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MASTER THESIS

# Multifractal analysis of Birkhoff averages over the symbolic space

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# Contents

# Chapter 1

# Introduction

## 1.1 Multifractal Analysis

Let $X$ be a metric space and $T : X \to X$ a continuous function. The pair $(X, T)$ is called a *dynamical system*. We review some definitions from ergodic theory that we will use throughout this work.

**Definition 1.1.1.** A Borel probability measure $\mu$ is called *$T$-invariant* if for every Borel set $A \subset X$ it satisfies $\mu(T^{-1}A) = \mu(A)$. The set of all $T$-invariant probability measures is denoted by $\mathcal{M}_T$.

**Definition 1.1.2.** A $T$-invariant probability measure $\mu$ is called *ergodic* for the system $(X, T)$ if for every Borel set $A \subset X$ such that $T^{-1}A = A$, it satisfies $\mu(A) \in \{0, 1\}$.

Associated to the dynamical system $(X, T)$ several local invariant quantities can be obtained, we will focus on one in particular, the Birkhoff averages:

**Definition 1.1.3.** Given a continuous function $\phi : X \to \mathbb{R}$ and a point $x \in X$. The *Birkhoff average of $\phi$ at $x$* is defined by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k x),$$

whenever the limit exists.

For $\alpha \in \mathbb{R}$, consider the following level sets,

$$J_\alpha := \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k x) = \alpha \right\}. \tag{1.1}$$

These level sets are pairwise disjoint and induce what is called a *multifractal decomposition*

$$X = \left( \bigcup_{\alpha \in \mathbb{R}} J_\alpha \right) \cup J', \tag{1.2}$$

where $J' := \{x \in X :$ the Birkhoff average of $x$ is not defined$\}$.

Briefly speaking, multifractal analysis studies the complexity of the level sets $J_\alpha$ by measuring the size of these sets, and establishing how it changes as $\alpha$ varies. We will compute the size of these sets in two different ways, one of them is obtained by means of the Hausdorff dimension (see Definition 2.1.2), and the other is dynamical in nature, it is obtained using the entropy restricted to subsets (see Definition 2.2.12). We define the functions that encodes the decomposition in (1.2):

**Definition 1.1.4.** The *Hausdorff dimension multifractal spectrum* and the *entropy multifractal spectrum* are defined respectively by

$$D(\alpha) := \dim_H J_\alpha,$$

$$E(\alpha) := h(T|J_\alpha).$$

The domain of these functions is the set $\{\alpha \in \mathbb{R} : J_\alpha \neq \varnothing\}$.

## 1.2 Main Theorem

In this section we state the main results of this thesis. We completely describe the multifractal analysis of Birkhoff averages in both a compact and a non-compact full shift $(X, T)$. In the compact case, Theorem 1.2.1, we provide a new proof of a result first obtained by Pesin and Weiss ([PW, Theorem 1]). In this setting the spectrum has bounded domain, it is real analytic and strictly concave.

Whereas in the non-compact case our complete characterization of the multifractal spectra is new (Theorem 1.2.2). The methods and techniques are based in work by Iommi and Jordan ([IJ]). Although the result in this setting is similar to the one in the compact case, in terms of the relation between the two spectra we analyze, it has remarkable differences. Interestingly, in the non-compact case new phenomena occurs. Indeed, as opposite to the compact setting the domain of the spectra is unbounded, the spectrum may or may not have phase transitions, and it is strictly decreasing.

First assume that $X$ is a compact full-shift with the metric $d$ defined in (3.1) for some $\lambda > 1$. The main theorem we are going to prove shows a relation between the Hausdorff dimension spectrum and the entropy spectrum, as well as their regularity.

**Theorem 1.2.1.** Let $\phi : X \to (-\infty, 0)$ be a Hölder continuous function defined on a compact full-shift $(X, T)$, and not cohomologous to a constant function. Let $\alpha \in \mathbb{R}$, $J_\alpha$ as in (1.1) and the functions $D, E$ from Definition 1.1.4. Then, the following hold:

- The domain of $D$ and $E$ is a compact interval $[\underline{\alpha}, \overline{\alpha}]$;
- For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, there exists a measure $\mu_\alpha \in \mathcal{M}_T$ such that $E(\alpha) = h(\mu_\alpha)$ and $D(\alpha) = \dim_H \mu_\alpha$ (see Definition 2.2.3 and Definition 2.1.3) ;
- For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, the set $J_\alpha$ is dense in $X$;
- The functions $D$ and $E$ are real analytic and strictly concave;
- For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$,
$$D(\alpha) = \frac{E(\alpha)}{\log \lambda}.$$

In the proof we use a tool called the *topological pressure* $P(\phi) \in \mathbb{R}$ (see Definition 3.2.1), and the good properties that the function $q \mapsto P(q\phi)$ has whenever $\phi$ is regular enough. As a corollary of the formula for $E(\alpha)$, we get the following result:

**Proposition 1.2.1.** For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$,

$$E(\alpha) = \sup\{h(\mu) : \mu \in \mathcal{M}_T, \mu(J_\alpha) = 1\}.$$

Now assume that $(X, T)$ is a non-compact full-shift. We will have several difficulties because of the non-compactness of the space $X$. If we ask for some regularity for the potential $\phi : X \to \mathbb{R}$, such as locally Hölder (see Definition 3.1.3) and zero pressure, the function $q \mapsto P(q\phi)$ has the good properties it had in the compact setting, however this holds in an interval $(q_*, \infty)$, whereas in $(-\infty, q_*)$ we get $P(q\phi) = \infty$.

Another difficulty we have in this setting is that we are no longer able to compute the same formula for $E(\alpha)$ as in Theorem 1.2.1, since we used strongly the compactness of the space $X$ and of the space $\mathcal{M}_T$ (with the weak* topology). However, Proposition 1.2.1 suggests a way to define in this setting the following spectrum:

**Definition 1.2.1.** The *variational entropy spectrum* is defined by

$$\tilde{E}(\alpha) := \sup\{h(\mu) : \mu \in \mathcal{M}_T, \mu(J_\alpha) = 1\}.$$

**Theorem 1.2.2.** Let $\phi : X \to (-\infty, 0)$ be a locally Hölder potential with $P(\phi) = 0$. For $\alpha \in \mathbb{R}$, $J_\alpha$ as in (1.1) and the functions $D, \tilde{E}$ from Definition 1.1.4 and Definition 1.2.1 respectively. Then, the following hold:

- The domain of $D$ and $\tilde{E}$ is an unbounded interval $(-\infty, \overline{\alpha}]$;
- For every $\alpha \in (-\infty, \overline{\alpha})$,
$$D(\alpha) = \frac{\tilde{E}(\alpha)}{\log \lambda};$$
- We have one of the following:
  1. $D$ (and hence $\tilde{E}$) is real analytic, strictly concave and strictly decreasing on $(-\infty, \overline{\alpha})$; or
  2. there exists $\alpha_* \in (-\infty, \overline{\alpha})$ such that $D$ (and hence $\tilde{E}$) is real analytic on $(\alpha_*, \overline{\alpha})$ and it is affine for $\alpha \in (-\infty, \alpha_*)$.

## 1.3 Working with symbolic space

In this section, as an illustration of the wide range of examples in which the results obtained in this thesis can be applied, we discuss several dynamical systems that admit symbolic codings. We present cases on compact and non-compact one-sided shifts, and one case of a compact two-sided shift. We will change the notation for the symbolic space to that we use in Chapter 3.

### 1.3.1 Symbolic coding

Let $f : M \to M$ be a differentiable map on a smooth Riemannian manifold $M$. We will see that sometimes we can study the dynamics of $f$ by looking at the the trajectory of the points given a certain partition. This relation will be called *coding*, and the trajectory of a point is given by an element of the symbolic space.

**Definition 1.3.1.** We say that $f$ is *expanding* on a compact $f$-invariant set $\Lambda \subset M$ if there exist constants $C > 0, \beta > 1$ such that

$$\|d_p f^n v\| \geq C\beta^n \|v\|$$

for every $n \in \mathbb{N}$, $x \in \Lambda$ and $v \in T_p M$.

**Definition 1.3.2.** A finite cover of $\Lambda$ by nonempty closed sets $\{R_1, \dots, R_N\}$ is called a *Markov Partition of* $\Lambda$ if

- $\overline{\operatorname{int} R_i} = R_i$ for every $i$;
- $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ if $i \neq j$;

- $R_j \subset f(R_i)$ if $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing$.

Let us assume also that for every $i, j$, $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing$. Then, it is possible to define a map $\chi : \Sigma_N := \{1, \ldots, N\}^{\mathbb{N}} \to \Lambda$ by

$$\chi(i_1 i_2 \ldots) := \bigcap_{k=0}^{\infty} f^{-k} R_{i_{k+1}}.$$

If we consider the shift map $\sigma : \Sigma_N \to \Sigma_N$ defined by $\sigma(i_1 i_2 i_3 \ldots) := (i_2 i_3 \ldots)$, then the coding map $\chi$ is such that $\chi \circ \sigma = f \circ \chi$.

$$\begin{array}{ccc} \Sigma_N & \xrightarrow{\sigma} & \Sigma_N \\ \chi \downarrow & & \downarrow \chi \\ \Lambda & \xrightarrow[f]{} & \Lambda \end{array}$$

Thus, in order to perform multifractal analysis over complicated spaces, sometimes it is very useful to work at symbolic level, apply the results obtained for symbolic space and then transfer them to the original setting. As an example, we have the following result from [Ba], which we state for the entropy spectrum.

**Theorem 1.3.1.** ([Ba, Theorem 9.4.1]) Let $f : \Lambda \to \Lambda$ be a $C^{1+\varepsilon}$ expanding transformation for some $\varepsilon > 0$. Assume that $f$ is conformal[1] and topologically mixing[2] on $\Lambda$. Let $\phi : \Lambda \to \mathbb{R}$ be a Hölder function, $\alpha \in \mathbb{R}$, $J_\alpha$ as in (1.1)[3] and the function $E$ from Definition 1.1.4. If $\phi$ is not cohomologous to a constant function, then

1. the function $E$ is defined on an interval $[\underline{\alpha}, \overline{\alpha}]$ and it is analytic in $(\underline{\alpha}, \overline{\alpha})$;
2. if $\alpha \in (\underline{\alpha}, \overline{\alpha})$,
$$E(\alpha) = \max \left\{ h(\mu) : \mu \in \mathcal{M}_f, \int \phi d\mu = \alpha \right\}.$$

**Remark 1.3.1.** There is an analogous way to define the coding of a partition $\{R_1, \ldots, R_N\}$ whenever $f$ is invertible. In this case the coding is over the space $\Sigma_N^{\pm} := \{1, \ldots, N\}^{\mathbb{Z}}$ and with the same shift map $\sigma : \Sigma_N^{\pm} \to \Sigma_N^{\pm}$, which is invertible as well.

Also, there exists countable Markov partitions, which coding is over the full-shift on countable many symbols $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{Z}}$, as we see in the following subsection.

[1] We say that $f$ is *conformal* in $\Lambda$ if $df_p$ is a multiple of an isometry for every $p \in \Lambda$.

[2] We say that $f$ is *topologically mixing* if for every $U, V$ open sets of $\Lambda$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for every $n \geq N$.

[3] Replacing $(X, T)$ by $(\Lambda, f)$.

### 1.3.2 Continued Fractions

This subsection is based on the work of Godofredo Iommi and Thomas Jordan in [IJ].

**Definition 1.3.3.** A map $T : [0,1] \to [0,1]$ is called an *EMR map* (expanding-Markov-Renyi) if there exists a countable family of closed intervals $I_i \subset [0,1]$ with pairwise disjoint interiors such that

- $T$ is of class $C^2$ on $\bigcup_i \operatorname{int} I_i$;
- there exists $\xi > 1$ and $N \in \mathbb{N}$ such that for every $x \in \bigcup_i I_i$ and $n \geq N$,

$$|(T^n)'(x)| > \xi^n;$$

- $T$ is Markov and it can be coded by a full-shift on a countable alphabet;
- (Renyi condition) there exists $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x,y,z\in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leq K.$$

The *repeller* of $T$ is defined by

$$\Lambda := \left\{ x \in \bigcup_i I_i : T^n x \text{ is well defined for every } n \in \mathbb{N} \right\}.$$

**Example 1.3.1.** The Gauss map $G : (0,1] \to (0,1]$ defined by $G(x) := 1/x - [1/x]$ ($[\cdot]$ is the integer part) is an EMR map.

For EMR maps, we can use the strategy we described in the previous subsection. This is, we can solve the problem at symbolic level, and then transfer the result to the original system. For example, consider the continued fraction expansion of a number

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} =: [a_1 a_2 a_3 \ldots] \in (0,1) \setminus \mathbb{Q},$$

where $a_i \in \mathbb{N}$ for every $i \in \mathbb{N}$. Then the Gauss map acts as the shift map in this expansion, i.e. if $x = [a_1 a_2 a_3 \ldots]$ then $G(x) = [a_2 a_3 \ldots]$.

Iommi and Jordan study the behavior of the limits

$$\lim_{n\to\infty} \log \sqrt[n]{a_1 a_2 \cdots a_n} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

where $x = [a_1 a_2 \dots]$, getting the following results:

**Proposition 1.3.1.** ([IJ, Proposition 6.5]) The function

$$D(\alpha) := \dim_H \left\{ x \in (0,1) : \lim_{n\to\infty} \log \sqrt[n]{a_1 \cdot a_2 \cdots a_n} = \alpha \right\}$$

is real analytic, it is strictly increasing and strictly concave in an interval $[\alpha_m, \alpha_*)$, and it is decreasing and has an inflection point in $(\alpha_*, \infty)$.

**Proposition 1.3.2.** ([IJ, Proposition 6.7]) The function

$$D(\alpha) := \dim_H \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \alpha \right\}$$

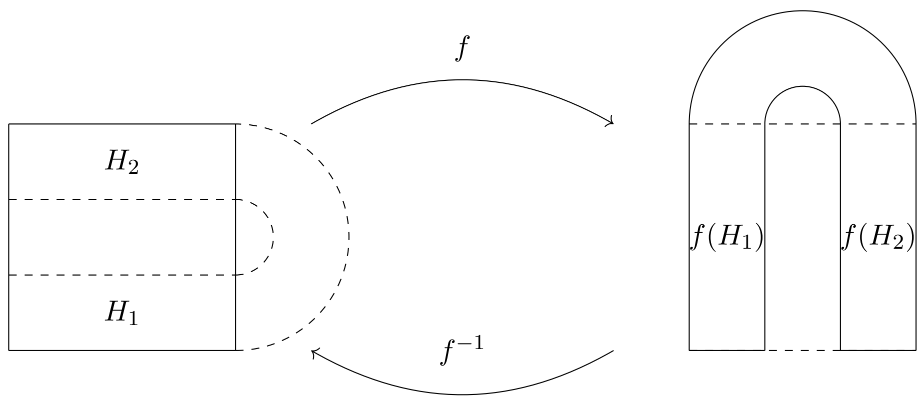
is real analytic and strictly increasing.

In Remark 1.3.1 we discussed that when the system $f : \Lambda \to \Lambda$ is invertible, it can be coded on a two-sided full shift, as we see in the following subsection.

