PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
FACULTAD DE MATEMATICAS

Optimal bounds for many T-singularities in stable surfaces

Por

Fernando José Figueroa Zamora

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Profesor Guía : Giancarlo Urzúa
Comisión Informante:
Prof. Robert Auffarth - Universidad de Chile
Prof. Jan Kiwi - Pontificia Universidad Católica de Chile
Prof. Pedro Montero - Universidad Técnica Federico Santa María

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Introduction

Analogously to the Deligne-Mumford compactificaction of the moduli space of curves of genus $g \geq 2$ [**DM69**], Kollár and Shepherd-Barron defined a compactification of the Gieseker moduli space of surfaces of general type with fixed K^2 and χ [**GIE**] [**KSB88**], whose boundary points correspond to surfaces with semi log canonical singularities and ample canonical class. The key ingredient to prove compactness of this moduli space of "stable surfaces" was found few years after by Alexeev [**A94**] (see also [**AM04**]). It implies that there is a bound on the index of the singularities that appear in these surfaces, i.e., there is a finite list of singularities after we fix K^2 and χ. Obtaining optimal bounds for these indices is a hard problem (see e.g. [**K17**] Problem 1.24.3).

Cyclic quotient singularities $\frac{1}{m}(1, q)$ are defined as the germ at the origin of the quotient of $\mathbb{C}^2$ by the action $(x, y) \mapsto (\mu x, \mu^q y)$, where μ is a primitive m-th root of 1 and $0 < q < m$ is an integer coprime with m. These singularities form a big family of the set of semi log canonical singularities. Among them, a special role is played by the singularities $\frac{1}{dn^2}(1, dna - 1)$, where n and a are coprime, since these are the singularities that appear in a normal degeneration of canonical surfaces in the KSBA compactification. Together with the Du Val singularities, they are called T-singularities. These singularities have a rich combinatorial structure, as their minimal resolutions are chains of $\mathbb{P}^1$s which can be described as the result of a very specific algorithm.

The purpose of this thesis is to optimally bound T-singularities in normal stable surfaces which are not rational. Let

$$\frac{dn^2}{dna - 1} = [b_1, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_r}}}.$$

be the Hirzebruch-Jung continued fraction associated to the T-singularity $\frac{1}{dn^2}(1, dna - 1)$. We define its length as r. The index of a T-singularity is n. We have $n \leq F_{r-d}$, where F_i is the i-th Fibonacci number, with $F_{-2} = F_{-1} = 1$. In this way, to bound the indices of these singularities, it is enough to bound $r-d$. Consider the diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ S & & W \end{tikzcd}$$

where W has l T-singularities and K_W is ample, X is the minimal resolution of W, and π a composition of blow-downs of (-1)-curves so that there are no (-1)-curves in S.

When we look at the pull-back divisors in X of a point blown-up through π, many more combinatorial properties arise which are thoroughly used in the present work. The techniques are mostly translating algebro-geometric properties of the exceptional divisors of π and φ, mainly their intersections, into graphs condensing the data. This will allow us to classify them in a suitable way to end up with bounds for the $r-d$.

A first attempt to find reasonable bounds for $r-d$ is due to Lee [**L99**, Theorem 23]. For the case $l=1$, $d=1$, and S of general type, he was able to show

$$r \leq 400(K_W^2)^4.$$

In [**RU17**] it is worked the case of one T-singularity, i.e. $l=1$. They get the optimal bounds $r-d \leq 4K_W^2$ when $\kappa(S)=0$, $r-d \leq 4K_W^2-2$ when $\kappa(S)=1$ and $r-d \leq \max(4(K_W^2-K_S^2)-4,1)$ when $\kappa(S)=2$, where $\kappa(S)$ is the Kodaira dimension of S. They classify the cases when equality holds. They also obtain bounds in the case where K_S is not nef, which turns out to be the case when S is rational, but those bounds depend on an extra unbounded degree. In [**ES17**] they obtain the bound $r-d \leq 4K_W^2+6$, where $d=1$ and the geometric genus is positive, using methods from symplectic topology. The bound is weaker than [**RU17**] and for a more restrictive set of surfaces, but it can be applied to a surface with many T-singularities individually. In this thesis, we obtain the bound

$$\sum_{i=1}^{l}(r_i-d_i) \leq 4l(K_W^2-K_S^2)+l-2lK_S\cdot\pi(C),$$

when W is not rational, where C is the exceptional divisor of ϕ. This is a better bound than adding up the bounds in [**ES17**], and it may allow classification of surfaces in particular situations.

Preliminaries

0.1. Algebraic varieties

This section is taken from [**Hart77**, Chapter 1], except where otherwise stated.

Our base field will be the complex numbers. Things can be done in more generality, but this will be enough for all the work that will be done later.

Let us denote by $\mathbb{A}^n$ or $\mathbb{A}^n(\mathbb{C})$ the affine space $\mathbb{C}^n$.

The zeroes of a set of polynomials $F \subset \mathbb{C}[x_1 \ldots, x_n]$ is defined as

$$Z(F) = \{x \in \mathbb{A}^n \mid f(x) = 0 \quad \forall f \in F\}.$$

A subset of $\mathbb{A}^n$ is called an **algebraic set** if it consists of the zeroes of a finite number of polynomials with coefficients in $\mathbb{C}$.

It is easy to see that this meets the properties of the closed sets of a topology. This topology is called the **Zariski topology**.

The ideal of a set $X \subset \mathbb{A}^n$ is defined as

$$I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \quad \forall x \in X\}.$$

DEFINITION 1. *An* **affine variety** *is an irreducible algebraic set, i.e. if $X = X_1 \cup X_2$, with X_1, X_2 algebraic sets, then $X = X_1$ or $X = X_2$.*

DEFINITION 2. *The* **coordinate ring** *of an affine variety X is defined as $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(X)$. The* **field of rational functions** *of X is defined as the field of fractions of $\mathbb{C}[X]$, which is denoted by $\mathbb{C}(X)$.*

A **morphism** of algebraic sets $f : X \to Y$ is the restriction of a function $\hat{f} : \mathbb{A}^n \to \mathbb{A}^m$, where $\hat{f}(x) = (f_1(x), \ldots, f_m(x))$, with $f_i \in \mathbb{C}[x_1, \ldots, x_n]$. As usual an **isomorphism** is a morphism $f : X \to Y$ such that there exists another morphism $g : Y \to X$ with $f \circ g = id_Y$ and $g \circ f = id_X$.

For any morphism $f : X \to Y$, we define the pullback of f, denoted $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, as $f^*(g) = g \circ f$, which is a ring homomorphism that preserves the base field $\mathbb{C}$. A morphism f is an isomorphism if and only if f^* is a ring isomorphism.

DEFINITION 3. *We define the* **automorphism group of** X*, denoted* $\mathrm{Aut}(X)$ *as the group of isomorphisms* $f : X \to X$*.*

Every coordinate ring is a finitely generated algebra over $\mathbb{C}$ with no nilpotent elements, by Hilbert's Basis Theorem and Hilbert's Nullstellensatz, respectively. The converse is also true [**Shaf13**, Chapter 1, theorem 1.3], which gives sense to the following.

DEFINITION 4. *Let* X *be an affine variety, and* G *a finite subgroup of* $\mathrm{Aut}(X)$*. Let* $\mathbb{C}[X]^G$ *be the sub-algebra consisting of the invariant elements of* $\mathbb{C}[X]$*, under the isomorphisms* $g^* : \mathbb{C}[X] \to \mathbb{C}[X]$ *induced by the morphism* $x \mapsto g(x)$ *in* X*, for every* $g \in G$*.* $\mathbb{C}[X]^G$ *is a finitely generated sub-algebra of* $\mathbb{C}[X]$ [**Shaf13**, Appendix 4]*. Then we define the* **quotient variety** X/G*, to be the affine variety that has coordinate ring* $\mathbb{C}[X]^G$*.*

EXAMPLE 1. *Consider* g *the automorphism of* $\mathbb{C}^2$ *given by* $g(x, y) = (-x, -y)$*. Then* $\mathbb{C}[X]^{\langle g\rangle} = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[x_1, x_2, x_3]/(x_1x_3 - x_2^2)$*. So,* $\mathbb{C}[X]/\langle g\rangle$ *is a cone in* $\mathbb{C}^3$*.*

DEFINITION 5. *Let* V *be a vector space of dimension* $n+1$ *over the field* $\mathbb{C}$*. The set of lines of* V *is called the* n*-dimensional* **projective space***, and denoted by* $\mathbb{P}^n$*. If we introduce coordinates* $p_0, \dots, p_n$ *in* V *then a point* $P \in \mathbb{P}^n$ *is given by* $n+1$ *elements* $(p_0 : \dots : p_n)$ *of the field* $\mathbb{C}$*, not all equal to* 0*; and two points* $(p_0, \dots, p_n)$ *and* $(q_0, \dots, q_n)$ *are equal in* $\mathbb{P}^n$ *if and only if there exists a constant* $\lambda \neq 0$ *such that* $\lambda p_i = q_i$ *for* $i \in \{0, \dots, n\}$*. Any set* $(p_0 : \dots : p_n)$ *defining the point* P *is called a set of homogeneous coordinates for* P*.*

DEFINITION 6. *We say a polynomial* $f \in \mathbb{C}[x_0, \dots, x_n]$ *vanishes at* $P \in \mathbb{P}^n$ *if* $f(p_0, \dots, p_n) = 0$ *for any choice of homogeneous coordinates* $(p_0 : \dots : p_n)$ *of* P*.*

DEFINITION 7. *A subset of* $\mathbb{P}^n$ *is* **algebraic** *if it consists of all points at which a finite number of polynomials with coefficients in* $\mathbb{C}$ *vanish.*

As in the affine case, the algebraic sets of $\mathbb{P}^n$ are the closed sets of a topology, which is again called the **Zariski Topology**. A **projective variety** is defined in the same way as in the affine case.

The Zariski topology is **Noetherian**, meaning that for every descending chain $F_0 \supset F_1 \supset \dots$ of closed subsets of X, there exists N such that $F_N = F_{N+1} = F_{N+2} = \dots$. An important consequence is that every subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ is quasi compact.

DEFINITION 8. *A* **quasi-projective variety** *is an open set of a projective variety with the induced topology.*

Any quasi-projective variety can be covered by finitely many affine open sets. The quasi-projective varieties cover the cases of affine and projective varieties, this is the most general context in which we will work.

DEFINITION 9. *For a quasi-projective variety $X \subset \mathbb{P}^n$, a function $f : X \to \mathbb{C}$ is said to be* **regular** *at a point p if there exists an open set $U \subset X$, such that there exist $g, h \in \mathbb{C}[x_0, \dots, x_n]$ of the same degree, with h never vanishing in U, with $f = g/h$ on U.*

DEFINITION 10. *For a quasi-projective variety X, we define its* **function field** *$\mathbb{C}(X)$, as follows: An element of $\mathbb{C}(X)$ is an equivalence class of pairs (U, f), where U is a non-empty open subset of X and f is a regular function at every point of U, and where we identify two such pairs (U, f) and (U', f') if $f = f'$ on $U \cap U'$. The elements of the function field are called* **rational functions**.

DEFINITION 11. *Let X and Y be quasi-projective varieties, a function $f : X \to Y$ is a* **morphism** *if we can take open affine covers $\{U_i\}, \{V_i\}$ of X and Y, such that $f\big|_{U_i} : U_i \to V_i$ are morphisms of affine varieties.*

We define finite maps as in [**Shaf13**, Chapter 1, section 5.3]. Let X and Y be affine varieties and $f : X \to Y$ a regular map such that $f(X)$ is dense in Y. Then f^* defines an inclusion $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]$. Therefore we can view $\mathbb{C}[Y]$ as a subring of $\mathbb{C}[X]$. We say f is a **finite map** if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$.

A regular map $f : X \to Y$ of quasiprojective varieties is **finite** if any point $y \in Y$ has an affine neighbourhood V such that the set $U = f^{-1}(V)$ is affine and the restriction $f : U \to V$ is finite map between affine varieties.

For a finite surjective morphism $f : X \to Y$ we define its **degree** as $[\mathbb{C}(X) : \mathbb{C}(Y)]$.

If two morphisms are equal in some non-empty open set, then they must be equal in the entire variety. So, we can define the following useful concept.

DEFINITION 12. *Let X and Y be quasi-projective varieties, a* **rational map** *$\phi : X \dashrightarrow Y$ is the existence of some non-empty open set $U \subset X$ and a morphism $\phi\big|_U : U \to Y$.*

If the image of rational map is dense, then it is called **dominant**.

DEFINITION 13. *A rational map $\phi : X \dashrightarrow Y$ is a* **birational map**, *if there exists a dominant rational map $\psi : Y \dashrightarrow X$, such that $\phi \circ \psi$, $\psi \circ \phi$ are the identity where they are defined.*

Proposition 1. *Let X and Y be quasi-projective varieties. It is equivalent to have:*

- *there exist a birational map between X and Y.*
- *there exist non-empty open sets $U \subset X$ and $V \subset Y$, such that U and V are isomorphic.*
- *$\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(Y)$ as $\mathbb{C}$-algebras.*

In any of these cases we say that X and Y are **birationally equivalent** *or simply* **birational**.

Definition 14. *The* **dimension** *of a quasi-projective variety is the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$.*

This is a birational invariant. Varieties of dimension one are called **curves**, varieties of dimension two are called **surfaces** and varieties of dimension three are called **threefolds**. Our main interest will be in surfaces and curves inside them.

Definition 15. *The* **local ring** *of a quasi-projective variety X at a point p is the subring of $\mathbb{C}(X)$ of regular functions at p. It is denoted by $\mathcal{O}_{X,p}$ or $\mathcal{O}_p$ when the context makes obvious the variety.*

We can extend that definition to any set $U \subset X$, getting $\mathcal{O}_{X,U}$, which will be especially important for open sets.

Definition 16. *Given an affine set U, containing the point p. Let m_p be the ideal defining the point p. Then m_p/m_{p^2} is a vector space over $\mathbb{C}$, if its dimension is the same as the dimension of X, then we say p is a* **non-singular point**. *Otherwise p is a* **singular** *point.*

Definition 17. *A variety is* **normal** *if for every point p, $\mathcal{O}_p$ is integrally closed.*

In normal varieties, the singular set is of co-dimension at least 2. So in the context of normal surfaces, which is our priority, there will only be isolated singularities. If X is normal, then so is the quotient variety X/G [**Shaf13**, Chapter 2, section 5.1, example].

0.2. Sheaves

This is taken from [**Hart77**, Chapter II].

Definition 18. *Let X be a topological space, a pre-sheaf $\mathcal{F}$ of abelian groups on X, consists of the data:*

(a) For every open subset $U \subset X$, an abelian group $\mathcal{F}(U)$.
(b) For every inclusion $V \subset U$ of open subsets of X, a morphism of abelian groups $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$.

Such that

(0) $\mathcal{F}(\emptyset) = 0$.
(1) $\rho_{U,U} = id_{\mathcal{F}(U)}$.
(2) *If* $W \subset V \subset U$ *are three open sets, then* $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

We may replace *abelian group* by rings, vector spaces or other categories.

Definition 19. *A pre-sheaf* $\mathcal{F}$ *on a topological space* X *is a sheaf if it also satisfies:*

(3) *if* U *is an open set,* $\{V_i\}$ *is an open covering of* U *and if* $s \in \mathcal{F}(U)$ *is an element such that* $\rho_{U,V_i}(s) = 0$ *for every* i, *then* $s = 0$.
(4) *if* U *is an open set,* $\{V_i\}$ *is an open covering of* U *and if we have* $s_i \in \mathcal{F}(V_i)$ *for each* i, *with the property that for each* i, j $\rho_{V_i,V_i \cap V_j}(s_i) = \rho_{V_j,V_i \cap V_j}(s_j)$, *then there is an element* $s \in \mathcal{F}(U)$ *such that* $\rho_{U,V_i}(s) = s_i$ *for each* i.

Example 2. *Let* X *be a variety, for each open set* $U \subset X$, *let* $\mathcal{O}(U)$ *be the ring of regular functions from* U *to* $\mathbb{C}$, *and for each* $V \subset U$, *let* $\rho_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ *be the restriction map, then* $\mathcal{O}$ *is a sheaf of rings on* X. *We call* $\mathcal{O}$ *the* **sheaf of regular functions** *on* X.

Definition 20. *Let* X *be a variety. A sheaf* $\mathcal{F}$ *of* $\mathcal{O}_X$*-modules is* **quasi-coherent** *if there exists an open affine covering* $X = \cup_i U_i$, *such that there are* $\mathbb{C}[U_i]$*-modules* M_i *with* $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$, *where* $\widetilde{M_i}$ *is the sheaf associated to the* $\mathcal{O}_X$*-module* M_i. *It is* **coherent** *if in addition each* M_i *can be taken to be finitely generated.*

A **morphism of sheaves** $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of morphisms of abelian groups

$$\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U),$$

such that the following diagram commutes for any $V \subset U$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{U,V}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V)
\end{tikzcd}$$

0.3. Sheaf Cohomology

This section follows [**Hart77**, Chapter 3.2].

