GEOMETRY OF HILBERT AND QUOT SCHEMES OF POINTS ON SMOOTH CURVES AND SMOOTH SURFACES

por

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Introduction

In this work we present a series of known results about the study of geometric and topological properties of the punctual Hilbert schemes and punctual Quot schemes. Furthermore in the case of the punctual Quot schemes we improve some results given by G. Ellingsrud and M. Lenh in [EL99] about smoothness, irreducibility and dimension of this kind of spaces. Following the techniques presented in [ES87] by G.Ellingsrud and S. Stromme, we gave a new formula to compute the Euler characteristic of some Quot schemes, see Theorem 3.8, which is a generalization of 2.12. Finally we introduce the enough theory about virtual classes to calculate as in [Sch12] the virtual Euler characteristic for a particular Quot scheme.

The thesis is divided in three parts. First background, then study of punctual Hilbert and Quot schemes and finally study of virtual classes to compute the final example.

The moduli problems can be classified in three standard types, such as Hartshorne says in [Har09]. These are: A) Subschemes of a fixed schemes X; B) Line bundle on a fixed scheme X, and C) Coherent sheaves, on a fixed scheme X. The moduli spaces that we study here are of the type A and C. Naturally the Hilbert schemes are of type A since they parametrize closed subschemes of a given scheme X. The existence of these schemes was presented originally by Grothendieck in[Gro60]. This proof was improved by Mumford [Fan05]. It was based in the notions of $k-$regular sheaf and Mumford-Casltenuovo's Theorem 1.42. Here we present Mumford's version following Stromme, [Str96].

The Quot schemes are a natural generalization of the Hilbert schemes, by its definition (see.1.8) they belong to type C of modulli spaces. The general study of these spaces is not easy, but using toric actions over them we can get results in some particular cases, thanks to Bialynicki-Birula's theorem presented in [BB73a]. Other techniques used to describe tangent spaces proceed from basic elementary deformation theory.

The simplest moduli spaces are the $n-$punctual Hilbert schemes of a given scheme X, denoted by $X^{[n]}$. This schemes parametrized $0-$dimensional subschemes of length n. When X is a projective and smooth curve C it is not difficult to see that $C^{[n]} = \mathrm{Sym}^n(C)$, so the properties as smoothness, irreducbility and dimension are completely determined. The next step is the understanding of these schemes, where $X = S$ is any smooth and projective surface. Fogarty in [Fog68] shows that $S^{[n]}$ is smooth, projective and irreducible with dimension $2n$. On the other hand G.Ellingsrud and S.Stromme in [ES87] find a formula to compute the Betty's numbers and the Euler characteristic of the Hilbert scheme $(\mathbb{P}^2)^{[n]}$ based on Byalinicki-Birula's theorem, along with the decomposition of the tangent space presented in 2.16 of a similar way to Nakayima in [Nak99, cap.V, Proposition 5.7].

The Quot schemes are more complicated and so we have to give various restrictions to work. Here we present these in 3.1 and we denoted by $M_{(S,\mathcal{E})}(n, q, d)$, where S is a smooth and projective surface and $\mathcal{E}$ is a coherent sheaf over S.

for schemes of the type $M_{(S,\mathcal{E})}(n, 0, d)$ G.Ellingsrud and M.Lenh on [EL99] find its dimensions $d(n+1)$ and they prove its irreducibility [EL99, Proposition 5]. In this thesis we present a generalization of this theorem in corollary3.6 where we prove in general that $M_{(S,\mathcal{E})}(n, q, d)$ is irreducible with dimension $(d + q)(n - q) + d$. We also prove that $M_{(\mathbb{P}^2,\mathcal{O})}(n, n-1, d)$ is smooth and show that in general these spaces aren't smooth. For example we show that $M_{(\mathbb{P}^2,\mathcal{O})}(2, 0, 2)$ is singular. For general spaces $M_{(\mathbb{P}^2,\mathcal{O})}(n, q, d)$ we find a formula to compute its Euler characteristic 3.8, that's not as clean as the formula for Hilbert schemes but is computable.

Finally, we present the Atiyah-Bott's classic and virtual formulas 4.10 and4.24, respectively [GP99]. For that was necessary give a short introduction about virtual classes. With Atiyah-Bott's formulas we compute the Euler virtual characteristic of the scheme $M(3, 2, 2)$. To see more of these kinds of examples the reader can consult the work done by D.Schulthesis in his doctoral thesis [Sch12]. For that computations we again use the decomposition of the tangent space, and is necessary do many small computations about Chern classes and use the $Grothendieck - Hirezebruch - Riemann - Roch'$ theorem. This shows that, in general, the use of the theory is not easy, but it can be used to compute some invariants in enumerative geometry such, as present Andre L. Meireles And Israel Vainsencher in [MV01].

1 Background

1.1 Hilbert Polynomials

Definition 1.1. A polynomial $P(z) \in \mathbb{Q}[z]$ is called numerical polynomial if , $P(n) \in \mathbb{Z}$ for all $n \gg 0$

The next proposition give a characterization of these kind of function.

Proposition 1.2. *1. If $P \in \mathbb{Q}[z]$ is a numerical polynomial, then there are integers $c_0, c_1, \cdots, c_r$,such that*

$$P(z) = c_0 \binom{z}{n} + c_1 \binom{z}{n-1} + \cdots + c_n,$$

where

$$\binom{z}{n} = \frac{1}{n!} z(z-1)\cdots(z-n+1)$$

2. If $f : \mathbb{Z} \to \mathbb{Z}$ is any function, and if there exists a numerical polynomial $q(z)$ such that the difference function $\Delta f = f(n+1) - f(n)$ is equal to $q(n)$ for all $n \gg 0$, then there exists a numerical polynomial $P(z)$ such that for all $f(n) = P(n)$, $n \gg 0$.

Proof. See [Har77, Ch.1,sec 7, pag 49] □

Let k be a field, and let M be a graded module over the polynomial ring $k[x_0, \ldots, x_n]$, we can define the function, $\varphi_M(l) = \dim_k M_l$, where M_l denotes the homogeneous part of M of degree l.

Example 1.3. Let $M = \mathbb{C}[x, y]/\langle xy - 1\rangle$, with the grading induces by the canonical grading in $\mathbb{C}[x, y]$. So $M_l = \mathbb{C}x^l \oplus \mathbb{C}y^l$, then $\varphi_M(l) = 2$, for any l.

Theorem 1.4 (Hilbert-Serre)**.** *Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]-$module. Then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore, $\deg P_M(z) = \dim (Z(\mathrm{Ann}\,(M)))$, where Z denotes the zero set in $\mathbb{P}^n$ of a homogeneous ideal.*

Sketch of proof. By the Proposition 7.4 on [Har77] we reduce to the case $M \cong (S/\mathfrak{p})$ where $\mathfrak{p}$ is a homogeneous prime ideal of S. If $\mathfrak{p} = (x_0, \ldots, x_n)$ there is nothing to do. Now if $\mathfrak{p} \neq (x_0, \ldots, x_n)$, there exists $x_i \notin \mathfrak{p}$ for some i. Then we consider the exact sequence

$$0 \to M \xrightarrow{x_i} M \to M'' \to 0,$$

where $M'' = M/x_iM$, so $\varphi_{M''}(l) = \varphi_M(l) - \varphi_M(l-1) = \Delta\varphi_M(l-1)$ and $Z(\mathrm{Ann}(M'')) = Z(\mathfrak{p}) \cap H$ where $H = \{x_i = 0\}$. Then $\dim(Z(\mathrm{Ann}(M''))) = \dim(Z(\mathfrak{p})) - 1$. Then by the Proposition 1.2 if $\varphi_{M''}$ is a polynomial function there exists a numerical polynomial P_M such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$ and $\deg(P_M) = \dim(Z(\mathfrak{p}))$. □

Definition 1.5. The polynomial given by last theorem is called *the Hilbert Polynomial of M.*

We know that for any subscheme Y of the projective space $\mathbb{P}^n$, we can assign a homogeneous ring $S(Y)$ the ring of coordinates, and this ring has an unique Hilbert polynomial by 1.4, then we can assign to Y the polynomial $P_Y = P_M$ of M, which is to be called the the Hilbert Polynomial of Y.

Example 1.6. Let $Y = \mathbb{P}^n_k$. Then the coordinate ring is $M = k[x_0, \ldots, x_n]$, so $\varphi_M(l) = \dim_k(k[x_0, \ldots, x_n])_l = \binom{l+n}{n} = P_Y(l)$.

1.2 Flat Morphisms.

The notion of flatness allows us to algebraically define a " continuous variation of a fibers ". This is important for giving the right definition of a family in algebraic geometry. Thanks to Theorem 1.16 we can decompose the Hilbert functor as a coproduct of functor indexed by Hilbert Polynomials. The notion of flat morphisms is locally given by the notion of flat modules. Here we present some theorems without proof but the reader can be find complete information in [Har77, , chapter III, section 9.]

Definition 1.7. Let A be a ring, and let M be an $A-$module, M is said to be *flat* if and only if for every finitely generated ideal $\mathfrak{a}$ of A, the map $\mathfrak{a} \otimes_A M \to M$ is injective, equivalently if the functor $\underline{(\ )} \otimes_A M$ is an exact functor. See [Eis13].

Proposition 1.8. 1. *Base extension: If M is a flat $A-$module, and $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat $B-$module.*

2. *Transitivity: If B is a flat A-algebra, and N is a flat B-module, then N is also flat as an A-module.*

3. *Localization: M is flat over A if and only if for all p prime ideal of A the localization M_p is flat over A_p.*

4. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are both flat then M is flat; if M and M'' are both flat, then M' is flat.*

5. *A finitely generated module M over a local noetherian ring A is flat if and only if is free.*

The last algebraic statement makes sense immediately with the following definition and proposition.

Definition 1.9. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *flat* over Y at point $x \in X$, if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x),Y}-$module. Consider $\mathcal{F}_x$ as an $\mathcal{O}_{f(x),Y}-$module via the map $f^{\#}\colon \mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$, we say that $\mathcal{F}$ is *flat* if it is flat for every point $x \in X$, and we say X is flat over Y if $\mathcal{O}_X$ is.

Proposition 1.10. 1. *An open inmmersion is flat.*

2. *Base change: let $f\colon X \to Y$ be a morphism, let $\mathcal{F}$ be an $\mathcal{O}_X-$module which is flat over Y, and let $g\colon Y' \to Y$ be any morphism. Let $X' = X \times_Y Y'$, and $f'\colon X' \to Y'$ be the second projection, and $\mathcal{F}' = p_1^*(\mathcal{F})$. Then $\mathcal{F}'$ is flat over Y'.*

3. *Transitivity: let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphism. Let $\mathcal{F}$ be an $\mathcal{O}_X-$module which is flat over Y, and assume also that Y is flat over Z. Then $\mathcal{F}$is flat over Z.*

4. *Let $A \to B$ be a ring homomorphism, and let M be a B-module. Let $f\colon X = \mathrm{Spec(B)} \to Y = \mathrm{Spec(A)}$ be the corresponding morphism of affine schemes, and let $\mathcal{F} = \tilde{M}$. Then $\mathcal{F}$ is flat over Y if and only if M is flat over A.*

5. *Let X be a noetherian scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat over X if and only if it is locally free.*

Proof. Use 1.8. ☐

Proposition 1.11. *Let $f\colon X \to Y$ be a separated morphism of finite type of noetherian schemes, $\mathcal{F}$ a quasi-coherent sheaf on X, and $u\colon Y' \to Y$ a flat morphim of noetherian schemes, such that the following diagram commutes:*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

Then for all $i \geq 0$ there are natural isomorphims

$$u^* R^i f_*(\mathcal{F}) \cong R^i g_*(v^* \mathcal{F}).$$

Corollary 1.12. *Let $f\colon X \to Y$ and $\mathcal{F}$ be as 1.11, and assume Y affine. For any point $y \in Y$, let X_y be the fiber over y, and $\mathcal{F}_y$ the induced sheaf. On the other hand, let $k(y)$ denote the constant sheaf $k(y)$ on the closed subset $\overline{\{y\}}$ of Y. Then for all $i \geq 0$ there are natural isomorphisms*

$$H^i(X_y, \mathcal{F}_y) \cong H^i(X, \mathcal{F} \otimes k(y)).$$

Flat families

A *family* X over Y is a morphism of schemes (varieties). As is customary given any element $y \in Y$ the fiber (pre-image) of y is denoted by X_y as above. We say that the family $X \xrightarrow{f} Y$ is flat if the morphism f is flat.

Proposition 1.13. *Let $f\colon X \to Y$ be a flat morphism of schemes of finite type over a field k. For any point $x \in X$, let $y = f(x)$. Then*

$$\dim_x(X_y) = \dim_x(X) - \dim_y(Y).$$

Here for any scheme X and any point $x \in X$. We denote by $\dim_x(X)$ the dimension of the local ring $\mathcal{O}_{x,X}$.

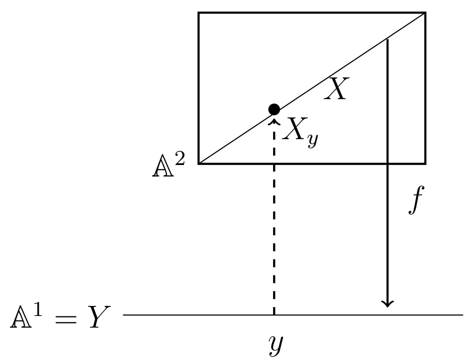
Corollary 1.14. *Let $f\colon X \to Y$ be a flat morphism of schemes of finite type over a field k, and assume that Y is irreducible. Then the followings conditions are equivalent:*

1. *every irreducible component of X has dimension $\dim(Y) + n$;*

2. *for any point $y \in Y$ (closed or not), every irreducible component of the fiber X_y has dimension n.*

Example 1.15.

1. An easy example is given by $X = \operatorname{Spec}(\mathbb{C}[x,y]/(x-y)) \xrightarrow{f} \operatorname{Spec}(\mathbb{C}[y]) = Y$, where f is induced by the natural map from $\mathbb{C}[y]$ to $\mathbb{C}[x,y]/(x-y)$, the fiber in any point of Y is a point on X. (See figure 1).



(1)

2. Let $X = \operatorname{Spec}(\mathbb{C}[x,y,t]/(xy-t)) \xrightarrow{f} \operatorname{Spec}(\mathbb{C}[t])$, and f the induced map by $\mathbb{C}[t] \to \mathbb{C}[x,y,t]/(xy-t)$. It is a flat family although the fiber X_0 is singular.

3. (Non example) Let $X = \mathbb{P}^2$ and let $\tilde{X} = \operatorname{Bl}_x(\mathbb{P}^2)$ be the bow-up of $\mathbb{P}^2$ at point x. The family $\tilde{X} \to X$ is not a flat family. Because the dimension of the exceptional divisor (curve) is one and for any other point $p \in \mathbb{P}^2$ the dimension of $\tilde{X}_p = 0$ (point), then Proposition 1.13, does not hold.

Finally we present the most important theorem of flatness for construction of the moduli spaces presented in this work.

Theorem 1.16. *Let T be an integral noetherian scheme. And $X \subset \mathbb{P}^n_T$ a closed subscheme. For each point $t \in T$, we consider the Hilbert polynomial $P_t \in \mathbb{Q}[z]$ of the fiber X_t considered as subscheme of $\mathbb{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

1.3 Representable Functors

In this section we present the necessary theory about representable functors to define and prove the existence of some moduli spaces, e.g. Hilbert schemes. For more information about these topics see [ML78] , [Str96] and [GW10].

Definition 1.17. Let $\mathbf{D}$ be a category and denote by Set the category of sets as is usual. A functor $H\colon \mathbf{D} \to \mathbf{Set}$ is said to be *representable* if there exist an object $d \in \mathbf{D}$, such that the functor of points $h_d(-) = \mathrm{Hom}_{\mathbf{D}}(-, d)$ is naturally isomorphic to F. d is called *the representing object* of F.

Definition 1.18. Now suppose $H\colon \mathbf{D} \to \mathbf{Set}$ is a representable functor, and let ϕ be the natural isomorphism between H and h_d, and let ϕ_d be the isomorphism between $h_d(d)$ and $H(d)$. Then we write by ξ the image of the identity map 1_d via ϕ_d, this element is called the *universal family.*

Example 1.19. (co-representable) Let **Top** be the category of topological spaces and continuous functions. Define $H((X, \tau)) = X$ to be the forgetful functor. Then the punctual space $\{x\}$ is a representing object of H, since $h_{\{x\}}(X) = \{\text{continuous functions } X \to \{x\}\} \cong X$, and the universal family is x.

Example 1.20 (Geometric example)**.** Let $\mathbf{sch}_k$ be the category of k-schemes. Define the global section functor by sending any k-scheme S to $\Gamma(S, \mathcal{O}_S)$. This functor is represented by $\mathbb{A}^1_k$, and this can be checked locally: for any unitary commutative$k-$algebra R the set of $k-$algebras homomorphism $\phi : k[x] \to R$ is isomorphic to R.

Henceforth we will work on the category $\mathbf{D}$ of S- schemes, denoted by $\mathbf{sch}_S$, and these kind of functors have as codomain the category of sets. We can define the concept of subfunctor using topological properties of the category $\mathbf{sch}_S$. So we may use notions of open and closed subfunctor, open coverings, closed covering of a given functor by a subfunctor and Zariski functors, which will be given below.

Definition 1.21. Let $F, H\colon \mathbf{sch}_S \to \mathbf{Set}$. We say that F is a subfunctor of H if for every $T \in \mathbf{sch}_S$ $F(T) \subseteq H(T)$ and given $t\colon R \to T$, the map $F(t)\colon F(T) \to F(R)$ is the restriction of $H(t)$. F is said to be a *closed subfunctor* (*resp.open*) of H if for any $T \to S$ and $\xi \in H(T)$, there exist a closed subscheme (*resp.open*), $U_\xi^F \subseteq T$, such that for any $f\colon R \to T$, we have

$$H(f)(\xi) = f^*\xi \in H(R) \text{ belongs to } F(R) \iff f \text{ factors through } U_\xi^F \subseteq T.$$

Proposition 1.22. *Consider the next diagram:*

$$\begin{array}{ccc} h_T \times_H F & \xrightarrow{\pi_1} & F \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle i} \\ h_T & \xrightarrow[\phi_\xi]{} & H \end{array}$$

Where $\xi \in H(T)$ and ϕ_ξ is given by sending any $f : R \to T$ to $f^*\xi$. Then F is a closed subfunctor of H if and only if the functor $h_T \times_H F$ is represented by a closed subscheme of T. Moreover if H is representable and F is a closed subfunctor, then F is represented by a closed subcheme of the scheme representing H.

Proof. Suppose F is a closed subscheme of H, and let $\xi \in H(T)$, then there exist U_ξ^F closed subscheme of T with the properties given in 1.21. Now let R be any S-scheme, then

$$h_T(R) \times_{H(R)} F(R) = \{(\phi : R \to T, x) | \phi^*\xi = x \in F(R)\} \stackrel{1-1}{\longleftrightarrow} \{f\colon R \to U_\xi^F\} = h_{U_\xi^F}(R).$$

Now if H is represented by T, let ξ be the universal family, then for every s-scheme R

$$F(R) \stackrel{1-1}{\longleftrightarrow} \{f\colon R \to U_\xi^F\} = h_{U_\xi^F}(R),$$

therefore U_ξ^F represents F, which proves the second part. □

Definition 1.23. Let $H\colon \mathbf{sch}_S \to \mathbf{Set}$ be a functor, this is called a *Zariski functor* if for any scheme T and any open covering $\{T_\alpha\}_\alpha$ of T, the sequence

$$0 \to H(T) \xrightarrow{f} \prod_{\alpha} H(T_\alpha) \underset{g_1}{\overset{g_2}{\rightrightarrows}} \prod_{\alpha,\beta} H(T_\alpha \cup T_\beta)$$

is exact; i.e. f is inyective and $\mathrm{Im}(f) = \{x | g_1(x) = g_2(x)\}$.

Definition 1.24. Let $H\colon \mathbf{sch}_S \to \mathbf{Set}$ be a functor and let $\{F_i\}$ be a collection of open subfunctors of H. This collection is an *open covering* if for all $T \to S$ and $\xi \in H(T)$, the collection $U_\xi^{F_i}$ is an open covering of T.

Proposition 1.25 (Zariski representable)**.** *For any Zariski functor $H\colon \mathbf{sch}_S \to \mathbf{Set}$, with open covering $\{H_i\}$, where every H_i is representable, then H is representable.*

Proof. Let H be a Zariski functor with $\{H_i\}$ an open covering by representable functors, and let X_i be the scheme that represent to H_i. The functor $H_i \times_H H_j$ is a subscheme of H_i and H_j, in fact For any $T \to S$, $H_i(T) \times_{H(T)} H_j(T) = H_i(T) \cap H_j(T)$. Moreover, let $f\colon R \to T$ and $\xi \in H_i(T)$, then $f^*\xi \in H_i(T) \cap H_j(T)$ if and only if f factors through $U_\xi^{H_i} \cap U_\xi^{H_j}$. Then $H_i \times_H H_j$ is an open subfunctor of H_i and H_j, therefore $X_i \cap X_j$ is an open subscheme of X_i and X_j. so we can glue together to a scheme X. But for any $T \to S$ and $\xi \in H(T)$, the collection $\{U_\xi^{H_k}\}$ is an open covering of T, and it is easy to see that for all $k, H(U_\xi^{H_k}) \cong H_k(U_\xi^{H_k}) \cong h_{X_k}(U_\xi^{H_k})$. Finally since H and h_X are Zariski functors we get the next exact sequences:

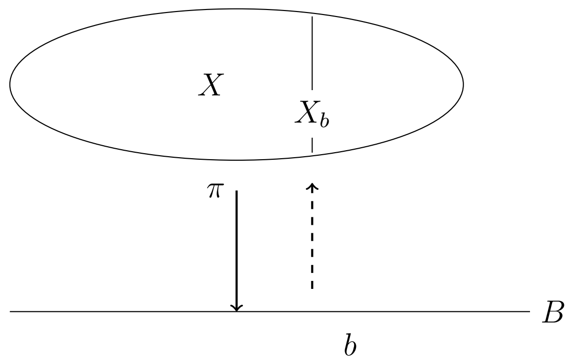
$$\begin{array}{ccccc}
H(T) & \longrightarrow & \prod_k H(U_\xi^{H_k}) & \longrightarrow & \prod_{i,j} H(U_\xi^{H_i} \cup U_\xi^{H_j}) \\
\downarrow & & \downarrow & & \downarrow \\
h_X(T) & \longrightarrow & \prod_k h_X(U_\xi^{H_k}) & \to & \prod_{i,j} h_x(U_\xi^{H_i} \cup U_\xi^{H_j})
\end{array}$$

So $H(T) \to h_X(T)$ is an isomorphism. ∎

1.4 Moduli spaces

A moduli problem is a problem of classification of some kind of objects (schemes) modulo some equivalence relation between these objects.

