PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

MASTER THESIS

# On a Helly-type problem concerning maximum-sum matchings, disks and ellipses

*Author:*
Oscar CHACÓN-RIVERA

*Supervisor:*
Pablo PÉREZ-LANTERO
(USACH)

*Examiners:*
Mario Ponce (Co-supervisor PUC)
Mircea Petrache (PUC)
Oscar Vásquez (USACH)

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# Declaration of Authorship

I, Oscar CHACÓN-RIVERA, declare that this thesis titled, "On a Helly-type problem concerning maximum-sum matchings, disks and ellipses" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date: Oct 20, 2020

*"A GREAT discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."*

George Pólya

PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

# *Abstract*

Faculty of Mathematics
Department of Mathematics

Master in Mathematics

**On a Helly-type problem concerning maximum-sum matchings, disks and ellipses**

by Oscar CHACÓN-RIVERA

Huemer et al. (Discrete Mathematics, 2019) proved that for any two point sets $R$ and $B$ with $|R| = |B|$, the perfect matching that matches points of $R$ with points of $B$, and maximizes the total squared Euclidean distance of the matched pairs, verifies that all the disks induced by the matching have a common point. Each pair of matched points $p \in R$ and $q \in B$ induces the disk of smallest diameter that covers $p$ and $q$. Following this research line, Bereg et al. (coauthored and submitted) considered the perfect matching that maximizes the total Euclidean distance and prove that this new matching for $R$ and $B$ does not always ensure the common intersection property of the disks. Furthermore, the study of this new matching is extended to sets of $2n$ uncolored points in the plane, where a matching is just a partition of the points into $n$ pairs. As the main result, it is proved that in this case all disks of the matching do have a common point. This implies a big improvement on a conjecture by Andy Fingerhut in 1995, about a maximum matching of $2n$ points in the plane. This thesis revises such results and further extend the study for ellipses, proving that pairwise intersection is always guaranteed, and laying the framework for an eventual proof of such conjecture.

# *Acknowledgements*

This thesis represents two years of an interesting mix of work, luck and effort. During those years studying my master, this discipline made me feel joy, sadness, gratitude and frustration, and if someone were to ask if I would live that again to get where I am, I would not hesitate to answer affirmatively. To say "thank you" to all who listened, proposed ideas, asked out of curiosity, pointed an error, or made a funny observation about this work is the least I can do, and even then it would not be enough. To every professor that laid the foundations of the path I have chosen, in particular Luis Disset, Pablo Pérez-Lantero, Mario Ponce and Daniel Vidal; to my classmates during these years, especially Esteban, Fernando, Gabriel, Ignacio, Matías, Nicolás, Pablo and Sebastián, for the good times, the laughs and the interesting chats, be them about mathematics or not; to professors Mircea Petrache and Oscar Vásquez for their willingness to examine this work and for their valuable input; to Soledad for helping me every time I needed help with the bureaucratic protocols; to every graduate/undergraduate student that showed their interest in what I had to say and fed me muffins whenever they needed help with a problem: to all of them a hearty "thank you". Of course, there are some people who contributed more that I could expect in their own fashion, so I would like to acknowledge them individually.

First and foremost I would like to thank my family, especially my parents, who have been there backing me up since I decided to study mathematics a decade ago. I cannot thank them enough for their love and support. They know how hard has it been for me, but they also know that if I were studying anything else, or working in any other area, I would not be the man who I grew up to be, and I would also be quite bored. I owe them my academic inheritance and thirst for knowledge. Mom, dad: mathematics is sometimes a harsh mistress, and your constant support helped me endure it and transform my rigid stubbornness into a healthy and methodical one. There is nothing like experiencing the feeling of conquering and solving a problem after hours, days or weeks fumbling with it, even if I sound like talking in tongues trying to explain it at dinner time. Part of this work is for you.

This work was typed and transcribed while living with my girlfriend Betzabé, whom I have stayed with since her accident in last Christmas, and her mother, who has accepted me with open arms like another son. Betzabé witnessed me juggling between this thesis, my everyday visits to the hospital, her rehabilitation process and then more visits to the hospital in a pandemic situation. It is not the best scenario to write a thesis, but she sit me to type anyway, anywhere, anytime. It is usual to thank your partner for "putting up" with you, but in this case it was she who encouraged me to keep working instead. Thank you for coming to my life in such an important stage, for your inputs, your drawings for the oral dissertation, for your company, your love, your support. This work is also for you.

I owe a great debt to Mario Ponce, my undergraduate advisor and the person who introduced me to the topic of Computational Geometry back in 2015. It was a time where I felt lost both personally and academically, and he helped me by straighten things out and renewing my passion for mathematics. Since then, he has been a source of both admiration and sometimes earned scoldings - all of them well earned, he would say-, and I cannot thank him enough for everything he has done, for being so patient, and for his always spot on questions that fosters one's own quest for knowledge. I consider him my mentor and he has set a great example to follow in my future academic endeavors.

It was luck that put Pablo Pérez-Lantero, my thesis advisor, on my path back on a SOMACHI talk in 2018. I still remember his talk on rectilinear convex hulls, since it was the moment I confirmed my interest in Discrete, Combinatorial and Computational Geometry. I practically owe him these pages: after all, he was the one that invited me over to work with him together with David Flores-Peñaloza, Clemens Huemer and Carlos Seara the summer of 2019, an instance that allowed me to learn more about the topic and introduced me to the main problem of this thesis, which in turn is mainly based on an article coauthored with them and Sergey Bereg. Even though our academic relationship has been somewhat unorthodox due to my country's social unrest and a world wide pandemic, Pablo has always been there to guide me, teaching me new topics, providing interesting approaches, answering my sometimes redundant questions, and most importantly putting me to work, be it in his office or via e-mail and instant messaging. I deeply thank Pablo for giving me this opportunity of researching and learning with him, and I look forward to his teachings in the near future.

Throughout my academic life I have made many friends and acquaintances, to whom I thank for their support, but there is one person that gets the spotlight and deserves some kudos. I met Sebastian Pavez years ago, and we have been close friends since our final years as undergraduates. We have exhausted all possible configurations: I have been his teaching assistant, he has been mine, I have graded his lab homework, he has graded my exams, we have both been classmates in most graduate courses, and we shared the same desk in our office these past couple of years, usually swapping each other's chairs to mess around. He is a brilliant young man, a superb friend and I am thankful for his geometric insights and fruitful discussions while having a sub sandwich and scribbling on napkins or pizza boxes. Hopefully we will keep doing it as colleagues in the future, so keep bringing it on, Sebastian.

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*To my family and my beloved Betzabé*

# Chapter 1

# Introduction

The framework of this thesis is the application of tools and techniques from Discrete and Combinatorial Geometry in order to prove a particular problem that arises in areas such as network design and the study of intersection of geometric graphs.

This chapter is devoted to briefly introduce the area of Discrete and Combinatorial Geometry, as well as to motivate the study of a conjecture posed by Andy Fingerhut in 1995, in conjunction with an exhaustive study of a Helly-type problem on diametric disks.

## 1.1 Discrete and Combinatorial Geometry

Discrete and combinatorial geometry is concerned with the study of combinatorial properties and constructive methods of discrete geometric objects, and focuses on finite sets of points, lines, triangles and the like, in contrast with the study of objects with a "continuous" geometry, for instance, smooth surfaces as in multi-variable calculus and Riemannian geometry. It is a rather recent area of research compared to others, given that mathematicians disregarded intuitive geometry in order to develop more abstract areas of geometry such as topology and differential geometry. It was during last century that classical questions studied by Newton, Gauss, Minkwoski, Hilbert, and Thue, started to garner a renewed attention, with new combinatorial approaches initiated by the works of László Tóth, Ambrose Rogers and Paul Erdös. In turn, many of these problems turned out to be crucially important in areas such as computational geometry, coding theory, combinatorial optimization, robotics and computer graphics, to name a few.

In the beginning, most questions in this area were focused on arrangements of points, lines, circles, spheres, as well as questions on packing, covering, and tiling. For instance, to study the number of incidences between $n$ points and $n$ lines in the plane. However, the advent of powerful computers and the explosion of activity in the field of computational geometry, allowed the term "discrete geometry" to also stand for convex polytopes and arrangements of other geometric objects in the plane and in higher dimensions.

## 1.2 Overview and motivation

Certain situations in the practical life can be nicely modeled by discrete geometric configurations of points, segments and other familiar geometric objects such as disks, polygons and angles, to name a few. Such mathematical models give rise to nice geometric properties and interesting questions concerning both the combinatorial and computational aspects of these geometric settings, sometimes bearing deep and elegant results such as Helly's theorem.

One such case, concerning the design of communication networks, was informed by Andy Fingerhut in 1995, as archived by David Eppstein on his *Geometry Junkyard* [18]. Fingerhut asked, given a maximum matching of six points in the Euclidean plane, whether there is a center point close to all matched edges (within distance a constant times the length of the edge). If this were true, then the previous statement could be generalized and proved for any even number of points by applying Helly's theorem. Thus, it would be possible to place the center of a star-shaped network close enough to every other node, or equivalently, with cost not too expensive compared to the cost of the optimal network. More rigorously, we have the following

**Conjecture 1.** *Let $P$ be any point set of $2n$ uncolored points in the Euclidean plane, and let $\{(a_i, b_i) : i = 1, \ldots, n\}$ be the maximum-sum matching of these points. There exists a point $o$ in the plane, not necessarily a point of $P$, such that*

$$\|a_i - x\| + \|x - b_i\| \leq \frac{2}{\sqrt{3}} \|a_i - b_i\| \quad \text{for all } i \in \{1, \ldots, n\}. \tag{1.1}$$

The previous statement will sometimes be referred to as "Fingerhut's conjecture", as in [6]. Geometrically speaking, the statement in (1.1) is equivalent to assert that the intersection $E_1 \cap E_2 \cap \cdots \cap E_n$ is not empty, where $E_i$ is the region bounded by the ellipse with foci $a_i$ and $b_i$, and semi-major axis of length $(1/\sqrt{3})\|a_i - b_i\|$ for all $i \in \{1, \ldots, n\}$ [18]. As noted by Fingerhut, the factor $2/\sqrt{3}$ (otherwise known as Hermite constant) is the minimum possible. So far, the only progress known to the author is an observation by Eppstein [18], who proved that the conjecture holds with a factor of 2.5 instead of $2/\sqrt{3}$ by a simple geometric argument, taking $x$ as the midpoint of the shortest edge in the matching.

This thesis is a revision of a coauthored article ([6]), filling in aditional details to several proofs, as well as adding some new results on the pairwise intersection of ellipses, conveyed via personal conversation by thesis advisor Pablo Pérez-Lantero and professor Carlos Seara [24]. Thus, the first goal of this thesis is to give a detailed revision on the progress on the numerical bound in inequality (1.1) in the bi-chromatic and monochromatic cases as done in [6]. The second goal is to complement said results with an analysis on the pairwise intersection of ellipses defined by a maximum-sum matching.

## 1.3 Related works

The study of monochromatic and bi-chromatic matchings with geometric objects is a well studied topic in discrete and computational geometry. The author identifies two branches of research concerning matchings: on one hand, the usual results establish whether a maximum strong-matching exists or not, while on the other hand, new results such as this work (as well as the paper it is based on) study whether the objects matched have a common intersection or not.

It is a classic result in discrete geometry that given $n$ red points and $n$ blue points there exists a strong perfect bi-chromatic segment-matching, which also happens to be a minimum-sum matching [22]. There are several variations of this problem using segments or segment-like shapes: Dumitrescu and Steiger devised algorithms for a partial strong segment-matching of points of the same color [17], later improved by Dumitrescu and Kaye [16], while Aloupis et al. considered strong matchings between point-object pairs joined by a segment [4].

On monochromatic triangle-matchings, Biniaz's doctoral thesis established bounds for algorithms concerning upward and downward equilateral triangles [7].

Study of the existence of a strong square-matching is introduced by Ábrego et al. ([1], [2]), while Bereg, Mutsanas and Wolff extended those results to axes-aligned rectangles [5], both articles constrained to a monochromatic setting; Caraballo et al. devoted their efforts in studying the existence of both a maximum monochromatic and bi-chromatic rectangle-matching, as well as the computational complexity of these problems [9]. This last result was further extended by Corujo et al., who studied same-color rectangle-matchings given a random bi-colored partition [10].

Existence of a strong perfect monochromatic circle-matching follows from a result on Delauney triangulations by Dillencourt, given an even number of points [15]. This result is generalized by Ábrego et al., who established bounds for such existence, but given an arbitrary number of points [1].

Recently, and directly related to this work, Huemer et al. proposed a different direction of research, studying the intersection of diametrical disks defined from a maximum bi-chromatic matching [21].

Finally, concerning the conjecture proposed by Fingerhut and in the language of network design, Fingerhut, Suri and Turner studied the existence of a star network depending on the network capacity and their transfer rate [19]; however, such analysis is completely focused on the properties of the networks involved instead of their geometric configuration.

## 1.4 Contribution of this Thesis

In the first part of this exposition, we revise the proof in [6] claiming that pairwise intersection of disks associated with the bi-chromatic matching that maximizes the total Euclidean distance is always not empty. However, a common intersection among all disks is not always guaranteed, in contrast to [21] where the matching maximizes the total squared Euclidean distance. Such pairwise intersection property is then used to give a subtle improvement to Eppstein's observation in [18] concerning Conjecture 1.

In the second part, we make a detailed revision of the case-by-case study in [6] of the configurations of monochromatic maximum-sum matchings of six points together with a geometric extensibility property. Such analysis, together with an application of Helly's theorem, gives in turn a much better improvement to (but still worse than) the numerical factor in (1.1).

In the last part, we introduce a proper study of the conjecture via analyzing the pairwise intersection of ellipses associated with the maximum-sum matching of four points.

The implications of these results are twofold. On one hand, considering the current pandemic scenario, we dust off an interesting unsolved problem that remained hidden on the internet, with a straightforward application in network design in these trying times, where an adequate disposal of distribution centers might help alleviate the goods' provisioning in the most affected locations of a state or administrative region. On the other hand, mathematically speaking, the case-by-case study on disks together with the difficulty of the study on ellipses may help devise better geometric techniques for this direction of research (as that of [21]) concerning the existence of a common intersection of whatever matching geometric objects are at play.

## 1.5 Outline

In what follows, we outline each chapter of this work for reference.

In Chapter 2, we state the main graph-theoretical and geometric definitions to be used throughout the thesis.

In Chapter 3, we first show that pairwise intersection of disks is always guaranteed in the bi-chromatic case. Then, we construct an example of three disks for which there is no common intersection, and generalize such construction for $2n$ points, with $n \geq 4$. We close the chapter by stating a first improvement related to Fingerhut's conjecture.

In Chapter 4, we prove via a case-by-case analysis that three disks always intersect in a monochromatic setting, and then extend this result for any number of disks by Helly's theorem. Then, we further improve the upper bound constant established in Chapter 3.