First we define cohomology for a $A-modules$, or groups, rings, etc. A cochain complex of $A-modules$ C^i is:

$$C^* : \ldots \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} \ldots$$

where $d_{i+1} \circ d_i = 0$. We define the i-th cohomology of C^* as

$$H^i(C^*) = ker(d_i)/Im(d_{i-1}).$$

To define the cohomology of sheaves we need the following definition.

Definition 21. *A sheaf $\mathcal{I}$ is* **injective** *if for every morphisms $f : \mathcal{M} \to \mathcal{I}$ and $g : \mathcal{M} \to \mathcal{N}$, exists $h : \mathcal{N} \to \mathcal{I}$, such that $h \circ g = f$.*

Given a topological space X and a sheaf $\mathcal{F}$, we define the a cochain complex in the following way. First, we need a resolution by injective sheaves:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \ldots$$

which is an exact sequence were each $\mathcal{I}^i$ is an injective sheaves. We then take global sections:

$$0 \to \mathcal{F}(X) \to \mathcal{I}^0(X) \xrightarrow{d_0} \mathcal{I}^1(X) \xrightarrow{d_1} \ldots$$

to obtain

$$\mathcal{I}^* : 0 \to \mathcal{I}^0(X) \xrightarrow{d_0} \mathcal{I}^1(X) \xrightarrow{d_1} \ldots$$

we finally define

$$H^i(X, \mathcal{F}) = H^i(\mathcal{I}^*).$$

The only cohomology that is clear from the definition is $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

The category of sheaves with abelian group values has enough injectives, meaning that there always exists the desired resolution by injectives, furthermore any resolution of injective sheaves yields the same cohomology groups. So, the sheaf cohomologies is well-defined.

For a projective variety X and $\mathcal{F}$ coherent, the $H^i(X, \mathcal{F})$ are vectorial spaces over $\mathbb{C}$ of finite dimension [**Hart77**, Theorem II.5.19]. We define $h^i(X, \mathcal{F}) = dim_{\mathbb{C}}(H^i(X, \mathcal{F}))$

DEFINITION 22. *Let X be a projective variety and $\mathcal{F}$ a coherent sheaf on X. We define the* **Euler characteristic of** $\mathcal{F}$ *as:*

$$\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F}).$$

The arithmetic genus of a curve C is $h^1(C, \mathcal{O}_C)$ or equivallently $1 - \chi(C, \mathcal{O}_C)$, and is denoted by $p_a(C)$.

The arithmetic genus is not a birational invariant. It is a known fact that an algebraic curve is isomorphic to $\mathbb{P}^1$ if and only if its arithmetic genus is 0. This will gives us a useful criterion, once we establish a relation between the arithmetic genus and the intersection theory in a surface containing the curve.

0.4. Divisors

This follows [**Shaf13**, Chapter 3.1].

DEFINITION 23. *Let X be a normal quasi-projective variety of dimension n. A* **Weil divisor** *is a formal linear combination of codimension one subvarieties. The set of all divisors with integer coefficients forms a group, which is the free abelian group on the irreducible and reduced divisors. These divisors are called the prime divisors. A $\mathbb{Q}-$divisor is a divisor with rational coefficients.*

DEFINITION 24. *Let X be a normal quasi-projective variety and let $f \in \mathbb{C}(X)$ be a rational function. We associate to f the divisor of the zero set of f minus the divisor of the zero set of $\frac{1}{f}$:*

$$(f) = (f)_0 - (f)_\infty = \sum_{V \subset X} mult_V f,$$

where the sum ranges over every irreducible subvariety $V \subset X$ of codimension one and $mult_V f$ is the multiplicity of f in V, which can be computed following [**Shaf13**, Chapter 3.1.1].

DEFINITION 25. *We say that two divisors D and D' are linearly equivalent, denoted $D \sim D'$, if $D = D' + (f)$ where f is a rational function.*

DEFINITION 26. *The group of Weil divisors modulo linear equivalence is called the Class group and it is denoted Cl(X).*

PROPOSITION 2. *Let X be a normal variety and let U be an open subset whose complement has codimension at least two. Then every Weil divisor on X is determined by its restriction to U.*

PROPOSITION 3. *Let X be a normal variety. We associate a divisor to X. Note that the singular locus of X has codimension at least two. Let ω be a rational n-form. Then the zeroes minus the poles of ω determine a divisor, K_X, called the canonical divisor. The canonical divisor is well-defined up to linear equivalence.*

PROOF. Suppose that η is any other rational n-form, with zeroes minus poles K'_X. The key point is that the ratio $f = \frac{\omega}{\eta}$ is a rational function. Thus $K_X = K'_X + (f)$. $\square$

DEFINITION 27. *Let X be a normal variety. We say that a divisor D is Cartier if D is locally defined by a single equation, i.e. if we have an open cover $X = \cup U_i$, a Cartier divisor is a collection of f_i rational invertible functions in U_i, such that for any $i \neq j$, f_i/f_j is regular at $U_i \cap U_j$.*

The key point of Cartier divisors is that given a morphism $\pi : Y \to X$ whose image does not lie in D, then we can pullback a Cartier divisor to Y . Indeed, we just pull back local defining equations. One can intersect a Cartier divisor with any subvariety and get a Cartier divisor on the subvariety, provided the subvariety is not contained in the Cartier divisor.

DEFINITION 28. *A Cartier divisor is* **principal** *if it is the divisor of a rational function on X.*

DEFINITION 29. *Given $D_1 = \{(f_i, U_i)\}$ and $D_2 = \{(g_i, V_i)\}$ Cartier divisors, we define $D_1 + D_2 = \{(f_i g_j, U_i \cap V_j)\}$ and $-D_1 = \{(f_i^{-1}, U_i)\}$. With this operation Cartier divisors form a group, two Cartier divisors are linearly equivalent if their difference is principal, denoted by $D_1 \sim D_2$.*

The group of Cartier divisors modulo linear independence is called the **Picard group** and it is denoted $\mathrm{Pic}(X)$.

PROPOSITION 4. *Let X be a non-singular variety. Then the group Div(X) of Weil divisors on X is isomorphic to the group of Cartier Divisors, and furthermore the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism. So, we have that Pic(X) $\cong$ Cl(X).*

This is [**Hart77**, Prop. II.6.11].

DEFINITION 30. *Let X be a non-singular variety. A family of divisors on X with base T is any map $f : T \to Div(X)$. We say that the family f is an* **algebraic family of divisors** *if there exists a divisor*

$C \in Div(X \times T)$ such that for any map $j_t : x \mapsto (x,t)$, with $t \in T$, $j_t^*(C)$ is defined and $j_t^*(C) = f(t)$.

Divisors D_1, D_2 on X are **algebraically equivalent** *if there exists an algebraic family of divisors f on X with base T and two points $t_1, t_2 \in T$, such that $f(t_1) = D_1$ and $f(t_2) = D_2$.*

It is not hard to see that algebraic equivalence is indeed an equivalence relation compatible with addition, and that the divisors which are algebraically equivalent to 0 are a group. This group will be denoted by $Div^a(X)$.

The group $Div(X)/Div^a(X)$ is called the Néron-Severi group and it is finitely generated [**Hart77**, Appendix B.5]

0.5. Intersection Theory

This section is taken from [**Bea78**, chapter 1].

We are particularly interested in the Picard group of a surface, since it has some type of intersection theory. In this section all surfaces are non-singular.

Definition 31. *Let C, C' be two different curves on a surface S, $x \in C \cap C'$. If f (respectively g) is an equation of C (respectively C') in $\mathcal{O}_x$, the intersection multiplicity of C and C' at x is defined as:*

$$m_x(C \cap C') = dim_{\mathbb{C}} \mathcal{O}_x/(f,g).$$

By the Nullstellensatz this is a finite number. This corresponds to the intuitive notion of the intersection number at a point (see [**Ful08**, Chapter 3]).

Definition 32. *If C, C' are two different curves on S, the intersection number $(C \cdot C')$ is defined by:*

$$(C \cdot C') = \sum_{x \in C \cap C'} m_x(C \cap C').$$

Since the intersection between two different curves is a finite number of points, this is a finite sum.

Definition 33. *Define $\mathcal{O}_{C \cap C'} = \mathcal{O}_S/(\mathcal{O}_S(-C) + \mathcal{O}_S(-C'))$. So we have $(C \cdot C') = h^0(S, \mathcal{O}_{C \cap C'})$.*

Proposition 5. *For L, L' in Pic(S), define:*

$$(L \cdot L') = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}).$$

Then (.) is a symmetric bilinear form on Pic(S), such that if C and C' are two different curves, then:

$$(\mathcal{O}_S(C) \cdot \mathcal{O}_S(C')) = (C \cdot C').$$

If D and D are divisors on S, we will write $D \cdot D'$ instead of $\mathcal{O}_S(D) \cdot \mathcal{O}_S(D')$ and D^2 instead of $D \cdot D$.

By [**Mum61**, II.b], this definition can be extended with the desired properties to the case of normal surfaces, this is the only intersection theory that we need for that case. For the case of non-singular surface we still need more properties.

Proposition 6. *The intersection number has the following properties:*

1. *Let C be a smooth curve, $f : S \to C$ a surjective morphism, F a fibre of f. Then $F^2 = 0$.*
2. *Let S' be a surface, $g : S \to S'$ a generically finite morphism of degree d, D and D' divisors on S. Then $g^*D \cdot g^*D' = d(D \cdot D')$.*

Proposition 7 (Riemann-Roch). *For all L in Pic(S) we have:*

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L \cdot K_S).$$

A consequence of the Riemann-Roch formula, which will be quite useful, is the genus formula:

Proposition 8 (Genus formula). *Let C be a curve on a surface S. Then:*

$$p_a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_S).$$

The genus formula will be used with the fact that genus 0 curves are isomorphic to $\mathbb{P}^1$.

Definition 34. *We say D in Pic S is* **nef** *if for every curve $C \subset S$, we have $D \cdot C \geq 0$.*

0.6. Blow-up

This section is taken from [**Bea78**, Chapter 2].

Proposition 9. *Let S be a non-singular surface and $p \in S$. Then there exists a non-singular surface $\hat{S}$ and a morphism $\sigma : \hat{S} \to S$, which are unique up to isomorphism, such that:*

1. *The restriction of σ to $\sigma^{-1}(S \setminus \{p\})$ is an isomorphism onto $S \setminus p$.*
2. *$\sigma^{-1}(p)$ is isomorphic to $\mathbb{P}^1$.*

We call σ the blow-up of S at p, and $\sigma^{-1}(p) = E$ the exceptional curve of the blow-up. Notice this is a birational map, which is not an isomorphism.

Let C be a curve on S that has multiplicity m on p. Then the closure of $\sigma^{-1}(C \setminus p)$ in $\hat{S}$ is a curve, called the **strict transform** of C, denoted by $\hat{C}$.

PROPOSITION 10. *We have that:*

$$\sigma^* C = \hat{C} + mE.$$

PROPOSITION 11. *Let S be a non-singular surface, $\sigma : \hat{S} \to S$ the blow-up of a point p, and $E \subset \hat{S}$ the exceptional curve. Then:*

- *There is an isomorphism $Pic(S) \oplus \mathbb{Z} \xrightarrow{\sim} Pic(\hat{S})$, defined by $(D, n) \mapsto \sigma^* D + nE$.*
- *$NS(\hat{S}) \cong NS(S) \oplus \mathbb{Z}[E]$.*
- *Let D, D' be divisors on S. Then $\sigma^* D \cdot \sigma^* D' = D \cdot D'$, $E \cdot \sigma^* D = 0$ and $E^2 = -1$.*
- *$K_{\hat{S}} = \sigma^* K_S + E$.*

We have the following corollary which will be useful when we compare intersections with canonical divisors in a blow-up.

COROLLARY 1. *For an irreducible curve C on S that has multiplicity m at p, we have:*

$$C \cdot K_S = \hat{C} \cdot K_{\hat{S}} - m.$$

0.7. Castelnuovo Theorem

Curves isomorphic to $\mathbb{P}^1$ with self-intersection (m), will be called (m)-curves. We have a special interest in (-1)-curves, because of the following criterion:

PROPOSITION 12 (Castelnuovo's contractibility criterion). *Let S be a non-singular surface and $E \subset S$ a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is the exceptional curve of a blow-up $\sigma : S \to S'$, where S' is a non-singular surface.*

For a proof see [**Bea78**, II.17].

This process of contracting (-1)-curves is called **blow-down**. Since the Néron-Severi rank is finite by the Néron-Severi theorem [**Hart77**, Appendix B.5], and after every blow-down this number goes down by one, we cannot do infinitely many blow-downs to a surface. Therefore in any surface we can contract all the (-1)-curves and end with a birationally equivalent surface without (-1)-curves. Such a surface is called a **minimal model**. A surface birational to $C \times \mathbb{P}^1$, where C is a curve, is called a **ruled surface**. For a surface that is not ruled we have a unique minimal model [**Bea78**, Theorem V.19], which is relevant for us.

0.8. Cyclic quotient Singularities

DEFINITION 35. *Given $p \in X$ a singular point in a variety. A* **resolution** *of p is a non-singular variety $\hat{X}$, with a morphism $\phi : \hat{X} \to X$, such that $X \setminus \phi^{-1}(p)$ is isomorphic to $X \setminus p$. The divisor $\phi^{-1}(p)$ is called the* **exceptional divisor of** *p.*

DEFINITION 36. *A* **minimal resolution** *of p in a surface, is a resolution, such that the exceptional divisor contains no (-1)-curve.*

By Castelnuovo criterion, any resolution of a singular surface gives rise to a minimal resolution, simply by contracting the (-1)-curves. Notice that a minimal resolution is not necessarily a minimal surface, as it can have (-1)-curves outside of the exceptional divisors. This will happen in most of our cases of interest.

Consider the automorhpism in $\mathbb{C}^2$, defined by

$$\phi_{m,q}(x, y) = (\mu x, \mu^q y),$$

where μ is a primitive m-th root of 1, and q is an integer with $0 < q < m$ and $\gcd(q, m) = 1$.

DEFINITION 37. *A* **cyclic quotient singularity** *is the germ of the singularity at $(0, 0)$ of the quotient $\mathbb{C}^2/\langle\phi_{m,q}\rangle$. This singularity is denoted by $\frac{1}{m}(1, q)$*

The minimal resolution of a singularity $\frac{1}{m}(1, q)$ has a chain of $\mathbb{P}^1$s as exceptional divisor. The chain is made of E_i, for $i \in \{1, \ldots, r\}$, with $E_i \cdot E_{i+1} = 1$, $E_i^2 = -b_i$ and $E_i \cdot E_j = 0$ for any other case, where $\frac{m}{q} = [b_1, \ldots, b_r]$ is the Hirzebruch-Jung continued fraction.

$$[b_1, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \frac{1}{b_r}}},$$

where each b_i is in integer bigger than 1. Notice this continued fraction always exists and is unique, so it is well-defined.

We define its **length** as r. In the rest of this work, the symbol $[b_1, \ldots, b_r]$ will correspond to the continued fraction or the singularity or the chain of curves $E_1, \ldots, E_r$ depending on the context. Also

$$[b_1, \ldots, b_r] - c - [b'_1, \ldots, b'_{r'}]$$

will represent a chain of $\mathbb{P}^1$'s with self-intersections either $-b_i$ or $-c$ or $-b'_i$ respectively. One can write the numerical equivalence

$$K_{\widetilde{Y}} \equiv \sigma^*(K_Y) + \sum_{i=1}^{r} \delta_i E_i$$

where $\delta_i \in]-1,0]$ is (by definition) the discrepancy at E_i. These numbers can be computed explicitly, as in [**S13**, section 1.3]

Define the following numbers:

$$0 = x_{r+1} \leq x_r = 1 < \ldots < x_1 = q < x_0 = m,$$

where $x_{i+1} = b_i x_i - x_{i-1}$. So,

$$\frac{x_{i-1}}{x_i} = [b_i, \ldots, b_r].$$

Also,

$$P_0 = 0 < P_1 = 1 < \ldots < P_{r+1} = m$$

where $P_{i+1} = b_i P_i - P_{i-1}$ and $Q_0 = -1, Q_1 = 0, Q_{i+1} = b_i Q_i - Q_{i-1}$. So,

$$\frac{P_i}{Q_i} = [b_1, \ldots, b_{i-1}].$$

We obtain:

$$K_{\widetilde{Y}} \equiv \sigma^*\Big(K_{\frac{1}{m}(1,q)}\Big) - \sum_{i=1}^{r} \Big(1 - \frac{x_i + P_i}{m}\Big) E_i.$$

So, $\delta_i = -(1 - \frac{b_i + P_i}{m})$.

