Lower bounds for the regulator

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Introduction

The understanding of units is a central problem in Algebraic Number Theory. It is remarkable, for example, that the main difficulty in calculating the class number of an algebraic number field is usually the calculation of the regulator. In fact the regulator and all the numerical invariants of number fields are intimately related, some beautiful illustrations of this matter are the Brauer-Siegel theorem and the class number formula.

The main object of study in this thesis are lower bounds for regulators. In chapter 1 we give a geometric bound inspired in the work of Remak [Re] and Pohst [Po]. In chapter 2 we apply analytic methods developed by Zimmert [Zi] and Friedman [Fr] to obtain new lower bounds for certain totally real fields.

The regulator was defined for the first time by Dedekind in the extended eleventh supplement for the fourth edition (1894) of Dirichlet's lectures in number theory *Vorlesungen über Zahlentheorie*. The first bound for the regulator of a field of arbitrary degree was obtained in 1918 by Landau [Lan], who found an inequality relating the regulator and the discriminant of the form $R_k < C_1\sqrt{D_k}(\log|D_k|)^{n-1}$. In 1952 Remak [Re] obtained $|D_k| < C_2\exp(C_3R_k)$ (See Lemma 7) for any non CM field k, the C_i are explicit constants depending only on n. This last inequality, together with the Hermite-Minkowski theorem, implies that there is a minimal regulator for each signature (r_1, r_2). In 1977 Pohst [Po] found the minimal regulator among all totally real cubic fields. In 1981 Zimmert [Zi], relying on work of Pohst [Po] and a new analytic method, showed that $\log((1+\sqrt{5})/2)$ is the minimal regulator among all totally real fields. Finally, in 1989, Friedman [Fr] found the minimal regulator for totally complex sextic fields and showed that it was the smallest regulator among all number fields. Recently, in 2016, Astudillo, Diaz y Diaz and Friedman published sharp lower bounds for regulators of number fields of all signatures up to degree seven [ADF], except for fields of degree seven having five real places.

In the first chapter we extend Pohst's geometric method from the totally real case to fields having one complex place. This refinement allows us to obtain the minimal regulator among number fields with signature (5,1). A shorter version of chapter 1 was published in the *Journal of Number Theory* [FRR].

In chapter 2 we obtain new lower bounds for regulators of totally real fields in degree 10 to 18. We apply these bounds to prove a conjecture published by Katok, Katok and Rodriguez-Hetz in 2014 [KKR] concerning the minimal entropy of certain dynamical systems.

Preliminary definitions and results

Preliminary definitions

We will follow the usual notation, k will be a number field of degree n, $\mathcal{O}_k$ the respective ring of integers, r_1 the number of real embeddings and r_2 the number of pairs of complex embeddings, D_k the discriminant, R_k the regulator, ω the number of roots of the unit in k and h_k the class number of k. We recall here the definition of the discriminant and the regulator and give some examples to motivate our work. The proofs can be found in classical books like [Sa] or [Neu].

Definition 1. *Let $b_1, \ldots, b_n$ be an integral basis of $\mathcal{O}_k$ and $\sigma_1, \ldots, \sigma_n$ the set of embeddings of k in $\mathbb{C}$. The discriminant is the square of the determinant of the n by n matrix whose (i, j)-entry is $\sigma_i(b_j)$.*

$$D_k := \det \begin{pmatrix} \sigma_1(b_1) & \sigma_1(b_2) & \cdots & \sigma_1(b_n) \\ \sigma_2(b_1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_n(b_1) & \cdots & \cdots & \sigma_n(b_n) \end{pmatrix}^2$$

Definition 2. *If $r = r_1 + r_2 - 1$, let $u_1, \ldots, u_r$ be a set of fundamental units of $\mathcal{O}_k$ and $\sigma_1, \ldots, \sigma_{r+1}$ a subset of the set of embeddings representing the archimedean places of k. Let N_j be 1 or 2 if the corresponding σ_j embedding is real or complex. Let M be the $r \times (r + 1)$ matrix*

$$M := \begin{pmatrix} N_1 \log|\sigma_1(u_1)| & N_2 \log|\sigma_2(u_1)| & \cdots & N_{r+1} \log|\sigma_{r+1}(u_1)| \\ N_1 \log|\sigma_1(u_2)| & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ N_1 \log|\sigma_1(u_r)| & \cdots & \cdots & N_{r+1} \log|\sigma_{r+1}(u_r)| \end{pmatrix}.$$

The regulator is the absolute value of the determinant of any submatrix formed by deleting one column of M.

Example Consider the field $k = \mathbb{Q}(\sqrt{5})$. We have that $\mathcal{O}_k = \mathbb{Z}[(1 + \sqrt{5})/2]$, $r_1 = 2$, $r_2 = 0$, $D_k = 5$, $R_k = \ln|(1 + \sqrt{5})/2| = 0.481211\ldots$, $\omega = 2$ and $h_k = 1$. Many examples can be found in the LMFDB (The L-functions and Modular Forms Database) at `www.lmfdb.org`.

Remark 3. *If the number field has a power integral basis, i.e. there exists an algebraic integer α such that $b_1 = 1, b_2 = \alpha, \ldots, b_n = \alpha^{n-1}$, then the discriminant satisfies $D_k = \prod_{1\le i\le j\le n}(\alpha_i - \alpha_j)^2$ where $\alpha_i = \sigma_i(\alpha)$ (determinant of a Vandermonde matrix).*

We state here the definition of Cartan action needed for chapter 2. Further properties and notions can be found at [KN, 2.2.5].