### 1.3.3 Horseshoes

This subsection is based on the work of Luis Barreira and Claudia Valls in [BV1].

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the Smale horseshoe map. It acts on the unit square $S := [0,1]^2$ as a strong contraction in the horizontal direction, followed by a strong expansion in the vertical direction, folding and placing back over $S$ (see [BV2, Section 5.2.2]):




The repeller of $f$

$$\Lambda := \{x \in S : f^k(x) \in S \text{ for every } k \in \mathbb{Z}\}$$

is the product of two middle third Cantor sets $C$.

Given continuous functions $\phi, \psi : \Lambda = C \times C \to \mathbb{R}$, consider the 'two sided' level sets of Birkhoff averages: for $\alpha, \beta \in \mathbb{R}$ set

$$J_{\alpha\beta} := \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \alpha \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^{-k} x) = \beta \right\},$$

and the following spectrum

$$D(\alpha, \beta) := \dim_H J_{\alpha\beta}.$$

From the behavior of $f$ along the vertical and horizontal directions, if we denote by $p_1$ and $p_2$ the orthogonal projections onto the horizontal and vertical axes respectively, then

$$p_1(J_{\alpha\beta}) \times C = \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^{-k} x) = \beta \right\},$$

$$C \times p_2(J_{\alpha\beta}) = \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^{k} x) = \alpha \right\}.$$

Thus, notice that

$$J_{\alpha\beta} = (p_1(J_{\alpha\beta}) \times C) \cap (C \times p_2(J_{\alpha\beta})) = p_1(J_{\alpha\beta}) \times p_2(J_{\alpha\beta}).$$

The main result in [BV1] is that a multifractal analysis on two variables becomes two independent multifractal analysis on one variable, since $p_1(J_{\alpha\beta})$ does not depend on $\alpha$ (neither on $\phi$) and $p_2(J_{\alpha\beta})$ does not depend on $\beta$ (neither on $\psi$).

**Theorem 1.3.2.** The spectrum $D(\alpha, \beta)$ is real analytic, and for every $(\alpha, \beta)$ in the domain of $D$

$$D(\alpha, \beta) = \dim_H p_1(J_{\alpha\beta}) + \dim_H p_2(J_{\alpha\beta}).$$

The system $(\Lambda, f)$ can be coded in the two-sided full shift on two symbols $\Sigma_2^{\pm} := \{1, 2\}^{\mathbb{Z}}$. The main idea is, through the coding map, to present the problem with functions $\tilde{\phi}, \tilde{\psi} : \Sigma_2^{\pm} \to \mathbb{R}$, use arguments on this space to prove that these functions are cohomologous (see Definition 3.1.5) respectively to functions $\phi^u : \Sigma_2^{\pm} \to \mathbb{R}$ and $\psi^s : \Sigma_2^{\pm} \to \mathbb{R}$, where $\phi^u$ depends only on the future of the points and $\psi^s$ depends only on the past of the points ([BV1, Lemma 1]). This allows to obtain an explicit formula for the multifractal spectrum as the sum of two multifractal spectra, and then transfer the result to the repeller $\Lambda$ ([BV1, Theorem 3]).

# Chapter 2

# Dimension Theory and Entropy

In multifractal analysis there are many ways of measuring the size of the level sets we are studying, in this chapter we will review some of them. Throughout the Dimension Theory section we define a notion of dimension, called the Hausdorff Dimension, and we present a technique to compute it using finite Borel measures. Then, in the Entropy section we review the classical definitions of topological entropy on compact topological spaces, Bowen's definition for uniformly continuous functions on metric spaces (not necessarily compact), and we finish with two equivalent dimensional-like definitions of entropy restricted to subsets given by Bowen in [Bo1] and by Pesin and Pitskel in [PP], which is the one we will use in multifractal analysis.

## 2.1 Dimension Theory

### 2.1.1 Hausdorff Dimension

Let $(X, d)$ be a separable metric space.

**Definition 2.1.1.** A collection of subsets $\{E_j\}_{j\in I}$ is called an *open cover of* $F \subset X$ if each $E_i$ is open and $F \subset \bigcup_{j\in I} E_i$.

For $F \subset X$, $\delta > 0$ and $s \geq 0$, define

$$\mathcal{H}^s_\delta(F) := \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} E_i)^s : \{E_i\}_i \text{ is open cover of } F \text{ and } \operatorname{diam} E_i < \delta\right\},$$

and $\mathcal{H}^s(F) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$.

For any $s \geq 0$, the function $\mathcal{H}^s$ is an outer measure on $X$ and induces a $\sigma$-additive measure on $X$ called the *$s$-dimensional Hausdorff measure*.

Given a set $F \subset X$ and $0 < \delta < 1$, observe that $\mathcal{H}^s_\delta(F)$ is a non-increasing function of $s$, and so is $\mathcal{H}^s(F)$. Moreover, if $t > s$ and $\{U_i\}$ is an open cover of $F$ with $\operatorname{diam} U_i < \delta$,

$$\sum_i (\operatorname{diam} U_i)^t \leq \sum_i (\operatorname{diam} U_i)^{t-s} (\operatorname{diam} U_i)^s \leq \delta^{t-s} \sum_i (\operatorname{diam} U_i)^s.$$

So, $\mathcal{H}^t_\delta(F) \leq \delta^{t-s} \mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$ notice that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. Thus, there exists a critical value $s_\star \geq 0$ such that $\mathcal{H}^s(F) = \infty$ for $s < s_\star$, $\mathcal{H}^s(F) = 0$ for $s > s_\star$ and $\mathcal{H}^{s_\star}(F) \in [0, \infty]$. This behavior is shown in Figure 2.1.

**Definition 2.1.2.** For $F \subset X$, the number $s_\star$ is called *the Hausdorff dimension of $F$*, and it is denoted by $\dim_H F$.

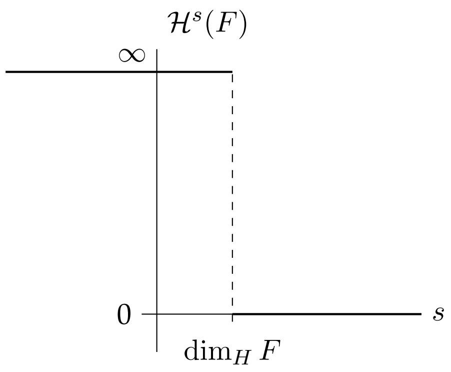



FIGURE 2.1: Graphic of $s \mapsto \mathcal{H}^s(F)$.

The Hausdorff dimension satisfy the following properties:

**Proposition 2.1.1.** (see [P, Theorem 6.1] and [P, Theorem 6.2])

1. $\dim_H \varnothing = 0$; $\quad \dim_H F \geq 0$ for any $F \subset X$.
2. If $F_1 \subset F_2$, then $\dim_H F_1 \leq \dim_H F_2$.
3. $\dim_H \bigcup_{i=1}^{\infty} F_i = \sup_{i \in \mathbb{N}} \dim_H F_i$.
4. If $F$ is finite or countable, then $\dim_H F = 0$.

**Remark 2.1.1.** In [F] this theory is developed in $\mathbb{R}^n$. For $m \in \mathbb{N}$, there is a relation between the $m$-dimensional Hausdorff measures and the classic $m$-Lebesgue measure. Hausdorff measures generalize the notions of length, area, volume, etc. in the following way: for $F \subset \mathbb{R}^n$

$$\mathcal{H}^m(F) = c_m^{-1} \operatorname{Leb}^m(F),$$

where $\mathrm{Leb}^m$ is the $m$-dimensional Lebesgue measure, and $c_m$ is the $\mathrm{Leb}^m$-measure of the $m$-dimensional ball of diameter 1. So, for lower-dimensional subsets of $\mathbb{R}^n$, $\mathcal{H}^0$ counts the number of points in the set, $\mathcal{H}^1$ gives the length (of a line or a curve for example), $\mathcal{H}^2$ gives the area of a smooth surface (or a 2-dimensional object), $\mathcal{H}^3$ is the volume, etc.

The scaling properties of length, area and volume are known. If we scale by a factor $k > 0$, the length of a curve is multiplied by $k$, the area of a plane section is multiplied by $k^2$, and the volume is multiplied by $k^3$. Hence, we can think the Hausdorff dimension $s$ of a set $F$ as the exponent in the scaling factor such that the $s$-dimensional Hausdorff measure is multiplied by $k^s$ when the set $F$ is scaled by a factor $k$. In other words, we have the following proposition.

**Proposition 2.1.2.** ([F, Scaling property 2.1]) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity transformation of scale factor $k > 0$, this is, image of sets can be obtained by uniformly scaling by $k$, possibly with additional translation, rotation and reflection. Then for $F \subset \mathbb{R}^n$ and $s \geq 0$,

$$\mathcal{H}^s(S(F)) = k^s \mathcal{H}^s(F).$$

**Example 2.1.1.** Let $C$ be the middle third Cantor set (see Figure 2.2), and decompose it into its left part $C_L := C \cap [0, 1/3]$ and its right part $C_R := C \cap [2/3, 1]$. Observe that both parts are geometrically the same as the original set $C$, but scaled by a factor $k = 1/3$. We also have that $C$ is the disjoint union of $C_L$ and $C_R$, then

$$\mathcal{H}^s(C) = \mathcal{H}^s(C_R) + \mathcal{H}^s(C_L) = \frac{1}{3^s}\mathcal{H}^s(C) + \frac{1}{3^s}\mathcal{H}^s(C) = \frac{2}{3^s}\mathcal{H}^s(C).$$

If we assume that $0 < \mathcal{H}^{\dim_H C}(C) < \infty$, then letting $s = \dim_H C$ and dividing by $\mathcal{H}^s(C)$, we have $\dim_H C = \dfrac{\log 2}{\log 3} = 0.6309...$.


FIGURE 2.2: Construction of the middle third Cantor set $C$.

In [OV, Example 12.4.1] is proven that actually $\mathcal{H}^{\log 2/\log 3}(C) = 1$, which implies that $\dim_H C = \log 2/\log 3$. However, to compute the Hausdorff dimension of the Cantor set we use the assumption $0 < \mathcal{H}^{\dim_H C}(C) < \infty$, but this is not always true. There exist sets $F$ for which $\mathcal{H}^{\dim_H F}(F)$ equals zero or infinity.

**Example 2.1.2.** An example is any Euclidian space $\mathbb{R}^n$. It has Hausdorff dimension $n$ and $\mathcal{H}^n(\mathbb{R}^n) = \infty$. Other example is any countable set: it has zero Hausdorff dimension, but when we count its points the result is infinity.

On the other hand, if we consider the family of entire functions $\{f(z) = \lambda e^z : \lambda \neq 0\}$ (the "exponential family"), we have that the Julia set (this is, the boundary of the set of points which converge to infinity under iteration) of any member of this family has Hausdorff dimension 2. However, for some values of $\lambda$ (for example $0 < \lambda < 1/e$), the area of the Julia set of $\lambda e^z$ is zero (see [Mc, Theorem 1.2] and [Mc, Theorem 1.3]).

Now, to compute the Hausdorff dimension of certain sets, we will need some tools related to the use of measures on the space $X$.

**Definition 2.1.3.** Let $\mu$ be a finite Borel measure on $X$. The *Hausdorff dimension of* $\mu$ is defined by

$$\dim_H \mu := \inf\{\dim_H F : \mu(F) = 1\}.$$

**Definition 2.1.4.** Let $\mu$ be a finite Borel measure on $X$. For $x \in X$, define the *lower and upper pointwise dimension of* $x$ *with respect to* $\mu$ respectively by

$$\underline{d_\mu}(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \overline{d_\mu}(x) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r};$$

where $B(x,r) := \{y \in X : d(x,y) < r\}$. When this limits coincide, we will denote it by $d_\mu(x)$.

**Remark 2.1.2.** Notice that the pointwise dimension of a point $x$ with respect to $\mu$ describes the behavior

$$\mu(B(x,r)) \sim r^{d_\mu(x)}$$

as $r \to 0$. It also quantifies how concentrated is a measure around a point.

**Example 2.1.3.** Let $x \in X$, and let $\mu$ be the atomic measure concentrated on $x$, i.e. $\mu(A) = 0$ if $x \notin A$ and $\mu(A) = 1$ if $x \in A$. Then $d_\mu(x) = 0$ and $d_\mu(x') = \infty$ whenever $x' \neq x$.

**Definition 2.1.5.** We say that $X$ is *a metric space of finite multiplicity* if the following condition holds: there exists $K > 0$ and $\varepsilon_0$ such that for any $0 < \varepsilon < \varepsilon_0$ one can find a cover of $X$ by balls of radius $\varepsilon$ such that every point in $X$ belongs to at most $K$ balls of the cover.

**Definition 2.1.6.** We say that a complete separable metric space $X$ is a *Besicovitch metric space* if the following condition holds: there exist $K > 0$ and $\varepsilon_0 > 0$ such that for any subset $Z \subset X$ and any cover $\{B(x,\varepsilon(x)) : x \in Z, 0 < \varepsilon(x) \leq \varepsilon_0\}$ one can find a subcover of $Z$ such that every point of $Z$ belongs to at most $K$ elements of the subcover.

The following theorems were proven in [P, Theorem 7.1] and [P, Theorem 7.2] in the case $X = \mathbb{R}^m$. However, in [P, Appendix I] there is a discussion about them with the hypotheses we will present.

**Theorem 2.1.1.** Let $X$ be a complete separable metric space of finite multiplicity and let $\mu$ be any Borel finite measure on $X$. Then the following statements hold:

1. if $\underline{d_\mu}(x) \geq d$ for $\mu$-almost every $x$ then $\dim_H \mu \geq d$;
2. if $\overline{d_\mu}(x) \leq d$ for $\mu$-almost every $x$ then $\dim_H \mu \leq d$.

Thus, if $\underline{d_\mu}(x) = \overline{d_\mu}(x) = d$ for $\mu$-almost every $x$, then $\dim_H \mu = d$.

**Theorem 2.1.2.** Let $X$ be a Besicovitch metric space and let $\mu$ be any Borel finite measure on $X$. Assume that there exists $d > 0$ such that $\underline{d_\mu}(x) \leq d$ for **every** $x \in Z \subset X$. Then $\dim_H Z \leq d$.

**Remark 2.1.3.** Theorem 2.1.1 and Theorem 2.1.2 give a technique to compute the Hausdorff dimension of a set $F \subset X$. If one can find a number $d$ and a finite measure $\mu$ such that $\mu(F) = 1$, $\underline{d_\mu}(x) \geq d$ for *$\mu$-almost every* $x$ and $\underline{d_\mu}(x) \leq d$ for *every* $x \in F$, then $\dim_H F = d$.

**Remark 2.1.4.** There is an alternative definition of dimension, called the *upper and lower Box-counting dimension* and denoted respectively by $\overline{\dim}_B$ and $\underline{\dim}_B$. Its relation with the Hausdorff dimension is that for every $F \subset \mathbb{R}^n$, $\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F$. Another property of the Box-counting dimension is that the dimension of a set equals the dimension of the closure of the set. This property shows the main disadvantage of the box-counting dimension to be used as multifractal spectrum, since the level sets considered in our computations of dimension are dense. Thus, when we compute the box dimension of one of these sets, we obtain the box dimension of the entire space.