Proposition 13. *Let Y be a surface with a unique singularity $\frac{1}{m}(1,q)$, then the following conditions are equivalent:*

- *$m = dn^2$ and $q = dna - 1$, for positive integers $a < n$ with $(n,a) = 1$.*
- *K_Y^2 is an integer.*
- *$(m, q+1)$ is divisible by $\frac{m}{(m,q+1)}$.*

A **T-singularity** is defined as a quotient singularity that admits a $\mathbb{Q}$-Gorenstein one parameter smoothing [**KSB88**, Definition 3.7]. They are precisely either ADE singularities or $\frac{1}{dn^2}(1, dna-1)$ with $d \geq 1$, $0 < a < n$ and $\gcd(n,a) = 1$ [**KSB88**, Proposition 3.10]. We call the exceptional divisor of a non-ADE T-singularity a **T-chain**.

The Problem

T-singularities have a particular combinatorial structure as it is shown in the following well-known proposition.

Proposition 14. *For non-ADE T-singularities* $\frac{1}{dn^2}(1, dna-1)$ *we have:*

(i) *If* $n = 2$ *then they are* $[4]$ *and* $[3, 2, \dots, 2, 3]$*, where the number of* 2*'s is* $d - 2$*. In this case all discrepancies are equal to* $-\frac{1}{2}$*.*
(ii) *If* $[b_1, b_2 \dots, b_r]$ *is a T-singularity, then so are* $[2, b_1, \dots, b_{r-1}, b_r + 1]$ *and* $[b_1 + 1, b_2, \dots, b_r, 2]$*.*
(iii) *Every non-ADE T-singularity can be obtained by starting with one of the singularities in (i) and iterating the steps described in (ii).*
(iv) *Consider a T-chain* $[b_1, \dots, b_r] = \frac{dn^2}{dna-1}$ *with discrepancies* $-1+\frac{t_1}{n}, \dots, -1 + \frac{t_r}{n}$*. Then* $[b_1 + 1, b_2, \dots, b_r, 2]$ *has discrepancies* $-1 + \frac{t_1}{n+t_1}, \dots, -1 + \frac{t_r}{n+t_1}, -1 + \frac{t_1+t_r}{n+t_1}$*, and* $[2, b_1, \dots, b_r + 1]$ *has discrepancies* $-1 + \frac{t_1+t_r}{n+t_r}, -1 + \frac{t_1}{n+t_r}, \dots, -1 + \frac{t_r}{n+t_r}$ *respectively.*
(v) *Given the T-chain* $[b_1, \dots, b_r]$*, the discrepancy of an ending* (-2)*-curve is* $> -\frac{1}{2}$*, and* $\delta_1 + \delta_r = -1$*, i.e.,* $t_1 + t_r = n$ *in (iv).*

Proof. The points (i), (ii) and (iii) are [**KSB88**, Proposition 3.11]. The point (iv) is [**St89**, Lemma 3.4]. The point (v) is a simple consequence of (iv). □

For a non-ADE T-singularity, we define its **center** as the collection of exceptional divisors which have the lowest discrepancies, this is, equal to $-\frac{n-1}{n}$. Hence, these divisors are the ones corresponding to (i) after we apply several times the algorithm (ii). The importance of the center is the following.

Proposition 15. *Let* $[b_1, \dots, b_r] - 1 - [b'_1, \dots, b'_{r'}]$ *be a chain of* $\mathbb{P}^1$*'s where* $[b_1, \dots, b_r]$ *and* $[b'_1, \dots, b'_{r'}]$ *are T-chains, and* $\delta_r + \delta'_1 < -1$*. After contracting the* (-1)*-curve and all new* (-1)*-curves after that, we obtain that there is no curve in the centers of any T-chain which is contracted.*

PROOF. We suppose by contradiction that at least one curve in the centers is contracted. Without loss of generality, let the center of $[b_1, \ldots, b_r]$ be the first to have a contracted curve. Then we have the following picture:

$$[b_1, \ldots, b_s, \text{center}, b_t, \ldots, b_r] - 1 - [b'_1, \ldots, b'_{r'}].$$

Using the preceding lemma, in order for b_{t-1} to be contracted, the curves $[b_t, \ldots, b_r]$ must be blown-down, and b_{t-1} must eventually become 1. As the blow-down process acts like the inverse of the algorithm (ii) in the preceding lemma, we must have $b_i = b'_i$ for $1 \leq i \leq s$, and both T-chains are made using the same steps of the algorithm (ii) in the preceding lemma: the first starting from $[x_1, \ldots, x_d]$, a T-chain with $n = 2$ (from (i)), and the other starting from $[\beta_1, \ldots \beta_k]$. Then an intermediate blow-down is:

$$[b_1, \ldots, b_{t-2}, x_d + y] - 1 - [\beta_1, b'_{s+2} \ldots, b'_{r'}]$$

where $y \geq 0$ and $x_d = 3$ or 4. Then $\beta_1 = 2$, because we must blow-down $x_d + y$.

By rewriting the formulas for discrepancies from the previous lemma, we obtain that in each step the discrepancy of the leftmost divisor goes from either x to $-1 + \frac{1}{x+2}$ or from x to $-1 + \frac{1}{1-x}$, which are both increasing functions. Therefore, by looking at $[x_1, \ldots, x_d]$ and $[\beta_1, \ldots \beta_k]$, we have that $\delta'(\beta_1) > -\frac{1}{2} = \delta(x_1)$, and so, after each step this remains true, meaning that we finish with $\delta'_1 > \delta_1$. Since by the previous Lemma part (v) $\delta_1 + \delta_r = -1$, we have $\delta'_1 + \delta_r > -1$, which contradicts with our hypothesis.

□

The following diagram represents a chain of $\mathbb{P}^1$s:

$$C_0 - [b_{1,1}, \ldots, b_{1,r_1}] - C_1 - \ldots - C_{x-1} - [b_{x,1}, \ldots, b_{x,r_x}] - C_x$$

where $[b_{i,1}, \ldots, b_{i,r_i}]$ is a T-chain and C_i is one of the following:

- A (-1)-curve, such that $\delta_{i,r_i} + \delta_{i+1,1} < -1$ if $0 < i < x$.
- A chain of $\mathbb{P}^1$s whose self-intersections are less than or equal to -2.

COROLLARY 2. *Let us consider the chain of $\mathbb{P}^1$s of the preceding paragraph:*

$$C_0 - [b_{1,1}, \ldots, b_{1,r_1}] - C_1 - \ldots - C_{x-1} - [b_{x,1}, \ldots, b_{x,r_x}] - C_x.$$

After contracting the (-1)-curves and all new (-1)-curves after that, we obtain that there is no curve in the center of any T-chain which is contracted.

PROOF. Let us consider

$$C_i - [b_{i+1,1}, \ldots, b_{i+1,r_{i+1}}] - C_{i+1}$$

where C_i, C_{i+1} are (-1)-curves. This is the worse case scenario for a T-chain $[b_{i+1,1}, \ldots, b_{i+1,r_{i+1}}]$. First assume it has a center with two or more curves. Then the contraction of the C_i, C_{i+1} and all the new (-1)-curves produced by them will not contract the center of the T-chain, by Proposition 15 applied to both ends. If the center has only one curve, then we replace it by a center with two curves. This will keep the discrepancies untouched by Proposition 14. Therefore, by the same previous reason, the center cannot be contracted. That means that the number of blow-downs from both directions are not enough to make disappear these two curves, and so for the case of a center with one curve. Hence we cannot contract centers of T-chains.

□

Let W be a normal projective surface with K_W ample and only T-singularities $\frac{1}{d_i n_i^2}(1, d_i n_i a_i - 1)$ where $i \in \{1, \ldots, l\}$. Let us consider the diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ S & & W \end{tikzcd}$$

where the morphism ϕ is the minimal resolution of W, and π is a composition of m blow-ups such that S has no (-1)-curves. We use the same notation as in [**R14, RU17**]. Let E_i be the pull-back divisor in X of the i-th point blown-up through π. Therefore, E_i is a tree of $\mathbb{P}^1$'s, $E_i^2 = -1$, and it may not be reduced. Let

$$C = \sum_{i=1}^{l} C_i = \sum_{i=1}^{l} \sum_{j=1}^{r_i} C_{i,j}$$

be the exceptional (reduced) divisor of ϕ, where $C_i = \sum_{j=1}^{r_i} C_{i,j}$ is the T-chain of the singularity $\frac{1}{d_i n_i^2}(1, d_i n_i a_i - 1)$. We have

$$K_S^2 - m + \sum_{i=1}^{l} (r_i - d_i + 1) = K_W^2.$$

REMARK 1. *Throughout this work, we will assume that $m > 0$, since otherwise $K_W^2 - K_S^2 = \sum_{i=1}^{l} (r_i - d_i + 1)$, and this case holds in our main theorems.*

LEMMA 1. *For any (-1)-curve Γ in X we have $\Gamma \cdot C \geq 2$. For any (-2)-curve Γ in X not in C we have $\Gamma \cdot C \geq 1$.*

PROOF. It is a simple computation using the pull-back of the canonical class, the discrepancies of the $C_{i,j}$, and that K_W is ample. □

LEMMA 2. *We have*

$$\Big(\sum_{i=1}^{m} E_i\Big) \cdot C = \sum_{j=1}^{l} (r_j - d_j + 2) - K_S \cdot \pi(C).$$

PROOF. Same as in [**RU17**, Lemma 2.4]. □

LEMMA 3. *For any i, we have* $E_i \cdot C \geq -1 + E_i \cdot \Big(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j}\Big)$.

PROOF. If $C_{k,j} \subset E_i$, then $C_{k,j} \cdot E_i = 0$ or $C_{k,j} \cdot E_i = -1$. The latter case can happen only for one $C_{k,j}$ in C. □

DEFINITION 38. *Let S_h be the number of E_i such that*

$$E_i \cdot \Big(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j}\Big) = h.$$

COROLLARY 3. *We have* $\Big(\sum_{i=1}^{m} E_i\Big) \cdot C \geq -m + \sum_{h \geq 0} h S_h$.

PROOF. This is adding up Lemma 3 for each E_i. □

Since $\sum_{h \geq 0} S_h = m$, the key for us will be to find an upper bound on S_h for small h, which in turn will give better and explicit lower bounds for $\Big(\sum_{i=1}^{m} E_i\Big) \cdot C$.

For each E_i we define the diagram Γ_{E_i} as in [**RU17**, Section 2]. First consider the dual graph of the l T-chains in X which consists of black dots (the $C_{k,j}$) together with segments representing intersections among the $C_{k,j}$'s. Now, if $C_{k,j} \subset E_i$, then we replace the k,j-th vertex of the dual graph by a box □, and in this way we obtain the graph Γ_{E_i}. Let us also denote as G_{E_i} the graph formed by the union of Γ_{E_i} and the dual graph of E_i, where we also join vertices from E_i and Γ_{E_i} if the corresponding curves intersect. In G_{E_i} the only intersections that might not be simple are those between a vertex in E_i not in Γ_{E_i} and a vertex in Γ_{E_i} not in E_i, but these will not appear in the cases that we are interested in, as we will see later.

REMARK 2. *A useful fact is that a (-1)-curve cannot intersect three different curves which will be blown down, since blowing down this (-1)-curve would yield a triple point, but each blow up gives only nodes. In particular a (-1)-curve cannot intersect three boxes in Γ_{E_i} or any succeeding blow down of it.*

LEMMA 4. *We always have $S_0 = 0$.*

PROOF. In order to have $S_0 > 0$ we must have some E_i such that each T-chain either has no curve in E_i and does not intersect it, or it is contained completely in E_i. We can assume that every T-chain is contained in E_i. So all the intersections in G_{E_i} are simple.

Hence we consider Γ_{E_i} inside of E_i, and so $G_{E_i} = E_i$. If no vertex has more than two neighbours, by Corollary 2 no divisor in a center can be contracted, a contradiction.

Now if a vertex A_1 has more than two neighbours, we can look at the connected components of $G_{E_i} \setminus A_1$. One of these components must be fully blown down, before the vertex A_1 is. This is because otherwise A_1 would become a (-1)-curve connected to more than two curves, which would produce a triple point in a blow-down of E_i and all blow-downs have only nodes. This component behaves independently of the rest of G_{E_i}, i.e. it can be blown down entirely without contracting (-1)-curves outside of it.

First suppose this component does not contain any vertex with more than two neighbours. Since it can be blown down, it must contain a (-1)-curve, and this curve must intersect two different T-chains, and so at least one of them must be completely contained in the connected component.

If A_1 is not part of a T-chain, then this component meets the hypotheses of Corollary 2, and this produces a contradiction. If A_1 is part of a T-chain but the component does not contain part of it, then Corollary 2 works as well with this component.

If A_1 is part of a T-chain and the component contains part of it, then we can look at the component joined with the rest of the T-chain that contains A_1, doing the corresponding blow downs here is the same as doing the corresponding blow downs in the independent component. Therefore we have the conditions for Corollary 2, so the T-chain which is completely contained will not be entirely contracted, a contradiction.

Now assume that the component contains a vertex A_2 which has more than two neighbours. We can look at the connected components of $G_{E_i} \setminus A_2$. As before, one of the components must be fully blown down before the vertex A_2 is. We note that the vertex A_1 is not contracted by the blow-downs we are looking at, so the component which fully blows-down does not contain A_1.

So we end up with an independent component not containing A_1 and A_2. If this component does not contain any vertex with more than two neighbours, we proceed as before. If it contains a vertex A_3 with more than two neighbours, we do as with A_2 and end up with an independent component not containing A_1, A_2 and A_3. This process must end since G_{E_i} is a finite graph, and so we obtain a contradiction. Therefore $S_0 = 0$.

$\square$

LEMMA 5. *We always have $S_1 = 0$.*

PROOF. Let us assume that $S_1 > 0$. Then there is E_i such that $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) = 1$, and, as in the preceding lemma, we can omit from this discussion all the T-chains with no curve in E_i and no curve intersecting E_i. So all the intersections in G_{E_i} are simple.

We consider the graph G_{E_i}. This graph is a tree, since any potential cycle would contain vertices in E_i and vertices not in E_i, which would give at least two points of intersection between curves in E_i and curves in T-chains but not in E_i, but this is not possible by our assumption on E_i. We now deal with two cases:

Case I): There is no vertex in E_i connected with more than two vertices in G_{E_i}. First we note that there is more than one T-chain, because otherwise a (-1)-curve would make a cycle in G_{E_i}. And so there is at least one T-chain contained in E_i. We apply Corollary 2 to G_{E_i}, so no center divisor can be contracted, which contradicts the fact that there is a T-chain contained in E_i.

Case II): There is a vertex $A \in E_i$ connected to (at least) three vertices in G_{E_i}. We can look at the connected components of $G_{E_i} \setminus A$. We have two subcases.

If one of these components contracts completely before A does, then we can apply the same argument as in Lemma 4 to arrive at a contradiction.

Hence none of these components contracts completely before A does. The final argument splits in two parts. We first blow down until A becomes a (-1)-curve. If all neighbours of the (-1)-curve A are in the image of E_i, then this produces a contradiction since A would be creating a triple point in a blow-down of E_i. So, one neighbour must be in the image of $\Gamma_{E_i} \setminus E_i$. In this case, since none of the components blow downs fully before A does, and we have at least two of them inside of E_i, we have that the (-1)-curve A in the divisor E_i has multiplicity bigger than or equal to 2. But, by pulling back, this would contradict our assumption $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) = 1$. Therefore $S_1 = 0$.

So, in each case we get a contradiction. $\square$

Classification

Now let us consider an E_i with

$$E_i \cdot \Big(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \Big) = 2.$$

This is the key case to analyse. As before, we can omit from the next discussion the T-chains with no curves in E_i and no curves intersecting E_i. The next goal is to find all the combinatorial possibilities for G_{E_i}. Also, if there was an intersection that is not simple in G_{E_i}, then the same G_{E_i}, but with that intersection being simple would have $E_i \cdot \Big(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \Big) < 2$, which contradicts Lemmas 4 and 5. So, all the intersections are simple in G_{E_i}.