Definition 4. *An action of $\mathbb{Z}^{n-1}$ on the tori $\mathbb{T}^n$ for $n \ge 3$ by ergodic automorphisms is called a Cartan action.*

Results in Chapter 1

Some thirty years ago, the number fields with smallest discriminant for signatures up to degree seven were all known [Od]. Recently [ADF] the same was established for regulators, except that no sharp lower bounds were proved for one signature in degree seven. In chapter 1 we close that gap.

Theorem 5. *Let k be a number field of degree seven having five real embeddings. Then its regulator R_k satisfies $R_k \ge R_{k_1} = 2.8846\ldots$, where k_1 is the unique field of discriminant $-2\,306\,599$ in this signature.*

More precisely, except for the three unique fields with discriminants $-2\,306\,599$, $-2\,369\,207$ and $-2\,616\,839$, in this signature all fields satisfy $R_k > 3.2$.

The idea in [ADF] is to first use analytic lower bounds for regulators. These are very good up to a certain value $D_{\text{anal}}(r_1, r_2)$ of the discriminant D_k, where (r_1, r_2) is the number of (real, complex) places of k. Then coarse geometric bounds due to Remak [Re] are used for $|D_k| \ge D_{\text{geom}}(r_1, r_2)$. This method works if $D_{\text{geom}}(r_1, r_2) \le D_{\text{anal}}(r_1, r_2)$, which holds for small degrees, but fails when the unit-rank reaches 5.

In fact, unit-rank 5, 6 and 7 are handled in [ADF], but only for totally real fields, where an improved inequality due to Pohst [Po] is available. To deal with signature $(5, 1)$, we extend Pohst's method, allowing one of the variables to be complex.

Results in Chapter 2

In 2014 Katok, Katok and Rodriguez Hetz [KKR, p. 1216] published the following

Conjecture. (*A. Katok, S. Katok and F. Rodriguez Hetz*) *"The Cartan action α corresponding to the quartic totally real number field of discriminant 725 and the defining polynomial $x^4 - x^3 - 3x^2 + x + 1$ minimizes the Fried average entropy $h^*(\alpha)$ among all Cartan actions α. For that action $h^*(\alpha) = 0.330027\ldots = h_{\min}$."*

Here we prove this and give the first six minima of $h^*(\alpha)$.

Theorem 6. *The above conjecture holds. Moreover, except for the six Cartan actions given in Table 1 below, all other Cartan actions α satisfy $h^*(\alpha) > 0.49$.*

The KKR conjecture applies to actions of $\mathbb{Z}^{n-1}$ by hyperbolic automorphisms on n-tori for $n \ge 3$ (Cartan actions). We refer the reader to their paper for motivation and for the definitions involved in the above conjecture, only noting that they were able to reduce the proof of their conjecture to a purely number-theoretic problem. Namely, to finding good

Table 1: The first six minima of the Fried average entropy (to six decimals).

$h^*(\alpha)$	Degree	Polynomial	Discriminant	Regulator
0.330027	4	$x^4 - x^3 - 3x^2 + x + 1$	725	0.825068
0.350303	3	$x^3 - x^2 - 2x + 1$	49	0.525454
0.373872	5	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$	14641	1.635694
0.416198	6	$x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$	300125	3.277562
0.466182	4	$x^4 - x^3 - 4x^2 + 4x + 1$	1125	1.165455
0.479301	6	$x^6 - x^5 - 5x^4 + 4x^3 + 6x^2 - 3x - 1$	371293	3.774500

lower bounds for regulators of totally real number fields, in this case $h^*(\alpha) = mR_K 2^{n-1}\big((n-1)!\big)^2/(2n-2)!$, where $m \geq 1$ is an integer [KKR, (3.8)]. They proved the lower bound $h_{\min} \geq 0.089$, and showed that their conjecture held for Cartan actions for $3 \leq n \leq 7$ and for $n \geq 17$, leaving $8 \leq n \leq 16$ open. This range was later narrowed to $10 \leq n \leq 16$ [ADF, p. 234].

As we claim a slightly stronger result than the KKR conjecture, in the proof we cannot restrict to dimensions 10 to 16. However, using results from [KKR] and [ADF] we will quickly reduce to $10 \leq n \leq 18$. We will deal with these nine cases by applying a variant of Zimmert's techniques [Zi] [Fr, §4]. Our innovation here is to use residues instead of integrals to handle otherwise heavy numerical calculations (Lemmas 17 and 18 below).

Chapter 1

Filling the gap in the table of smallest regulators up to degree 7

1.1 Proof of Theorem 5

If ε is a unit in k, let

$$m_k(\varepsilon) := \sum_{\omega} (\log \|\varepsilon\|_\omega)^2, \tag{1.1}$$

where ω runs over the set of archimedean places of k and $\|\cdot\|_\omega$ denotes the corresponding absolute value, normalized so that $|\mathrm{Norm}_{k/\mathbb{Q}}(a)| = \prod_{\omega \in \infty_k} \|a\|_\omega$. A proof of the following inequality can be found in [Re, §6] or [Fr, Lemma 3.4].