In Chapter 5, we introduce the study of common intersection of ellipses related to a monochromatic matching, proving that they intersect at least pairwise.

In Chapter 6, we summarize our results, and propose several directions of research based on our findings.

# Chapter 2

# Preliminaries

We assume the reader is familiar with the basics of geometry and graph theory, and only give an overview of the most relevant terms and concepts. For more detailed introductions to graph theory, we refer to [8], [14], [26]. For a detailed exposition on Euclidean geometry, see [11], [12] and [27]. Finally, we refer to [20], [13] and [23] for a vast exposition on topics in discrete and combinatorial geometry.

## 2.1 Graph theory

We start with some basic definitions and notations of graph theory.

**Definition 1.** *A* ***graph*** *$G$ is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of* ***vertices*** *and a set $E(G)$, disjoint from $V$, of* ***edges****, together with a relation that associates with each edge of $G$ two vertices of $G$ (not necessarily distinct), called its* ***endpoints****.*

*An edge with identical endpoints is called a* ***loop****.* ***Multiple edges*** *are edges having the same pair of endpoints.*

*A* ***simple graph*** *is a graph having no loops or multiple edges.*

*A* ***matching*** *or* ***segment-matching*** *in a graph $G$ is a set of non-loop edges with no shared endpoints.*

*A graph is said to be* ***bipartite*** *if its vertex set can be partitioned into two subsets $X$ and $Y$ so that every edge has one end in $X$ and one end in $Y$.*

Let $G$ be a bipartite graph such that $G = R \cup B$ models a union of disjoint point sets in the plane $\mathbb{R}^2$, where the vertices in $R$ are *red* points, and those in $B$ are *blue* points. Let $|R| = |B| = n$, $n \geq 2$.

**Definition 2.** *A* ***red-blue matching*** *or* ***bi-chromatic matching*** *of $G = R \cup B$ is a matching such that each edge consists of a red point and a blue point. Thus, a point $p \in R$ and a point $q \in B$ are said to be* ***matched*** *if the edge $(p, q)$ belongs to such matching.*

*If $G$ consists of uncolored points, then the matching is said to be* ***monochromatic****.*

In what follows, the terms *monochromatic matching* and *bi-chromatic matching* will be used only to emphasize, and the term *matching* will be favoured instead: the reader will therefore have to infer the nature of such matching according to the given setting.

## 2.2 Euclidean and discrete geometry

We now define the main geometric objects we will use throughout this thesis, for reference.

**Definition 3.** *For every $p, q \in \mathbb{R}^2$, let $pq$ denote the segment connecting $p$ and $q$, and let $\|p - q\|$ denote its length, which is the Euclidean norm of the vector $p - q$. Furthermore, $x(p)$ will denote the x-coordinate of $p$, and $y(p)$ the y-coordinate of $p$.*

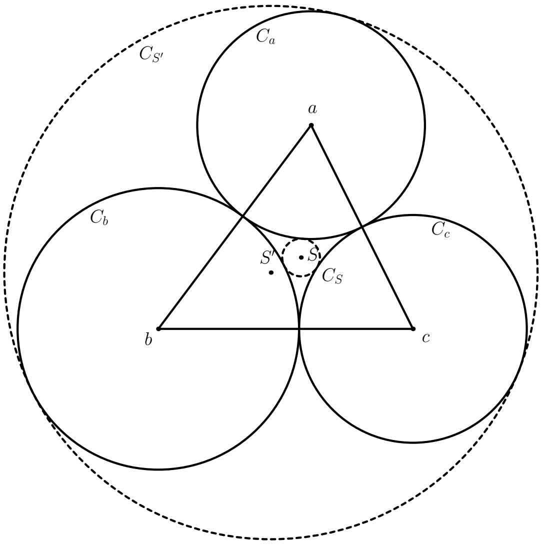



FIGURE 2.1: Soddy circles. $C_{S'}$ is the outer Soddy circle with outer Soddy center $S'$. $C_S$ is the inner Soddy circle with inner Soddy center $S$

*Let $D_{pq}$ denote the (diametrical) disk with diameter equal to $\|p-q\|$, centered at the midpoint $\frac{p+q}{2}$ of the segment $pq$.*

*Let $E_{pq}$ denote (the region bounded by) the ellipse with foci $p$ and $q$ and semi-major axis length $(1/\sqrt{3})\|p-q\|$.*

**Definition 4.** *Let $\mathcal{M}$ be a matching. The **segments** of $\mathcal{M}$ is the segment set $\{pq : (p,q) \in \mathcal{M}\}$. The set of diametrical disks associated with $\mathcal{M}$ is denoted by $D_{\mathcal{M}}$, that is, $D_{\mathcal{M}} = \{D_{pq} : (p,q) \in \mathcal{M}\}$.*

*Let $\mathrm{cost}(\mathcal{M})$ denote the sum $\sum_{(p,q)\in\mathcal{M}} \|p-q\|$. A **maximum-sum matching** $\mathcal{E}$ is the matching that maximizes the total Euclidean distance of the matched points. That is, $\mathcal{E}$ is such that $\mathrm{cost}(\mathcal{E})$ is maximum among all matchings.*

In what follows, a matching will be assumed to be a maximum-sum matching unless stated otherwise. Also, note that $D_{\mathcal{M}}$ is just a reformulation of the definition of circle-matching found in [1], [2].

**Definition 5.** *Let $p, q$ and $r$ be three points in the plane. We denote by $\Delta pqr$ the triangle with vertices $p, q$ and $r$; by $\ell(p,q)$ the straight line through $p$ and $q$ oriented from $p$ to $q$; by $\tau(p,q)$ the ray with apex $p$ that goes through $q$; by $\vec{pq}$ the segment $pq$ oriented from $p$ to $q$, and by $C_{pq}$ the circle bounding $D_{pq}$.*

*If $s$ is a fourth point, we say that $\vec{pq}$ **points to** $rs$ if $q$ is in the interior of $\Delta prs \cap D_{rs}$.*

**Definition 6.** *Let $\alpha$ be a planar (open or closed) curve that splits the plane into two open regions. Given a point $p$ not in $\alpha$, let $H(\alpha, p)$ denote the region (between the two above ones) that contains $p$.*

**Definition 7.** *Let $a, b$ and $c$ be the vertices of the triangle $\Delta abc$. Let $C_a, C_b$ and $C_c$ be the three mutually exterior tangent circles centered at vertices $a, b$ and $c$, respectively. The **inner Soddy circle** of $\Delta abc$ is the exterior tangent circle to each $C_a, C_b$ and $C_c$. The center of the inner Soddy circle is known as the **inner Soddy center**. See Figure 2.1.*

# Chapter 3

# Bi-chromatic circle-matchings

Let $R$ and $B$ be two disjoint point sets defined as in the previous chapter, where $|R| = |B| = n$, $n \geq 2$. In [21], Huemer et al. proved that if $\mathcal{M}$ is any segment-matching that maximizes the sum of the squared Euclidean distances of the matched points, i.e. maximizes $\sum_{(p,q)\in\mathcal{M}} \|p-q\|^2$, then all disks of $D_{\mathcal{M}}$ have a point in common. They proceeded as follows: they first study the pairwise intersection of disks, following with a case analysis on the configuration of four to six points, and finishing with a proof on the intersection of three disks. Then, an application of Helly's theorem extends the result to any pair of sets $R$ and $B$ of $n$ points each.

In this chapter we prove that, while pairwise intersection is always possible, such intersection property for three or more disks in $D_{\mathcal{M}}$ does not necessarily hold for the perfect bi-chromatic matching $\mathcal{M}$ of $R$ and $B$ that maximizes $\text{cost}(\mathcal{M})$. Then, we apply the pairwise intersection property to study the bi-chromatic case of Fingerhut's conjecture, improving Eppstein's observation in [18].

## 3.1 Common intersection property of disks in $D_{\mathcal{M}}$

As in [21], we first observe that pairwise intersection is always possible.

**Proposition 1.** *Every pair of disks in $D_{\mathcal{M}}$ have a non-empty intersection.*

*Proof.* Let $(a,a')$ and $(b,b')$ be two different pairs in $\mathcal{M}$, with $a,b \in R$ and $a',b' \in B$. Then

$$\|a-b'\| + \|a'-b\| \leq \|a-a'\| + \|b-b'\|$$

since $\mathcal{M}$ is a maximum-sum matching: otherwise, we would have that

$$\text{cost}((\mathcal{M} \setminus \{(a,a'),(b,b')\}) \cup \{(a,b'),(a',b)\}) > \text{cost}(\mathcal{M}),$$

which contradicts the definition of $\mathcal{M}$. Note that equality may hold: consider a square with consecutive vertices $a, a', b$ and $b'$.

Now, observe that two disks have a common point if and only if the distance between their corresponding centers is at most the sum of their radii. Since the disks $D_{aa'}$ and $D_{bb'}$ have centers and radii $\frac{a+a'}{2}$ and $\frac{\|a-a'\|}{2}$, and $\frac{b+b'}{2}$ and $\frac{\|b-b'\|}{2}$, respectively, it is enough to show that $\|\frac{a+a'}{2} - \frac{b+b'}{2}\| \leq \frac{\|a+a'\|}{2} + \frac{\|b+b'\|}{2}$. This condition follows from

$$\begin{aligned} \|(a+a')-(b+b')\| &= \|(a-b')+(a'-b)\| \\ &\leq \|a-b'\| + \|a'-b\| \\ &\leq \|a-a'\| + \|b-b'\| \end{aligned}$$

Hence, $D_{aa'} \cap D_{bb'} \neq \emptyset$ for every pair of disks $D_{aa'}$ and $D_{bb'}$ of $D_{\mathcal{M}}$. $\square$

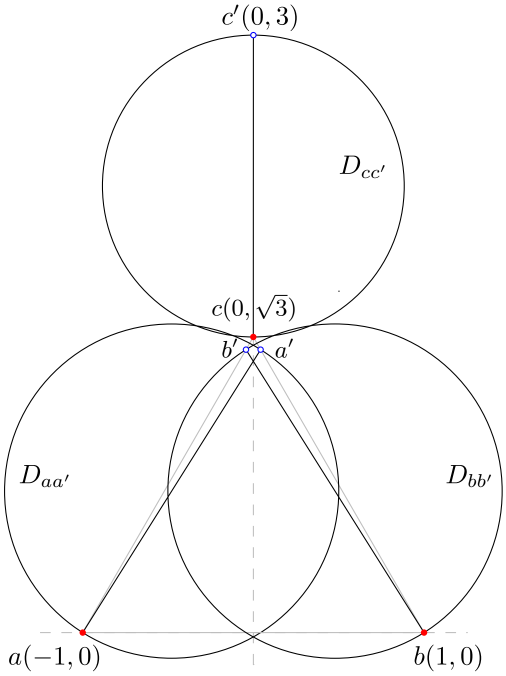



FIGURE 3.1: Proof of Theorem 1.

So, pairwise intersection also holds when we consider Euclidean distances, a fact that will turn useful when making a first improvement on Eppstein's observation concerning inequality (1.1). However, such intersection property may not hold when we consider three disks, as we now show.

**Theorem 1.** *There exist point sets $R \cup B$, with $R \cap B = \emptyset$ and $|R| = |B| = 3$, such that, for any maximum-sum matching $\mathcal{M}$ of $R$ and $B$, the intersection of the disks of $D_{\mathcal{M}}$ is empty.*

*Proof.* Let $R = \{a, b, c\}$ and $B = \{a', b', c'\}$, with $a = (-1, 0)$, $b = (1, 0)$, $c = (0, \sqrt{3})$, and $a' \in bc$ and $b' \in ac$ such that $\|c - a'\| = \|c - b'\| = \epsilon$, where $\epsilon > 0$ ensures that $\mathcal{M} = \{(a, a'), (b, b'), (c, c')\}$ is the only maximum matching of $R \cup B$ (see Figure 3.1).

We first focus our study on finding values of $\epsilon > 0$ such that $\mathcal{M}$ is in fact the only maximum matching of $R \cup B$, and then we prove that no common intersection is possible.

Note that

$$\|a - b'\| = \|a - c\| - \|b' - c\| = 2 - \epsilon.$$

Secondly, observe that, by symmetry of our point configuration, we have

$$\begin{aligned} \|a - b'\| + \|b - c'\| + \|c - a'\| &= \|a - c'\| + \|b - a'\| + \|c - b'\| \\ &= \sqrt{10} + (2 - \epsilon) + \epsilon \\ &= 2 + \sqrt{10}. \end{aligned}$$

Since $\mathcal{M}$ is the only maximum matching, we need to ensure that

$$2 + \sqrt{10} < \|a - a'\| + \|b - b'\| + \|c - c'\| = \text{cost}(\mathcal{M}).$$

In other words, the matching $\{(a, a'), (b, b'), (c, c')\}$ must have a larger total Euclidean distance than the matchings $\{(a, b'), (b, c'), (c, a')\}$ and $\{(a, c'), (b, a'), (c, b')\}$. Now, on one hand,

$$\begin{aligned} \|a-a'\| + \|b-b'\| + \|c-c'\| &= 2\|a-a'\| + \|c-c'\| \\ &= 2\|a-a'\| + (3-\sqrt{3}) \\ &> 2(\|a-c\| - \epsilon) + (3-\sqrt{3}) \\ &= 7-\sqrt{3}-2\epsilon, \end{aligned}$$

so it suffices to ensure

$$2+\sqrt{10} < 7-\sqrt{3}-2\epsilon \iff \epsilon < \frac{5-\sqrt{10}-\sqrt{3}}{2} \approx 0.0528. \tag{3.1}$$

On the other hand, again by symmetry,

$$\begin{aligned} \|a-c'\| + \|b-b'\| + \|c-a'\| &= \|a-a'\| + \|b-c'\| + \|c-b'\| \\ &< (2+\epsilon) + \sqrt{10} + \epsilon \\ &= 2+\sqrt{10}+2\epsilon, \end{aligned}$$

so to guarantee that the matching $\{(a,a'),(b,b'),(c,c')\}$ has in fact larger total Euclidean distance than $\{(a,c'),(b,b'),(c,a')\}$ and $\{(a,a'),(b,c'),(c,b')\}$, it suffices to ensure that

$$2+\sqrt{10}+2\epsilon < 7-\sqrt{3}-2\epsilon \iff \epsilon < \frac{5-\sqrt{10}-\sqrt{3}}{4} \approx 0.0264. \tag{3.2}$$

Thus, any $\epsilon > 0$ satisfying (3.2) (and therefore (3.1)) is such that the matching $\mathcal{M} = \{(a,a'),(b,b'),(c,c')\}$ is in fact the only maximum matching of $R \cup B$.