Remark 3. *From now on we will omit T-chains in G_{E_i} with no curve in E_i and no curve intersecting E_i and assume that G_{E_i} is a* ***tree*** *to facilitate our analysis of the possibilities. At the end, we will show how to classify all the cases when G_{E_i} is not a tree via a suitable combinatorial reduction to the case of a tree. The notation $\bigcirc$ in the next figures will mean (-1)-curve in E_i.*

Proposition 16. *If G_{E_i} is a tree and there is no vertex in E_i having three neighbours in G_{E_i}, then G_{E_i} corresponds to one of the Figures 0.1 to 0.3.*

Proof. Since G_{E_i} is a tree and $E_i \cdot \Big(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \Big) = 2$, the number of T-chains is one more than the number of (-1)-curves. We can apply the Corollary 2 for G_{E_i} after removing each T-chain which does not have curves in E_i. So, no divisor in a center could be contracted. Therefore no T-chain can be in E_i, and so there are at most 2 T-chains, and only one (-1)-curve. According to the number of T-chains contained in E_i, we get the possibilities in Figures 0.1 to 0.3

$\square$

Proposition 17. *If G_{E_i} is a tree, then we have that any vertex in E_i has at most three neighbours.*

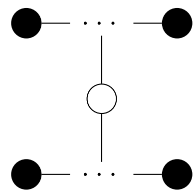

FIGURE 0.1. Case C.1

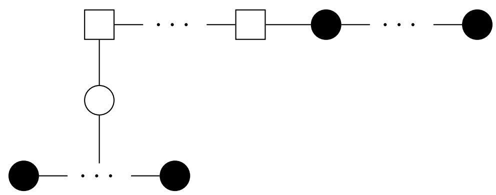

FIGURE 0.2. Case C.2

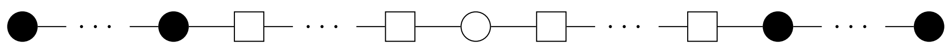

FIGURE 0.3. Case C.3

PROOF. Suppose there is a vertex A in E_i with more than three neighbours. If a connected component of $G_{E_i} \setminus A$ was blown-down before A, then we can apply the same argument as in Lemma 4 to arrive at a contradiction. So, after doing the corresponding blow-downs A becomes a (-1)-curve with at least four neighbours. It cannot have three neighbours inside the blow-down of E_i, because there cannot be a triple point in a blow-down of E_i. It also cannot have three neighbours outside the blow-down of E_i or we would have $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 3$. Therefore the only possibility is to have two neighbours in the blow-down of E_i, and two neighbours outside of it. Then the (-1)-curve A would have multiplicity at least 2 in the divisor E_i, and so by pulling-back we would get $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 4$, a contradiction. $\square$

PROPOSITION 18. *If G_{E_i} is a tree, then there is at most one vertex in E_i with three neighbours.*

PROOF. Suppose there is a vertex A_1 in E_i with three neighbours. Since we have that $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) = 2$, at least one of the components of $G_{E_i} \setminus A_1$ is contained in E_i. If there are more vertices with three neighbours inside a component of $G_{E_i} \setminus A_1$ fully contained in E_i, then we can take one of these vertices, call it A_2, and check whether there are vertices with three neighbours inside a component fully contained in E_i of $G_{E_i} \setminus A_2$. We iterate this process. In each step A_{j+1} is in a component of $G_{E_i} \setminus A_j$ not containing any other A_k, with $k < j$. So all those vertices are different and, since the graph is finite, the process ends with a vertex A such that any component of $G_{E_i} \setminus A$ fully contained in E_i has no vertex with three neighbours. We now divide the analysis into two cases:

Case A: Two components of $G_{E_i} \setminus A$ have curves outside of E_i. If there is a vertex $B \in E_i$ with three neighbours in a component of $G_{E_i} \setminus A$ containing curves which are not in E_i, then, by the same argument as in Lemma 4, no components of $G_{E_i} \setminus B$ can be completely blow down before B becomes a (-1)-curve. We blow-down until B becomes a (-1)-curve, which is connected to: a component containing A, a component completely contained in E_i, and a component containing curves that are not in E_i (this is because our Case A assumption). So it is a (-1)-curve either connected to three curves in E_i or connected to two curves in E_i and a curve C not in E_i. In the first case, we get a triple point in a blow-down of E_i, a contradiction. In the second case, the (-1)-curve B has multiplicity bigger than or equal to 2 in the image divisor of E_i, by pulling-back the intersection of C with E_i will give us at least 2. Thus, $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 3$, a contradiction.

Case B: Exactly one component of $G_{E_i} \setminus A$ has curves outside of E_i. No component of $G_{E_i} \setminus A$ can be fully blow down before A becomes a (-1)-curve, by the same argument as in Lemma 4. We do blow-downs until A becomes a (-1)-curve. It cannot be connected to three curves in E_i, so it must be connected to one curve not in E_i and two curves in E_i. Hence, we have that the (-1)-curve A in the divisor E_i has multiplicity at least 2, and so do all curves in E_i in the component of $G_{E_i} \setminus$ containing curves outside of E_i. Now, if there is a vertex B with three neighbours in the component containing curves that are not in E_i, then, by the same argument as in Lemma 4, no component of $G_{E_i} \setminus B$ can be fully blow down before B becomes a (-1)-curve. We blow-down until B becomes a (-1)-curve, which is connected to: a component containing A, a component completely contained in E_i, and a component containing curves that are not in E_i. Since a (-1)-curve cannot be connected to three curves in E_i, B is either connected to two curves in E_i and a curve not in E_i, or it is connected to one curve in E_i and two curves not in E_i. In the first case, the (-1)-curve B in the blow down of the divisor E_i has multiplicity at least 4 (since each of the curves in E_i connected to the (-1)-curve B have multiplicity at least 2). Therefore by pulling-back, we obtain $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 4$, a contradiction. In the second case, the (-1)-curve B in the blow down of the divisor E_i has multiplicity at least 2 and is connected to two curves outside of E_i, hence by pulling-back this data, we obtain $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 4$, a contradiction. $\square$

Now let us consider all the possible cases with one vertex in E_i having three neighbours.

PROPOSITION 19. *If G_{E_i} is a tree and there is one vertex in E_i having three neighbours in G_{E_i}, then G_{E_i} corresponds to one of the Figures 0.4 to 0.15.*

PROOF. Let us denote this special vertex by V. We consider the following cases.

Case A: Only one component of $G_{E_i} \setminus V$ is fully contained in E_i. When V becomes a (-1)-curve, no component in $G_{E_i} \setminus V$ has been contracted, otherwise we would have a contradiction as in Lemma 4. This (-1)-curve cannot have three neighbours in the blow-down of E_i, as this would be a triple point. If this (-1)-curve had two neighbours in the blow-down of E_i, then it would have multiplicity at least 2 in the image divisor of E_i, and we would have $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 3$.

Now if we re-order the blow-downs, so that we do all possible blow-downs except for blowing down the (-1)-curve that V becomes, then we end up with only one (-1)-curve, connected to two curves not in the blow-down of E_i, and a component which is a chain of $\mathbb{P}^1$s in E_i. For this chain to be blown-down, they need to be all (-2)-curves.

If to the original component of $G_{E_i} \setminus V$ contained in E_i, we add a (-1)-curve to the vertex connected to just one other vertex (i.e. “the ending curve”), then by Corollary 2 inside this component no divisor in a center can be blown-down. After doing the blow-downs in the new order, we end up with a chain of (-2)-curves connected to a (-1)-curve. So everything is blow-down, and therefore there were no divisors in a center in that component. If there was a (-1)-curve in the component, then there would be some centers. So in this component there is no (-1)-curve, and so it was a chain of (-2)-curves before doing any blow-down.

Only one (-2)-curve can be outside of E_i. If this is the only (-2)-curve, then removing it does no change which curves are contracted. So, we have a case as in Proposition 16 with an extra (-2)-curve. These case are shown in Figure 0.4 and 0.5

So we are left to analyse the case when some of these (-2)-curves are in a T-chain, and therefore V is in that same T-chain.

Case A.I: These curves form a complete T-chain. In any of the other two connected components of $G_{E_i} \setminus V$, there must be curves in

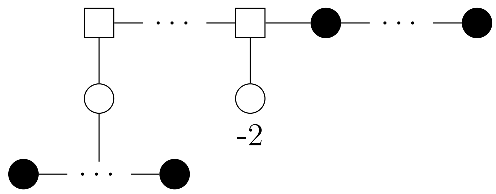

FIGURE 0.4. Case A.1

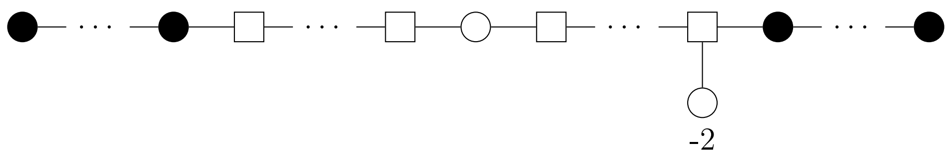

FIGURE 0.5. Case A.2

E_i, or V would not intersect it. We showed before that this curves are contracted before V is. So, by Corollary 2 there cannot be a contracted center divisor, so there can only be one T-chain in each component. The vertex V must be connected by a (-1)-curve, or there would be no contracted curves. These cases are shown in Figure 0.6 and 0.7.

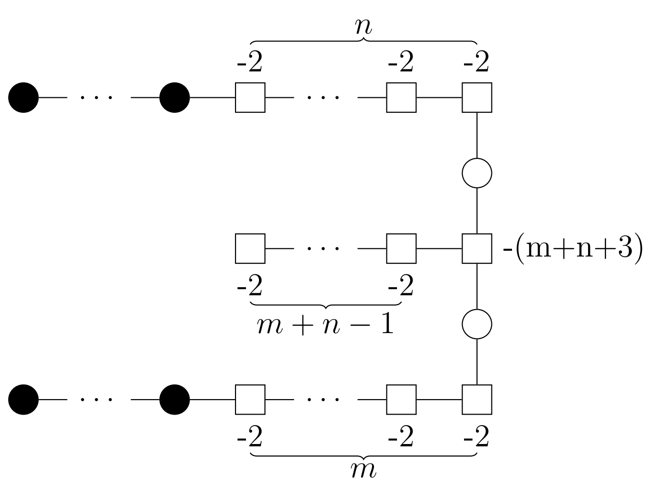

FIGURE 0.6. Case A.I.1

Case A.II: These curves do not form a complete T-chain. In one of the connected components of $G_{E_i} \setminus V$, there are no curves of the T-chain containing V. There must be curves in E_i, or V would not intersect it. We showed before that these curves are contracted before V is. So, by Corollary 2 there cannot be a contracted center divisor,

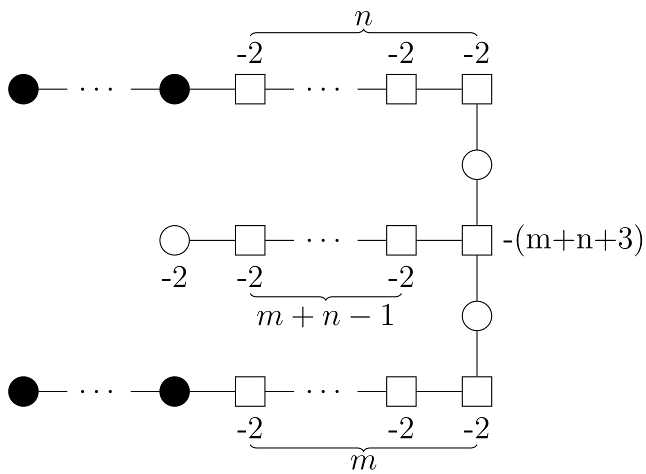

FIGURE 0.7. Case A.I.2

so there can only be one T-chain. The vertex V must be connected by a (-1)-curve, or there would be no contracted curves.

Since there is no center divisor contracted in the other component, there cannot be more than one T-chain completely contained in this component. If there were curves contracted, then the T-chain containing V would need to be connected by a (-1)-curve to a T-chain, or there would be no contracted curve. We end up with the cases in Figures 0.8 to 0.11.

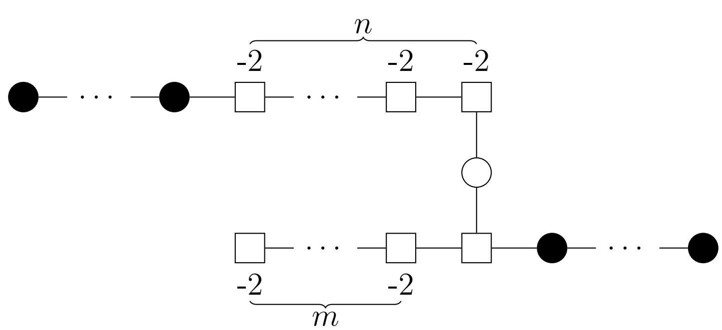

FIGURE 0.8. Case A.II.1

Case B: Exactly two components are fully contained in E_i. When V becomes a (-1)-curve, no component in $G_{E_i} \backslash V$ can be contracted, or we would get a contradiction via Lemma 4. The (-1)-curve V cannot have three neighbours in the blow-down of E_i, as this would create a triple point.

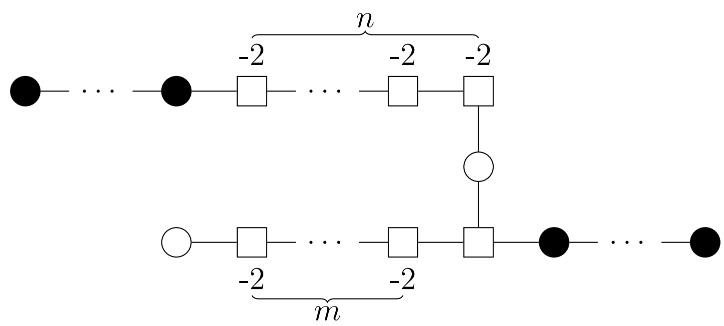

FIGURE 0.9. Case A.II.2

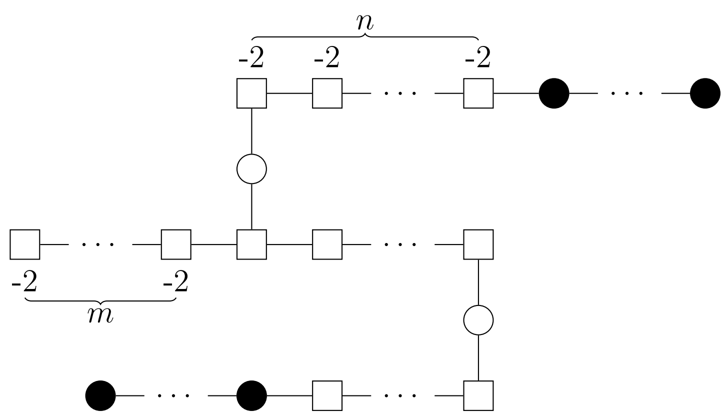

FIGURE 0.10. CASE A.II.3

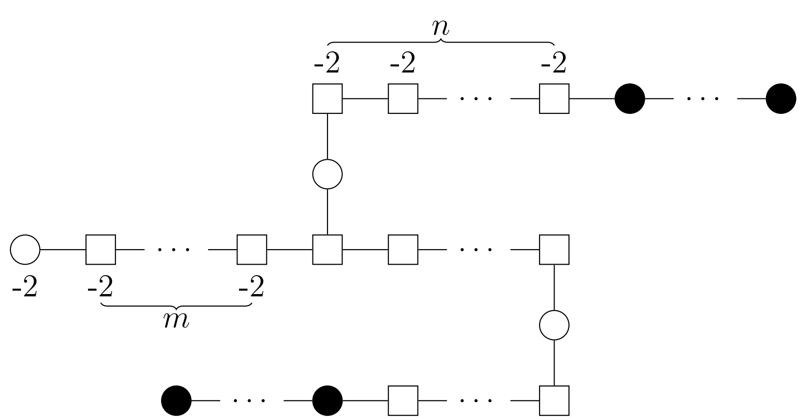

FIGURE 0.11. CASE A.II.4

If we re-order the blow-downs, so that we do all possible blown-downs except for blowing down the (-1)-curve that V becomes, then we end up with only one (-1)-curve connected to one curve not in the blow-down of E_i, and two chains of $\mathbb{P}^1$s contained in the blow-down of E_i.

