_k) \in G$, then $0 \le g_i \le n_i$. We use the multi-index notation:

$$\alpha^g = \alpha_1^{g_1} \cdots \alpha_k^{g_k}.$$

For $f = \prod p_i^{a_i}$, we will also use the notation:

$$[f_i] := \prod p_i^{[a_i]},$$

where $[a_i]$ denotes the biggest integer less than or equal to a_i .

Proposition 7.1 (See Proposition 3.1 of [10]). *An element $h = \sum_{g \in G} k_g \alpha^g \in K$, with the $k_g \in F$ for $g \in G$, is integral over A if and only if $k_g \alpha^g$ is integral over A for every $g \in G$.*

Theorem 7.2 (See Theorem 3.1 of [10]). *The integral closure $\widetilde{B}$ of B in K is freely generated over A by the set:*

$$\left\{ \alpha_g = \frac{\prod_{i=1}^k \alpha_i^{g_i}}{\left[\prod_{i=1}^k f_i^{g_i/n_i}\right]} \mid g = (g_1, \ldots, g_k) \in G \right\}.$$

Corollary 7.1 (See Corollary 3.2 of [10]). *Assume X is a smooth surface. Denote:*

$$L_g = -\sum_{i=1}^k g_i L_i + \left[\sum_{i=1}^k \frac{g_i}{n_i} D_i\right].$$

Then:

$$\chi(\mathcal{O}_Y) = |G|\chi(\mathcal{O}_X) + \sum_{g\in G} \frac{1}{2}(L_g^2 - L_g K_X).$$

Theorem 7.3 (See Theorem 3.2 of [10] and Lemma 3.19 in section 3 of [8])**.** *Let P be an irreducible and reduced hypersurface in X, let $\overline{P} = \pi^{-1}(P)$ be the reduced preimage of P in Y, and let a_i be the multiplicity of P in $D_i = \operatorname{div}(f_i)$. Then:*

$$\pi^* P = \frac{|G|}{d_P}\overline{P},$$

where:

$$d_P = \gcd\left(|G|, |G|\frac{a_1}{n_1}, \ldots, |G|\frac{a_k}{n_k}\right)$$

is the number of points in the preimage of a generic point on P.

Proof. Let p be a generic point of P. At this point, P is smooth and is locally defined by $x = 0$. Then the cover is defined locally as $z_i^{n_i} = x^{a_i}$, $i = 1, \ldots, k$.

The proof now proceeds by induction on k. The case $k = 1$ is obtained from Lemma 3.19 in section 3 of [8]. So assume that the statement holds for $k-1$. Factorize the cover $\pi : Y \to X$ as $\pi : Y \xrightarrow{\pi'} Z \xrightarrow{\phi} X$, where ϕ is defined by $z_1^{n_1} = x^{a_1}$, and π' is defined by $z_i^{n_i} = \phi^*(x)^{a_i}$, $i = 2, \ldots, k$.

As we chose p generic, $\phi^{-1}(p)$ consists of d_1 smooth points of $P' = \phi^{-1}(P)$. So, the cardinality of $\pi^{-1}(p)$ is $d_1 d_{P'}$, where $d_{P'}$ is given by the induction hypothesis:

$$d_{P'} = \gcd\left(\frac{|G|}{n_1}, \frac{|G|a_2}{d_1 n_2}, \ldots, \frac{|G|a_k}{d_1 n_k}\right),$$

and so $d_P = d_1 d_{P'}$. ☐

Example 5 (Line arrangements, see section 2 of [15])**.** In [15], abelian covers are used to create surfaces of general type from line arrangements on the projective plane. Let X be the complex projective plane. If $\Lambda = \{L_1, \ldots, L_d\}$ is a line arrangement, then we consider the lines in Λ to be the divisors on X. If $[z_0 : z_1 : z_2]$ are homogeneous coordinates on X, then the function field of X is: $K(X) = \mathbb{C}\,(z_1/z_0, z_2/z_0)$.

Consider now the field extension of $K(X)$:

$$K' = \mathbb{C}\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)\left(\left(\frac{L_2}{L_1}\right)^{\frac{1}{n}}, \ldots, \left(\frac{L_d}{L_1}\right)^{\frac{1}{n}}\right).$$

The choice of putting L_1 in the denominator is arbitrary as clearly for every i, j we have that $(L_i/L_j)^{1/n} \in K'$. K' is a Galois field extension of $K(X)$ of degree n^{d-1} with Galois group $(\mathbb{Z}/n\mathbb{Z})^{d-1}$.

We have shown how to construct a surface Σ and a morphism $\pi : \Sigma \to X$ such that the branch locus of π if Λ and $K(\Sigma) = K'$. Note that we have chosen all n_i to be equal to n. Let $\rho : Y \to \Sigma$ be the normalization of Σ and let $\tau : Z \to X$ be the blow up of all points p in X with $r_p \geq 3$ (this choice will be explained at the end of the example). We then have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Z \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ \Sigma & \xrightarrow[\pi]{} & X \end{array}$$

where $\sigma : Y \to Z$ is the map induced by π. The previous theorem now says that for any point $x \in Z$, the cardinality of $\sigma^{-1}(x)$ is n^{d-1-r_x}, so the map σ is of degree n^{d-1}. Given a point $p \in X$ such that $r_p \geq 3$, let E_p be its associated exceptional divisor in Z. Also $\sigma^* E_p$ is a divisor on Y consisting of n^{d-1-r_p} disjoint curves C, each with multiplicity n. In other words $\sigma^* E_p = n \sum C$, where the sum is taken over a set of cardinality n^{d-1-r_p}. Using the Hurwitz formula one can calculate:

$$2 - 2g(C) = n^{r_p - 1}(2 - r_p) + r_p n^{r_p - 2}.$$

Notice that:

$$(\sigma^* E_p)^2 = n^{d-1}(E_p^2) = -n^{d-1}.$$

Each curve C has the same self-intersection, call it b, therefore:

$$-n^{d-1} = \left(n \sum C\right)^2 = n^{d-1-r_p} n^2 b = n^{d-r_p+1} b,$$

which means that for each curve C we have $C.C = -n^{r_p-2}$. Notice now that if we had included double points in the blow-up to form Z, the curves C would have self-intersection equal to -1, and so Y would not be minimal. Double points are the only points of X with this property.

Part II

Line Arrangements in $\mathbb{P}^2$

In this part we consider arrangements of lines in $\mathbb{P}^2$. Our aim is to study methods developed in [15], [16] and [31], and expand on them.

Proposition 7.2. *For a line arrangement Λ in $\mathbb{P}^2$, the expression for the log Chern slope is:*

$$\overline{S}(\mathbb{P}^2, \Lambda) = \frac{9 - 5d + \sum_{k\geq 2}(3k-4)t_k}{3 - 2d + \sum_{k\geq 2}(k-1)t_k}.$$

Proof. We use Proposition 6.1 to calculate the log Chern slope. First $c_1^2(\mathbb{P}^2) = 9$ and $c_2(\mathbb{P}^2) = 3$. Also, for any line L in the projective plane we have that $L^2 = 1$ and $g(L) = 0$. □

Just like Hirzebruch does in [15], we will only consider line arrangements satisfying the following restrictions:

(a) The amount of lines is at least 4.

(b) $t_d = t_{d-1} = 0$.

These restrictions include those of the definition of curve arrangements. Also, for any pair of lines L, L' we have that $L.L' = 1$, therefore line arrangements are regular curve arrangements of index 1. So $f_0 \geq d$ by Lemma 6.1. In fact we can say more:

Theorem 7.4 (De Bruijn-Erdös Theorem, see Theorem 14.1.13 in [17])**.** *If a line arrangement Λ satisfies $f_0 = d$, then one of the following situations occurs:*

(a) Λ satisfies $t_{d-1} = 1$.

(b) Λ is a finite projective plane.

This version of the theorem is stronger than the one given in [4], and we have used it for that reason. Theorem 1 of [4] assumes that for every pair of multiple points, there is a line

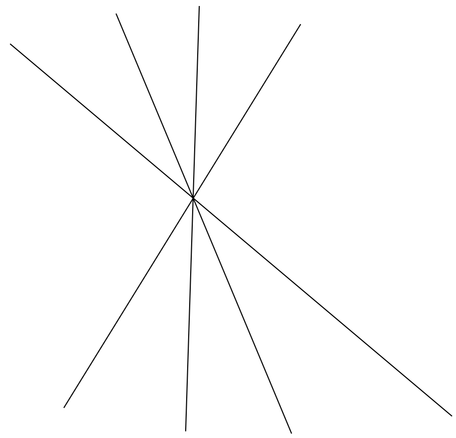

Figure 1: Pencil of lines.

in the arrangement going through them, which is not necessarily the case for us. The source from which we have taken this stronger version, is written in the language of Ryser designs.

In our definition of line arrangement we added an additional restricition not included in the definition of curve arrangement, namely $t_{d-1} = 0$. Up to isomorphism, the restrictions $t_d = t_{d-1} = 0$ only exclude two types of arrangements:

1. A pencil of lines (excluded by $t_d = 0$), see Figure 1.

2. A *near-pencil*, that is, a pencil of $d-1$ lines, with the last line intersecting the others transversally (excluded by $t_{d-1} = 0$), see Figure 2.

We have added this last extra restriction because the log Chern slope is not defined for the near-pencil. As Hirzebruch shows in section 2 of [15], a great deal of algebraic machinery (i.e. abelian covers) can be used if for line arrangements satisfying these conditions. The most important results of this part of the thesis depends on this.

8 On Single Line Arrangements

We start with the simplest case.

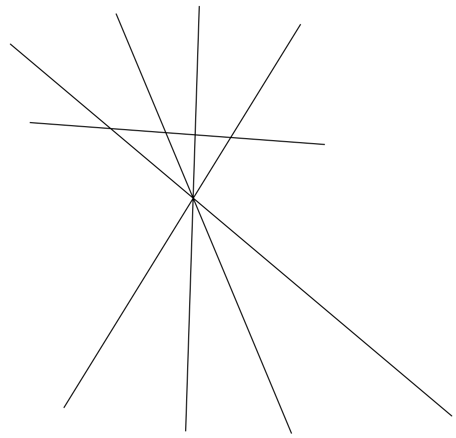

Figure 2: Near-pencil.

Example 6 (d Lines in General Position)**.** Consider d lines in $\mathbb{P}^2$ in general position, then $t_2 = d(d-1)/2$ and $t_k = 0$ for all $k \geq 3$. Then the log Chern slope is:

$$\frac{9 - 5d + d(d-1)}{3 - 2d + \frac{d(d-1)}{2}} = \frac{2d-6}{d-2}$$

which converges to 2 as d tends to infinity.

The biggest possible value the log Chern slope of a line arrangement on a projective plane can have is 3, since $\overline{S}(\mathbb{P}^2, \Lambda) \leq 3$ is equivalent to $f_0 \geq d$, which we know holds by Lemma 6.1.

Lemma 8.1. *If Λ is a line arrangement on $\mathbb{P}^2$ such that $t_2 = 0$, then $\overline{S}(\mathbb{P}^2, \Lambda) > \frac{5}{2}$.*

Proof. From the formula given for the log Chern slope, we can deduce that $\overline{S}(\mathbb{P}^2, \Lambda) > \frac{5}{2}$ is equivalent to $3 + f_1 > 3f_0$. Since $t_2 = 0$, then $f_k \geq 3f_0$, and so $3 + f_1 > f_0$. □

The following results follow section 5 of [31]. Even though the proofs imitate those of [31], we give them because there are some minor errors in the original.

Proposition 8.1 (See Theorem 5.1 of [31])**.** *Let Λ be an arrangement of d lines on $\mathbb{P}^2$. Then*

$$\overline{S}(\mathbb{P}^2, \Lambda) \geq \frac{2d-6}{d-2}$$

Proof. Suppose

$$\overline{S}(\mathbb{P}^2, \Lambda) \leq \frac{2d-6}{d-2}.$$

This is equivalent to:

$$\sum_{k=2}^{d-2}(dk - 2d + 2)t_k \leq d^2 - d,$$

which is the same as:

$$\sum_{k=2}^{d-2}(k^2 - k)t_k + \sum_{k=2}^{d-2}(k - k^2 + dk - 2d + 2)t_k \leq d^2 - d,$$

or equivalently:

$$\sum_{k=2}^{d-2}(k - k^2 + dk - 2d + 2)t_k \leq 0.$$

Notice that as $d \leq k + 2$, we have $k - k^2 + dk - 2d + 2 = k - k^2 + d(k-2) + 2 \geq k - k^2 + (k-2)(k+2) + 2 = k - 2$. As $k \geq 3$ implies $k - 2 \geq 1$, we have that $t_k = 0$ for $k \geq 3$. In other words, Λ is an arrangement of lines in general position. We already know what the value of $\overline{S}(\mathbb{P}^2, \Lambda)$ is in this case:

$$\overline{S}(\mathbb{P}^2, \Lambda) = \frac{2d-6}{d-2}.$$

This completes the proof. □

Remark 10. This proof says something stronger, namely that if $\overline{S}(\mathbb{P}^2, \Lambda) = (2d-6)/(d-2)$, then the lines are in general position. So, in $\mathbb{P}^2$ a line arrangement is in general position if and only if $\overline{S}(\mathbb{P}^2, \Lambda) = (2d-6)/(d-2)$. For any other Λ, $\overline{S}(\mathbb{P}^2, \Lambda) > (2d-6)/(d-2)$.

Remark 11. Let Λ be an arrangement of d lines on $\mathbb{P}^2(\mathbb{C})$. For $4 \leq d \leq 6$ we can construct all possible line arrangements and give the values for $\overline{S}(\mathbb{P}^2, \Lambda)$ in each case. The question here is, given d and values for the t_k, when is it possible to construct a line arrangement with these values? As we know, one restricition is Lemma 6.2. This allows us to eliminate various combinations (for example $t_k = 0$ for $k \geq 5$), but not all the impossible ones. The rest have to be treated case by case. For example, for $d = 6$, the combination $t_2 = 3, t_3 = 2, t_4 = 1$ is one that could be allowed. But the value for $\overline{S}(\mathbb{P}^2, \Lambda)$ in this case is 1, which is less than

$^3/_2$, which is the minimum possible value for $\overline{S}(\mathbb{P}^2, \Lambda)$ by Proposition 8.1. Another possible combination is $t_2 = 0, t_3 = 5, t_4 = 0$, but in this case the value for $\overline{S}(\mathbb{P}^2, \Lambda)$ is 4, which is greater than 3, which would contradict $f_0 \geq d$. One can rule out all remaining impossible combinations in this manner.