To show that $D_{aa'} \cap D_{bb'} \cap D_{cc'} = \emptyset$, note that all points of $D_{aa'} \cap D_{cc'}$ have negative $x$-coordinates. Indeed, let $p \in D_{aa'} \cap D_{cc'}$ be such that $x(p) \geq 0$. Note that $x(a') > 0$ and $y(a') < \sqrt{3}$, since $a' \in bc$ and $R \cap B \neq \emptyset$. Thus,

$$\|a-a'\| = \sqrt{(1+x(a'))^2 + y(a')^2} < \sqrt{(1+x(a'))^2+3} < 2.$$

Now, it must be that $y(p) \geq \sqrt{3}$, since $p \in D_{cc'}$; however, since $p \in D_{aa'}$, we have that

$$2 \leq \sqrt{(x(p)+1)^2+3} \leq \sqrt{(x(p)+1)^2+y(p)^2} = \|a-p\| \leq \|a-a'\| < 2,$$

a contradiction. Hence, all points of $D_{aa'} \cap D_{cc'}$ have negative $x$-coordinates. Similarly, all points of $D_{bb'} \cap D_{cc'}$ have positive $x$-coordinates. Therefore, $D_{aa'} \cap D_{bb'} \cap D_{cc'} = \emptyset$. □

Let $|R| = |B| = n$. Until now, Proposition 1 guarantees a common intersection when $n = 2$, but Theorem 1 shows a counterexample where such intersection is empty when $n = 3$. What about the intersection of the disks of $D_{\mathcal{M}}$ when $n$ is arbitrarily large? One could expect that the common intersection property holds for sufficiently large values of $n$. However, we now show that we can adapt the construction in Theorem 1 to find a configuration of disks of $D_{\mathcal{M}}$ that have no common intersection. The main difference with Theorem 1 is that we need to further analyze the possible matching configurations, since now some of the points defined in Theorem 1 might now be matched to some of the extra $n-3$ points to consider.

**Theorem 2.** *For any $n \geq 4$, there exist point sets $R \cup B$, with $R \cap B = \emptyset$ and $|R| = |B| = n$, such that, for any maximum-sum matching $\mathcal{M}$ of $R$ and $B$, the intersection of the disks of $D_{\mathcal{M}}$ is the empty set.*

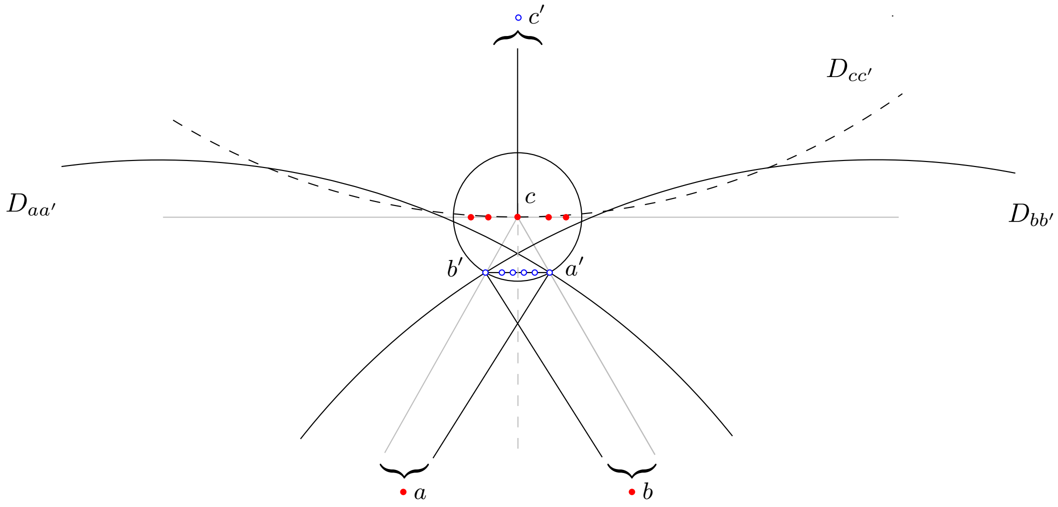



FIGURE 3.2: Proof of Theorem 2.

*Proof.* Consider the following construction of a set $R$ of $n$ red points, and a set $B$ of $n$ blue points. Take six points $a, b, c \in R$ and $a', b', c' \in B$ as in Theorem 1, and add $n-3$ red points and $n-3$ blue points in the $\epsilon$-neighbourhood as explained below, where $\epsilon > 0$ is a sufficiently small number that will be specified later.

Refer to Figure 3.2. Add $n-3$ blue points, denoted $a'_1, a'_2, \ldots, a'_{n-3}$, on the segment $b'a'$. As in Theorem 1, we have $\|c-b'\| = \|c-a'\| = \epsilon$, so $\|c-a'_i\| < \epsilon$ for $i \in \{1, \ldots, n-3\}$. Add $n-3$ red points, denoted $a_1, a_2, \ldots, a_{n-3}$, on the horizontal line through $c$ (perpendicular to $cc'$) and such that $\|c-a_i\| < \epsilon$ for $i \in \{1, \ldots, n-3\}$. So, in particular we have $\|c'-a_i\| \geq \|c'-c\|$ for all $i \in \{1, \ldots, n-3\}$.

We proceed to study some possible configurations of matchings. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matchings such that in $\mathcal{M}_1$ point $c'$ is matched to $c$ or to some $a_i$, $i \in \{1, \ldots, n-3\}$, and in $\mathcal{M}_2$ point $c'$ is matched to $a$ or $b$. Given $p \in R \cup B$ and a matching $\mathcal{M}$ of $R \cup B$, denote by $\mathcal{M}(p)$ the point such that $p$ and $\mathcal{M}(p)$ are matched in $\mathcal{M}$. Then,

$$\|c' - \mathcal{M}_1(c')\| \geq \|c'-c\| = 3-\sqrt{3} \text{ and } \|a-\mathcal{M}_1(a)\| \geq \|a-c\| - \epsilon = 2-\epsilon.$$

Similarly,

$$\|b-\mathcal{M}_1(b)\| \geq \|b-c\| - \epsilon = 2-\epsilon.$$

Hence, we have

$$\mathrm{cost}(\mathcal{M}_1) \geq 3-\sqrt{3}+2(2-\epsilon) = 7-\sqrt{3}-2\epsilon.$$

By symmetry, we can assume that $(a, c') \in \mathcal{M}_2$. In $\mathcal{M}_2$, $b$ is matched to $a'$, to $b'$ or to some $a'_i$, $i \in \{1, \ldots, n\}$. In particular, we have that

$$\|b-\mathcal{M}_2(b)\| \leq \|b-b'\| \leq \|b-c\| + \epsilon = 2+\epsilon.$$

Furthermore, the remaining $n-2$ pairs of $\mathcal{M}_2$ are in an $\epsilon$-neighbourhood of $c$. So in particular,

$$\mathrm{cost}(\mathcal{M}_2) \leq \sqrt{10} + (2+\epsilon) + 2(n-2)\epsilon.$$

We choose $\epsilon > 0$ such that

$$7-\sqrt{3}-2\epsilon > \sqrt{10}+2+\epsilon+2(n-2)\epsilon \iff 5-\sqrt{3}-\sqrt{10} > (2n-1)\epsilon.$$

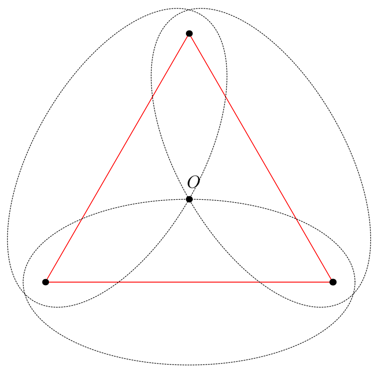



FIGURE 3.3: Fingerhut's observation

Since $5 - \sqrt{3} - \sqrt{10} \approx 0.10567 > \frac{1}{10}$, it suffices to choose $\epsilon < \frac{1}{10(2n-1)}$ to guarantee $\text{cost}(\mathcal{M}_1) > \text{cost}(\mathcal{M}_2)$. Therefore, $\mathcal{M} \neq \mathcal{M}_2$, so we can assume that $\mathcal{M} = \mathcal{M}_1$.

Finally, we show that there is no common intersection between the disks of $D_{\mathcal{M}}$. To this end, it suffices to show that $D_{a\mathcal{M}_1(a)} \cap D_{b\mathcal{M}_1(b)} \cap D_{c'\mathcal{M}_1(c')} = \emptyset$. This follows from the fact (as in Theorem 1) that all points of $D_{a\mathcal{M}_1(a)} \cap D_{c'\mathcal{M}_1(c')}$ have negative $x$-coordinates, and all points of $D_{b\mathcal{M}_1(b)} \cap D_{c'\mathcal{M}_1(c')}$ have positive $x$-coordinates. As a result, the intersection of the disks of $D_{\mathcal{M}}$ is the empty set. □

## 3.2 On a bi-chromatic Fingerhut's conjecture

While Fingerhut did not know whether inequality (1.1) was true for a factor of $2/\sqrt{3}$, he did know that it was false by a smaller constant, as we now show. Consider six points, $a_1, a_2, a_3, b_1, b_2, b_3$ in the plane such that $a_i$ are red and $b_i$ are blue for $i \in \{1, 2, 3\}$, $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ is the maximum-sum bi-chromatic matching of these points, and an equilateral triangle where at each vertex two of these points are located. Refer to Figure 3.3. Observe that the maximum-sum matching would then be made of vertex-opposed points, and the regions bounded by the ellipses $E_{a_i b_i}$, $i \in \{1, 2, 3\}$, have exactly one point in common, say $o$: clearly, any other point different from $o$ fails to satisfy at least one of the loci defining the ellipses. Also note that $R \cap B \neq \emptyset$.

Instead of proving the result directly for $2/\sqrt{3}$, Fingerhut was interested in proving it with a worse constant as close to $2/\sqrt{3}$ as possible. Eppstein [18] proved, without applying Helly's theorem, that the result holds for a factor of 3 as follows: let $\mathcal{M} = \{(a_i, b_i) : i \in \{1, \ldots, n\}\}$ be the maximum-sum bi-chromatic matching, with $a_i \in R$ red points and $b_i \in B$ blue points. Let $(a_1, b_1)$ the shortest edge in $\mathcal{M}$, and let $x$ be any point on that edge. Suppose by contradiction that

$$\|a_i - x\| + \|x - b_i\| > 3\|a_i - b_i\| \quad \text{for some } i \in \{1, \ldots, n\}.$$

Say, $i = 2$. Note that, without loss of generality, the matching defined by $\mathcal{M}_0 = \{(a_1, b_2), (a_2, b_1), (a_3, b_3), \ldots, (a_n, b_n)\}$ is such that $\text{cost}(\mathcal{M}_0) < \text{cost}(\mathcal{M})$. However, by using the triangle inequality and our previous assumption, we have that

$$\begin{aligned}
cost(\mathcal{M}_0) &= \|a_1 - b_2\| + \|a_2 - b_1\| + \|a_3 - b_3\| + \cdots + \|a_n - b_n\| \\
&= \|a_1 - b_2\| + \|a_2 - b_1\| + \|a_3 - b_3\| + \\
&\quad \|a_1 - x\| + \|x - b_1\| - \|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&\geq \|x - b_2\| + \|a_2 - x\| + \|a_3 - b_3\| - \|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&> 3\|a_2 - b_2\| + \|a_3 - b_3\| - \|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&> 2\|a_1 - b_1\| + \|a_2 - b_2\| + \|a_3 - b_3\| - \|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&= \mathrm{cost}(\mathcal{M}),
\end{aligned}$$

which cannot be. So, we have the following

**Observation 1.** *For any $n \geq 2$, and any point sets $R$ and $B$ with $|R| = |B| = n$ and such that $R \cap B = \emptyset$, let $\mathcal{M} = \{(a_i, b_i) : i \in \{1, \dots, n\}\}$ be any maximum-sum matching of $R \cup B$. There exists a point $x$ in the plane such that*

$$\|a_i - x\| + \|x - b_i\| \leq 3\|a_i - b_i\| \quad \textit{for all } i \in \{1, \dots, n\}. \tag{3.3}$$

However, Proposition 1 allows us to further improve the constant in inequality (3.3) to $\sqrt{5} \approx 2.236$. We start by proving the next technical lemma:

**Lemma 1.** *Let $p$ and $q$ be two points in the plane, and consider the disk $D_{pq}$ with radius $r_{pq}$. Let $D$ be a second disk with center $o$ and radius $r \leq r_{pq}$ such that $D \cap D_{pq} \neq \emptyset$. Then,*

$$\|p - o\| + \|q - o\| \leq \sqrt{5}\|p - q\|.$$

*Proof.* Let $o_{pq}$ denote the center of $D_{pq}$. Since $D \cap D_{pq} \neq \emptyset$, we have that $\|o - o_{pq}\| \leq r + r_{pq} \leq 2r_{pq} = \|p - q\|$. Then,

$$\begin{aligned}
(\|p - o\| + \|q - o\|)^2 &\leq 2(\|p - o\|^2 + \|q - o\|^2) && \text{(Cauchy-Schwarz)} \\
&= 2\left(\tfrac{1}{2}\|p - q\|^2 + 2\|o - o_{pq}\|^2\right) && \text{(Apollonius)} \\
&= \|p - q\|^2 + 4\|o - o_{pq}\|^2 \\
&\leq 5\|p - q\|^2,
\end{aligned}$$

from where the lemma follows. □

Combining Lemma 1 with Proposition 1 we get the following

**Theorem 3.** *For any $n \geq 2$, and any point sets $R$ and $B$ in the plane such that $R \cap B = \emptyset$, let $\mathcal{M} = \{(a_i, b_i) : i \in \{1, \dots, n\}\}$ be any maximum-sum matching of $R \cup B$. Let $o$ be the midpoint of the shortest segment in $\mathcal{M}$. Then,*

$$\|a_i - o\| + \|o - b_i\| \leq \sqrt{5}\|a_i - b_i\| \quad \textit{for all } i \in \{1, \dots, n\}. \tag{3.4}$$

*Proof.* Without loss of generality, we can assume that $(a_1, b_1)$ is the shortest segment in the matching $\mathcal{M}$. Denote by $D_{a_1b_1}$ the corresponding disk. Proposition 1 implies that $D_{a_1b_1}$ intersects pairwise with every disk $D_{a_ib_i} \in D_{\mathcal{M}}$. Lemma 2 then guarantees inequality (3.4) for each $D_{a_ib_i}$. The result follows. □

Observe that the bound of $\sqrt{5}$ is tight if the point $o$ is always considered as the midpoint of the shortest segment in the matching. In particular, consider 2 red points and 2 blue points as vertices of a square, such that diagonal-opposed vertices have the same color. Without loss of generality, assume that $o$ is the midpoint of the segment $(a_1, b_1)$. Then, $\|a_2 - o\| = \sqrt{5}\|a_1 - o\| = \sqrt{5}\|o - b_1\| = \|o - b_2\|$, so equality in (3.4) holds.