- We have that one of these chains is exactly a (-2)-curve. For these chains to be blown down, there needs to be a (-2)-curve intersecting the blow-down of V, call it V'. If V' intersects another curve, then after blowing down the (-1)-curve V, the curve V' becomes a (-1)-curve intersecting two curves in the blow-down of E_i and a curve not in E_i. So the divisor V' has multiplicity at least two in the divisor E_i. Hence, the blow-down of the divisor V that is a (-1)-curve has multiplicity at least 3 in the divisor E_i. So, by pulling-back we obtain $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j} \right) \geq 3$, a contradiction.
- We have that the other chain is $(-3)-(-2)-\ldots-(-2)$ (which shows the self-intersections of curves in the chain). Because after blowing down the (-1)-curve V, the blow-down of E_i is only one (-1)-curve, connected to a chain of $\mathbb{P}^1$s. So all the remaining curves have to be (-2)-curves.
- We have that the component of $G_{E_i} \setminus V$ that contracts into V' is exactly a (-2)-curve. Because, if to this component we add a (-1)-curve to the vertex connected to just one other vertex, then by Corollary 2 inside the component no divisor in a center can be blow-down. After doing the blow-downs in the new order, we end up with a (-2)-curve connected to a (-1)-curve. So everything is blown-down, and therefore there were no divisors in a center in the component. If there was a (-1)-curve in the component, then there would be some centers. So in this component there is no (-1)-curve, and it remains unchanged after the blow-downs.
- We have that the curve V is part of a T-chain. Because V' must be intersecting a T-chain (or be inside one) and a (-2)-curve cannot be a T-chain.
- We have that the curve V' must be part of a T-chain. Otherwise, we can remove V' and add a (-1)-curve to the vertex in E_i that is connected to one vertex. In this situation we can apply Corollary 2 to G_{E_i}, so no center gets contracted. But when V becomes a (-1)-curve, E_i becomes:

$$(-1)-(-3)-(-2)-\ldots-(-2)-(-1).$$

Since there are at least two T-chains and only one has curves outside of E_i, we get a contracted T-chain, a contradiction.

- We have that there is at most one divisor inside a center in the component of $G_{E_i} \setminus V$ that is not V'. Because, if we add a (-1)-curve to the vertex in this component connected to just one other vertex, then by Corollary 2 in the component no divisor in a center can be blown-down. After doing the blow-downs in the new order, we end up with a (-3)-curve connected to a chain of (-2)-curves connected to a (-1)-curve. So only one curve is not contracted after doing just the blow-downs inside this component, and therefore there was at most one curve in a center in the component.

Now we divide in cases, according to which components have curves of the T-chain containing V.

Case B.I: The component of $G_{E_i} \setminus V$ that is contained in E_i, that is not V', does not contain curves from the T-chain that contains V.

If this component does not contain any (-1)-curve, then originally it must be only the (-3)-curve (as the (-2)-curves must intersect some T-chain). This curve must be part of a T-chain. Otherwise, we can remove it and add a (-1)-curve to the vertex in E_i that is connected to one vertex. In this situation we can apply Corollary 2, so no center gets contracted. But after doing the blow-downs in the new order, V can be blow-down and then V' can be blow-down. Since there are at least two T-chains and only one has curves outside of E_i, we get a contracted T-chain, a contradiction. Since a (-3)-curve is not a T-chain, there must be a (-1)-curve in the component of $G_{E_i} \setminus V$ that is contained in E_i, that is not V'.

The (-1)-curve in this component must be connected to V, so the component must be of the form:

$$(-1) - (-2) - \ldots - (-2) - (-4) - (-2) - \ldots - (-2).$$

Since the (-1)-curve has to intersect two curves in T-chains, the only possibilities for it are to be $(-1) - [4]$ or $(-1) - [4] - (-2)$.

Now, suppose there is a (-1)-curve in the component of $G_{E_i} \setminus V$ containing curves outside of E_i. Then the T-chain containing V cannot have a center in any component of $G_{E_i} \setminus V$. So, V is a center divisor. But after doing all the blow-downs in the component of $G_{E_i} \setminus V$ containing curves outside of E_i, the vertex V cannot become a (-2)-curve. Because we could change the T-chain for the T-chain generated by the same algorithm, but starting from $[3, 3]$, and if the vertex V where to become a (-2)-curve, then one of the new center divisors would get contracted, contradicting Corollary 2. Therefore, after doing all blow-downs in the new order V cannot become a (-1)-curve, a contradiction. So the component of $G_{E_i} \setminus V$ with curves outside of E_i

contains no (-1)-curve. Therefore V is a (-2)-curve, and we would get the situation of Figure 0.12 or Figure 0.13.

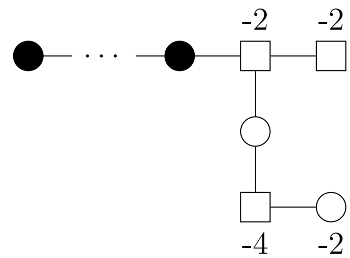

FIGURE 0.12. CASE B.I.1

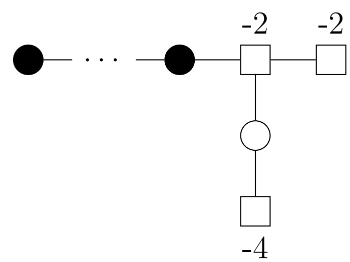

FIGURE 0.13. CASE B.I.2

Case B.II: The component of $G_{E_i} \setminus V$ that is contained in E_i, that is not the one containing V', contains curves from the T-chain that contains V. If this component does not contain any (-1)-curve, then originally it must be $(-3)-(-2)-\ldots-(-2)$. so the T-chain containing V is

$$[2, X, 3, 2, \ldots, 2]$$

So the only possibility for the T-chain containing V is $[2, 5, 3]$ which gives us the case in Figure 0.14 or Figure 0.15.

Now if the component that becomes $[3, 2, \ldots, 2]$ contains a (-1)-curve, then there is a T-chain contained in it. Since only one center divisor can be in the component, then there is at most one T-chain and its center divisor becomes the (-3)-curve, so the chain of (-2)-curves is unchanged by the blow-downs. Therefore this T-chain has only (-2)-curves at one side of its center, so it is of one of the following forms $[4+n, 2, \ldots, 2]$, $[2+n, 5, 2 \ldots, 2]$, $[2+n, 2, \ldots, 2, 5+m, 2, \ldots, 2]$ or $[2, \ldots, 2, 4+n]$.

Since no center divisor of the T-chain containing V can be in any component of $G_{E_i} \setminus V$, V is its center. In order to have the $3, 2, \ldots, 2$

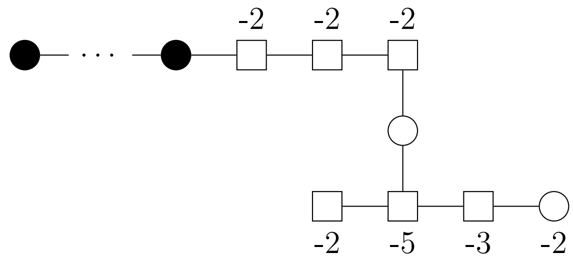

FIGURE 0.14. CASE B.II.0

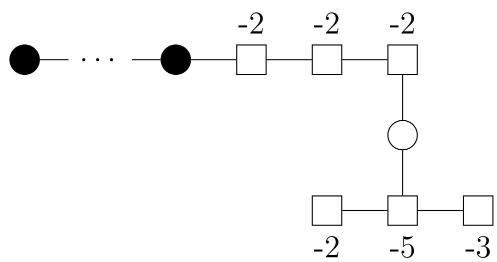

FIGURE 0.15. CASE B.II.1

chain after some blow-downs, the curves to the side of V, which is not V', have to be $[2,\dots,2]$, $[3,2,\dots,2]$, $[2.\dots,2,m+3,2,\dots,2]$ or $[2,\dots,2,n+2]$ corresponding to the possibilities of the T-chain that becomes $[3,2,\dots,2]$. The only case that yields a T-chain would give us that the T-chain containing V is $[2,5,3]$. This case gives us $[2,5,3]-(-1)-[2,5]$, where the discrepancies of the curves intersecting the (-1)-curve add up to more than -1, a contradiction. $\square$

It is easy to verify that the set of all T-chains which are not contained completely in E_i is one of seven cases in Figures 0.16–0.22. Assuming $E_i\cdot\left(\sum_{C_{k,j}\not\subseteq E_i} C_{k,j}\right)=1$, we get that in cases 2, 3, 4 and 6 the graph G_{E_i} is a tree. In the other cases there could be cycles inside G_{E_i}. We now explain how the classification for the cases when G_{E_i} is a tree gives a classification for the cases when G_{E_i} is not a tree. Assume that G_{E_i} is not a tree. Then we are in case 1, 5 or 7. We analyse each case separately.

Case 1: We construct in a combinatorial way a new graph G'_{E_i} in the following way. In G_{E_i} there are one or two curves in E_i connected to the T-chain which is not contained in E_i. Disconnect one of these intersections to this T-chain and reconnect it, instead, to the corresponding vertex in a new equal T-chain. Then G'_{E_i} is a tree. After doing the corresponding blow-downs, the same curves as in G_{E_i} are

FIGURE 0.16. Case 1

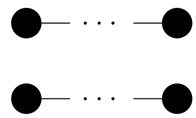

FIGURE 0.17. Case 2

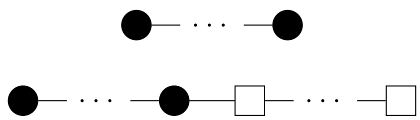

FIGURE 0.18. Case 3

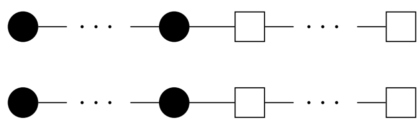

FIGURE 0.19. Case 4

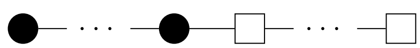

FIGURE 0.20. Case 5

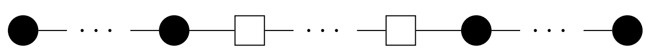

FIGURE 0.21. Case 6

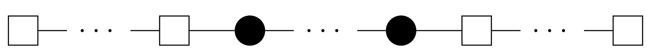

FIGURE 0.22. Case 7

contracted, and it fulfills the same combinatorial restrains. Hence G'_{E_i} satisfies the classification in Proposition 17 or Proposition 19. So it must be as in Figure 0.1, and so the original G_{E_i} has to be as in Figure 0.23.

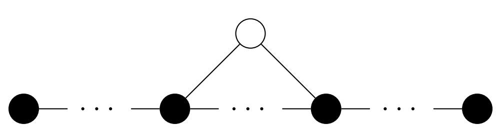

FIGURE 0.23. Case C.1*

Case 5: We again construct in a combinatorial way a new graph G'_{E_i}. In G_{E_i}, there is a curve in E_i connected to a vertex not in E_i

in the T-chain not contained in E_i. Disconnect this curve to this T-chain and connect it, instead, to the corresponding vertex of a new equal T-chain. Then G'_{E_i} is a tree. After doing the corresponding blow-downs, the same curves as in G_{E_i} are contracted, and it fulfills the same combinatorial restrains. Then G'_{E_i} satisfies the classification in Proposition 17 or 19. So it must be as in Figure 0.2, 0.4, 0.8 or 0.9 so the original G_{E_i} has to be as in Figures 0.24, 0.25, 0.26 or 0.27.

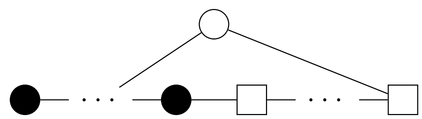

FIGURE 0.24. Case C.2*

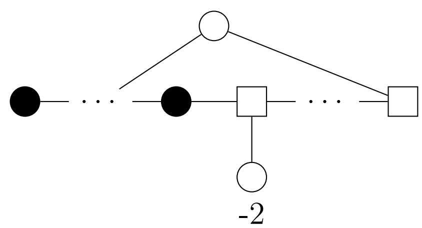

FIGURE 0.25. Case A.1*

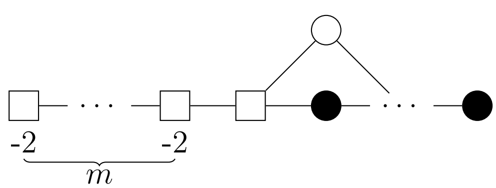

FIGURE 0.26. Case A.II.1*

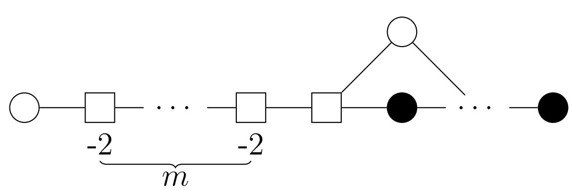

FIGURE 0.27. Case A.II.2*

Case 7: Once more, we construct in a combinatorial way a new graph G'_{E_i} in the following way. We change the T-chain not contained in E_i for two equal T-chains, changing from Figure 0.28 to Figure 0.29.

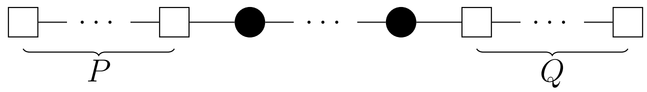

FIGURE 0.28

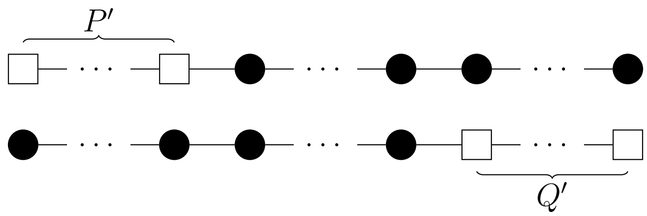

FIGURE 0.29

We now connect to a curve in P' (respectively Q') whichever was connected to the corresponding curve in P (respectively Q). So G'_{E_i} is a tree. After doing the corresponding blow-downs, the same curves as in G_{E_i} are contracted and it fulfills the same combinatorial restrains. Then G'_{E_i} satisfies the classification in Proposition 17 or Proposition 19. So it must be as in Figure 0.3, 0.5, 0.6, 0.7, 0.8, 0.9, 0.10 or 0.11. For cases as in Figures 0.3 or 0.5 the discrepancies of the contracted end-curves add up less than -1, but in the original T-chain this would yield a contradiction, as the discrepancies of end-curves add up -1 In each of the other cases both T-chains not contained in E_i have chains of (-2)-curves which will produces a contradiction to the original G_{E_i}, since a T-chain does not have (-2)-curves in both ends. So, this case does not yield any possibilities.

All in all, we now can prove the following.

PROPOSITION 20. *If $E_i \cdot C = 1$, then G_{E_i} is one of the graphs in Figures 0.30 to 0.38.*

PROOF. In order to have $E_i \cdot C = 1$, we need to have either $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j}\right) = 1$ and $C_{k,j} \cdot E_i = 0$ for every $C_{k,j} \subset E_i$, or $E_i \cdot \left(\sum_{C_{k,j} \not\subseteq E_i} C_{k,j}\right) = 2$ and $C_{k,j} \cdot E_i = -1$ for a $C_{k,j} \subset E_i$. The first case is impossible by Proposition 5. The second case case gives figure 0.24 and 0.26 as the only possible graphs with cycles, and the possibilities from Propositions 17 and 19 in which the last curve of E_i to be blown-down is inside a T-chain. This gives us the desired figures.