Given any base B, a *family of objects* over B is a pair (X, π) where π is a morphism from X to B, $X \xrightarrow{\pi} B$, such that for all b in B, the fiber $\pi^{-1}(b) = X_b$ is an object of the type we are classifying.



A *moduli space* for a moduli problem is an scheme (in general some space) M such that for all elements m of M there exists a unique element corresponding to the type that we are classifying.

Suppose that M is a moduli space for some moduli problem, we say that M is a *fine moduli space* if there is a universal family ξ over M, i.e, exists a morphism $\xi \xrightarrow{\pi} M$ such that any other family over a scheme B is obtained, up equivalence, pulling back ξ by unique morphism $\phi\colon B \to M$.

These terminology can be formalized using category theory as follow.

Let $F\colon \mathbf{sch} \to \mathbf{Set}$ be a contravariant functor, and given any $X \in \mathbf{sch}$ the image via $F(X)$ is the set of equivalence class of families (these families generally are flat families) over X.

Definition 1.26. Let $F\colon \mathbf{sch} \to \mathbf{Set}$ be a contravariant functor. we say that F is a *fine moduli functor* if F is a representable functor by an scheme M, and the scheme M is called the *moduli space* associated to F.

Remark 1.27. By representability we know that there exist a natural isomorphism which means $\phi : \hom(\bullet, M) \to F(\bullet)$, it say that for any morphism $T \xrightarrow{f} T'$ of schemes the following diagram commute

$$\begin{array}{ccc} \mathrm{Hom}(T', M) & \xrightarrow{\phi_{T'}} & F(T') \\ \Big\downarrow \mathrm{Hom}(f, M) & & \Big\downarrow F(f) \\ \mathrm{Hom}(T, M) & \xrightarrow{\phi_T} & F(T), \end{array}$$

and $\phi_{T'}, \phi_T$ are isomorphisms.

In particular if there exists some morphism $T \xrightarrow{f} M$ we have the diagram

$$\begin{array}{ccc} \mathrm{Hom}(M, M) & \xrightarrow{\phi_{M}} & F(M) \\ \Big\downarrow \mathrm{Hom}(f, M) & & \Big\downarrow F(f) \\ \mathrm{Hom}(T, M) & \xrightarrow{\phi_T} & F(T), \end{array}$$

Let $1_M : M \to M$ be the identity map of M, we denote by $\xi = \phi_M(1_M) \in F(M)$, this element is called the *Universal family* of F, this because $\phi_T(\mathrm{Hom}(f, M)(1_M)) = F(f)(\xi) = f^*\xi$, so $f = \phi_T^{-1}(f^*\xi)$, then any family $T \xrightarrow{f} M$ can be recovered as a pullback of ξ.

In this work we are interested in three classical moduli spaces which are the *Grasmman scheme*, the *Hilbert scheme* and the *Quot scheme.* Here we prove the existence of these spaces and in the next chapter we show some properties of Hilbert and Quot schemes.

1.5 Grasmman Schemes

The Grassmanian scheme is the generalization of the Grassmanian space $Gr_V(r, n)$ that parameterises the vector subspaces of dimension r for a given vector space V of dimension n. e.g. $Gr_{\mathbb{C}^n}(1, n) \cong \mathbb{P}^{n-1}_{\mathbb{C}}$.

Definition 1.28. Let S be a scheme and $\mathcal{E}$ a locally free sheaf on S. The functor

$$\mathrm{Grass}_S(r,\mathcal{E}) : \mathbf{sch}_S \to \mathbf{Set}$$

given by

$$T \mapsto \{\mathcal{V}|\mathcal{V} \subseteq \mathcal{E}_T \text{ is a subbundle of rank } r\}$$

where $\mathcal{E}_T$denotes the pull-back of $\mathcal{E}$ via $T \to S$, and for any map $T \xrightarrow{\phi} T'$,

$$\mathrm{Grass}_S(r,\mathcal{E})(\phi)\colon \ \mathrm{Grass}_S(r,\mathcal{E})(T') \to \mathrm{Grass}_S(r,\mathcal{E})(T)$$

$$. \qquad \mathcal{V}' \to \mathcal{V} := \mathcal{V}'_T = \phi^*(\mathcal{V})$$

is called *the r-Grassmannian funtor of $\mathcal{E}$ over S.*

We can reformulate the functor $\mathrm{Grass}_S(r,\mathcal{E})$ changing $\mathcal{V}$ by its quotients, i.e.

$$\mathrm{Grass}_S(r,\mathcal{E})(T) = \{[\mathcal{E}_T \xrightarrow{q} \mathcal{Q} \to 0]|\text{such that } \mathcal{Q} \text{ is a sheaf on } T \text{ with rank } rank(\mathcal{E}_T) - r.\}$$

Theorem 1.29. *The functor* $\mathrm{Grass}_S(r,\mathcal{E})$ *is represented by a projective S-scheme $\mathcal{G}rass_S(r,\mathcal{E})$ and a universal subbundle (quotient) $\mathcal{U} \subseteq \mathcal{E}_{Grass_S(r,\mathcal{E})}$ of rank r.*

Definition 1.30. For any $r \in \mathbb{N}$, locally free sheaf $\mathcal{E}$ on S , the scheme $\mathcal{G}rass_S(r,\mathcal{E})$ is called *the r-Grassmannian scheme* of $\mathcal{E}$ over S. When $\mathcal{E} = \mathcal{O}_S^n$ we write $\mathcal{G}rass_S(r,n)$.

We present the proof of a particular case of Theorem 1.29.

Proposition 1.31. *Let $S = \mathrm{Spec}(\mathbb{Z})$. Then the fucntor* $\mathrm{Grass}_{\mathbb{Z}}(r,n)$ *is represented by a projective scheme.*

Proof. The idea is to find an open covering of functor $\{H_i\}$ for $\mathrm{Grass}_{\mathbb{Z}}(r,n)$ and show that each of these functors is representable; to conclude we use the Theorem 1.25.

Let I be the set of subsets of cardinality $n-r$ of $\{1,2,\dots,n\}$. Denote by $e_j = (0,\dots,0,\underset{j\text{-th}}{1},\dots,\underset{n\text{-th}}{0})$ for $1 \le j \le n$ and $f_j = (0,\dots,0,\underset{j\text{-th}}{1},\dots,\underset{n-r\text{-th}}{0})$ for $1 \le j \le n-r$, the canonical global section for $\mathcal{O}_S^n$ and $\mathcal{O}_S^{n-r}$ respectively.

For any set $i = \{i_1 < i_2 < \dots < i_{n-r}\}$ define $s_i\colon \mathcal{O}^{n-r} \to \mathcal{O}^n$ by $s_i(f_j) = e_{i_j}$.

Now define $H_i(T) = \{q\colon \mathcal{O}^n \to \mathcal{Q} \in Grass(r,n) | q \circ s_i \text{is surjective } \} \subseteq Grass(r,n)$. By the right exactness of the pullback every H_i is a subfunctor of $Grass(r,n)$.

Suppose now that $i = \{1, 2, \ldots, n-r\}$, then the map s_i is the inclusion map on the first $n-r$-coordinates, so $q \circ s_i$ is an isomorphism for any $q\colon \mathcal{O}^{n-r} \to \mathcal{Q} \to 0$. Therefore we can think $q\colon \mathcal{O}^n \to \mathcal{O}^{n-r}$ such that $q(e_j) = f_j$, for any $1 \leq j \leq n-r$, and let $\{q(e_{n-r+k})\}_{k=1}^{r}$ the subset of $\Gamma(S, \mathcal{O}^{n-r})$ the set that finish to determine q. Then $H_i(S) \cong \prod_{j=nr+1}^{n} \Gamma(S, \mathcal{O}^{n-r})$. By the example 1.20 the global section functor is represented by $\mathbb{A}^1_{\mathbb{Z}}$, so H_i is represented by $\mathbb{A}^{r(n-r)}_{\mathbb{Z}}$. Finally $Grass(r,n)$ is represented by the $\binom{n}{r}$ coproducts of $\mathbb{A}^{r(n-r)}$ affine spaces. $\square$

Corollary 1.32. *For any $n \geq 1$, the scheme $\mathcal{G}rass(n, n+1) \cong \mathbb{P}^n_{\mathbb{Z}}$.*

1.6 Hilbert Schemes

Given any projective scheme X its Hilbert scheme parametrizes its closed subschemes. This scheme will be defined in a similar way to the Grassmannian scheme, i.e it will be defined as the object that represent some functor. The proof of its existence is nontrivial and here we show this using [Str96] as main reference.

Definition 1.33. Let X be a projective k-scheme, where k is any algebraically closed field. An algebraic family parametrized by T is a closed subscheme $Z \subseteq X \times_k T = X_T$. This family is called flat if the morphism $\iota \circ \pi_T : Z \to T$ is flat.

Definition 1.34. Let X be a projective k-scheme. The *Hilbert functor* of X is defined as follows:

$$\mathrm{Hilb}_{X/k}\colon \mathbf{sch}_k \to \mathbf{Set}$$

$$T \mapsto \{Z \subseteq X_T | Z \text{ is a flat family parametrized by } T\}$$

and given any morphism $\phi : T \to S$,

$$\mathrm{Hilb}_{X/k}(\phi) : \mathrm{Hilb}_{X/k}(S) \to \mathrm{Hilb}_{X/k}(T),$$

$$Z \mapsto Z' = (1_X \times \phi)^* Z.$$

The goal of this section is to show that the functor defined above is representable. With this in mind, we have the next definition:

Definition 1.35. Let X be a projective k-scheme, and $\mathrm{Hilb}_{X/k}$ its Hilbert functor, the k-scheme representing this functor is called the *Hilbert scheme* of X and is denoted by $\mathcal{H}ilb_{X/k}$.

From the flat properties of the subschemes Z of X_T, and the use of Theorem 1.16, the Hilbert scheme can be written as a disjoint union of subschemes, each of these indexed by numerical polynomials $P(z) \in \mathbb{Q}[z]$. In fact we define the next subfunctor of $\mathrm{Hilb}_{X/k}$.

Definition 1.36. Let $P(z) \in \mathbb{Q}[z]$ be a numerical polynomial. Define the functor $\mathrm{Hilb}^{P(z)}_{X/k}(T)$ given by the flat families $Z \subseteq X_T$ with Hilbert polynomial $P(z)$ in all geometric fibers.

Proposition 1.37. *For any numerical polynomial p, the functor* $\mathrm{Hilb}^{P(z)}_{X/k}$ *is a closed and open subfunctor of* $\mathrm{Hilb}_{X/k}$ *and*

$$\mathrm{Hilb}_{X/k} = \coprod_P \mathrm{Hilb}^{P(z)}_{X/k}.$$

Furthermore if every functor $\mathrm{Hilb}^{P(z)}_{X/k}$ *is a representable functor represented by the scheme* X_P *then* $\coprod_P X_P = \mathcal{H}ilb_{X/k}$.

Our objective is to prove that for every numerical polynomial $P(z) \in \mathbb{Q}[z]$, the functor $Hilb^{P(z)}_{X/k}$ is a representable functor.

Here we recall two important theorems of Serre. Their proof can be found in [Har77].

Theorem 1.38 (Serre 1)**.** *Let X be a projective scheme over a noetherian ring A, $\mathcal{O}(1)$ a very ample invertible sheaf on X, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

Theorem 1.39 (Serre 2)**.** *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample sheaf on X over* $\mathrm{Spec}(A)$. *Let $\mathcal{F}$ be a coherent sheaf on X .Then:*

- *for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module;*

- *there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.*

The following is the most important theorem of this section.

Theorem 1.40 (Grothendieck)**.** *Let X be a projective scheme over S and let $P \in \mathbb{Q}[z]$ be a numerical polynomial. Then $\mathrm{Hilb}^p_{X/S}$ is representable.*

Before giving a proof of this theorem we need some previous results; we give the same presentation of these results as in [Str96].

Boundesness.

Let k be a field, denote the projective n- space over k by $\mathbb{P}^n$.

Definition 1.41. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if $H^i(\mathcal{F}(m-i)) = 0$ for all $i > 0$.

Proposition 1.42 (Mumford-Castelnuovo)**.** *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}^n$. Then*

1. *$H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{F}(k+1))$ is surjective for $k \geq m$.*
2. *$H^i(\mathcal{F}(k)) = 0$ whenever $k + i \geq m$ and $i > 0$. Equivalently, $\mathcal{F}$ is n-regular for all $n \geq m$.*
3. *$\mathcal{F}(k)$ is generated by global sections if $k \geq m$.*

Proof. We use induction on n and prove at the same time (1) and (2). If $n = 0$ there is nothing to prove. Now suppose that for any $k \leq n$, (1) and (2) hold. Let $H \subseteq \mathbb{P}^n$ be a hyperplane. Then there exists an exact sequence

$$0 \to \mathcal{F}(k-1) \to \mathcal{F}(k) \to \mathcal{F}_H(k) \to 0.$$

Taking a long exact sequence of cohomology we get:

$$\cdots \to H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}_H(m-i)) \to H^{i+1}(\mathcal{F}(m-1-i)) \to \dots$$

by hypothesis the right and the left groups are zero, hence $\mathcal{F}_H$ is m-regular, so by induction, (1) and (2) are valid for $\mathcal{F}_H$. Consider the next sequence of cohomology

$$\cdots \to H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}(m-i+1)) \to H^{i+1}(\mathcal{F}_H(m-(i+1)))$$

If $i > 0$, by (2) for $\mathcal{F}_H$, the last group is zero, then $\mathcal{F}$ is $m+1$-regular and iterating the process we get (2) for $\mathcal{F}$. Now consider the exact sequences

$$0 \to \mathcal{F}(k-1) \to \mathcal{F}(k) \to \mathcal{F}_H(k) \to 0$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_H(1) \to 0.$$

Then taking long sequence of cohomology and tensoring we get the morphism

$$H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}) \stackrel{\sigma}{\to} H^0(\mathcal{F}_H(k)) \otimes H^0(\mathcal{O}_H(1))$$

which is surjective for $k \geq m$ since $H^1(\mathcal{F}(k-1)) = 0$. Consider the next diagram:

$$\begin{array}{ccc} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & \stackrel{\sigma}{\to} & H^0(\mathcal{F}_H(k)) \otimes H^0(\mathcal{O}_H(1)) \\ \downarrow \mu & & \downarrow \tau \\ H^0(\mathcal{F}(k+1)) & \xrightarrow{\nu} & H^0(\mathcal{F}_H(k+1)) \end{array}$$

τ and σ are surjetcive if $k \geq m$. Therefore $\nu \circ \mu$ is surjective. Since $\operatorname{Ker}(\nu) \subseteq \operatorname{im}(\mu)$, it follows that μ is surjective which prove (1) for $\mathcal{F}$.

For (3), we know by Theorem 1.39 that $\mathcal{F}(k)$ is generated by its global sections for all $k \gg 0$, but (1) says that these global sections can be expressed using global sections of $\mathcal{F}(m)$. □

We want to relate the regularity of the ideal sheaf associated to some closed subscheme Z of $\mathbb{P}^m$, to its numerical polynomial. The next proposition says that there is an integer number m_0 such that the ideal sheaf of Z is m_0-regular. This will be very useful for finding an embedding of the Hilbert scheme of $\mathbb{P}^n$ to some Grassmannian Scheme.

Proposition 1.43. *Let P be a numerical polynomial. Then there exist an integer $m_0 = m_0(P)$ (depending on P) such that for any closed subscheme $Z \subseteq \mathbb{P}^n$ with Hilbert polynomial P, the ideal sheaf $\mathcal{I}_Z$ is m_0-regular.*

Proof. We use induction on n. If $n = 0$ there is nothing to prove. Now suppose $n > 0$, and let H be an hyperplane, and consider the exact sequence

$$0 \to \mathcal{I}(-1) \to \mathcal{I} \to \mathcal{I}_H \to 0,$$

where $\mathcal{I} \subseteq \mathcal{O}_H$ is an ideal sheaf. By induction on $\mathcal{I}_H$, there is an integer $m_1(P) = m_1$, such that $\mathcal{I}_H$ is m_1-regular. If $i \geq 2$ we have the next sequence

$$\cdots \to H^{i-1}(\mathcal{I}_H((m_1-1)-(i-1))) \to H^i(\mathcal{I}(m_1-i-1)) \to H^i(\mathcal{I}(m_1-i)) \to H^i(\mathcal{I}_H(m_1-i)) \to \ldots$$

where $H^{i-1}(\mathcal{I}_H((m_1-1)-(i-1))) = H^i(\mathcal{I}_H(m_1-i)) = 0$. Then for all $k \geq m_1 - i$ and $i \geq 2$ we get that $H^i(\mathcal{I}(k-1)) \cong H^i(\mathcal{I}(k))$, so I is almost m_1-regular except posibly for the vanishing of $H^1(\mathcal{I})$, but we use the following lemma.

Lemma 1.44. *The sequence $\{\dim_k(H^1(\mathcal{I}(m)))\}_{m \geq m_1 - 1}$ decreases strictly to zero.*

Proof. We use the next following exact sequence if $m \geq m_1 - 1$

$$H^0(\mathcal{I}(m+1)) \overset{\rho_m}{\to} H^0(\mathcal{I}_H(m+1)) \to H^1(\mathcal{I}(m)) \to H^1(\mathcal{I}(m+1)) \to 0,$$

then $0 \leq h^1(\mathcal{I}(m+1)) \leq h^1(\mathcal{I}(m))$. If we suppose that for some natural number m we have $h^1(\mathcal{I}(m)) = h^1(\mathcal{I}(m+1))$, then ρ_m is suryective and using the following commutative diagraman.

$$\begin{array}{ccc} H^0(\mathcal{I}(m+1)) & \xrightarrow{\rho_m} & H^0(\mathcal{I}_H(m+1)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{I}(m+2)) & \xrightarrow{\rho_{m+1}} & H^0(\mathcal{I}_H(m+2)) \end{array}$$

We conclude that the morphism ρ_{m+1} is surjective. This implies that for all $k \geq 1$ we have $h^1(\mathcal{I}(m+1)) = h^1(\mathcal{I}(m+k))$ but by Theorem 1.39, these are all zero. □

The last lemma says that for all $k \geq m_1 - 1 + h^1(\mathcal{I}(m_1 - 1))$ the first homology $H^1(\mathcal{I}(k))$ is zero and so $\mathcal{I}$ is m_0-regular for all $m_0 \geq m_1 + h^1(\mathcal{I}(m_1 - 1))$. Now if we consider the exact sequence

$$0 \to \mathcal{I}(m_1 - 1) \to \mathcal{O}_Z(m_1 - 1) \to \mathcal{O}_Z(m_1 - 1)/\mathcal{I}(m_1 - 1) \to 0$$

and the large sequence of cohomology, we get that the morphism $H^0(\mathcal{O}_Z(m_1 - 1)) \to H^1(\mathcal{I}(m_1 - 1))$ is surjective, i.e, $h^0(\mathcal{O}_Z(m_1 - 1)) \geq h^1(\mathcal{I}(m_1 - 1))$.

Therefore $P(m_1 - 1) = \chi(\mathcal{O}_Z(m_1 - 1)) + \chi(\mathcal{I}(m_1 - 1)) \geq h^1(\mathcal{I}(m_1 - 1)) - 1$ and so $m_1 + P(m_1 - 1) \geq m_1 - 1 + h^1(\mathcal{I}(m_1 - 1))$, then $m_0 = m_1 + P(m_1 - 1)$ that is the integer wanted to find. □

Base Change.

In section 1.16 we talked about some properties of flat families. We showed that given the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{P}^n_T & \xrightarrow{h = 1_{\mathbb{P}^n} \times g} & \mathbb{P}^n_S \\ \downarrow p & & \downarrow q \\ T & \xrightarrow{g} & S \end{array} \qquad (2)$$

Base change diagram.

If $\mathcal{F}$ is coherent sheaf on $\mathbb{P}^n \times S$ by Theorem 1.11 there exist base change maps

$$b_i \colon g^* R^i p_* \mathcal{F} \to R^i q_* h^* \mathcal{F},$$

which are natural isomorphism if $\mathcal{F}$ is flat over S. But if we replace this for twists by a large integer m, $\mathcal{F}(m)$ the b_i with $i \geq 1$ are isomorphism. In fact by [Har77, cap.III, Theorem 8.8], the higher direct images are zero, so we only are interested in the case $i = 0$.

Proposition 1.45. *Let $T \xrightarrow{g} S$ be a morphism of noetherian schemes on the base change diagram. Suppose $\mathcal{F}$ is a coherent sheaf over $\mathbb{P}^n_S$ and consider the diagram as above. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, the base change map $b_0 : g^* p_* \mathcal{F}(m) \to q_* h^* \mathcal{F}(m)$ is an isomorphism.*

Proof. By the noetherian hypothesis it is possible cover S by finite affine open sets $U_i$and for any $g^{-1}(U_i)$ find a finite cover by affine open sets $V_{i,j}$, then is enough consider the case where S and T are affine.

We know that for any $i \in \mathbb{Z}$, the map $g^* q_{s_*} \mathcal{O}_{\mathbb{P}^n_S}(i) \to p_*(1 \times g)^* \mathcal{O}_{\mathbb{P}^n_S}(i) = p_* \mathcal{O}_{\mathbb{P}^n_T}(i)$, is an isomorphism, these maps are called base change maps.