In the following table, we show all combinations of values which are possible (compare with Table 1 of [31]):

d	t_2	t_3	t_4	$\overline{S}(\mathbb{P}^2, \Lambda)$
4	6	0	0	1
5	10	0	0	$^4/_3$
5	7	1	0	$^3/_2$
5	4	2	0	2
6	15	0	0	$^3/_2$
6	12	1	0	$^8/_5$
6	9	0	1	$^5/_3$
6	9	2	0	$^7/_4$
6	6	3	0	2
6	6	1	1	2
6	3	4	0	$^5/_2$

Notice that this table shows us examples of line arrangements with $\overline{S}(\mathbb{P}^2, \Lambda) < 2$ which are not in general position.

Corollary 8.1 (See Corollary 5.2 of [31]). *Let Λ be an arrangement of d lines such that and $t_k \neq 0$ for some $k \geq 3$. Then $\overline{S}(\mathbb{P}^2, \Lambda) \geq {}^3/_2$ with equality if and only if $d = 5$, $t_2 = 7$ and $t_3 = 1$. Excluding this case, $\overline{S}(\mathbb{P}^2, \Lambda) \geq {}^8/_5$, with equality if and only if $d = 6$, $t_2 = 12$ and $t_3 = 1$.*

As Sommese noted in [31], this corollary could be more extense. For every $\epsilon > 0$ one can try to catalogue all possible line arrangements with $\overline{S}(\mathbb{P}^2, \Lambda) \leq 2 - \epsilon$ using the method we have shown in the previous remark, but of course the amount of arrangements increases dramatically with ϵ.

8.1 Line Arrangements on $\mathbb{P}^2(\mathbb{C})$

When our projective plane is defined over $\mathbb{C}$, we can use abelian coverings. We will show the details of how to use this machinery in Part III, which is basically the same type of use needed to prove the difficult Proposition 8.2 and Theorem 8.1. So we will not give full proofs of these results. However, will spend some time discussing certain aspects of the proof of Proposition 8.2 because of the interesting questions and results that arise from them. The overall theory can be examined in [15] and [16].

Proposition 8.2 (Hirzebruch-Sakai inequality, see [29] and the second remark added in proof of [15])**.** *Let Λ be an arrangement of d lines on $\mathbb{P}^2_{\mathbb{C}}$. Then:*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(k-4)t_k.$$

This proposition is highly non-trivial. Note that the statement says that every line arrangement defined on $\mathbb{P}^2(\mathbb{C})$ necessarily has either double or triple points. The arrangement of lines in general position has only double points, whereas the arrangement of Remark 13 only has triple points. Regarding the issue of double and triple points we can prove:

Lemma 8.2. *For any line arrangement on the complex projective plane we have that:*

$$t_2 + \frac{1}{4}t_3 \geq 3.$$

Proof. By contradiction. Assume $t_2 + \frac{1}{4}t_3 < 3$. Then one can find the possible tuples of possitive integers (t_2, t_3) that satisfy this: $(0, k)$, $(1, \ell)$, $(2, m)$, where $k = 0, \ldots, 11$, $\ell = 0, \ldots, 7$ and $m = 0, \ldots, 3$. Each case is discarded using Lemma 6.2 and proposition 8.2. Remark 13 gives an example of a line arrangement with $t_2 + \frac{1}{4}t_3 = 3$. □

Proposition 8.2 can be improved using Theorem 14.2 to obtain:

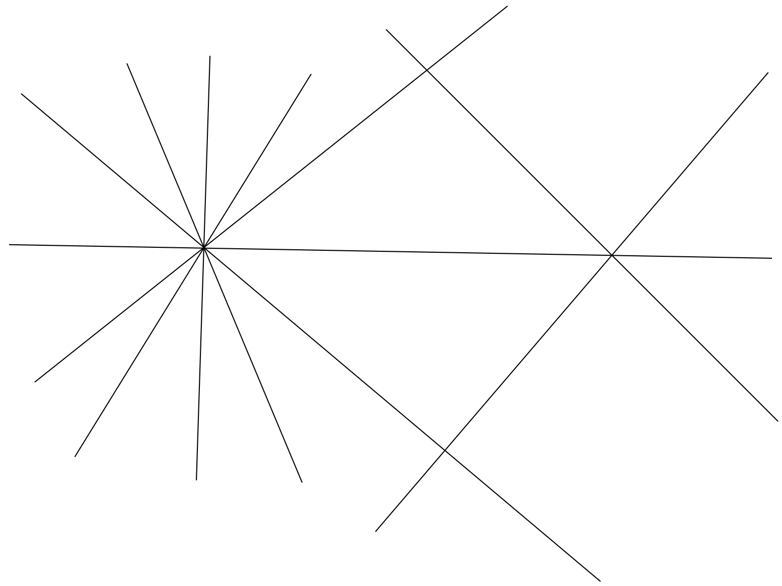

Figure 3: Example showing that $t_{d-2} = 0$ is necessary in Theorem 8.1.

Theorem 8.1 (See [16], equation 9)**.** *For an arrangement of d lines on the complex projective plane satisfying $t_d = t_{d-1} = t_{d-2} = 0$, we have:*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k \geq 5}(2k-9)t_k.$$

Remark 12. Aside from the pencil and the near-pencil, restriction $t_d = t_{d-1} = t_{d-2} = 0$ leaves out two types of line arrangements which can be seen in Figures 3 (which has a triple point) and 4 (which, for $d > 5$, has no triple points). As $2k - 9 \geq k - 4$ for $k \geq 5$, then Theorem 8.1 implies Proposition 8.2, except for line arrangements having $t_{d-2} \neq 0$, which can be treated separately. For this reason we say that Theorem 8.1 is an improvement on Proposition 8.2. As Figure 3 shows, the hypothesis $t_{d-2} = 0$ is necessary for the Theorem 8.1 to work. The figure shows a line arrangement of 8 lines with $t_2 = 10$, $t_3 = 1$ and $t_6 = 1$.

Now we will discuss aspects of the proof of Proposition 8.2. For every line arrangement on $\mathbb{P}^2(\mathbb{C})$, Hirzebruch associates the following quadratic polynomial:

$$F(x) = x^2(f_0 - d) - 2x(f_1 - 2f_0) + 4(f_0 - t_2) \tag{5}$$

and proves the following, using Theorem 5.2:

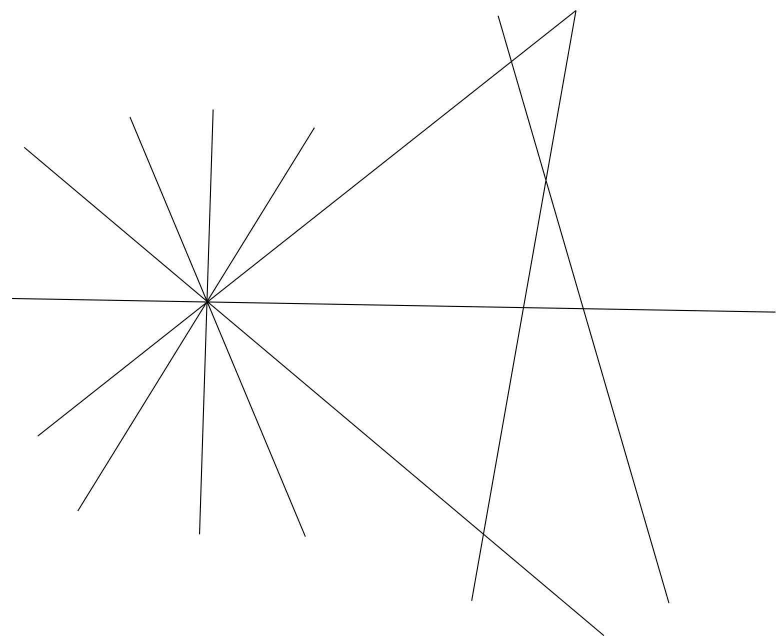

Figure 4: A line arrangement with $t_{d-2} = 1$ and no triple points.

Theorem 8.2 (See the theorem of section 3.1 of [15])**.** *For an arrangement of d lines in the complex projective plane with $t_d = t_{d-1} = 0$, we have that $F(x) \geq 0$ for all $x \in \mathbb{Z} \setminus \{1\}$. If $t_d = t_{d-1} = t_{d-2} = 0$, then $F(x) \geq 0$ for all $x \in \mathbb{Z}$.*

This theorem gives us a family of inequalities for line arrangements, one for each possible value of x. For $x < 0$, the fact that $F(x) > 0$ is simply a consequence of the terms $f_0 - d$, $f_1 - 2f_0$ and $f_0 - t_2$ being non-negative, so this gives no new information. On the other hand $f_0 > d$ (by Theorem 7.4, equality is only possible on finite projective planes), so $F(0) > 0$.

For $x \geq 2$, here are some of the inequalities:

$$\begin{aligned}
x = 2 \quad &\to \quad t_2 + t_3 \geq d + \sum_{k\geq 5}(k-4)t_k \\
x = 3 \quad &\to \quad 9t_2 + 7t_3 + t_4 \geq 9d + \sum_{k\geq 5}(6k-25)t_k \\
x = 4 \quad &\to \quad 4t_2 + 3t_3 + t_4 \geq 4d + \sum_{k\geq 5}(2k-9)t_k \\
x = 5 \quad &\to \quad 25t_2 + 19t_3 + 9t_4 \geq 25d + \sum_{k\geq 5}(10k-49)t_k \\
x = 6 \quad &\to \quad 9t_2 + 7t_3 + 4t_4 + t_5 \geq 9d + \sum_{k\geq 6}(3k-16)t_k
\end{aligned}$$

A priori it is not clear whether one of these inequalities implies another, so one should ask the question: do these inequalities provide new information? The following proposition answers this.

Proposition 8.3. *If $x \geq 2$, then Proposition 8.2 implies $F(x) \geq 0$.*

Proof. The first thing is to notice that the statement $F(x) \geq 0$ is equivalent to:

$$x^2t_2 + (x^2 - 2x + 4)t_3 + (x-2)^2t_4 + 4x^2\sum_{k\geq 5} t_k \geq x^2d + \sum_{k\geq 5}(2xk + 3x^2 - 4x - 4)t_k.$$

We will prove the following inequalities for $x \geq 2$:

$$\begin{aligned}
x^2t_2 + \frac{3}{4}x^2t_3 + 4x^2\sum_{k\geq 5} t_k \quad &\leq \quad x^2t_2 + (x^2 - 2x + 4)t_3 + (x-2)^2t_4 + 4x^2\sum_{k\geq 5} t_k \\
x^2d + \sum_{k\geq 5}(2xk + 3x^2 - 4x - 4)t_k \quad &\leq \quad x^2d + x^2\sum_{k\geq 5} kt_k.
\end{aligned}$$

Once we have proven this, then the result follows from multiplying the inequality in Proposition 8.2 by x^2.

For the first inequality it suffices to prove that $\frac{3}{4}x^2 \leq (x^2 - 2x + 4)$. But this is equivalent to $0 \leq (x-4)^2$.

For the second inequality it suffices to prove that for $k \geq 5$ we have that $x^2k \geq 2xk + 3x^2 - 4x - 4$. This is equivalent to $0 \leq x^2(k-3) + x(4-2k) + 4$. The roots of this quadratic polynomial are 2 and $\frac{2}{k-3}$. As $x \geq 2$, it follows that $0 \leq x^2(k-3) + x(4-2k) + 4$. □

Proposition 8.2 is obtained from improving inequality $F(3) \geq 0$ using a result by Sakai. This result was improved by Miyaoka to obtain Theorem 14.2. The same theorem could be used to improve all inequalities $F(x) \geq 0$, in fact, Theorem 8.1 is obtained from $F(2) \geq 0$ plus studying a few special cases. However, when doing the same for $F(x) \geq 0$, $x \geq 4$, no new results are obtained. Therefore, in the sense of previous proposition and taking into account Remark 12, Theorem 8.1 has all the information we can obtain from the polynomial $F(x)$.

8.2 Upper-Bound of the Log Chern Slope in $\mathbb{P}^2(\mathbb{C})$

Theorem 8.3 (See Theorem 5.3 of [31]). *Let Λ be an arrangement of d lines on $\mathbb{P}^2(\mathbb{C})$. Then $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \leq \frac{8}{3}$. Also $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = \frac{8}{3}$ if and only if $d = 9$, $t_3 = 12$ and $t_k = 0$ for $k \neq 3$.*

Proof. Suppose $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \geq \frac{8}{3}$. Then

$$3 + d + \sum_{k \geq 5} (k-4)t_k \geq 2t_2 + t_3$$

Combining this with the inequality 8.2, we obtain:

$$3 + d + \sum_{k \geq 5} (k-4)t_k \geq \frac{2}{3}t_2 + \frac{4}{3}\left(d + \sum_{k \geq 5} (k-4)t_k\right)$$

or equivalently

$$9 \geq 2t_2 + d + \sum_{k \geq 5} (k-5)t_k$$

By looking at the table from the previous remark, we can deduce that $7 \leq d \leq 9$ and $t_2 = 0$. Now we study case by case the possible values for d.

- Suppose $d = 9$. By inequality 8.2, $t_3 \geq 12$. Also:

$$36 = \frac{9(9-1)}{2} = \sum_{k \geq 2} k(k-1)t_k = 3t_3 + \sum_{k \geq 4} k(k-1)t_k$$

  so $t_3 = 12$ and $t_k = 0$ for $k \neq 3$. In this case, $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = \frac{8}{3}$.

- Suppose $d = 8$. By inequality 8.2, $t_3 \geq 11$. But also we must have:

$$28 = 3t_3 + \sum_{k \geq 4} k(k-1)t_k$$

which is impossible.

- Suppose $d = 7$. By inequality 8.2 $t_3 \geq 10$. But also we must have:

$$21 = 3t_3 + \sum_{k \geq 4} k(k-1)t_k$$

which is impossible.

□

Remark 13. There exists exactly one line arrangement (up to isomorphism) of 9 lines with $t_3 = 12$, called the *dual Hesse arrangement*. It can be represented by $(x^3 - y^3)(y^3 - z^3)(x^3 - z^3)$ (it is a special case of Ceva arrangement, which will be defined in the next section). To see that it is unique, let Λ be a line arrangement of 9 lines with $t_3 = 12$. Then by Lemma 6.2 we know then that $t_k = 0$ for $k \neq 3$. The same proposition gives that for any line $L \in \Lambda$ we have that:

$$\sum_{p \in L} (r_p - 1) = 8.$$

Since the arrangement only has triple points we get that the number of triple points of the arrangement that are on L is 4. In other words, every line of the arrangement passes through exactly 4 triple points.