Lemma 7. *(Remak) Suppose $k = \mathbb{Q}(\varepsilon)$, where $\varepsilon \in k$ is a unit. Then the discriminant D_k satisfies*

$$\log |D_k| \leq m_k(\varepsilon) A(k) + \log P_n,$$

where

$$A(k) := \sqrt{(n^3 - n - 4r_2^3 - 2r_2)/3}, \qquad P_n = P_n(\varepsilon_1, ..., \varepsilon_n) := \prod_{1 \leq i < j \leq n} \left|1 - \frac{\varepsilon_i}{\varepsilon_j}\right|^2,$$

$n := [k : \mathbb{Q}]$, r_2 is the number of complex places of k, and the ε_i are the conjugates of ε arranged so that $|\varepsilon_1| \leq |\varepsilon_2| \leq \cdots \leq |\varepsilon_n|$.

Lemma 8. $\big($Remak, Pohst *[Re, (18)][Po, Satz IV]*$\big)$ *If $z_1, ..., z_n$ are non-zero complex numbers arranged so that $|z_1| \leq \cdots \leq |z_n|$, then*

$$P_n(z_1, ..., z_n) := \prod_{1 \leq i < j \leq n} \left|1 - \frac{z_i}{z_j}\right|^2 \leq n^n. \tag{1.2}$$

If, in addition, $n \leq 11$ and $z_i \in \mathbb{R}$ $(1 \leq i \leq n)$, then

$$P_n(z_1, ..., z_n) \leq 4^{\lfloor n/2 \rfloor}, \tag{1.3}$$

where $\lfloor n/2 \rfloor := (n-1)/2$ if n is odd, $\lfloor n/2 \rfloor := n/2$ if n is even.

Our main task will be to improve on Remak's bound $P_7 \le 7^7$ when 5 of the z_i's are real and the remaining two are complex conjugates. We begin more generally, assuming henceforth that $n-2$ of the z_i's are real and the remaining two are complex conjugates. We shall denote the real elements by r_i $(1 \le i \le n-2)$ and the complex conjugate pair by $x\mathrm{e}^{i\theta}$ and $x\mathrm{e}^{-i\theta}$ $(\theta \in (0,\pi),\ x>0)$, arranging them so that

$$0 < |r_1| \le |r_2| \le \cdots \le |r_{n-2}|, \qquad |r_t| \le x \le |r_{t+1}|, \tag{1.4}$$

where if $x \ge |r_{n-2}|$ we mean $t = n-2$, and if $x \le |r_1|$ we mean $t = 0$.

Grouping the factors $|1-\frac{z_i}{z_j}|^2$ in (1.2) according to whether both, none or one of $z_i, z_j \in \mathbb{R}$, P_n factors as

$$P_n = P_{n-2}(r_1, ..., r_{n-2}) \cdot |1-\mathrm{e}^{-2i\theta}|^2 \cdot \prod_{m=1}^{n-2} |1-c_m\mathrm{e}^{i\theta}|^4, \qquad c_m := \begin{cases} r_m/x & \text{if } m \le t, \\ x/r_m & \text{if } m > t. \end{cases} \tag{1.5}$$

Note that $c_m \in [-1,1],\ c_m \ne 0\ \ (1 \le m \le n-2)$.

Lemma 9. *If* $0 \le c \le 1$, *then*

$$\left|1 - c\,\mathrm{e}^{i\theta}\right|^2 \le \begin{cases} 1 & \text{if } 0 \le \theta \le \pi/3, \\ 2\big(1-\cos(\theta)\big) & \text{if } \pi/3 \le \theta \le \pi. \end{cases}$$

If $-1 \le c \le 0$, *then*

$$\left|1 - c\,\mathrm{e}^{i\theta}\right|^2 \le \begin{cases} 1 & \text{if } 2\pi/3 \le \theta \le \pi, \\ 2\big(1+\cos(\theta)\big) & \text{if } 0 \le \theta \le 2\pi/3. \end{cases}$$

Proof. Let $g(c) := |1-c\,\mathrm{e}^{i\theta}|^2 = 1 + c^2 - 2c\cos(\theta)$. The critical point of g is a minimum, so we just compare the values of g at the endpoints of the intervals involved. □

Lemma 10. *For* $a, b > 0$ *and* $\theta \in \mathbb{R}$, *we have*

$$\big(1-\cos^2(\theta)\big)^a \big(1-\cos(\theta)\big)^b \le \frac{2^{2a+b}a^a(a+b)^{a+b}}{(2a+b)^{2a+b}}. \tag{1.6}$$

Proof. For $-1 \le x \le 1$, let $g(x) := (1-x^2)^a(1-x)^b$. Elementary calculus shows that g assumes its maximum value M at $x = -b/(2a+b)$, and that M is given by the right-hand side of (1.6). □

Lemma 11. *Assume* $\theta \in \mathbb{R}$ *and* $-1 \le c_m \le 1$ *for* $1 \le m \le r$. *Let* d_+ *be the number of* c_m *with* $c_m > 0$, *let* d_- *be the number of* c_m *with* $c_m < 0$, *and define*

$$B_r = B_r(\theta, c_1, ..., c_r) := |1-\ \mathrm{e}^{-2i\theta}|^2 \prod_{m=1}^{r} |1-c_m\,\mathrm{e}^{i\theta}|^4. \tag{1.7}$$

Then

$$B_r \le \max\Big(\frac{4^{2a+b}a^a(a+b)^{a+b}}{(2a+b)^{2a+b}}\,,\ \frac{4^{2+f}(1+f)^{1+f}}{(2+f)^{2+f}}\Big), \tag{1.8}$$

where $a := 1 + 2\min(d_+, d_-)$, $b := 2|d_+ - d_-|$ *and* $f := 2\max(d_+, d_-)$.