## 2.2 Entropy

Throughout this section we will review the notion of entropy of a dynamical system with respect to an invariant probability measure and the notion of topological entropy. Finally, we will introduce a dimensional definition of entropy of a system restricted to arbitrary subsets, which is more useful in multifractal analysis.

### 2.2.1 Metric Entropy

Let $T : X \to X$ be a measurable transformation preserving a probability measure $\mu$, i.e. $\mu(T^{-1}A) = \mu(A)$ for every measurable set $A \subset X$. Recall that a partition $\mathcal{P}$ of a the probability space $(X, \mathcal{B}, \mu)$ is a finite or countable collection of pairwise disjoint subsets of $X$ such that their union has full measure.

**Definition 2.2.1.** The *entropy* of the partition $\mathcal{P}$ with respect to the measure $\mu$ is defined by

$$H_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where $0 \log 0 := 0$.

Given a partition $\mathcal{P}$ and $n \in \mathbb{N}$, set

$$\mathcal{P}^n := \{P_0 \cap T^{-1}P_1 \cap \cdots \cap T^{-n+1}P_{n-1} : P_j \in \mathcal{P}\}.$$

Since $T$ preserves $\mu$, the sequence $H_\mu(\mathcal{P}^n)$ is subadditive, so we have the following definition.

**Definition 2.2.2.** The *entropy* of the measure $\mu$ with respect to the partition $\mathcal{P}$ is defined by

$$h(\mu, \mathcal{P}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

**Definition 2.2.3.** The *entropy* of $\mu$ is defined by

$$h(\mu) = h(\mu, T) := \sup\{h(\mu, \mathcal{P}) : \mathcal{P} \text{ partition of } X, H_\mu(\mathcal{P}) < \infty\}.$$

We also have results to compute the entropy of a measure. Given a partition $\mathcal{P}$, denote by $\mathcal{P}(x)$ the element of the partition which contains the point $x$. For $n \geq 1$, let

$$\mathcal{P}^n(x) := \mathcal{P}(x) \cap T^{-1}\mathcal{P}(Tx) \cap \cdots \cap T^{-n+1}\mathcal{P}(T^{n-1}x)$$

be the points with trajectory close to that of $x$ until time $n - 1$.

**Theorem 2.2.1.** (Shannon-McMillan-Breiman) Given a partition with finite entropy $\mathcal{P}$, for $\mu$-almost every $x \in X$ there exists the limit

$$h(\mu, \mathcal{P}, x) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(x)).$$

The function $x \mapsto h(\mu, \mathcal{P}, x)$ is $\mu$-integrable and the convergence holds also in $L^1(\mu)$. Moreover,

$$h(\mu, \mathcal{P}) = \int h(\mu, \mathcal{P}, x) d\mu(x)$$

and if $\mu$ is an ergodic measure with respect to $T$, then $h(\mu, \mathcal{P}, x) = h(\mu, \mathcal{P})$ for $\mu$-almost every $x \in X$.

We also have another way to compute the metric entropy of a dynamical system $T$ with respect to a $T$-invariant probability measure $\mu$:

**Proposition 2.2.1.** ([OV, Corolary 9.2.5]) Let $\mathcal{P}$ be a partition of finite entropy of $X$ such that the union of the iterates $\mathcal{P}^n$ generates the $\sigma$-algebra of measurable sets, up to measure zero. Then $h(\mu) = h(\mu, \mathcal{P})$.

**Example 2.2.1.** Let $X := \{1, 2, \dots, N\}^{\mathbb{N}}$ with the dynamical system $T : X \to X$ defined by $T((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}$. Consider a product measure $\mu := \nu^{\mathbb{N}}$ and set $p_i := \nu(\{i\})$ for $1 \leq i \leq N$. Such a measure $\mu$ is called a *Bernoulli measure*, and it is ergodic with respect to $T$ [OV, Proposition 4.2.7]. For $n \in \mathbb{N}$, $i_1, \dots, i_n \in \{1, \dots, N\}$, define the cylinder

$$[i_1 \dots i_n] := \{(x_n)_{n\in\mathbb{N}} \in X : x_j = i_j, 1 \leq j \leq n\}.$$

Then, we have

$$\mu([i_1 \dots i_n]) = \prod_{j=1}^{n} p_{i_j}.$$

We will use the Shannon-McMillan-Breiman Theorem to compute the entropy of $T$ with respect to this measure $\mu$ and the partition $\mathcal{P} := \{[i] : 1 \leq i \leq N\}$ of cylinders of length one. Notice that for every $n \geq 1$ and $x = (x_n)_{n\in\mathbb{N}} \in X$,

$$\mathcal{P}^n(x) = [x_1 \dots x_n].$$

Now, for $1 \leq i \leq N$, $x \in X$ and $n \geq 1$ define

$$f_i(x, n) := \#\{1 \leq j \leq N : x_j = i\} = \sum_{j=0}^{n-1} \chi_{[i]}(T^j x),$$

and observe that

$$\mu(\mathcal{P}^n(x)) = \prod_{i=1}^{N} p_i^{f_i(x,n)}.$$

Since $\mu$ is an ergodic measure with respect to $T$, by Birkhoff's Ergodic Theorem we obtain $\lim_n \frac{f_i(x,n)}{n} = \mu([i]) = p_i$ for $\mu$-almost every $x \in X$.

Then, for $\mu$-almost every $x \in X$

$$\begin{aligned} h(\mu, \mathcal{P}, x) &= \lim_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{P}^n(x)) = \lim_{n\to\infty} -\frac{1}{n}\log\prod_{i=1}^{N} p_i^{f_i(x,n)} \\ &= \lim_{n\to\infty} -\frac{1}{n}\sum_{i=1}^{N} f_i(x,n)\log p_i = -\sum_{i=1}^{N} p_i \log p_i. \end{aligned}$$

By Shannon-McMillan-Breiman Theorem, for $\mu$-almost every $x \in X$

$$h(\mu, \mathcal{P}) = h(\mu, \mathcal{P}, x) = -\sum_{i=1}^{N} p_i \log p_i.$$

Since cylinders generate the topology on $X$, and hence the measurable Borel sets, by Proposition 2.2.1 we have $h(\mu) = -\sum_{i=1}^{N} p_i \log p_i$.

### 2.2.2 Topological Entropy

The topological entropy of a topological dynamical system $T$ is a number $h(T) \in [0, \infty]$ which measures the complexity of the system. First, we will present the definition introduced by Adler, Konheim and McAndrew on a compact topological space; then we will present a definition of topological entropy by Bowen on a metric space, not necessarily compact. Finally, we conclude this chapter presenting a dimensional-like definition of entropy of a dynamical system restricted to arbitrary subsets of the space.

#### 2.2.2.1 Definition by open covers

Let $X$ be a compact topological space and $T : X \to X$ a continuous function. Given an open cover $\alpha$, for $n \geq 1$ let

$$\alpha^n := \{A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-n+1}A_{i_{n-1}} : A_{i_j} \in \alpha\}, \tag{2.1}$$

which is also an open cover of $X$.

Denote by $N(\alpha^n)$ the number of sets in a finite subcover of $\alpha^n$ with smallest cardinality, and define the *entropy of $T$ with respect to the cover $\alpha$* by

$$h(T, \alpha) := \lim_{n\to\infty} \frac{1}{n}\log N(\alpha^n).$$

**Definition 2.2.4.** The *topological entropy of $T$* is defined by

$$h(T) := \sup\{h(T,\alpha) : \alpha \text{ is open cover of } X\}.$$

**Example 2.2.2.** Let $(X,T)$ be the dynamical system as in the Example 2.2.1 and consider $\alpha := \{[i] : 1 \le i \le N\}$ the open cover by cylinders of length one. In order to compute $h(T,\alpha)$, notice that

$$\alpha^n = \{[i_1 \dots i_n] : 1 \le i_j \le N\}$$

and that if we remove an element of $\alpha^n$, it will no longer be a cover of $X$. Therefore, $N(\alpha^n) = \#\alpha^n = N^n$.

Finally,

$$h(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} \log N^n = \log N.$$

Moreover, it can be proven that actually $h(T) = \log N$ (see [OV, Example 10.1.2] and [OV, Corollary 10.1.13]).

Now we present an important relationship between the two notions of entropy that we have discussed, the metric entropy and the topological entropy (see for example [W, Theorem 8.6]).

**Theorem 2.2.2.** (Variational Principle) Let $T : X \to X$ be a continuous map of a compact metric space $X$. Then

$$h(T) = \sup\{h(\mu) : \mu \in \mathcal{M}_T\}, \tag{2.2}$$

where $\mathcal{M}_T$ is the set of all Borel $T$-invariant probability measures on $X$.

#### 2.2.2.2 Bowen's definition

Let $(X,d)$ be a metric space (not necessarily compact) and $T : X \to X$ a uniformly continuous map. For $n \ge 1$, $x \in X$ and $\varepsilon > 0$ define the *dynamic ball of center $x$, length $n$ and radius $\varepsilon$* by

$$B_n(x,\varepsilon) := \{y \in X : d(T^j x, T^j y) < \varepsilon \text{ for every } 0 \le j \le n-1\}.$$

**Definition 2.2.5.** Let $n \in \mathbb{N}$, $\varepsilon > 0$ and let $K$ be a compact subset of $X$. We say that a subset $F \subset X$ is a $(n,\varepsilon)$-*spanning set* for $K$ if for every $x \in K$, there exists $y \in F$ such that $d(T^j x, T^j y) < \varepsilon$ for every $0 \leq j \leq n-1$. This is, if

$$K \subset \bigcup_{y \in F} B_n(y, \varepsilon).$$

**Definition 2.2.6.** Denote by $r_n(\varepsilon, K)$ the smallest cardinality of any $(n, \varepsilon)$-spanning set for $K$. Notice that this number is finite because of the compactness of $K$, and that is decreasing as a function of $\varepsilon$.

**Definition 2.2.7.** Define

$$r(T) := \sup_K \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K),$$

where the supremum is taken over the collection of all compact subsets of $X$.

**Definition 2.2.8.** Let $n \in \mathbb{N}$, $\varepsilon > 0$ and let $K$ be a compact subset of $X$. We say that a subset $E \subset K$ is $(n, \varepsilon)$-*separated* if for every $x, y \in E$, there exists $0 \leq j \leq n-1$ such that $d(T^j x, T^j y) \geq \varepsilon$. That is, if for every $x \in E$ the dynamic ball $B_n(x, \varepsilon)$ contains no other point of $E$.

**Definition 2.2.9.** Denote by $s_n(\varepsilon, K)$ the largest cardinality of any $(n, \varepsilon)$-separated subset of $K$.

It can be proven that this number $s_n(\varepsilon, K)$ is finite, and that it is decreasing as a function of $\varepsilon$.

**Definition 2.2.10.** Define

$$s(T) := \sup_K \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, K),$$

where the supremum is taken over the collection of all compact subsets of $X$.

The following propositions are proven in [OV, Chapter 10].

**Proposition 2.2.2.** $r(T) = s(T)$.

**Proposition 2.2.3.** If $X$ is a compact metric space, then $h(T) = r(T) = s(T)$.

Thus, we can define the *topological entropy* of a uniformly continuous map $T : X \to X$ of a metric space by $h(T) := r(T) = s(T)$. Proposition 2.2.3 shows that this definition is compatible with the definition by open covers when $X$ is compact.

**Remark 2.2.1.** A relevant difference between the definition of topological entropy from subsection 2.2.2.1 and the one given in this subsection is that in the compact case it depends only on the topology ($h(T)$ is defined using open covers), but in the non-compact case the definition depends upon the metric $d$. Sometimes we write $h_d(T)$ to show this dependence.

**Example 2.2.3.** Let $T : X \to X$ be an isometry of the metric space $(X, d)$. Notice that for every $n \geq 1$ and $x \in X$, $B_n(x, \varepsilon) = B_1(x, \varepsilon)$ is the usual ball of center $x$ and radius $\varepsilon$. Then for every compact $K \subset X$, $s_n(\varepsilon, K) = s_1(\varepsilon, K)$ and therefore $h_d(T) = 0$.

**Example 2.2.4.** (Dependence on the metric) Consider $T : (0, \infty) \to (0, \infty)$ defined by $T(x) := 2x$. Define the metric $d'$ on $(0, \infty)$ by

$$d'(x, y) := |\log x - \log y|.$$

Notice that $T$ is an isometry of the metric space $((0, \infty), d')$ and by the Example 2.2.3 $h_{d'}(T) = 0$.

Now let $d$ be the Euclidian metric. Notice that

$$\begin{aligned} B_n(x, \varepsilon) &= \{y \in (0, \infty) : d(T^j x, T^j y) < \varepsilon \text{ for } 0 \leq j \leq n-1\} \\ &= \{y \in (0, \infty) : |x - y| < \varepsilon / 2^{n-1}\} \\ &= \left(x - \frac{\varepsilon}{2^{n-1}}, x + \frac{\varepsilon}{2^{n-1}}\right). \end{aligned}$$

The length of each one of these intervals is $\varepsilon / 2^{n-2}$. If we sum $k$ of these lengths the result is $k\varepsilon / 2^{n-2}$, so to cover for example the interval $[1, 2]$ we need $k\varepsilon / 2^{n-2} > 1$ and thus, $k \geq 2^{n-2} / \varepsilon$. Therefore,

$$\begin{aligned} h_d(T) &= \sup_K \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K) \\ &\geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, [1, 2]) \\ &\geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\frac{2^{n-2}}{\varepsilon}\right) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left(\frac{1}{n} \log 2^{n-2} - \frac{1}{n} \log \varepsilon\right) \\ &= \log 2. \end{aligned}$$

So, even though the metrics $d'$ and $d$ induce the same topology, $h_{d'}(T) \neq h_d(T)$. This is because these two metrics are not uniformly equivalent: being uniformly equivalent is a sufficient condition for to metrics to have the same topological entropy on a non-compact space (see [W, Theorem 7.4]).

#### 2.2.2.3 Entropy restricted to subsets

Let $X$ be a separable metric space (not necessarily compact) and $T : X \to X$ a continuous function. In order to study the level sets in multifractal analysis (see (1.1)), we will need a dimensional definition of entropy restricted to subsets of the space. The usual definition is not useful because these level sets are dense and non-compact.

Let $A \subset X$ be a non-compact dense set and assume for a moment that $X$ is compact. We will try to define the topological entropy of $T$ restricted to $A$, $h(T|A)$, in a similar way it was done in subsection 2.2.2.1. In order to do this, we have to consider an open cover $\alpha$ of $A$, and the cover $\alpha^n$ as in (2.1).

The first problem is that the number $N(\alpha^n)$ is not necessarily well defined for each $n$, since $A$ is non-compact. We can try to avoid this problem considering open covers of $X$, which always have finite sub-covers since $X$ is compact. So if $\alpha$ is an open cover of $X$, then $\alpha_A := \{U \in \alpha : U \cap A \neq \varnothing\}$ is open cover of $A$. Now for $n \in \mathbb{N}$ we can define $N(\alpha_A^n)$ as its usual definition if $\alpha_A^n$ has a finite sub-cover of $A$, and as $N(\alpha^n)$ otherwise. However, since $A$ is dense, we have that $\alpha_A = \alpha$ and we conclude $h(T|A) = h(T)$. Then, since the level sets $J_\alpha$ are dense and non-compact, this is not a good definition because our spectrum $E(\alpha) := h(T|J_\alpha)$ would be the constant $h(T)$.