Suppose now that $L_1, L_2, L_3 \in \Lambda$ intersect in a triple point. Given that Λ has 12 triple points, there has to be a triple point p that does not belong to $L_1 \cup L_2 \cup L_3$. Suppose $L_4, L_5, L_6 \in \Lambda$ intersect at p. Note that L_4 has to intersect L_1, L_2 and L_3, and that L_4 only has 4 triple points. Therefore the three remaining lines of the arrangement L_7, L_8 and L_9 intersect at a point not in $L_1 \cup \cdots \cup L_6$.

By the analysis we made, we can say that it is unique up to combinatorial isomorphism. In fact it is unique up to projective equivalence, because the dual line arrangement it defines is a $(4, 3)$-net (see [34], Example 3.1). See Section 9 for definition of net.

Remark 14. The inequality $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \leq 8/3$ is equivalent to:

$$2t_2 + t_3 \geq 3 + d + \sum_{k\geq 5}(k-4)t_k.$$

8.3 On The Interval $[2.65, 2.\overline{6}]$

Notice that the term t_4 does not appear in Theorem 8.2. It is the only t_k for which Theorem 8.2 does not give a bound (either upper or lower). In Section 11 we will see that this is a problem when trying to understand the behaviour of the log Chern slopes in the interval $[2.5, 2.\overline{6}]$. What kind of bound on t_4 would be suitable for this? The following proposition offers one.

Proposition 8.4. *Let $\epsilon > 0$. The number of line arrangements Λ that satisfy the following conditions:*

(a) $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \geq 2.65 + \epsilon$

(b) $t_4 < 2d$

is finite.

Proof. By Remark 12 we can assume $t_{d-2} = 0$, because in this restriction we are only leaving out two types of line arrangements. Let $\frac{a}{b} \in \left(\frac{53}{20}, \frac{8}{3}\right)$. Suppose that $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \geq \frac{a}{b}$. This is equivalent to:

$$9b - 3a + d(2a-5b) + b\sum_{k\geq 4}(3k-4)t_k \geq (a-2b)t_2 + (2a-5b)t_3 + a\sum_{k\geq 4}(k-1)t_k.$$

Note that $9b - 3a$, $2a - 5b$ and $a - 2b$ are positive numbers. We now use Theorem 8.1 to replace t_3 to get:

$$9b-3a+d(2a-5b)+b\sum_{k\geq 4}(3k-4)t_k \geq (a-2b)t_2+\frac{4}{3}(2a-5b)(d-t_2+\sum_{k\geq 5}(2k-9)t_k)+a\sum_{k\geq 4}(k-1)t_k.$$

Now note that $b(3k-4) \leq \frac{4}{3}(2a-5b)(2k-9) + a(k-1)$ for $k \geq 5$ (here is where it is crucial that $\frac{a}{b}$ be greater than 2.65). We are left then with:

$$27b - 9a + (24b-9a)t_4 \geq (14b-5a)t_2 + d(2a-5b) + \sum_{k\geq 5}\left((19a-49b)k - 75a + 192b\right)t_k.$$

Note that $14b - 5a > 0$. Given that $t_4 < 2d$ we conclude:

$$27b - 9a \geq (14b - 5a)t_2 + d(20a - 53b) + \sum_{k \geq 5} ((19a - 49b)k - 75a + 192b)\, t_k,$$

so d is bounded by $\frac{27b-9a}{20a-53b}$. □

The only example known of a line arrangement with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) > 2.65$ is the dual Hesse arrangement. Two arrangements are known with log Chern slope equal to 2.65: the Klein arrangement, and the Ceva arrangement with $n = 4$. These examples will be discussed in the following sections. On the other hand, I do not have any examples of line arrangements with $t_4 \geq 2d$.

9 Complex Nets

This section describes a special type of line arrangements in $\mathbb{P}^2(\mathbb{C})$ called nets. They are a matter of study on their own (for the various interests they arise see [6] [34], [39]). Our interest in them is to study two results which prohibit the existence of certain nets in the complex projective plane. As we will see, these constraints exhibit properties of the complex projective plane that are not always satisfied on projective planes of positive characteristic.

Definition 38. Let $p, q \geq 3$ be integers. A line arrangement Λ is called a (p, q)-*net* if it satisfies the following conditions:

(a) Λ can be written as the disjoint union of p sets (called *blocks*) $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_p$, where each Λ_i is a set of q lines. So, the number of lines in Λ is $d = pq$.

(b) There exists a set $B \subset \mathbb{P}^2$, called the set of *base points*, such that for every point $x \in B$ and for every block Λ_i, there exists exactly one line $L \in \Lambda_i$ passing through x. In other words, viewing each block as a curve, then B is the set of points of intersection of these curves. In particular this means that $|B| = q^2$.

Notice that if $p > a \geq 3$, then the existance of a (p, q)-net implies the existence of a (a, q)-net, simply by considering a of the given p blocks. There is an infinite family of examples of $(3, q)$-nets in the complex projective plane (see section 4.2 of [32]). There are also (p, q)-nets with p arbitrarily large on projective planes of positive characteristic (see Example 2.1 of [32]).

The existence of $(4, q)$-nets is related to a question we have already presented in 8.3: are there line arrangements with $t_4 \geq 2d$? Note that every point in B of a $(4, q)$-net is a point of multiplicity 4. So, viewing the net as a line arrangement, we have that $t_4 \geq q^2$. On the other hand, the amount of lines is $d = pq$. So, if we could prove that for every line arrangement $t_4 \leq 2d$, then we would know that there are no $(4, q)$-nets with $q \geq 8$. The cases for $4 \leq q \leq 8$ have been studied and are known not to exist.

Example 7. As it was explained in Remark 13, the dual arrangement of $(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$ defines a $(4, 3)$-net called the *Hesse arrangement*. In fact, it is the only $(4, q)$-net known to exist in the complex projective plane.

9.1 Cross-Ratio

The first result we are going to look at is from [19] and speaks about the cross-ratio of four concurrent lines.

Theorem 9.1 (See Propositions 3.1, 4.1 and Theorem 4.2 of [19])**.** *Let Λ be a $(4, q)$-net on the complex projective plane, and let B be the set of base points of Λ. Then the cross-ratio of the four lines going through any point in B is constant. Moreover, $q \not\equiv 2 \mod 3$.*

Using that the cross-ratio is constant and that the existence of a $(5, q)$-net implies the existence of at least two different $(4, q)$-nets, the authors of [19] conclude the following:

Theorem 9.2 (See Theorem 5.2 of [19])**.** *There are no $(5, q)$-nets on the complex projective plane (and hence, there are no (p, q)-nets, $p \geq 5$).*

This theorem shows then a special property of the complex projective plane because, like we said, there are nets with an arbitrarily large number of blocks on projective planes of positive characterisitic. We are left then with the following:

Question: Are there $(4, q)$-nets on the complex projective plane other than the Hesse arrangement?

9.2 The Hessian of a Line Arrangement

In this section we will look at the polynomials defining the lines of an arrangement. We will not distinguish between a plane curve and the polynomial defining it. Let k be the field over which $\mathbb{P}^2$ is defined.

Definition 39. The *Hessian* of a polynomial $F(X_0, X_1, X_2)$ is given by the 3×3 determinant:

$$H_F = \det\left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{i,j=0,1,2}$$

The next result is one of the reasons why it is interesting to study the Hessian of a line arrangement.

Theorem 9.3 (See Proposition 1.1.22 of [7]). *Let F be a plane curve of degree $n \geq 3$. Assume $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > n$. Then H_F is a multiple of F if and only if F is completely reducible (i.e. a union of lines).*

This is saying that if F is a completely reducible polynomial then there exists a polynomial Q_F such that $H_F = Q_F F$. The next proposition tells us how to calculate Q_F.

Proposition 9.1 (See Exercise 1.6 of chapter 1 of [7] and page 660 of [22]). *Suppose $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > d$. Let $F = \prod_{i=1}^{d} L_i$ where each L_i is a linear form. Let A be the $3 \times d$ matrix whose i-th column consists of the coefficients of L_i. Let M be the set of all subsets of $\{1, \dots, d\}$ with exactly three elements. Given $I \in M$, let Δ_I be the 3×3 minor of A whose columns are given by the elements of I. Also, let $F_I = \prod_{i \notin I} L_i$. Then:*

$$H_F = (d-1)F \sum_{I \in M} \Delta_I^2 F_I^2$$

Remark 15. Notice that the previous proposition says that if F is completely reducible, then:

$$Q_F = (d-1)\sum_{I\in M} \Delta_I^2 F_I^2$$

So, if p is a singular point of multiplicity greater than 2 in F, then it is a point of Q_F. This is because if the multiplicity of p in F is at least 3, then there are at least three lines in F that pass through p. Suppose these lines are L_1, L_2 and L_3. Then they all belong to the pencil $\lambda L_1 + \mu L_2$, $[\lambda : \mu] \in \mathbb{P}^1_k$. Therefore, if $I = \{1, 2, 3\}$, then $\Delta_I = 0$. For any other I, the polynomial F_I consists of at least one of the lines L_1, L_2, L_3, so it will be zero at p. Thus $Q_F(p) = 0$.

Not only that, p will have to be a singular point of Q_F, as all the F_I are squared in the expression for Q_F. So all derivatives of Q_F can be written as sums of products of lines, and each term in the sum has at least one line L_1, L_2, L_3 in them. So any point of multiplicity greater than 3 in F, is a singular point of Q_F.

This remark motivates the following converse question: if p is a singular point of Q_F, is p a singular point of F? Not necessarily, as the following example shows.

Example 8. Consider:

$$F = X(Z+Y-X)(Z-Y-X)(Z-iY+X)(Z+iY+X)$$

As the reader can check, the point $p = [0 : 1 : 0]$ is a singular point of Q_F that is not a singular point of F.

Now we apply this knowledge of Hessians to line arrangements defining complex nets. For this, we will follow [39]. Let Λ be a (p, q)-net and let F_1, F_2 be two of the blocks of Λ. Let $\mathcal{P}$ be the pencil of curves $\{\lambda F_1 + \mu F_2 \mid [\lambda : \mu] \in \mathbb{P}^1(\mathbb{C})\}$. Let $H(\lambda, \mu)$ denote the Hessian of $\lambda F_1 + \mu F_2$.

Proposition 9.2 (See Proposition 2.14 of [39])**.** *There exists a form $Q \in \mathbb{C}[X_0, X_1, X_2]$ of degree $2d-6$, and quadratic forms $R_1, R_2 \in \mathbb{C}[\lambda, \mu]$ such that:*

$$H(\lambda, \mu) = Q\left(\lambda R_1(\lambda, \mu) F_1 + \mu R_2(\lambda, \mu) F_2\right)$$

for all $(\lambda, \mu) \in \mathbb{C}^2$.

As Theorem 3.1 of [39] shows, this result can also be used to argue the non-existence of $(5, q)$-nets in $\mathbb{P}^2(\mathbb{C})$ using a completely different method from the one we had shown before. What is most interesting about the previous proposition is the fact that the polynomial Q does not depend on λ and μ. In particular, this says that for every block L_i in Λ, the polynomial $Q_{L_i} = Q$ is constant (does not depend on i).

Conjecture: For these reasons I believe that the Hesse arrangement is the only $(4, q)$-net in the complex projective plane.

10 Finite Reflection Groups

Here we will introduce the basic notions of finite reflection groups. We are interested in using these groups to obtain line arrangements on the projective plane. For more details see the chapter of Reflection Arrangements of [23].

Definition 40. Let V be a vector space. An element $s \in GL(V)$ is a *reflection* if it has finite order and its fixed point set is a hyperplane H_s of V. H_s is called the reflecting hyperplane of s. A finite subgroup $G \leq GL(V)$ is called a *reflection group* if it is generated by reflections.

A line in $\mathbb{P}^2(\mathbb{C})$ can be seen as a hyperplane passing through the origin in the complex vector space $\mathbb{C}^3$. $GL(\mathbb{C}, 3)$ acts on the set of hyperplanes plassing through the origin, so it acts on the set of lines of $\mathbb{P}^2(\mathbb{C})$.

Definition 41. Let $G \leq GL(V)$ be a finite reflection group. The set $\mathcal{A}(G)$ of reflecting hyperplanes of G is called the *reflection arrangement* of G.

Example 9 (Klein's Arrangement)**.** The simple group of order 168 acts on $\mathbb{P}^2(\mathbb{C})$ and the fixed set of this action is a line arrangement with 21 lines in which, $t_3 = 28$, $t_4 = 21$, and $t_k = 0$ otherwise. The Chern slope of this arrangement is:

$$\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = \frac{53}{20} = 2.65$$

In [30] Shephard and Todd classified finite unitary reflection groups. This classification has three infinite families (called A_ℓ, $G(m,p,n)$ and $C(r)$), and 34 exceptional groups (which are referred to as G_r, where $4 \leq r \leq 37$). Hirzebruch examines this classification in [15] and shows tables of the line arrangements obtained by non real groups (which are found in the family $G(m,p,n)$ and the exceptional groups G_{24}, G_{25}, G_{26} and G_{27}). The Klein arrangement of the previous example is obtained from group G_{24}. The Ceva family is obtained from the groups $G(m,p,n)$. These groups give rise to a family of examples of line arrangements on the complex projective plane which have no isomorphic counterpart on the real projective plane.

11 On Families of Line Arrangements

In this section we are interested in families of line arrangements such that the sequence of their Chern slopes converges as the number of lines tends to infinity. We already saw an example, the arrangement on lines in general position. Here is a more intricate one:

Example 10 (Ceva Arrangement)**.** Consider the completely reducible curve $(x^n - y^n)(y^n - z^n)(x^n - z^n)$ in $\mathbb{P}^2(\mathbb{C})$, $n \geq 4$. This equation describes a $(3,n)$-net. In this case we have $t_n = 3$, $t_3 = n^2$ and $t_k = 0$ otherwise. The log Chern slope of this family is:

$$\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = \frac{9 - 15n + 5n^2 + 3(3n-4)}{3 - 6n + 2n^2 + 3(n-1)} = \frac{5n^2 - 6n - 3}{2n^2 - 3n},$$

which converges to $5/2$ as n tends to infinity.