Proof. Replacing θ by $-\theta$ if necessary, we can assume $0 \le \theta \le \pi$. We shall first show that if $\pi/3 \le \theta \le 2\pi/3$, then B_r is bounded by the first element inside the max in (1.8). Say $d_+ > d_-$, so that $a = 1 + 2d_-$ and $b = 2(d_+ - d_-)$. Then, using Lemma 9 and $\pi/3 \le \theta \le 2\pi/3$,

$$\begin{aligned}
B_r &= 4\big(1-\cos^2(\theta)\big)\Big(\prod_{m=1}^{r} |(1-c_m e^{i\theta})|^2\Big)^2 \\
&\le 4\big(1-\cos^2(\theta)\big)\Big(\prod_{\substack{m \\ c_m>0}} 2\big(1-\cos(\theta)\big)\Big)^2\Big(\prod_{\substack{m \\ c_m<0}} 2\big(1+\cos(\theta)\big)\Big)^2 \\
&= 2^{2+2(d_+ + d_-)}\big(1-\cos^2(\theta)\big)\big(1-\cos(\theta)\big)^{2d_+}\big(1+\cos(\theta)\big)^{2d_-} \\
&= 2^{2a+b}\big(1-\cos^2(\theta)\big)^{1+2d_-}\big(1-\cos(\theta)\big)^{2(d_+-d_-)} \\
&= 2^{2a+b}\big(1-\cos^2(\theta)\big)^{a}\big(1-\cos(\theta)\big)^{b} \\
&\le \frac{2^{2(2a+b)}a^a(a+b)^{a+b}}{(2a+b)^{2a+b}} \qquad \text{(see Lemma 10)},
\end{aligned}$$

proving (1.8) in this case. If $d_+ < d_-$, a similar argument gives

$$B_r \le 2^{2a+b}\big(1-\cos^2(\theta)\big)^{1+2d_+}\big(1+\cos(\theta)\big)^{2(d_- - d+)},$$

and (1.8) follows as above from Lemma 10 (with θ replaced by $\theta + \pi$). The case $d_+ = d_-$ is clear, since then $b = 0$ and we get $B_r \le 2^{2a}\big(1-\cos^2(\theta)\big)^{1+2d_+} \le 2^{2a}$, proving (1.8) when $\pi/3 \le \theta \le 2\pi/3$.

If $0 \le \theta < \pi/3$, we again use Lemmas 9 and 10 to get

$$\begin{aligned}
B_r &\le 4\big(1-\cos^2(\theta)\big)\Big(\prod_{\substack{m \\ c_m<0}} 2\big(1+\cos(\theta)\big)\Big)^2 = 2^{2+2d_-}\big(1-\cos^2(\theta)\big)\big(1+\cos(\theta)\big)^{2d_-} \\
&\le 2^{2+f}\big(1-\cos^2(\theta)\big)\big(1+\cos(\theta)\big)^{f} \le 2^{2+f}\frac{2^{2+f}(1+f)^{1+f}}{(2+f)^{2+f}}.
\end{aligned}$$

A similar argument proves (1.8) in the remaining case, *i. e.* when $2\pi/3 < \theta \le \pi$. $\square$

Lemma 12. *(Pohst) For $\alpha, \beta \in [-1, 1]$, the following hold.*

- (i) If $\alpha \ge 0$, then $(1-\alpha)(1-\alpha\beta) \le 1$.
- (ii) $(1-\alpha)(1-\beta)(1-\alpha\beta) \le 2$.
- (iii) If $|\alpha| \le |\beta|$ and $\beta \ne 0$, then $\big(1-\alpha\big)\big(1-\beta\big)\big(1-(\alpha/\beta)\big) \le 2$.

Proof. Inequalities (i) and (ii) [Po, p. 468] can be proved by checking for critical points and the boundary. The last one follows from (ii), on replacing α by α/β. $\square$

We now specialize to $n = 7$.

Lemma 13. *Suppose $n = 7$ and $c_1 > 0$ in* (1.5)*, then $P_7 < \mathrm{e}^{12} < 162755$.*

We note that $7^7 = 823543 \approx \mathrm{e}^{13.62}$, so we have gained a factor of a little over 5 compared with Remak's bound (1.2).

Proof. We begin with (1.5),

$$P_7 = B_5 P_5 = B_5(\theta, c_1, ..., c_5) P_5(r_1, ..., r_5) \qquad \big(\text{see (1.2) and (1.7)}\big). \tag{1.9}$$

Depending on the signs of the c_m, we will show that B_5 or P_5 is small. There are 16 possibilities for the signs of $c_2, ..., c_5$, which we divide into three cases:

(1) Three of the c_m are of one sign and two have the opposite sign ($1 \leq m \leq 5$). Hence, in the notation of Lemma 11, $a = 5$, $b = 2$ and $f = 6$.

(2) One of the c_m is of one sign and four have the opposite sign. Hence $a = 3$, $b = 6$ and $f = 8$.