Now, if $X$ is non-compact we may use a definition of entropy restricted analogous to Bowen's definition, but in this case as we saw in Example 2.2.4, it depends on the metric, and we do not want this either.

We will give two definitions, the first one was presented by Pesin and Pitskel in [PP] and it coincides with the second one presented by Bowen in [Bo1] (see [PP, Proposition 4]).

**Definition 2.2.11.** ([PP]) Let $\mathcal{U}$ be a finite open cover of $X$, and set

$$\mathcal{W}_m(\mathcal{U}) := \{(U_{i_0}, \ldots, U_{i_{m-1}}) : U_{i_j} \in \mathcal{U}\}, \quad \mathcal{W}(\mathcal{U}) := \bigcup_{m \geq 0} \mathcal{W}_m(\mathcal{U}).$$

For $J \subset X$ and $\underline{U} \in \mathcal{W}_m(\mathcal{U})$, set

$$J(\underline{U}) := \{x \in J : T^k x \in U_{i_k}, 0 \leq k \leq m-1\}.$$

We say that $\Gamma \subset \mathcal{W}(\mathcal{U})$ covers $J$ if $J \subset \bigcup_{\underline{U} \in \Gamma} J(\underline{U})$, and denote by $m(\underline{U})$ the length of the vector $\underline{U}$, this is, the unique integer $m \geq 0$ such that $\underline{U} \in \mathcal{W}_m(\mathcal{U})$.

Let us define

$$M(\mathcal{U}, \lambda, J, N) := \inf\left\{\sum_{\underline{U}\in\Gamma} \exp(-\lambda m(\underline{U}))\right\},$$

where the infimum is taken over all $\Gamma \subset \mathcal{W}(\mathcal{U})$ covering $J$ such that $m(\underline{U}) \geq N$ for all $\underline{U} \in \Gamma$. Notice that $M$ increases monotonically when $N$ increases. Thus, the following limit exists

$$m_{\mathcal{U}}(\lambda, J) := \lim_{N\to\infty} M(\mathcal{U}, \lambda, J, N).$$

For $J$ fixed, the function $m_{\mathcal{U}}$ has the following property: there exists $\lambda_0$ such that $m_{\mathcal{U}}(\lambda, J) = 0$ for $\lambda > \lambda_0$ and $m_{\mathcal{U}}(\lambda, J) = \infty$ for $\lambda < \lambda_0$. Now define

$$h_{\mathcal{U}}(J) := \inf\{\lambda : m_{\mathcal{U}}(\lambda, J) = 0\}.$$

The following properties hold:

1. $h_{\mathcal{U}}(\varnothing) = 0$;
2. if $J_1 \subset J_2 \subset X$, then $h_{\mathcal{U}}(J_1) \leq h_{\mathcal{U}}(J_2)$;
3. if $J = \bigcup_{i\geq 1} J_i \subset X$, then $h_{\mathcal{U}}(J) = \sup_{i\geq 1} h_{\mathcal{U}}(J_i)$.

Finally, define the entropy of the map $T$ restricted to the set $J$ by

$$h(J) = h(T|J) := \sup\{h_{\mathcal{U}}(J) : \mathcal{U} \text{ is finite open cover of } X\}.$$

**Definition 2.2.12.** ([Bo1]) Let $\mathcal{U}$ be a finite open cover of $X$. We write $E \prec \mathcal{U}$ if $E$ is contained in some member of $\mathcal{U}$ and $\{E_i\}_i \prec \mathcal{U}$ if $E_i \prec \mathcal{U}$ for every $i$. Denote by $n_{\mathcal{U}}(E)$ the largest nonnegative integer such that $T^k E \prec \mathcal{U}$ for every $0 \leq k < n_{\mathcal{U}}(E)$; $n_{\mathcal{U}}(E) = 0$ if $E \not\prec \mathcal{U}$ and $n_{\mathcal{U}}(E) = \infty$ if $T^k E \prec \mathcal{U}$ for every $k \geq 1$. Now set

$$D_{\mathcal{U}}(E) := \exp(-n_{\mathcal{U}}(E)).$$

For $\lambda \in \mathbb{R}$, define a measure by

$$m_{\mathcal{U},\lambda}(J) := \lim_{\varepsilon\to 0} \inf\left\{\sum_{i=1}^{\infty} D_{\mathcal{U}}(E_i)^{\lambda} : J \subset \bigcup_i E_i \text{ and } D_{\mathcal{U}}(E_i) < \varepsilon\right\}.$$

Observe that for $J$ fixed, this function of $\lambda$ satisfy the same property of the function $m_{\mathcal{U}}(\lambda, J)$ from Definition 2.2.11. That is, $0 < m_{\mathcal{U},\lambda}(J) < \infty$ for at most one $\lambda$.

Define now $h^*_{\mathcal{U}}(J) := \inf\{\lambda : m_{\mathcal{U},\lambda}(J) = 0\}$ and finally

$$h^*(J) = h^*(T|J) := \sup\{h^*_{\mathcal{U}}(J) : \mathcal{U} \text{ is finite open cover of } X\}.$$

Pesin and Pitskel prove in [PP, Proposition 4] that these definitions coincide, and we denote it by $h(J)$ or $h(T|J)$. This definition is compatible with the usual definition of topological entropy discussed in Subsection 2.2.2.1 whenever $X$ is a compact topological space: Bowen prove in [Bo1, Proposition 1] that if $X$ is compact, then $h(T|X) = h(T)$.

Now we present two results that show relations between this definition of entropy and the set of Borel $T$-invariant probability measures.

**Theorem 2.2.3.** ([Bo1, Theorem 1]) Assume that $X$ is compact and let $J \subset X$. If $\mu \in \mathcal{M}_T$ is such that $\mu(J) = 1$, then $h(\mu) \leq h(T|J)$.

For $X$ compact, the set $\mathcal{M}_T$ is a compact topological space with the weak* topology (see for example [OV, Chapter 2]). For $x \in X$, we define $V_T(x)$ as the set of all limit points of the sequence

$$\left(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{T^j x}\right)_{n\in\mathbb{N},}$$

where $\delta_a$ is the atomic measure concentrated at the point $a$. Then, $V_T(x)$ is non-empty and by [OV, Lemma 2.2.4] we have $V_T(x) \subset \mathcal{M}_T$.

**Theorem 2.2.4.** ([Bo1, Theorem 2]) Assume $X$ is compact, and set

$$QR(t) := \{x \in X : \exists \mu \in V_T(x) \text{ with } h(\mu) \leq t\}.$$

Then $h(T|QR(t)) \leq t$.

# Chapter 3

# Symbolic Dynamics

In this chapter we will introduce the symbolic space over a countable alphabet, where later we will do the multifractal analysis. Also we are going to define notions of topological entropy of the shift map, and thermodynamic formalism tools which will be used in our arguments. We will use definitions and notation mainly from [MaU].

## 3.1 Countable full-shift

**Definition 3.1.1.** Consider the countable alphabet of natural numbers $\mathbb{N}$. Denote by

$$\Sigma := \{x = (x_n)_{n\geq 1} : x_n \in \mathbb{N} \text{ for every } n\}$$

the space of sequences with terms in $\mathbb{N}$.

The set $\Sigma$ is a topological space with the product topology, generated by the cylinders

$$[i_1 \cdots i_n] := \{x \in \Sigma : x_j = i_j \text{ for } 1 \leq j \leq n\}, \quad n \in \mathbb{N}.$$

The finite sequence $w = i_1 i_2 \cdots i_n \in \mathbb{N}^n$ is called a *word* of length $n$. Sometimes, we will use the notation $[w]$ for the cylinders, where $w$ is a word. Also for general elements $x, y \in \Sigma$, we will asume they have the form $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$.

Over this space, consider the following metrics: for $\beta > 0$ and $x, y \in \Sigma$, define

$$d_\beta(x, y) := \exp(-\beta \max\{n \geq 1 : x_i = y_j \text{ for } 1 \leq j \leq n\}).$$

These metrics are all equivalent and they induce the product topology. A function is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all of them. The same property holds for Hölder continuity, so we can define a notion of locally Hölder continuity in general, but with dependence in the metric.

When we work with Hausdorff dimension we will use the following metric, which is essentially one of the metrics presented: fix $\lambda > 1$ and for $x, y \in \Sigma$ define

$$d(x,y) := \lambda^{-\min\{k \geq 1 : x_k \neq y_k\}}. \tag{3.1}$$

This metric satisfies

$$[i_1 \cdots i_n] = B(x, \lambda^{-n}) \quad \forall x \in [i_1 \cdots i_n], \tag{3.2}$$

where $B(x, \lambda^{-n})$ is the usual open ball of center $x$ and radius $\lambda^{-n}$.

**Definition 3.1.2.** Consider the dynamical system $\sigma : \Sigma \to \Sigma$ defined by

$$\sigma((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}.$$

This is a continuous function, and it is called *the shift map*.

**Definition 3.1.3.** A function $\phi : \Sigma \to \mathbb{R}$ is said to be *locally Hölder* if there exist constants $C, \beta > 0$ such that for every $x, y \in \Sigma$ with $x_1 = y_1$ we have

$$|\phi(x) - \phi(y)| \leq C d_\beta(x,y).$$

**Remark 3.1.1.** This definition of locally Hölder continuity is called "Hölder continuity" in [MaU], but it is weaker than the usual definition of Hölder continuity, since we do not ask anything for sequences $x, y$ with $x_1 \neq y_1$. Thus, locally Hölder continuous functions can be unbounded.

**Definition 3.1.4.** A function $\phi : \Sigma \to \mathbb{R}$ is said to be *summable* if

$$\sum_{i\in\mathbb{N}} \exp(\sup_{[i]} \phi) < \infty.$$

**Definition 3.1.5.** We say that two functions $\phi, \psi : \Sigma \to \Sigma$ are *cohomologous* in a class $\mathcal{D}$ if there exists a function $g \in \mathcal{D}$ such that

$$\phi - \psi = g - g \circ \sigma.$$

Given a set $F \subset \mathbb{N}$, set

$$\Sigma_F := \{x \in \Sigma : x_i \in F \text{ para todo } i \in \mathbb{N}\} \subset \Sigma,$$

and observe that $\sigma(\Sigma_F) \subset \Sigma_F$. Thus, for every subset $F \subset \mathbb{N}$ it is allowed to consider $\sigma : \Sigma_F \to \Sigma_F$. Also notice that the space $\Sigma_F$ is compact if and only if the set $F$ is finite.

**Definition 3.1.6.** We will use the following notation: given a natural number $N \in \mathbb{N}$, set $\Sigma_N := \Sigma_{\{1,2\ldots,N\}}$.

Recall that we already have seen the shift map acting on this space in Example 2.2.1 and Example 2.2.2.

## 3.2 Thermodynamic formalism

In this section we present some aspects and results of the thermodynamic formalism of continuous functions (also called potentials) on the symbolic space over a countable alphabet. We will define an important generalization of the concept of topological entropy. The topological pressure is a weighted version of the topological entropy, where the 'weights' are given by a potential $\phi : \Sigma_F \to \mathbb{R}$, $F \subset \mathbb{N}$. Using a subadditivity argument (see [MaU, Lemma 2.1.2]) we have the following definition.

**Definition 3.2.1.** Given a continuous function $\phi : \Sigma_F \to \mathbb{R}$, the *topological pressure* of $\phi$ with respect to the shift map $\sigma : \Sigma_F \to \Sigma_F$ is defined by

$$P_F(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in F^n} \exp\left( \sup_{x\in[w]\cap\Sigma_F} \sum_{j=0}^{n-1} \phi(\sigma^j x) \right).$$

If $F = \mathbb{N}$, denote $P(\phi) := P_{\mathbb{N}}(\phi)$.

**Remark 3.2.1.** Notice that it can happen that $P_F(\phi) = -\infty$ or $\infty$. Also observe that if $E \subset F \subset \mathbb{N}$ then $P_E(\phi) \le P_F(\phi)$, and that if $\phi \le \psi$ then $P_F(\phi) \le P_F(\psi)$.

**Example 3.2.1.** Let us compute the topological pressure of the locally constant potential $\phi : \Sigma \to \mathbb{R}$ defined by $\phi \mid_{[i]} \equiv \log a_i$:

$$
\begin{aligned}
P(\phi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathbb{N}^n} \exp\left( \sup_{x\in[w]} \sum_{i=0}^{n-1} \phi(\sigma^i x) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathbb{N}^n} \exp\left( \sum_{i=0}^{n-1} \log a_{w_i} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathbb{N}^n} a_{w_1} \cdots a_{w_n} \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left( \sum_{i\in\mathbb{N}} a_i \right)^n \\
&= \log \sum_{i=1}^{\infty} a_i.
\end{aligned}
$$

**Remark 3.2.2.** Let $F \subset \mathbb{N}$. Then for every continuous function $\phi : \Sigma_F \to \mathbb{R}$ and $C \in \mathbb{R}$, we have $P_F(\phi + C) = P_F(\phi) + C$. In fact,

$$
\begin{aligned}
\frac{1}{n} \log \sum_{w\in F^n} \exp\left( \sup_{[w]\cap\Sigma_F} S_n\phi + nC \right) &= \frac{1}{n} \log \sum_{w\in F^n} e^{nC} \exp\left( \sup_{[w]\cap\Sigma_F} S_n\phi \right) \\
&= \frac{1}{n} \log e^{nC} + \frac{1}{n} \log \sum_{w\in F^n} \exp\left( \sup_{[w]\cap\Sigma_F} S_n\phi \right),
\end{aligned}
$$

so letting $n \to \infty$ we get the desired result.

**Proposition 3.2.1.** ([MaU, Proposition 2.1.9]) Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder potential. Then $\phi$ is summable if and only if $P(\phi) < \infty$.

**Definition 3.2.2.** We define the *topological entropy* of $\sigma : \Sigma_F \to \Sigma_F$ as the topological pressure of the constant potential $\phi \equiv 0$. That is,

$$
h_F(\sigma) := P_F(0).
$$

**Example 3.2.2.** Let us compute the topological entropy of $\sigma : \Sigma_F \to \Sigma_F$ when $F$ is a finite subset of $\mathbb{N}$. Notice that

$$
h_F(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in F^n} 1 = \log \#F.
$$

Observe that this number coincides with our previous computation of the topological entropy on Example 2.2.2, with $F = \{1, 2, \ldots, N\}$. Also, by Remark 3.2.1, we have $h(\sigma) \geq h_F(\sigma)$ for every finite set $F \subset \mathbb{N}$, in particular $h(\sigma) \geq \log N$ for every $N \in \mathbb{N}$. Thus, we conclude that the countable full shift has infinite entropy.

**Definition 3.2.3.** A function $\phi : \Sigma \to \mathbb{R}$ is said to be *acceptable* if it is uniformly continuous and

$$\mathrm{osc}(f) := \sup_{i\in\mathbb{N}}\{\sup_{[i]}\phi - \inf_{[i]}\phi\} < \infty.$$

**Remark 3.2.3.** Notice that each locally Hölder function is acceptable.

Now, we have an approximation of the topological pressure of an acceptable function by its pressure on compacts full-shifts contained in $\Sigma$ (see Theorem 2.1.5 in [MaU])

**Theorem 3.2.1.** If $\phi : \Sigma \to \mathbb{R}$ is acceptable, then

$$P(\phi) = \sup\{P_F(\phi) : F \subset \mathbb{N} \text{ finite}\}.$$

Recall that we denote by $\mathcal{M}_\sigma$ the set of all $\sigma$-invariant Borel probability measures on $\Sigma$.