The following Lemma is crucial for showing that the log Chern slopes of line arrangements are dense in some interval. The way we will do this is this: we will take two families of line arrangements $\{\Lambda_n\}, \{\Lambda'_m\}$ such that their sequences of log Chern slopes converges to numbers a and b respectively. If the expression of the log Chern slope of each Λ_n and each Λ'_m satisfy a certain condition (which will be made precise in the Lemma as an order of convergence condition), then the family of line arrangements obtained from taking Λ_n and intersecting it transversally with Λ'_m (where n and m are chosen appropiately) will converge to a dense subset of numbers between a and b.

Lemma 11.1 (Density Lemma)**.** *Let $p, q, r, s \in \mathbb{R}[t]$ be polynomials of the same degree $h \geq 2$ such that:*

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} = a, \qquad \lim_{t\to\infty} \frac{r(t)}{s(t)} = b$$

where $0 < a < b$. Fix $K_1, K_2 \in \mathbb{R}$. Define $Q : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ as:

$$Q(n, m) = \frac{p(n) + r(m) + 2nm + K_1}{q(n) + s(m) + nm + K_2}$$

Then there exist injections $\phi, \psi : \mathbb{N} \hookrightarrow \mathbb{N}$ such that the image of $R(k) = Q(\phi(k), \psi(k))$ is dense in the interval $[a, b]$.

Proof. Let the injections be given as multiples of some number k, so that $n = \alpha k$ and $m = \beta k$. Lets fix some notation for the polynomials:

$$p(n) = \sum_{i=0}^{h} a_i n^i \qquad q(n) = \sum_{i=0}^{h} b_i n^i$$
$$r(m) = \sum_{i=0}^{h} c_i m^i \qquad s(m) = \sum_{i=0}^{h} d_i m^i$$

By hypothesis we know that:

$$\frac{a_h}{b_h} = a < \frac{c_h}{d_h} = b$$

We now separte the proof in cases. If $h = 2$ then:

$$R(k) = Q(\alpha k, \beta k) = \frac{k^2(\alpha^2 a_2 + \beta^2 c_2 + 2\alpha\beta) + \cdots}{k^2(\alpha^2 b_2 + \beta^2 d_2 + \alpha\beta) + \cdots}$$

where the $\cdots$ denote terms of lower degree on k. So

$$\lim_{k\to\infty} R(k) = \frac{\alpha^2 a_2 + \beta^2 c_2 + 2\alpha\beta}{\alpha^2 b_2 + \beta^2 d_2 + \alpha\beta}$$

Let $x = {}^{\alpha}/_{\beta}$, so that x is any rational number greater than 0. The previous quotient can be re-written as:

$$f(x) = \frac{a_2 x^2 + 2x + c_2}{b_2 x^2 + x + d_2}$$

Note that $f(0) = b$, and when x tends to infinity, $f(x)$ tends to a. Since $f(x)$ is continuous over $(0, \infty)$, then by the intermediate value theorem, we have that the set $[a, b] \subseteq \overline{f(\mathbb{Q}^+)}$.

Now we study the case $h > 2$. This situation is basically the same one as before, with the following difference:

$$\lim_{k\to\infty} R(k) = \frac{\alpha^2 a_2 + \beta^2 c_2}{\alpha^2 b_2 + \beta^2 d_2}$$

So that:

$$f(x) = \frac{a_2 x^2 + c_2}{b_2 x^2 + d_2}$$

which still satisfies that $f(0) = b$, $f(\infty) = a$ and is continuous over $(0, \infty)$. Therefore we obtain the desired result just as before. □

Corollary 11.1 (How to use the Density Lemma)**.** *There exist families of line arrangements such that the limits of their log Chern slopes form a dense subset of the interval* $[2, 2.5]$.

Proof. This proof will show how to use the density lemma. Let $\{\Lambda_d\}$ be the family of d lines in general position. Let $\{\Lambda'_n\}$ be the family of the Ceva arrangement: $(x^n - y^n)(y^n - z^n)(x^n - z^n)$. Λ_d has d lines and $t_2 = d(d-1)/2$. On the other hand, Λ'_n has $3n$ lines, $t_3 = n^2$ and $t_n = 3$.

Let $\mathcal{A}(d, n)$ be the line arrangement obtained from joining Λ_d and Λ'_n such that every line of Λ_d intersects every line Λ'_n transversally and vice-versa. We say that $\mathcal{A}(d, n)$ is obtained from combining Λ_d and Λ'_n transversally. So $\mathcal{A}(d, n)$ has $d + 3n$ lines, $t_2 = d(d-1)/2 + 3dn$, $t_3 = n^2$ and $t_n = 3$. The log Chern slope of $\mathcal{A}(d, n)$ is:

$$\frac{9 - 5(d + 3n) + 2(d(d-1)/2 + 3dn) + 5n^2 + 3(3n-4)}{3 - 2(d + 3n) + (d(d-1)/2 + 3dn) + 2n^2 + 3(n-1)}.$$

Now we use the density lemma and obtain the desired result. Notice that to be able to use the density lemma, we need the polinomials of the log Chern slope to be of the same degree both in d and in n. □

Corollary 11.2. *Let Λ_n be a family of line arrangements on $\mathbb{P}^2(\mathbb{C})$. Then the limit of the log Chern slopes lies in the interval* $[2, 8/3]$.

Proof. By the previous results we know that:

$$\frac{2d-6}{d-2} \leq \overline{S}(\mathbb{P}^2, \Lambda) \leq \frac{8}{3}$$

From the definition, we know that $d \to \infty$ as $n \to \infty$. Taking limits on all sides of the previous inequality we obtain:

$$2 \leq \lim_{n\to\infty} \overline{S}(\mathbb{P}^2, \Lambda) \leq \frac{8}{3}$$

□

From Proposition 8.1 we know that for $\epsilon > 0$, the amount of line arrangements with $\overline{S}(\mathbb{P}^2, \Lambda) < 2 - \epsilon$ is finite. So far the biggest limit point we have found is 2.5. But Proposition 8.3 says that the maximum of $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda)$ is $8/3$. We are left with the following:

Question: Is there a family of line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda)$ tending to something bigger than 2.5?

This is an open question. The problem can be seen in various ways. One way is to look at the proof of Proposition 8.3 and ask: why did it work? The reason is that when one takes the statement $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \geq 8/3$ and expands it, one gets:

$$27 - 15d + \sum_{k\geq 2}(9k-12)t_k \geq 24 - 16d + \sum(8k-8)t_k.$$

Notice that the coefficient of t_4 is the same on both sides, so they cancel each other out. Also, the inequality of Proposition 8.2 doesn't consider t_4. This allows everything else to work. But if one tried to imitate the proof with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \geq a/b$, with $a/b < 8/3$ then the term t_4 will show up, and it will appear on the big side of the inequality. This is a problem as it doesn't allow you to bound the value of d. In other words:

the problem is that we do not have a good upper bound for t_4.

In fact, the only one that I know is the one resulting from Lemma 6.2:

$$t_k \leq \frac{d(d-1)}{k(k-1)},$$

but this is too big.

Another way of looking at the problem is through finite reflection groups. Hirzebruch classified the lines arrangements resulting from these groups in [15]. It is from these arrangements that we have obtained the only examples of lines arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda)$ superior to 2.5. So the question of whether or not 2.5 is the biggest limit point could be answered if we could prove that a line arrangement with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) > 2.5$ induces a finite reflection group. It is mainly because of this finite reflection group perspective that I make the following:

Conjecture: The behaviour of line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) > 2.5$ mimics that of line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) < 2$, namely, that given $\epsilon > 0$, the amount of line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) > 2.5 + \epsilon$ is finite.

11.1 Arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) = 2.5$ in $\mathbb{P}^2(\mathbb{R})$

If the polynomials defining a line arrangement only have real coefficients, then the arrangement defines a cellular decomposition of $\mathbb{P}^2(\mathbb{R})$; each cell is bounded by an r-gon, $r \geq 3$ as we are assuming $t_d = 0$. Let p_r be number of cells bounded by r-gons. Note that:

$$f_1 = \sum_{k \geq 2} k t_k = \frac{1}{2} \sum_{r \geq 3} r p_r.$$

This means that f_1 is the number of edges of the decomposition. Also, it is clear that $f_0 = \sum t_k$ is the amount of vertices of the decomposition. Call f_2 the number of faces, that is $f_2 = \sum p_r$. Since the Euler-Poincaré characteristic of $\mathbb{P}^2(\mathbb{R})$ is 1, then $f_0 - f_1 + f_2 = 1$. This implies that:

$$0 = 3 + \sum_{k \geq 2} (k-3) t_k + \sum_{r \geq 3} (r-3) p_r. \tag{6}$$

Definition 42. A cellular decomposition of $\mathbb{P}^2(\mathbb{R})$ is called *simplicial* if $p_r = 0$ for $r \geq 4$.

Proposition 11.1. *A line arrangement on $\mathbb{P}^2(\mathbb{R})$ satifies $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) \leq 2.5$ with equality if and only if it defines a simplicial decomposition.*

Proof. The condition $\overline{S}(\mathbb{P}^2(\mathbb{R}),\Lambda) \leq 2.5$ is equivalent to saying $t_2 \geq 3+\sum_{k\geq 4}(k-3)t_k$. From equation 6 we have:

$$t_2 = 3 + \sum_{k\geq 4}(k-3)t_k + \sum_{r\geq 4}(r-3)p_r \geq 3 + \sum_{k\geq 4}(k-3)t_4$$

and we have equality if and only if $p_r = 0$ for $r \geq 4$, that is, the decomposition is simplicial.

□

Corollary 11.3. *Every real line arrangement satisfies* $t_2 \geq 3+\sum_{k\geq 4}(k-3)t_k$. *In particular every real line arrangement satisfies* $t_2 \geq 3$ [1].

Remark 16. Note that this corollary is not true over $\mathbb{P}^2(\mathbb{C})$ because there exist line arrangements that do not have double points ($t_2 = 0$), like the Ceva arrangements or the Klein arrangement. The previous corollary implies that any line arrangement Λ in $\mathbb{P}^2(\mathbb{C})$ with $\overline{S}(\mathbb{P}^2(\mathbb{C}),\Lambda) > 2.5$ cannot be realized in $\mathbb{P}^2(\mathbb{R})$, i.e. there is no line arrangement in $\mathbb{P}^2(\mathbb{R})$ combinatorially isomorphic to Λ. In this sense the Ceva and Klein arrangements are "purley complex" arrangements. This poses a couple of open questions:

(a) What is the minimum positive amount of double points that a line arrangement can have?

(b) If Λ is a purely complex line arrangement, is it true that $\overline{S}(\mathbb{P}^2(\mathbb{C}),\Lambda) > 2.5$?

There is an infinite amount of line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{R}),\Lambda) = 2.5$, as the following example shows.

Example 11 (Simplicial Arrangements)**.** For example, take an m-gon on the Euclidean plane and add the m lines of symmetry. These arrangement has $2m$ lines and for $m > 3$:

$$t_2 = m, t_3 = \frac{m(m-1)}{2}, t_m = 1$$

[1] In [15], Hirzebruch cites a result by Sten Hansen from his paper of 1981 *Contributions to the Sylvester-Gallai-Theory* which claims that for real arrangements $t_2 \geq [d/2]$. However, in a survery paper of 1990 by P. Borwein and W.O.J. Moser called *A survery of Sylvester's problem and its generalizations*, the authors say that Hansen's proof of his claim is so long (100 pages) that nobody has read it completely and verified it.

and $t_k = 0$ otherwise. These arrangements all have $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) = 2.5$. For this and more examples see Grünbaum's work: [11] and [12].

Corollary 11.4. *The set of Chern slopes of line arrangements on* $\mathbb{P}^2(\mathbb{R})$ *is dense in* $[2, 2.5]$.

Proof. Use the density lemma with line arrangements on general position and the family described in Example 11. □

11.2 Arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$

The condition $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$ is equivalent to $d = 3 + \sum_{k \geq 3}(k-2)t_k$.

Proposition 11.2. *Given* $d \geq 5$ *and a tuple of number* $(t_3, \ldots, t_{d-2})$ *that satisfy* $d = 3 + \sum_{k\geq 3}(k-2)t_k$, *one can find by induction a real line arrangement having the imposed combinatorial information.*

Proof. To do this we define a well-order on the set of all finite tuples of non-negative integers. We define:

$$(a_1, a_2, \ldots, a_d) < (b_1, b_2, \ldots, b_{d'})$$

if one of the following happens:

(a) $d' > d$

(b) $d' = d$ there exists i such that $b_i > a_i$ and $b_j = a_j$ for all $i < j \leq d$.

Use this well-order to well-order the subset T of all tuples $(t_3, \ldots, t_{d-2})$ that satisfy $d = 3 + \sum_{k\geq 3}(k-2)t_k$.

First notice that no line configuration in general position has $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$. Now notice that there is exactly one line arrangement of five lines with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$, call it A. For $d = 6$ we have two tuples: $(3, 0)$ and $(1, 1)$. To achieve $d = 6$ and $(3, 0)$, take A and add a line passing through a double point of A, crossing transversally everywhere else. To achieve $(1, 1)$, add a line to A passing through a triple point and transversally everywhere else.

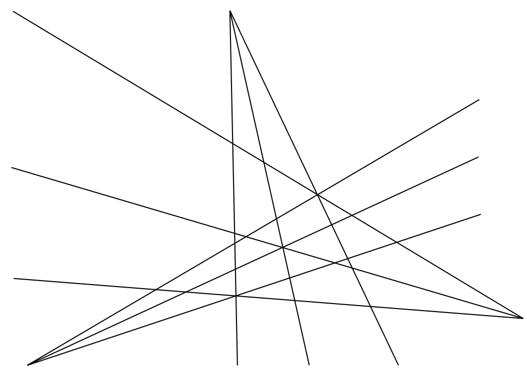

Figure 5: A line arrangement with no “almost transversal” lines and $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$.

Now for the inductive step. Suppose all tuples in T less than $t = (t_3, \ldots, t_{d-2}) \in T$ are realizable. Let n be the biggest index of this tuple such that $t_n \neq 0$. Consider the tuple $t' = (t_3, \ldots, t_{n-1} + 1, t_n - 1, \ldots, t_{d-3})$. Note that $t \in T$ implies $t' \in T$. Clearly $t' < t$, so there is a line arrangement L with t' as its combinatorial information. Take an $n-1$ point of L and pass a new line through it that intersects transversally elsewhere. You get a new line configuration with t as combinatorial information. □

Remark 17. All line arrangements obtained with the preceding proof can be realized on $\mathbb{P}^2(\mathbb{R})$.