□

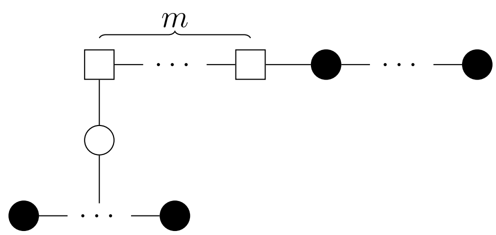

Figure 0.30. Case C.2

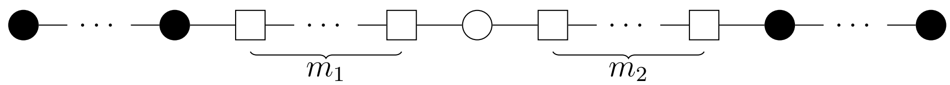

Figure 0.31. Case C.3

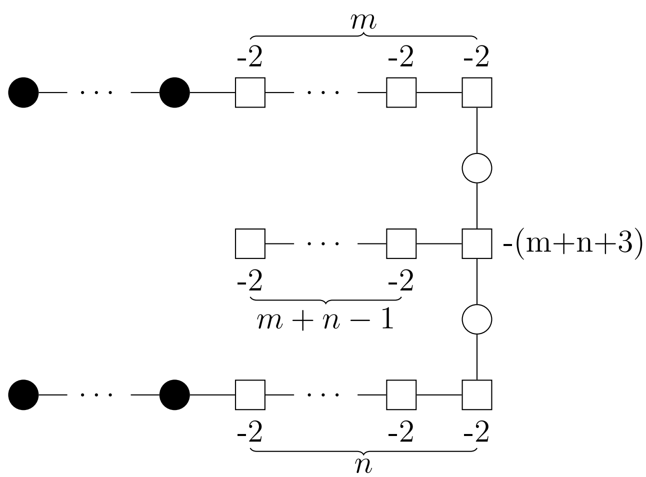

Figure 0.32. Case A.I.1

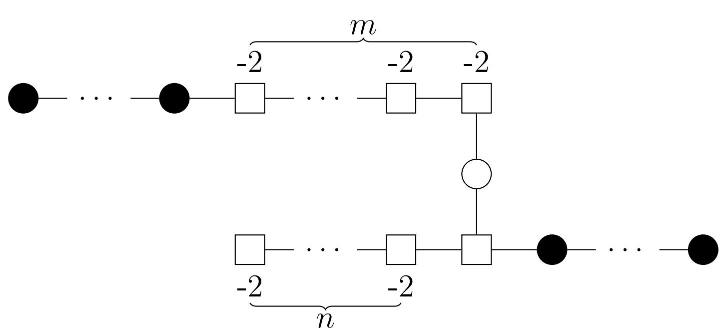

Figure 0.33. Case A.II.1

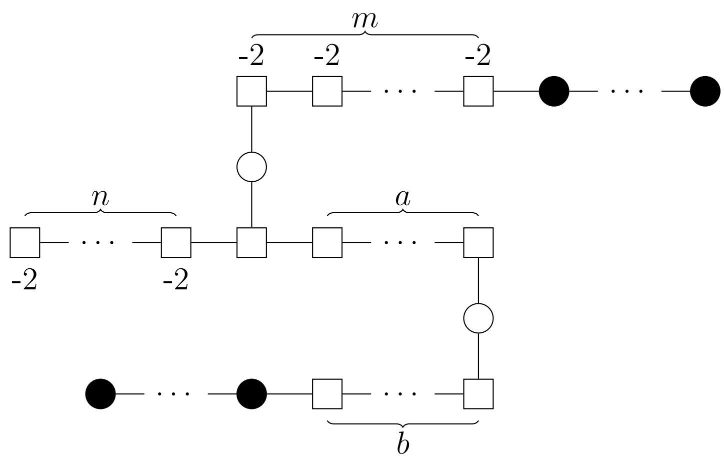

FIGURE 0.34. CASE A.II.3

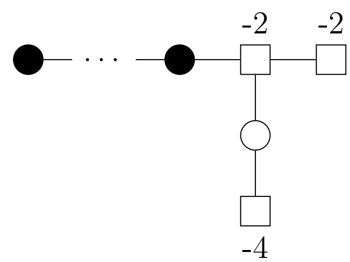

FIGURE 0.35. CASE B.I.2

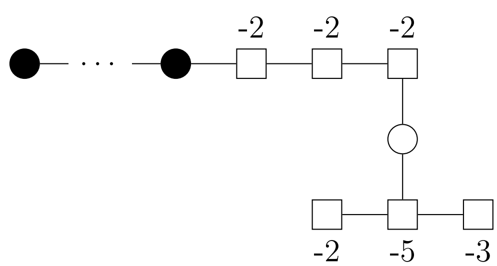

FIGURE 0.36. CASE B.II.1

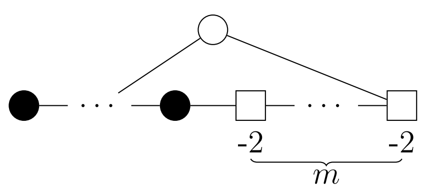

FIGURE 0.37. Case C.2*

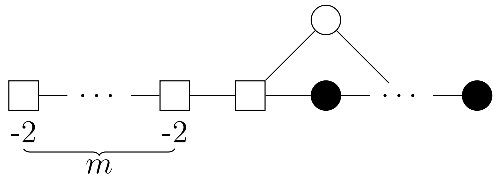

FIGURE 0.38. Case A.II.1*

Bounding

Let us call an E_i **maximal** if $E_i \cdot C = 1$ and it is not contained in any other E_j with $E_j \cdot C = 1$. We are now going to study the number of maximal E_i's in a given situation. Using the fact that two distinct E_js are either disjoint or one is contained in the other, and that by Proposition 20 any maximal E_i has an ending (-2)-curve of some T-chain, we obtain that there are at most l maximal E_is.

We now define the following directed graph. We have one vertex corresponding to each T-chain. The idea is to assign as many edges as end-curves with discrepancies greater than or equal to $-\frac{1}{2}$ which are contained in some maximal E_i, and decorate each edge with the number of $E_j \subset E_i$ with $E_j \cdot C = 1$. For this we do the following construction.

- For every maximal E_i as in Figure 0.30 or Figure 0.31, we construct an edge between the vertices corresponding to the T-chains connected to the (-1)-curve in E_i. Make the edge to point away from the T-chain with the (-2)-curve connected to the (-1)-curve. To the edge assign the number m or $m_1 + m_2$.
- For every maximal E_i as in Figure 0.32, we construct three edges. The first edge connecting the T-chain that is completely contracted in E_i to itself, and assign to it the number 1. The second and third edges connecting the T-chain that is completely contracted in E_i to each of the other T-chains with contracted curves in E_i. Make each of this edges pointing to the T-chain that is completely contracted. Assign to each of these edges the number of (-2)-curves contracted in the vertex from which they point away, i.e. m and n.
- For every maximal E_i as in Figure 0.33, if $n \neq 0$, then we construct two edges. The first edge connecting the T-chains connected to the (-1)-curve in E_i, pointing away from the T-chain with the (-2)-curve connected to the (-1)-curve. Assign to this edge the number m. The second edge connecting the T-chain with a triple point in E_i to itself. Assign to this edge the number 1.

If $n = 0$, then we only construct an edge connecting the T-chain which has contracted curves in E_i to itself. Assign the number 1 to this edge.

- For every maximal E_i as in Figure 0.34, we construct three edges. The first edge connecting the T-chain that is completely contracted in E_i to itself and assign to it the number 1. The second and third edges connecting the T-chain that is completely contracted in E_i to each of the other T-chains with contracted curves in E_i. Make each of these edges pointing to the T-chain that is completely contracted. Assign to the edge corresponding to the T-chain with m (-2)-curves the number m and to the other edge assign the number $a + b$.
- For every maximal E_i as in Figure 0.35, we construct two edges. The first edge connecting the T-chain with only one curve to itself and assign to it the number 1. The second connecting the T-chains that are connected to the (-1)-curve in E_i, pointing to the T-chain with only one curve. Assign to it the number 1.
- For every maximal E_i as in Figure 0.36, we construct two edges. The first edge connecting $[2, 5, 3]$ T-chain to itself and assign to it the number 2. The second connecting the T-chains that is connected to the (-1)-curve in E_i, pointing to the $[2, 5, 3]$ T-chain. Assign to it the number 3.
- For every maximal E_i as in Figure 0.37, we construct an edge connecting the T-chain with contracted curves in E_i to itself. Assign to it the number m.
- For every maximal E_i as in Figure 0.38, we construct and edge connecting the T-chain with contracted curves in E_i to itself. Assign to it the number 1.

Each of these cases is shown in Figures 0.39 to 0.46

FIGURE 0.39. Case C.2/ Case C.3

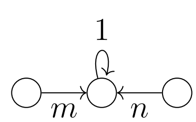

FIGURE 0.40. Case A.I.1

$m \neq 0$ m=0

FIGURE 0.41. Case A.II.1

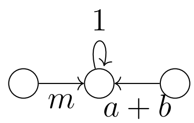

FIGURE 0.42. Case A.II.3

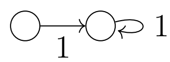

FIGURE 0.43. Case B.I.2

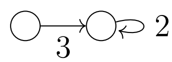

FIGURE 0.44. Case B.II.1

For every maximal E_i the sum of the numbers assigned to the edges constructed corresponding to it is equal to the number of $E_j \subset E_i$ with $E_j \cdot C = 1$. So, if we add the numbers assigned to the edges of this directed graph, then we obtain the number of E_is with $E_i \cdot C = 1$.

Now we consider every connected component of the directed graph, and bound the sum of the numbers assigned to the edges for each of them.

In a connected component with l' vertices, there are at least $l' - 1$ edges (not including those that connect only one vertex). By our restrictions, there are at most l' edges, including those that connect only one vertex. So, there is at most one edge connecting only one vertex. Therefore there is at most one case from Figures 0.32 to 0.38.

PROPOSITION 21. *Let $D = \sum d_j$, $R = \sum r_j$, and $\lambda = K_S \cdot \pi(C)$. We have*

$$R - D \leq 2(K_W^2 - K_S^2) + Z - \lambda,$$

where Z is the number of E_i with $E_i \cdot C = 1$.

PROOF. By Lemma 2, Corollary 3 and Lemmas 4 and 5, we have

$$R - D + 2l = \sum E_i \cdot C + \lambda \geq 2m - Z + \lambda.$$

FIGURE 0.45. Case C.2*

FIGURE 0.46. Case A.II.1*

The result follows since $K_S^2 - m + R - D + l = K_W^2$. $\square$

Now our plan will be to study the number of E_j with

$$E_j \cdot C = 1,$$

and $E_j \subset E_i$, for every possible G_{E_i} as in Proposition 20.

REMARK 4. *Let Γ be a $\mathbb{P}^1$ in X. By the adjunction formula, we have $K_X \cdot \Gamma = -2 - \Gamma^2$. Let Δ be a (-1)-curve, and assume $\Delta \cdot \Gamma = m$. Then after blowing-down Δ we obtain that the intersection of the canonical class with the image of Γ is $-2 - \Gamma^2 - m$. Therefore, if K_S is nef then $\Gamma^2 \leq -(\sum m_i) - 2$, where the m_is are the multiplicities corresponding to the various blow-downs.*

REMARK 5. *For any T-chain $[x_1, x_2, \ldots, x_n]$, we have $r - d + 2 = \sum(x_i - 2)$ (see* [**RU17**]*). We will use this several times in the next propositions.*

PROPOSITION 22. *If we have a maximal E_i as in Figure 0.30, assuming K_S is nef, we have $r_1 - d_1 \geq m$ and $r_2 - d_2 \geq m$, where r_x, d_x are the values in the T-chains that intersect curves in E_i.*

PROOF. The T-chain which has curves in E_i is one of three possibilities:

$$[2, \ldots, 2, 4 + m]$$

$$[2, \ldots, 2, 3, 2, \ldots, 2, 3 + m]$$

$$[2, \ldots, 2, x_1, \ldots, x_h, 2 + m]$$

So, by Remark 5, we have $r_1 - d_1 + 2 \geq m + 2$, with equality only in the first two cases, and the curve in the other T-chain, intersecting the (-1)-curve must have self-intersection less than or equal to $-(m + 3)$. So, by Remark 5, we have $r_2 - d_2 + 2 \geq m + 1$, with equality only if the T-chain is $[2, \ldots, 2, m + 3]$. If we replace this T-chain by the T-chain made by the same algorithm, but starting from $[3, 3]$, then a center-divisor would be contracted, a contradiction by Proposition 15. So $r_2 - d_2 \geq m$. $\square$

PROPOSITION 23. *If we have a maximal E_i as in Figure 0.31, assuming K_S is nef, we have $r_1 - d_1 \geq m_1 + m_2$ and $r_2 - d_2 \geq m_1 + m_2$, wherer$_x, d_x$ are the values in the T-chains that intersect curves in E_i.*

PROOF. By the Proposition 15, no curve in a center is blown-down, so we can replace the T-chains by those made with the same algorithms, but starting from $[3, 3]$. So r_x, d_x remains the same and we can calculate more easily some self-intersections. If the parts of the T-chains that get blow-down are

$$[w_1, \dots, w_{m_1}] - (-1) - [y_1, \dots, y_{m_2}],$$

then the T-chains are

$$[y_1, \dots, y_{m_2}, c_1 \dots, c_h, w_1, \dots, w_{m_1}]$$

$$[y_1, \dots, y_{m_2}, c'_1 \dots, c'_{h'}, w_1, \dots, w_{m_1}].$$

We have that $\sum(y_i - 2) + c_1 - 2 \geq m_1$ and $\sum(w_i - 2) + c_h - 2 \geq m_2 + 1$. So adding everything and using Remark 5, we obtain the result. With $\sum(y_i - 2) + c_1 - 2 = m_1$, only if there is no center divisor in $\{y_1, \dots, y_{m_2}, c_1\}$, and since no center can be in $\{w_1, \dots, w_{m_1}\}$, we have $r_1 - d_1 > z$. We can do the exact same analysis for the other T-chain and obtain the desired result. □

PROPOSITION 24. *If we have a maximal E_i as in Figure 0.32, assuming K_S is nef, we have $r_1 - d_1 = m + n - 1$, $r_2 - d_2 \geq 2m$, $r_3 - d_3 \geq 2n$, where r_x, d_x are the values in the T-chains that intersect curves in E_i, r_1, d_1 for the T-chain with m (-2)-curves, r_2, d_2 for the T-chain with $m + n - 1$ (-2)-curves and r_3, d_3 for the T-chain with n (-2)-curves.*

PROOF. By Remark 5, we have $r_2 - d_2 + 2 = m + n + 1$. In both T-chains that are not contained in E_i, the curve next to the chain of (-2)-curves must have self-intersection less or equal to $-(3+m+n)$, or they would be blown-down, a contradiction. If one of these T-chains has only one curve outside of E_i, without loss of generality $[4+n, 2, \dots, 2]$, then $4 + n = 3 + m + n$, so $m = 1$, and therefore the T-chain in E_i is $[4 + n, 2, \dots, 2]$. So,the discrepancies in the curves intersecting a (-1)-curve add up to exactly -1, a contradiction. Therefore, the end-curves have self-intersections at most $-(n+2)$ and $-(m+2)$. Hence, by Remark 5, $r_1 - d_1 + 2 \geq (m+n+1)+m$ and $r_3 - d_3 + 2 \geq (m+n+1)+n$. Using the fact that $m > 0$, $n > 0$ we get the desired results. Since if $m = 0$, then the discrepancies in the (-1)-curve connecting the T-chains with contracted curves, do not add up to less than -1, a contradiction. □

Proposition 25. *If we have a maximal E_i as in Figure 0.33, assuming K_S is nef, we have $r_1 - d_1 \geq m$, $r_2 - d_2 \geq m + 1$, where r_x, d_x are the values in the T-chains that intersect curves in E_i, r_1, d_1 for the T-chain with m (-2)-curves and r_2, d_2 for the T-chain with n (-2)-curves.*

Proof. The curve in the T-chain with no triple point in E_i, intersecting the (-2)-curve in E_i, must have self-intersection less than or equal to $-(4+n)$. If this is the only curve, then it has self-intersection $-(4+m)$, otherwise the end-curve must have self-intersection $-(m+2)$. So, by Remark 5, $r_1-d_1+2 \geq m+2$. In the other T-chain, the curve intersecting the (-1)-curve has self intersection $-(m+2)$, the curve next to it must have self-intersection less than or equal to $-(n+3)$ and the end-curve $-(n+2)$. So, by Remark 5, $r_2-d_2+2 \geq m+(n+1)+n \geq m+3$ and we get the desired result. $\square$

Proposition 26. *If we have a maximal E_i as in Figure 0.34, assuming K_S is nef, we have $r_1 - d_1 \geq m$, $r_2 - d_2 \geq m + n - 1$ and $r_3 - d_3 \geq m + 2n - 2$, where r_x, d_x are the values in the T-chains that intersect curves in E_i, r_1, d_1 for the T-chain with m (-2)-curves, r_2, d_2 for the T-chain with n (-2)-curves and r_3, d_3 for the other T-chain.*

Proof. If we re-order the blow-downs, and do not do any blow-down in the component with m (-2)-curves, then the center curve of the T-chain in E_i intersects only one blow-down curve. Because we can remove the T-chain with m (-2)-curves and the (-1)-curve intersecting it, and change the T-chain with n (-2)-curves, by the T-chain made by the same algorithm, but starting from $[3, 3]$. So we would have $[2, \ldots, 2, t + 3, 3, 2, \ldots, 2, 2 + n] - (-1) - [c_1, \ldots, c_{r_3}]$. By Corollary 2, no center curve could be blown-down. So, the $-(3)$-curve in the T-chain cannot be blown-down, meaning it intersects only one blow-down curve. Therefore in the original situation the center curve of the T-chain in E_i intersects only one blow-down curve. So we can calculate all self-intersections in E_i, which are showed in Figure 0.47.

The vertex V, has self-intersection less than or equal to $-(m+n+2)$, or it would be contracted. So, the vertex V' has self-intersection $-(n+2)$, therefore $r_3-d_3+2 \geq m+2n$. The vertex V'' could have have self-intersection $-(m+4)$, $-(m+3)$ or $-(m+2)$, in the second and third case, the curve in this T-chain intersecting the $-(2)$-curves would have self-intersection less than or equal to $-(n+4)$. So, by Remark 5, $r_1-d_1+2 \geq m+2$. By adding up these inequalities we get the desired result.