# Chapter 4

# Monochromatic circle-matchings

In the previous chapter, we focused our analysis to disks in $D_{\mathcal{M}}$ for when $\mathcal{M}$ is a bi-chromatic maximum-sum matching, which led us to establish some results related to Fingerhut's conjecture, improving Eppstein's observation stated in Observation 1. The following question arises naturally: what happens if we drop the bi-chromatic condition? That is, instead of two disjoint planar $n$-point sets $R$ and $B$, let us consider any point set $P$ of $2n$ points.

Thus, this chapter is devoted to prove that there is a common intersection property of three or more disks in $D_{\mathcal{M}}$ when $\mathcal{M}$ is a perfect monochromatic matching that maximizes $\mathrm{cost}(\mathcal{M})$. Later, we apply such property to further improve the constant in Theorem 3, and therefore getting an approximation closer to Fingerhut's conjecture.

## 4.1 Common intersection property of disks in $D_{\mathcal{M}}$

We start by noting that $P$ might contain different points with the same coordinates. Moreover, the common intersection of all disks in $D_{\mathcal{M}}$ might be a singleton. For example, consider six points $a, b, c, a', b'$, and $c'$, where $a, b$, and $c$ are the vertices of a non-empty triangle, and $a', b'$, and $c'$ coincide with a point $z$ in the interior of $\Delta abc$. By the triangle inequality, $\{(a, a'), (b, b'), (c, c')\}$ is a maximum-sum matching, and $z$ is the only point in the common intersection $D_{aa'} \cap D_{bb'} \cap D_{cc'}$.

Recall from Chapter 2 that if $p, q, r$, and $s$ are four points of the plane, we say that $\vec{pq}$ *points to* $rs$ if $q$ is in the interior of $\Delta prs \cap D_{rs}$. Refer to Figure 4.1 (left), where segment $\vec{cd}$ points to $ab$.

We already know by Proposition 1 that if $\mathcal{M}$ is a maximum-sum bi-chromatic matching, then the disks of $D_{\mathcal{M}}$ intersect pairwise, so in a monochromatic setting those disks still satisfy such property. With that in mind, one might wonder about how the segments that arise from these four points interact with each other. It is clear that the four vertices of any two segments cannot be in convex position, otherwise the matching

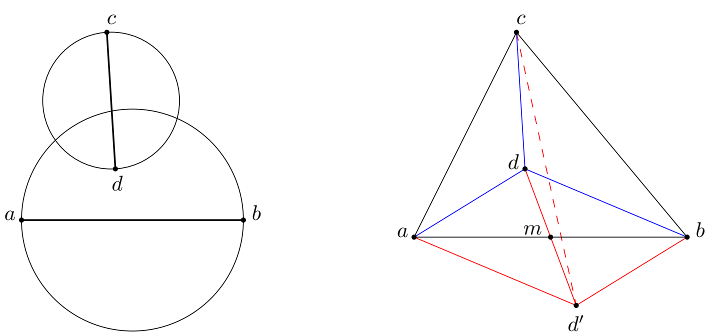



FIGURE 4.1: Proof of Lemma 2.

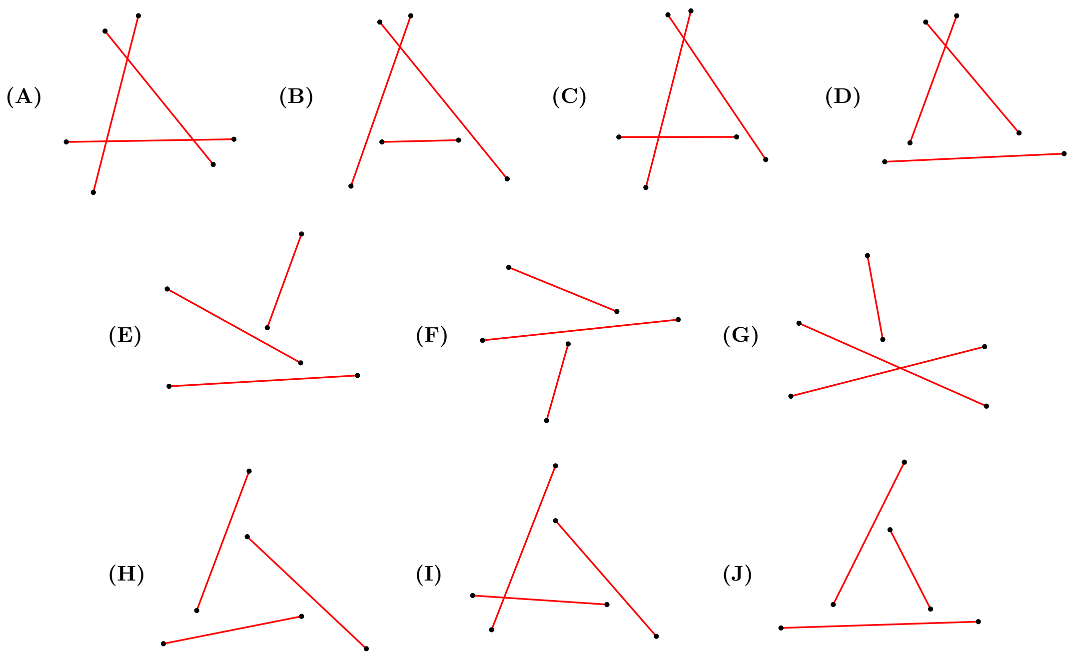



FIGURE 4.2: The ten different relative positions of three segments.

would not be a maximum-sum one by the triangle inequality. Hence, the segments of $\mathcal{M}$ either cross or one oriented segment points to the other, as we now show.

**Lemma 2.** *Let $\{a, b, c, d\}$ be a set of four points such that $\{(a, b), (c, d)\}$ is a maximum-sum matching of $\{a, b, c, d\}$ and $d$ belongs to the interior of $\Delta abc$. Then, $d$ belongs to the interior of disk $D_{ab}$. In other words, $\vec{cd}$ points to $ab$.*

*Proof.* Let $d'$ be the reflection of $d$ about the midpoint $m = (a + b)/2$ of segment $ab$, and assume without loss of generality that $d$ belongs to triangle $\Delta acd'$. Note that $m$ is also the center of $D_{ab}$. Refer to Figure 4.1. The perimeter of triangle $\Delta dcd'$ is smaller than the perimeter of triangle $\Delta acd'$, hence

$$\|c - d\| + \|d - d'\| < \|c - a\| + \|a - d'\|.$$

On one hand, since the quadrilateral with vertex set $\{a, d', b, d\}$ is a parallelogram, we have that $\|a - d'\| = \|d - b\|$. On the other hand, given that $\{(a, b), (c, d)\}$ is a maximum-sum matching, we have $\|c - a\| + \|d - b\| \le \|a - b\| + \|c - d\|$. Thus, the inequality above can be extended to

$$\begin{array}{rcl} \|c - d\| + \|d - d'\| & < & \|c - a\| + \|a - d'\| \\ & = & \|c - a\| + \|d - b\| \\ & \le & \|a - b\| + \|c - d\|, \end{array}$$

which in turn implies that $\|d - d'\| < \|a - b\|$. In other words, $\|d - m\| = \|d - d'\|/2$ is smaller than the radius $\|a - m\| = \|a - b\|/2$ of $D_{ab}$, whose center is $m$. Therefore, $D_{dd'} \subsetneq D_{ab}$, and the result follows. □

To prove that three disks intersect, we must examine the possible configurations of matchings of three segments. As we proved above, every pair of segments of a maximum-sum matching either cross or one of them points to the other one; hence, we can distinguish ten cases of relative position of the three segments, as shown in Figure

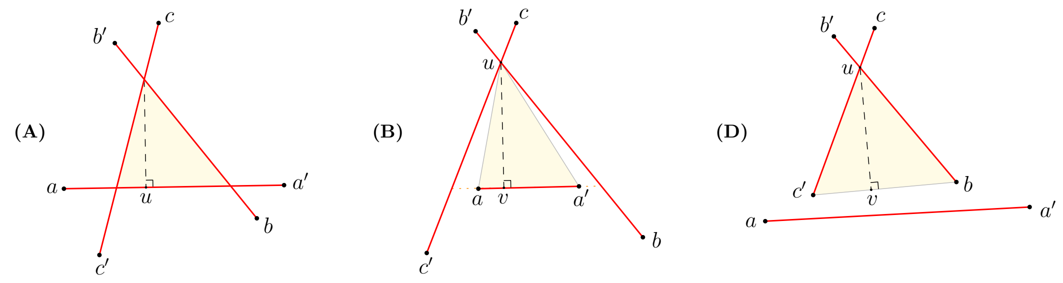



FIGURE 4.3: Proof of Lemma 3.

4.2, enumerated from **(A)** to **(J)**. In the rest of this chapter, we devote ourselves to prove that in every case the three disks have a common point.

Proofs for cases **(A)** to **(G)** are somewhat similar and most of them rely directly on Thales's theorem, so we present them together under the following lemma.

**Lemma 3.** *If the segments of a maximum-sum matching of six points fall in one of the cases from* **(A)** *to* **(G)***, then the three disks of said matching have a common intersection.*

*Proof.* Let $\{a, b, c, a', b', c'\}$ be a six point set, and let $\mathcal{M} = \{(a, a'), (b, b'), (c, c')\}$ be a maximum-sum matching. In any case, refer to Figure 4.3 for the location of each point.

Case **(A)**: Note that at least one altitude of the triangle $T$ bounded by the three segments goes through the interior of $T$. Let $u$ be the vertex of such an altitude in a side of $T$. By Thales's theorem, each of the three disks $D_{aa'}, D_{bb'}$ and $D_{cc'}$ must contain $u$. Hence, $u \in D_{aa'} \cap D_{bb'} \cap D_{cc'}$.

Case **(B)**: Let $u$ be the intersection point between $bb'$ and $cc'$. If $a \in D_{bb'}$, then we are done since Lemma 2 implies $a \in D_{cc'}$ because $\vec{a'a}$ points to $cc'$. Similarly, if $a' \in D_{cc'}$, then we are done since $a' \in D_{bb'}$ because $\vec{aa'}$ points to $bb'$. Suppose then that $a \notin D_{bb'}$ and $a' \notin D_{cc'}$: in that case, the triangle $\Delta aa'u$ is such that the interior angles at $a$ and $a'$, respectively, are both acute. Hence, the altitude $h$ from vertex $u$ goes through the interior of $\Delta aa'u$; let $v \in aa'$ be the other vertex of $h$. Since $\vec{a'a}$ points to $cc'$ and $\vec{aa'}$ points to $bb'$, Thales's theorem implies that each of the disks $D_{aa'}, D_{bb'}$ and $D_{cc'}$ contains $v$.

Case **(C)**: Let $u$ be the intersection point between $bb'$ and $cc'$, and $w$ be the common point between $aa'$ and $cc'$. If $a \in D_{bb'}$, then we are done since $w$ would be in $D_{bb'}$ too. If $a' \in D_{cc'}$, then again Lemma 2 implies $a' \in D_{bb'}$ because $\vec{aa'}$ points to $bb'$. Suppose then that $a \notin D_{bb'}$ and $a' \notin D_{cc'}$: in that case, the triangle $\Delta wa'u$ is such that the interior angles at $w$ and $a'$, respectively, are both acute. The proof then follows as in case **(B)**.

Case **(D)**: Again, let $u$ be the intersection point between $bb'$ and $cc'$. If $c' \in D_{bb'}$, then by Lemma 2 $c' \in D_{aa'}$ because $\vec{cc'}$ points to $aa'$. Similarly, if $b \in D_{cc'}$, then we are done since $b \in D_{aa'}$ because $\vec{b'b}$ points to $aa'$. Suppose then that $c' \notin D_{bb'}$ and $b \notin D_{cc'}$: then the triangle $\Delta c'bu$ is such that the interior angles at $c'$ and $b$, respectively, are both acute. Hence, the altitude $h$ from vertex $u$ goes through the interior of $\Delta c'bu$; let $v \in c'b$ be the other vertex of $h$. By Thales's theorem, we have that $v \in D_{bb'} \cap D_{cc'}$. Furthermore, since Lemma 2 implies that $c', b \in D_{aa'}$, we have in particular that the segment $c'b$ is entirely contained in $D_{aa'}$. Therefore, $v \in D_{aa'} \cap D_{bb'} \cap D_{cc'}$.

Cases **(E)**, **(F)**, and **(G)**: In each of these cases, one of the oriented segments points to the other two ones. Namely, segment $\vec{aa'}$ points to both $bb'$ and $cc'$. By Lemma 2, $a' \in D_{bb'} \cap D_{cc'}$, so in particular $a' \in D_{aa'} \cap D_{bb'} \cap D_{cc'}$.

The proof is complete. □

Interestingly enough, proofs for cases **(H)** to **(J)** are not as direct as the previous ones. Instead, we will need several technical lemmas regarding each of the remaining cases, and then we will prove by contradiction that the disks of $D_{\mathcal{M}}$ must intersect. The first and most crucial observation (one which the proofs by contradiction will depend on) is that if $\mathcal{M}$ is a maximum-sum matching, then extending one of the segments by moving one of the points results in a maximum-sum matching of the resulting point set.

**Lemma 4.** *Let $\mathcal{M} = \{(a_i, b_i) : i = 1, \dots, n\}$ denote a maximum-sum matching of the set $P$ of $2n$ uncolored points, and let $c \notin P$ be a point such that $b_1$ belongs to the interior of the segment $a_1c$. Then, $\mathcal{M}^* = (\mathcal{M} \setminus \{a_1, b_1\}) \cup \{a_1, c\}$ is a maximum-sum matching of $(P \setminus \{b_1\}) \cup \{c\}$.*

*Proof.* Let $\mathcal{M}'$ be any matching of $(P \setminus \{b_1\}) \cup \{c\}$, and note that the matching $(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}) \cup \{(b_1, \mathcal{M}'(c)\}$ is also a matching of $P$. Then,

$$\begin{aligned}
\text{cost}(\mathcal{M}') &= \text{cost}(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}) + \|\mathcal{M}'(c) - c\| \\
&\leq \text{cost}(\mathcal{M}' \setminus \{(c, \mathcal{M}'(c)\})) + \|\mathcal{M}'(c) - b_1\| + \|b_1 - c\| \\
&= \text{cost}((\mathcal{M}' \setminus \{(c, \mathcal{M}'(c))\}) \cup \{(b_1, \mathcal{M}'(c))\}) + \|b_1 - c\| \\
&\leq \text{cost}(\mathcal{M}) + \|b_1 - c\| \\
&= \text{cost}(\mathcal{M}^*).
\end{aligned}$$

Hence, the lemma follows. □

Since in cases **(H)** to **(J)** there is at least one segment pointing to another, let us prove some technical facts regarding that situation. Lemma 5 deals with the non crossing segments case and will be referred often given the cases we will deal with, while Lemma 6 deals with the crossing segments one.