Remark 18. Not all arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2$ are obtained with the method of the previous proof. Note that with the method shown, all the arrangements obtained have the following property: there is a line that crosses transversally or “almost transversally” (i.e. a line such that it passes through through exactly one point of multiplicity greater than 2). However, the figure 5 shows a real line arrangement with 9 lines in which each line has exactly two triple points, so it does not have the property.

11.3 Arrangements on Projective Planes Over Finite Fields

Let p be a primer number. We consider line arrangements now on $\mathbb{P}^2\left(\overline{\mathbb{F}_q}\right)$. Lemma 6.2 is still true so the lower bound given by Proposition 8.1 still holds, but results like Proposition

8.2 or Corollary 11.3 are no longer valid, as the next well-known example shows.

Example 12. Consider the surface $\mathbb{P}^2(\mathbb{F}_q)$. If $q = p^m$, and p prime, then $\mathbb{P}^2(\mathbb{F}_q)$ has $p^{2m} + p^m + 1$ points and the same number of lines. Each line has $p^m + 1$ points and through each point pass $p^m + 1$ lines. Notice that $\mathbb{P}^2(\mathbb{F}_q)$ can be seen as a line arrangement on $\mathbb{P}^2\left(\overline{\mathbb{F}_q}\right)$. So, seen as a line arrangement, $\mathbb{P}^2(\mathbb{F}_q)$ is described by: $d = p^{2m} + p^m + 1$, $t_{p^m+1} = p^{2m} + p^m + 1$ and $t_k = 0$ otherwise. The Chern slope then is:

$$\overline{S}(\mathbb{P}^2(\overline{\mathbb{F}}_q), \Lambda) = \frac{9 - 5(p^{2m} + p^m + 1) + (3(p^m + 1) - 4)(p^{2m} + p^m + 1)}{3 - 2(p^{2m} + p^m + 1) + p^m(p^{2m} + p^m + 1)} = 3.$$

By Theorem 7.4 then, $\mathbb{P}^2(\mathbb{F}_q)$ are the only line arrangements which satisfie $\overline{S}(\mathbb{P}^2, \Lambda) = 3$.

Proposition 11.3. *The log Chern slopes of line arrangements on* $\mathbb{P}^2(\overline{\mathbb{F}_q})$ *is dense in* $[2, 3]$.

Proof. We will replicate the idea of the density lemma using the arrangement of lines in general position and the finite projective planes. Write $q = p^m$, where p is prime. Let $\{\Lambda_d\}$ be the arrangement of d lines in general position in $\mathbb{P}^2(\overline{\mathbb{F}_q})$, and let $\{\mathbb{P}^2(\mathbb{F}_{p^m})\}_{m=1}^{\infty}$ be the collection of finite projective planes. Define $\mathcal{A}(d, m)$ as the line arrangement obtained from combining Λ_d and $\mathbb{P}^2(\mathbb{F}_{p^m})$ transversally in $\mathbb{P}^2(\overline{\mathbb{F}_q})$. Then $\mathcal{A}(d, m)$ has $d + p^{2m} + p^m + 1$ lines, $t_2 = d(d-1)/2 + d(p^{2m} + p^m + 1)$ and $t_{p^m+1} = p^{2m} + p^m + 1$. The log Chern slope of $\mathcal{A}(d, m)$ is:

$$\frac{9 - 5(d + p^{2m} + p^m + 1) + 2(d(d-1)/2 + d(p^{2m} + p^m + 1)) + (3p^m - 1)(p^{2m} + p^m + 1)}{3 - 2(d + p^{2m} + p^m + 1) + d(d-1)/2 + d(p^{2m} + p^m + 1) + p^m(p^{2m} + p^m + 1)}.$$

Write $d = ap^n$, $m = n + b$ and $x = a/p^n$. Let n tend to infinity in the previous expression. What we get is:

$$\frac{2x + 3}{x + 1}.$$

Given that numbers of the form a/p^n are dense in $[0, \infty)$, we obtain the desired result. □

Part III

Section Arrangements in Σ_e

12 Hirzebruch Surfaces

12.1 Rational Scroll Description

Our description of the Hirzebruch surfaces will be made as in chapter 2 of [26], that is, we will see them as rational scrolls.

Let $e \geq 1$ be an integer. Let k be the base field (algebraically closed). Consider the group action of $k^* \times k^*$ on the set $(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ defined by:

$$\begin{aligned} (\lambda, 1)(t_1, t_2; x_1, x_2) &= (\lambda t_1, \lambda t_2; x_1, \lambda^{-e} x_2) \\ (1, \mu)(t_1, t_2; x_1, x_2) &= (t_1, t_2; \mu x_1, \mu x_2), \end{aligned}$$

where $\lambda, \mu \in k^*$. The action of an element $(\lambda, \mu) \in k^* \times k^*$ is understood as the composition of the actions $(\lambda, 1)(1, \mu)$. Let Σ_e be the quotient of $(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ by this action. For an element $(t_1, t_2; x_1, x_2) \in (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$, let $[t_1 : t_2; x_1 : x_2]$ denote the corresponding equivalence class in Σ_e.

Definition 43. Σ_e is called a *Hirzebruch surface.*

Note that there is a natural projection from $\pi : \Sigma_e \to \mathbb{P}^1$, given by $[t_1 : t_2; x_1 : x_2] \mapsto [t_1 : t_2]$. This projection is well defined because the action by $(\lambda, 1)$ multiplies both t_1 and t_2 by λ, while the action by $(1, \mu)$ leaves them intact.

Lemma 12.1. *Fix $[t_1 : t_2] \in \mathbb{P}^1$ and consider the fiber $\pi^{-1}([t_1 : t_2])$. This fiber is a copy of $\mathbb{P}^1$.*

Proof. To see this, we choose a special representative (u, v) of $[t_1 : t_2]$ as follows: if $t_1 = 0$, choose $(u, v) = (0, 1)$; otherwise choose $(u, v) = (1, t)$. Given $[t_1 : t_2; x_1 : x_2]$ in the fiber, choose a representative of the form $(u, v; x_1, x_2)$. Define then $\psi(u, v; x_1, x_2) = [x_1 : x_2]$. Note that because of the way we chose (u, v), any other representative of $[t_1 : t_2; x_1 : x_2]$ of our selected form will look like $(u, v; \mu x_1, \mu x_2)$, for some $\mu \in k^*$. Therefore ψ defines an isomorphism between a fiber of π and $\mathbb{P}^1$. □

Lemma 12.2. *The projection $\pi : \Sigma_e \to \mathbb{P}^1$ defines in fact a fiber bundle.*

Proof. Set the charts of $\mathbb{P}^1$ to be $U_0 = \{[t_1 : 1] : t_1 \in k\}$ and $U_\infty = \{[1 : t_2] : t_2 \in k\}$. Take $[t_1 : 1] \in U_0 \cap U_\infty$. Note that:

$$[t_1 : 1; x_1 : x_2] = \left[1 : \frac{1}{t_1}; x_1 : t_1^e x_2\right].$$

This means that Σ_e is the union of two copies of $\mathbb{A}^1 \times \mathbb{P}^1$ glued togehter by $t_1 \mapsto t_1^{-1}$ in the first factor, and $\mathrm{diag}(1, t_1^e)$ in the second factor. □

The two previous Lemmas say that the surface Σ_e can be seen as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In this sense some authors, like [2], define these surfaces as:

$$\Sigma_e = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)\right).$$

12.2 Map Into Projective Space

Let $\mathcal{S}^e(t_1, t_2)$ denote the set of all monomials in t_1 and t_2 of degree e. Let $g \in \mathcal{S}^e(t_1, t_2)$. The action of $(\lambda, 1)$ on g gives $\lambda^e g$, so the action of $(\lambda, 1)$ on the monomial gx_2 leaves it invariant. The ratios of the monomials $S^e(t_1, t_2)x_2$ along with x_1, will give us a map of Σ_e into projective space. Set $B = \{[t_1 : t_2; x_1 : 0]\} \subset \Sigma_e$.

Proposition 12.1. *Define $\phi : \Sigma_e \to \mathbb{P}^{e+1}$ be given by:*

$$[t_1 : t_2; x_1 : x_2] \mapsto [t_1^e x_2 : t_1^{e-1} t_2 x_2 : \cdots : t_2^e x_2 : x_1].$$

This map satisfies that it is an embedding when restricted to the fibers of π. The image is the subvariety of $\mathbb{P}^{e+1}$ defined by the determinantal equations (that is, take the 2×2 determinants *of the following matrix):*

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_e \\ u_2 & u_3 & \cdots & u_{e+1} \end{bmatrix},$$

where $[u_1 : \cdots : u_{e+1} : u_{e+2}]$ *are the homogeneous coordinates of* $\mathbb{P}^{e+1}$*. The image* $\phi(\Sigma_e)$ *is a cone.*

Proof. Clearly ϕ is well defined. Note that $\phi(B) = \{[0 : 0 : \cdots : 0 : 1]\}$. Now we will shows that ϕ is injective outside of B. Points outside of B satisfy $x_2 \neq 0$. Suppose $\phi([t_1 : t_2; x_1 : x_2]) = \phi([v_1 : v_2; w_1 : w_2])$. From this we get that there exists $\alpha \in k^*$ such that:

$$\begin{aligned} t_1^e x_2 &= \alpha v_1^e w_2 \\ t_1^{e-1} t_2 x_2 &= \alpha v_1^{e-1} v_2 w_2 \\ x_1 &= \alpha w_1. \end{aligned}$$

From the first two equations we get $[t_1 : t_2] = [v_1 : v_2]$ (here we used $x_2 \neq 0$). Therefore:

$$\begin{aligned} [t_1 : t_2; x_1 : x_2] &= \left[t_1 : t_2; \alpha w_1 : \alpha \frac{v_1^e}{t_1^e} w_2\right] \\ &= \left[t_1 : t_2; w_1 : \frac{v_1^e}{t_1^e} w_2\right] \\ &= \left[v_1 : v_1 \frac{t_2}{t_1}; w_1 : w_2\right] \\ &= [v_1 : v_2; w_1 : w_2], \end{aligned}$$

so ϕ is injective outside B. Each fiber of π contains only one point of B, so ϕ restricted to the fiber is an embedding.

Let $C = \{[t_1 : t_2; 1 : 1]\} \subset \Sigma_e$. Since C does not intersect B, then ϕ embeds C in $\mathbb{P}^{e+1}$. Then $\phi(\Sigma_e)$ can be seen as the cone over $\phi(C)$ with vertex point $[0 : 0 : 0 : 1]$. For this, note that the line joining $[t_1^e : t_1^{e-1} t_2 : \cdots : t_2^e : 1]$ to $[0 : 0 : 0 : 1]$ is given as the image under ϕ of $\{[t_1 : t_2; a + b : a] : [a : b] \in \mathbb{P}^1\} \subset \Sigma_e$. □

Example 13. In this example we will work explicitly the equations defining the image on Σ_2 in $\mathbb{P}^3$. The morphism $\phi : \Sigma_2 \to \mathbb{P}^3$ is given by:

$$[t_1 : t_2; x_1 : x_2] \mapsto [t_1^2 x_2 : t_1 t_2 x_2 : t_2^2 x_2 : x_1].$$

The image of B under ϕ is just the point $[0 : 0 : 0 : 1]$, so that the image of Σ_2 in $\mathbb{P}^3$ is a cone, as ϕ is injective outside B. Let $[u_1 : u_2 : u_3 : u_4]$ be homogeneous coordinates for $\mathbb{P}^3$. Then the image of Σ_e is the cone defined by the equation $u_1 u_3 = u_2^2$.

12.3 The Picard Group of Σ_e

Lemma 12.3. *The Picard group of Σ_e is the free abelian group $\mathbb{Z}F \oplus \mathbb{Z}\Gamma$, where F is the class of a fiber of π and Γ is the class of B.*

Proof. Let α and β be linear forms in t_1 and t_2. Denote by $[\alpha_0 : \alpha_1]$ the zero of α and $[\beta_0 : \beta_1]$ the zero of β. Note that $\alpha(t_1, t_2)/\beta(t_1, t_2)$ is a rational function on Σ_e whose corresponding divisor is: $[\alpha_0 : \alpha_1] \times \mathbb{P}^1 - [\beta_0 : \beta_1] \times \mathbb{P}^1$. Therefore, the difference of any two fibers of π is a principal divisor, so all fibers are on the same divisor class, which we call F. This shows that F is well defined.

Suppose now that $aF + b\Gamma$ is linearly equivalent to zero. Restricting this divisor to any fiber of π shows that $b\Gamma$ should be linerealy equivalent to zero, which means $b = 0$. Therefore aF is linearly equivalent to zero, and so $a = 0$. This shows that F and Γ are lineraly independent.

Let $C \subset \Sigma_e$ be an irreducible codimension 1 subvariety. C is defined by a single bihomogeneous polynomial equation f. Choose a monomial of the form $t_1^{ea+b} x_2^a$, so that it has the same bidegree as f. Then $t_1^{ea+b} x_2^a / f$ is a rational function on Σ_e, and so $C \sim bF + a\Gamma$, finishing the proof. □

Lemma 12.4. *If F and Γ are as in the previous lemma, then $F^2 = 0$, $F.\Gamma = 1$ and $\Gamma^2 = -e$.*

Proof. From the previous Lemma, we know that any two fibers of π are linearly equivalent, so F^2 can be seen as the intersection of two distinct fibers, which is clearly zero. The intersection of any fiber of π with B gives exactly one point, so $F.\Gamma = 1$.

Consider now the curve $C \subset \Sigma_e$ given by $x_1 = 0$. Notice that $t_1^e x_2 / x_1$ is a rational function on Σ_e, so that $C \sim aF + \Gamma$. Since $B \cap C = \emptyset$, then $\Gamma.(aF + \Gamma) = 0$, from which we obtain $\Gamma^2 = -a$. □

Proposition 12.2. *A canonical class of* Σ_e *is* $-2\Gamma - (e+2)F$.