**Definition 3.2.4.** A measure $\mu \in \mathcal{M}_\sigma$ is said to be *compactly supported* if there exists a finite set $F \subset \mathbb{N}$ such that $\mu(\Sigma_F) = 1$.

We have a variational principle for pressure and compactly supported measures (see [Bo2], [R], [W]), which says that if $F \subset \mathbb{N}$ is finite,

$$P_F(\phi) = \sup\left\{h(\mu) + \int \phi d\mu : \mu \in \mathcal{M}_\sigma, \mu(\Sigma_F) = 1\right\}.$$

We need a variational principle for functions defined on the whole space $\Sigma$, and also we are interested in Borel probability measures which attain that supremum. With some hypotheses on $\phi$, there is a special kind of measures which have this property.

**Definition 3.2.5.** Let $\phi : \Sigma \to \mathbb{R}$ be a potential. We say that $\mu \in \mathcal{M}_\sigma$ is a *Gibbs state of* $\phi$ if there exist constants $C \geq 1$ and $P \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, every word $w$ of length $n$ and every $x \in [w]$,

$$C^{-1} \leq \frac{\mu([w])}{\exp(-nP + \sum_{j=0}^{n-1}\phi(\sigma^j x))} \leq C. \tag{3.3}$$

**Remark 3.2.4.** Notice that the Birkhoff sum $S_n\phi := \sum_{j=0}^{n-1}\phi\circ\sigma^j$ in (3.3) can be replaced by $\sup_{[w]} S_n\phi$. In fact, it is clear that for $x \in [w]$

$$\frac{\mu([w])}{\exp(-nP + \sup_{[w]} S_n\phi)} \leq \frac{\mu([w])}{\exp(-nP + S_n\phi(x))} \leq C.$$

On the other hand, notice that for every $x \in [w]$ we have $S_n\phi(x) \leq \log(C\mu([w])) + nP$, so we can take supremum over $x \in [w]$ and get

$$C^{-1} \leq \frac{\mu([w])}{\exp(-nP + \sup_{[w]} S_n\phi)} \leq C. \tag{3.4}$$

**Proposition 3.2.2.** If a potential $\phi : \Sigma \to \mathbb{R}$ admits a Gibbs state $\mu$ with constants $C$ and $P$, then it has finite pressure and $P(\phi) = P$.

*Proof.* Fix $n \geq 1$. Sum in (3.4) over all words $w \in \mathbb{N}^n$. Since $\sum_{w\in\mathbb{N}^n} \mu([w]) = 1$ we have

$$C^{-1}e^{-nP} \sum_{w\in\mathbb{N}^n} \exp(\sup_{[w]} S_n\phi) \leq 1 \leq Ce^{-nP} \sum_{w\in\mathbb{N}^n} \exp(\sup_{[w]} S_n\phi).$$

Taking logarithm, dividing by $n$ and taking limit as $n \to \infty$ we get

$$-P + P(\phi) \leq 0 \leq -P + P(\phi).$$

■

See [MaU] for the proofs of the following results:

**Theorem 3.2.2.** Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder summable potential. Then there exists a unique Gibbs state of $\phi$, and this measure is ergodic.

**Theorem 3.2.3.** (Variational Principle) Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder summable potential. Then

$$P(\phi) = \sup\left\{h(\nu) + \int \phi d\mu : \nu \in \mathcal{M}_\sigma, \int \phi d\nu > -\infty\right\} = h(\mu) + \int \phi d\mu,$$

where $\mu$ is the unique Gibbs state for $\phi$.

**Definition 3.2.6.** A measure which attains the supremum is called an *equilibrium state for $\phi$.*

**Theorem 3.2.4.** Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder summable potential. Then the Gibbs state of $\phi$ is its unique equilibrium state.

**Remark 3.2.5.** The three previous results also hold in a compact full-shift $\Sigma_N$ and for a Hölder function $\phi : \Sigma_N \to \Sigma_N$ (see [Bo2, Chapter 1]).

Later, we will use the function $q \mapsto P(q\phi)$. In [MiU, Theorem 2.10] it is explained how to use transfer operator theory to prove the following theorem.

**Theorem 3.2.5.** Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder function and consider the set of real numbers $q$ such that $P(q\phi) < \infty$, $D(\phi) := \{q \in \mathbb{R} : q\phi \text{ is summable}\}$. Then the function $D(\phi) \ni q \mapsto P(q\phi) \in \mathbb{R}$ is real analytic, and for $q_0 \in D(\phi)$

$$\frac{d}{dq}P(q\phi)\Big|_{q=q_0} = \int \phi d\mu_{q_0},$$

where $\mu_{q_0}$ is the equilibrium state of $q_0\phi$.

**Remark 3.2.6.** If in addition $\phi \leq 0$, the function $q \mapsto P(q\phi)$ when finite, it is real analytic and decreasing. It is also a convex function. It is strictly convex unless $\phi$ is cohomologous to a constant function. This also holds in compact full-shifts for a Hölder potential $\phi : \Sigma_N \to \Sigma_N$.

Also, we have a good approximation of the function $q \mapsto P(q\phi)$ by the topological pressure of $q\phi : \Sigma_N \to \Sigma_N$, that is, the pressure restricted to the compact full-shifts. Denote by $P_N(q\phi) := P_{\{1,2,\dots,N\}}(q\phi)$ the topological pressure of $q\phi$ restricted to $\Sigma_N$.

**Proposition 3.2.3.** Let $\phi : \Sigma \to \mathbb{R}$ be an acceptable potential. Then for every $q \in \mathbb{R}$,

$$\lim_{N\to\infty} P_N(q\phi) = P(q\phi).$$

*Proof.* Let $\varepsilon > 0$ and fix $q \in \mathbb{R}$. Observe that the function $q\phi$ is also acceptable, so $P(q\phi) = \sup\{P_F(q\phi) : F \subset \mathbb{N} \text{ finite}\}$. Therefore, there exists a finite set $F \subset \mathbb{N}$ such that $P(q\phi) - \varepsilon < P_F(q\phi)$. Now set $n_* := \max F$ and observe that $F \subset \{1, 2, \dots, n_*\}$. Thus, by Remark 3.2.1 for every $N \geq n_*$

$$P_F(q\phi) \leq P_{n_*}(q\phi) \leq P_N(q\phi) \leq P(q\phi).$$

Hence, for every $N \geq n_*$

$$P(q\phi) - \varepsilon < P_N(q\phi) < P(q\phi) + \varepsilon.$$

■

## 3.3 Relation between Hausdorff dimension and entropy

In the full-shift there is a relationship between Hausdorff dimension and entropy. For example, when it is computed with respect to an ergodic probability measure, or computed over the compact full-shifts contained in the whole space $\Sigma$. We will review them before performing multifractal analysis, which may give us this same relation on the level sets we will be studying.

**Remark 3.3.1.** Observe that $\Sigma$ with the metric

$$d(x,y) := \lambda^{-\min\{k\geq 1: x_k \neq y_k\}}, \quad \lambda > 1$$

is a Besicovitch metric space with finite multiplicity (see Definition 2.1.5 and Definition 2.1.6), considering that balls in this space correspond to cylinders (see (3.2)). This allows us to use Theorem 2.1.1 and Theorem 2.1.2 on our arguments when we are computing Hausdorff dimension.

**Proposition 3.3.1.** Let $\mu$ be an ergodic Borel $\sigma$-invariant probability measure on $\Sigma_N$. Then,

$$\dim_H \mu = \frac{h(\mu)}{\log \lambda}.$$

*Proof.* Denote by $C_n(x) := \{y \in \Sigma_N : x_i = y_i \text{ for } 1 \leq i \leq n\}$ the cylinder of length $n$ which contains $x$. Since $\mu$ is ergodic, by Shannon-McMillan-Breiman Theorem (Theorem 2.2.1), for $\mu$-almost every $x \in \Sigma_N$

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \lim_{n\to\infty} \frac{\log \mu(C_n(x))}{\log \lambda^{-n}} = \frac{1}{\log \lambda} \lim_{n\to\infty} -\frac{1}{n} \log \mu(C_n(x)) = \frac{h(\mu)}{\log \lambda}.$$

Thus, by Theorem 2.1.1 we conclude that $\dim_H \mu = \dfrac{h(\mu)}{\log \lambda}$. ∎

Now, the same holds for Hausdorff dimension and entropy on the entire space. Recall the notation introduced on Definition 3.1.6 for the full-shift on finite symbols, and denote by $h_N(\sigma)$ the topological entropy of $\sigma : \Sigma_N \to \Sigma_N$.

**Proposition 3.3.2.** $\dim_H \Sigma_N = \dfrac{h_N(\sigma)}{\log \lambda}$.

*Proof.* Let $\mu$ be the Bernoulli measure (recall the definition in Example 2.2.1) on $\Sigma_N$ such that $\mu([i]) = 1/N$ for $1 \leq i \leq N$. By Example 2.2.1, we know that

$$h(\mu) = -\sum_{i=1}^{N} \frac{1}{N} \log \frac{1}{N} = \log N.$$

Since $\mu$ is ergodic, by Proposition 3.3.1 and using that $\mu(\Sigma_N) = 1$ we have

$$\frac{\log N}{\log \lambda} = \dim_H \mu = \inf\{\dim_H Z : \mu(Z) = 1\} \leq \dim_H \Sigma_N.$$

For the other inequality, set $s := \frac{\log N}{\log \lambda}$ and notice that $\operatorname{diam}[i_1 \dots i_n] = \frac{1}{\lambda^{n+1}}$ for every cylinder of length $n$. Fix $\delta > 0$ and choose $n \geq 1$ such that $1/\lambda^{n+1} < \delta$. Then

$$\begin{aligned}
\mathcal{H}^s_\delta(\Sigma_N) &= \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} U_i)^s : \{U_i\} \text{ is open cover of } \Sigma_N, \operatorname{diam} U_i < \delta\right\} \\
&\leq \sum_{\text{cylinders of length } n} (\operatorname{diam}[i_1 \dots i_n])^s \\
&= \frac{N^n}{\lambda^{(n+1)s}} = \frac{1}{\lambda^s}.
\end{aligned}$$

Letting $\delta \to 0$ we get $\mathcal{H}^s(\Sigma_N) < \infty$. Thus, $\dim_H \Sigma_N \leq s$ and we conclude

$$\dim_H \Sigma_N = \frac{\log N}{\log \lambda}.$$

Recall that we already computed the topological entropy of a full-shift over a finite alphabet $h_N(\sigma) = \log N$ (see Example 2.2.2 or Example 3.2.2).

■

# Chapter 4

# Multifractal Analysis: Compact case

## 4.1 Introduction and notation

Let us fix some notation. Throughout this chapter, fix $N \in \mathbb{N}$ and let $\Sigma$ be the full-shift on $N$ symbols $\{1, 2, \ldots, N\}$. Denote by $P(\phi)$ the topological pressure of a potential $\phi : \Sigma \to \mathbb{R}$ (in order to lighten the notation, we will not use the index $N$) and by $\mathcal{M}_\sigma$ the space of Borel $\sigma$-invariant probability measures on $\Sigma$. For $\lambda > 1$, we use the metric $d$ as in (3.1).

Let $\phi : \Sigma \to \mathbb{R}$ be a Hölder potential such that $\phi < 0$, $P(\phi) \leq 0$ and not cohomologous to a constant function. For $\alpha \in \mathbb{R}$, define

$$J_\alpha = \left\{ x \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j x) = \alpha \right\}. \tag{4.1}$$

**Definition 4.1.1.** The *Hausdorff dimension spectrum* and the *entropy spectrum* are defined respectively by

$$D(\alpha) := \dim_H J_\alpha$$

$$E(\alpha) := h(\sigma | J_\alpha),$$

and the domain of each one of them is the set $\{\alpha \in \mathbb{R} : J_\alpha \neq \varnothing\}$.

## 4.2 Theorem: compact setting

In this section we will prove the main theorem in the compact case. Yakov Pesin and Howard Weiss proved this result ([PW, Theorem 1]) for the Hausdorff dimension spectrum. Their proof is a little different than ours, since they use the level sets

$K_\alpha := \{x \in \Sigma : d_{\mu_\phi}(x) = \alpha\}$, where $\mu_\phi$ is the equilibrium measure of the potential $\phi$. In [PW, Proposition 1] they showed a relation between the Birkhoff average and the pointwise dimension for a special equilibrium measure. It follows from this proposition that results on the spectrum $f(\alpha) := \dim_H K_\alpha$ can be translated into results on the spectrum $D(\alpha)$.

**Theorem 4.2.1.**

Let $\phi : \Sigma \to (-\infty, 0)$ be a Hölder potential with $P(\phi) \leq 0$ and not cohomologous to a constant function. For $\alpha \in \mathbb{R}$, let $J_\alpha$ as in (4.1) and the functions $D, E$ from Definition 4.1.1. Then, the following hold:

1. The domain of $D$ and $E$ is a compact interval $[\underline{\alpha}, \overline{\alpha}]$.
2. For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, there exists a measure $\mu_\alpha$ such that $E(\alpha) = h(\mu_\alpha)$ and $D(\alpha) = \dim_H \mu_\alpha$.
3. For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, the set $J_\alpha$ is dense in $\Sigma$.
4. The functions $D$ and $E$ are real analytic and strictly convex.
5. For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$,
$$D(\alpha) = \frac{E(\alpha)}{\log \lambda}.$$

*Proof.* By [J, Proposition 2.1], we have that

$$\overline{\alpha} := \sup_{\mu \in \mathcal{M}_\sigma} \int \phi d\mu \quad \text{and} \quad \underline{\alpha} := \inf_{\mu \in \mathcal{M}_\sigma} \int \phi d\mu$$

are respectively the supremum and infimum of possible Birkhoff averages reached by points in $\Sigma$. So, if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ we have $J_\alpha = \varnothing$ and if $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $J_\alpha \neq \varnothing$.

We introduce the following function, which will be useful with both spectra. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(q) := P(q\phi)$. We already know some properties of the function $T$ (see Theorem 3.2.5 and Remark 3.2.6): it is real analytic, strictly convex and for every $q \in \mathbb{R}$

$$T'(q) = \int \phi d\mu_q,$$

where $\mu_q$ is the equilibrium state of the function $q\phi$.

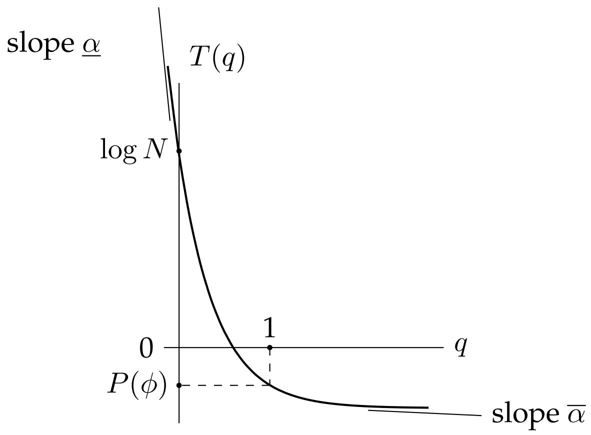



FIGURE 4.1: Graphic of $q \mapsto T(q)$.