**Lemma 5.** *Let $p, p', q$, and $q'$ be four points such that $\vec{pp'}$ points to $qq'$, and $q$ is to the right of $\ell(p, p')$. Let $z$ be a point to the left of both $\ell(p, p')$ and $\ell(q, q')$ such that: (i) $q$ is to the left of $\ell(z, p)$; (ii) vectors $p - z$ and $p' - z$ are orthogonal, and (iii) vectors $q - z$ and $q' - z$ are also orthogonal. Refer to Figure 4.4(A). Then, we have that*

$$\|p - z\| - \|q - z\| < \|p - q'\| - \|q - q'\|. \tag{4.1}$$

*Proof.* Rearranging terms in equation (4.1), we only need to prove that

$$\|p - z\| + \|q - q'\| < \|p - q'\| + \|q - z\|. \tag{4.2}$$

Note that conditions (*i*), (*ii*) and (*iii*), the fact that $\vec{pp'}$ points to $qq'$, and the location of $z$, imply that $q$ is to the right of $\ell(z, p')$ if and only if segments $pq'$ and $qz$ have a common point.

Suppose that $q$ is to the right of $\ell(z, p')$ (see Figure 4.4(B)), that is, the case where segments $pq'$ and $qz$ have a common point. Then, points are $p, q, q'$, and $z$ are the vertices of a convex quadrilateral with non-empty interior and diagonals $pq'$ and $zq$. Hence, by triangle inequality

$$\|p - z\| + \|q - q'\| < \|p - q'\| + \|q - z\|.$$

Suppose now that $q$ is not to the right of $\ell(z, p')$ (see Figure 4.4(C)), then segments $pq'$ and $qz$ do not intersect. Let $z'$ be the reflection of $z$ about the center of segment $qq'$, also the center of $D_{qq'}$. On one hand, Thales's theorem and conditions (*ii*) and (*iii*) implies $\|q - z\| = \|q' - z'\|$ and $\|q - q'\| = \|z - z'\|$. On the other hand, since $\vec{pp'}$ points to $qq'$ we have $p' \in D_{qq'}$, which in turn implies that $z'$ must be to the right of line $\ell(p, z)$. Then, since $pq'$ and $qz$ do not intersect, it must be that $z$ belongs to triangle

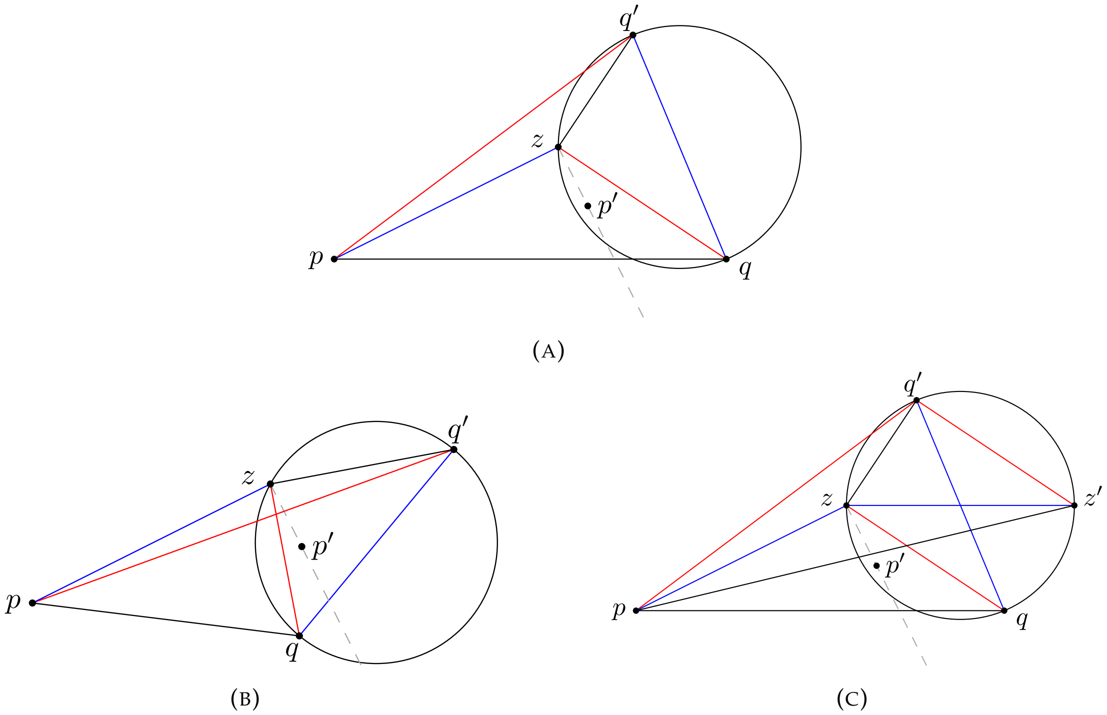



FIGURE 4.4: (A) Statement of Lemma 5. (B) - (C) Proof of Lemma 5.

$\Delta pz'q'$. Thus

$$\|p - z\| + \|q - q'\| = \|p - z\| + \|z - z'\| < \|p - q'\| + \|q' - z'\| = \|p - q'\| + \|q - z\|.$$

The result follows. □

**Lemma 6.** *Let $p, p', q$, and $q'$ be four points in convex position such that $q$ and $q'$ are to the right and left of line $\ell(p, p')$, respectively. Let $z$ be a point to the left of both $\ell(p, p')$ and $\ell(q, q')$ such that: vectors $p - z$ and $p' - z$ are orthogonal, and vectors $q - z$ and $q' - z$ are also orthogonal. Then, as in Lemma 5, we again have that $\|p - z\| - \|q - z\| < \|p - q'\| - \|q - q'\|$.*

*Proof.* Note that, in this case, segments $pq'$ and $qz$ have a common point. The proof then continues as that of the first case of Lemma 5 by triangle inequality. □

We now proceed to generalize both previous lemmas to segment configurations related to cases **(H)** (Lemma 7) and **(I)** (Lemma 8), which in turn will be useful in our proof by contradiction mentioned earlier.

**Lemma 7.** *Let $a, b, c, a', b', c'$, and $z$ be seven points such that: $c$ is to the left of line $\ell(a, b)$; segments $\vec{aa'}, \vec{bb'}$, and $\vec{cc'}$ point to $bb', cc'$ and $aa'$, respectively; and for each $u \in \{a, b, c\}$, point $z$ is to the left of line $\ell(u, u')$, and vectors $u - z$ and $u' - z$ are orthogonal. Refer to Figure 4.5(A). Then, $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching of point set $\{a, b, c, a', b', c'\}$.*

*Proof.* Observe that the conditions of the lemma ensure (three times) the conditions of Lemma 5. Namely, applying Lemma 5 for $a, a', b, b'$, and $z$ (where $a$ and $b$ play the role of $p$ and $q$, respectively) we obtain

$$\|a - z\| - \|b - z\| < \|a - b'\| - \|b - b'\|;$$

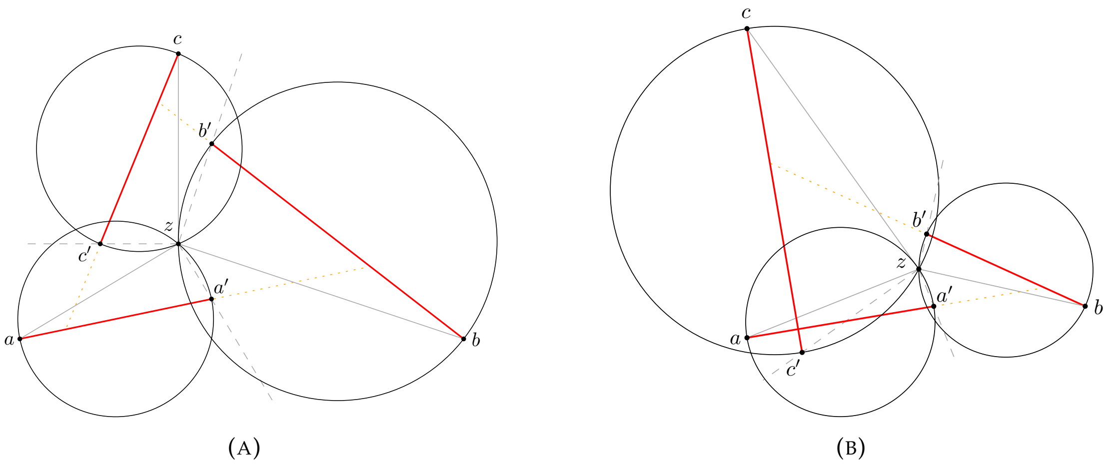



FIGURE 4.5: (A) Statement of Lemma 7. (B) Statement of Lemma 8.

for $b, b', c, c'$, and $z$ (where $b$ and $c$ play the role of $p$ and $q$, respectively) we obtain

$$\|b - z\| - \|c - z\| < \|b - c'\| - \|c - c'\|;$$

and for $c, c', a, a'$, and $z$ (where $c$ and $a$ play the role of $p$ and $q$, respectively) we obtain

$$\|c - z\| - \|a - z\| < \|c - a'\| - \|a - a'\|.$$

Adding the above three inequalities, we obtain

$$\|a - a'\| + \|b - b'\| + \|c - c'\| < \|a - b'\| + \|b - c'\| + \|c - a'\|,$$

finishing the proof. □

**Lemma 8.** *Let $a, b, c, a', b', c'$, and $z$ be seven points such that: $c$ is to the left of line $\ell(a, b)$; segments $\vec{aa'}$ and $\vec{bb'}$ points to $bb'$ and $cc'$, respectively; segments $aa'$ and $cc'$ have a common point with $a$ and $a'$ to the right and left of line $\ell(c, c')$, respectively; and for each $u \in \{a, b, c\}$, point $z$ is to the left of line $\ell(u, u')$, and vectors $u - z$ and $u' - z$ are orthogonal. Refer to Figure 4.5(B). Then, $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching of point set $\{a, b, c, a', b', c'\}$.*

*Proof.* Observe that the conditions of the lemma ensure (two times) the conditions of Lemma 5. Namely, applying Lemma 5 for $a, a', b, b'$, and $z$ (where $a$ and $b$ play the role of $p$ and $q$, respectively) we obtain

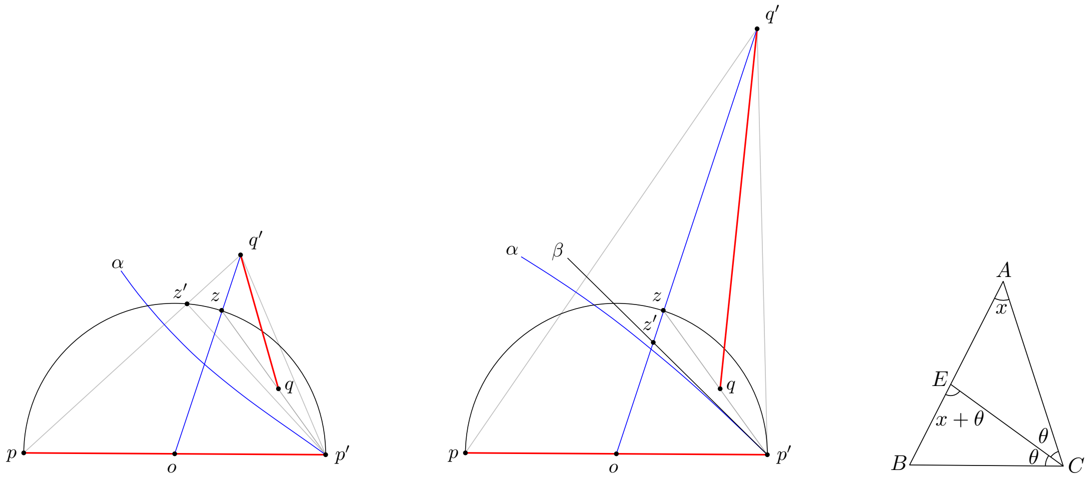
$$\|a - z\| - \|b - z\| < \|a - b'\| - \|b - b'\|;$$

for $b, b', c, c'$, and $z$ (where $b$ and $c$ play the role of $p$ and $q$, respectively) we obtain

$$\|b - z\| - \|c - z\| < \|b - c'\| - \|c - c'\|.$$

They also guarantee the conditions of Lemma 6, that is, applying Lemma 6 for $c, c', a, a'$, and $z$ (where $c$ and $a$ play the role of $p$ and $q$, respectively) we obtain

$$\|c - z\| - \|a - z\| < \|c - a'\| - \|a - a'\|.$$




(A) (B) (C)

FIGURE 4.6: Proof of Lemma 9.

Adding the above three inequalities, we obtain

$$\|a-a'\| + \|b-b'\| + \|c-c'\| < \|a-b'\| + \|b-c'\| + \|c-a'\|,$$

finishing the proof. □

Before generalizing Lemma 5 for case **(J)**, we establish a couple of technical facts related to that segment configuration. Recall that $H(\alpha, p)$ denotes the open region, between the two arisen after splitting the plane by some curve $\alpha$, that contains $p$ (not in $\alpha$).

**Proposition 2.** *Let $a, b, a'$, and $b'$ be four points such that $\{(a,a'),(b,b')\}$ is a maximum-sum matching of $\{a,b,a',b'\}$. Let $\alpha$ be the arc of the hyperbola with foci $a$ and $b$ that goes through $b'$. Then, we have that $a' \in \alpha \cup H(\alpha, b)$.*

*Proof.* Note that, by construction of the given hyperbola using circular directrices, the arc $\alpha$ is the locus of the points $x$ of the plane such that $\|a-x\| + \|b-x\| = \|a-b'\| - \|b-b'\|$. Since $\{(a,a'),(b,b')\}$ is a maximum-sum matching, we have that $\|a-b'\| + \|b-a'\| \leq \|a-a'\| + \|b-b'\|$; that is,

$$\|a-b'\| - \|b-b'\| \leq \|a-a'\| - \|b-a'\|,$$

which in turn implies the proposition. □

**Lemma 9.** *Let $p, p'$, and $o$ be three points such that $o$ is the midpoint of segment $pp'$. Let $z$ be a point of the circle $C_{pp'}$ to the left of line $\ell(p,p')$, $q$ a point of segment $zp'$ with $q \neq p'$, and $q'$ a point of ray $\tau(o,z)$ not in segment $oz$. Then, $\|p-p'\| + \|q-q'\| < \|p-q\| + \|p'-q'\|$.*

*Proof.* We have two cases: $\|p-p'\| \geq \|q'-p'\|$; and $\|p-p'\| < \|q'-p'\|$. In both cases, $\alpha$ will denote the arc of the hyperbola with foci $p$ and $q'$ and goes through $p'$.