Given $a, b \in \mathbb{Z}$ and $f : \mathcal{O}_{\mathbb{P}^n_S(a)} \to \mathcal{O}_{\mathbb{P}^n_S(b)}$ and denoting $f_T = (1 \times g)^* f$ the pull-back of f via $(1 \times g)$, we have the following commutative diagram:

$$\begin{array}{ccc} g^* \pi_{S_*} \mathcal{O}_{\mathbb{P}^S_n}(a+i) & \xrightarrow{g^* q_* f(i)} & g^* \pi_{S_*} \mathcal{O}_{\mathbb{P}^S_n}(b+i) \\ \downarrow 1 & & \downarrow 2 \\ \pi_{T_*} \mathcal{O}_{\mathbb{P}^T_n}(a+i) & \xrightarrow{p_* f_T(i)} & \pi_{T_*} \mathcal{O}_{\mathbb{P}^T_n}(b+i), \end{array}$$

where the maps 1 and 2 are the base change ismomoprhism. By the noetherian property of S and [Har77, cap.II, Corollary 5.18], exist some positive integers a, b, r_1, r_2 such that the following sequence is exact:

$$\mathcal{O}^{\oplus r_1}_{\mathbb{P}^n_S}(a) \xrightarrow{u} \mathcal{O}^{\oplus r_2}_{\mathbb{P}^n_S}(b) \xrightarrow{v} \mathcal{F} \to 0$$

and pulling-back by $(1 \times g)$ we obtain

$$\mathcal{O}^{\oplus r_1}_{\mathbb{P}^n_T}(a) \xrightarrow{u_T} \mathcal{O}^{\oplus r_2}_{\mathbb{P}^n_T}(b) \xrightarrow{v_T} \mathcal{F}_T \to 0,$$

Call $\mathcal{G} = \mathrm{Ker}(v)$ and $\mathcal{H} = \mathrm{Ker}(v_T)$, so for any $m \in \mathbb{Z}$, we get the following exact sequences:

$$q_* \mathcal{O}^{\oplus r_1}_{\mathbb{P}^n_S}(a+m) \xrightarrow{q_* u(m)} q_* \mathcal{O}^{\oplus r_2}_{\mathbb{P}^n_S}(b+m) \xrightarrow{q_* v(m)} q_* \mathcal{F} \to R^1 q_* \mathcal{G}(m) \to 0$$

and

$$p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_1}(a+m) \overset{p_*u_T(m)}{\longrightarrow} p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_2}(b+m) \overset{p_*v_T(m)}{\longrightarrow} p_*\mathcal{F}_T \to R^1p_*\mathcal{H}(m) \to 0,$$

where $R^1q_*\mathcal{G}(m)$ and $R^1p_*\mathcal{H}(m)$ denote the image of the first higher direct image functor of $q_*\mathcal{G}(m)$ and $p_*\mathcal{H}(m)$. Applying [Har77, cap.III, Theorem 8.8] , there exist $m_0 \in \mathbb{Z}$ such that $R^1q_*\mathcal{G}(m) = R^1p_*\mathcal{H}(m)$ for all $m \geq m_0$, then we have the following exact sequences:

$$q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_1}(a+m) \overset{q_*u(m)}{\longrightarrow} q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_2}(b+m) \overset{q_*v(m)}{\longrightarrow} q_*\mathcal{F} \to 0$$

and

$$p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_1}(a+m) \overset{p_*u_T(m)}{\longrightarrow} p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_2}(b+m) \overset{p_*v_T(m)}{\longrightarrow} p_*\mathcal{F}_T \to 0.$$

Now we pull-back these exact sequences by g and obtain:

$$g^*q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_1}(a+m) \overset{g^*q_*u(m)}{\longrightarrow} g^*q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_2}(b+m) \overset{g^*q_*v(m)}{\longrightarrow} g^*q_*\mathcal{F} \to 0$$

and

$$g^*p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_1}(a+m) \overset{g^*p_*u_T(m)}{\longrightarrow} g^*p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_2}(b+m) \overset{g^*p_*v_T(m)}{\longrightarrow} g^*p_*\mathcal{F}_T \to 0,$$

Finally connecting these exact sequences with base change maps and using the five lemma on the resulting diagram

$$\begin{array}{ccccccc} g^*q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_1}(a+m) & \longrightarrow & g^*q_*\mathcal{O}_{\mathbb{P}^n_S}^{\oplus r_2}(b+m) & \longrightarrow & g^*p_*\mathcal{F}_T & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ g^*p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_1}(a+m) & \longrightarrow & g^*p_*\mathcal{O}_{\mathbb{P}^n_T}^{\oplus r_2}(b+m) & \longrightarrow & g^*p_*\mathcal{F}_T & \longrightarrow & 0 \end{array}$$

we get that the third row is an isomorphism. □

In the next proposition we present a criterion for flatness if the base S on the base change diagram 2 is noetherian.

Proposition 1.46. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_S$ is flat if and only if there exist an m_0 such that $q_*\mathcal{F}(m)$ is locally free for all $m \geq m_0$.*

Proof. The first implication is given by 1.10 part 2. $q_*\mathcal{F}(m)$ is flat and then by part 5. of 1.10 again, this is locally free.

Conversely, Let $M_r = q_*\mathcal{F}(r)$ and denote by $M = \bigoplus_{r\geq m_0} M_r$. Then the sheaf $\mathcal{F}$ over $\mathbb{P}^n_S = \operatorname{Proj}_S \mathcal{O}_S[x_0,\dots,x_n]$ is isomorphic to $\tilde{M}$ since $\Gamma_*(\mathcal{O}_{\mathbb{P}^n_S}) = S$. Since by Hypothesis every M_r is flat then M it is. By 1.8 part 3. for any variable x_i the localization M_{x_i} is flat over $\mathcal{O}_S$. We can give a $\mathbb{Z}-$graduation on M_{x_i} such that for any $\theta = \dfrac{v_p}{x_i^q}$ its degree is p_q. On $M_{x_i} = \bigoplus_{r\geq m_0} M_{r,x_i}$ the part $(M_{x_i})_0$ of degree 0 is flat over $\mathcal{O}_S$. And we know that any affine piece $U_i = \operatorname{Spec}_S(S[\dfrac{x_0}{x_i},\dots,\dfrac{x_n}{x_i}])$ of $\mathbb{P}^n_S$ we have that $\Gamma(U_i) = \tilde{M}(U_i)$, then $\mathcal{F}|_{\mathbb{P}^n_{U_i}}$ is flat over U_i, therefore as $\{U_i\}$ form an open covering of $\mathbb{P}^n_S$ we get that $\mathcal{F}$ is flat over S. $\square$

Fitting ideals.

If $\mathcal{F}$ is a coherent sheaf on S there exist sheaves $\mathcal{E}_0, \mathcal{E}_1$ such that $\mathcal{F} \cong \mathcal{E}_0/\mathcal{E}_1$ where $\mathcal{E}_0, \mathcal{E}_1$ are locally free sheaves of finite rank e_0, e_1 respectuvely. Given any morphism $f\colon \mathcal{E}_1 \to \mathcal{E}_0$ and any $r \in \mathbb{Z}$ we define the *r-th* fitting ideal of f and more generally the *r-th* fitting ideal of $\mathcal{F}$ as follow.

Definition 1.47. The sheaves $\mathcal{E}_0, \mathcal{E}_1$ are called a *local presentation* of the sheaf $\mathcal{F}$ if $\mathcal{F} \cong \mathcal{E}_0/\mathcal{E}_1$.

Let r be an integer. The *r-th Fitting ideal* $F_r(f)$ is the image of the map

$$\wedge^{e_0-r}\mathcal{E}_1 \otimes \wedge^{e_0-r}\mathcal{E}_0^\vee \to \mathcal{O}_S,$$

induced by the map $\wedge^{e_0-r} f\colon \wedge^{e_0-r}\mathcal{E}_1 \to \wedge^{e_0-r}\mathcal{E}_0$. We agree that $F_r(f) = \mathcal{O}_S$ if $r \geq e_0$ and if $r < 0$ then $F_r(f) = 0$. If $\mathcal{F}$ is a coherent sheaf on S, we define the *r-th Fitting ideal* $F_r(\mathcal{F})$ of $\mathcal{F}$ to be the *r-th* Fitting ideal of a locally free presentation of $\mathcal{F}$.

Remark 1.48. The last definition is well defined. It says for any local presentation $\mathcal{E}_1 \xrightarrow{f} \mathcal{E}_0$ of $\mathcal{F}$ the *r-th* ideal $F_r(f)$ is the same. In fact, suppose f is any local presentation of $\mathcal{F}$, and as this is local, let $S = \operatorname{Spec}(A)$ where A is a local ring, and E_i free $A-$modules. Let

$g: A^n \to A^m$be a minimal presentation of $\mathcal{F}$. Then there exist a commutative diagram of $A-$modules:

$$\begin{array}{ccccccc}
E_1 & \xrightarrow{f} & E_0 & \longrightarrow & F & \longrightarrow & 0 \\
{\scriptstyle j}\uparrow & & {\scriptstyle i}\uparrow & & \| & & \\
A^n & \xrightarrow{g} & A^m & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

Where i and j are split monomorphisms. Then we have a monomorphism $\varphi : \wedge^{m-r} A^n \otimes \wedge^{m-r}(A^m)^\vee \to \wedge^{e_0-r}\mathcal{E}_1 \otimes \wedge^{e_0-r}\mathcal{E}_0^\vee$, then $\mathrm{Im}(g) = \mathrm{Im}(f \circ \varphi)$, but since φ is monomorphism $\mathrm{Im}(f) = \mathrm{Im}(f \circ \varphi)$ and therefore $F_r(g) = F_r(f)$.

Proposition 1.49. *Let $\mathcal{F}$ be a coherent sheaf on S, and let r be an integer. Then $\mathcal{F}$ is locally free of rank r if and only if $F_{r-1}(\mathcal{F}) = 0$ and $F_r(\mathcal{F}) = \mathcal{O}_s$.*

Proof. $\Rightarrow$] Clear.

$\Leftarrow$]Assume that $S = \mathrm{Spec(A)}$ for a local ring A. Let $f: A^n \to A^m$ be a local representation of $\mathcal{F}$. Let M_f the matrix of f. Since $F_r(\mathcal{F}) = A$, there exists an invertable minor of M_f of $(m-r) \times (m-r)$. For this invertible submatrix we obtain a new presentation of $\mathcal{F}$ say $g : A^{n-m+r} \to A^r$ but $F_{r-1}(\mathcal{F}) = 0$, so $g = 0$ and therefore $\mathcal{F} \cong A^r/A^{n-m+r} \cong A^r$. □

Corollary 1.50. *Let $\mathcal{F}$ be a coherent sheaf on S, and let r be an integer Let $\mathrm{S_r}(\mathcal{F})$ be the locally closed subscheme $\mathrm{V}(F_{r-1}(\mathcal{F})) - \mathrm{V}(F_r(\mathcal{F}))$ of S. Then for any morphism $g: T \to S$, the pullback $\mathcal{F}_T = g^*(\mathcal{F})$ is locally free of rank r if and only if g factors through the inclusion $S_r(\mathcal{F}) \subseteq S$.*

Proof. Apply 1.49 to the coherent sheaf $\mathcal{F}_T = g^*(\mathcal{F})$. □

Flattening stratification.

In the last part we see under which conditions a coherent sheaf $\mathcal{F}$ over S and a morphism $g: T \to S$ are such that the pullback sheaf $g^*(\mathcal{F})$ is locally free over T. Here we come back to the situation of Base change diagram, and want to know when a sheaf $\mathcal{F}$ over $\mathbb{P}^n_S$ not necessarily flat over S, is such that the pullback sheaf $(1 \times g)^*\mathcal{F}$ on $\mathbb{P}^n_T$ is flat over T.

A beautiful result of the section 1.54 says that for any coherent sheaf $\mathcal{F}$ over $\mathbb{P}^n_S$ there exists only a finite number of Hilbert polynomials for the various geometric fibers $\mathcal{F}_s$ for $s \in S$. Comparing this with theorem 1.16 says that we can find a finite disjoint descomposition of S, $\{S_i\}$ such that over any S_i the sheaf $\mathcal{F}$ is flat. In fact this will be proved using the concept of *flattening stratification* for a sheaf.

Remember the diagram in question

$$\begin{array}{ccc} \mathbb{P}^n_T & \xrightarrow{h = 1_{\mathbb{P}^n} \times g} & \mathbb{P}^n_S \\ \downarrow p & & \downarrow q \\ T & \xrightarrow{g} & S \end{array}$$

Base change diagram.

Definition 1.51. A *flattening stratification* for $\mathcal{F}$ over S depending of Base change diagram is a finite disjoint collection $\{S_i\}$ of locally closed subeschemes of S, such that $S = \bigcup_i S_i$ as a set, with the following property:

$$(1 \times g)^* \mathcal{F} \text{ is flat} \Leftrightarrow \text{ each } g^{-1}(S_i) \text{ is open and closed in } T.$$

the theorem we need the followings results:

Lemma 1.52. *Let $f : T \to S$ be a morphism of finite type of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a non-empty open set $U \subset S_{\text{red}}$ such that $\mathcal{F}_U$ is flat over U.*

Proposition 1.53 (Generic flatness)**.** *Let A an integral domain with field of fractions F, and let B be a finitely generated $A-$algebra contained in $F \otimes_A B$. Then for some nonzero elements a of A and b of B, the homomorphism $A_a \to B_b$ is flat.*

Proof. As $F \otimes_A B$ is finitely generated as $F-$algebra, by Noether normalization's lemma there exists elements $x_1, \dots, x_m \in F \otimes_A B$ such that $F[x_1, \dots, x_n]$ is a polynomial ring over

F and $F \otimes_A B$ is finite $F[x_i \dots, x_n]-$algebra. After multiplying each element x_i by an element of A, we may suppose that it lies in B. Let $b_1 \dots, b_n$ generated B as an $A-$algebra. Each b_i satisfies a monic polynomial equation with coefficients in $F[x_1, \dots, x_n]$. Let $a \in A$ be a common denominator for the coefficients of these polynomials. Then each b_i is integral over A_a. As the b_i generate as a A_a-algebra, this shows that B_a is a finite $A_a[x_1, \dots, x_n]-$algebra. Therefore, after replacing A with A_a and B with B_a, we may suppose that B is a finite $A[x_1, \dots, x_n]-$algebra.

$$\begin{array}{ccccc}
B & \xrightarrow{\text{injective}} & F \otimes_A B & \longrightarrow & E \otimes_{A[x_1,\dots,x_n]} B \\
\text{finite}\uparrow & & \text{finite}\uparrow & & \text{finite}\uparrow \\
A[x_1, \dots, x_n] & \longrightarrow & F[x_1, \dots, x_n] & \rightarrow & E \overset{def}{=} F(x_1, \dots, x_n) \\
\uparrow & & \uparrow & & \\
A & \longrightarrow & F & &
\end{array}$$

Let $E = F(x_1, \dots, x_n)$ be the field of fraction of $A[x_1, \dots, x_n]$, and let $b_1, \dots, b_r$ be elements of B that their form a basis for $E \otimes_{A[x_1,\dots,x_n]} B$ as a E vectorial space. Each element of B can be expressed as a linear combination of b_i with coefficients on E. Let q be a common denominator for the coefficient arising from a set of generators for B as an $A[x_1, \dots, x_n]-$module. Then $b_1, \dots, b_r$ generate B_q as an $A[x_1, \dots, x_n]_q-$module is equivalent to the fact that the map

$$A[x_1, \dots, x_n]_q^r \rightarrow B_q$$
$$(c_1, \dots, c_r) \mapsto \sum_{i=1}^{r} c_i b_i$$

is surjective. This map becomes is an isomorphism when tensored with E over $A[x_1, \dots, x_n]_q$, which implies that each element of its kernel that is killed by a nonzero element of $A[x_1, \dots, x_n]_q$ is zero. This because $A[x_1, \dots, x_n]_q$ is an integral domain. Hence the last map is an isomorphism, and so B_q is free of of finite rank over $A[x_1, \dots, x_n]_q$. Let a be a nonzero coefficient of the polynomial q, and consider the composition map

$$A_a \rightarrow A_a[x_1, \dots, x_n] \rightarrow A_a[x_1, \dots, x_n]_q \rightarrow B_{aq}$$

The first and third arrows realize their targets as nonzero free modules over their sources, and so are faithfully flat,and the middle is flat because is the canonical map of localization. Let $\mathfrak{m}$ be the maximal ideal of A_a. Then $\mathfrak{m}A_a[x_1,\dots,x_n]$ does not contain the polynomial q because the coefficient a of q is invertible in A_a. Hence $\mathfrak{m}A_a[x_1,\dots,x_n]_q$ is a propper ideal of $A_a[x_1,\dots,x_n]_q$, and so the map $A_a \to A_a[x_1\dots,x_n]_q$ is flat. □

Corollary 1.54. *There is a finite set of locally closed reduced subschemes Y_i of S such that their set-theoretic union is S and such that $\mathcal{F}_{Y_i}$ is flat over Y_i for all i. In particular, there is only a finite number of Hilbert polynomials for the various geometric fibers $\mathcal{F}_s$ for $s \in S$, and we may, if is necessary after collecting all Y_i with the same Hilbert polynomial in the fibers, index Y_i by Hilbert polynomilas and write Y_P instead.*

Theorem 1.55. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$. Then there exist a flattening stratification $\{S_P\}$ for $\mathcal{F}$, indexed by numerical polynomials P, such that for all $g\colon T \to S$, we have*

$$\mathcal{F}_T \textit{ is } T-\textit{flat with Hilbert polynomial } P \Leftrightarrow g \textit{ factors as } T \to S_P \to S.$$

Sketch of proof. In the case $n = 0$, say that $\mathcal{F}$ is a coherent sheaf on S and by 1.50 we know that the set $\{S_r(\mathcal{F})\}$ forms a flattening stratification. For the general case with $n \geq 1$, let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$ and $q : \mathbb{P}^n_S \to S$ the natural projection. By 1.54 there is only a finite numbers of locally closed subschemes of S, Y_p such that $\mathcal{F}_{Y_P}$ is flat over Y_P. Then applying 1.45 for every sheaf $\mathcal{F}_{Y_P}$ we get a number $m_0(P)$ (depending of P) such that the fibers over points of Y_P are $m_0(P)-$regular. Taking the maximum of this number we find a number m_0 such that $\mathcal{F}_s$ is m_0-regular for all $s \in S$.

Therefore given any $s \in S$ the Hilbert polynomial of $\mathcal{F}_s$ is determined by the number $h^0(\mathcal{F}_s(m))$ for $m_0 \leq m \leq m_0 + n$ (see the proof of 1.43). Then

$$\{g^*q_*\mathcal{F}(i) \text{ is flat over T } \forall i \geq m_0\} \Leftrightarrow \{g^*\mathcal{F} \text{ is flat over } T\}$$

For each $m \geq m_0+n$, put $\mathcal{M}_m = \bigoplus_{i=m_0}^{m} q_*\mathcal{F}(i)$. Every $\mathcal{M}_m$ is a sheaf on S, and if $m \geq m_0+n$, given any flattening stratification for it, this is such that the Hilbert polynomial is constant over fibers on each stratum, then as m grow, the flattening stratification for the $\mathcal{M}_m$ form a locally sequence of locally closed subschemes of S with support on Y_P. Then for large m by 1.46 the flattening strata for M_m is an strata for $\mathcal{F}$. For a complete proof of this important theorem see [Fan05] □

Remark 1.56. The last theorem says that if we have a family $T \to S$, then the base change of $\mathcal{F}$ is flat with Hilbert polynomial P if and only if the family was actually $T \to Y_P \subseteq S$. So there is a subscheme Y_P depending only on the Hilbert polynomial P for which $\mathcal{F}_{Y_P}$ is flat over Y_P.

1.7 Existence of the Hilbert scheme

Proof of 1.40. The proof is divided by steps. The idea is reduced to the case $X = \mathbb{P}^n_S$ and prove for that case there exist a natural map $\mathrm{Hilb}^p_{\mathbb{P}^n_S/S} \to Grass_S(r, \mathcal{E})$ of functors that induce a closed immersion between some scheme H_p and $\mathcal{G}rass_S(r, \mathcal{E})$ and finally show that H_p is in fact the representing scheme of $\mathrm{Hilb}^p_{\mathbb{P}^n_S/S}$.

1. Reduce to the case $\mathbb{P}^n_S$.

   Let X be a scheme, and let $X \xrightarrow{\iota} \mathbb{P}^n_S$ be a closed immersion for some natural number n.

   Suppose that $\mathrm{Hilb}^p_{\mathbb{P}^n_S/S}$ is representable by a projective scheme H_p and denote as V_p its universal family.

   Let $U_p = V_p \cap (X \times_s H_p)$ the schematic theory intersection inside $\mathbb{P}^n_S \times_S H_p$. Now by 1.55 there exist a closed subcheme $\tilde{H}_p \xrightarrow{j} H_p$ such that for any $g : Z \to H_p$ the pull-back $g^*(U_p \times_{H_p} Z) \subset X \times_S Z$ is flatt over Z with Hilbert polynomial p if and only if g factor through j.

   We claim that $\tilde{H}_p$ is the representing scheme of $\mathrm{Hilb}^p_{X/S}$ and U_p is its the universal family.

In fact; let $W \in \mathrm{Hilb}^p_{X/S}(Z) \subset \mathrm{Hilb}^p_{\mathbb{P}^n_S/S}(Z)$. There exist a classifying morphism $\phi : Z \to H_p$ corresponding to W, such that $W = (1_{\mathbb{P}^n_S} \times \phi)^* V_p$. Finally we have:

$$(1_{\mathbb{P}^n_S} \times \phi)^{-1} V_p = (1_{\mathbb{P}^n_S} \times \phi)^{-1}(V_p \cap (X \times_S H_p)) = (1_{\mathbb{P}^n_S} \times \phi)^{-1} U_p.$$

Since $(1_{\mathbb{P}^n_S} \times \phi)^{-1} U_p$ is flat over Z with Hilbert polynomial p, then we can factor $(1_{\mathbb{P}^n_S} \times \phi)$ through $j : \tilde{H}_p \to H_p$.

2. Morphism of funtors.

Let $Z \xrightarrow{p} S$ be any $S-$scheme. We want to define a natural map

$$\phi_Z : \mathrm{Hilb}^p_{\mathbb{P}^n_S/S}(Z) \to Grass_S(Q, \mathcal{E})(Z),$$

for some parameters $Q \in \mathbb{N}$ and $\mathcal{E}$ locally free sheaf on S only depending of the Hilbert polynomial p.

Consider the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\iota} & \mathbb{P}^n_S \times_S Z & \longrightarrow & \mathbb{P}^n_S \\ & & \downarrow g^*p & & \downarrow p \\ & & Z & \xrightarrow{g} & S \end{array} \qquad (3)$$

For some $Y \in \mathrm{Hilb}^p_{\mathbb{P}^n_S}(Z)$. Let $\mathrm{Spec}(k) \to Z$ any geometric point, pulling back we get $Y_k \overset{iota}{\to} \mathbb{P}^n_k$. Denoted by $\mathcal{I}_k$ the ideal sheaf of $\mathcal{I}_k$, then :

$$\chi(\mathcal{I}_k(m)) = \chi(\mathbb{P}^n_k, \mathcal{O}(m)) - \chi(Y_k, \mathcal{O}(m)) = \binom{m+n}{n} - p(m) = Q(m).$$

The polynomial Q only depend of n and p, and by 1.43 there exist a natural number N such that $\mathcal{I}_Y$ is $N-$regular.

Now using the sequence $0 \to \mathcal{I}_Y(N) \to \mathcal{O}_{\mathbb{P}^n_S}(N) \to \mathcal{O}_Y(N) \to 0$, we obtain by pushing forward:

$$0 \to (g^*p)_*\mathcal{I}_Y(N) \to (g^*p)_*\mathcal{O}_{\mathbb{P}^n_S}(N) \to (g^*p)_*\mathcal{O}_Y(N) \to R^1(g^*p)_*\mathcal{I}_Y(N).$$

The last term is zero by the flatness of $\mathcal{I}_Y$ and since $H^i(\mathbb{P}^n_k, \mathcal{I}_k) = 0$ for $i \geq 1$ and for any fiber. By the $N-$regularity, we know that $h^0(Y_k, \mathcal{O}(N)) = p(N)$ and $h^i(Y_k, \mathcal{O}(N)) = 0$ for $i \geq 1$. Then we obtain $q_Y = [(g^*p)_*\mathcal{O}_{\mathbb{P}^n_S} \to (g^*p)_*\mathcal{O}_Y(N) \to 0]$, where $(g^*p)_*\mathcal{O}_{\mathbb{P}^n_S}$ is a locally free sheaf of rank $p(N)$. So we define

$$\phi_Z : \mathrm{Hilb}^p_{\mathbb{P}^n_S}(Z) \to Grass_S(Q(N), p^*\mathcal{O}_{\mathbb{P}^n_S})(Z)$$

by

$$Y \to q_Y.$$

Since the number N and the polynomial Q depend only of the Hilbert polynomial p, ϕ_Z is well defined.