Now define $\alpha(q) := T'(q) = \int \phi d\mu_q$.

**Lemma 4.2.1.** For each $\alpha \in (\underline{\alpha}, \overline{\alpha})$, there exists $q \in \mathbb{R}$ such that $\alpha(q) = \alpha$.

*Proof.* Define the function

$$S(q) = \int \phi d\mu_q - \alpha.$$

For $q > 0$, by the Variational Principle we have

$$\begin{aligned} S(q) &= \frac{1}{q}(P(q\phi) - h(\mu_q)) - \alpha \\ &= \sup_{\nu \in \mathcal{M}_\sigma} \left( \int \phi d\nu - \overline{\alpha} + \frac{h(\nu) - h(\mu_q)}{q} \right) + \overbrace{\overline{\alpha} - \alpha}^{>0} \\ &> 0 \end{aligned}$$

for $q > 0$ large enough, since the entropies are bounded. A similar argument follows when $q < 0$:

$$\begin{aligned} S(q) &= \frac{1}{q}(P(q\phi) - h(\mu_q)) - \alpha \\ &= \inf_{\nu \in \mathcal{M}_\sigma} \left( \int \phi d\nu - \underline{\alpha} + \frac{h(\nu) - h(\mu_q)}{q} \right) + \overbrace{\underline{\alpha} - \alpha}^{<0} \\ &< 0, \end{aligned}$$

thus we conclude that for $q < 0$ negative enough ($|q|$ large), we have $S(q) < 0$.

Then, since $S(q) = T'(q) - \alpha$, in particular is a continuous function. By the Intermediate Value Theorem, there exists some $q_\star \in \mathbb{R}$ such that $S(q_\star) = 0$. ∎

**Lemma 4.2.2.** For every $q \in \mathbb{R}$,

$$D(\alpha(q)) = \frac{T(q) - q\alpha(q)}{\log \lambda} = \frac{h(\mu_q)}{\log \lambda}.$$

*Proof.* Fix $q \in \mathbb{R}$. Recall that $\mu_q$ is an ergodic measure and a Gibbs state for $q\phi$. By Birkhoff's ergodic theorem, for $\mu_q$-almost every $x \in \Sigma$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\sigma^k x) = \int \phi d\mu_q = \alpha(q),$$

and thus $\mu_q(J_{\alpha(q)}) = 1$. This implies that $J_{\alpha(q)}$ is dense in $\Sigma$, since $\mu_q$ gives positive measure to open sets.

Now denote by $S_n\phi$ the $n$th Birkhoff sum of $\phi$. By the Gibbs property, there exists $C \geq 1$ such that for every $x \in \Sigma$ and $n \geq 1$

$$C^{-1} \leq \frac{\mu_q(C_n(x))}{\exp(-nT(q) + qS_n\phi(x))} \leq C.$$

Taking logarithm and then multiplying by $\frac{-1}{n \log \lambda}$ we get

$$\frac{\log C}{n \log \lambda} + \frac{T(q)}{\log \lambda} - \frac{qS_n\phi(x)}{n \log \lambda} \geq \frac{\log \mu_q(C_n(x))}{\log \lambda^{-n}} \geq \frac{\log C^{-1}}{n \log \lambda} + \frac{T(q)}{\log \lambda} - \frac{qS_n\phi(x)}{n \log \lambda}.$$

Letting $n \to \infty$, by Birkhoff's ergodic theorem, and by the Variational Principle applied to $q\phi$, we have that for $\mu_q$-almost every $x \in \Sigma$

$$d_{\mu_q}(x) = \frac{T(q) - q\alpha(q)}{\log \lambda} = \frac{h(\mu_q)}{\log \lambda} = \dim_H \mu_q.$$

This equality also holds for every $x \in J_{\alpha(q)}$, so by Remark 2.1.3 we conclude the result. ∎

**Lemma 4.2.3.** For every $q \in \mathbb{R}$,

$$E(\alpha(q)) = h(\mu_q) = T(q) - q\alpha(q).$$

*Proof.* Fix $q \in \mathbb{R}$. Since $\mu_q$ is ergodic, by Birkhoff's Ergodic Theorem $\mu_q(J_{\alpha(q)}) = 1$. By Theorem 2.2.3, and using the variational principle applied to the function $q\phi$ we have $T(q) - q\alpha(q) = h(\mu_q) \leq E(\alpha(q))$.

Now we claim that $J_{\alpha(q)} \subset QR(h(\mu_q))$ (recall the definition of $QR(t)$ from Theorem 2.2.4). In fact, let $x \in J_{\alpha(q)}$ and $\nu \in V_\sigma(x)$. Notice that there exists a subsequence $(n_k)_k$

such that $\frac{1}{n_k}\sum_{j=0}^{n_k-1}\delta_{\sigma^j x}$ converges to $\nu$ in the weak* topology. Then,

$$\int \phi d\nu = \lim_{k\to\infty}\int \phi d\left(\frac{1}{n_k}\sum_{j=0}^{n_k-1}\delta_{\sigma^j x}\right) = \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1}\phi(\sigma^j x) = \alpha(q).$$

By the variational principle applied to the function $q\phi$,

$$T(q) = P(q\phi) \geq h(\nu) + q\int \phi d\nu = h(\nu) + q\alpha(q),$$

which implies $h(\nu) \leq T(q) - q\alpha(q) = h(\mu_q)$. Thus, $x \in QR(h(\mu_q))$. Therefore, by the monotonicity of the entropy restricted to subsets and Theorem 2.2.4,

$$E(\alpha(q)) \leq h(\sigma|QR(h(\mu_q))) \leq h(\mu_q) = T(q) - q\alpha(q),$$

which completes the proof of the Lemma 4.2.3. ■

Now we have another proof of the following conditional variational principle, which avoids the use of the topological pressure restricted to subsets in the most general case (see [Ba, Theorem 9.2.1]):

**Corollary 4.2.1.** For every $\alpha \in (\underline{\alpha}, \overline{\alpha})$,

$$E(\alpha) = h(\sigma|J_\alpha) = \sup\{h(\mu) : \mu \in \mathcal{M}_\sigma, \mu(J_\alpha) = 1\}. \tag{4.2}$$

*Proof.* By Theorem 2.2.3 for every $\sigma$-invariant measure $\mu$ such that $\mu(J_\alpha) = 1$ we have $h(\mu) \leq E(\alpha)$. However, by Lemma 4.2.1, there exists $q \in \mathbb{R}$ such that $\alpha(q) = \alpha$ and $\mu_q(J_\alpha) = 1$. By Lemma 4.2.3 $E(\alpha) = E(\alpha(q)) = h(\mu_q)$ and we conclude the result. ■

Lemma 4.2.2 and Lemma 4.2.3 imply that for every $q \in \mathbb{R}$ we have

$$D(\alpha(q)) = \frac{E(\alpha(q))}{\log\lambda}.$$

We know by Lemma 4.2.1 that every $\alpha \in (\underline{\alpha}, \overline{\alpha})$ can be expressed as $\alpha(q)$ for some $q \in \mathbb{R}$.

We want to write $D$ and $E$ as functions of $\alpha$. In order to do this, notice that the function $\alpha(q)$ satisfies $\alpha'(q) = T''(q) > 0$ since $T$ is strictly convex. Then, by the Inverse Function Theorem , we can write $q = q(\alpha)$ as the inverse function of $\alpha(q)$:

$$E(\alpha) = T(q(\alpha)) - \alpha q(\alpha), \quad D(\alpha) = \frac{T(q(\alpha)) - \alpha q(\alpha)}{\log\lambda} = \frac{E(\alpha)}{\log\lambda}.$$

Since $E$ and $D$ only differ by a factor $\log \lambda$, we analyze only one of them, $E(\alpha)$. Observe that

$$E'(\alpha) = T'(q(\alpha))q'(\alpha) - \alpha q(\alpha) - q(\alpha) = -q(\alpha).$$

So $E'(\alpha(q)) = 0$ if and only if $q = 0$, this is on the value

$$E(\alpha(0)) = h(\mu_0) = \log N,$$

because $\mu_0$ is the equilibrium state of the function $0$, i.e. the measure of maximal entropy. Also, $\alpha$ is an increasing function of $q$ (recall $\alpha' = T'' > 0$) and so is $q$ as a function of $\alpha$. Thus, if $\alpha < \alpha(0)$ then $E'(\alpha) > 0$ and if $\alpha > \alpha(0)$, $E'(\alpha) < 0$. Finally, we have $E''(\alpha) = -q'(\alpha) < 0$, so $E$ is concave. The real analyticity comes inherited from that of $T$.

Notice that $E(\alpha)$ is tangent to the line $y = -\alpha + P(\phi)$, since $E'(\alpha(1)) = -1$ at the point $(\alpha(1), E(\alpha(1))) = (\alpha(1), -\alpha(1) + P(\phi))$.

The same properties hold for the function $D$, and we sketch their behaviors on the following graphics.

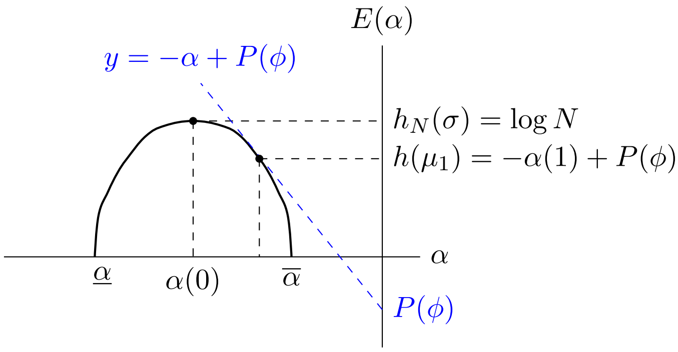



FIGURE 4.2: Graphic of $\alpha \mapsto E(\alpha)$.

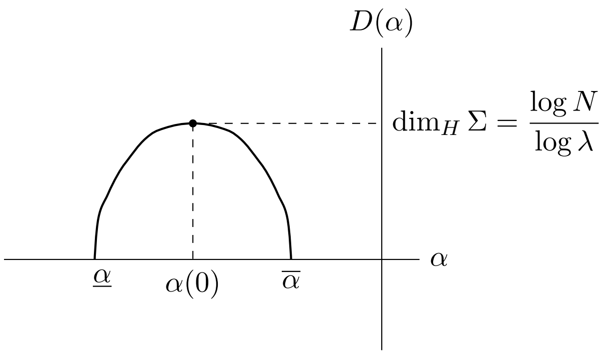



FIGURE 4.3: Graphic of $\alpha \mapsto D(\alpha)$.

■

# Chapter 5

# Multifractal Analysis: Non-compact case

## 5.1 Introduction and notation

In this chapter we go back to the non-compact space $(\Sigma, d)$ with the notation used on Chapter 2: $P(\cdot) = P_{\mathbb{N}}(\cdot)$ and for every $N \in \mathbb{N}$, $P_N(\cdot)$ is the topological pressure of a potential defined on $\Sigma_N$. If a potential $\phi$ is defined over the whole space $\Sigma$, then $P_N(\phi)$ will denote the topological pressure of $\phi|_{\Sigma_N}$. Also recall that now $h(\sigma) = \dim_H \Sigma = \infty$. This will have consequences for the function $T$ and for the function $D$ as well.

Let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder potential with $\phi < 0$ and $P(\phi) = 0$. For $\alpha \in \mathbb{R}$, we consider the level sets of the Birkhoff averages of the potential $\phi$ again

$$J_\alpha = \left\{ x \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j x) = \alpha \right\}. \tag{5.1}$$

In the non-compact setting, we cannot compute the formula for $E(\alpha)$ as we did in the previous chapter, since it was strongly used the compactness of the space and of $\mathcal{M}_\sigma$. For example, we cannot ensure that the set $V_\sigma(x)$ is non-empty for every $x$. However, Corollary 4.2.1 suggests a way to define a new entropy spectrum, which may satisfy the relation between Hausdorff dimension and entropy, and in fact it does.

**Definition 5.1.1.** The *variational entropy spectrum* is defined by

$$\tilde{E}(\alpha) := \sup\{h(\mu) : \mu \in \mathcal{M}_\sigma, \mu(J_\alpha) = 1\}.$$

## 5.2 Theorem: non-compact setting

In this section we will prove the theorem on the non-compact space. We will prove analogous results for the functions $D$ and $\tilde{E}$, with certain differences depending on the function $\phi$.

We use again the function $T(q) = P(q\phi)$, but notice that now it has a different behavior.

**Remark 5.2.1.** Observe that $T(0) = h(\sigma) = \infty$ and that $T(1) = P(\phi) = 0$, so we can define

$$q_* := \inf\{q \in \mathbb{R} : P(q\phi) < \infty\} \in [0, 1].$$

Then for every $q > q_*$ we have $T(q) < \infty$, so by Theorem 3.2.2 and Theorem 3.2.4 there exists a unique Gibbs state $\mu_q \in \mathcal{M}_\sigma$ of $q\phi$, which is also its equilibrium state. Also, we have the same properties of $T$ on $(q_*, \infty)$ (see Remark 3.2.6 and Theorem 3.2.5) that we had in the previous case: it is real analytic, strictly convex and

$$T'(q) = \int \phi d\mu_q.$$

The behavior of the function $D$ and $\tilde{E}$ depends on the behavior of the function $T$: there are several cases, $\lim_{q\to q_*^+} T'(q) = -\infty$ or $> -\infty$, and $\lim_{q\to q_*^+} T(q) = \infty$ or $< \infty$.

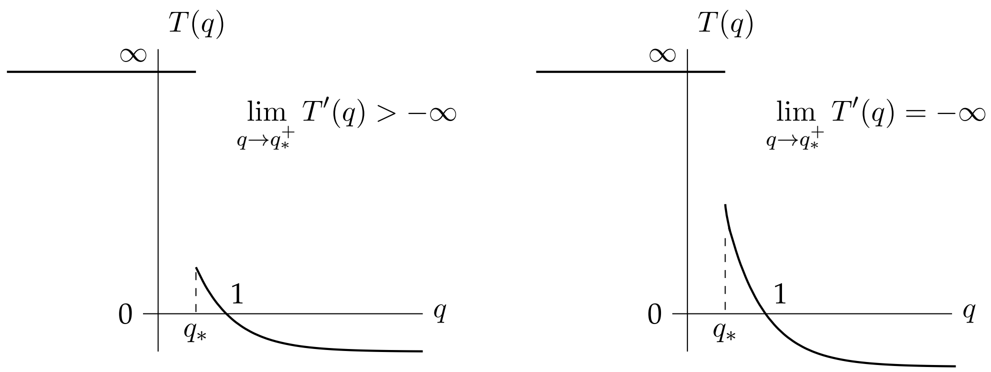



FIGURE 5.1: Some behaviors of the function $T(q)$.

**Remark 5.2.2.** Despite it is not in the pictures, it is also possible that $\lim_{q\to q_*^+} T(q) = \infty$, and this implies $\lim_{q\to q_*^+} T'(q) = -\infty$.