(3) All of the c_m are positive.

In case (1), Lemma 11 gives $B_5 < 4842.63$ and Pohst's inequality (1.3) gives $P_5 \leq 16$. Now (1.9) yields $P_7 < 77483$, proving the Lemma in case (1).

In case (2), Lemma 11 only gives

$$B_5 < 40624, \tag{1.10}$$

but we will improve Pohst's bound to $P_5 \leq 4$. This just suffices to prove the Lemma in this case. Following Pohst [Po, p. 467], for $1 \leq i, \ell, \ell' \leq 4$ let

$$x_i := \frac{r_i}{r_{i+1}}, \qquad y_{\ell,\ell'} := 1 - \prod_{i=\ell}^{\ell'} x_i = 1 - \frac{r_\ell}{r_{\ell'}},$$

and

$$A = A(x_1, x_2, x_3, x_4) := \prod_{1 \leq \ell \leq \ell' \leq 4} y_{\ell,\ell'} = \sqrt{P_5(r_1, ..., r_5)}.$$

Note that $-1 \leq x_i \leq 1$, $0 \leq y_{\ell,\ell'} \leq 2$ and that the signs of the x_i's are determined from those of the c_m's and vice-versa, as we are assuming $c_1 > 0$ in (1.5). All 5 possible signs of $c_1, ..., c_5$ in case (2) are shown in Table 1.1.

Table 1.1: All sign patterns in case (2)

c_1	c_2	c_3	c_4	c_5	x_1	x_2	x_3	x_4
+	+	+	+	−	+	+	+	−
+	−	−	−	−	−	+	+	+
+	+	+	−	+	+	+	−	−
+	−	+	+	+	−	−	+	+
+	+	−	+	+	+	−	−	+

Since $A(x_1, x_2, x_3, x_4) = A(x_4, x_3, x_2, x_1)$, it suffices to deal with the first, middle and last lines in Table 1.1.

We factor

$$\begin{aligned} A &= y_{1,1} y_{2,2} y_{3,3} y_{4,4} y_{1,2} y_{2,3} y_{3,4} y_{1,3} y_{2,4} y_{1,4} \\ &= (y_{1,1} y_{2,2} y_{1,2})(y_{3,3} y_{3,4})(y_{2,3} y_{2,4})(y_{1,3} y_{1,4})(y_{4,4}) \end{aligned} \tag{1.11}$$

For the first line in Table 1.1, $x_1, x_2, x_1x_2 \geq 0$, so we have trivially that $y_{1,1}y_{2,2}y_{1,2} \leq 1$. By Lemma 12 (i), using $x_3, x_2x_3, x_1x_2x_3 \geq 0$, we have $y_{3,3}y_{3,4} \leq 1$, $y_{2,3}y_{2,4} \leq 1$ and $y_{1,3}y_{1,4} \leq 1$. Finally $y_{4,4} \leq 2$, and so $A \leq 2$ for the signs on the first line of Table 1.1.

We consider now the third line in Table 1.1. Then, grouping (1.11) differently,

$$A = (y_{1,1}y_{1,4}y_{2,4})(y_{2,2}y_{2,3})(y_{1,2}y_{1,3})(y_{3,3}y_{4,4}y_{3,4}).$$

Trivially, $y_{1,1}y_{1,4}y_{2,4} \leq 1$. By Lemma 12 (i), since $x_2, x_1x_2 \geq 0$, we have $y_{2,2}y_{2,3} \leq 1$ and $y_{1,2}y_{1,3} \leq 1$. By Lemma 12 (ii), $y_{3,3}y_{4,4}y_{3,4} \leq 2$, and so again $A \leq 2$.

For the last line in Table 1.1 we write

$$A = (y_{1,3}y_{1,4}y_{2,4})(y_{1,1}y_{1,2})(y_{4,4}y_{3,4})(y_{2,2}y_{3,3}y_{2,3}).$$

Again trivially, $y_{1,3}y_{1,4}y_{2,4} \leq 1$. By Lemma 12 (i), since $x_1, x_4 \geq 0$, $y_{1,1}y_{1,2} \leq 1$ and $y_{4,4}y_{3,4} \leq 1$. Finally, by Lemma 12 (ii), we have $y_{2,2}y_{3,3}y_{2,3} \leq 2$. Thus, in case (2) we are done proving $A \leq 2$, , *i. e.* $P_5 \leq 4$. As indicated after (1.10), this implies the Lemma in case (2).

In case (3) we have $c_m > 0$, and so $r_m > 0$ for $m = 1, \dots, 5$. Thus

$$0 \leq 1 - \frac{r_\ell}{r_{\ell'}} \leq 1 \qquad \big(\ell < \ell'\big). \tag{1.12}$$

We shall need

$$R_{\ell,\ell'} := (1 + c_\ell)(1 + c_{\ell'})\big(1 - (r_\ell/r_{\ell'})\big) \leq 2 \qquad \big(\ell < \ell'\big). \tag{1.13}$$

To prove (1.13), we consider three possibilities according to the position of t in (1.4). If $\ell' \leq t$, then by (1.5), $c_\ell = r_\ell/x,\ c_{\ell'} = r_{\ell'}/x$. Hence $|c_\ell| \leq |c_{\ell'}|$ and so Lemma 12 (iii) yields (1.13) (on setting $\alpha := -c_\ell,\ \beta := -c_{\ell'}$). Similarly, if $\ell > t$, $c_\ell = x/r_\ell,\ c_{\ell'} = x/r_{\ell'}$, so $|c_{\ell'}| \leq |c_\ell|$ and Lemma 12 (iii) yields (1.13) (with $\alpha := -c_{\ell'},\ \beta := -c_\ell$). Lastly, if $\ell \leq t < \ell'$, then $c_\ell = r_\ell/x,\ c_{\ell'} = x/r_{\ell'}$. Now (1.13) follows from Lemma 12 (ii).