$\square$

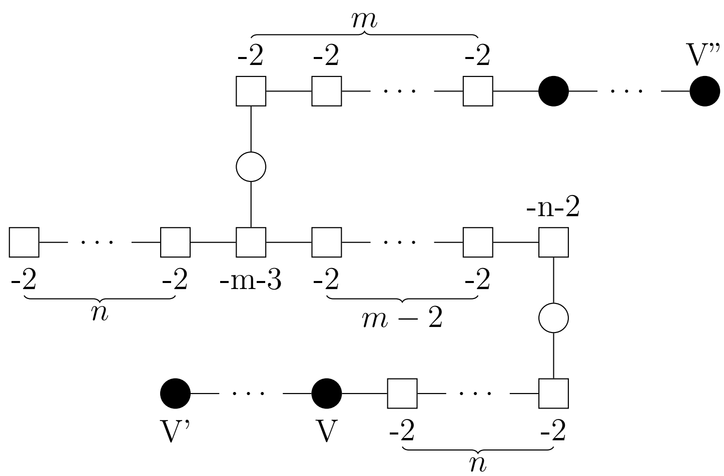

FIGURE 0.47. CASE A.II.3

PROPOSITION 27. *If we have a maximal E_i as in Figure 0.35, assuming K_S is nef, we have $r_1 - d_1 \geq 2$, where r_1, d_1 are the values in the T-chain that intersect curves in E_i, which is not* [4].

PROOF. The curve intersecting the first (-2)-curve to be contracted, must have self-intersection less or equal to (-5). If this is the only curve, then it has self-intersection (-6), otherwise, the end-curve has self-intersection -4. So, by Remark 5 we have the desired formula. □

PROPOSITION 28. *If we have a maximal E_i as in Figure 0.36, assuming K_S is nef, we have $r_1 - d_1 \geq 3$. $r_2 - d_2 = 2$, where r_1, d_1 are the values in the T-chains that intersect curves in E_i which is not* $[2,5,3]$ *and r_2, d_2 are the values in* $[2,5,3]$.

PROOF. The curve that is not contracted and intersects a (-2)-curve in E_i must have self-intersection less or equal to -6. If this is the only curve, then it has self-intersection (-7), otherwise the end-curve must have self-intersection (-5). So by Remark 5 we have the desired formula. □

PROPOSITION 29. *If we have a maximal E_i as in Figure 0.37, assuming K_S is nef, we have $r_1 - d_1 \geq 2m$, where r_1, d_1 are the values in the T-chains that intersect curves in E_i.*

PROOF. The (-1)-curve intersects a curve not in E_i. It cannot be the end curve, as the discrepancies of the curves intersecting the (-1)-curve would add up to exactly (-1), a contradiction. If it is the curve intersecting a (-2)-curve contained in E_i, then it must have self-intersection less than or equal to $-(m+4)$, and the end-curve must have self-intersection $-(m+2)$. Otherwise, the curve intersecting the (-2)-curve contained in E_i must have self-intersection less than or equal to $-(3)$ and the curve intersecting the (-1)-curve must have self-intersection less than or equal to $-(m+3)$, and the end-curve must have self-intersection $-(m+2)$. In both cases, by Remark 5, we obtain the desired result. $\square$

REMARK 6. *In Figure 0.38 there is no other $E_j \subset E_i$ with $E_j \cdot C = 1$ and $r - d \geq m + 1$ by Remark 5, if $m = 0$, the discrepancies of the curves intersecting the (-1)-curve add exactly -1, a contradiction. So $r - d \geq 2$.*

We now analyse some special properties of graphs, which will come in handy to join all the information from the bounds in each possible maximal E_i.

LEMMA 6. *Let G be a finite graph which is a tree, with a fixed vertex V_1. Then there is a bijection from the rest of the vertices to the edges, such that every vertex correspond to an edge that is connected to it.*

PROOF. We can do this inductively on the number of vertices, for the case of two vertices is trivial. Now if the lemma holds for $p-1$ vertices, then we consider a tree G with p vertices. A leaf is a vertex with only one edge, in a tree with more than one vertex there are always two or more leaves. There is a leaf V in G that is not V_1, we send V to the edge connected to it L. Now in $G \setminus \{V, L\}$ the lemma holds and we obtain the bijection for G. $\square$

LEMMA 7. *Let G be a finite graph with p vertices which is a tree. Then it is possible to assign to each edge $L_{VV'}$ two natural numbers $L_{VV'}(V)$, $L_{VV'}(V')$ such that the following equations hold:*

$$L_{VV'}(V) + L_{VV'}(V') = p,$$
$$\sum_{V'} L_{VV'}(V) = p - 1.$$

PROOF. For a graph with 2 vertices, we can simply put $L_{V_1V_2}(V_1) = L_{V_1V_2}(V_2) = 1$. We will do induction on the number p of vertices of G. Assume that the Lemma is true for graphs with p-1 vertices. Let G have p vertices. We take a leaf V_1 in G connected to vertex V_2. We put $L_{V_1V_2}(V_1) = p - 1$ and $L_{V_1V_2}(V_2) = 1$. We now consider the graph $G \setminus \{V_1, L_{V_1V_2}\}$. Thanks to Lemma 6 we can associate each edge to a vertex different than V_2. By the induction hypothesis we get the numbers $L'_{VV'}(V), L'_{VV'}(V')$ for each edge. We define $L_{VV'}(V) = L'_{VV'}(V) + i$, where $i = 1$ if $L_{VV'}$ is associated to V and $i = 0$ otherwise.

This way we have the desired properties for the numbers assigned to the edges. $\square$

COROLLARY 4. *Let G be a finite graph with p vertices which is a tree. Assume we have assigned to each vertex V and edge $L_{VV'}$ the real numbers a_V and $b_{VV'}$ respectively such that $a_V \geq b_{VV'}$ and $a_{V'} \geq b_{VV'}$. Then $(p-1)\sum a_V \geq p \sum b_{VV'}$.*

PROOF. We get the numbers associated to the edges in Lemma 7 and add the inequalities $L_{VV'}(V)a_V \geq L_{VV'}(V)b_{VV'}$, $L_{VV'}(V')a_{V'} \geq L_{VV'}(V')b_{VV'}$ for every edge to get the desired inequality. $\square$

COROLLARY 5. *Let G be a finite graph with p vertices which is a tree and a fixed vertex V_1. Assume we have assigned to each vertex V and edge $L_{VV'}$ the real numbers a_V and $b_{VV'}$ respectively such that $a_V \geq b_{VV'}$ and $a_{V'} \geq b_{VV'}$. Then*

$$(p-1)a_{V_1} + (2p-1)\sum_{V \neq V_1} a_V \geq 2p \sum b_{VV'}.$$

PROOF. By Lemma 6 we obtain the numbers $i_{VV'}(V), i_{VV'}(V')$ where $i_{VV'}(V) = 1$ if the bijection sends V to $L_{VV'}$ and $i_{VV'}(V) = 0$ otherwise. Now we add the inequalities $i_{VV'}(V)a_V \geq i_{VV'}(V)b_{VV'}$, $i_{VV'}(V')a_{V'} \geq i_{VV'}(V')b_{VV'}$ for every edge to obtain the inequality:

$$\sum_{V \neq V_1} a_V \geq \sum b_{VV'}.$$

Adding this inequality $(p-1)$ times to the inequality from Corollary 5 we obtain the desired result. $\square$

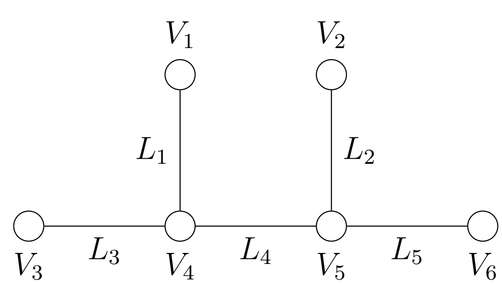

FIGURE 0.48

EXAMPLE 3. *If we have a graph as in Figure 0.48, then the numbers in Lemma 7 can be computed following the induction process and they are $L_1(V_1) = 5$, $L_1(V_4) = 1$, $L_2(V_2) = 5$, $L_2(V_5) = 1$, $L_3(V_3) = 5$,*

$L_3(V_4) = 1,\ L_4(V_4) = 3,\ L_4(V_5) = 3,\ L_5(V_5) = 1,\ L_5(V_6) = 5.$ *If we fix V_6 the bijection in Lemma 6 would send V_i correspond to L_i.*

Proposition 30. *If K_S is nef and in a connected component of the directed graph there is no cycle, where we include a vertex connected to itself as a cycle, then $(l'-1)(R'-D') \geq l'Z'$, where l' is the number of T-chains associated to the vertices in the component, R', D' are the sums of the r_i, d_i of these T-chains, and Z' is the sum of the values in the edges of the component of the directed graph.*

Proof. There can only be maximal E_i as in Figure 0.30 or 0.31. So, using the Propositions 22 to 23, we have that $r_V - d_V \geq z_{L_{VV'}}$, where r_V, d_V are the values in the T-chain corresponding to vertex V in the directed graph, and $z_{L_{VV'}}$ is the value in an edge connected to V. So it is enough to use Corollary 4 with $a_V = r_V - d_V$ and $b_{VV'} = z_{L_{VV'}}$. $\square$

Proposition 31. *If K_S is nef, and in a connected component of the directed graph there is a maximal E_i as in Figure 0.32 to 0.35 or Figure 0.38, then $(l'-1)(R'-D') \geq l'Z' - l'$, where l' is the number of T-chains associated to the vertices in the component, R', D' are the sums of the r_i, d_i of these T-chains, and Z' is the sum of the values in the edges of the component of the directed graph.*

Proof. By Propositions 24 to 27 we have $r_V - d_V \geq z_{L_{VV'}}$, where r_V, d_V are the values in the T-chain corresponding to vertex V in the directed graph, and $z_{L_{VV'}}$ is the value in an edge connected to V. So we can use Corollary 4 on the graph after removing the cycle, with $a_V = r_V - d_V$ and $b_{VV'} = z_{L_{VV'}}$. So it is enough to notice that $\sum b_{VV'} = Z' - 1$, because we are missing the cycle.

$\square$

Proposition 32. *If K_S is nef, and in a connected component of the directed graph there is a maximal E_i as in Figure 0.36, then $(l'-1)(R'-D') \geq l'Z' - 3l' + 1$, where l' is the number of T-chains associated to the vertices in the component, R', D' are the sums of the r_i, d_i of these T-chains, and Z' is the sum of the values in the edges of the component of the directed graph.*

Proof. Let V_1 be the vertex corresponding to the vertex with a cycle. We have $r_V - d_V \geq z_{L_{VV'}}$, where r_V, d_V are the values in the T-chain corresponding to vertex $V \neq V_1$ in the directed graph, and $z_{L_{VV'}}$ is the value in an edge connected to V. For V_1 we have $r_{V_1} - d_{V_1} = 2 \geq z_{L_{VV_1}} - 1$ So we can use Corollary 4 with $a_V = r_V - d_V$ for $V \neq V_1$ and $a_{V_1} = r_{V_1} - d_{V_1} + 1$ and $b_{VV'} = z_{L_{VV'}}$. Noticing that $\sum b_{VV'} = Z' - 2$, because we are missing the cycle, we obtain

$$(l'-1)(R'-D'+1) \geq l'Z' - 2l'.$$

□

Proposition 33. *If K_S is nef and in a connected component of the directed graph there is a cycle in the directed graph, only having maximal E_is as in Figures 0.30, 0.31 or 0.37, then there is a special vertex V_i and an edge L_j that is part of the cycle connected to it, such that $r_{V_i} - d_{V_i} \geq 2z_{L_j} - 1$, where r_{V_i}, d_{V_i} are the values in the T-chain associated to the vertex V_i and z_j is the value of the edge L_j. The inequality is strict if there is a maximal E_i as in Figure 0.37.*

Proof. If there is a maximal E_i as in Figure 0.37, then by Proposition 29 the vertex connected to itself has the desired property. So, we are left only with the case where every maximal E_i is as in Figure 0.30 or 0.31. We look at the cycle in this context, it must be a directed cycle, since no two edges can point away from the same vertex. To every vertex assign the length of the chain of (-2)-curves that the T-chain has at one of its ends.

If there are vertices with different numbers in the cycle, then we name as V a vertex with the smallest number, such that the vertex (V') which is connected to V with an edge pointing at V' has a bigger number. Then the maximal E_i which gives the edge connecting V and V' has to be as in Figure 0.30, since in Figure 0.31 both T-chains have chains of (-2)-curves of the same length. Also the (-1)-curve connecting the T-chains corresponding to V and V', must intersect the T-chain corresponding to V' at a curve that is not an end-curve. So, $r_{V'} - d_{V'} + 2 \geq y_{V'} + y_V + 1 \geq 2y_V + 2$, where y_V is the length of the chain of (-2)-curves in the T-chain corresponding to vertex V. So, it is enough to notice that y_V is also the number assigned to the edge joining V and V'.

If each vertex in the cycle has the same number, then call V the vertex in the cycle such that the discrepancy at the (-2)-end curve is the biggest possible, such that the vertex (V') which is connected to V with an edge coming out of V, has lower discrepancy at the (-2)-end curve. Then, in the maximal E_i, which gives the edge connecting V and V', the (-1)-curve connecting the T-chains corresponding to V and V', must intersect the T-chain corresponding to V' at a curve that is not an end-curve. Otherwise the discrepancies in the curves intersecting the (-1)-curve would not add up to less than (-1). So it must be as in Figure 0.30. Thus $r_{V'} - d_{V'} + 2 \geq y_{V'} + y_V + 1 = 2y_V + 1$, where y_x is the length of the chain of (-2)-curves in the T-chain corresponding to vertex x.It is enough to notice that we have that y_V is also the number assigned to the edge joining V and V'. $\square$

THEOREM 1. *If K_S is nef, then for any connected component of the directed graph we have $(2l'-1)(R'-D') \geq 2l'Z'-l'$, where l' is the number of T-chains associated to the vertices in the component, R', D' are the sums of the r_i, d_i of these T-chains, and Z' is the sum of the values in the edges of the component of the directed graph. The inequality is strict if in the component there are maximal E_is other than those from Figures 0.30 and 0.31.*

PROOF. For the case when there are no cycles we have a better bound, so we only have to take care of the case when the number of edges is the same as the number of vertices.

If there is a maximal E_i as in Figure 0.32 to 0.36 or Figure 0.38, then by Propositions 31 and 32 we have a better bound, so we can discard these cases.

Let V_1 and L_1 be the vertex and edge from Proposition 33. Using the Propositions 22 to 29, we have that $r_{V_1} - d_{V_1} \geq 2z_{L_1} - 1$ and $r_V - d_V \geq z_{L_{VV'}}$, for all vertices, where r_V, d_V are the values in the T-chain corresponding to vertex V in the directed graph, $L_{VV'}$ is an edge connected to V, and z_L is the value in the edge L. So we can use Corollary 5 removing L_1 from the component with fixed vertex V_1, $a_V = r_V - d_V$ and $b_{VV'} = z_{L_{VV'}}$. So, it is enough to add $l'(r_{V_1} - d_{V_1}) \geq 2l'z_1 - l'$ to the inequality. If we had a maximal E_i as in Figure 0.37, then we would have to add $l'(r_{V_1} - d_{V_1}) \geq 2l'z_1$ instead, to get $(2l'-1)(R-D) \geq 2l'Z'$.

$\square$

PROPOSITION 34. *If $\alpha(R-D) \geq \beta Z - \gamma$, then*

$$R - D \leq 2\frac{\beta}{\beta-\alpha}(K_W^2 - K_S^2) + \frac{1}{\beta-\alpha}\gamma - \frac{\beta}{\beta-\alpha}\lambda.$$

PROOF. This is a direct consequence of Proposition 21. $\square$

THEOREM 2. *If K_S is nef, then*

$$R - D \leq 4L(K_W^2 - K_S^2) + l - 2LK_S \cdot \pi(C),$$

where L is the maximum number of vertices in a connected component of the directed graph, in particular $L \leq l$.

Proof. For any connected component of the directed graph with l' vertices, by Theorem 1 we have $(2l'-1)(R'-D') \geq 2l'Z'-l'$. Since Z' is at least the number of edges we obtain that $(2l'-1)(R'-D') \geq (2l'-1))Z'$, adding this inequality with the right coefficient to the inequality from Theorem 1, we obtain $(2L-1)(R'-D') \geq 2LZ'-l'$. If we do this for every connected component and add all the inequalities, then we obtain

$$(2L-1)(R-D) \geq 2LZ-l.$$

Finally we put together this inequality with Proposition 21.