In the first case (see Figure 4.6(A)), let $z'$ be the intersection point of $C_{pp'}$ and $pq'$. Since $\|p-p'\| \geq \|q'-p'\|$, we have that the region $H(\alpha, q')$ is convex. Furthermore, line $\ell(p', z')$ is perpendicular to the line $\ell(p, q')$ through the foci of $\alpha$, so $z' \in H(\alpha, q')$, which in turn implies that $z \in H(\ell(p', z'), q') \cap H(\alpha, q')$, a convex region. Now, $z$ belongs in particular to the interior of $\alpha \cup H(\alpha, q')$, $p'$ is on the boundary of $\alpha \cup H(\alpha, q')$, and $q \in zp'$ with $q \neq p'$; therefore, we also have that $q \in H(\alpha, q')$, which is equivalent

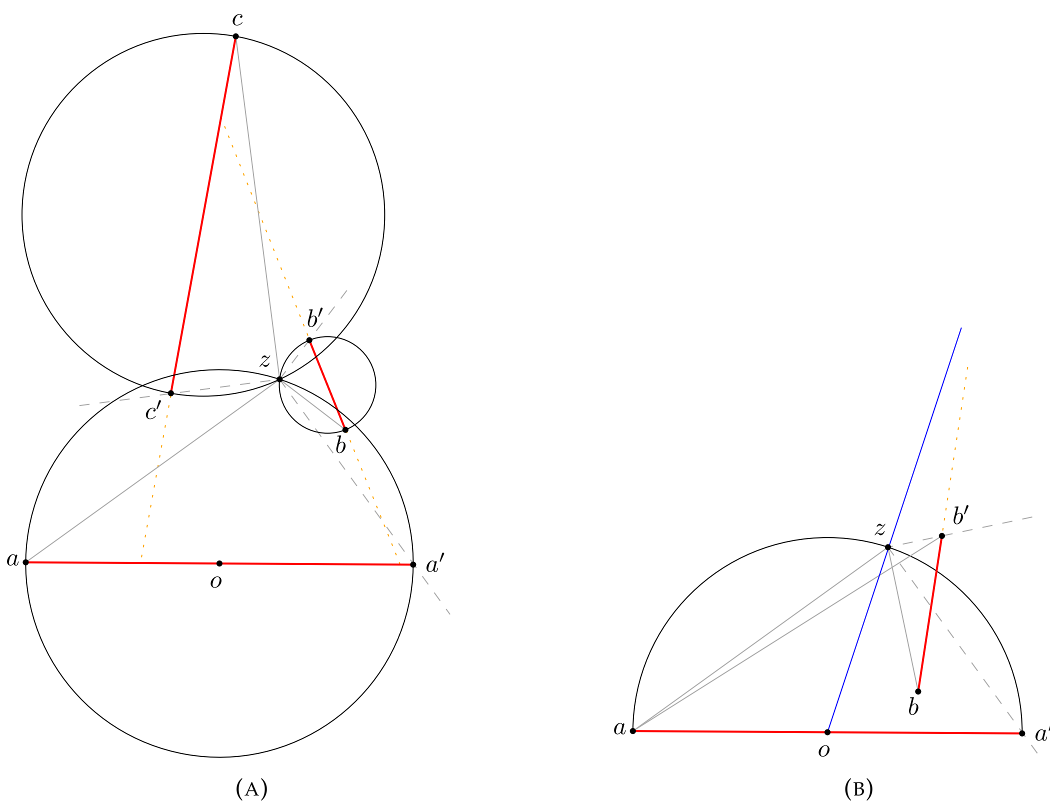



FIGURE 4.7: Proof of Lemma 10.

to saying that $q \notin \alpha \cup H(\alpha, p)$. By Proposition 2, this last fact implies $\|q' - q\| - \|p - q\| < \|q' - p'\| - \|p - p'\|$ (where $q, q'$ and $p$ play the role of $a', a$ and $b$, respectively). The result follows.

Consider now the second case, $\|p - p'\| < \|q' - p'\|$ (see Figure 4.6(B)). Let $\beta$ be the bisector of the interior angle at $p'$ of triangle $\Delta op'q'$. Then, by geometric properties of hyperbolas, $\beta$ is the tangent of $\alpha$ at $p'$. Furthermore, since $\|p - p'\| < \|q' - p'\|$, it holds that $\beta$ separates $\alpha$ from vertex $q'$. Let $z'$ be the intersection point between $oq'$ and $\beta$.

**Claim.** *If $A, B$, and $C$ are the vertices of a triangle, and point $E$ belongs to side $AB$, such that the line $\ell(C, E)$ is the bisector of the interior angle at $C$, then $\|C - B\| > \|B - E\|$.*

The claim follows from the fact that given any triangle, precisely in $\Delta BCE$, larger sides correspond to larger opposed interior angles (see Figure 4.6(C)). Applying the claim to $\Delta op'q'$, we have that $\|o - z'\| < \|o - p'\|$, which implies that $\beta$ also separates point $z$ and arc $\alpha$. Since $q \in zp' \setminus \{p'\}$, it also holds that $\beta$ separates point $q$ and arc $\alpha$. Thus, $q$ is to the left of $\alpha$ in the direction from $q'$ to $p$, that is, $q \notin \alpha \cup H(\alpha, p)$. Proposition 2 then implies that $\|q - q'\| - \|p - q\| < \|q' - p'\| - \|p - p'\|$, which in turn implies the result. $\square$

We are now ready to generalize Lemma 5 for case **(J)** as proposed above.

**Lemma 10.** *Let $a, b, c, a', b', c'$, and $z$ be seven points such that: none of them is to the right of line $\ell(a, a')$; segments $\vec{b'b}, \vec{bb'}$, and $\vec{cc'}$ point to $aa', cc'$ and $aa'$, respectively; and for each $u \in \{a, b, c\}$, point $z$ is to the left of line $\ell(u, u')$, and vectors $u - z$ and $u' - z$ are orthogonal. Refer to Figure 4.7(A). Then, $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching of $\{a, b, c, a', b', c'\}$.*

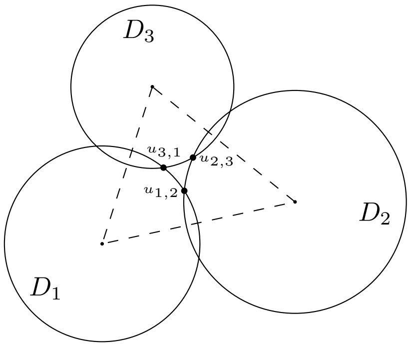



FIGURE 4.8: Proof of Lemma 11.

*Proof.* Applying Lemma 5 for $b, b', c, c'$, and $z$ (where $b$ and $c$ play the role of $p$ and $q$, respectively), we obtain

$$\|b - z\| - \|c - z\| < \|b - c'\| - \|c - c'\|;$$

and for $c, c', a, a'$, and $z$ (where $c$ and $a$ play the role of $p$ and $q$, respectively), we obtain

$$\|c - z\| - \|a - z\| < \|c - a'\| - \|a - a'\|.$$

Now, let $o$ be the midpoint of segment $aa'$, also the center of $D_{aa'}$. Since $z - c$ and $z - c$ are orthogonal, and $cc'$ points to $aa'$, we have that $c$ is to the left of line $\ell(o, z)$. Similarly, given that $z - b$ and $z - b'$ are orthogonal, and $\vec{bb'}$ points to $aa'$, it follows that $b'$ is to the right of line $\ell(o, z)$. Since $\vec{bb'}$ also points to $cc'$, we then have that rays $\tau(b, b')$ and $\tau(o, z)$ must intersect.

Suppose that $b \in \Delta aa'z$ (see Figure 4.7(B)). Since $z - b$ and $z - b'$ are orthogonal, and $\tau(b, b') \cap \tau(o, z) \neq \emptyset$, it holds that segments $bz$ and $ab'$ have a common point. Hence, triangle inequality implies $\|a - z\| + \|b - b'\| < \|b - z\| + \|a - b'\|$. In other words,

$$\|a - z\| - \|b - z\| < \|a - b'\| - \|b - b'\|.$$

Adding the three inequalities above, we obtain

$$\|a - a'\| + \|b - b'\| + \|c - c'\| < \|a - b'\| + \|b - c'\| + \|c - a'\|,$$

implying the result.

Suppose now that $b \notin \Delta aa'z$ (see Figure 4.7(A)). Let us assume by contradiction that the matching $\{(a, a'), (b, b'), (c, c')\}$ is in fact a maximum-sum matching. Then, the matching $\{(a, a'), (b, b')\}$ is also a maximum-sum matching of $\{a, a', b, b'\}$. Let $w$ and $w'$ be the intersection points of $\ell(b, b')$ with $\tau(o, z)$ and $\tau(z, a')$, respectively. Clearly $bb' \subset ww'$, and Lemma 4 implies that $\{(a, a'), (w, w')\}$ is a maximum-sum matching of $\{a, a', w, w'\}$. However, applying Lemma 9 for $a, a', w, w'$ (where $a$ and $w$ play the role of $p$ and $q$, respectively) implies that $\{(a, a'), (w, w')\}$ is not a maximum-sum matching, a contradiction. Therefore, $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching, as we wanted to show. □

We can finally prove the common intersection property for cases **(H)** to **(J)**.

**Lemma 11.** *If the segments of a maximum-sum matching of six points fall in one of the cases from* **(H)** *to* **(J)***, then the three disks of the matching have a common intersection.*

*Proof.* Suppose by contradiction that the three disks, denoted $D_1, D_2$, and $D_3$, intersect pairwise, but without a common intersection (see Figure 4.8). Let $u_{1,2}, u_{2,3}$, and $u_{3,1}$ be the vertices of the pairwise disjoint lenses $D_1 \cap D_2, D_2 \cap D_3$, and $D_3 \cap D_1$, respectively, located inside the triangle with vertices at the centers of $D_1, D_2$, and $D_3$, respectively.

The idea is to use Lemma 4, combined with Lemmas 7, 8, and 10, such that the point $z$ of such lemmas is one of $u_{1,2}, u_{2,3}$, and $u_{3,1}$. Therefore, we need to guarantee that point $z$ is not an extreme point of some segment of the matching at play, as we show below.

It is clear that two vertices among $u_{1,2}, u_{2,3}$, and $u_{3,1}$ cannot be extreme points of a same segment of the matching or else there would be one of them belonging to the three disks, which cannot be. Furthermore, if each of the three vertices is an extreme point of some segment of the matching, then at least one pair of disjoint segments would violate Lemma 2. That is, the extreme point of one segment, in the interior of the convex hull of the four involved points, is not in the interior of the disk corresponding to the other segment. Hence, we can assume that at least one vertex among $u_{1,2}, u_{2,3}$, and $u_{3,1}$ is not an extreme point of a segment of the matching: say vertex $u_{1,2}$. This in turn implies that we can extend the segment of disk $D_3$ by moving one of it extreme points such that the new three corresponding disks have a singleton common intersection at $u_{1,2}$. Let $z = u_{1,2}$, where $z$ is distinct from all the new six points.

Denote the six new points by $a, b, c, a', b'$, and $c'$ in such a way that the new segments are precisely $aa'$, $bb'$, and $cc'$, and for each $u \in \{a, b, c\}$ point $z$ is to the left of line $\ell(u, u')$. Consequently, Lemma 4 implies that $\{(a, a'), (b, b'), (c, c')\}$ is a maximum-sum matching of $\{a, b, c, a', b', c'\}$.

Now, it is important to remark the following:

- If the original segments are in case **(H)**, then by extending one segment we can either stay in case **(H)**, or change into case **(I)**.
- If the original segments are in case **(I)**, then by extending one segment we can either stay in case **(I)**, or change into case **(C)** with a non-singleton common intersection of the three disks by Lemma 3.
- If the original segments are in case **(J)**, then by extending one segment we can either stay in case **(J)**, or change into case **(B)** or **(D)** with a non-singleton common intersection of the three disks by Lemma 3.

Hence, since we extended the original matching in such a manner that the common intersection of the new three disks $D_{aa'}, D_{bb'}$, and $D_{cc'}$ is singleton, we can ensure that the new segments $aa', bb'$, and $cc'$ are again in a case from **(H)** to **(J)**, and the proof continues as follows.

If $aa', bb'$, and $cc'$ fall in case **(H)**, then Lemma 7 implies that $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching. If the segments fall in case **(I)**, then Lemma 8 implies that $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching. Otherwise, if they fall in case **(J)**, then Lemma 10 implies that $\{(a, a'), (b, b'), (c, c')\}$ is not a maximum-sum matching. In each of the cases there exists a contradiction, so the original three disks must have a common intersection, proving the result. □

We finish this section by generalizing this common intersection property for any $n \geq 1$.

**Theorem 4.** *Let $P$ be a set of $2n$ (uncolored) points in the plane, $n \geq 1$. Any maximum-sum matching $\mathcal{M}$ of $P$ is such that all disks of $D_{\mathcal{M}}$ have a common intersection.*

*Proof.* If $n = 1$ there is nothing to prove. Lemma 2 proves the case $n = 2$. Lemmas 3 and 11 prove the result for $n = 3$. Helly's theorem then implies the result for $n \geq 3$. □

## 4.2 On a monochromatic Fingerhut's conjecture

While not proving it thoroughly, Eppstein [18] did note that the bound in Observation 1 could be improved to a factor of 2.5 as follows: let $\mathcal{M} = \{(a_i, b_i) : i \in \{1, \dots, n\}\}$ be the maximum-sum monochromatic matching. Let $(a_1, b_1)$ the shortest edge in $\mathcal{M}$, and let $x$ be the midpoint of $a_1b_1$. Suppose by contradiction that

$$\|a_i - x\| + \|x - b_i\| > 2.5\|a_i - b_i\| \quad \text{for some } i \in \{1, \dots, n\}.$$

Say, $i = 2$. Note that, without loss of generality, the matching defined by $\mathcal{M}' = \{(a_1, b_2), (a_2, b_1), (a_3, b_3), \dots, (a_n, b_n)\}$ is such that $\text{cost}(\mathcal{M}') < \text{cost}(\mathcal{M})$. Furthermore, one of $a_1, b_1$ is always farther from $a_2$ or $b_2$ than $x$; say $\|b_1 - a_2\| > \|a_2 - x\|$. Therefore, just as in Observation 1, we have that

$$\begin{aligned}
cost(\mathcal{M}') &= \|a_1 - b_2\| + \|a_2 - b_1\| + \|a_3 - b_3\| + \cdots + \|a_n - b_n\| \\
&= \|a_1 - b_2\| + \|a_2 - b_1\| + \|a_3 - b_3\| + \\
&\quad\ \|a_1 - x\| - 0.5\|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&> \|x - b_2\| + \|a_2 - x\| + \|a_3 - b_3\| - 0.5\|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&> 2.5\|a_2 - b_2\| + \|a_3 - b_3\| - 0.5\|a_1 - b_1\| + \cdots + \|a_n - b_n\| \\
&> 2\|a_2 - b_2\| + \|a_3 - b_3\| + \cdots + \|a_n - b_n\| \\
&> \|a_1 - b_1\| + \|a_2 - b_2\| + \|a_3 - b_3\| + \cdots + \|a_n - b_n\| \\
&= \text{cost}(\mathcal{M}),
\end{aligned}$$

which cannot be. Thus, we have the following

**Observation 2.** *For any $n \geq 1$, and a point set $P$ of $2n$ (uncolored) points, let $\mathcal{M} = \{(a_i, b_i) : i \in \{1, \dots, n\}\}$ be any maximum-sum matching of $P$. There exists a point $x$ in the plane such that*

$$\|a_i - x\| + \|x - b_i\| \leq 2.5\|a_i - b_i\| \quad \text{for all } i \in \{1, \dots, n\}. \tag{4.3}$$

However, Theorem 4 allows us to greatly improve the constant in inequality (4.3) to $\sqrt{2} \approx 1.4142$, hence refining the overall results on Fingerhut's conjecture presented so far.