3. Existence of $\mathcal{H}ilb^p_{\mathbb{P}^n_S}$.

Call $\mathcal{E} = p^*\mathcal{O}_{\mathbb{P}^n_S}(N)$, and denote by $Grass := \mathcal{G}rass_S(Q(N), \mathcal{E})$. Consider the following diagram:

$$\begin{array}{ccc} Grass \times_S \mathbb{P}^n_S & \xrightarrow{\pi_2} & \mathbb{P}^n_S \\ \downarrow \pi_1 & & \downarrow p \\ Grass & \xrightarrow{f} & S \end{array} \quad (4)$$

Let $\mathcal{Q}$ be the universal rank d quotient of $f^*\mathcal{E}$ and $\mathcal{K} := \mathrm{Ker}(f^*\mathcal{E} \to \mathcal{Q})$. consider the map

$$\pi_1^*\mathcal{K} \to \pi_1^* f^* p^* \mathcal{O}_{\mathbb{P}^n_S}(N) = \pi_2^* p^* p_* \mathcal{O}_{\mathbb{P}^n_S}(N) \to \pi_2^* \mathcal{O}_{\mathbb{P}^n_S}(N),$$

and call $\mathfrak{G}$ its kernel.

By 1.55, there is a flattening strata of $Grass$ for $\mathfrak{G}(-N)$. Let $H_p \xrightarrow{\iota} Grass$ the locally closed subscheme corresponding to the Hilbert polynomial p. i.e. For any sheaf on $H_p \times_{Grass} Grass \times_S \mathbb{P}^n_S = H_p \times_S \mathbb{P}^n_S$ is such that $i^*\mathfrak{G}(-N)$ is flat over H_p and all its fibers have Hilbert polynomial p.

Since $\mathfrak{G} = \pi_2^*\mathcal{O}_{\mathbb{P}^*_S}(N)/\operatorname{image}$, then $i^*\mathfrak{G}(-N)$ is a quotient on $i^*\pi_2^*\mathcal{O}_{\mathbb{P}^n_S}(N)(-N) = i^*\pi_2^*\mathcal{O}_{\mathbb{P}^n_S} = \mathcal{O}_{H_p \times_S \mathbb{P}^n_S}$.

Then we can consider the exact sequence $[0 \to \mathcal{I} \to \mathcal{O}_{H_p \times_S \mathbb{P}^n_S} \to i^*\mathfrak{G}(-N) \to 0]$, therefore exist a closed subscheme V_p of $H_P \times_S \mathbb{P}^n_S$ associated to the sheaf $\mathcal{I}$.

We claim that (H_p, V_p) represents $\mathrm{Hilb}^p_{\mathbb{P}^n_S}$. In fact; Let $Y \in \mathrm{Hilb}^P_{\mathbb{P}^n_S}(Z)$, by the second step there is an element $q_Y \in Grass_S(Q(N), \mathcal{E})(Z)$, so using the representability of $Grass_S(Q(N), \mathcal{E})$ there exist a map $\varphi : Z \to Grass$ such that $\varphi^*(f^*\mathcal{E} \to \mathcal{Q}) = g^*\mathcal{E} \to (g^*p)_*\mathcal{O}_Y(N)$. Since $\mathcal{Q}$ is universal then $\varphi^*\mathcal{Q} \cong (g^*p)_*\mathcal{O}_Y(N)$ as quotients of $g^*\mathcal{E}$, so $(1_{\mathbb{P}^n_S} \times \varphi)^*\mathfrak{G} \cong \mathcal{O}_Y(N)$. Then $(1_{\mathbb{P}^n_S} \times \varphi)^*\mathfrak{G}(-N)$ is flat over Z with Hilbert polynomial p, but H_p is such that φ factor through $\iota \colon H_p \to Grass$, then $Y \mapsto \varphi|_Z : Z \to H_p$ is a functorial map from $\mathrm{Hilb}^p_{\mathbb{P}^n_S}(Z)$ to $\hom(Z, H_p)$. $\therefore$ H_p represents $\mathrm{Hilb}^p_{\mathbb{P}^n_S}$.

□

1.8 Quot Schemes

Definition 1.57. Let S be a noetherian scheme, X be a projective $S-$scheme and $\mathcal{E}$ a coherent sheaf on X. We define the *Quot scheme* associated to $X, \mathcal{E}$ as the representing object of the following functor:

$$Quot_{\mathcal{E},X/S} : \mathbf{sch}_S \to \mathbf{Set}$$

$$T \mapsto \left\{ [0 \to \mathcal{I} \to \mathcal{E}_T \overset{q}{\to} \mathcal{Q} \to 0] \;\middle|\; \begin{array}{l} \mathcal{Q} \text{ sheaf on } X_T = X \times_S T \text{ flat over T,} \\ \mathcal{E}_T \text{ is the pullback of } \mathcal{E} \text{ over the projection} \\ p\colon X_T \to X. \end{array} \right\} \Big/ \text{isomorph.}$$

Where two sequences $[0 \to \mathcal{I} \to \mathcal{E}_T \overset{q}{\to} \mathcal{Q} \to 0]$ and $[0 \to \mathcal{I}' \to \mathcal{E}_T \overset{q'}{\to} \mathcal{Q}' \to 0]$ are isomorphic if $\mathcal{I} = \mathcal{I}'$ as submodules sheaves of $\mathcal{E}_T$.

Theorem 1.58. *The functor $Quot_{\mathcal{E},X/S}$ is a representable functor by a projective scheme.*

Remark 1.59. When $\mathcal{E} = \mathcal{O}_X$, the Quot functor (scheme) is the Hilbert functor (scheme), if $\mathcal{E} = \mathcal{O}_X^r$ the Quot scheme is the natural generalization for the Hilbert scheme and its closed points are in correspondence with quotients sheaves of $\mathcal{O}_X^r$. Furthermore the Grassman functor (scheme) is a particular of some Quot functor (scheme). In fact for any $1 \leq d \leq r$ the Grassmannian scheme Grass(r, d) is the representing object of $\mathcal{Q}uot^{d,\mathcal{O}^r\mathcal{O}_{\mathbb{Z}}}_{\mathcal{O}^r\mathcal{O}_{\mathbb{Z}}} = \mathcal{G}rass(r, d)$.

1.9 Bialynicki-Birula's Theorem

The Bialynicki-Birula Theorem is an important tool in algebraic geometry which give a decomposition of a smooth projective variety X over $\mathbb{C}$ with some $\mathbb{G}_m-$action or an $\mathbb{C}^*-$action. In the especial case where the set of fixed points of the action is finite this decomposition it allows us to calculate the Betti number and the topological Euler characteristic of X. For some similar results on these topics and the proof of the Bialynicki-Birula's theorem see [BB73b],[BB76].

Definition 1.60. Let x_i be a fixed point of the $\mathbb{G}_m-$action on X, then set

$$X_i^+ := \{x \in X \,|\, \lim_{t \to 0} t.x = x_i\}$$

the *Plus cell* associated to x_i, and denote by T_i^+ the $\mathbb{G}_m-$submodule where $\mathbb{G}_m$ acts with positive weights.

Remark 1.61. Let x be an element of X, and suppose that there exist some $\mathbb{C}^*-$action on X. Define the map $(-).x : \mathbb{C}^* \to X, t \mapsto t.x$. By the evaluation criterion there exist a morphism $\phi : \mathbb{P}^1 \to X$ such that, for any $t \in \mathbb{C}^*$, $\phi(t) = t$, $\phi(0) := \lim_{t\to 0} t.x$ and $\phi(\infty) := \lim_{t\to\infty} t.x$.

Theorem 1.62 (Bialynicki-Birula)**.** *Let X be an smooth projective variety over $\mathbb{C}$ with a $\mathbb{G}_m-$action, and suppose that the set of fixed point $X^{\mathbb{G}_m} := \{x_1, \dots, x_n\}$ is finite.*

1. *The collection $\{X_i^+\}$ form a locally closed filtrable decomposition of X, i.e., X is filtered by closed subsets $\emptyset = F_{-1} \subseteq F_1 \subseteq \cdots \subseteq F_{p-1} \subseteq F_p = X$ such that $F_j - F_{j-1} = X_i$ for some i.*

2. *Each X_i is isomorphic to $\mathbb{A}^{n_i}$ for some $n_1 \in \{0, 1 \dots, \dim(X)\}$, and $T_{x_i}(X_i) \cong T_i^+$ as subspace of $T_{x_i}(X)$. In particular X equal to some union of affine spaces, so X is rational.*

3. *The Chow ring $A(X)$ is the free abelian group generated by the classes of $\overline{X_i^+}$, and numerical and rational equivalence of cycles on X coincide.*

Definition 1.63. Let $k \in \{0, 1 \dots, \dim(X)\}$, the $2k-$Betti number of X denoted by $\mathrm{b}_{2k}(X) = \dim(H^{2k}(X;\mathbb{Q}))$, these numbers here match with the number of $i \in \{1, \dots, p\}$ such that $\dim_{\mathbb{C}} T_i^+ = k$. In particular this counts the numbers of Plus cells of dimension k on the decomposition.

As corollary we get the following important result for our work.

Corollary 1.64.

1. $b_{2k}(X) = \operatorname{rank}_{\mathbb{Z}} A^k(X)$.

2. $\chi_{top}(X) = \sum_{k=0}^{\dim(X)} b_{2k}(X) =$ *number of fixed points.*

The second part of the last corollary is given by the equality over $\mathbb{C}$,

$$A^k(X) = H^{2k}(X^h, \mathbb{Z}),$$

and the odd cohomology are zero. X^h denote the complex manifold associated to the algebraic variety over $\mathbb{C}$.

Remark 1.65. The second statement of the last corollary still happens if X is not necessarily a smooth variety, this is proved by A. Bialynicky-Birula on [BB73a] Corollary 2.

1.10 Some topics on Deformation theory

Deformation theory typically studies the "infinitesimal" changes of flat families $X \xrightarrow{f} B$ around neighborhoods of any point $b \in B$. These infinitesimal changes are given by extensions over rings of the type $D_n := k[t]/t^{n+1}$.

Here we are only interested in the basic case of deformations as say Hartshorne in his book [Har09]. It is : Deformations of some kind of subschemes of a given scheme X. (The notation in this book say case type A.)

Tangent space of Hilbert schemes.

Definition 1.66.
- Given any field k, we define the ring of *dual numbers* as the quotient $D := D_1 = k[t]/t^2$.
- If X is any closed scheme over k and $Y \subseteq X$ is a closed subcheme flat over k, we define the first order deformation of Y as a closed subscheme $Y' \subseteq X' := X \times_k D$ such that is flat over D and $Y' \times_D k = Y$.

We want to classify the first order deformation as above, this is basically because this describes the first order deformation of any subscheme inside the Hilbert scheme. We study the affine case.

Let B be a $k-$algebra and $X = \operatorname{Spec}(B)$, then give some subscheme $Y \subseteq X$ is equivalent to taking some ideal $I \subseteq B$. So we are looking for giving ideals $I' \subseteq B \otimes_k k[t]/t^2 = B[t]/t^2 := B'$ such that I' inside $B'/tB' = B$ is exactly I and is flat over $K[t]/t^2$. By the flatness condition of B'/I' over D we get the exact sequence

$$0 \to B/I \xrightarrow{t} B'/I' \to B/I \to 0,$$

now suppose I' is one of these ideals and consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{t} & I' & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{t} & B' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B/I & \xrightarrow{t} & B'/I' & \rightarrow & B/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the last two rows are exact then the top row is exact.

Proposition 1.67. *To give some ideal $I' \subseteq B'$ with all the properties required is equivalent to giving some $\varphi \in \mathrm{Hom}_B(I, B/I)$. In particular if $\varphi = 0$ this correspond to the trivial deformation given by $I' = I \oplus tI$ inside $B' \cong B \oplus tB$.*

Proof. Let $\pi : B \oplus tB \to B$ be the usual projection to B and let $\sigma : B \to B'$ be the section map $\sigma(b) = b + t.0$, so B' is a $B-$module with the product induce by σ. Let $I' \subseteq B'$ be some ideal with all the properties required. Given an element $x \in I$, let $x' = x + ty$ for some $y \in B$ be an element lifting x. If x has another lifting $x'' = x + ty'$ with $y' \in B$, then $x' - x'' = (y - y')t = zt \in tI$, therefore we can define a map $\varphi : I \to B/I$ by $\varphi(x) = y \mathrm{mod} I$, where $x' = x + ty \in B'$; so $\varphi \in \mathrm{Hom}_B(I, B/I)$.

For the other side, let $\psi \in \mathrm{Hom}_B(I, B/I)$ be a morphism and define the set

$$I' := \{x + ty | x \in I, y \in B, \psi(x) = y \mathrm{mod} I\}.$$

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & I \times_{B/I} B & \longrightarrow & I & \longrightarrow & 0 \\
 & & \Big| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & I & \longrightarrow & B & \xrightarrow{\pi} & B/I & \longrightarrow & 0
\end{array}$$

where π is the projection, so $I' = I \times_{B/I} B$, then given $(x + ty) \in I$ and $(x' + ty') \in B'$, then $(x+ty)(x'+ty') = x'x+t(x'y+xy')$, and the difference $\varphi(x'x)-(x'y+xy') = x'(\varphi(x)-y)+xy' \in I'$ since $\varphi(x) - y \in I'$ and $x \in I'$, therefore I' is an ideal.

If π denotes the usual projection from B' to B, we have that $\pi(I') = I \subseteq B$, so the image of I' inside B is I and then $\pi_{|_{I'}}(I') = I$ with $\mathrm{Ker}(\pi_{|_{I'}}) = I$ therefore the next sequence is exact

$$0 \to I \xrightarrow{t} I' \to I \to 0,$$

so considering the diagram above we get the exact sequence

$$0 \to B/I \xrightarrow{t} B'/I' \to B/I \to 0,$$

then B'/I' is flat over D by the local criterion of flatness, [Har09, cap.I, Proposition 2.2].

Finally note that given any ideal I' as before the map $\psi : I \to B/I$ is exactly φ; then these constructions are inverse, and the case $\varphi = 0$ implies $I' = I$. □

Now remembering that any $Y \subseteq X$ which is closed and flat over k, define an exact sequence

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0,$$

then locally we have that any morphism $\psi \in \mathrm{Hom}(I, \mathcal{O}_X/I)$ correspond to some deformation I' of I over the dual numbers D,i.e a deformation of first order Y' of Y. Then by the last discussion and [Har77, cap.II, Theorem 2.8] we get the following important proposition.

Proposition 1.68. *Let X be any projective scheme over a field k, and let $[Y] \in \mathrm{Hilb}_{X/k}$. Then the Zariski tangent space of $\mathrm{Hilb}_{X/k}$ at a point $[Y]$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(I, \mathcal{O}_X/I) = \Gamma(\mathcal{N}_Y, X)$ where I is the ideal sheaf defined by Y,*

2 Hilbert scheme of points

Given any projective scheme X and for any constant numerical polynomial P of value n, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is called the Hilbert scheme of n points of X. This name make sense because every $[z] \in \mathrm{Hilb}^n(X)$ is a collection of n points of X, formally;

Let $\mathcal{H}ilb^n : \mathbf{Sch}_S \to \mathbf{Set}$ be the functor that associates to every $S-$scheme T the set of all closed flat families $Z \subseteq X_T$ with a Hilbert polynomial constant equal n. If we denoted $\mathrm{Hilb}^n(X)$ by $X^{[n]}$ as the representing scheme of the last functor, we get a one-to-one correspondence between the geometric points of $X^{[n]}$ and the closed subschemes of X with Hilbert polynomial n. Let Z be one of these closed subschemes, then its Hilbert polynomial is the same as the Hilbert polynomial of its sheaf of ideal $\mathcal{I}_Z$ and therefore $P_{\mathcal{I}_Z}(k) = n$. This says that $\mathrm{Supp}\,\mathcal{I}_Z = \{z_1 \ldots, z_k\}$ is such that $n = \sum_i \mathrm{length}(Z_i)$ and so Z can be thought as a set of n points of X with multiplicities.

The study of the Hilbert scheme, $X^{[n]}$, which parametrizes the $0-$dimensional subschemes of X is difficult in general. Here we focus on the cases X is a smooth curve or a smooth surface using the methods in [ES87].

Now we turn to study the general case with base S for varieties over the complex number. These cases motivated by the use of Bialynicki-Birula's theorem.

2.1 Hilbert scheme of points over smooth curves and smooth surfaces.

The easiest case of study is when X is a smooth projective curve C. In this case it is not difficult to see that $C^{[n]} = \mathrm{Sym}^n(C) = C \times \cdots \times C/\Sigma_n$, where Σ_n is the symmetric group in n letters. In particular $(\mathbb{P}^1)^{[n]} \cong \mathbb{P}^n$. The case of projective smooth surface S is more complicated and we show some properties of $S^{[n]}$ following the treatment of Fogarty presented in [Fog68].

To give some results it is necessary to define unipotent algebraic groups G and look at how the fixed locus of X by some $G-$actions can be.

For more information about unipotent affine groups see [Mil].

Definition 2.1. 1. A group G is said unipotent if it is a subgroup of a unitary ring and for any element $g \in G$, there is some $n \in \mathbb{N}$ such that $(r-1)^n = 1$.

2. A group G is called an unipotent affine group if every nonzero representation of G has a nonzero fixed vector.

If we denote by $\mathbb{U}(n)$ the set of all upper triangular matrices of dimension n^2 with diagonal entries equal to 1, then $\mathbb{U}(n) \subseteq \mathbb{PGL}(n)$. There exist a characterization of unipotent affine groups given by the next theorem.

Theorem 2.2. *A group G is unipotent if and only if G is isomorphic to an algebraic subgroup of $\mathbb{U}(n)$ for some n.*

Remark 2.3. For any unipotent group G there is a morphism $\sigma : G \to \mathbb{PGL}(n)$.

Corollary 2.4. *Subgroups, quotients and extension of unipotent group are unipotent.*

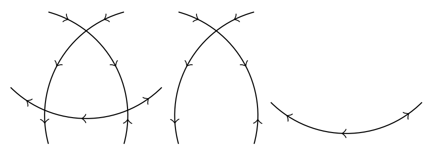
With the last description of unipotent groups we can start to work in geometry.

Given any closed connected subscheme X of $\mathbb{P}^n$ over a closed algebraic field k, and given a unipotent algebraic group G, if $f\colon G \to \mathbb{PGL}_n$ is any $k-$homorphism, it induces a natural action of G in $\mathbb{P}^n$ given by $g.[x_0, \ldots, x_n] = [f(g)_{ij}]_{i,j=0}^n.[x_0, \ldots, x_n] = [x'_0 \ldots, x'_n]$, where $x'_i = \sum_{j=0}^n f(g)_{i-1,j}x_j$. Then if X is stable under this action, f induces an action of G on X, and we have the following result.

Proposition 2.5. *Let G be a unipotent group acting on X. If X^G denote the set of fixed points of X under this action, then X^G is connected.*

Proof. The proof is given by induction on the dimension of X. If $\dim(X) = 0$, then there is nothing to prove because $X = X^G$.

If X is a curve C, we use induction on the numbers of irreducible components. If C is irreducible then the $G-$action is trivial or C only have one fixed point, this because every simple $G-$module is trivial. Let C be a reducible curve and write $C = \bigcup_{i=1}^{n} C_i$, where every C_i is an irreducible component of C, and denote by $C' = \bigcup_{i=1}^{n} C_i$, (see the picture 5).



(5)

The intersection $C_0 \cap C'$ has only fixed points, let C_i^G be the fixed locus of C_i, by induction this is connected, so if C_0 has some point $c \notin C_0^G$, then C_0 only has one fixed point and therefore the fixed locus of C_0 intersect the fixed locus of C' which is connected and therefore so is C^G.

Let X be a projective scheme with $\dim(X) \geq 2$. Suppose that X^G is disconnected. Then there exist two points x_0, x_1 living in different components of X^G, since X is projective there exists a curve C intersecting x_0 and x_1. Denote by Q the Hilbert polynomial of C, and call G' the action on Hilb_X^Q induced by f. Let $z \in \mathrm{Hilb}_X^Q$ be the point corresponding to the curve C, denote by U the isotropy group of z, then $U \cong G$ or $U \cong \{z\}$. In the first case, there is a point $z_0 \in \overline{U}$, and then z_0 is a fixed point. Let C_0 be the curve associated to z_0. For any point $z' \in U$, its corresponding curve C' is connected (Hilbert polynomial Q) and intersect the points x_0, x_1. Then C_0 is a limit of connected curves intersecting x_0 and x_1 so this is connected and intersect x_0 and x_1, but these points are fixed, then C_0^G is not connected. This finished the proof. □

Proposition 2.6. *For any finite dimensional local $k-$algebra A, the Hilbert scheme* $\mathrm{Hilb}_{X/k}^n$ *is connected where* $X = \mathrm{Spec}(A)$.

Proof. Let $\mathbb{G}$ be the Grassmanian scheme $Grass_{A/k}(d, d-n)$ where $d = \dim_k A$. By construction $\mathrm{Hilb}_{X/k}^n$ is a closed subscheme of $\mathbb{G}$. If $\mathcal{M}$ is the maximal ideal of A, we will induce a $(1+\mathcal{M})-$action on $\mathbb{G}$ using Plücker coordinates as follows. We consider the multiplicative action of $(1+\mathcal{M})$ by multiplication on A, which give us a representation $\rho : (1+\mathcal{M}) \to \mathbb{S}^d$, similarly $(1+\mathcal{M})$ act on the exterior product $\wedge^n A$, for that reason we find a representation

$\wedge^n \rho : (1+\mathcal{M}) \to \mathbb{S}^{\binom{d}{n}}$. We know that $\mathbb{G}$ is a closed subscheme of $\mathbb{P}(\wedge^n A)$ by the Plücker embedding, and moving the columns of the matrices on $\mathbb{PGL}(\binom{d}{n}-1)$ by elements of $\mathbb{S}^{\binom{d}{n}}$, we induce a $(1+\mathcal{M})-$action on $\mathbb{P}(\wedge^n A)$ such that $\mathbb{G}$ is invariant. So $(1+\mathcal{M})$ act on $\mathbb{G}$. Then any quotient A/V on $\mathbb{G}$ is invariant if $(1+\mathcal{M})V = V$ therefore by Nakayama's Lemma V is an ideal of A, then any invariant element by the action induce an exact sequence $0 \to V \to A \to A/V \to 0$. Therefore the fixed locus of this action on $\mathbb{G}$ is $\text{Hilb}_{X/k}$ then by 2.5 $\text{Hilb}_{X/k}$ is connected. □

In what follows, we will see some results in the case $X = S$ is a smooth projective surface.