Proof. Consider the rational 2-form $s = \mathrm{d}t_2 \wedge \mathrm{d}x_2$ on the affine chart $\mathbb{A}^2_{1,1} = (t_1 \neq 0, x_1 \neq 0)$. First note that the coordinate change form $\mathbb{A}^2_{1,2} = (t_1 \neq 0, x_2 \neq 0)$ to $\mathbb{A}^2_{1,1}$ can be given by $y = 1/x_2$ and $t' = t_1$, so that the Jacobian of this transformation is x_2^{-2}. Therefore s has a pole of order 2 along B. On the other hand, the change of coordinates with $\mathbb{A}^2_{2,1}$ can be described by $t' = 1/t_1$ and $y = x_2 t_1^{-e}$, whose Jacobian is t_1^{-e-2}. Therefore s has a pole of order $-e-2$ along the fiber $\pi^{-1}([0:1])$. $\square$

Remark 19. Hirzebruch surfaces are minimal for $e \geq 2$, as they contain no exceptional divisors.

12.4 Another Description

There is another way of characterising these surfaces using blow ups and blow downs. Set Σ_1 as $\mathbb{P}^2$ with a point blown up. Let E be the corresponding exceptional divisor of Σ_1. Let $\phi : S \to \Sigma_1$ be the blowing up of Σ_1 in a point $p \in E$. Let F be the fibre of Σ_1 passing through p. Then we can write $\phi^*(E) = E' + D$ and $\phi^*(F) = F' + D$, where D is the exceptional divisor on S over p. Note that $E'^2 = -2$ and $F'^2 = -1$, so we can blow down S along F' by Castelnuovo's Theorem. Let $\psi : S \to S'$ be this blow down. Let $E'' = \psi_*(E')$ and $D' = \psi_*(D)$. Note that $E''^2 = -2$ and that $D'^2 = 0$. Now S' is again a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In fact $S' = \Sigma_2$. This method can be repeated to obtain all Σ_e via blow ups and downs starting from $\mathbb{P}^2$.

13 Section Arrangements

13.1 General Results and Examples

Definition 44. A *section arrangement* on Σ_e, $e \geq 2$, is a curve arrangement which satisfies:

(a) The arrangement consists of d global sections that do not intersect Γ. This means that as divisors, the sections are represented by $\Gamma + eF$.

(b) $d \geq 4$

(c) $t_d = t_{d-1} = 0$.

We can use Lemma 6.1 to deduce that $f_0 \geq d$ for section arrangements on Σ_e. The log Chern slope in this case can be written as:

$$\overline{S}(\Sigma_e, \Lambda) = \frac{8 - 4d - de + \sum_{k\geq 2}(3k-4)t_k}{4 - 2d + \sum_{k\geq 2}(k-1)t_k}.$$

Section arrangements on Σ_e are regular curve arrangements of index e.

Remark 20. Given that $e \geq 2$, then $2d < 4 + de + f_0$, which is equivalent to $\overline{S}(\Sigma_e, \Lambda) < 3$.

Lemma 13.1. *For a section arrangement on Σ_e we have that $f_0 > e + 2$.*

Proof. We proceed by contradiction. Suppose $f_0 \leq e + 2$. Then by Lemma (6.2) we get:

$$ed(d-1) = \sum k(k-1)t_k \leq (d-2)(d-3)f_0 \leq (d-2)(d-3)(e+2).$$

From this we obtain $e(4d-6) \leq 2(d-2)(d-3)$. As $d \leq f_0 \leq e+2$, we have:

$$d - 2 \leq \frac{2(d-2)(d-3)}{4d-6},$$

or in other words:

$$0 \leq \frac{d(4-2d)}{4d-6},$$

which is a contradiction for $d \geq 4$. □

Remark 21. The previous result can be improved using the same method of proof and verifying the statement for a few specific cases. However, we will not need this stronger result.

Example 14 (Arrangements of sections in general position)**.** In the case where the sections are in general position (that is, $t_k = 0$ for $k \geq 3$), the Chern slope is:

$$\overline{S}(\Sigma_e, \Lambda) = \frac{16 + 2ed^2 - 4ed - 8d}{8 + ed^2 - ed - 4d}.$$

Taking the limit of $\overline{S}(\Sigma_e, \Lambda)$ as d tends to infinity gives 2. But if one takes the limit of $\overline{S}(\Sigma_e, \Lambda)$ as e tends to inifity, one gets:

$$\lim_{e\to\infty} \overline{S}(\Sigma_e, \Lambda) = \frac{2d-4}{d-1}.$$

Notice that for $d = 4$ you get $4/3 < 2$. This example then gives various families of section arrangements with log Chern slopes converging to numbers less than 2 (something that can not be done on $\mathbb{P}^2$). This is the primary motivation for studying section arrangments on Σ_e.

Proposition 13.1. *Let Λ be a section arrangement on Σ_e. Then its log Chern slope $\overline{S}(\Sigma_e, \Lambda)$ satisfies:*

$$\frac{16 + 2ed^2 - 4ed - 8d}{8 + ed^2 - ed - 4d} \le \overline{S}(\Sigma_e, \Lambda),$$

with equality if and only if the arrangement is in general position.

Proof. Suppose there exists an arrangement for which:

$$\frac{16 + 2ed^2 - 4ed - 8d}{8 + ed^2 - ed - 4d} \ge \overline{S}(\Sigma_e, \Lambda).$$

Using the formula for $\overline{S}(\Sigma_e, \Lambda)$, this is equivalent to:

$$\sum_{k\ge2}(8k + 8d - 4dk - 2ed^2 + edk + ed^2k - 16)t_k \le eded(d-1).$$

Using Lemma 6.2, we get:

$$\sum_{k\ge2}(8k + 8d - 4dk - 2ed^2 + 2edk + ed^2k - edk^2 - 16)t_k \le 0.$$

Notice that $ed^2k - edk^2 + 2edk - 2ed^2 - 4dk + 8d = (k-2)(ed(d-k) - 4d)$, which is non-negative as $k \ge 2$, $e \ge 2$ and $d \ge k + 2$. □

Example 15 (Arrangements obtained from line arrangements in $\mathbb{P}^2$)**.** Consider a line arrangement Λ in $\mathbb{P}^2$ with multiplicity information $(t_3, \dots, t_{d-2})$. Let $\rho : \Sigma_1 \to \mathbb{P}^2$ be the blow up of a point p that is not on the line arrangement. Define the morphism $f : \Sigma_e \to \mathbb{P}^1 \times \mathbb{P}^1$ as:

$$f([t_1 : t_2; x_1 : x_2]) = [t_1^e : t_2^e; \psi([t_1 : t_2; x_1 : x_2])],$$

where ψ is the fiber isomorphism defined in Lemma 12.1. The morphism f is of degree e and is branched upon two fibers of Σ_1. In fact, f is such that the following diagram is commutative:

$$\begin{array}{ccc} \Sigma_e & \xrightarrow{f} & \Sigma_1 \\ \pi \big\downarrow & & \big\downarrow \sigma \\ \mathbb{P}^1 & \xrightarrow[g]{} & \mathbb{P}^1 \end{array}$$

where π and σ are the standard projections and $g([u:v]) = [u^e : v^e]$. This gives yet another description of Σ_e; Σ_e is the fibered product of Σ_1 and $\mathbb{P}^1$ over $\mathbb{P}^1$.

Choosing appropiate coordinates on $\mathbb{P}^2$, we can choose two lines passing through p that intersect Λ transversally such that the pull back of these lines by ρ gives the two fibers of the branch locus of f. Let Λ' be the pull back of Λ by $\rho \circ f$. In this way, every line of Λ is pulled back to a section of Σ_e, and multiple point of Λ will be repeated e times in Λ'.

The amount of sections in Λ' is the same as the amount of lines, but each multiple point is repeated e times, so the log Chern slope looks likes:

$$\overline{S}(\Sigma_e, \Lambda') = \frac{8 - 4d - de + e\sum_{k\geq 2}(3k-4)t_k}{4 - 2d + e\sum_{k\geq 2}(k-1)t_k}.$$

If we took the Ceva arrangement as Λ, then after pulling back, the Chern slope would be:

$$\frac{e(5n^2 + 6n - 12) - 12n + 8}{e(2n^2 + 3n - 3) - 6n + 4}.$$

As n tends to infinity, this number tends to $5/2$. If e tends to infinity, we get:

$$2 + \frac{n^2 - 6}{2n^2 + 3n - 3} \in \left[2, \frac{5}{2}\right[.$$

Note that Ceva's behaviour changes completely from $\mathbb{P}^2(\mathbb{C})$ to Σ_e. In the projective plane, Ceva arrangements had Chern slopes above $5/2$, even reaching the maximum slope possible for $n = 3$. In Σ_e instead the arrangement is below $5/2$.

Consider now the simplicial family of Example 11. The log Chern slope after pulling back is:

$$\overline{S}(\Sigma_e, \Lambda) = \frac{8 - 8m - 2me + 2me + \frac{5}{2}m(m-1)e + (3m-4)e}{4 - 4m + me + m(m-1)e + (m-1)e}.$$

When e tends to infinity, we get:

$$\frac{5m^2+m-8}{2m^2+2m-2},$$

which belongs to the interval $[2, 2.5[$ when $m \geq 4$.

Remark 22. From these examples we can show density of the log Chern slopes of section arrangements in the interval $[2, 2.5]$. For this, combine the general position arrangement transversally with the Ceva arrangement for example, let e tend to infinity, and then use the density lemma.

Lemma 13.2. *If Λ is a section arrangement on Σ_e obtained from a line arrangement Λ' such that $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda') \leq 2.5$. Then $\overline{S}(\Sigma_e, \Lambda) < 2.5$.*

Proof. $\overline{S}(\mathbb{P}^2(C), \Lambda') \leq 2.5$ is equivalent to $3 + \sum_{k\geq 4}(k-3)t_k \leq t_2$, while $\overline{S}(\Sigma_e, \Lambda) < 2.5$ is equivalent to $e\sum_{k\geq 4}(k-3)t_k < 4 + 2d(e-1) + et_2$ (where $t_k(\Lambda) = t_k(\Lambda')$). So the implication is immediate. □

13.2 Real Hirzebruch Surfaces

Just like we did for the real projective plane, we can use the Euler-Poincare characteristic of real Hirzebruch surfaces to obtain inequalities. The Euler-Poincare characteristic of $\Sigma_e(\mathbb{R})$ is 0. So, maintaining the notation from real line arrangements we have:

$$\sum_{k\geq 3}(k-3)t_k + \sum_{r\geq 3}(r-3)p_r = t_2,$$

from where we get:

$$t_2 \geq \sum_{k\geq 3}(k-3)t_k, \tag{7}$$

which says that every section arrangement on a real Hirzebruch surface has double points.

Lemma 13.3. *If Λ is a section arrangement on $\Sigma_e(\mathbb{R})$, then $\overline{S}(\Sigma_e(\mathbb{R}), \Lambda) < 2.5$.*

Proof. $\overline{S}(\Sigma_e(\mathbb{R}), \Lambda) < 2.5$ is equivalent to:

$$2d + \sum_{k\geq 4}(k-3)t_k \leq 2de + t_2 + 4.$$

Using inequality (7) and that $2d < 2de + 4$, we get the result. As we saw in Example 15, there are families of section arrangements with log Chern slope tending to 2.5. $\square$

13.3 Hirzebruch Surfaces Over $\overline{\mathbb{F}_p}$

If we pull back a finite projective plane, the section arrangement produced has log Chern slope:

$$\frac{8-(4+e)(p^{2m}+p^m+1)+e(3p^m-1)(p^{2m}+p^m+1)}{4+(ep^m-2)(p^{2m}+p^m+1)}.$$

If we let e tend to infinity we get:

$$\frac{(3p^m-2)(p^{2m}+p^m+1)}{p^m(p^{2m}+p^m+1)} = 3 - \frac{2}{p^m},$$

which tends to 3 as m tends to infinity.

13.4 Section Arrangements and Ryser Designs ($f_0 = d$)

Now we will look at the question: are there section arrangements such that $f_0 = d$? The answer of this question in the case of line arrangements was: yes, only the finite projective planes and the near-pencil have that property. The answer for section arrangements is also yes, but the family of these section arrangements is not known completely. At first we will address this question from a purely combinatorial perspective.

The first question then is, given a set X of d points, is there a set $\mathcal{A}$ of d subsets of X such that for every $A, B \in \mathcal{A}$, if $A \neq B$ then $A \cap B = e$, and $|A| > e$? Due to Theorem 6.3 we know there are only two posibilities: either all points have the same multiplicity, or there are only two types of multiplicities. Both cases are of interest and have special names which we now define.

Definition 45. Let X be a finite set and let Λ be a set of subsets of X such that $|X| = |\mathcal{A}|$.

(a) If all points of X have the same multiplicity (i.e. are contained in the same number of elements of $\mathcal{A}$), then $(X, \mathcal{A})$ is called a *symmetric design.*

(b) If there are only two distinct multiplicities and the elements of $\mathcal{A}$ do not all have the same cardinality, then $(X, \mathcal{A})$ is called a *Ryser design*.

For a detailed study of symmetric and Ryser designs see [17]. We will focus on a special family of Ryser designs and see if these can be used to produce sections arrangements with $f_0 = d$. As we will see, the family we will choose is inherently related to the Cremona transformations of finite projective planes.

First of all, observe that any symmetric design can be used to produce a Ryser design in the following way[2]: if $(X, \mathcal{A})$ is a symmetric design and $A \in \mathcal{A}$, let $\mathcal{A}' = \{A\} \cup \{A \triangle B : B \in \mathcal{A}, A \neq B\}$ (here $\triangle$ denotes the symmetric difference of sets), then $(X, \mathcal{A}')$ is a Ryser design. For the combinatorial information of $(X, \mathcal{A}')$ see Proposition 14.2.4 of [17]. We know a family of symmetric designs, namely the finite projective planes, so we can produce a family of Ryser designs of which we know all of its combinatorial information. This is the family we will work with.

Example 16 (Ryser designs obtained from finite projective planes)**.** Let X be the set of points of the projective plane over the finite field $\mathbb{F}_{p^m}$ (where p is a prime number), and let $\mathcal{A}$ be the set of all lines on X. Then $|X| = |\mathcal{A}| = p^{2m} + p^m + 1$. Also for every $x \in X$ we have that $r_x = p^m + 1$. Choose $L \in \Lambda$ and let $\mathcal{A}' = \{L\} \cup \{L \triangle L' : L' \in \mathcal{A}, L \neq L'\}$. Proposition 14.2.4 of [17] says that (X, Λ') is a Ryser design. It also says that:

(a) For any $A, B \in \mathcal{A}'$ such that $A \neq B$, we have that $|A \cap B| = p^m$.