**Theorem 5.** *Let $P$ be a set of $2n$ (uncolored) points in the plane, and let $\{(a_i, b_i) : i \in \{1, \dots, n\}\}$ be a maximum-sum matching of $P$. There exists a point $o$ of the plane such that for all $i \in \{1, \dots, n\}$ we have*

$$\|a_i - o\| + \|o - b_i\| \leq \sqrt{2}\|a_i - b_i\|.$$

*Proof.* Let $\mathcal{M}$ be a maximum-sum matching of $P$. By Theorem 4, all disks of $D_{\mathcal{M}}$ have a common intersection; hence, there exists a point $o$ in the plane such that $o \in D_{a_ib_i}$ for all $i \in \{1, \dots, n\}$. Let $x_i$ be the midpoint of each circular arc $a_ib_i$ that goes through the common intersection. Then, it holds

$$\|a_i - o\| + \|o - b_i\| \leq \|a_i - x_i\| + \|x_i - b_i\| \leq \sqrt{2}\|a_i - b_i\|$$

for all $i \in \{1, \dots, n\}$. The result follows. □

# Chapter 5

# Monochromatic ellipse-matchings

As mentioned in the Introduction, Fingerhut noted that inequality (1.1) is the set of all such points $x$ that lie inside an ellipse with foci $a_i$ and $b_i$, for all $i \in \{1, \ldots, n\}$. Thus, the conjecture says that the three corresponding ellipses have a common intersection; therefore, applying Helly's theorem, all the ellipses $E_{a_i b_i}$ intersect in a common region of the plane.

In this chapter, we focus on showing that the ellipses do intersect pairwise. The reader will eventually realize that this is no trivial task; in fact, we will establish a couple of very technical lemmas to accomplish our goal.

## 5.1 Common intersection property of disks in $E_{\mathcal{M}}$

We first note that we will be dealing with a maximum-sum matching of four points, say $a, b, c, d$. It is straightforward to prove that if those four points are in convex position, then the ellipses clearly intersect; hence, we direct our efforts towards the case in which one of the segments points to the other, say $\vec{cd}$ points to $ab$, as in Figure 5.1 (A). In this case, finding a common point between the ellipses can then be interpreted as finding a common detour point $x$ with respect to $ab$ and $cd$. Now, Veldkamp introduced the equal detour point for any triangle [25], which is equivalent to the inner Soddy center $s$. Therefore, given $\Delta abc$ with $d$ an interior point, it is natural to ask how can we relate the detour from $a$ to $b$ via $d$ in contrast to taking a detour through $s$. Lemma 12 shows that the inner Soddy center is an upper bound for detour points in such maximum-sum matching configurations.

**Lemma 12.** *Let $a, b, c$, and $d$ be four points such that $d \in \Delta abc$ and $\{(a,b),(c,d)\}$ is a maximum-sum matching of $\{a,b,c,d\}$. Then, we have that*

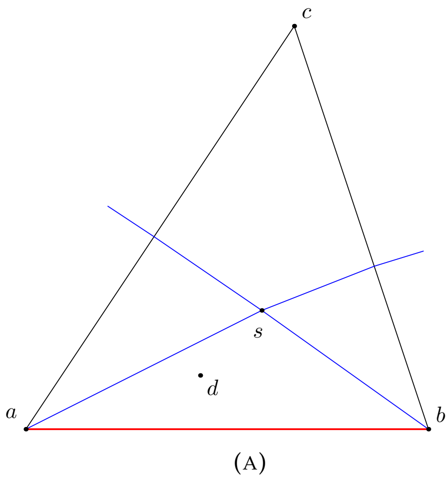
$$\|a - d\| + \|b - d\| \leq \|a - s\| + \|b - s\|,$$

*where $s$ is the Soddy center of $\Delta abc$.*

*Proof.* Refer to Figure 5.1(A). Note that if $d$ belongs to $\Delta abs$, then the claim follows from the fact that the perimeter of $\Delta abd$ is at most the perimeter of $\Delta abs$. Otherwise, we can assume without loss of generality that $d$ belongs to $\Delta asc$. Since in this case the perimeter of $\Delta adc$ is at most the perimeter of $\Delta asc$, we have that

$$\|a - d\| + \|c - d\| \leq \|a - s\| + \|c - s\|.$$

Denote by $\alpha_a$ the arc of hyperbola with foci $b$ and $c$ that goes through $a$, and by $\alpha_b$ the arc of hyperbola with foci $a$ and $c$ that goes through $b$. Since $\{(a,b),(c,d)\}$ is a maximum-sum matching, Proposition 2 implies on one hand that $d \in H(\alpha_a, b)$, while



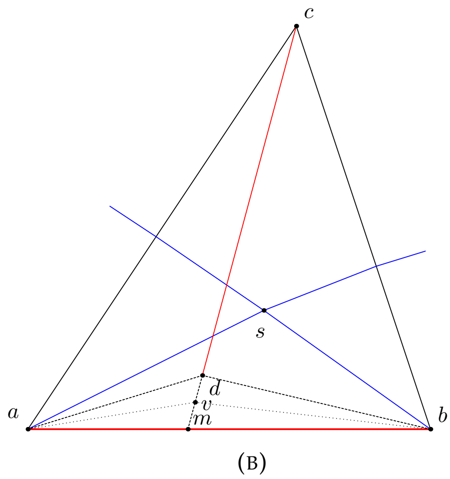




FIGURE 5.1: (A) Proof of Lemma 12. (B) Proof of Lemma 13.

on the other $d \in H(\alpha_b, a)$, therefore $d \in H(\alpha_a, b) \cap H(\alpha_b, a)$; thus, it holds that

$$\|b - d\| - \|c - d\| \leq \|b - s\| - \|c - s\|,$$

since $s$ is the intersection point between arcs $\alpha_a$ and $\alpha_b$ [27], so $s \in \alpha_a$. Adding the above inequalities gives us the inequality of the lemma. □

We now proceed to show the main result of this chapter, that is, ellipses in $E_{\mathcal{M}}$ always intersect pairwise. To this end, we approach the search for a common point between the ellipses in terms of the observation made in Lemma 12 with respect to the inner Soddy center together with Descartes' theorem applied to the mutually tangent circles that define the inner Soddy circle.

**Lemma 13.** *Let $a, b, c$, and $d$ be four points such that $d \in \Delta abc$ and $\{(a, b), (c, d)\}$ is a maximum-sum matching of $\{a, b, c, d\}$. Then, the intersection $E_{ab} \cap E_{cd}$ is not empty.*

*Proof.* Let $C_a, C_b$, and $C_c$ be three mutually exterior tangent circles centered at the vertices $a, b$, and $c$, respectively. Let $r_a, r_b$, and $r_c$ be the radii of circles $C_a, C_b$, and $C_c$, respectively. Let $r_s$ denote the radius of the inner Soddy circle, centered at $s$ (the inner Soddy center of $\Delta abc$), which is tangent to each of $C_a, C_b$, and $C_c$. By Descartes' theorem, $r_s$ satisfies

$$\frac{1}{r_s} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + 2\sqrt{\frac{1}{r_a r_b} + \frac{1}{r_b r_c} + \frac{1}{r_a r_c}}. \tag{5.1}$$

Refer to Figure 5.1(B). Let $m$ be the intersection point between side $ab$ and ray $\tau(c, d)$, and let $v$ be the vertex of $E_{cd}$ that belongs to ray $\tau(d, m)$. If $m \in dv$, then $E_{ab} \cap E_{cd}$ is clearly non-empty. Let us assume then that $v$ belongs to the interior of $dm$. To show that $v \in E_{ab}$, we need to prove that

$$\|a - v\| + \|b - v\| \leq \frac{2}{\sqrt{3}}\|a - b\| = \frac{2}{\sqrt{3}}(r_a + r_b). \tag{5.2}$$

Let $\delta = \frac{1}{2}\left(\frac{2}{\sqrt{3}} - 1\right)$, and

$$\lambda = \frac{\|d-v\|}{\|d-m\|} = \frac{\delta\|c-d\|}{\|d-m\|} = \delta\left(\frac{\|c-m\|}{\|d-m\|} - 1\right),$$

for which $v = \lambda m + (1-\lambda)d$. Since the Euclidean distance is a convex function, the function $F : \ell(c,d) \to \mathbb{R}$ defined by $F(x) = \|a-x\| + \|b-x\|$ for all $x \in \ell(c,d)$ is also convex, so Jensen's inequality implies that

$$F(v) = F(\lambda m + (1-\lambda)d) \leq \lambda F(m) + (1-\lambda)F(d).$$

Since $F(m) = \|a-b\| = r_a + r_b$ and, by Lemma 12, $F(d) = \|a-d\| + \|b-d\| \leq \|a-s\| + \|b-s\| = r_a + r_b + 2r_s$, previous inequality translates to

$$\begin{aligned} \|a-v\| + \|b-v\| &\leq \lambda(r_a + r_b) + (1-\lambda)(r_a + r_b + 2r_s) \\ &\leq r_a + r_b + 2(1-\lambda)r_s. \end{aligned}$$

Then, to show inequality (5.2), it suffices to prove the inequality

$$(1-\lambda)r_s = \left(1 - \delta\left(\frac{\|c-m\|}{\|d-m\|} - 1\right)\right) r_s \leq \frac{1}{2}\left(\frac{2}{\sqrt{3}} - 1\right)(r_a + r_b) = \delta(r_a + r_b). \quad (5.3)$$

Let $h_c$ be the length of the altitude of the triangle $\Delta abc$ from vertex $c$, and let $h_d$ be the length of the altitude of the triangle $\Delta abd$ from vertex $d$. By the SAS similarity criterion, it holds that $\Delta mbc \sim \Delta mbd$; hence,

$$\frac{\|c-m\|}{\|d-m\|} = \frac{h_c}{h_d}.$$

Then, inequality (5.3) can be rewritten as

$$\left(1 - \delta\left(\frac{h_c}{h_d} - 1\right)\right) r_s \leq \delta(r_a + r_b),$$

which in turn translates to

$$\frac{1+\delta}{\delta} = 7 + 4\sqrt{3} \leq \frac{r_a + r_b}{r_s} + \frac{h_c}{h_d}. \quad (5.4)$$

By Heron's formula, we have that the area $A_c$ of triangle $\Delta abc$ satisfies

$$A_c = \sqrt{p(p - \|a-b\|)(p - \|b-c\|)(p - \|c-a\|)},$$

where $p = r_a + r_b + r_c$ is the semiperimeter of $\Delta abc$. Since

$$\|a-b\| = r_a + r_b, \qquad \|b-c\| = r_b + r_c, \qquad \|c-a\| = r_c + r_a,$$

we have that $A_c = \sqrt{r_a r_b r_c (r_a + r_b + r_c)}$, which implies that

$$h_c = \frac{2A_c}{\|a-b\|} = \frac{2\sqrt{r_a r_b r_c (r_a + r_b + r_c)}}{r_a + r_b}.$$

Similarly, we have that the area $A_d$ of $\Delta abd$ satisfies

$$A_d = \sqrt{q(q - \|a-b\|)(q - \|b-d\|)(q - \|d-a\|)},$$

where $q$ is the semi-perimeter of $\Delta abd$. Note that Lemma 12 implies that $q$ is at most the semiperimeter of $\Delta abs = r_a + r_b + r_s$. Then, on one hand we have that

$$q(q - \|a-b\|) = q(q - r_a - r_b) \leq (r_a + r_b + r_s)r_s.$$

On the other hand, it holds that

$$(q - \|b-d\|) + (q - \|d-a\|) = \|a-b\| = r_a + r_b,$$

which in turn implies that $q - \|b-d\| = r_a - t$ and $q - \|d-a\| = r_b + t$, for some $t \in \mathbb{R}$. If we consider the polynomial $(r_a - t)(r_b + t) = -t^2 + (r_a - r_b)t + r_a r_b$, then it is straightforward to check that it attains a global maximum at $t = (r_a - r_b)/2$, where it equals to $(r_a + r_b)^2/4$. Then, we that

$$(q - \|b-d\|)(q - \|d-a\|) \leq \frac{(r_a + r_b)^2}{4},$$

which allows us to obtain an upper bound for $h_d$:

$$h_d = \frac{2A_d}{\|a-b\|} \leq \frac{2\sqrt{(r_a + r_b + r_s)r_s \cdot \frac{(r_a+r_b)^2}{4}}}{r_a + r_b} = \sqrt{r_s(r_a + r_b + r_s)}.$$

Hence, to prove inequality (5.4), it now suffices to prove that

$$7 + 4\sqrt{3} \leq \frac{r_a + r_b}{r_s} + 2\sqrt{\frac{r_a r_b}{(r_a + r_b)^2} \cdot \frac{r_c(r_a + r_b + r_c)}{r_s(r_a + r_b + r_s)}}. \tag{5.5}$$

Let $z = (r_a + r_b)/r_c > 0$ and $w = \frac{r_a}{r_b} + \frac{r_b}{r_a} \geq 2$. Let $x = \sqrt{z+1}$, which satisfies $x > 1$, and $y = \sqrt{w+2}$, which satisfies $y \geq 2$. Replacing on equation (5.1), we have that

$$\frac{1}{r_s} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{z}{r_a + r_b} + 2\sqrt{\frac{1}{r_a r_b} + \frac{z}{r_a + r_b}\left(\frac{1}{r_a} + \frac{1}{r_b}\right)}.$$

Then,

$$\begin{aligned}
\frac{r_a + r_b}{r_s} &= 2 + \frac{r_a}{r_b} + \frac{r_b}{r_a} + z + 2\sqrt{\frac{(r_a + r_b)^2}{r_a r_b} + z(r_a + r_b)\left(\frac{1}{r_a} + \frac{1}{r_b}\right)} \\
&= 2 + \frac{r_a}{r_b} + \frac{r_b}{r_a} + z + 2\sqrt{(z+1)\left(2 + \frac{r_a}{r_b} + \frac{r_b}{r_a}\right)} \\
&= 2 + w + z + 2\sqrt{(z+1)(w+2)} \\
&= (x+y)^2 - 1.
\end{aligned}$$

Furthermore,

$$\sqrt{\frac{r_ar_b}{(r_a+r_b)^2}\cdot\frac{r_c(r_a+r_b+r_c)}{r_s(r_a+r_b+r_s)}} = \sqrt{\frac{1}{w+2}\cdot\frac{\frac{(r_a+r_b)}{2}\left(1+\frac{r_c}{r_a+r_b}\right)}{r_s\left(1+\frac{r_s}{r_a+r_b}\right)}}$$

$$= \frac{1}{y}\sqrt{\frac{\frac{(r_a+r_b)}{r_s}\left(1+\frac{r_c}{r_a+r_b}\right)}{z\left(1+\frac{r_s}{r_A+r_b}\right)}}$$

$$= \frac{1}{y}\sqrt{\frac{((x+y)^2-1)\left(1+\frac{1}{z}\right)}{z\left(1+\frac{1}{(x+y)^2-1}\right)}}$$

$$= \frac{(x+y)^2-1}{y}\cdot\frac{\sqrt{z-1}}{z(x+y)}$$

$$= \frac{x\left((x+y)^2-1\right)}{y(x^2-1)(x+y)}.$$

Therefore, to prove inequality (5.5) it is enough to prove that

$$7+4\sqrt{3} \leq (x+y)^2-1+\frac{2x\left((x+y)^2-1\right)}{y(x^2-1)(x+y)}$$

for all $x>1$ and $y\geq 2$, which we prove in Lemma 14. The result follows. □

**Lemma 14.** *Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the function defined as*

$$f(x,y) = (x+y)^2-1+\frac{2x\left((x+y)^2-1\right)}{y(x^2-1)((x+y))}.$$

*Then, $f(x,y)>7+4\sqrt{3}$ for all $x>1$ and $y\geq 2$.*

*Proof.* We divide the proof into two cases: $x\geq 11/10$ and $1<x<11/10$.