Proposition 2.7. *$S^{[d]}$ is irreducible*

Proof. Consider the Chow morphism

$$\text{ch} : S^{[n]} \to \text{Sym}^n(S),$$

given by $\text{ch}(Z = \{x_1, \ldots, x_n\}) = \sum_{i=1}^{k} \text{length}(\mathcal{O}_{x_i})x_i$. It is enough to show that every fiber of this morphism is irreducible. Any point on $\text{Sym}^n(S)$ has the form $\sum_{i=1}^{k} n_i x_i$, where $\sum_i n_i = n$ and $x_i \in S$. Then $\text{ch}^{-1}(\sum_{i=1}^{k} n_i x_i) = \prod_{i=1}^{k} \text{Hilb}^{n_i}_{X_i/k}$, where $X_i = \text{Spec}(\mathcal{O}_{X,x_i}/\mathcal{M}^{n_i}_{X,x_i})$ with $\mathcal{M}_{X,x_i}$ the maximal ideal of the local ring $\mathcal{O}_{X,x_i}$.But the algebra $A = \mathcal{O}_{X,x_i}/\mathcal{M}^{n_i}_{X,x_i}$ is local therefore for any n_i the Hilbert scheme $\text{Hilb}^{n_i}_{X_i/k}$ is connected by 2.6. Then $S^{[n]}$ is irreducible. □

The following important propositions will be presented without proof but the idea of these are to use Proposition 2.7 and compare the dimension using Proposition 1.68 to express the tangent space as $\text{Hom}(I, O/I)$ where O is a two dimensional local ring and I is an ideal of O, and finally use the following algebraic lemma:

Lemma 2.8. *Let O be a two dimensional regular local ring and let I be an O ideal primary for the maximal ideal, $\mathcal{M}$. If the length of O/I is n, then* $\text{length}(O/A) \leq 2n$. *(Geometrically this says that* $\dim(T_x\text{Hilb}^{[n]})$ *is lest than or equal to $2n$.)*

Proof. See [Fog68] or [Eis13]. □

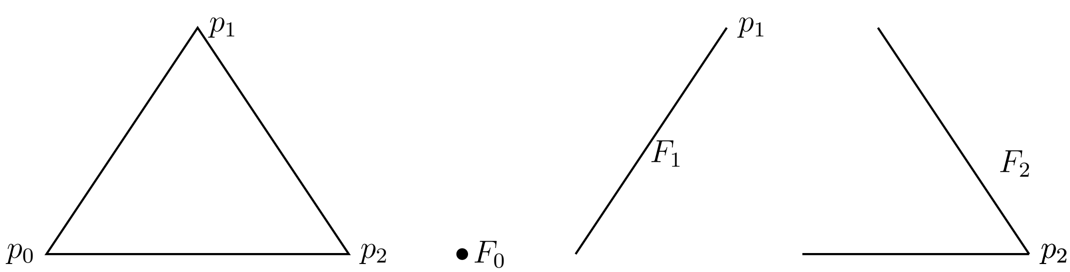
Proposition 2.9. *$S^{[d]}$ is a smooth and projective variety.*

Proposition 2.10. $\dim(S^{[d]}) = 2d$

2.2 Euler characteristic and Betti numbers of $\mathrm{Hilb}^d(\mathbb{P}^2)$

At first we described a $\mathbb{C}^*$ action on $\mathrm{Hilb}^d(\mathbb{P}^2)$ and the following step is to show that this action only has finitely many fixed points morder to use Bialynicki-Birula 1.62, and then use the Young tableaux to count the number of fixed point and get the Euler characteristic of $(\mathbb{P}^2)^{[d]}$. Finally, using the cellular decomposition we find its Betti numbers. For this we follow [ES87].

Let $G \subset SL(3, \mathbb{C})$ be the subgroup of diagonal matrices, and let w_0, w_1, w_2 be integers such that $w_0 < w_1 < w_2$ and $w_0 + w_1 + w_2 = 0$. For any element $t \in \mathbb{C}^*$ denote by $\Delta(t^{w_0}, t^{w_1}, t^{w_2})$ the diagonal matrix with entries nonzero $t^{w_0}, t^{w_1}, t^{w_2}$. Denoted by x_0, x_1, x_2 the homogeneous coordinates of $\mathbb{P}^2$. Given any element $g \in G$, it acts on the point $[x_0 : x_1 : x_2]$ by multiplication that is $g.[x_0 : x_1 : x_2] = [g_{11}x_0 : g_{22}x_1 : g_{33}x_2]$, and then $\mathbb{C}^*$ acts on $\mathbb{P}^2$ with weights w_0, w_1, w_2 by $t.[x_0 : x_1 : x_2] = [t^{w_0}x_0 : t^{w 1}x_1 : t^{w_2}x_2]$, this action only has as a set of fixed points the set of 'corners' $[1 : 0 : 0] = p_0, [0 : 1 : 0] = p_1, [0 : 0 : 1] = p_2$ of $\mathbb{P}^2$ and this induce a cellular decomposition of $\mathbb{P}^2$,given by $F_0 = p_0$, $F_1 = L - p_0$ and $F_2 = \mathbb{P}^2 - L$, where L is the line $x_2 = 0$. Then $F_i \cong \mathbb{A}^i$. See 6



(6)

This action induce an action on $\mathrm{Hilb}^n(\mathbb{P}^2)$, as follow. Given any $g \in G$ and any point $z = [0 \to \mathcal{I} \to \mathcal{O} \overset{\phi}{\to} \mathcal{Q} \to 0]$, we define $g.z = [0 \to \mathcal{I}' \to \mathcal{O} \overset{\phi \circ g^*}{\to} \mathcal{Q} \to 0]$ where g^* is the pullback of functions from $\mathcal{O}$ to $\mathcal{O}$.

Given any closed subscheme Z of d points of $\mathbb{P}^2$ which is fixed by this action, it is clear that $\mathrm{Supp}(Z) \subseteq \{p_0, p_1, p_2\}$. Then we can decompose Z as a union of $Z_0, Z_1, Z_{,2}$ where $\mathrm{Supp}(Z_i) = p_i$ with $\mathrm{length}(\mathcal{O}_{Z_i}) = d_i$ and $d_0 + d_1 + d_2 = n$.

In orders to use 1.62 we only need to prove the next lemma, since $\mathrm{Hilb}^n(\mathbb{P}^2)$ is projective and smooth as we showed in 2.9

Lemma 2.11. *The number of fixed points of* $\mathrm{Hilb}^n(\mathbb{P}^2)$ *under the* $\mathbb{C}^*-$*action described above is finite.*

Proof. Let $z = [0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{Q} \to 0]$ be an element of $\mathrm{Hilb}^n(\mathbb{P}^2)$. Locally this looks like a chain of modules of the type $[0 \to I \to \mathbb{C}[x_0, x_1, x_2] \to Q \to 0]$. Then z is fixed by the $\mathbb{C}^*-$ action if and only if I is fixed by the action on the coordinates x_0, x_1, x_2, and since the action on any polynomial $p(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2]$ is of the form $t.p(x_0, x_1, x_2) = p(t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2)$, then I is fixed under the action if and only if it is generated by monomials and the set of monomials of degree n is a finite set, therefore the set of fixed points $((\mathbb{P}^2)^{[n]})^{\mathbb{C}^*}$ is a finite set. $\square$

Our next goal is to count the number of fixed points.

Let $U_0 = \{x_0 \neq 0\}$ be an affine neighborhood of the point p_0, calling $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$ in U, then we have that any fixed point on $(\mathbb{P}^2)^{[n]}$ supported only on p_0 has the form $z = [0 \to I \to \mathbb{C}[x, y] \to Q \to 0]$ where $Q = \mathbb{C}[x, y]/I = \bigoplus_{k=1}^{n} \mathbb{C}x^{i_k}y^{j_k}$. On the corner p_0 we put boxes with the elements $x^j y^j$ that appear in the decomposition of Q following the next rules: on the first row put only powers of x growing to the right, in the first column put only powers of y growing up, and the other letters $x^j y^j$ put as a multiplicative table, for example if $Q = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}x^2 \oplus \mathbb{C}y \oplus \mathbb{C}y^2 \oplus \mathbb{C}xy$ then we draw the figure 7 :

y^2		
y	xy	
1	x	x^2

(7)

We claim that the Young tableaux of length n and the fixed points supported only in p_0 are in $1-1$ correspondence. In fact, if $Q = \mathbb{C}[x,y]/I = \bigoplus_{k=1}^{n} \mathbb{C}x^{i_k}y^{j_k}$, the letter $x^0y^0 = 1$ must appear since I is a proper ideal, and if x^jy^i is one of the letters on the decomposition of Q but $x^{j'}y^{i'}$ is not one of these, with $i' < i$ or $j' < j$, then multiplying by an appropriated power of x and y the expression $x^{j'}y^{i'}$ we get that x^jy^i is in the ideal I. But this can not happen. Then every fixed point induces a Young tableaux and obviously every Young tableaux induce a fixed point. This enables us to state the following theorem.

Theorem 2.12. *If $\chi(X)$ denotes the Euler characteristic of a given topological space X, then*

$$\chi(\mathrm{Hilb}^n(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=n} p(d_0)p(d_1)p(d_2),$$

where $p(d)$ denotes the number of partitions of d.

Proof. Let $Z \in (\mathrm{Hilb}^n(\mathbb{P}^2))^{\mathbb{C}^*}$. Then $Z = Z_0 \cup Z_1 \cup Z_2$, where $\mathrm{Supp}(Z_i) = p_i$ and $\mathrm{length}(\mathcal{O}_{Z_i}) = d_i$ with $d_0 + d_1 + d_2 = n$. Then from the last discussion, the set of fixed points of type Z_i is in correspondence with the set of Young tableaux of length d_i which count the number of partitions of d_i. Then

$$\#(\mathrm{Hilb}^n(\mathbb{P}^2))^{\mathbb{C}^*} = \sum_{d_0+d_1+d_2=n} p(d_0)p(d_1)p(d_2).$$

Finally by 1.62 we get the result. □

Given any $Z \in \mathrm{Hilb}^n(\mathbb{P}^2)$, this can be decomposed as the union $Z_0 \cup Z_1 \cup Z_2$ where $\mathrm{Supp}(Z_i) \subseteq F_i$ and $\mathrm{length}(\mathcal{O}_{Z_i}) = d_i$ and writing $W(d_0, d_1, d_2)$ as the set of all subschemes of $\mathrm{Hilb}^n(\mathbb{P}^2)$ of $\mathrm{length}(\mathcal{O}_{Z_i}) = d_i$. We can write the Hilbert scheme of d points of $\mathbb{P}^2$ as the following union:

$$\mathrm{Hilb}^d(\mathbb{P}^2) = \bigcup_{d_0+d_1+d_2=d} W(d_0, d_1, d_2).$$

If Z is expressed as $Z_0 \cup Z_1 \cup Z_2$, each of these pieces are such that $\lim_{t\to 0} \mathrm{Supp}(t.Z_1) = p_i$. Then $W(d_0, d_1, d_2)$ is a union of elements of the cellular decomposition of $\mathrm{Hilb}^n(\mathbb{P}^2)$. i.e. $W(d_0, d_1, d_2) = W(d_0, 0, 0) \times W(0, d_1, 0) \times W(0, 0, d_2)$ To calculate the $2k-$Betti number of

$\text{Hilb}^n(\mathbb{P}^2)$ we have to count the number of pieces in the decomposition of dimension k, but this is the same as counting the number of these pieces that appear on $W(d_0, d_1, d_2)$. Then we have the next lemma:

Lemma 2.13.

$$b_{2k}(\text{Hilb}^n(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=n} \sum_{p+q+r=k} b_{2p}(W(d_0,0,0))b_{2q}(W(0,d_1,0))b_{2r}(W(0,0,d_2)).$$

For giving a more explicit formula we need to calculate the Betti numbers

$$b_{2k}(W(d,0,0)), b_{2k}(W(0,d,0)) \text{ and } b_{2k}(W(0,0,d)).$$

For this we need to count the number of cells of the cellular decomposition of dimension k, but D is some cell inside $W(d,0,0)$ (resp.W(0,d,0),W(0,0,d)) if and only if $\text{Supp}(D) = p_0$. Therefore we are interested in subschemes of $\mathbb{P}^2$ with only one fixed point by G. Each of these subschemes are inside an appropriate affine plane $U_i = \{x_i \neq 0\}$. As we see above, any subscheme of $\mathbb{P}^2$ which is fixed by the torus action supported only in a point p_i, is in correspondence with some ideal I of $\mathbb{C}[x,y]$ (x and y appropriated quotients), and fixed by the maximal torus of diagonal matrices $\Gamma \subseteq SL(2,\mathbb{C})$, but this action can be seen from $\mathbb{C}^*$ using a $1-$parameter subgroup $t \mapsto \Delta(t^\lambda, t^\mu)$ where λ and μ are some weights. Then each of these ideals are fixed by the $\mathbb{C}^*-$action $t.p(x,y) = p(t^\lambda x, t^\mu y)$, which says that I is generated by monomials in the coordinates x, y and colength(I) is finite.

Let I be a monomial ideal and let Y_I its Young tableaux, the set $\{y^{b_0}, xy^{b_1}, \ldots, x^i y^{b_i}, \ldots, x^r\}$, where $b_j = \inf\{k \in \mathbb{N} | x^j y^k \in I\} = \inf\{k \in \mathbb{N} | x^j y^k \notin Y_I\}$ is a non$-$minimal set of generators for I. The following re clear properties:

- $b_r = 0$ for $r \gg 0$
- $\{b_j\}_{j \in \mathbb{N}}$ is a non-increasing sequence
- $\displaystyle\sum_{j=0}^{r} \text{length}(Y_I) = \text{length}(\mathbb{C}[x,y]/I)$.

The proof of these properties is given by the properties of the Young tableaux and the correspondence between I and Y_I.

Example 2.14. Suppose Y_I isthe Young tableaux 7;

y^2		
y	xy	
1	x	x^2

(8)

Then, $b_0 = 3, b_1 = 2, b_2 = 1, b_3 = 0$ and $I = \langle y^3, xy^2, x^2y, x^3\rangle$.

In [ES87] they introduce the following notation: Denote $\mathbb{C}[x, y]$ as R and for any pair $\mathbf{a} = (\alpha, \beta) \in \mathbb{Z}^2$, let $R[\mathbf{a}] = R[\alpha, \beta]\colon = \mathbb{C}[x, y][\alpha, \beta]$. i.e. the double-graded module with $(R[\alpha, \beta])_{\mathbf{d}} = R_{\mathbf{d}+\mathbf{a}}$, given by the action $t.x^m y^n = t^{-\lambda(m-\alpha)} x t^{-\mu(n-\beta)} y$. The symbols λ and μ can be interpreted as characters of the $\mathbb{C}^*-$action (by diagonal matrix). Then we can write $R[\alpha, \beta] = \sum_{\substack{p \geq -\alpha, \\ q \geq -\beta}} \lambda^p \mu^q$, in the case where $p = -\alpha$ and $q = -\beta$ we find the elements of degree 0.

We want to find some expression of the tangent space T_i^+ for computing the Betti numbers, using deformation theory we know that $T \cong \mathrm{Hom}_R(I, R/I)$. Using some facts of homological algebra we can compute $\mathrm{Hom}_R(I, R/I)$ in the representation ring of $\mathbb{C}^*$. Ellingsrud and Stromme in [ES87] prove the following lemmas.

Lemma 2.15. *There is a $\mathbb{C}^*-$equivariant resolution*

$$0 \to \bigoplus_{i=1}^{r} R[-\boldsymbol{n}_i] \xrightarrow{M} \bigoplus_{i=0}^{r} R[-\boldsymbol{d}_i] \xrightarrow{\varphi} I \to 0$$

where $\boldsymbol{n}_i = (i, b_{i-1})$ and $\boldsymbol{d}_i = (i, b_i)$ and the map φ is defined by

$$\varphi(P_0(x, y), \ldots, P_r(x, y)) = (y^{b_0}, xy^{b_1}, \ldots, x^r).(P_0(x, y), \ldots, P_r(x, y))^T \in I.$$

If $e_i = b_{i-1} - b_i$ *for* $1 \leq i \leq r$ *then*

$$M = \begin{bmatrix} x & 0 & \dots & \dots & 0 \\ y^{e_1} & x & 0 & \dots & \\ 0 & y^{e_2} & x & \dots & \\ & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & y^{e^r} \end{bmatrix}$$

Proof. It is enough to show the maximal minors of M are precisely $y^{b_0}, xy^{b_1}, \dots, x^r$ and that easy to check. ☐

For the example 2.14 we have:

$$d_0 = (0,3), d_1 = (1,2), d_2 = (2,1), d_3 = (3,0),$$

$$\varphi : \mathbb{C}[x,y](0,3) \oplus \mathbb{C}[x,y](1,2) \oplus \mathbb{C}[x,y](2,1) \oplus \mathbb{C}[x,y](3,0) \to I = \langle x^3, x^2y, xy^2, y^3 \rangle$$

is given by

$$\varphi(P_0(x,y), P_1(x,y), P_2(x,y), P_3(x,y)) \mapsto y^3P_0(x,y) + xy^2P_1(x,y) + x^2yP_2(x,y) + x^3P_3(x,y)$$

and

$$M = \begin{bmatrix} x & 0 & 0 \\ y^1 & x & 0 \\ 0 & y^1 & x \\ 0 & 0 & y^1 \end{bmatrix},$$

for this matrix M the maximal minors are y^3, xy^2, x^2y and x^3. The map φ is given by

$$\begin{bmatrix} x & 0 & 0 \\ y^1 & x & 0 \\ 0 & y^1 & x \\ 0 & 0 & y^1 \end{bmatrix} \cdot \begin{bmatrix} P_1(x,y) \\ P_2(x,y) \\ P_3(x,y) \end{bmatrix} = \begin{bmatrix} xP_1(x,y) \\ yP_1(x,y) + xP_2(x,y) \\ yP_2(x,y) + xP_3(x,y) \\ yP_3(x,y) \end{bmatrix}.$$

The following lemma will be useful to compute the Betti numbers and for the last part of this work it will be used to calculate some Chern roots as will be shown with an example.

Lemma 2.16. *In the representation ring of* $\Gamma = \mathbb{C}^*$ *we have the identity*

$$\mathrm{Hom}_R(I, R/I) = \sum_{1\leq i\leq j\leq r} \sum_{s=b_j}^{b_{j-1}-1} (\lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \lambda^{j-1}\mu^{s-b_{i-1}}).$$

Example 2.17. Let $\begin{array}{|c|c|} \cline{1-1} y & \multicolumn{1}{c}{} \\ \hline 1 & x \\ \hline \end{array} = Y_I$, then $I = \langle y^2, xy, x^2\rangle$, the numbers b_i are $b_0 = 2, b_1 = 1, b_2 = 0$. So by the Lemma 2.16 we can compute the tangent space $T = \mathrm{Hom}_R(I, R/I)$ as:

$$\sum_{1\leq i\leq j\leq 2} \sum_{s=b_j}^{b_{j-1}-1} (\lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \lambda^{j-1}\mu^{s-b_{i-1}}) = T,$$

if we call $E(i,j,s)$ the expression $(\lambda^{i-j-1}\mu^{b_{i-1}-s-1} + \lambda^{j-1}\mu^{s-b_{i-1}})$, then we have

$$\begin{aligned}
T &= \sum_{j=1}^{2} \sum_{s=b_j}^{b_{j-1}-1} E(i=1, j, s) + \sum_{s=b_2}^{b_1-1} E(i=2, j=2, s) \\
&= \sum_{s=b_1}^{b_0-1} E(i=1, j=1, s) + \sum_{s=b_2}^{b_1-1} E(i=1, j=2, s) + E(i=2, j=2, s=0) \\
&= E(i=1, j=1, s=1) + E(i=1, j=2, s=0) + E(i=2, j=2, s=0) \\
&= (\lambda^{-1}\mu^0 + \lambda^0\mu^{-1}) + (\lambda^{-2}\mu^1 + \lambda^1\mu^{-2}) + (\lambda^{-1}\mu^0 + \lambda^1\mu^{-1}),
\end{aligned}$$

therefore

$$T \cong (\mathbb{C}\lambda^{-1}\otimes\mathbb{C}\mu^0)\oplus(\mathbb{C}\lambda^0\otimes\mathbb{C}\mu^{-1})\oplus(\mathbb{C}\lambda^{-2}\otimes\mathbb{C}\mu^1)\oplus(\mathbb{C}\lambda^1\otimes\mathbb{C}\mu^{-2})\oplus(\mathbb{C}\lambda^{-1}\otimes\mathbb{C}\mu^0)\oplus(\mathbb{C}\lambda^1\otimes\mathbb{C}\mu^{-1}),$$

so T has dimension 6 as we hope and its Chern roots are $-\lambda h, -\mu h, (-2\lambda + \mu)h, (\lambda - 2\mu)h, -\lambda h$, and $(\lambda - \mu)h$, where h is the generator of the $\mathbb{C}^*-$equivariant cohomology, see section 4.1.

In the next propositions we compute the Betti numbers.

Proposition 2.18.

$$b_{2k}(W(d, 0, 0)) = p(k, d-k).$$

Where $p(n, m) :=$ *partitions of* n *using only positive integers that are less than or equal to* m.

Proof. Every $z \in W((d,0,0))$ is such that $\mathrm{Supp}(z) = p_0$, then z is inside $U_0 = \mathrm{Spec}(\mathbb{C}[x,y])$, where $x = \dfrac{x_1}{x_0}, y = \dfrac{x_2}{x_0}$, and the action is induced by $\lambda = w_1 - w_0$ and $\mu = w_2 - w_0$ with $w_0 < w_1 < w_2$. If we denote T the tangent space of $\mathrm{Hilb}^n(\mathbb{P}^2)$ at point z, then

$$T^+ = \sum_{1 \le i < j \le r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{j-i} \mu^{s-b_i-1},$$

and $\dim(T^+) = \displaystyle\sum_{i=1}^{r} \sum_{j=i+1}^{r} (b_{j-1} - b_j) = \sum_{i=1}^{r} b_i = d - b_0$, and $b_0 + b_1 + \cdots + b_r = d$. If $z \in W(d,0,0)$ has dimension k, this implies that $k = d - b_0$, then $b_1 + b_2 + \cdots + br = k$. This proves the proposition. ☐

Proposition 2.19.

$$b_{2k}(W(0,d,0)) = \begin{cases} 0, & \textit{if } k \neq d \\ p(d), & \textit{if } k = d. \end{cases}$$

Proof. If $z \in W(0,d,0)$ this is inside $U_1 = \mathrm{Spec}(\mathbb{C}[x,y])$, where $x = \dfrac{x_0}{x_1}, y = \dfrac{x_2}{x_1}$, the $\mathbb{C}^*-$action is given by $\lambda = w_0 - w_1 < 0$ and $\mu = w_2 - w_1 > 0$. Then the positive part of the tangent space T at the point z is:

$$T^+ = \sum_{1 \le i \le j \le r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1} \mu^{b_i-s-1},$$

and so $\dim(T^+) = \sum_{j=1}^{r} \sum_{b_j}^{r} (b_{j-1} - b_j) = \sum_{i=1}^{r} b_{i-1} = d$. This implies that

$$\#\{z \in W(0,d,0) |\ \text{z is a cell with dimension } k\} = \begin{cases} 0, & \text{if } k \neq d \\ p(d), & \text{if } k = d. \end{cases}$$

☐

Proposition 2.20.

$$b_{2k}(W(0,0,d)) = p(2d-k, k-d).$$

Proof. The idea is the same as the last two proof, but the weights are $\lambda = w_0 - w_1 < 0$ and $\mu = w_1 - w_2 < 0$, then

$$T^+ = \sum_{1 \le i \le j \le r} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1} \mu^{b_i-s-1} + \sum_{j=1}^{r} \sum_{s=b_j}^{b_{j-1}-1} \mu^{s-b_j-1},$$

so $\dim(T^+) = d + b_0$. Then if z is inside $W(0,0,d)$ is a cel of dimension k, we have the equality $b_0 = d - k$ and hence $b_1 + b_2 \cdots + b_r = 2d - k$. $\square$

This give us a better formula for the Betti number:

$$b_{2k}(\mathrm{Hilb}^n(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=n} \sum_{q+r-k=-d1} p(q, d_0 - q)p(d_1)p(2d_2 - r, r - d_2).$$

Finally we present the next tables with some values of the Euler characteristic for various parameters n.