See [CI, Section 2] for analytic tools to construct examples of these different behaviors of the function $q \mapsto P(q\phi)$ depending on the potential.

For $q > q_*$, define

$$\alpha(q) := T'(q) = \int \phi d\mu_q, \quad \alpha_* := \lim_{q\to q_*^+} \alpha(q), \tag{5.2}$$

**Theorem 5.2.1.**

Let $\phi : \Sigma \to (-\infty, 0)$ be a locally Hölder potential with $P(\phi) = 0$. For $\alpha \in \mathbb{R}$, let $J_\alpha$ as in (5.1) and the functions $D, \tilde{E}$ from Definition 4.1.1 and Definition 5.1.1 respectively. Let $\alpha_*$ as in (5.2). Then, the following hold:

1. If $\alpha_* = -\infty$, then the functions $D$ and $\tilde{E}$ are real analytic, strictly decreasing and concave. Moreover, for every $\alpha \in (-\infty, \overline{\alpha})$
$$D(\alpha) = \frac{\tilde{E}(\alpha)}{\log \lambda}.$$

2. If $\alpha_* > -\infty$, then the functions $\tilde{E}$ and $D$ are real analytic and strictly concave for $\alpha > \alpha_*$. For $\alpha < \alpha_*$ $\tilde{E}$ is affine with slope $-q_*$, and $D$ is affine with slope $-q_*/\log \lambda$. Moreover, for every $\alpha \in (-\infty, \overline{\alpha})$
$$D(\alpha) = \frac{\tilde{E}(\alpha)}{\log \lambda}.$$

In order to prove this theorem, we will need some previous lemmas.

**Lemma 5.2.1.** The domain of $D$ and $\tilde{E}$ is unbounded.

*Proof.* Jenkinson, Mauldin and Urbanski proved in [JMU, Theorem 1] that

$$\overline{\alpha} := \sup_{\mu \in \mathcal{M}_\sigma} \int \phi d\mu = \lim_{q \to \infty} \alpha(q). \tag{5.3}$$

Analogously, let

$$\underline{\alpha} := \inf_{\mu \in \mathcal{M}_\sigma} \int \phi d\mu,$$

we claim that $\underline{\alpha} = -\infty$. Assume by contradiction that $\underline{\alpha} > -\infty$. Since $P(\phi) = 0$, by the Variational Principle for every $\nu \in \mathcal{M}_\sigma$

$$h(\nu) = h(\nu) + \int \phi d\nu - \int \phi d\nu \leq P(\phi) - \int \phi d\nu \leq -\underline{\alpha},$$

which is a contradiction since $\sup\{h(\mu) : \mu \in \mathcal{M}_\sigma\} = \infty$. So our domain in this case is the interval $(\underline{\alpha}, \overline{\alpha}) = (-\infty, \overline{\alpha}) \subseteq (-\infty, 0]$. ■

**Lemma 5.2.2.** For every $q > q_*$,

$$\tilde{E}(\alpha(q)) = T(q) - q\alpha(q).$$

*Proof.* Fix $q > q_*$. Recall that $\mu_q$ is ergodic, so by Birkhoff's Ergodic Theorem we have $\mu_q(J_{\alpha(q)}) = 1$, and that implies $h(\mu_q) \leq \tilde{E}(\alpha(q))$. We claim $\tilde{E}(\alpha(q)) = h(\mu_q)$.

Let $\nu \in \mathcal{M}_\sigma$ such that $\nu(J_{\alpha(q)}) = 1$. If we denote by $\overline{\phi}$ the Birkhoff average of $\phi$ (defined $\nu$-almost everywhere), then

$$\int \phi d\nu = \int \overline{\phi} d\nu = \int_{J_{\alpha(q)}} \overline{\phi} d\nu = \alpha(q). \tag{5.4}$$

Therefore, by the Variational Principle with the potential $q\phi$,

$$h(\nu) = h(\nu) + q \int \phi d\nu - q \int \phi d\mu_q \leq P(q\phi) - q \int \phi d\mu_q = h(\mu_q).$$

Taking supremum over all $\nu \in \mathcal{M}_\sigma$ such that $\nu(J_{\alpha(q)}) = 1$, we get $\tilde{E}(\alpha(q)) \leq h(\mu_q)$ and therefore

$$\tilde{E}(\alpha(q)) = h(\mu_q) = T(q) - q\alpha(q).$$

∎

**Lemma 5.2.3.** For every $q > q_*$,

$$D(\alpha(q)) = \frac{T(q) - q\alpha(q)}{\log \lambda}.$$

*Proof.* Fix $q \in \mathbb{R}$. Recall that $\mu_q$ is an ergodic measure and a Gibbs state for $q\phi$. By Birkhoff's ergodic theorem, for $\mu_q$-almost every $x \in \Sigma$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\sigma^k x) = \int \phi d\mu_q = \alpha(q),$$

and thus $\mu_q(J_{\alpha(q)}) = 1$.

Now denote by $S_n\phi$ the $n$th Birkhoff sum of $\phi$. By the Gibbs property, there exists $C \geq 1$ such that for every $x \in \Sigma$ and $n \geq 1$

$$C^{-1} \leq \frac{\mu_q(C_n(x))}{\exp(-nT(q) + qS_n\phi(x))} \leq C.$$

Taking logarithm and then multiplying by $\frac{-1}{n \log \lambda}$ we get

$$\frac{\log C}{n \log \lambda} + \frac{T(q)}{\log \lambda} - \frac{qS_n\phi(x)}{n \log \lambda} \geq \frac{\log \mu_q(C_n(x))}{\log \lambda^{-n}} \geq \frac{\log C^{-1}}{n \log \lambda} + \frac{T(q)}{\log \lambda} - \frac{qS_n\phi(x)}{n \log \lambda}.$$

Letting $n \to \infty$, by Birkhoff's ergodic theorem we have that for $\mu_q$-almost every $x \in \Sigma$

$$d_{\mu_q}(x) = \frac{T(q) - q\alpha(q)}{\log \lambda}.$$

This equality also holds for every $x \in J_{\alpha(q)}$, so by Remark 2.1.3 we conclude the result. ∎

**Corollary 5.2.1.** For every $q > q_*$,

$$D(\alpha(q)) = \frac{\tilde{E}(\alpha(q))}{\log \lambda}.$$

*Proof.* The result follows from Lemma 5.2.2 and Lemma 5.2.3. ∎

*Proof of Theorem 5.2.1.* For the part 1, assume that $\alpha_* = -\infty$ and observe that since $\alpha(q) = T'(q)$ is a continuous function on $(q_*, \infty)$, by (5.3) for every $\alpha \in (-\infty, \overline{\alpha})$ there exists $q > q_*$ such that $\alpha(q) = \alpha$. It follows from Corollary 5.2.1 that

$$D(\alpha) = \frac{\tilde{E}(\alpha)}{\log \lambda}$$

for every $\alpha \in (-\infty, \overline{\alpha})$.

Since $\alpha'(q) = T''(q) > 0$ on $(q_*, \infty)$, by the Inverse Function Theorem we can write $q = q(\alpha)$, and thus

$$D(\alpha) = \frac{T(q(\alpha)) - q(\alpha)\alpha}{\log \lambda}, \quad \tilde{E}(\alpha) = T(q(\alpha)) - q(\alpha)\alpha.$$

The real analyticity of both functions $D$ and $\tilde{E}$ are inherited of that of $T$. Since $D$ and $\tilde{E}$ only differ by a factor $\log \lambda$, we analyze only one of them, $\tilde{E}(\alpha)$. Notice that for $\alpha \in (-\infty, \overline{\alpha})$

$$E'(\alpha) = -q(\alpha) < -q_* \leq 0,$$

so $\tilde{E}$ is strictly decreasing. Moreover, since $\alpha(q)$ is strictly increasing, so its inverse $q(\alpha)$ is. Thus, $\tilde{E}''(\alpha) = -q'(\alpha) < 0$ and we conclude that $\tilde{E}$ is concave. Also, it is clear from the formula that

$$\lim_{\alpha \to -\infty} D(\alpha) = \lim_{\alpha \to -\infty} \tilde{E}(\alpha) = \infty.$$

Notice that the tangent line at $\alpha_1 := \alpha(1)$ is given by the slope

$$\tilde{E}'(\alpha_1) = -q(\alpha(1)) = -1$$

and the point $(\alpha_1, \tilde{E}(\alpha_1)) = (\alpha_1, h(\mu_1))$. But since $P(\phi) = 0$, by the Variational Principle we have that

$$h(\mu_1) = -\int \phi d\mu_1 = -\alpha_1,$$

and we conclude that $\tilde{E}$ is tangent to the line $y = -\alpha$, similarly to the compact case.

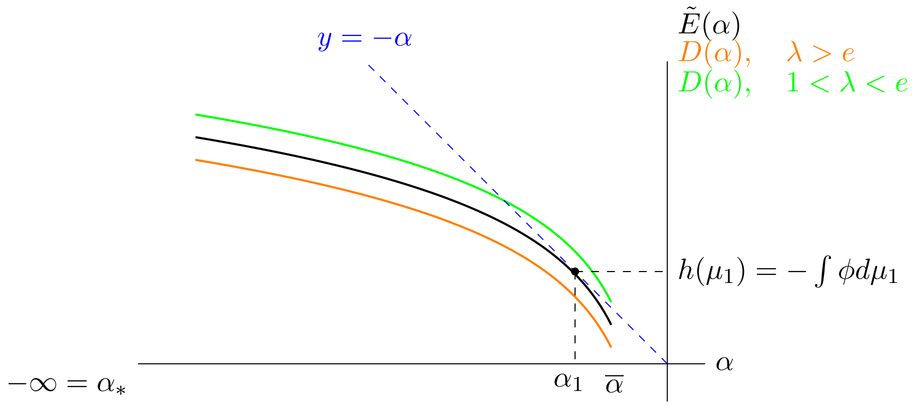



FIGURE 5.2: Graphic of $\alpha \mapsto D(\alpha)$ and $\alpha \mapsto \tilde{E}(\alpha)$.

For the part 2, assume that $\alpha_* > -\infty$ and observe that for $\alpha \in (\alpha_*, \overline{\alpha})$, there exists $q > q_*$ such that $\alpha(q) = \alpha$. It follows from Corollary 5.2.1 that

$$D(\alpha) = \frac{\tilde{E}(\alpha)}{\log \lambda}$$

for every $\alpha \in (\alpha_*, \overline{\alpha})$. The fact that the functions $D$ and $\tilde{E}$ are real analytic, strictly concave and strictly decreasing on $(\alpha_*, \overline{\alpha})$ are concluded with exactly the same arguments as in part 1.

Before we go to the case $\alpha < \alpha_*$, we prove the following lemma:

**Lemma 5.2.4.** The function $q_*\phi$ is such that $T(q_*) = P(q_*\phi) < \infty$.

*Proof.* First notice that since $0 \geq \lim_{q \to q_*^+} T'(q) > -\infty$ and since the function $T'$ is increasing, then

$$L := \lim_{q \to q_*^+} T(q) < \infty,$$

because if not, $T$ would be an unbounded uniformly continuous (since $T'$ is bounded) function on a bounded interval, say $(q_*, q_* + 1)$, and such function cannot exist. Moreover, since $T$ is strictly decreasing, $T(q) < L$ for every $q > q_*$.

Assume by contradiction that $T(q_*) = \infty$ and set $T_N(q) := P_N(q\phi)$. By Proposition 3.2.3, there exists $N \in \mathbb{N}$ such that $T_N(q_*) > L + 1$. Since $T_N$ is continuous on $q_*$, there exists $\delta > 0$ such that if $|q - q_*| < \delta$, then $|T_N(q_*) - T_N(q)| < 1$. In particular, if $q_* < q < q_* + \delta$, then $T_N(q_*) - T_N(q) < 1$ ($T_N$ is strictly decreasing as well). Thus,

$$L + 1 < T_N(q_*) < T_N(q) + 1.$$

This implies

$$L < T_N(q) \leq T(q) < L,$$

which is a contradiction. Therefore $T(q_*) < \infty$. ∎

In order to verify that $\tilde{E}$ is affine on $(-\infty, \alpha_*)$, fix $\alpha \in (-\infty, \alpha_*)$. Let $\nu \in \mathcal{M}_\sigma$ such that $\nu(J_\alpha) = 1$. Then, by the same argument as in (5.4) we get $\int \phi d\nu = \alpha$. Hence, by the Variational Principle with the potential $q_*\phi$

$$h(\nu) \leq P(q_*\phi) - q_* \int \phi d\nu = T(q_*) - q_*\alpha.$$

Taking supremum over $\nu \in \mathcal{M}_\sigma$ such that $\nu(J_\alpha) = 1$ we get $\tilde{E}(\alpha) \leq T(q_*) - q_*\alpha$.

For the inverse inequality, recall that by Proposition 3.2.3 for every $q \in \mathbb{R}$

$$\lim_{N\to\infty} T_N(q) = T(q).$$

Also, we can use the result on the compact case defining

$$\tilde{E}_N(\alpha) = E_N(\alpha) := h(\sigma | J_\alpha \cap \Sigma_N) \leq \tilde{E}(\alpha) \quad \forall N \in \mathbb{N}.$$

**Remark 5.2.3.** Here $\tilde{E}_N$ is the variational entropy spectrum on the compact space $\Sigma_N$.

Notice that $T(q_*) < \infty$ implies that $q_* > 0$ because $T(0) = h(\sigma) = \infty$. Now choose a sequence $q_k \in (0, q_*)$ such that $q_k \to q_*$. Since for every $k$, $\lim_N T_N(q_k) = \infty$, we choose $N_k$ such that $T_{N_k}(q_k) > k$ and hence $T_{N_k}(q_k) \to \infty$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \frac{T_{N_k}(q_k) - T_{N_k}(q_*)}{q_k - q_*} = -\infty.$$

Since $T_{N_k}$ is analytic, combining the Mean Value Theorem and the Intermediate Value Theorem, there exists $q_k' \in (q_k, q_*)$ such that $T_{N_k}'(q_k') = \alpha$. Then,

$$\begin{aligned} \tilde{E}(\alpha) \geq \tilde{E}_{N_k}(\alpha) &= \tilde{E}_{N_k}(T_{N_k}'(q_k')) \\ &= T_{N_k}(q_k') - q_k'\alpha \\ &\geq T_{N_k}(q_*) - q_k'\alpha. \end{aligned}$$

Taking limit as $k \to \infty$ we get $\tilde{E}(\alpha) \geq T(q_*) - q_*\alpha$. Thus we conclude that for $\alpha < \alpha_*$, $\tilde{E}(\alpha) = T(q_*) - q_*\alpha$. Observe that $T(q_*) = P(q_*\phi) < \infty$ implies that there exists an equilibrium measure and Gibbs state $\mu_*$, so by the same arguments as in the proof of Lemma 5.2.2 we conclude $\tilde{E}(\alpha_*) = T(q_*) - q_*\alpha_*$.

By the same arguments as in the proof of part 1, and noticing that $\alpha(1) \geq \alpha_*$, we deduce again that the line $y = -\alpha$ is tangent to the function $\tilde{E}$.