Let us consider first the case $x\geq 11/10$. We have that

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= 2x+2y+\frac{2x}{x^2-1}\cdot\frac{(2x+2y)y(x+y)-(2y+x)\left((x+y)^2-1\right)}{y^2(x+y)^2}\\ &= 2(x+y)+\frac{2x}{x^2-1}\cdot\frac{2y+x-x(x+y)^2}{y^2(x+y)^2}.\end{aligned}$$

We will show that $\frac{\partial f}{\partial y}(x,y)>0$ for all $x\geq 11/10$ and $y\geq 2$. To show this statement, it suffices to prove that

$$(x^2-1)y^2(x+y)^3+x\left(2y+x-x(x+y)^2\right)>0.$$

The previous inequality follows from the the equations below:

$$
\begin{aligned}
&(x^2-1)y^2(x+y)^3 + x\left(2y + x - x(x+y)^2\right) \\
=&(x^2-1)y^2(x+y)^3 - x^2(x+y)^2 + x^2 + 2xy \\
=&(x+y)^2\left((x-1)^2y^2(x+y) - x^2\right) + x^2 + 2xy \\
>&(x+y)^2\left(4(x-1)^2(x+2) - x^2\right) && \text{(since } y \geq 2\text{)} \\
=&(x+y)^2(4x^3 + 7x^2 - 4x - 8) \\
=&(x+y)^2(4x(x^2-1) + 7x^2 - 8) \\
>&0 && \text{(since } x \geq 11/10\text{)}
\end{aligned}
$$

It is worth noting that the last inequality holds since $x^2 - 1 > 0$, and $7x^2 - 8 > 0$ because $x^2 \geq 121/100 > 8/7$.

Now, since $\frac{\partial f}{\partial y}(x,y) > 0$ for all $x \geq 11/10$ and $y \geq 2$, we have that $f(x,y)$ attains its minimum in this domain at $y = 2$. Then, since $7 + 4\sqrt{3} < 14$, to prove the lemma it suffices to show that the inequality

$$F(x) = f(x,2) = (x+1)(x+3) + \frac{(x+3)x}{(x-1)(x+2)} = \frac{(x+3)(x^3+2x^2-2)}{(x-1)(x+2)} > 14 \quad (5.6)$$

holds for all $x \geq 11/10$. Note that

$$F'(x) = \frac{2(x^5 + 4x^4 + x^3 - 11x^2 - 6x + 5)}{(x-1)^2(x+2)^2}.$$

We claim that $F'(x)$ has at most one root, denoted $x_0$, in the interval $[11/10, +\infty)$. Indeed, let $P(x) = x^5 + 4x^4 + x^3 - x^2 - 6x + 5$, with $P'(x) = 5x^4 + 16x^3 + 3x^2 - 22x - 6$. For uniqueness, note that $P'(x) = (5x^4 - 6) + x(16x^2 + 3x - 22)$ and, for $x \geq 11/10$, we have $5x^4 - 6 > 0$, and also $16x^2 + 3x - 22 > 0$ since the largest (real) root of this quadratic equals $(-3 + \sqrt{1417})/32 \approx 1.0826 < 11/10$; thus, $P'(x) > 0$, so the root, if exists, must be unique. To prove its existence, note that $F'(1.43) \approx -0.407771 < 0$ and $F'(1.45) \approx 0.259627 > 0$. Then, Bolzano's theorem guarantees the existence of $x_0 \in (1.43, 1.45)$ such that $F'(x_0) = 0$. In particular, $x_0$ is a (local) minimum of $F(x)$ in $[11/10, +\infty)$. Now, observe that the numerator $\operatorname{num}(F)(x) = (x+3)(x^3+2x^2-2)$ and denominator $\operatorname{den}(F)(x) = (x-1)(x+2)$ of $F(x)$ are both increasing functions in $[1, +\infty) \subset [11/10, \infty)$, since $(\operatorname{num}(F))'(x) = 4x^3 + 15x^2 + 12x - 2 > 0$ for all $x \geq 1$, and $(\operatorname{den}(F))'(x) = 2x + 1 > 0$ for all $x \geq 1$. Therefore, we calculate the following lower bound to $F(x_0)$:

$$F(x_0) = \frac{\operatorname{num}(F)(x_0)}{\operatorname{den}(F)(x_0)} \geq \frac{\operatorname{num}(F)(x_0)}{\operatorname{den}(F)(1.45)} \geq \frac{\operatorname{num}(F)(1.43)}{\operatorname{den}(F)(1.45)} \approx 14.30728 > 14,$$

which validates inequality (5.6).

We now consider the case $1 < x < 11/10$. Then, we have that

$$\begin{aligned} f(x,y) &> (1+y)^2 - 1 + \frac{2\left((1+y)^2-1\right)}{y\left(\frac{121}{100}-1\right)\left(\frac{11}{10}+y\right)} \\ &= (2y+y^2)\left(1+\frac{200}{21}\cdot\frac{1}{y\left(\frac{11}{10}+y\right)}\right) \\ &> (2y+y^2)(1+\tfrac{200}{21}) \\ &> 10(2y+y^2) \end{aligned}$$

which is greater than $7+4\sqrt{3}$ for $y \geq 2$. The result follows. $\square$

# Chapter 6

# Conclusions and future research directions

Throughout this work, we have shown that the study of maximum-sum matchings together with the Euclidean distance does not generalize the results of Huemer et al. in [21], and a common intersection property of diametric disks fails to hold in a bi-chromatic setting. However, considering a monochromatic matching we managed to prove that such property does hold via an exhaustive case-by-case analysis, with the extensibility property of the monochromatic matchings being a key observation. Then, we approach this conjecture properly, by studying the common intersection of the ellipses defined by the conjecture's condition, and we prove that such ellipses do intersect at least pairwise; this is no trivial task, since we had to resort to a special triangle center as the Soddy inner center is, and change tracks from Euclidean geometry to multi-variable calculus. This ponders the question of how hard can this conjecture be: it is easily stated but, as the chapters progress, we can see the increasing difficulty and technicality of our reasoning. This is summarized in Table 6.1. A direct corollary of these results on disks is an improvement to the known bounds for Fingerhut's conjecture: improving Eppstein's factor of 2.5 to ours of $\sqrt{2}$, and getting even closer to the yet elusive factor of $2/\sqrt{3}$. Figure 6.1 is a graphical, drawn to scale representation of the improvements obtained, compared to those made by Eppstein.

Directions for future research are many. There is, of course, the task of proving or disproving Fingerhut's conjecture, and we hope that our results, submitted in [6], can turn some heads towards this question, which has remained hidden from the last two decades in a geometry junkyard of problems (literally speaking) until now. Our approach regarding the relative positions of segments in the monochromatic matching suggests that a similar case analysis might be done for ellipses.

**Question 1.** *Study whether a case analysis similar to the one done in Chapter 4 leads to a common intersection property of ellipses and, thus, proving Fingerhut's conjecture.*

One can also wonder whether our results, together with those in [21], hold in higher dimensions or in different geometries, and the implications of such results in other

| Geometric object | Metric | $n = 2$ | $n \geq 3$ | Reference |
|---|---|---|---|---|
| Disks | $\|\cdot\|^2$ | ✓ | ✓ | [21] |
| | $\|\cdot\|$ | ✓ | ✗ | Proposition 1, Theorems 2 and 3 |
| Ellipses | $\|\cdot\|$ | ✓ | ? | Lemma 13 |

TABLE 6.1: Summary of results on common intersections for disks and ellipses.

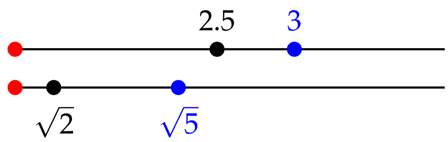



FIGURE 6.1: Graphical representation on Fingerhut's conjecture bound improvements drawn to scale. Top figure represents the bounds established by Eppstein, while bottom figure represents bounds established in this work. Blue points represent bounds for the bi-chromatic version, black points represent bounds for the monochromatic version, and the red point represents the conjectured constant $2/\sqrt{3}$.

discrete and computational geometry problems. For instance, it is remarked in [6] that the results reviewed in this thesis give rise to an improvement in a result of Adiprasito et al. ([3]) regarding a no-dimension version of Tverberg's theorem.

**Question 2.** *Do the common intersection property hold for $\mathbb{R}^3$ considering diametric spheres instead of disks for both the squared Euclidean distance and the Euclidean distance? What about $\mathbb{R}^d$, for $d \geq 4$, again for both metrics? What if we change the geometry of the problem for more general metrics, for example, where the shortest distance between two points is no longer the segment joining them but the geodesic between them?*

Although the conditions of Fingerhut's conjecture naturally suggest the use of the Euclidean distance, it might be of interest to study what happens when the metric used in the definition of $\text{cost}(\mathcal{M})$ changes to another $p$-norm, thus changing the geometry of our disks.

**Question 3.** *Study whether the common intersection property holds for the disks in $D_{\mathcal{M}}$ when considering a $p$-norm, for $p \neq 2$.*

Related to Question 3, there is also the question of the geometric shapes involved. Namely, Huemer et al. ([21]) observed that replacing circles with similar shapes, such as hexagons and decagons, does not preserve the common intersection property for the squared Euclidean distance.

**Question 4.** *Do the common intersection property hold for geometric shapes other than disks, also defined diametrically, when considering the Euclidean distance, for both monochromatic and bi-chromatic cases?*

Note that we could also generalize the conditions on the number of colours when considering a natural extension to $\mathbb{R}^3$.

**Question 5.** *Let $S = R \cup B \cup G \subset \mathbb{R}^3$ (where $G$ is a set of green points) with $|R| = |B| = |G| = n$. Does there exists a partition of $S$ into triplets with a point of each color in each triplet, such that if for each triplet we draw the minimum ball containing the three points, the resulting intersection graph of disks is the complete graph $K_n$.*

Finally, it might be of interest the study colourful Carathéodory-type questions considering disks (or balls) instead of convex hulls, thus generalising Question 4. The Colourful Carathéodory Theorem states that if $P_1, P_2, \ldots, P_{d+1}$ are $d+1$ sets in $\mathbb{R}^d$ such that there exists a point $q \in \text{conv}(P_1) \cap \text{conv}(P_2) \cap \cdots \cap \text{conv}(P_{d+1})$, then there exists points $a_1 \in P_1, a_2 \in P_2, \ldots, a_{d+1} \in P_{d+1}$ such that $q \in \text{conv}(\{a_1, a_2, \ldots, a_{d+1}\})$. Here, $\text{conv}(X)$ denotes the convex hull of $X$, and every set $P_i$ is assumed to be coloured with a different colour, so there is a clear relation with the results shown in this work.

**Question 6.** *Given $d$ finite point sets $P_1, P_2, \ldots, P_{d+1} \subset \mathbb{R}^d$, each with at least two points, is it true that if for all $P_i$ and every two points $p, q \in P_i$ the ball $\mathcal{B}_{pq}$ contains the point $o \in \mathbb{R}^d$, then there exist points $a_1 \in P_1, a_2 \in P_2, \ldots, a_{d+1} \in P_{d+1}$ such that the minimum enclosing ball of $\{a_1, a_2, \ldots, a_{d+1}\}$ also contains $o$?*

# Appendix A

# Helly's Theorem

We present the following proof of Helly's theorem, which is vital in the main theorems of the thesis.

**Theorem.** *Let $\mathcal{F}$ be a finite collection of closed, convex sets in $\mathbb{R}^d$. Every $d+1$ of the sets have a non-empty common intersection if and only if they all have a non-empty common intersection.*

*Proof.* Assume that every $d+1$ of the sets have a non-empty common intersection, and let us prove by induction over the dimension $d$ and the number of sets, $n = |\mathcal{F}|$. If $d = 1$ and any $n$, then the implication is clearly true; it also holds for $n = d+1$. So, let us suppose that we have a minimal counterexample consisting of $n > d+1$ closed, convex sets in $\mathbb{R}^d$, denoted by $X_1, X_2, \ldots, X_n$. By minimality of the counterexample, the set $Y_n = \bigcap_{i=1}^{n-1} X_i$ is non-empty and disjoint from $X_n$. Since both $Y_n$ and $X_n$ are closed and convex, we can find a $(d-1)$-dimensional plane $h$ that separates both sets, and is disjoint from them. Let $\mathcal{F}'$ be the collection of sets $Z_i = X_i \cap h$, for $i \in \{1, \ldots, n-1\}$, each a non-empty, closed, convex set in $\mathbb{R}^{d-1}$. By assumption, any $d$ of the first $n-1$ sets $X_i$ have a common intersection with $X_n$. It follows that the common intersection of the $d$ sets contains points located on both sides of $h$, from where it follows that any $d$ sets $Z_i$ have a non-empty common intersection. By minimality of the counterexample, we then have $\bigcap \mathcal{F}' \neq \emptyset$. Namely,

$$\bigcap \mathcal{F}' = \bigcap_{i=1}^{n-1} (X_i \cap h) = Y_n \cap h.$$

But this contradicts the choice of $h$ as a $(d-1)$-plane disjoint from $Y_n$. $\square$

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