$d = \text{length}$	$\chi(\text{Hilb}^{[d]}(\mathbb{P}^2))$
1	3
2	9
3	22
4	51
5	108
6	221
7	429
8	810
9	1479
10	2640
11	4599
12	7868
13	13209
14	21843
15	35581
16	57222
17	90585
18	142175
19	220425
20	338679

Table 1: Some Euler characteristic for $Hilb^{[d]}$

3 Quot scheme of points

3.1 Over smooth Surfaces

Let S be a smooth and projective surface, denote by $M_(S,\mathcal{E})(n,q,d)$ the Quot scheme

$$Quot_{(S,\mathcal{E})}(n,q,d) = \left\{ [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0] \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a fixed locally free sheaf on } S \text{ of rank } n, \\ \text{rank}(Q) = q, c_1(Q) = 0 \text{ and } c_2(Q) = d \end{array} \right\}.$$

The purpose of this chapter is the study of geometric properties of $M_{(S,\mathcal{E})}(n,q,d)$ such as smoothness, ireducibility, dimension, Betti numbers and Euler characteristic for different values of parameters n, q, d.

Some of these properties have been studied before in different papers such as [EL99],[ES98] and [Str81]. We present a generalization of theorems about irreducibility and prove some new results on smoothness.

3.2 Irreducibility

Ellingsrud and Lehn in [EL99] prove Theorem 3.1, calculate the dimension of the scheme $M_{(S,\mathcal{E})}(n,0,d)$ and show its irreducibility. We give a generalization of this result, compute the dimension of the scheme $M_{(S,\mathcal{E})}(n,q,d)$ and show its irreducibility. The technique for the proof is the same of Ellingsrud and Lenh, are the elementary modifications, calculation of size of fibers for some specials morphism and induction.

Theorem 3.1. *The scheme $M_{(S,\mathcal{E})}(n,0,d)$ is an irreducible scheme of dimension $d(n+1)$.*

The last theorem is a generalization of Propositions 2.10 and 2.7, because when $n = 1$ the scheme $M_{(S,\mathcal{E})}(1,0,d)$ is precisely the Hilbert scheme $S^{[d]}$ with dimension $2d$.

Let $p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0] \in M_{(S,\mathcal{E})}(n,q,d)$. We will construct an element $p' \in M_{(S,\mathcal{E})}(n,q,d+1)$ via push-out and pullback diagram.

Let $s \in S$, and suppose there exists a morphism $\mathcal{K} \stackrel{\lambda}{\to} k(S) \to 0$, then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & k(s) & \xrightarrow{\mu} & \mathcal{Q}' & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \lambda\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{K}' & \longrightarrow & \mathcal{K}' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{9}
$$

$p' = [0 \to \mathcal{K}' \to \mathcal{E} \to \mathcal{Q}' \to 0] \in M_{(S,\mathcal{E})}(n, q, d+1)$, since $c_2(\mathcal{Q}') = c_2(k(s)) + c_2(\mathcal{Q}) = 1 + c_2(\mathcal{Q})$. We say that p' is an *elementary modification* of p or simply $\mathcal{Q}'$ is an *elementary modification* of $\mathcal{Q}$. This new element will be very important for the induction step on d.

Definition 3.2. 1. Let $\mathcal{K}$ be a coherent $\mathcal{O}_S-$sheaf, we denote by $e(\mathcal{K}_s) := \hom_S(\mathcal{K}, k(s))$ the dimension of the fiber $\mathcal{K}(s) = \mathcal{K}_s \otimes_{\mathcal{O}_s} k(s)$.

By Nakayama's Lemma $e(\mathcal{K}_s)$ is the minimal numbers of generators of the stalk $\mathcal{K}_s$.

2. Let $\mathcal{Q}$ be a coherent sheaf with zero-dimensional support, we denote by $i(\mathcal{Q}_s) := \hom_S(k(s), \mathcal{Q})$ the *socle* dimension of $\mathcal{Q}_s$.

Lemma 3.3. *Given any closed point* $p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0]$ *of* $M_{(S,\mathcal{E})}(n, q, d)$. *We have the relation:*

$$e(\mathcal{K}_s) = i(\mathcal{Q}_s) + (n - q).$$

For a proof of see [EL99]

Lemma 3.4. $|i(\mathcal{Q}_s) - i(\mathcal{Q}'_s)| \leq 1$, *for any* $s \in S$.

Proof. Applying the functor $\mathrm{Hom}(k(s), \bullet)$ to the top row of diagram (9) we obtain the exact sequence

$$0 \to k(s) \to \mathrm{Hom}(k(s), \mathcal{Q}') \to \mathrm{Hom}(k(s), \mathcal{Q}) \to \mathrm{Ext}^1(k(s), k(s)) \cong k(s)^{\oplus 2},$$

We verify by looking at the dimensions. □

Now we describe a global version of the elementary modifications.

Let n, q be fixed parameters, consider the sequence of schemes $\{Y_d\}_d$ where every Y_d is equal to $M_S(n, q, d) \times S = M_d \times S$ and consider the universal sequence

$$0 \to \underline{\mathcal{K}} \to \mathcal{O}_{M_d} \otimes \mathcal{E} \to \underline{\mathcal{Q}} \to 0.$$

Denote by $\mathcal{Z}$ the projectivization of $\underline{\mathcal{K}}$, then we have a the natural projection $\varphi = (\varphi_1, \varphi_2) : \mathcal{Z} \to Y_d$, where $\varphi_1 : \mathcal{Z} \to M_d$ and $\varphi_2 : Z \to S$.

On the scheme $\mathcal{Z} \times S$ there exist a natural epimorphism Λ which is the composition map

$$(\varphi_1, 1_S)^*\underline{\mathcal{K}} \to (1_{\mathcal{Z}}, \varphi_2)_*(1_{\mathcal{Z}}, \varphi_2)^*(\varphi_1, 1_S)^*\underline{\mathcal{K}} \to (1_{\mathcal{Z}}, \varphi_2)_*\varphi^*\underline{\mathcal{K}} \to (1_{\mathcal{Z}}, \varphi_2)_*\mathcal{O}_{\mathcal{Z}}(1) := \overline{\mathcal{K}}.$$

Then given the family $\underline{\mathcal{Q}}$ on Y_d we can obtained a family $\underline{\mathcal{Q}'}$ on Y_{d+1} by push-out and pull-back the following diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\mathcal{K}} & \longrightarrow & \underline{\mathcal{Q}'} & \longrightarrow & (\varphi_1, 1_S)^*\underline{\mathcal{Q}} & \longrightarrow & 0 \\
 & & \Lambda \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\varphi_1, 1_S)^*\underline{\mathcal{K}} & \to & \mathcal{O}_Z \otimes \mathcal{E} & \to & (\varphi_1, 1_S)^*\underline{\mathcal{Q}} & \longrightarrow & 0
\end{array} \tag{10}$$

For every $i \geq 0$ we define the closed subscheme

$$Y_{d,i} = \{(p, s) \in Y_d | i(\mathcal{Q}_s) = i, \text{ and } p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0]\}.$$

These sets form an stratification of Y_d.

Now we are ready to prove the main theorem of this section.

Theorem 3.5. *For any d the scheme Y_d is irreducible with dimension equal to $(d + q)(n - q) + d + 2$ and for any $i \geq 0$ we have that* $\operatorname{codim}(Y_{d,i}, Y_d) \geq 2i$.

As immediately corollary we have:

Corollary 3.6. *For any smooth projective surface S and parameters n, q, d the Quot scheme $M_{(S,\mathcal{E})}(n,q,d)$ is irreducible with dimension $(d+q)(n-q)+d$, unless if the Y_d is empty.*

Proof of theorem 3.5. We do induction on d.

Case $d = 0$. If $d = 0$ then every $p \in M_{(S,\mathcal{E})}(n,q,0)$ is an exact sequence of the form $p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0]$ where $\text{rank}(\mathcal{Q}) = q$ and $\text{length}(\text{Tor}(\mathcal{Q})) = 0$, then $M_S(n,q,d) \cong \mathcal{G}rass(q,\mathcal{E})$, so $\dim(Y_0) = \dim(\mathcal{G}rass(q,n) \times S) = (n-q)q + 2$.

Case $d+1$. Consider $\psi_1 : \mathcal{Z} \to M_{(S,\mathcal{E})}(n,q,d+1)$ the classifying morphism for the family $\underline{\mathcal{Q}'}$ on the diagram (10) and define $\psi = (\psi_1, \varphi_2) : \mathcal{Z} \to M_{(S,\mathcal{E})}(n,q,d+1) \times S = Y_{d+1}$. Then

$$\psi(\mathcal{Z}) = \left\{ (p,s) \in Y_{d+1} \;\middle|\; \begin{array}{c} \text{there exists } j \geq 1 \text{ such that } i(\mathcal{Q}_s) = j, \text{ where} \\ p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0] \in M_S(n,q,d+1) \end{array} \right\} = \bigcup_{j \geq 1} Y_{d+1,j}.$$

Let (p,s) be an element of $Y_{d,i}$, then by Lemma 3.3 the fiber of φ over (p,s) is $\mathbb{P}(\mathcal{K}_s)$ where $p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0]$, and so $\dim(\mathbb{P}(\mathcal{K}_s)) = i(\mathcal{Q}_s) + (n-q) - 1 = i + (n-q) - 1$. in a similar way for any element $(p', s) \in Y_{d+1,j}$ the fiber via the morphism ψ is $\mathbb{P}(\text{Soc}(\mathcal{K}'_{s'})^\vee)$ and then $\dim(\psi^{-1}(p', s)) = j - 1$. Now if $p' = [0 \to \mathcal{K}' \to \mathcal{E} \to \mathcal{Q}' \to 0]$ is obtained by elementary modifications of $p = [0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{Q} \to 0]$. Then by 3.4 $|i(\mathcal{Q}'_s) - i(\mathcal{Q}_s)| \leq 1$. It can be expressed in terms of the fibers of ψ and ϕ as:

$$\psi^{-1}(Y_{d+1,j}) \subset \bigcup_{|i-j| \leq 1} \varphi^{-1}(Y_{d,i}).$$

Now using the induction step and the dimension of the fibers we find the relation

$$\begin{aligned}
\dim(Y_{d+1,j}) &\leq \max_{|i-j|\leq 1}\{(d+q)(n-q) + d + 2 - 2j + (n-q+i-1)\} + 1 - j \\
&\leq \max_{|i-j|\leq 1}\{(d+q)(n-q) + d + 1 + 2 - 2j + (n-q) + i - j + 1\} \\
&\leq \dim(Y_{d+1}) - 2j - \min_{|i-j|\leq 1}\{i - j + 1\} \\
&\leq \dim(Y_{d+1}) - 2j.
\end{aligned}$$

The last inequality holds because $\min_{|i-j|\leq 1}\{i-j+1\} \geq 0$, and then $\text{cod}(Y_{d+1,j}, Y_{d+1}) \geq 2j$.

To prove the irreducibility of Y_{d+1} it is enough to show that $\mathcal{Z}_d = \mathbb{P}(\underline{\mathcal{K}})$ is irreducible, where $\underline{\mathcal{K}}$ is the kernel of the universal sequence associated to Y_{d+1}. Since S is smooth and projective, we can consider the finite resolution of locally free sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{O}_{M_S(n,q,d+1)} \otimes \mathcal{E} \to \underline{\mathcal{Q}} \to 0,$$

with $\mathrm{rank}(\mathcal{A}) = m$ and $\mathrm{rank}(\mathcal{B}) = m + (n-q)$ for some $m \in \mathbb{N}$. Then $\mathcal{Z} \subset \mathbb{P}(\mathcal{B})$ is defined as the zero-locus of the composition map $b \circ a \colon \varphi^*(\mathcal{A}) \xrightarrow{a} \varphi^*\mathcal{B} \xrightarrow{b} \mathcal{O}_{\mathbb{P}\mathcal{B}}(1)$, and, by induction, Y_d is irreducible and $\mathcal{Z}$ is locally defined by an irreducible variety of dimension $(n-q)(d+q)+d+2+(n-q+m-1)$ using m equations. In others words, every irreducible subvariety of $\mathcal{Z}$ has dimension greater than or equal to $(n-q)(d+q+1)+d+1$. On the other hand, the dimension of the fibers of $Y_{d,i}$ via φ is

$$\begin{aligned}
\dim(\varphi^{-1}(Y_{d,i})) &\leq (n-q)(d+q)+d+2-2i+(n-q)+i-1 \\
&= (n-q)(d+q)+d+1+(n-q)-i \\
&= (n-q)(d+q+1)+d+1-i.
\end{aligned}$$

Then, if $i \geq 1$ the dimension of the fiber of $Y_{d,i}$ is less than the dimension of the irreducible components of $\mathcal{Z}$, so $\varphi^{-1}(Y_{d,0}) \subset \mathcal{Z}$ is dense. Finally, since we know the dimension of the fiber of $Y_{d+1,i}$, we get that $\dim(Y_{d+1}) = \dim(Y_{d+1,i}) + 2 = \dim(\mathcal{Z}) + 2 = (d+1+q)(n-q) + (d+1) + 2$. □

Finally we present a table with some dimensions for $M(n,q,d)$.

n	q	d	$\dim(M(n,q,d))$
1	0	1	2
1	0	2	4
1	0	3	6
1	1	2	2
1	1	3	3
2	0	1	3
2	0	2	6
2	0	3	9
2	1	1	3
2	1	2	5
2	1	3	7
2	2	2	2
2	2	3	3
3	0	1	4
3	0	2	8
3	0	3	12
3	1	1	5
3	1	2	8
3	1	3	11
3	2	1	4
3	2	2	6
3	2	3	8
3	3	2	2

Table 2: Dimension

Euler characteristic of M(n,q,d) and torus action

Here we are interested in obtaining a formula for the topological Euler characteristic of

$$M_{(\mathbb{P}^2,\mathcal{O}^n_{\mathbb{P}^2})}(n,q,d) = M(n,q,d).$$

For that we show how to construct a $\mathbb{C}^*-$action on it with finitely many fixed points, of course at this moment we do not know if $M(n,q,d)$ is smooth or not but by Remark 1.65 in any case we can compute the Euler characteristic as the number of fixed points by a torus action.

Torus action

Let T_1 be the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$, $t.(a_1,\ldots,a_n) = (t^{u_1}a_1,\ldots,t^{u_n}a_n)$ for some weights $u_1,u_2,\ldots,u_n \in \mathbb{Z}$. Let w_0, w_1 and w_2 be integers such that $w_0+w_1+w_2 = 0$. Then we define the torus action T_2 (as before) on $\mathbb{P}^2$ by $t[a_0 : a_1 : a_2] = [t^{w_0}a_0 : t^{w_1}a_1 : t^{w_2}a_2]$. This action can be extended to $\mathcal{O}_{\mathbb{P}^2}$ and called the extension T_2'.

Let T be the product action $T_1 \times T_2'$ on $M(n,q,d)$, given any $p \in M(n,q,d)$ and for any $t \in \mathbb{C}^*$, the action is $t.p = t.[0 \to \mathcal{K} \to \mathbb{C}^n \otimes \mathcal{O} \xrightarrow{f} \mathcal{Q} \to 0] = [0 \to \mathcal{K} \to \mathbb{C}^n \otimes \mathcal{O} \xrightarrow{f\circ t^*} \mathcal{Q} \to 0]$, where the function t^* locally ($U_0 = \{x_0 \neq 0\}$) looks like $t^* : \mathbb{C}^n\otimes\mathbb{C}[x,y] \to \mathbb{C}^n\otimes\mathbb{C}[x,y], (a_i)_i\otimes p(x,y) \mapsto (t^{u_i}a_i)_i \otimes p(t^{w_1-w_0}x, t^{w_2-w_0}y)$.

The fixed locus of $M(n,q,d)$ by T are the collection of short exact sequences $p = [0 \to \mathcal{K} \to \mathbb{C}^n \otimes \mathcal{O} \xrightarrow{f} \mathcal{Q} \to 0]$, where

$$\mathcal{K} = \mathcal{I}_{s_1} \oplus \cdots \oplus \mathcal{I}_{s_k} \oplus \mathcal{O}^{n-q-k},$$

the support of every $\mathcal{I}_{s_i}$ is contained in one of the corners of $\mathbb{P}^2$, p_0, p_1, p_2. Every ideal sheaf $\mathcal{I}_{s_i}$ is a monomial ideal and $\sum_{i=1}^{k} \text{length}(\mathcal{O}_{s_i}) = d$, this is for k in the set $\{1, 2, \ldots, n-q\}$.

Note that every possible permutation of the ideal sheaf $\mathcal{I}_{s_i}$ of $\mathcal{K}$ gives us a new fixed point, because we count different submodules of $\mathcal{O}^n$, and not simply abstract isomorphic ideal sheaves.

By simplicity we denote the fixed locus $M(n,q,d)^T$ by Λ.

Lemma 3.7. *The set*

$$\Lambda = \left\{ [0 \to \mathcal{I}_{s_1} \oplus \cdots \oplus \mathcal{I}_{s_k} \oplus \mathcal{O}^{n-q-k} \to \mathcal{O}^n \to \mathcal{Q} \to 0] \;\middle|\; \begin{array}{l} s_1, \ldots, s_k \in \mathbb{P}^2, d = \sum_{i=1}^k \operatorname{length} s_i, \\ \operatorname{Supp} s_i \subseteq p_0, p_1, p_2, \\ k = 1, \ldots n-q. \end{array} \right\}$$

Is finite.

We have seen in the last chapters that the monomial ideals $\mathcal{I}_{s_i}$ are in correspondence with the Young tableaux with length equal to length $\mathcal{O}_{s_i} = L_i$.

Our purpose is to find some formula for the cardinality of Λ and with this get a way to compute the Euler characteristic of $M(n, q, d)$.

First we fix some $k \in \{1, \ldots, n-q\}$, and let $x = [0 \to \mathcal{I}_{s_1} \oplus \cdots \oplus \mathcal{I}_{s_k} \oplus \mathcal{O}^{n-q-k} \to \mathcal{O}^n \to \mathcal{Q} \to 0]$ be an element in Λ with a fixed immersion. This has to be such that $\sum_{i=1}^k L_i = d-k$ because if some ideal sheaf $\mathcal{I}_{s_i}$ appears then its length is at least 1. Now suppose that $L_1 + \cdots + L_k = d-k$ is one of these possibles configurations, since every ideal sheaf can be supported in p_0, p_1, p_2 each L_i will be distributed in triples of non-negative integers (d_i^0, d_i^1, d_i^2) such that $L_i = d_i^0 + d_i^1 + d_i^2$. By the discussion given before to present Theorem 2.12 we know that we have $P(d_i^j)$ possibilities to organize the support of the ideal $\mathcal{I}_{s_i}$ at the point p_j, so we obtain the formula $\displaystyle\sum_{L_1+\cdots+L_k=d-k} \prod_{i=1}^{k} \sum_{L_i=d_i^0+d_i^1+d_i^2} \prod_{t=0}^{2} P(d_i^j)$. Until now, we only have to count the possibles elements x with a fixed immersion of $\mathcal{I}_{s_1} \oplus \cdots \oplus \mathcal{I}_{s_k} \oplus \mathcal{O}^{n-q-k} \to \mathcal{O}^n$ to $\mathcal{O}^n$, then we can vary these immersions of $\binom{n}{k}\binom{n-k}{n-k-q}$ forms. The first combinatorial number says the possibilities to choose the k immersions of the ideal $\mathcal{I}_{s_i}$ on one of the n copies of $\mathcal{O}$, similarly the second combinatorial number counts the possibilities to send the $n-k-q$ copies of $\mathcal{O}$ on the $n-k$ free copies. Finally we vary k we get the formula :

$$\chi(M(n, q, d)) = \sum_{k=1}^{n-q} \binom{n}{k} \binom{n-k}{n-k-q} \sum_{L_1+\cdots+L_k=d-k} \prod_{i=1}^{k} \sum_{L_i=d_i^0+d_i^1+d_i^2} \prod_{t=0}^{2} P(d_i^j) \qquad (11)$$

Let $\mathbb{P}$ be the set of all homogeneous polynomials over $\mathbb{C}$ of degree $d-k$ in k variables. They are in correspondence one-to one with the set of all $k-$tuples such that $L_1 + \cdots + L_k = d-k$, then denoting by $\eta_{p,i} = \sum_{L_i=d_i^0+d_i^1+d_i^2} \prod_{t=0}^{2} P(d_i^p)$ and $\eta_p = \prod \eta_{p,i}$ we reorganize the formula 11 as in the following theorem.

Theorem 3.8.
$$\chi(M(n,q,d)) = \sum_{k=1}^{n-q} \binom{n}{k} \binom{n-k}{n-k-q} \sum_{p \in \mathbb{P}} \eta_p.$$
where $\mathbb{P}$ is the set of all homogeneous polynomials over $\mathbb{C}$ of degree $d-k$ in k variables.

Remark 3.9.

The case $\chi(M(1,0,d)) = \chi(\text{Hilb}^d(\mathbb{P}^2)) = \sum_{d_0+d_1+d_2=d} p(d_0)p(d_1)p(d_2)$ shown in Theorem 2.12 is a particular case where $k=1$ is the unique possibility for numbers of points s_i.

Smoothness of $M_{(\mathbb{P}^2,\mathcal{O}^n)}(n,q,d) = M(n,q,d)$

In general the scheme $M(n,q,d)$ is singular for various parameters n,q,d. Here we find some conditions on the parameters to get smoothness and show an example of the singular case. For these we use the techniques of deformation theory presented on the chapter 1 section 1.10.