Using (1.12) and (1.13), we estimate

$$\begin{aligned}
\sqrt{P_7} &= |1 - e^{-2i\theta}| \ \cdot \prod_{1\leq\ell<\ell'\leq5} \big(1 - \frac{r_\ell}{r_{\ell'}}\big) \ \cdot\ \prod_{m=1}^{5} |1 - c_m e^{i\theta}|^2 \\
&\leq 2 \prod_{1\leq\ell<\ell'\leq5} \big(1 - \frac{r_\ell}{r_{\ell'}}\big) \ \cdot\ \prod_{m=1}^{5} (1 + c_m)^2 \\
&= 2R_{1,2}R_{2,3}R_{3,4}R_{4,5}R_{1,5}\big(1 - \frac{r_1}{r_3}\big)\big(1 - \frac{r_1}{r_4}\big)\big(1 - \frac{r_2}{r_4}\big)\big(1 - \frac{r_2}{r_5}\big)\big(1 - \frac{r_3}{r_5}\big) \\
&\leq 2R_{1,2}R_{2,3}R_{3,4}R_{4,5}R_{1,5} \leq 2^6.
\end{aligned}$$

Hence $P_7 \leq 2^{12}$. □

We can now prove our final geometric bound.

Lemma 14. *Suppose k is a number field of degree 7 having five real places and regulator $R_k \leq 3.2$. Then the discriminant D_k of k satisfies* $\log|D_k| < 31.492$.

Proof. Let ε yield the positive minimum value of m_k in (1.1) on the units of k. As $[k:\mathbb{Q}]=7$, we have $k=\mathbb{Q}(\varepsilon)$. Using the value $\gamma_5=\sqrt[5]{8}$ for Hermite's constant in dimension 5, we find $m_k \leq \big(3.2\sqrt{6}\big)^{1/5}\sqrt{\gamma_5} < 1.85847$ [ADF, (5)]. Let $r_1,...r_5$ be the five real conjugates of ε, ordered so that $|r_1|\leq\cdots\leq|r_5|$, and let $x\mathrm{e}^{\pm i\theta}$ be the two complex conjugates $\big(x>0,\ \theta\in(0,\pi)\big)$. Replacing ε by $-\varepsilon$ if necessary, we may assume that $r_1>0$, so $c_1>0$ with notation as in (1.5). Lemmas 7 and 13 yield $\log|D_k|<31.4918$. □

We shall need the following analytic tool [ADF, Lemmas 4 and 5].

Lemma 15. *Let k be a number field having r_1 real and r_2 complex places, and define*

$$g(x) := \frac{1}{2^{r_1}4\pi i}\int_{2-i\infty}^{2+i\infty}(\pi^n 4^{r_2}x)^{-s/2}(2s-1)\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\,ds \quad (x>0,\ n:=r_1+2r_2).$$

Suppose $0<d_1\leq|D_k|\leq d_2\leq d_3$, and assume $g(4/d_3)\geq 0$. Then for any $N\in\mathbb{N}$ we have $R_k\geq 2G(d_1,d_2,N)$, where

$$G(d_1,d_2,N) := \sum_{j=1}^{N}\min\big(g(j^{2n}/d_1),g(j^{2n}/d_2)\big).$$

If the ideal class of the different of k is trivial, then $R_k\geq 4G(d_1,d_2,N)$.

We now prove Theorem 5. So assume $(r_1,r_2)=(5,1)$ and $R_k\leq 3.2$. We shall first show that $|D_k|<3\,030\,000$. Since $R_k\leq 3.2$, Lemma 14 shows that $|D_k|\leq \mathrm{e}^{31.492}$. We deal separately with various subintervals of $[3\,030\,000,\mathrm{e}^{31.492}]$, always taking $d_3=\mathrm{e}^{31.492}$ in Lemma 15, noting that $g(4/\mathrm{e}^{31.492})=8.5631...>0$. If $|D_k|\leq\mathrm{e}^{20}$, then the ideal class of the different of k is trivial [ADF, Table 2]. A calculation shows that $R_k\geq 4G(3\,030\,000,\mathrm{e}^{20},1)=3.23...>3.2$. Hence this range of discriminant is ruled out by Lemma 15. We subdivide the remaining interval $[\mathrm{e}^{20},\mathrm{e}^{31.492}]$ into four subintervals and calculate $2G$ for them.

$$2G(\mathrm{e}^{31.4},\mathrm{e}^{31.492},3)=3.511..., \qquad 2G(\mathrm{e}^{31},\mathrm{e}^{31.4},3)=4.195...,$$
$$2G(\mathrm{e}^{28},\mathrm{e}^{31},3)=3.257..., \qquad 2G(\mathrm{e}^{20},\mathrm{e}^{28},3)=13.295....$$

Thus, Lemma 15 rules out discriminants in the interval $[\mathrm{e}^{20},\mathrm{e}^{31.492}]$, and so $|D_k|<3\,030\,000$. We conclude with Table 1.2, listing R_k for all fields k with $|D_k|<3\,030\,000$ [DyD].