(b) For every $x \in X$, either $r_x = p^m + 1$ or $r_x = p^{2m} + 1$.

(c) $t_{p^m+1} = p^{2m}$ and $t_{p^{2m}+1} = p^m + 1$.

(d) All points $x \in X$ with $r_x = p^{2m} + 1$ lie on L.

Part (a) says that if the Ryser design obtained from the projective plane over $\mathbb{F}_{p^m}$ can be realized as a section arrangement, then the section arrangement lives in Σ_{p^m}.

[2]All known Ryser designs are obtained via this method (which is called *block complementation*) from symmetric designs, and it is conjectured that all Ryser designs arise in this way.

This example gives de intersection information of the section arrangement we are trying to obtain. The following proposition will give us the key step for finding it, as it transforms the question from a problem on section arrangements, to a problem on curve arrangements on the plane.

Proposition 13.2. *Asking whether a Ryser design obtained from a finite projective plane is realizable in a Hirzebruch surface, is equivalent to asking whether a special configuration of lines and conics exists on the projective plane. Specifically, if the Ryser design has $p^{2m}+p^m+1$ points, then the configuration of lines and conics has to satisfy:*

(a) There are only two points, x_1 and x_2, of multiplicity p^{2m+1}. These points lie on the same line and $r_{x_1} = r_{x_2} = p^m$.

(b) There are $2p^m + 1$ lines and $p^m(p^m - 1)$ conics. This, combined with (a), means that every conic passes through x_1 and x_2.

(c) There is a third point x_3 different from x_1 and x_2, such that all conics pass through x_3 and no lines pass through x_3.

(d) The set of conics can be divided into $p^m - 1$ sets of p^m conics such that the conics of each set are tangent to each other in x_3.

Proof. We will assume that the section arrangement exists and through a series of blow ups and downs we will get to the projective plane and show what types of curves are left. Suppose Λ is a section arrangement on Σ_{p^m} which realizes the given Ryser design. From Example 16 we know a few things about how the sections of Λ intersect each other. First, there is a section $s_1 \in \Lambda$ that goes through all points of multiplicity $p^{2m} + 1$. There are $p^m + 1$ of these points. This means that s_1 has no other multiple points.

Let x be a multiple point of s_1. Let F be the fiber through x. Note that Λ has $p^{2m}+p^m+1$ sections, and the only multiplicities are $p^{2m} + 1$ and $p^m + 1$. Since x has multiplicity $p^{2m} + 1$, the fiber has to go through the remaining p^m sections transversally, that is, x is the only multiple point of Λ in F.

Choose $p^m - 1$ of the multiple points on s_1. Go all Rambo on them and blow them all up. The fibers through of these points turn to (-1)-curves after the blow ups. Now blow down these curves. After this, the $(-p^m)$-curve Γ on Σ_{p^m} is changed to a (-1)-curve. Blow it down so as to get to the projective plane.

Let us analyze what was left after all these explosions. Let x_1 and x_2 be the two points of s_1 that were not blown up. Note that all curves that passed through x_1, still pass through this point after the transformations. Every section that does not pass through x_1, passes through every other multiple point of s_1, so it has gone through $p^m - 1$ blow ups. After this it is left a curve of self intersection 1, i.e. a line. The amount of sections not passing through x_1 is p^m. The same holds for x_2. Also s_1 has gone through $p^m - 1$ blow ups, so the image of s_1 is the line going through x_1 and x_2.

Let s be a section of Λ passing through both x_1 and x_2 that is not s_0. There is exactly one multiple point in s_1 through which s does not pass, call it x. Let F_x be the fiber at x. Note that s first passed through $p^m - 2$ blow ups, so it was left a curve of self intersection 2. When F was blown down, s changed to a curve of self intersection 3 and it now intersected Γ. When Γ was blown down, s changed to a curve of self intersection 4, i.e. a conic. If s' is another section of Γ not passing through x, then after blowing down F_x, s, s' and Γ, intersect transversally. After blowing down Γ, s and s' become tangent in one point.

Note that after blowing down the fibers and before blowing down Γ, all the curves that will be made conics, intersect Γ. This means that all conics pass through a point x_3, through which none of the lines pass. $\square$

The next theorem gives us exactly the existence of this special configuration, and the proof uses a Cremona transformation.

Theorem 13.1. *The Ryser designs obtained from finite projective planes are realizable on a Hirzebruch surface of the same characteristic as the finite projective plane.*

Proof. Let $\mathbb{K}$ be the clousure of $\mathbb{F}_{p^m}$. Consider $\mathbb{P}^2(\mathbb{F}_{p^m})$ as a line arrangement on $\mathbb{P}^2(\mathbb{K})$. We will see that after applying a specific Cremona transformation, we will obtain the configuration of lines and conics necessary for the realization of the Ryser designs.

Choose two points $x_1, x_2 \in \mathbb{P}^2(\mathbb{F}_{p^m})$ and let L be the line joining them. Choose $x_3 \in \mathbb{P}^2(\mathbb{K})$ not in $\mathbb{P}^2(\mathbb{F}_{p^m})$ such that, if Let L' and L'' are the lines joining x_3 to x_1 and x_2 respectively, then L' and L'' cross $\mathbb{P}^2(\mathbb{F}_{p^m})$ transversally. Blow up x_1, x_2 and x_3, and then blow down L, L' and L''. The image of $\mathbb{P}^2(\mathbb{F}_{p^m})$ under this Cremona transformation is the desired configuration of lines and conics.

To see this, let us look at what happens to each of the lines of $\mathbb{P}^2(\mathbb{F}_{p^m})$ through the transformation. Let x be a multiple point on L that is not x_1 or x_2. Every line going through x different from L does not pass through x_1 nor x_2. So it remains unaltered after the blow ups. But it crosses all the curves that are being blown down. So after the transformation, these lines are left as conics. Since L is one of the curves that will be blown down, then all the lines going through x will become conics tangent at x.

On the other hand, a line that goes through x_1 and is not L, will go through the blowin up of x_1 and through the blowing down of L''. So it will remain a line. □

Remark 23. The log Chern slopes of the section arrangement representing a Ryser designs described in the previous theorem are:

$$\frac{3p^{3m} + p^{2m} - 7p^m + 1}{p^{3m} + p^{2m} - 3p^m}.$$

This number can be calculated by calculating the log Chern slope of the line arrangement used in the proof of the previous theorem, that is, the finite projective plane plus two special lines. But there is a catch. One needs to be careful and notice that the quotient above does not equal the log Chern slope of the section arrangement obtained in Σ_{p^m}, because the process we described for realizing the section arrangement also produces other curves. For instante, some fibers and the special curve Γ have to be taken into consideration.

Note that the section arrangements we have developes all live in surfaces defined on positive characteristic. These arrangements are not realizable over the complex field because the steps of the proof (blow ups, blow downs and Cremona) are all birrational transformations.

Later on, in Proposition 14.2 we will see that section arrangements defined over $\mathbb{C}$ satisfy $f_0 > d$.

14 Obtaining Inequalities

Here we focus on complex Hirzebruch surfaces. Our aim is to obtain a result like Theorem 8.1. For this will apply the same abelian cover technique as in [15]. Let $\Lambda = \{s_1, \ldots, s_d\}$ be a section arrangement on Σ_e. Let K be the function field of Σ_e. Let $n \geq 2$ and consider the field $L = K\left((s_2/s_1)^{\frac{1}{n}}, \ldots, (s_d/s_1)^{\frac{1}{n}}\right)$, which is a normal separable extension of K. There exists a (normalized) surface X with function field L and a morphism $\pi : X \to \Sigma_e$ whose branch locus is Λ. Notice that $\deg(\pi) = n^{d-1}$, more specifically, given $p \in \Sigma_e$ we have that $|\pi^{-1}(p)| = n^{r_p - 1}$.

We desingularize X to obtain $\rho : Y \to X$. Note that $q \in X$ is a singular point if and only if $r_{\pi(q)} \geq 3$. Let $\tau : Z \to \Sigma_e$ be obtained from blowing up all points $p \in \Sigma_e$ such that $r_p \geq 3$. Then there exists a morphism $\sigma : Y \to Z$ such that $\pi\rho = \tau\sigma$.

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Z & \xrightarrow[\tau]{} & \Sigma_e \end{array}$$

Let $q \in X$ be singular, then $C = \rho^{-1}(p)$ is a curve and $p = \pi(q)$ satisfies that $r_p \geq 3$. Let E_p be the exceptional curve in Z over p. Then σ restricts to a covering $\phi : C \to E_p$ of degree $n^{r_p - 1}$. Suppose $s_1, \ldots, s_{r_p}$ are the sections passing through p. Let s_i' be the strict tranform of s_i, $i = 1, \ldots, s_{r_p}$. The branch points of ϕ are those $x \in E_p$ that are in the intersection of E_p with some s_i'. Using Hurwitz's formula we get:

$$2 - 2g(C) = n^{r_p-1}(2 - 2g(E_p)) - n^{r_p-2} r_p(n-1) = n^{r_p-1}(2 - r_p) + n^{r_p-2} r_p,$$

which allows us to conclude the following:

Proposition 14.1. *For $n = 2$, C is rational if and only if $r_p = 3$, and C is elliptic if and only if $r_p = 4$. If $n = 3$, then C is never rational, and C is elliptic if and only if $r_p = 3$. If $n \geq 4$, then C is neither rational nor elliptic.*

Remark 24. Note that $\sigma^* E_p$ consists of n^{d-r_p-1} disjoint curves C, each of multiplicity n. Therefore, as $(\sigma^* E_p)(\sigma^* E_p) = -n^{d-1}$, we get that for each curve C, $C^2 = -n^{r_p-2}$.

Consider now a section s in the arrangement Λ. Consider s' the strict transform of s in Z and let S be the strict transform of s' under σ. Restricting σ to S gives a morphism $\psi : S \to s'$ of degree n^{d-2}. Using Hurwitz's formula, we get:

$$2 - 2g(S) = 2n^{d-2} - \sum_{p \in s' : r_p \geq 2} n^{d-r_p-1}(n^{r_p-1} - 1).$$

I claim that this number is negative. To see this, let $a = |\{p \in s' : r_p \geq 2\}|$. Then:

$$\begin{aligned} 2n^{d-2} - \sum_{p \in s' : r_p \geq 2} n^{d-r_p-1}(n^{r_p-1} - 1) &= 2n^{d-2} - an^{d-2} + \sum_{p \in s' : r_p \geq 2} n^{d-r_p-1} \\ &\leq n^{d-2}(2 - a) + an^{d-3}. \end{aligned}$$

It suffices to show that $n^{d-2}(2 - a) + an^{d-3} < 0$, or equivalently $a > 2\frac{n}{n-1}$. Given that we have that $d \geq 4$, $e \geq 2$ and $t_d = t_{d-1} = 0$, then $a > 4$, which implies $a > 2\frac{n}{n-1}$, proving the claim. Therefore: $g(S) \geq 2$, so S is neither rational nor elliptic.

14.1 Calculating the Chern Numbers of Y

We are interested in obtaining the Chern numbers of Y in terms of the chern numbers of Σ_e and Λ. For this, note that since we already calculated the Euler number of the curves C that are over singular points of X we get:

$$e(Y) = e(X - \text{sing}X) + \sum_{k \geq 3} n^{d-1-k} t_k (n^{k-1}(2 - k) + kn^{k-2}).$$

We also have that:

$$\begin{aligned} e(X - \text{sing}X) &= n^{d-1}(e(\Sigma_e) + 2\sum(g(C_i) - 1) + \sum_{k \geq 2}(k-1)t_k) \\ &\quad -n^{d-2}(2\sum(g(C_i) - 1) - \sum_{k \geq 2} kt_k) + n^{d-3}t_2, \end{aligned}$$

which allows us to state:

$$\frac{e(Y)}{n^{d-3}} = n^2(4 - 2d + f_1 - f_0) + 2n(d + f_0 - f_1) + f_1 - t_2,$$

where $f_0 = \sum_{k\geq 2} t_k$ and $f_1 = \sum_{k\geq 2} kt_k$.

Now we want to find a useful expression for the canonical divisor of Y. Consider the divisor D on Z given by:

$$D = \tau^* K_{\Sigma_e} + \sum E_p + \frac{n-1}{n}\left(\sum E_p + \tau^*\Lambda - \sum r_p E_p\right).$$

So:

$$D^2 = K_{\Sigma_e}^2 + 2\frac{n-1}{n} K_{\Sigma_e}\Lambda - \sum_{k\geq 3} t_k + 2\frac{n-1}{n}\sum_{k\geq 3}(k-1)t_k + \frac{(n-1)^2}{n^2}\left(\Lambda^2 - \sum_{k\geq 3}(k-1)^2 t_k\right).$$

Notice that the canonical divisor of Y can be written as $K_Y = \sigma^*(D)$, so that $c_1^2(Y) = n^{d-1}D^2$. Then:

$$\frac{c_1^2(Y)}{n^{d-3}} = n^2(8 - (e+4)d + 3f_1 - 4f_0) + 2n((e+1)d - 2f_1 + 2f_0) + d + f_1 - f_0 + t_2.$$

In this way we obtain a *quadratic polynomial associated to* Λ:

$$f(n) = \frac{(3e(Y) - c_1^2(Y))}{n^{d-3}} = n^2(4+(e-2)d+f_0)+2n(f_0-f_1+(2-e)d)+2f_1+f_0-d-4t_2. \quad (8)$$

Now we give a couple of properties of the divisor D.