Lemma 3.10. *Let x be an element of the singular locus of $M(n,q,d)$. Then $\tilde{x} = \lim_{t\to 0} t.x$ is a fixed point.*

Proof. Since the singular locus of M is a closed subscheme, then $\tilde{x} \in Sing(M)$. Define the the map $(-)\cdot x : \mathbb{C}^* \to M$ by $t \mapsto t\cdot x$, by the valuation criterion there exist a morphism $\phi : \mathbb{C} \to M$, such that for any $t \in \mathbb{C}^*$, $\phi(t) = t\cdot x$ and $\phi(0) = \tilde{x}$. (As in Remark 1.61.)

Now define $\psi : \mathbb{C}^* \times \phi(\mathbb{C}) \to M$ by $(t,y) \mapsto t.y$, then $\psi(\overline{\mathbb{C}^* \times \phi(\mathbb{C})}) \subseteq \overline{\psi(\mathbb{C}^* \times \phi(\mathbb{C}))} = \overline{\phi(\mathbb{C}^*)} = \phi(\mathbb{C})$, so $\phi(\mathbb{C})$ is a union of orbits, then $\phi(\mathbb{C}^*)$ is a whole orbit, therefore $\tilde{x}$ is a fixed point. □

By the last lemma all the possible singular points on $M(n,q,d)$ are fixed points by the $\mathbb{C}^*-$action. We are ready to present the next result of this work.

Theorem 3.11. *For any parameters n, d the scheme $M(n, n-1, d)$ is smooth.*

Proof. Let p be a fixed point by the $\mathbb{C}^*-$action T, then by the last discussion p is of the form

$$p = [0 \to \mathcal{I}_Z \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{O}_Z \oplus \mathcal{O}^{(n-1)} \to 0,]$$

where Z is a subscheme of $\mathbb{P}^2$ of length d supported on torus fixed points of $\mathbb{P}^2$. And by Proposition 1.68 we know that the tangent space of $M(n, n-1, q)$ at a point p is isomorphic to

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z \oplus \mathcal{O}^{(n-1)}) \cong \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \oplus \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O})^{(n-1)}.$$

Then $\dim(T_pM) = 2d + (n-1)$, the number $2d$ is the dimension of the smooth scheme $\mathrm{Hilb}^d(\mathbb{P}^2)$ (proposition 2.10). On the other hand we know by Theorem 3.5 that $\dim(M(n, n-1, d) = (n-1+d)(n-n+1) + d = 2d + (n-1)$. Then for any point the dimension of the tangent space at this point is the same of the dimension of the scheme $M(n, n-1, d)$. Therefore $M(n, n-1, d)$ is smooth. □

As a counterpart of the previous theorem we have that for any $0 \leq q \leq n-2$ the scheme $M(n, q, d)$ is singular.

Example 3.12. Consider the scheme $M = M(2, 0, 2)$. Then by 3.6 we that see $\dim(M(2, 0, 2)) = 2(2+1) = 6$. On the other hand every point p is the form $[0 \to \mathcal{K} \to \mathcal{O}^2 \to \mathcal{Q} \to 0]$ where $\mathrm{rank}(\mathcal{Q}) = 0$ and $c_2(\mathcal{Q}) = 2$. These can be classified in three types, since $\mathcal{Q}$ is a sheaf supported in some subscheme Z of $\mathbb{P}^2$ of length 2.

Types:

1. $p = [\mathcal{O}^2 \to \mathcal{Q} \to 0]$, where $\mathcal{Q} = \mathcal{O}_{s_1} \oplus \mathcal{O}_{s_2}$ and $s_1 \neq s_2$;

2. $p = [\mathcal{O}^2 \to \mathcal{Q} \to 0]$, where $\mathcal{Q} = \mathcal{O}_Z$ and $\mathrm{Supp}(Z) = \xi$ and $\mathrm{length}(\xi) = 2$.

3. $p = [\mathcal{O}^2 \to \mathcal{Q} \to 0]$, where $\mathcal{Q} = \mathcal{O}_{s_1} \oplus \mathcal{O}_{s_1}$;

For any of these kind of points we compute the tangent space of M at these points.

1. Let $p = [0 \to \mathcal{I}_{s_1} \oplus \mathcal{I}_{s_2} \to \mathcal{O}^2 \to \mathcal{O}_{s_1} \oplus \mathcal{O}_{s_2} \to 0]$ be any point of the first type. Then

$$\begin{aligned} T_pM \cong & \mathrm{Hom}(\mathcal{I}_{s_1} \oplus \mathcal{I}_{s_2}, \mathcal{O}_{s_1} \oplus \mathcal{O}_{s_2}) \\ & \cong \mathrm{Hom}(\mathcal{I}_{s_1}, \mathcal{O}_{s_1}) \oplus \mathrm{Hom}(\mathcal{I}_{s_1}, \mathcal{O}_{s_2}) \oplus \mathrm{Hom}(\mathcal{I}_{s_2}, \mathcal{O}_{s_1}) \oplus \mathrm{Hom}(\mathcal{I}_{s_2}, \mathcal{O}_{s_2}), \end{aligned}$$

   so $\dim(T_pM) = 2 + 1 + 1 + 2 = 6$.

2. Let $p = [0 \to \mathcal{I}_Z\mathcal{O}_Z \to \mathcal{O}^2 \to \mathcal{O}_Z \to 0]$, where $\mathrm{Supp}(Z) = \xi$ and $\mathrm{length}(\xi) = 2$. Then

$$T_pM \cong \mathrm{Hom}(\mathcal{I}_Z \oplus \mathcal{O}_Z, \mathcal{O}) \cong \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}) \oplus \mathrm{Hom}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}),$$

   so $\dim(T_pM) = 2(2) + 2 = 6$.

3. Let $p = [0 \to \mathcal{I}_{s_1} \oplus \mathcal{I}_{s_1} \to \mathcal{O}^2 \to \mathcal{Q} \to 0]$ be some point of the third type. Then

$$T_pM \cong \mathrm{Hom}(\mathcal{I}_{s_1} \oplus \mathcal{I}_{s_1}, \mathcal{O}_{s_1} \oplus \mathcal{O}_{s_1}) \cong \mathrm{Hom}(\mathcal{I}_{s_1}, \mathcal{O}_{s_1})^4,$$

   so $\dim(T_pM) = 2(4) = 8 \neq 6$.

Then the points of the third type are the singulars points on M, and clearly this imply that M is not smooth.

In the proof of Theorem 3.11 we see that the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}^2)$ appears in the computation of the dimension of the tangent space, this is not a coincidence since we can show that $M(n, n-1, d)$ is a $\mathbb{P}^{n-1}-$bundle of $\mathrm{Hilb}^d(\mathbb{P}^2)$.

Let $p = [0 \to \mathcal{K} \to \mathcal{O}^n \to \mathcal{Q} \to 0]$ be an element of $M(n, n-1, d)$, at least one of the compositions $\mathcal{K} \xrightarrow{\iota} \mathcal{O}^n \xrightarrow{\pi_i} \mathcal{O}$ is not the zero map. Then $\mathcal{K} \xrightarrow{\pi_i \circ \iota} \mathcal{O}$ is an inclusion for some $i \in \{1, 2, \ldots, n\}$, because $\mathcal{K}$ is a torsion-free sheaf of rank 1, and so it is isomorphic to some ideal sheaf $\mathcal{I}_Z$ where $Z \subseteq \mathbb{P}^2$ is a closed subscheme of length d. We define the map $\pi\colon M(n, n-1, d) \to \mathrm{Hilb}^d(\mathbb{P}^2)$ by:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}^{\oplus n} & \longrightarrow & Q & \longrightarrow & 0 & \in & M_{(n,n-1,d)} \\ & & & & \downarrow & & & & & & \downarrow \pi \\ 0 & \longrightarrow & \mathcal{I}_Z & \to & \mathcal{I}_Z^{\vee\vee} = \mathcal{O} & \to & \mathcal{O}_Z & \longrightarrow & 0 & \in & Hilb^{[d]}(\mathbb{P}^2), \end{array}$$

where $\mathcal{K}^\vee$ denote the dual sheaf $\mathrm{Hom}(\mathcal{K}, \mathcal{O}_X)$.

Remark 3.13. The map π is such that every fiber is isomorphic to $\mathbb{P}^n$. Then we can show that $M(n.n-1, d)$ is connected because the base of π and every fiber is connected.

4 Atiyah-Bott formulas and virtual Atiyah-Bott formulas

4.1 Equivariant cohomology

For the understanding of the geometry of quotient spaces X/G of schemes X by an algebraic group G action, i the *equivariant cohomology* $H^*_G(X)$ is defined. The trick is to exchange the space X for a new space X_G and relate the cohomology of these two spaces.

Definition 4.1. Let G be reductive algebraic group. We call X a $G-$space if there exists some action of G on X.

Not every $G-$space X is such that G acts freely. However G can be made to act " freely p to homotopy". We explain how this is done.

Definition 4.2. A scheme E is called a *universal $G-$space* if it is a $G-$space with free action of G and is contractible.

It is not difficult to see that when this universal space exists, it is unique up to homotopy, for that reason we write it as E_G and refer to it as *the universal* $G-$space of G.

Definition 4.3. The *$G-$equivariant cohomology* of the $G-$space X is simply the cohomology of the space $X_G = (X \times E_G)/G$ i.e.

$$H^*_G(X) := H^*(X_G).$$

The quotient space $E_G/G := B_G$ is called the *Classifying* $G-$space. This space classifies the principal G-bundles, that is B_G is the moduli space associated to the functor $Bun_G(\bullet)$.

Example 4.4 (Classical example)**.** Let $T : \mathbb{C}^*$ be the 1−dimensional torus. The space $\mathbb{C}^\infty - \{0\} = \lim\limits_{\rightarrow} \mathbb{C}^n - 0$ is contracible and T acts freely on it, then $E_T = \mathbb{C}^\infty - \{0\}$. Furthermore $B_T = E_T/T = \lim\limits_{\rightarrow} \mathbb{C}^n - \{0\}/\mathbb{C}^* = \lim\limits_{\rightarrow} \mathbb{P}^n := \mathbb{P}^\infty$. The $T-$equivariant cohomology for a point pt can be compute as:

$$H^*_T(pt) = H^*((pt \times E_T)/T) = H^*(B_T) = H^*(\mathbb{P}^\infty) = \mathbb{Q}[\lambda],$$

where $\lambda = -c_1(E_T) \in H^2(B_T)$, in other words is the polynomial ring with coefficients in the rational number with indeterminate λ of degree 2.

Here are some facts above equivariant cohomology:

1. Given any $G-$space X, the equivariant cohomology $H_G^*(X)$ is a $H_G^*(pt)-$module and there exist a map $\sigma^* : H^*(X/G) \to H_G^*(X)$.

2. If the action of G on X is free, then $H_G^*(X) = H^*((X \times E_G)/G) = H^*(X/G \times E_G) = H^*(X/G)$.

3. The map $\pi : X_G \to B_G$ is a fibration with fiber X.

4. Let V a $G-$equivariant vector bundle over the $G-$space X. Then $V_G = (V \times E_G)/G$ is vector bundle over X_G. We define the define the $G-$equivariant chern classes of V as $c_i^G(V) := c_i(V_G) \in H^{2i}(X_G) = H_G^{2i}(X)$.

4.2 Localization and integration Atiyah-Bott formulas

The group G will be a torus T at this moment. Let X be a $T-$space, and suppose that the fixed locus X^T can be written as $\bigcup_i X_i$, where every X_i is irreducible.

Th inclusion maps $\iota_{X_i} : X_i \to X$ allows us to define the pull- back and push-forward maps:

$$\iota^*_{X_i^T} : H_T^*(X) \to H_T^*(X_i) = H^*(X) \otimes H_T^*(pt),$$

and

$$\iota_{X_i^T,*} : H_T^k(X) \to H_T^{k+r}(X),$$

with $r = \mathrm{cod}(X_i, X)$.

We do not say anything about the proof of the following important formula which can be found in [Hus66, cap.II, Theorem 2.8].

Proposition 4.5. *The composition map*

$$\iota^*_{X_i^T} \circ \iota_{X_i^T,*} : H_T^k(X) \to H_T^{k+r}(X)$$

is exactly the cup product with the $T-$equivariant Euler class of the normal bundle of X at X_i, i.e.

$$\iota^*_{X_i^T} \circ \iota_{X_i^T,*}(\alpha) = \alpha \cup e^T(\mathcal{N}_{X_i}(X)).$$

The example 4.4 can be extend, changing $T = \mathbb{C}^*$ by $(\mathbb{C}^*)^n$, and then the $T-$equivariant cohomology of a point is $H_T := H^*_T(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_n]$.

Notation 4.6. We denote by F_T the function field of H_T.

Proposition 4.7 (Atiyah-Bott)**.** *The class $e^T(\mathcal{N}_{X_i}(X)) \in H^*(X_i) \otimes H_T$, has inverse on $H_T(X_i) \otimes_{\mathbb{Q}} F_T$.*

Sketch of proof. We can write the normal bundle as the direct sum of tensor of eingensub-bunbles $\mathcal{V}_\rho$ with line bundles L_ρ associated to characters ρ,

$$\mathcal{N}^T_{X_i}(X) = \bigoplus_{\rho \in \mathrm{Hom}(T,\mathbb{C}^*)} \mathcal{V}_\rho \otimes L_p.$$

Denote by $x_{\rho,j}$ the $j-$th Chern roots of $\mathcal{V}_\rho$, then

$$e^T(\mathcal{N}_{X_i}(X)) = \prod_\rho \prod_j (x_{\rho,j} + \lambda_\rho),$$

therefore

$$(e^T(\mathcal{N}_{X_i}(X)))^{-1} = \prod_\rho \lambda_\rho^{-\operatorname{rank} \mathcal{V}_\rho} \prod_j \left(\sum_i (-1)^i \left(\frac{x_{\rho,j}}{\lambda_\rho} \right)^i \right) \in H^*_T(X_i) \otimes_{\mathbb{Q}} F_T.$$

□

Proposition 4.8. *The association map $\phi\colon \bigoplus_i H(X_i) \otimes F_T \to H_T(X) \otimes_{H_T} F_T$, given by $\phi(\{a_i\}) = \sum_i \iota_{X_i^T,*}(a_i)$ is an isomorphism of F_T-modules.*

Proof. Use directly the proposition 4.7. □

Finally we present two important formulas to evaluate integral of the form $\int_X \alpha := [\alpha] \cup \mu_X$, where μ_X is the fundamental class of X, in terms of irreducible components X_i of the fixed locus and the class $(e^T(\mathcal{N}_{X_i}(X)))^{-1}$.

Definition 4.9. A class $\alpha \in H^*(X)$ has an *equivariant extension* if it is the image of some $\tilde{\alpha} \in H_T^*(X)$ via the pull-back map ι^*. i.e. $\iota^*(\tilde{\alpha}) = \alpha$.

Theorem 4.10. *1.* Atiyah-Bott localization formula. *Given any $\tilde{\alpha} \in H_T^*(X)$, then:*

$$\tilde{\alpha} = \sum_i \iota_{X_i^T,*}\left(\frac{\iota^*_{X_i^T}}{e^T(\mathcal{N}_{X_i}(X))}\right)$$

2. Atiyah-Bott integration formula. *For any $\alpha \in H^*(X)$ with a equivariant extension $\tilde{\alpha}$ we have:*

$$\int_X \alpha = \int_{X_T/B_T} \tilde{\alpha} = \sum_i \int_{X_i^T/B_T}\left(\frac{\iota^*_{X_i^T}}{e^T(\mathcal{N}_{X_i}(X))}\right).$$

Theorem 4.10 is proving by calculation using Propositions 4.5 and 4.7.

As example of the use of the theorem 4.10 we show how compute the topological Euler characteristic of some $T-$space X.

Proposition 4.11. *Let X be a $T-$space, and suppose that $X^T = \bigcup_i X_i$. Then*

$$\chi(X) = \sum_i \chi(X_i).$$

Proof. Recall that $\chi(X) = \int_X e(\mathcal{T}X)$. Then by 4.10 we have

$$\begin{aligned}
\chi(X) = \int_X e(\mathcal{T}X) &= \int_{X_T/B_T} e^T(\mathcal{T}X) \\
&= \sum_i \int_{X_i^T/B_T} \frac{\iota^*_{X_i^T} e^T(\mathcal{T}X)}{e^T(\mathcal{N}_{F_i}X)} \\
&= \sum_i \int_{X_i} e(\mathcal{T}X_i) \\
&= \sum_i \chi(X_i).
\end{aligned}$$

□

Remark 4.12. Note that 4.11 give a proof of 1.62, because under the hypothesis that every X_i is a point, we have $\chi(X_i) = 1$ and then $\chi(X) = \sum_{\text{fixed points}} 1 = \#$ Fixed points.

4.3 Virtual Fundamental class.

The virtual fundamental class of some scheme X is the substitute of the fundamental class for singular schemes. The virtual fundamental class $[X]^{\mathrm{Vir}} \in H^*(X)_{d_{\mathrm{Vir}}(X)}$, where $d_{\mathrm{Vir}}(X)$ is the virtual dimension of X, thus the virtual fundamental class a cohomology class in the expected dimension of X. If X is such that its real and virtual dimension are the same we say that X has correct dimension and in this case the $[X] = [X]^{\mathrm{Vir}}$. To have a correct definition of virtual dimension it is necessary to introduce a perfect obstruction theory for X. See [GP99].

Suppose here that X can be embedding in Y, where Y is a smooth variety over $\mathbb{C}$.

Definition 4.13. *A perfect obstruction Theory* for X is a map $\phi : [E^{-1} \to E^0] \to L_X^\bullet$, where E^i is a sheaf on X and $L_X^\bullet = [\mathcal{N}_{X/Y}^\vee \to \Omega_Y|_M]$ the $2-$truncated cotangent complex, such that ϕ induce a isomorphism on $0-$cohomology and a surjection on $(-1)-$cohomology.

Definition 4.14. Given a Perfect obstruction E for X. The *Virtual dimension* of X (depending of E) is defined by $d_{\mathrm{Vir}}(X) = \mathrm{rank}[E^0] - \mathrm{rank}[E^{-1}]$.

Proposition 4.15. *The virtual dimension is independent of the perfect obstruction for X and only depends on the cohomology of $L_X^\bullet$.*

Definition 4.16. Using the last proposition we can define the *Virtual dimension* of X as:

$$\mathrm{rank}\, h^0 - \mathrm{rank}\, h^{-1}.$$

Proposition 4.17. *With the conditions above the following inequality holds:*

$$d_{\mathrm{Vir}}(X) \leq \dim(X).$$

Definition 4.18. We say that X has the *correct dimension* if the inequality in 4.17 is an equality, and we say that X is unobstructed if the $(-1)-$cohomology is trivial, i.e. $h^{-1} = 0$.

To construct a perfect obstruction theory we will assume that a group G is acting in X, Y and the embedding from X to Y is $G-$equivariant. Since we always use $\mathbb{C}^*-$actions, in this work we can assume $G = \mathbb{C}^*$.

Under these hypothesis the cotangent complex of X is $L^\bullet_X = [I/I^2 \to \Omega_Y]$, where I is the ideal sheaf of X as closed subscheme of Y. Using the fact there are enough locally-fee sheaves [Har77][ex.6.8, cap III], it can be how hat there is an equivariant perfect obstruction theory $\phi : E^\bullet \to [I/I^2 \to \Omega_Y]$, where ϕ is a map of $2-$terms complexes.

Using the commutative diagram

$$\begin{array}{ccc} E^{-1} & \xrightarrow{\delta} & E^0 \\ \downarrow \phi^{-1} & & \downarrow d \\ I/I^2 & \xrightarrow{\phi^0} & \Omega_Y \end{array} \tag{12}$$

We get the exact sequence of sheaves

$$E^{-1} \overset{(\phi^{-1},\delta)}{\longrightarrow} I/I^2 \oplus E^0 \overset{\gamma}{\to} \Omega_Y \to 0,$$

where $\gamma(i,e) = d(i) - \phi^0(e)$.

Let $Q = \ker(\gamma)$. Taking cones there, exists an exact sequence $0 \to T_Y \to C(I/I^2) {\times_X} E_0 \to (Q) \to 0$, $(E^i = E_i^\vee)$. Note that $C(Q)$ is a closed sub-cone of E_1.

Definition 4.19. Let $D = C(X)|_Y \times_X E_0$ this is a closed subcone of $C(I/I^2) \times_X E_0$. With the notation above we define the *virtual fundamental class* of D by $[D]^{\text{Vir}} := D/T_Y$, this is a subcone of $C(Q)$ and hence of E_1, and the *Virtual fundamental class* of X as the refined intersection $[D]^{\text{Vir}} \cap [0_{E_1}]$, where 0_{E_1}, is the zero section of the vector bundle E_1.

Notation 4.20. The notation X^G denote the scheme theoretic fixed point locus, i.e. If $X = \text{Spec}(A)$, then $X^G = \mathcal{Z}(I)$, where $I = \langle \mathbb{C}^* - \text{eigenfunction with nontirvial characters} \rangle$.

Is easy to see that $X^G = Y^G \cap X$ and if $Y^G = \bigcup_i Y_i$ irreducible decomposition then $X_i = X \cap Y_i$ form a decomposition of X.

Given any coherent sheaf S on X_i, this can write as $S = \bigoplus_{k\in\mathbb{Z}} S^k$, where $S^k = \mathbb{C}^* -$ eigensheaf of degree k. Then $S^0 = S^G$ is the fixed part of S and $S^{mov} = \bigoplus_{k\neq 0} S^k$, is the moving part of S.

With the last notation we have that if $\Omega_Y|_{Y_i}^G = \Omega_{Y_i}$ then $\Omega_X|_{X_i}^G = \Omega_{X_i}$.

Proposition 4.21. *Let $E_i^\bullet = E^\bullet|_{X_i}$, we have a map $\varphi_i : E_i^{\bullet,G} \to L_{X_i}^\bullet$, given by*

$$E_i^{\bullet,G} \overset{\phi_i^G}{\to} L^\bullet|_{X_i}^G \overset{can}{\to} L_{X_i}^\bullet.$$

Then φ_i is a perfect obstruction theory on X_i.

By the last proposition 4.21, we can construct a virtual structure over every X_i.

Definition 4.22. The virtual normal class is by definition $N_i^{\mathrm{Vir}} : (E_\bullet, i)^{\mathrm{mov}}$.

Definition 4.23. Let $[B_0 \to B_1]$ be a $2-$complex, the top Chern class is given by $c_{top}([B_0 \to B_1]) = e([B_0 \to B_1])\colon = e(B_0)/e(B_1)$, in the cases where it can be defined.