$\square$

Optimality

EXAMPLE 4. *For arbitrary n, l positive numbers let us have l T-chains, $C_i = [2, \ldots, 2, 3, n+3, X_i, 2, \ldots, 2, 3, n+2]$, where each $2, \ldots, 2$ represents a chain of n twos, $X_1 = 5$ and for $i > 1$ $X_i = 3+i, 2, \ldots, 2, 3$ which has $i-2$ twos. Let us also have a (-1)-curve intersecting transversely the $(-n-3)$-curve and the (-2)-end-curve in C_1 and for $i < l$ let us have a (-1)-curve intersecting transversely the $(-n-2)$-end-curve in C_i and the (-2)-end-curve in C_{i+1}. With no other curve being in the pull-back divisor of π. Which is represented in Figure 0.49, by a dual graph where a white box represents any curve in C that is blow-down by π, a white circle represents a curve that is blow-down by π and is not part of C and a black circle represents a curve in C which is not blow-down by π.*

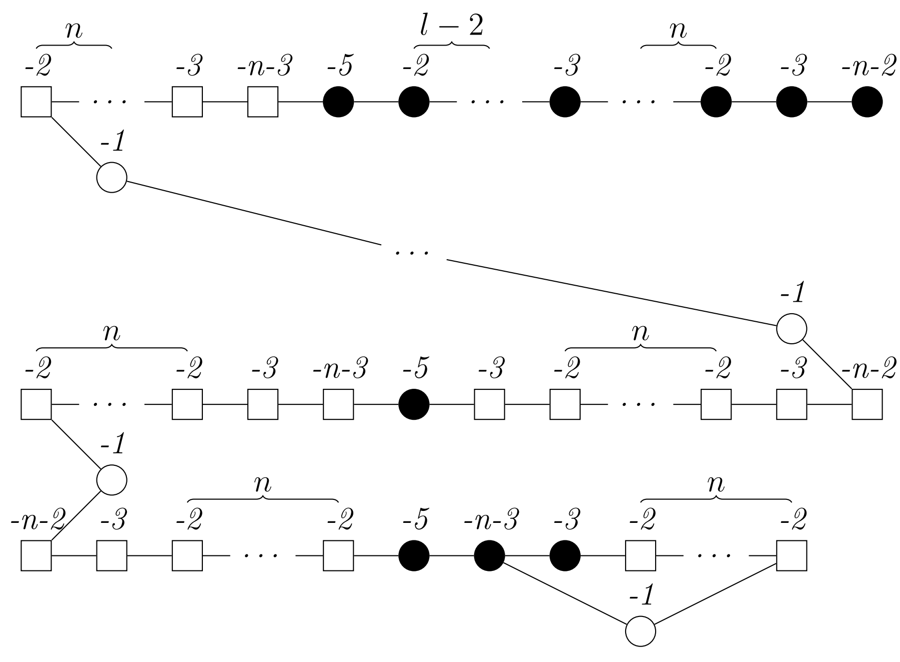

FIGURE 0.49. Optimal Example

For each T-chain, we have $d_i = 1$ *and* $r_i = 2n+4+i$*, so* $R-D = 2ln + \frac{l^2}{2} + \frac{9l}{2}$*. In the directed graph we have a loop with the number* n *and the edge between* C_i *and* C_{i+1} *has the number* $2n+3+i$ *on it, so* $Z = n + (2n+3)(l-1) + \frac{(l-1)(l)}{2} = n(2l-1) + \frac{l^2}{2} + \frac{5l}{2} - 3$*. Therefore* $(2l-1)(R-D) = 2lZ + \frac{7l^2}{2} + \frac{3l}{2}$*. So, we have* $\lim_{n\to\infty} \frac{R-D}{Z} = \frac{2l}{2l-1}$*, which implies that Theorem 1 is maximal in the sense of Proposition 35a). By Lemma 2 we have that* $K_S \cdot \pi(C) = R-D+2l-Z-2l = n+2l+3$ *and* $K_W^2 - K_S^2 = R-D-m+l = R-D-Z = n+2l+3$*. So, we have* $\lim_{n\to\infty} \frac{R-D}{K_W^2-K_S^2} = lim_{n\to\infty} \frac{R-D}{2(K_W^2-K_S^2-K_S\cdot\pi(C))} = 2l$*, which implies that Theorem 2 is maximal in the sense of Proposition 35b).*

Proposition 35. *For fixed* l *and any* $\varepsilon \in \mathbb{R}_{>0}$*, there exists a combinatorial configuration of* l *T-chains such that*

a) $(2l-1)(R-D) < (2l+\varepsilon)Z$.
b) $R-D > 4(L-\varepsilon)(K_W^2-K_S^2) - 2(L-\varepsilon)K_S\cdot\pi(C)$.
c) $R-D > 2(L-\varepsilon)(K_W^2-K_S^2)$.

Remark 7. *Proposition 35a) and 35b), with Theorems 1 and 2 give us optimality in an asymptotic sense, but we do not have a counterpart for Proposition 35c) that bounds* $R-D$ *only in terms of* $K_W^2 - K_S^2$ *optimally.*

What we can do now is to find some properties of the families of combinatorial T-chains that give the optimums in our bounds.

Proposition 36. *If* K_S *is nef and the T-chain corresponding to vertex* V *is contained only in maximal* E_i*s as in Figures 0.30, 0.31 or 0.37, then in the directed graph we have that* $2(r_V - d_V) \geq z_V$*, where* z_V *is the sum of the values in all the edges connected to* V *and* r_V, d_V *are the values in the T-chain corresponding to* V*.*

Proof. This is just combining Propositions 22, 23 and 29, using the fact that the maximal E_is are pairwise disjoint and for two maximal E_is as in Figure 0.31 the $[w_1, \ldots, w_{m_1}]$ and $[y_1', \ldots, y_{m_2'}']$ from Proposition 23 in the same T-chain from different E_is are also disjoint, because the center divisors separate them. □

Proposition 37. *If* K_S *is nef and in a connected component of the directed graph there is no cycle, where we include a vertex connected to itself as a cycle, then* $(Y+1)(R'-D') \geq (Y+2)Z'$*, where* Y *is the number of vertices which are not leaves in the connected component of the directed graph,* R'*,* D' *are the sums of the* r_i*,* d_i *of the T-chains in the component, and* Z' *is the sum of the values in the edges of the component of the directed graph.*

Proof. Let R_1, D_1 and R_2, D_2 be the sums of r_i, d_i in the vertices corresponding to leaves and the rest of them, respectively. Let Z_1 and Z_2 be the sums of the values in the edges connected to leaves and the rest of them, respectively. There can only be maximal E_is as in Figure 0.30 or 0.31. So, using Propositions 22 and 23, we have that $r_V - d_V \geq z_{L_{VV'}}$, where r_V, d_V are the values in the T-chain corresponding to vertex V in the directed graph, and $z_{L_{VV'}}$ is the value in an edge connected to V. We use Corollary 4 with $a_V = r_V - d_V$ and $b_{VV'} = z_{L_{VV'}}$ on the tree that is obtained by removing all the leaves, and obtain

$$(Y-1)(R_2 - D_2) \geq (Y)(Z_2).$$

By adding the inequalities $r_V - d_V \geq z_L$ for every leaf Y times, we obtain

$$Y(R_1 - D_1) \geq YZ_1.$$

By adding the inequality obtained from Proposition 36 at each non-leaf vertex, we obtain

$$2(R_2 - D_2) \geq Z_1 + 2Z_2.$$

By adding this three inequalities, and noticing that $R - D = R_1 - D_1 + R_2 - D_2$ and $Z = Z_1 + Z_2$, we obtain the desired inequality. $\square$

Remark 8. *This is generally better than Proposition 30, since $Y \leq l' - 2$, where Y is the number of non-leaf vertices and l' is the number of vertices in the connected component of the directed graph. Using Proposition 37, we can change l' to $Y + 2$ in Propositions 31 and 32, where Y is the number of non-leaf vertices in the connected component after removing the loop.*

Proposition 38. *If K_S is nef and in a connected component of the directed graph there is a cycle in the directed graph, only having maximal E_is as in Figures 0.30, 0.31 or 0.37, then we have $(2Y + 3)(R' - D') \geq (2Y + 4)Z' - Y - 2$, where Y is the number of non-leaf vertices in the connected component after removing the edge L_j from Proposition 33, R', D' are the sums of the r_i, d_i of the T-chains in the connected component, and Z' is the sum of the values in the edges of the component of the directed graph.*

Proof. By Proposition 37 on the component without the edge L_j, we have

$$(Y+1)(R' - D') \geq (Y+2)(Z' - Z_{L_j}).$$

By Proposition 33, we have

$$(Y+2)(r_{V_i} - d_{V_i}) \geq (Y+2)(2Z_{L_j} - 1).$$

And by the bijection from Lemma 6, removing L_j and fixing V_i, we obtain

$$(Y+2)(R'-D'-(r_{V_i}-d_{V_i})) \geq (Y+2)(Z'-Z_{L_j}).$$

By adding this 3 inequalities, we obtain the result. □

We call **cycle** a connected graph where every vertex has two edges and **line** a tree where no vertex has more than two edges.

PROPOSITION 39. *For any connected component of the directed graph that is not a cycle or a line with a loop at one end, we have*

$$(2l'-2)R'-D' \geq (2l'-1)Z'-l'.$$

PROOF. By Remark 8 and Propositions 37 and 38, we only need to look at components which are the result of adding an extra edge (or loop) to a tree with exactly two leaves. So, we only need to look at the cases of a cycle with two lines connected to neighbouring vertices of the cycle, and the case of a line with a loop at a vertex in the middle.

For the case of the cycle. Let us call L the edge from Proposition 33. Let us label the vertices V_1 to $V_{l'}$, so that after removing the edge L the neighbouring vertices have consecutive indices and label L_i the edge between V_i and V_{i+1}. We call V_A, V_B, $A<B$ the vertices connected to L. Without loss of generality suppose V_A is the vertex in Proposition 33. For each vertex with $i<A$, add up

$$(2l'-2)(r_{V_i}-d_{V_i}) \geq (i-1)z_{L_{i-1}}+(2l'-i-1)z_{L_i}.$$

For each vertex with $i>B$, add up

$$(2l'-2)(r_{V_i}-d_{V_i}) \geq (l'+i-2)z_{L_{i-1}}+(l'-i)z_{L_i}.$$

Also add up

$$\begin{gathered}2(A-1)(r_{V_A}-d_{V_A}) \geq (A-1)(z_{L_{A-1}}+z_{L_A}+z_L),\\ 2(l'-B)(r_{V_B}-d_{V_B}) \geq (l'-B)(z_{L_{B-1}}+z_{L_B}+z_L),\\ (B-A+1)(r_{V_A}-d_{V_A}) \geq 2(B-A+1)z_L-(B-A+1).\end{gathered}$$

If $l'>B$, we add up

$$\begin{gathered}(2l'-B-A-1)(r_{V_A}-d_{V_A}) \geq (l'-B-1)z_L+(l'-A)z_{L_A},\\ (2B-2)(r_{V_B}-d_{V_B}) \geq (A-1)z_L+(2B-A-1)z_{L_{B_1}},\\ (2l'-2)(r_{V_i}-d_{V_i}) \geq (l'+i-A-1)z_{L_{i-1}}+(l'-i+A-1)z_{L_i},\end{gathered}$$

where i takes all the values bigger than A and less than B.

If $l'=B$, we must have $A>1$, in which case we add up

$$(2l'-B-A-1)(r_{V_A}-d_{V_A}) \geq (l'-B)z_L+(l'-A-1)z_{L_A},$$

$$(2B-2)(r_{V_B}-d_{V_B}) \geq (A-2)z_L + (2B-A)z_{L_{B_1}},$$

$$(2l'-2)(r_{V_i}-d_{V_i}) \geq (l'+i-A)z_{L_{i-1}} + (l'-i+A-2)z_{L_i},$$

where i takes all the values bigger than A and less than B. In any case, after adding up we obtain the desired inequality.

For the case of the line, let us label the vertices V_1 to $V_{l'}$, where neighbouring vertices have consecutive indices and label L_i the edge between V_i and V_{i+1}. We call V_A the vertex with a loop L. For each vertex with $i < A$, add up

$$(2l'-2)(r_{V_i}-d_{V_i}) \geq (i-1)z_{L_{i-1}} + (2l'-i-1)z_{L_i}$$

For each vertex with $i > A$, add up

$$(2l'-2)(r_{V_i}-d_{V_i}) \geq (l'+i-2)z_{L_{i-1}} + (l'-i)z_{L_i}.$$

Without loss of generality suppose $A-1 \leq l'-A$, i.e. there are more vertices to the left of V_A than to its right. Add up

$$(l'-2A+1)(r_{V_A}-d_{V_A}) \geq (l'-2A+1)z_{L_A},$$

$$(l'-1)(r_{V_A}-d_{V_A}) \geq (2l'-2)z_L.$$

By Proposition 36 we can add

$$2(A-1)(r_{V_A}-d_{V_A}) \geq (A-1)(z_{L_{A-1}} + z_{L_A} + z_L).$$

Adding all these inequalities and noting that $A-1 \geq 1$, we obtain the desired result. □

REMARK 9. *Proposition 39 says that the optimality in the sense of Proposition 35a), for fixed l and ε small enough can only be obtained when the directed graph is a cycle or a line with a loop at one end.*

Open questions

In this short chapter we briefly explain some few open questions for future research.

(1) In this thesis we say nothing about the case when K_S is not nef. In this case S must be a rational surface (see [**RU17**, Prop. 2.2]). As the possible "bad graphs" are classified independently of K_S nef, it would only remain to bound $r-d$ with respect to the number of E_i with $E_i \cdot C = 1$ for each case separately, as it was done in this thesis using that K_S is nef. This might be possible following what was done in [**RU17**], whose main result is: Let C be the exceptional divisor of ϕ. If K_S is not nef, then S must be rational, and

$$r - d \leq \begin{cases} 2(K_W^2 - K_S^2) - K_S \cdot \pi(C) & \text{if no long diagram} \\ 2(K_W^2 - K_S^2) + 1 - K_S \cdot \pi(C) & \text{if long diagram of type I} \\ 4(K_W^2 - K_S^2) - 2K_S \cdot \pi(C) & \text{if long diagram of type II} \end{cases}$$

where long diagrams are the bad graphs when considering only one T-singularity. Notice that the integer $K_S \cdot \pi(C)$ is negative in this case, and so the bound for $r-d$ depends on that degree as well. In [**RU17**] it is shown that for the same fixed surface W, one can make $K_S \cdot \pi(C)$ arbitrarily negative by changing the morphism π via suitable "Cremona transformations". But by Alexeev boundedness the "minimal Cremona degree" should be bounded. It is an open problem to find such explicit and optimal bound.

(2) Another problem for future work would be to do some similar procedure to bound other types of relevant singularities. For example, the singularities that appear in normal stable surfaces, i.e., log canonical singularities. There is a well known list of them, but again it is not clear which of them appear after we fix K^2 and χ, and how to bound them explicitly. We note that a bound of the index in relation to a function of K^2 is not possible, since in general K^2 for stable surfaces is rational and can (and do) accumulate at certain rational points. So this problem would be more subtle. In the case of T-singularities, the K_W^2 is always an integer, and so we can bound.

(3) In this thesis the main results bound the sum of all $r_i - d_i$, but we do not get any bound for each $r_i - d_i$ independently. Under certain conditions [**RU17**, Remark 1.2] the bound for one T-chain works for each T-chain, so it could be expected that in other cases some restriction on algebraic surfaces prohibit having a huge $r - d$ for a singularity and the rest relatively small. It is possible that using the classification of "bad graphs" we could get some bounds for each $r_i - d_i$ or at least that special things happen when some $r_i - d_i$ pass a threshold. For example if some singularity has $r_i - d_i$ three times bigger than the rest of the singularities, then a bound as in Proposition 33 could be improved to $r_i - d_i \geq 3z$ for some cases, which could lead to doing a different process to the directed graph like in Proposition 1 and obtaining a better bound.

(4) There could be a similar classification for E_is with fixed $C \cdot E_i = 2$ or even bigger. As we increase $C \cdot E_i$ the possibilities should increase a lot, since most of the arguments we used to discard possible cases, fail outright when we increase $C \cdot E_i$. Finding this classifications should give bounds on the quantities of E_is for small $C \cdot E_i$ (bigger than one), and this would give better bounds for the sum of all $r_i - d_i$, at least in some cases. In the case of one singularity this was unnecessary to get optimal bounds, as the optimal cases did not have E_i with $C \cdot E_i > 2$. There may be room for improvement in the bounds for many singularities, even finding the best bounds only for small cases can be useful. For example $\mathbb{Q}$-homology projective planes with quotient singularities and K_S nef have at most 4 singularities except for one case [**HK**]. So, to reduce the possibilities for these surfaces, at least when there are only non ADE T-singularities, it is only necessary to bound the cases of 2, 3 and 4 T-singularites.

(5) In this thesis we only obtain asymptotically optimal cases. Optimal remain unknown. A first step would be to construct all the combinatorial configurations of T-chains which achieve asimptotically optimal bounds. A second step would be to get combinatorial configurations of T-chains achieving the bounds, this is so far not clearly possible. It may even be the case that the current bounds cannot be achieved and the bounds can be improved. A final step would be to see which of these optimal configurations are actually realizable in a smooth projective surface, which is an even harder task.

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