Table 1.2: All fields of degree 7 having 5 real places and |discriminant| $<3\,030\,000$.

Discriminant	Polynomial	Regulator
$-2\,306\,599$	$x^7-3x^5-x^4+x^3+3x^2+x-1$	2.88465
$-2\,369\,207$	$x^7-x^5-5x^4-x^3+5x^2+x-1$	2.93325
$-2\,616\,839$	$x^7-x^6-5x^5-x^4+4x^3+3x^2-x-1$	3.13684
$-2\,790\,047$	$x^7+x^6-2x^5-3x^4-2x^3+3x^2+4x-1$	3.26802
$-2\,790\,551$	$x^7-5x^5-x^4+7x^3+3x^2-3x-1$	3.27113
$-2\,894\,039$	$x^7-4x^5-2x^4+4x^3+4x^2-x-1$	3.34402
$-2\,932\,823$	$x^7-x^6-4x^3+2x^2+2x-1$	3.36846

Chapter 2

Lower bounds for regulators and the Fried average entropy

2.1 Proof of Theorem 6

We assume throughout $h^*(\alpha) \leq 0.49$ and show in the end that this restricts the Cartan action α to one of the six in Table 2.1. From [KKR, Prop. 3.2 and (3.8)], we know that associated to the action α on an n-torus there is a totally real field K of degree $n := [K : \mathbb{Q}]$ for which the Fried average entropy $h^*(\alpha)$ satisfies

$$h^*(\alpha) \geq R_K 2^{n-1} \big((n-1)!\big)^2 / (2n-2)!, \tag{2.1}$$

where R_K is the regulator of K. Using Zimmert's lower bound [KKR, (3.13)]

$$R_K \geq 0.000376 \exp(0.9371n) \tag{2.2}$$

and an estimate of the factorials, the inequality $h^*(\alpha) \geq 0.000752 \exp(0.244n)$ was proved in [KKR, (3.15)]. Hence we find that $h^*(\alpha) \leq 0.49$ implies $n \leq 26$. Using (2.1) and (2.2), this range narrows further to $n \leq 18$.

Inequality (2.1) and $h^*(\alpha) \leq 0.49$ imply $R_K \leq 0.49(2n-2)!/(2^{n-1}(n-1)!^2)$. This bound is shown in Table 2.1 for $3 \leq n \leq 18$.

Table 2.1: **Upper bounds for R_K implied by $h^*(\alpha) \leq 0.49$**

n	3	4	5	6	7	8	9	10
$R_K \leq$	0.735	1.225	2.144	3.86	7.075	13.139	24.634	46.531

n	11	12	13	14	15	16	17	18
$R_K \leq$	88.42	168.79	323.5	622.11	1199.8	2319.6	4494.2	8724

Table 2.1 and the lower bounds for R_K given in [ADF, p. 234] are in contradiction for $n = 7$, 8 and 9. Hence we may discard these values of n. Moreover, all fields in degrees 3 to 6 with regulators in the range of Table 2.1 are listed in [ADF, Theorems 7, 10, 8, 11]. These are exactly the fields in Table 1. For example, according to [ADF, Theorem 11], all but three totally real fields of degree 6 satisfy $R_K > 4.39$. The exceptions have regulators

3.277..., 3.774... and 4.187.... The first two appear in Table 1, as they are associated to actions with $h^*(\alpha) \le 0.49$. Thus, to prove Theorem 6 it remains to show that the upper bounds in Table 2.1 are impossible for $10 \le n \le 18$.

We now summarize the regulator lower bounds in [Fr, p. 619].

Lemma 16. *Let K be a totally real field of degree n, let $0 < \beta < \gamma < \kappa$ and define*

$$T(s) := \Big(\frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{1+s+2\gamma}{2}\big)}\Big)^n, \qquad R(s) := \frac{s}{(s+\beta)(s+\kappa)^2}. \tag{2.3}$$

Assume that for some $y, \delta \in \mathbb{R}$ with $0 < \delta < \beta$ and some $M \in \mathbb{N}$ we have

$$\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} (my)^{s-1} T(s) R(s)\, ds \le 0 \qquad (m = 1, 2, \dots, M-1), \tag{2.4}$$

and

$$\frac{\beta T(-\beta)}{(\kappa-\beta)^2} \ge \frac{(My)^{\beta-\gamma}}{2\pi}\int_{-\gamma-i\infty}^{-\gamma+i\infty} |R(s)|\, ds. \tag{2.5}$$

Then

$$R_K \ge \frac{-\big(\Gamma(1+\gamma)\big)^n}{R(1) 2^n \pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} y^{s-1} T(s) R(s)\, ds. \tag{2.6}$$

We note that the integral in (2.4) is independent of δ, as long as $0 < \delta < \beta$. Computing numerically around 10^7 integrals of this type, as we will need to do below, seems difficult as the integrand oscillates like $\mathrm{e}^{itn\log(t)+it\log(y)}$, where $t = |\mathrm{Im}(s)| \gg 0$. Instead, in Lemma 18 we will approximate the integrals by a short sum of residues.