4.4 Localization and integration virtual Atiyah-Bott formulas

With all the terminology given above we have a natural structure to give a virtual generalization of theorem 4.10. In fact

Theorem 4.24. *Let $\iota : X \to Y$ the $\mathbb{C}^*-$equivariant embedding of X into a smooth scheme Y. Then*

$$[X]^{\mathrm{Vir}} = \iota_* \sum_i \frac{[X_i]^{\mathrm{Vir}}}{e(N_i^{\mathrm{Vir}})},$$

and

$$\int_{[X]^{\mathrm{Vir}}} e(A) = \sum_i \int_{[X_i]^{\mathrm{Vir}}} \frac{e(A_i)}{e(N_i^{\mathrm{Vir}})},$$

where A is bundle of rank equal to $d_{\mathrm{Vir}}(X)$.

Remark 4.25.

- Since N_i^{Vir} is a complex with nonzero $\mathbb{C}^*-$weights, $e(N_i^{\text{Vir}})$ is invertible in

$$H^*_{\mathbb{C}^*,t}(X) := H^*_{\mathbb{C}^*}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, {}^1\!/\!_t]$$

- The second formula should be a consequence of a localization formula in equivariant $H^*(X)-$groups. The key result on a nonsingular Y is the formula given in 4.10

$$[Y] = \iota_* \sum_i \frac{[Y_i]}{e(N_i)} \in H^*_{\mathbb{C}^*,t}(X).$$

- Here we only present a proof of 4.24 in the most basic case, i.e. when Y is a nonsingular variety with a $\mathbb{C}^*-$action, and given any $\mathbb{C}^*-$bundle V on Y we take some equivariant section $v \in \Gamma(Y, V)^{\mathbb{C}^*}$ and define X as the zero section $Z(v)$ of v inside Y. The general proof can be found in [GP99].

- In the final section of this work we will go to present a concrete example of the use of these results.

Proof of 4.24. First considering the diagram

$$\begin{array}{ccc} E^\bullet = [V^\vee|_X & \xrightarrow{d(.v)} & \Omega_Y|_X] \\ \Big\downarrow (.v) & & \Big\downarrow 1_{\Omega_Y|_X} \\ L^\bullet = [I/I^2 & \longrightarrow & \Omega_Y|_X], \end{array}$$

We have a perfect obstruction theory for X. In this case the virtual class of X is just the refined Euler class of V, so

$$[X]^{\text{Vir}} = e_{ref}(V), \tag{13}$$

where the expression of the right hand is the refined product between the graph of v and the zero section,i.e. $\Gamma_v \cap 0_v$.

Now observe that since v is a $\mathbb{C}^*-$invariant section then $v \in \Gamma(Y_i, V_i^G)$ and $X_i = \mathcal{Z}(v) \cap Y_i$, and by the proposition 4.21 we obtain a perfect obstruction theory for the pair V_i^G and $v \in H^0(Y_i, V_i^G)$:

$$[(V_i^G)^\vee \to \Omega_{Y_i}]$$

and therefore

$$[X_i]^{\text{Vir}} = e_{ref}(V_i^G). \tag{14}$$

The virtual normal bundle is by definition the moving part of the complex $[T_{Y_i} \to V_i]$, but the moving part T_{Y_i} is just the normal bundle of Y_i, i.e. $N_i^{\text{Vir}} = [N_{Y_i|_Y} \to V_i^{\text{mov}}]$, thus

$$e(N_i^{\text{Vir}}) \colon = \frac{e(N_{Y_i|_Y})}{e(V_i^{\text{mov}})} \tag{15}$$

Now substituting in the expression

$$[X]^{\text{Vir}} = \iota_* \sum_i \frac{[X_i]^{\text{Vir}}}{e(N_i^{\text{Vir}})}$$

the expressions given by 13,14 and 15 we are reduced to proving

$$e_{ref}(V) = \iota_* \sum_i \frac{e_{ref}(V_i^G) \cap e(V_i^{\text{mov}})}{e(N_{Y_i}|_Y)} = i_* \sum \frac{e_{\text{ref}(V_i)}}{e(N_{Y_i}|_Y)}$$

But by the localization formula on Y we have that

$$[Y] = \iota_* \sum_i \frac{[Y_i]}{e(N_{Y_i}|_Y)}$$

and capping with $e_{ref}(V)$ we obtain

$$e_{ref}(V) = \iota_* \sum_i \frac{e_{ref}(V) \cap [Y_i]}{e(N_{Y_i}|_Y)} \tag{16}$$

this because pullback commutes with take $e_{ref}(.)$.

But, $V_i = V_i^G \oplus V_i^{\text{mov}}$, and since the section is entirely in V_i^G, hence

$$e_{ref}(V_i) = e_{ref}(V_i^G) \cap e_{ref}(V_i^{\text{mov}}),$$

finally substituting the last equality in the equation 16 we obtain the theorem. □

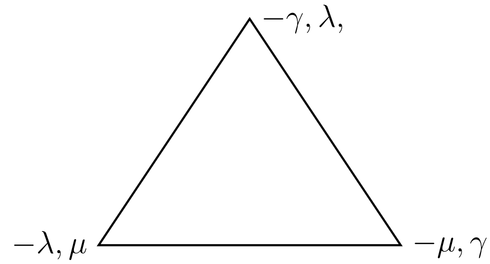
5 Final example

Finally in this thesis we show an example of how to use the Atiyah-Bott's formulas for compute the virtual Euler characteristic of $M = M(3, 2, 2)$.

Example 5.1.

$$\int_{[M]^{Vir}} 1 = 270.$$

The rest of this chapter is devoted to proof this statement. Let $w_0 < w_1 < w_2$ the weights of the action of $\mathbb{C}^*$ on $\mathbb{P}^2$, we call $\mu = w_2 - w_0$; $\lambda = w_0 - w_1$ and $\gamma = w_1 - w_2$ the weights of this action around the corners of $\mathbb{P}^2$. We have the following picture:



(17)

and $\mu + \lambda + \gamma = 0$. Then the $\mathbb{C}^*-$action T on M depend of the weights $u_1, \ldots, u_n$ given by the $\mathbb{C}^*-$action on $\mathbb{C}^n$ and λ, μ since $\gamma = -(\mu + \lambda)$.

Using the virtual Atiyah-Bott's formulas we can write

$$Y = \int_{[M]^{Vir}} 1 = \sum_{\text{fixed points}} \int_{[pt]} \frac{1}{e_T(N^{vir})}.$$

Now recall that

$$[N]^{Vir} = \oplus_{i=1}^{s} \oplus_{j=1}^{s} \mathrm{Ext}^\bullet(I_{z_i}, \mathcal{O}_{z_i}) \oplus_{i=1}^{s} \mathrm{Ext}^\bullet(I_{z_i}, \mathcal{O}^q) \oplus_{j=1} \mathrm{Ext}^\bullet(\mathcal{O}^{n-q-k}, \mathcal{O}_{z_j}) \oplus \mathrm{Ext}^\bullet(\mathcal{O}^{n-q-k}, \mathcal{O}),$$

and the fixed points of M are short exact sequences of the form $[0 \to \mathcal{I}_z \oplus \mathcal{O}^0 \to \mathcal{O}^3 \to \mathcal{O}_z \oplus \mathcal{O}^2 \to 0]$, in our case $n - q - k = 0$ and so

$$[N]^{Vir} = \mathrm{Ext}^\bullet(I_z, \mathcal{O}_z) \oplus \mathrm{Ext}^\bullet(I_z, \mathcal{O}^2).$$

Then

$$Y = \sum \int_{[pt]} \frac{1}{e_T(\mathrm{Ext}^\bullet(I_z, \mathcal{O}_z)) e_T(\mathrm{Ext}^\bullet(I_z, \mathcal{O})) e_T(\mathrm{Ext}^\bullet(I_z, \mathcal{O}))}. \quad (18)$$

The last formula shows us that we have to do three things to compute the integral: (1) compute the dimensions of all different Ext group involved, (2) find all the pictorial configurations of the possibles fixed points and (3) calculate the Chern roots of the Ext groups depending of the type of points in the configurations.

For (1), we identify $\operatorname{Ext}^0(I_z, \mathcal{O}_z)$ as the tangent space of $(\mathbb{P}^2)^{[2]}$ at point I_z and $\operatorname{Ext}^1(I_z, \mathcal{O}_z)$ as the obstruction space at the same point, so $\dim(\operatorname{Ext}^0(I_z, \mathcal{O}_z)) = 4$ and $\dim(\operatorname{Ext}^1(I_z, \mathcal{O}_z)) = 2$.

The group $\operatorname{Ext}^0(I_z, \mathcal{O}_z)$ has dimension 1; in fact, let $f\colon I_z \to \mathcal{O}$ any homomorphism, then $\overline{f}\colon I_z|_{\mathbb{P}^2-\{z\}} \to \mathcal{O}_{\mathbb{P}^2-\{z\}} \cong \mathcal{O}$, is such that $\overline{f}(1) = \sigma \in \Gamma(\mathbb{P}^2 - \{z\}, \mathcal{O})$, but since $\operatorname{cod}_{\mathbb{P}^2}(\{z\}) = 2$ the section σ can be extended to $\Gamma(\mathbb{P}^2, \mathcal{O}) \cong \mathbb{C}$, call such extension c. Let $0 = \overline{f} - c\colon I_z|_{\mathbb{P}^2-\{z\}} \to \mathbb{O}_{\mathbb{P}^2-\{z\}}$, then $f - c = 0$ everywhere but no one morphism from I_z to $\mathcal{O}$ has kernel because I_z is torsion free, therefore f is give by scalar multiplication.

Finally we use the $Grothendieck - Hirzebruch - Riemann - Roch's$ theorem G-H-R-R (see [Har77], Appendix A.) The Euler characteristic by definition is

$$\chi(I_z, \mathcal{O}) = \sum_{i=0}^{2} (-1)^i \dim(\operatorname{Ext}^i(I_z, \mathcal{O})) = \dim(\operatorname{Ext}^0(I_z, \mathcal{O})) - \dim(\operatorname{Ext}^1(I_z, \mathcal{O})),$$

and by G-H-R-R we have

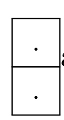
$$\chi(I_z, \mathcal{O}) = \int_{[\mathbb{P}^2]} ch(I_z) td(\mathcal{T}) = \int_{[\mathbb{P}^2]} (1 - 2\omega)(1 + \omega) = -1,$$

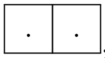
where ω is the virtual class of a point in $\mathbb{P}^2$, then $\dim \operatorname{Ext}^1(I_z, \mathcal{O}) = 2$.

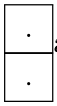
Given these dimensions, we have:

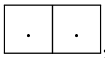
$$Y = \sum_{\text{Fixed locus}} \int_{[pt]} \frac{c_2(\operatorname{Ext}^1(I_z, \mathcal{O}_z))}{c_4(\operatorname{Ext}^0(I_z, \mathcal{O}_z))} . \frac{c_2(\operatorname{Ext}^1(I_z, \mathcal{O}))}{c_1(\operatorname{Ext}^0(I_z, \mathcal{O}))} . \frac{c_2(\operatorname{Ext}^1(I_z, \mathcal{O}))}{c_1(\operatorname{Ext}^0(I_z, \mathcal{O}))} . \tag{19}$$

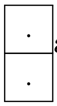
Step (2) configurations. The unique ways to distribute 2 boxes (in Young Tableaux) in three corners are:

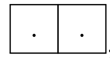
1. On p_0 :  and

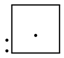
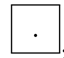
2. ,

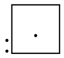
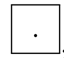
3. On p_1 :  and

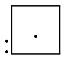
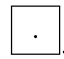
4. ,

5. On p_2 :  and

6. ,

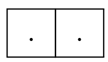
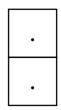
7. On p_0 and p_1 :  ; ,

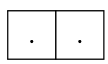
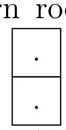
8. On p_0 and p_2 :  ; ,

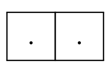
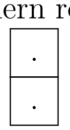
9. On p_1 and p_2 :  ; .

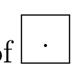
Each of these nine configurations of fixed points can be injected in three different copies of $\mathcal{O}$, so the number of fixed points is 27.

Step (3) Chern roots. First we will compute the top Chern class $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z))$, for this we use lemma 2.16 as in the Example 2.17.

- On p_0 the configuration  has Chern roots are $-2\mu h, \lambda h, -\mu h, (\mu+\lambda)h$, then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = 2\mu^2\lambda(\mu+\lambda)h^4$ and  has Chern roots $-\mu h, 2\lambda h, \lambda h, -(\mu+\lambda)h$, then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = 2\mu\lambda^2(\mu+\lambda)h^4$.

- On p_1 the configuration  has Chern roots are $2(\mu+\lambda)h, \mu h, (\mu+\lambda)h, -\lambda h$, then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = 2(\mu+\lambda)^2\mu\lambda h^4$ and  has Chern roots $(\mu+\lambda)h, 2\mu h, -(\mu+\lambda)h, \lambda h$,then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = 2(\mu+\lambda)\mu^2\lambda h^4$.

- On p_2 the configuration  has Chern roots are $-2\lambda h, -(\mu+\lambda)h, -\lambda h, -\mu h$, then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = 2\lambda^2(\mu+\lambda)\mu h^4$ and  has Chern roots $-\lambda h, -2(\mu+\lambda)h, -(\mu+\lambda)h, \mu h$, then $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = -2\lambda(\mu+\lambda)^2\mu h^4$.

- At the different points p_0, p_1 and p_2 the Chern roots of  are $-\mu h, \lambda h; -(\mu+\lambda)h, -\mu h$ and $-\lambda h, (\mu+\lambda)h$ respectively and then for the configuration 7. $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = -\mu\lambda h^2$, for 8. $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = \mu^2 + \lambda\mu h^2$ and for 9, $c_4(\mathrm{Ext}^0(I_z, \mathcal{O}_z)) = -(\lambda\mu + \lambda^2)h^2$.

We can use without lost of generality that I_z injects on the $i-$ th copy of $\mathcal{O}$, then we have some relations between the weights $u_i's$ and λ, μ on the Ext groups and its top Chern classes. The Group $\text{Ext}(A,B)$ depends on the $u_i's$ if and only if A and B are subsheaves and quotients of different copies of $\mathcal{O}$, because if they are in the same copy we only act with weight $u_iu_i^{-1} = 1$; furthermore $\text{Ext}(A,B)$ does not depend on μ and λ if and only if the morphisim are given by scalar multiplications.

By the last discussion we see that $c_1(\text{Ext}(I_z,\mathcal{O}))$ only depend on $u_i's$. Then $c_1(\text{Ext}(I_z,\mathcal{O})) = (u_k - u_i)h$.

Now to find $c_2(\text{Ext}^1(I_z,\mathcal{O}))$ we consider the short exact sequence

$$0 \to I_z \to \mathcal{O} \to \mathcal{O}_z \to 0,$$

and apply the functor the functor $\text{Hom}(I_z,\bullet)$ to get the long exact sequence

$$0 \to \text{Hom}(I_z,I_z) \to \text{Hom}(I_z,\mathcal{O}) \to \text{Hom}(I_z,\mathcal{O}_z) \to$$

$$\to \text{Ext}^1(I_z,I_z) \to \text{Ext}^1(I_z,\mathcal{O}) \to \text{Ext}^1(I_z,\mathcal{O}_z) \to$$

$$\to \text{Ext}^2(I_z,I_z) \to \text{Ext}^2(I_z,\mathcal{O}) \to \text{Ext}^2(I_z,\mathcal{O}_z) \to 0.$$

Since $\dim(\text{Ext}^0(I_z,\mathcal{O})) = 1$, then $\dim(\text{Ext}^0(I_z,I_z)) = 1$. Considering $\text{Ext}^0(I_z,\mathcal{O}_z)$ and $\text{Ext}^1(I_z,I_z)$ as tangents spaces they both have dimension 4. By G-H-R-R $\text{Ext}^1(I_z,\mathcal{O})$ has dimension 2 and $\dim(\text{Ext}^1(I_z,\mathcal{O}_z)) = 2$ because is an obstruction space.

Finally since $\text{Ext}^2(I_z,I_z) \cong \text{Ext}^2(I_z,\mathcal{O}_z) \cong \text{Ext}^2(I_z,\mathcal{O}) \cong 0$, by Serre Duality we obtain the isomorphism

$$\text{Ext}^1(I_z,\mathcal{O}) \cong \text{Ext}^1(I_z,\mathcal{O}_z) \cong \text{Ext}^2(\mathcal{O}_z,\mathcal{O}_z) \cong \text{Ext}^0(\mathcal{O}_z,\mathcal{O}_z \otimes K),$$

where K is the canonical sheaf.

This show us that we have to compute the Chern rooots of K, for the different points p_0, p_1 and p_2 and for all of the nine configuration of fixed points. In the next tables we present these Chern roots.

Sheaves over p_0	Chern Roots
T	$\lambda h, -\mu h$
$T^\vee$	$-\lambda h, \mu h$
$K = \wedge^2 T^\vee$	$(\mu - \lambda)$h

Table 3: Chern roots.

Sheaves over p_1	Chern Roots
T	$-(\mu + \lambda)h, \mu h$
$T^\vee$	$(\mu + \lambda)h, -\mu h$
$K = \wedge^2 T^\vee$	$-(2\mu + \lambda)h$

Table 4: Chern roots.

Sheaves over p_2	Chern Roots
T	$-\lambda h, -(\mu + \lambda)h$
$T^\vee$	$\lambda h, (\mu + \lambda)h$
$K = \wedge^2 T^\vee$	$(\mu + 2\lambda)h$

Table 5: Chern roots.

Then;

Points	Sheaf	Chern roots.
	K	$(\mu-\lambda)h$
	$\mathcal{O}_z \otimes K$	$(\mu-\lambda)h, (2\mu-\lambda)h$
p_0	$\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K)$	$(\mu-\lambda)h, (2\mu-\lambda)h$
	$\mathrm{Ext}^1(I_z, \mathcal{O}) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$	$(-\mu+\lambda)h, (\lambda-2\mu)h$
	K	$-(2\mu+\lambda)h$
	$\mathcal{O}_z \otimes K$	$-(2\mu+\lambda)h, -(3\mu+2\lambda)h$
p_1	$\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K)$	$-(2\mu+\lambda)h, -(3\mu+2\lambda)h$
	$\mathrm{Ext}^1(I_z, \mathcal{O}) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$	$(2\mu+\lambda)h, (3\mu+2\lambda)h$
	K	$(2\lambda+\mu)h$
p_2	$\mathcal{O}_z \otimes K$	$(2\lambda+\mu)h, (3\lambda+\mu)h$
	$\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K)$	$(2\lambda+\mu)h, (3\lambda+\mu)h$
	$\mathrm{Ext}^1(I_z, \mathcal{O}) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$	$-(2\lambda+\mu)h, -(3\lambda+\mu)h$

Table 6: Chern roots of $\mathrm{Ext}^1(I_z, \mathcal{O})$.

Since the groups $\mathrm{Ext}^1(I_z, \mathcal{O})$ depend on the $u_i's$ we have three possibles top Chern classes which are summarized in the next table:

Points	$c_2(\mathrm{Ext}^1(I_z, \mathcal{O}))$
p_0	$(u_k - u_i - \mu + \lambda)(u_k - u_i - 2\mu + \lambda)h^2$
p_1	$(u_k - u_i + 2\mu + \lambda)(u_k - u_i - \mu + 2\lambda)h^2$
p_2	$(u_k - u_i - \mu - 2\lambda)(u_k - u_i + \lambda + 2\mu)h^2$

Table 7: top Chern Classes of $\mathrm{Ext}^1(I_z, \mathcal{O})$.

Note that in the expression (19) the variable h has degree 6 in the denominator and the enumerator, so this shows that the value of (18) is in fact a number. Putting all the information together in (19) we get nine integrals because we have precisely nine possible configurations of points. The result of this computation have to be multiplied by 3 (number of possible injections). Then

$$
\begin{aligned}
Y =&3\Big(\int_{(1)} H(E^1,E^0) + \int_{(2)} H(E^1,E^0) + \int_{(3)} H(E^1,E^0) + \int_{(4)} H(E^1,E^0) + \\
&\int_{(5)} H(E^1,E^0) + \int_{(6)} H(E^1,E^0) + \int_{(7)} H(E^1,E^0) + \int_{(8)} H(E^1,E^0) + \int_{(9)} H(E^1,E^0)\Big),
\end{aligned}
$$

where every integral is computed over the configurations of points indicated within parenthesis $H(E^1,E^0)$ is the expression $\frac{c_2(\mathrm{Ext}^1(I_z,\mathcal{O}_z))}{c_4(\mathrm{Ext}^0(I_z,\mathcal{O}_z))}\cdot\frac{c_2(\mathrm{Ext}^1(I_z,\mathcal{O})}{c_1(\mathrm{Ext}^0(I_z,\mathcal{O}))}\cdot\frac{c_2(\mathrm{Ext}^1(I_z,\mathcal{O}))}{c_1(\mathrm{Ext}^0(I_z,\mathcal{O}))}$. We compute the first of these integrals to demonstrate how the calculation is performed.

A computation shows that $Y = 3(90) = 270$.

Example of computation:

$$\int_{(1)} H(E^1,E^0) = \int_{\substack{\square\\\square},p_0} H(E^1,E^0) \tag{20}$$

For the data $\substack{\square\\\square}, p_0$ we have:

- $c_1(\mathrm{Ext}^0(I_z,\mathcal{O})) = (u_k - u_i)h$
- $c_2(\mathrm{Ext}^1(I_z,\mathcal{O})) = (u_k - u_i - \mu + \lambda)(u_k - u_i - 2\mu + \lambda)h^2$
- $c_2(\mathrm{Ext}^1(I_z,\mathcal{O}_z)) = (-\mu + \lambda)(-2\mu + \lambda)h^2$
- $c_4(\mathrm{Ext}^0(I_z,\mathcal{O}_z)) = 2\mu\lambda^2(\mu + \lambda)h^4$.

So

$$\int_{\substack{\square\\\square},p_0} H(E^1,E^0) = \tag{21}$$

$$\int_{\substack{\square\\\square},p_0} \frac{(-\mu+\lambda)(-2\mu+\lambda)(u_k-u_i-2\mu+\lambda)^2h^6}{2\mu\lambda^2(\mu+\lambda)(u_k-u_i)^2h^6} = \tag{22}$$

$$\int_{\substack{\square\\\square},p_0} \frac{(-\mu+\lambda)(-2\mu+\lambda)(u_k-u_i-2\mu+\lambda)^2}{2\mu\lambda^2(\mu+\lambda)(u_k-u_i)^2} \tag{23}$$

$$\tag{24}$$

If we take weights $w_0 = -1, w_1 = 0, w_2 = 1$ and $u_1 = 1, u_2 = 2, u_3 = 3$, the expression (24) become in:

$$\sum_{k=1}^{3} \sum_{i \neq k} \int_{p_0} \frac{-15(k-i-5)}{4(k-i)^2} = -\frac{15}{4} \sum_{k=1}^{3} \sum_{i \neq k} \frac{(k-i-5)}{(k-i)^2} \int_{p_0} 1 = \frac{585}{4}. \tag{25}$$

This is one of the $9 \times 3 = 27$ computations which are necessary to get the value $Y = 270$.

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