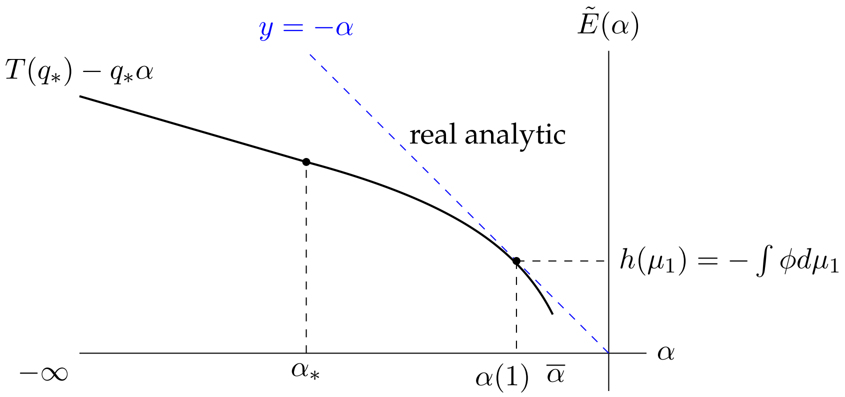



FIGURE 5.3: Graphic of $\alpha \mapsto \tilde{E}(\alpha)$.

Now we study the Hausdorff dimension spectrum on $(-\infty, \alpha_*)$. Let $\alpha < \alpha_*$ and set $D_N(\alpha) := \dim_H(J_\alpha \cap \Sigma_N)$ the dimension spectrum on the compact space $\Sigma_N$. Using the results on the compact case and the sequences $\{N_k\}$ and $\{q'_k\}$ we get

$$D(\alpha) \geq D_{N_k}(\alpha) = \frac{\tilde{E}_{N_k}(\alpha)}{\log\lambda} \geq \frac{T_{N_k}(q_*) - q'_k\alpha}{\log\lambda}.$$

Letting $k \to \infty$,

$$D(\alpha) \geq \frac{T(q_*) - q_*\alpha}{\log\lambda}$$

for $\alpha < \alpha_*$.

For the inverse inequality we use an auxiliar function similar to the one used in (4.2). For $\alpha \in (-\infty, \alpha_*)$, define

$$F(\alpha) := \frac{1}{\log\lambda} \sup\left\{h(\mu) : \mu \in \mathcal{M}_\sigma, \int \phi d\mu = \alpha\right\}.$$

An important property of the function $F$ is that it is continuous. This fact was proven in [IJ, Lemma 3.3], and we do the same argument.

**Lemma 5.2.5.** The function $F$ is continuous in $(-\infty, \alpha_*)$.

*Proof.* Let $\{\mu_n\}_n$ be a sequence of measures in $\mathcal{M}_\sigma$ such that $\int \phi d\mu_n =: \alpha_n$ converges to $\alpha \in (-\infty, \alpha_*)$. Let $\overline{\mu}, \underline{\mu} \in \mathcal{M}_\sigma$ such that

$$-\infty < \int \phi d\underline{\mu} < \alpha < \int \phi d\overline{\mu}.$$

Then for every $n \in \mathbb{N}$, there exists $p_n \in [0,1]$ such that the following convex combination $\nu_n := p_n\mu_n + (1-p_n)\mu^n$, where $\mu^n \in \{\underline{\mu}, \overline{\mu}\}$, satisfies $\int \phi d\nu_n = \alpha$.

Since $P(\phi) = 0$ and the sequence $\{\alpha_n\}_n$ is bounded, the sequence of entropies $\{h(\mu_n)\}_n$ is bounded as well. In fact, if $|\alpha_n| \leq M$, then by the Variational Principle we have

$$0 \leq h(\mu_n) \leq -\int \phi d\mu_n = -\alpha_n \leq M.$$

It follows from the definition of the measures $\nu_n$ that $\alpha = p_n\alpha_n + (1 - p_n)\alpha^n$, where $\alpha^n := \int \phi d\mu^n \in \{\int \phi d\underline{\mu}, \int \phi d\overline{\mu}\}$. Thus, we get

$$(1 - p_n)(\alpha - \alpha^n) = p_n(\alpha_n - \alpha),$$

which implies that $p_n \to 1$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} |h(\nu_n) - h(\mu_n)| = \lim_{n\to\infty} |1 - p_n| \cdot \underbrace{|h(\mu^n) - h(\mu_n)|}_{\text{bounded}} = 0,$$

and hence we conclude that

$$F(\alpha) \geq \limsup_{n\to\infty} F(\alpha_n).$$

For the other direction let $\mu, \nu \in \mathcal{M}_\sigma$ such that $\int \phi d\nu = \beta < \alpha = \int \phi d\mu$. Letting $\nu_p := p\nu + (1 - p)\mu$ we observe that

$$\liminf_{x\to\alpha^-} F(x) \geq \lim_{p\to 0} \frac{h(\nu_p)}{\log \lambda} = \frac{h(\mu)}{\log \lambda}$$

and

$$\liminf_{x\to\beta^+} F(x) \geq \lim_{p\to 1} \frac{h(\nu_p)}{\log \lambda} = \frac{h(\nu)}{\log \lambda},$$

since $\int \phi d\nu_p = p\beta + (1-p)\alpha$ takes values on every point of the interval $(\beta, \alpha)$ as $p$ varies. Thus, we deduce

$$F(\alpha) \leq \liminf_{n\to\infty} F(\alpha_n).$$

■

Let $\mu \in \mathcal{M}_\sigma$ such that $\int \phi d\mu = \alpha$. By the Variational Principle with the summable potential $q_*\phi$,

$$h(\mu) = h(\mu) + q_* \int \phi d\mu - q_* \int \phi d\mu \leq T(q_*) - q_*\alpha.$$

Taking supremum over $\mu \in \mathcal{M}_\sigma$ such that $\int \phi d\mu = \alpha$ we get

$$F(\alpha) \le \frac{T(q_*) - q_*\alpha}{\log \lambda},$$

so to conclude the theorem it is enough to prove that for every $\alpha < \alpha_*$, $D(\alpha) \le F(\alpha)$. This inequality is also proven in [IJ], but we adapted it to our case. For $\alpha < \alpha_*$, $N \in \mathbb{N}$ and $\varepsilon > 0$ consider

$$J_\alpha(N,\varepsilon) := \left\{ x \in \Sigma : \left| \frac{1}{k} S_k\phi(x) - \alpha \right| < \varepsilon \quad \forall k \ge N \right\}.$$

Observe that $J_\alpha \subset \bigcup_{n=1}^{\infty} J_\alpha(N,\varepsilon)$, so by Proposition 2.1.1, in order to prove the desired inequality it is enough to get an upper bound on the dimension of the sets $J_\alpha(N,\varepsilon)$. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$. For $k \in \mathbb{N}$ define

$$\mathcal{C}_k := \{ C = [i_1 \cdots i_k] : C \cap J_\alpha(N,\varepsilon) \neq \varnothing \},$$

this is, the cover of $J_\alpha(N,\varepsilon)$ by cylinders of length $k$.

**Lemma 5.2.6.** The cardinality of $\mathcal{C}_k$ is finite for every $k \ge N$.

*Proof.* Let $k \ge N$. Since $P(\phi) = 0$, $\phi$ is summable by Proposition 3.2.1, so

$$\lim_{i \to \infty} \sup_{[i]} \phi = -\infty.$$

Thus, we choose $i \in \mathbb{N}$ such that if $x \in [j]$ with $j \ge i$ we have $\phi(x) < k(\alpha - \varepsilon)$. Let $[i_1 \cdots i_k] \in \mathcal{C}_k$ and assume by contradiction that there exists $0 \le j \le k-1$ such that $i_{j+1} \ge i$. By definition of $\mathcal{C}_k$, there exists $x \in [i_1 \cdots i_k] \cap J_\alpha(N,\varepsilon)$ which satisfies $\phi(\sigma^j x) < k(\alpha - \varepsilon)$ and

$$\alpha - \varepsilon < \frac{S_k\phi(x)}{k} < \alpha + \varepsilon.$$

Now since $\phi < 0$

$$k(\alpha - \varepsilon) < S_k\phi(x) \le \phi(\sigma^j x) < k(\alpha - \varepsilon),$$

which is a contradiction. Then, $\mathcal{C}_k$ contains cylinders $[i_1 \cdots i_k]$ such that $i_j < i$, and there are finitely many of them. ∎

For $k \ge N$, let $s_k \in \mathbb{R}$ be the unique real number such that

$$\sum_{C \in \mathcal{C}_k} (\operatorname{diam} C)^{s_k} = 1,$$

and define

$$s := \limsup_{k\to\infty} s_k.$$

**Lemma 5.2.7.** The following holds:

$$\dim_H J_\alpha(N,\varepsilon) \le s,$$

and there exists $\{\mu_k\} \subset \mathcal{M}_\sigma$ with $\lim_{k\to\infty}\left(s_k - \frac{h(\mu_k)}{\log\lambda}\right) = 0$ and $\int \phi d\mu_k \in (\alpha-2\varepsilon, \alpha+2\varepsilon)$.

*Proof.* Observe that $\operatorname{diam} C = \lambda^{-k-1} < \lambda^{-k}$ for every $C \in \mathcal{C}_k$. For $\delta > 0$ and $k$ sufficiently large,

$$\mathcal{H}^{s+\delta}_{\lambda^{-k}}(J_\alpha(N,\varepsilon)) \le \sum_{C\in\mathcal{C}_k} (\operatorname{diam} C)^{s+\delta} \le 1.$$

Letting $k \to \infty$ we get $\mathcal{H}^{s+\delta}(J_\alpha(N,\varepsilon)) \le 1$ and thus $\dim_H J_\alpha(N,\varepsilon) \le s+\delta$. Since $\delta$ was arbitrary we conclude the first part of the Lemma.

For the second part, denote by $\nu_k$ the $\sigma^k$-invariant Bernoulli measure which gives to a cylinder $C \in \mathcal{C}_k$ the probability $(\operatorname{diam} C)^{s_k}$. Then, the entropy of this measure with respect to $\sigma^k$ is

$$h(\nu_k, \sigma^k) = -s_k \sum_{C\in\mathcal{C}_k} (\operatorname{diam} C)^{s_k} \log(\operatorname{diam} C) = s_k \log \lambda^{k+1},$$

thus $\dfrac{h(\nu_k,\sigma^k)}{\log\lambda^k} = \dfrac{s_k(\log\lambda^k + \log\lambda)}{\log\lambda^k}$ and $\lim_{k\to\infty}\left(\dfrac{h(\nu_k,\sigma^k)}{\log\lambda^k} - s_k\right) = 0$.

**Remark 5.2.4.** For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that for every cylinder of length $k$, $C = [i_1 \cdots i_k]$ and every $x, y \in C$

$$\left|\frac{1}{k}S_k\phi(x) - \frac{1}{k}S_k\phi(y)\right| < \epsilon.$$

In fact, given $x, y \in C = [i_1 \cdots i_k]$ notice that since $\phi$ is locally Hölder there exist constants $C, \beta > 0$ such that

$$|\phi(\sigma^j x) - \phi(\sigma^j y)| \le C\frac{1}{\lambda^{(k+1-j)\beta}}$$

for $0 \le j \le k-1$. Then,

$$\left|\frac{1}{k}S_k\phi(x) - \frac{1}{k}S_k\phi(y)\right| \le \frac{C}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda^\beta}\right)^{j+1}. \tag{5.5}$$

Since $\beta > 0$ and $\lambda > 1$ the series $\sum_n (1/\lambda^\beta)^{n+1}$ converges, therefore the right side of (5.5) tends to zero.

Let $C = [i_1 \cdots i_k] \in \mathcal{C}_k$ and $x \in C$. By definition of $\mathcal{C}_k$, there exists $y \in C \cap J_\alpha(N, \varepsilon)$ and

$$\left| \frac{1}{k} S_k\phi(x) - \alpha \right| \leq \left| \frac{1}{k} S_k\phi(x) - \frac{1}{k} S_k\phi(y) \right| + \left| \frac{1}{k} S_k\phi(y) - \alpha \right|.$$

Then, by Remark 5.2.4, for $k$ sufficiently large each cylinder in $\mathcal{C}_k$ only contains points $x$ such that $S_k\phi(x)/k \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon)$, and this implies that

$$\frac{1}{k} \sum_{j=0}^{k-1} \int \phi \circ \sigma^j d\nu_k = \int \frac{S_k\phi}{k} d\nu_k \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon).$$

To complete the proof set

$$\mu_k := \frac{1}{k} \sum_{j=0}^{k-1} \nu_k \circ \sigma^{-i}.$$

Clearly $\mu_k$ is $\sigma$-invariant, since $\nu_k$ is $\sigma^k$-invariant. Also we have $kh(\mu_k) = h(\nu_k, \sigma^k)$, for this see [JJOP, Section 2], they work on the compact full-shift, however the proof of this fact ([JJOP, Lemma 2]) is based on Abramov's Theorem (see [W, Theorem 4.13]), which does not ask for the compactness of the space. Let us verify the rest of the properties.

Observe that

$$\int \phi d\mu_k = \frac{1}{k} \sum_{j=0}^{k-1} \int \phi d(\nu_k \circ \sigma^{-j}) = \frac{1}{k} \sum_{j=0}^{k-1} \int \phi \circ \sigma^j d\nu_k \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon),$$

and finally

$$\lim_{k\to\infty} \left( s_k - \frac{h(\mu_k)}{\log \lambda} \right) = \lim_{k\to\infty} \left( s_k - \frac{kh(\mu_k)}{k \log \lambda} \right) = \lim_{k\to\infty} \left( s_k - \frac{h(\nu_k, \sigma^k)}{\log \lambda^k} \right) = 0.$$

■

Then, we get

$$D(\alpha) \leq \lim_{\varepsilon\to 0} \sup\{F(\xi) : \xi \in (\alpha - \varepsilon, \alpha + \varepsilon)\}.$$

It follows from Lemma 5.2.5 that $D(\alpha) \leq F(\alpha)$, and we conclude the desired inequality.

Observe that $T(q_*) = P(q_*\phi) < \infty$ implies that there exists an equilibrium measure and Gibbs state $\mu_*$, so by the same arguments as in the proof of Lemma 5.2.3 we conclude $D(\alpha_*) = \frac{T(q_*) - q_*\alpha_*}{\log \lambda}$. Thus, the proof of Theorem 5.2.1 is finished.

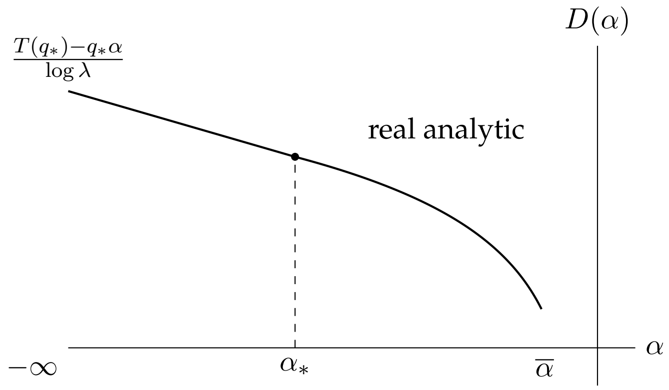



FIGURE 5.4: Graphic of $\alpha \mapsto D(\alpha)$.

■

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