Lemma 17. *Let $f(t) := |\Gamma(a+it)/\Gamma(b+it)|$, and assume $b > a > 0$. Then f assumes its maximum value for $t \in \mathbb{R}$ at $t = 0$.*

Proof. Writing $\big(f(t)\big)^2 = \Gamma(a+it)\Gamma(a-it)/\big(\Gamma(b+it)\Gamma(b-it)\big)$ we find $f'(t)/f(t) = \mathrm{Im}\big(\psi(b+it) - \psi(a+it)\big)$, where [AAR, (1.2.13)]

$$\psi(z) := \Gamma'(z)/\Gamma(z) = -\gamma - \sum_{n=0}^{\infty}\big(\frac{1}{z+n} - \frac{1}{n+1}\big) \qquad (\gamma = 0.5772\dots).$$

We now calculate

$$\begin{aligned}\psi(b+it) - \psi(a+it) &= \sum_{n=0}^{\infty}\Big(\frac{1}{n+a+it} - \frac{1}{n+b+it}\Big) = \sum_{n=0}^{\infty}\frac{b-a}{(n+a+it)(n+b+it)} \\ &= (b-a)\sum_{n=0}^{\infty}\frac{(n+a-it)(n+b-it)}{\big((n+a)^2+t^2\big)\big((n+b)^2+t^2\big)} \\ &= (b-a)\sum_{n=0}^{\infty}\frac{(n+a)(n+b) - t^2 - it(2n+a+b)}{\big((n+a)^2+t^2\big)\big((n+b)^2+t^2\big)}.\end{aligned}$$

We therefore have $f'(t) < 0$ for $t > 0$ and $f'(t) > 0$ for $t < 0$. $\square$

Lemma 18. *Assume $0 < \delta < \beta < \gamma < \kappa$, and let T and R be as in Lemma 16 for some $n \in \mathbb{N}$. Then, for any $q = 2k \in 2\mathbb{N}$ and any $x > 0$ we have*

$$\Big|\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} x^{s-1}T(s)R(s)\,ds + \sum_{r=1}^{k}\mathrm{Res}_{s=2r-1}\big(x^{s-1}T(s)R(s)\big)\Big| \leq \frac{x^{q-1}}{2(q+\kappa)}\Big(\frac{\Gamma(\frac{1+q}{2})2^q}{\Gamma(\frac{1+q+2\gamma}{2})\prod_{j=0}^{q-1}|q-1-2j|}\Big)^n. \tag{2.7}$$

Proof. The Stirling estimate as $|T| \to \infty$ [AAR, Cor. 1.4.4], uniform in a vertical strip, $|\Gamma(\sigma + iT)| = \sqrt{2\pi}|T|^{\sigma-1/2}\mathrm{e}^{-\pi|T|/2}(1 + \mathrm{O}(1/|T|)$, allows us tor replace the line $\mathrm{Re}(s) = -\delta$ in (2.7) by $\mathrm{Re}(s) = q$, subtracting the residues. Using $\Gamma(z) = \Gamma(z + 1)/z$ successively, we find for $t \in \mathbb{R}$,

$$|T(q+it)| = \left|\frac{\Gamma(\frac{1+q-it}{2})}{\Gamma(\frac{1+q+2\gamma+it}{2})\prod_{j=0}^{q-1}(\frac{1-q}{2}+j-i\frac{t}{2})}\right|^n \leq \left|\frac{\Gamma(\frac{1+q}{2})}{\Gamma(\frac{1+q+2\gamma}{2})\prod_{j=0}^{q-1}(\frac{1-q}{2}+j)}\right|^n,$$

where in the last inequality we used Lemma 17. Lastly,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|R(q+it)|\,dt \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(q+\kappa)^2+t^2}\,dt = \frac{1}{2(q+\kappa)}. \tag{2.8}$$

□

The sum over residues in Lemma 18 is a polynomial in x and $\log(x)$, which can be readily calculated, and then quickly evaluated for millions of values of x. As a result, we can implement the numerical verification of the assumptions in Lemma 16. The resulting lower bounds are shown in Table 2.2. Using $q = 12$ and $\delta := \beta/2$ in Lemma 18 to approximate the integrals in (2.4) and (2.6). In verifying (2.5), to keep $M < 10^7$, we calculate $\int_{\mathrm{Re}(s)=-\gamma}|R(s)|\,ds$ numerically.

Table 2.2: Lower bound for the regulator using Lemma 16

$[K:\mathbb{Q}]$	$R_K \geq$	γ	y	β	κ	M
10	47.2298	0.46	0.00021	0.37	3.8	9 938 152
11	111.142237	0.42	0.00005	0.33	3.46	9 039 502
12	266.819052	0.375	0.00001	0.285	3.1	9 281 932
13	611.881830	0.545	0.00001	0.465	3.5	9 304 919
14	1291.090237	0.72	0.00001	0.64	4.1	9 704 058
15	2686.034353	0.885	0.00001	0.805	4.4	9 304 299
16	5600.694261	1.055	0.00001	0.965	5.3	9 908 560
17	11769.783217	1.205	0.00001	1.115	5.2	9 492 575
18	24936.817837	1.355	0.00001	1.255	5.7	9 872 459

Comparing the upper bounds for regulators in Table 2.1 with the lower bounds in Table 2.2, we see that $h^*(\alpha) > .49$ if $10 \leq n \leq 18$, proving Theorem 6.

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