Lemma 14.1. *Given a section arrangement Λ on Σ_e, we have that for each p such that $r_p \geq 3$, $D.E_p \geq 0$ with equality if and only if $n = 2$ and $r_p = 3$.*

Proof.

$$D.E_p = -1 + \frac{n-1}{n}(-1 + r_p) \geq 0,$$

and $D.E_p = 0$ if and only if $n = 2$ and $r_p = 3$. □

Lemma 14.2. *Given a section arrangement Λ on Σ_e, let $s' = \tau^* s - \sum E_p$ be the strict transform of a section $s \in \Lambda$. Then $D.s'_j > 0$.*

Proof.

$$D.s'_j = -e - 2 + \frac{n-1}{n}de - \sum_{p \in s_j : r_p \geq 3} D.E_p.$$

Using Lemma 6.2 we conclude:

$$\sum_{p \in s_j : r_p \geq 3} D.E_p = -\left|\{p \in s_j : r_p \geq 3\}\right| + \frac{n-1}{n}(d-1)e - \frac{n-1}{n}\left|\{p \in s_j : r_p = 2\}\right|.$$

Therefore:

$$D.s'_j = -2 - \frac{e}{n} + \left|\{p \in s'_j : r_p \geq 2\}\right| - \frac{1}{n}\left|\{p \in s'_j : r_p = 2\}\right| = -2 - \frac{e}{n} + f_0 - \frac{t_2}{n}.$$

Now recall that $f_0 > e + 2$ by Lemma 13.1. □

Theorem 14.1. *K_Y is nef if any of the following conditions is satisfied:*

(a) $e > 2$

(b) $n > 3$

(c) $n = e = 2$ *and* $d > 4$

Proof. Notice that D can be written as:

$$D = \tau^* K_{\Sigma_e} + \left(2 - \frac{1}{n}\right) \sum E_p + \frac{n-1}{n} \sum s'_j$$

From our initial conventions we have that $t_d = t_{d-1} = 0$ and $d \geq 4$. This means that there exist four sections in the arrangement Λ (call them $s_1, \ldots, s_4$) such that no more than two pass through a given point. Let F denote a general fibre in Σ_e. Then:

$$(e-2)F - \frac{1}{2}(s_1 + s_2 + s_3 + s_4)$$

is a canonical divisor of Σ_e. Using this in the expression for D, we can see that all the coeficients in D are non-negative. The only case for which D vanishes is when $n = e = 2$ and $d = 4$ (and so the arrangement is in general position). Aside from this case, D is effective and positive, so $K_Y = \sigma^* D$ is an effective canonical divisor of Y, so it contains all exceptional curves of Y.

If G is an exceptional curve of Y, then $K_Y.G = -1$. G is a component of some divisor $\sigma^* E_p$ or $\sigma^* s'_j$. Therefore, using lemma 14.1 and lemma 14.2 we have that $K_Y.G < 0$ is impossible. From these two lemmas it also follows that $D^2 > 0$. □

The previous theorem gives us conditions under which Y has Kodaira dimension ≥ 0. This means that we can use Theorem 5.2 to conclude that the quadratic polynomial $f(n)$ from equation (8) is non-negative for every $n \geq 2$. For example, choosing $n = 2$ we get the inequality:

$$t_2 + 3t_3 + t_4 + 16 \geq d + \sum_{k \geq 5} (2k - 9) t_k. \tag{9}$$

14.2 Improving the Inequality

In [16], Hirzebruch shows a way to improve the inequalities obtained by the quadratic polynomial. The idea is this: we know from Remark 8 that if Y is of general type, then $c_1^2(Y) = 3c_2(Y)$ if and only if the universal cover of Y is the unit ball. So if Y is of general type and contains rational or elliptic curves, then the universal cover of Y cannot be the unit ball, so $c_1^2(Y) < 3c_2(Y)$. Then, if H is a configuration of elliptc or rational curves of Y, there exists a positive number $m(H)$ such that: $3c_2(Y) - c_1^2(Y) \geq m(H)$.

If H consists of a single rational curve on Y of self-interesection $-b$, then:

$$m(H) = 2 + b + \frac{1}{b} \tag{10}$$

(see [18] and formula (4) in [16]). On the other hand, Sakai showed in [29] that if H is a disjoint union of elliptic curves $C_1, \ldots, C_p$, then:

$$3c_2(Y) - c_1^2(Y) \geq \sum_{j=1}^{p} (-C_j^2). \tag{11}$$

The numbers C_j^2 are negative as Y is of general type. There exists in fact the following stronger statement:

Theorem 14.2 (See [16], [21]). *Let X be a smooth surface with non-negative Kodaira dimension. Let $H_1, \ldots, H_k$ be configurations (disjoint to each other) of rational curves (arising* *from quotient singularities). Let $C_1, \dots, C_p$ be smooth elliptic curves (disjoint to each other and disjoint to the E_i). Let c_1^2 and c_2 be the Chern numbers of X. Then:*

$$3c_2 - c_1^2 \geq \sum_{i=1}^{k} m\left(H_i\right) + \sum_{j=1}^{p}\left(-C_j^2\right). \tag{12}$$

This theorem allows us to improve our inequality (9). Later on, in Lemma 14.3, we will see that improving inequality (9) using Theorem 14.2 will give us an inequality that implies any other inequality obtained from applying Theorem 14.2 to the quadratic polynomial $f(n)$ defined in (8).

By Proposition 14.1 we know that Y has at least $t_3 2^{d-4}$ rational curves H_i of self-intersection number -2, and has at least $t_4 2^{d-5}$ elliptic curves C_j of self-interesection -4. Also Y has 2^{d-1} copies of Γ (the rational curve of self intersection $-e$). These copies are all rational curves. Therefore:

Proposition 14.2. *For an arrangement of d sections on Σ_e we have:*

$$t_2 + \frac{3}{4}t_3 \geq 4\left(e + \frac{1}{e}\right) - 8 + d + \sum_{k\geq 5}(2k-9)t_k.$$

Proof. We want to use Theorem 14.2. But by Theorem 14.1, we can do so for almost all sections arrangements, except the arrangment of 4 sections in Σ_2 (which is an arrangement in general position). Certainly this special can be seen to satisfy the inequality. For all other arrangements we use Theorem 14.2 and we have that:

$$0 \leq \frac{3c_2(Y) - c_1^2(Y) - \sum m(H_i) - \sum(-C_j)^2}{2^{d-3}}.$$

The H_i represent the rational curves. By equation (10) we know that $m(H_i) = \frac{9}{2}$ for all the rational curves of self-intersection -2, while $m(H_i) = (2 + e + \frac{1}{e})$ for the rational curves of self-intersection $-e$. Therefore, Theorem 14.2 says that the numerator of the previous expression is at most:

$$2^{d-3}(t_2 + 3t_3 + t_4 + 16 - d - \sum_{k\geq 5}(2k-9)t_k) - \frac{9}{2}t_3 2^{d-4} - \left(2 + e + \frac{1}{e}\right)2^{d-1} - 4t_4 2^{d-5}.$$

This gives the statement. □

Given that for $e \geq 2$ we have that $4(e + 1/e) \geq 10$, we will generally use the less sharp inequality of the following corollary.

Corollary 14.1. *For an arrangement of d sections on Σ_e we have:*

$$t_2 + \frac{3}{4}t_3 \geq 2 + d + \sum_{k \geq 5}(2k - 9)t_k.$$

Therefore, every section arrangement on Σ_e has either double or triple points.

Remark 25. In the case of line arrangements, Theorem 8.1 required $t_{d-2} = 0$ and Figure 3 showed why this was necessary. Theorem 14.2 has no exceptions. For instance, pulling back the line arrangement of Figure 3, gives as a section arrangement with $d = 8$, $t_2 = 10e$, $t_3 = e$ and $t_6 = e$. Clearly:

$$10e + \frac{3}{4}e \geq 4\left(e + \frac{1}{e}\right) + 3e.$$

Proposition 14.3. *For an arrangement of d sections on Σ_e we have that $\overline{S}(\Sigma_e, \Lambda) < {}^8\!/_3$.*

Proof. We proceed by contradiction. Suppose $\overline{S}(\Sigma_e, \Lambda) \geq {}^8\!/_3$. This is equivalent to:

$$4d - 3de + \sum_{k \geq 5}(k - 4)t_k \geq 2t_2 + t_3 + 8.$$

Now we use Corollary 14.1 to replace t_3 in the previous expression and obtain:

$$0 \geq 2t_2 + 32 + 9de - 8d + \sum_{k \geq 5}(5k - 24)t_k > 0,$$

which is a contradiction. □

This method of proof does not allow us to improve the upper bound because, once again, we cannot control t_4. But we can imitate the proof of Proposition 8.4 to obtain the following result.

Proposition 14.4. *Let $\epsilon > 0$. The number of section arrangements on Σ_e that satisfy:*

(a) $\overline{S}(\Sigma_e, \Lambda) \geq 2.65 + \epsilon$

(b) $t_4 < 2(10e - 9)d$

is finite.

Now we look at the family of inequalities given by $f(n) \geq 0$, where $f(n)$ is given by (8). To apply Theorem 14.2 to $f(3) \geq 0$, note that blowing up triple points produces elliptic curves of self-intersection -3, while blowing up points of multiplicity greater than 3 produces curves of genus greater than 1. We also have the 3^{d-1} rational curves covering the curve Γ. Thus we obtain:

$$t_2 + 3t_3 + (3e-7)d + 18 \geq 9\left(e + \frac{1}{e}\right) + 4\sum_{k\geq 5}(k-4)t_k.$$

This inequality is implied by Proposition 14.2. To see this, divide the previous expression by 4 and treat the cases $e = 2$ and $e > 2$ separately.

For $n \geq 4$ we have that blowing up multiple points will not result in obtaining any rational or elliptical curves, so the only correction to inequality $f(n) \geq 0$ we can make is given by the rational curves covering the curve Γ. This means that for $n \geq 4$ what we have is: $f(n) - n^2(2 + e + 1/e) \geq 0$.

Lemma 14.3. *For $n \geq 2$, Proposition 14.2 implies $f(n) - n^2(2 + e + 1/e) \geq 0$.*

Proof. The cases $n = 2, 3$ have already been seen to be obtained from Proposition 14.2. So let $n \geq 4$. Now $f(n) - n^2(2 + e + 1/e) \geq 0$ can be written as:

$$\begin{aligned} (n-1)^2 t_2 + (n^2 - 4n + 7)t_3 + (n-3)^2 t_4 + d(n(n-2)(e-2) - 1) + (n+1)^2 \textstyle\sum_{k\geq 5} t_k \\ \geq n^2 \left(e + \tfrac{1}{e} - 2\right) 2(n-1) \textstyle\sum_{k\geq 5} k t_k \end{aligned}.$$

Notice that $(n^2 - 2n)(e-2) - 1 > 0$ if $n, e > 3$. If $e = 2$, then it gives -1. Now write Proposition 14.2 as:

$$t_2 + \frac{3}{4}t_3 + 9\sum_{k\geq 5} t_k \geq 4\left(e + \frac{1}{e} - 2\right) + d + 2\sum_{k\geq 5} k t_k.$$

Multiply it by $(n^2 - 4n + 7)$. Now note that if $n \geq 3$, then $(n-1)^2 \geq (n^2 - 4n + 7)$, and that $4(n^2 - 4n + 7) > n^2$. So now the Lemma is proved by direct comparison. □

15 Conclusions

15.1 Brief Summary of Results From Part II

The results that we covered in Part II for $\overline{S}(\mathbb{P}^2, \Lambda)$ can be summarized as follows. Let Λ be a line arrangement on $\mathbb{P}^2(\mathbb{K})$. Then:

(a) For $\mathbb{K} = \mathbb{C}$ we know that $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) \in \left[\frac{4}{3}, \frac{8}{3}\right]$, the smalles limit point of $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda)$ is 2, and there is density in $[2, 2.5]$. There is exactly one line arrangement with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = \frac{8}{3}$ (the dual Hesse arrangement), an infinite amount with $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) = 2.5$ and an infinite amount with $\overline{S}(\mathbb{P}^2\mathbb{C}, \Lambda) = 2$. These two infinite families can be realized over $\mathbb{R}$.

(b) For $\mathbb{K} = \mathbb{R}$ we know that $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) \in [\frac{4}{3}, 2.5]$, the smallest limit point is 2 and there is density in $[2, 2.5]$. There are infinite real line arrangements with $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) = 2$ and with $\overline{S}(\mathbb{P}^2(\mathbb{R}), \Lambda) = 2.5$.

(c) For $\mathbb{K} = \overline{\mathbb{F}_p}$ we know that $\overline{S}(\mathbb{P}^2(\overline{\mathbb{F}_p}), \Lambda) \in [1, 3]$, the smallest limit point being 2 and the biggest limit point being 3. The log Chern slopes are dense in $[2, 3]$.

15.2 Brief Summary of Results From Part III

To summarize what we have done in Part III, let Λ be a section arrangement on Σ_e. Then:

(a) $\frac{4}{3} \leq \overline{S}(\Sigma_e, \Lambda) < \frac{8}{3}$.

(b) Line arrangements in $\mathbb{P}^2$ induce section arrangements on each Σ_e.

(c) The Ryser designs produced by finite projective planes can be realized as section arrangements on Hirzebruch surfaces on positive characteristic.

(d) The smallest limit point of the log Chern slopes on Σ_e is 2.

(e) There is density of the log Chern slopes on $[2, 2.5]$.

If one takes into account section arrangements over all Σ_e, then we have:

(a) All numbers of the form $(2d-4)/(d-1)$ are obtained as the limit of log Chern slopes of section arrangements over the family Σ_e.

(b) The biggest limit point that can be obtained from the family of real Hirzebruch surfaces is 2.5.

(c) The biggest limit point from the family of Hirzebruch surfaces on positive characteristic is 3.

15.3 Open Questions For Future Work

Parts II and III have left us with some open questions, many of which can be the subject of future work in this area.

- The most interesting question about the log Chern slope of line arrangement in the complex projective plane is: what happens in the interval $[2.5, 2.\overline{6}]$? This question is interesting because it speaks about purely complex line arrangements. Like we explained, this questions can be seen from two perspectives. First, one can approach it from the perspective of finite reflection groups, and here the question translates to: given a line arrangement with log Chern slope bigger than 2.5, is the reflection group generated by those lines finite? On the other hand, we saw that this question was strongly related to the existence of a good upper bound on t_4, which is something that was eluded by Hirzebruch-Sakai inequality.
- If Λ is a purely complex line arrangement, is it true that $\overline{S}(\mathbb{P}^2(\mathbb{C}), \Lambda) > 2.5$?
- We know from looking at the Ceva arrangement, that complex line arrangements need not have double points. On the other hand, real line arrangements need to have at least three points. So naturally one is inclined to ask: what is the minimum number of double points a complex line arrangement can have? Can it have just one double point?

- A different question about a line arrangements, but related to the existence of an upper bound on t_4, is the one about the existence of $(4, p)$-nets on the complex projective plane, different from the dual Hesse arrangement.

- For the case of section arrangements on complex Hirzebruch surfaces, the main question about the log Chern slopes is: are there section arrangements with log Chern slope bigger than or equal to 2.5? Again the problem stems from a poor understanding on the behaviour of t_4.

- For the case of Hirzebruch surfaces defined over fields of positive characteristic we have the highly combinatorial problem: what are all the section arrangements on Hirzebruch surfaces that satisfy $f_0 = d$? These section arrangements are interesting, as they tend to have very large log Chern slopes.

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