Overdetermined Elliptic Problems in Annular Domains

by

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Para Ángela, aunque no le gusten las EDP

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Contents

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded, *connected* C^2-domain of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1}$, where Ω_0 is a bounded domain, and Ω_1 is a finite union of bounded domains in $\mathbb{R}^n$, $n \geq 2$, with $\Omega_1 \Subset \Omega_0$. The present work is devoted to the study of the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0, & u > 0 \quad \text{in} \quad \Omega, \\ u = 0, & u_\nu = c_0 \quad \text{on} \quad \partial\Omega_0, \\ u = a, & u_\nu = c_1 \quad \text{on} \quad \partial\Omega_1, \end{cases} \tag{1.1}$$

where f is a C^1 function, ν denotes the *inner* unit normal to Ω and $a \geq 0$, c_0 and c_1 are real constants. Note that the problem above is *overdetermined* in the sense that both Dirichlet and Neumann boundary data are provided, and therefore a solution might not exist even if the separated Dirichlet problem and Neumann problem have solutions. Note that whenever (1.1) admits a solution $u \in C^2(\overline{\Omega})$, then the Hopf lemma implies that the constant $c_0 > 0$.

Problem (1.1) arises in classical models in the theory of elasticity, fluid mechanics and electrostatics – we refer to [Sir02] for a discussion of applications to physics. The special case in which $a = 0$ and $c_0 = c_1$ was treated by Serrin in his seminal 1971 paper [Ser71]. He showed that a strong property of *rigidity* was forced upon any solution $u \in C^2(\overline{\Omega})$ and upon the shape of the domain Ω supporting it: namely, Ω has to be a ball ($\Omega_1 = \emptyset$) and u has to be radially symmetric and monotonically decreasing along the radius. Serrin's proof is based on the *moving planes method*, pioneered earlier by Alexandrov [Ale62] in a geometric context, and has ever since been a powerful tool for establishing symmetry of positive solutions to second-order elliptic problems. An important artifact of the method is that proving radial symmetry comes hand in hand with proving the monotonicity of solutions in the radial direction.

Reichel [Rei95] adapted Serrin's method to analyze (1.1) in the case when

$$u|_{\partial\Omega_1} = a > 0 \quad \text{and} \quad u_\nu|_{\partial\Omega_1} = c_1 \leq 0.$$

Under the additional assumption that $0 < u < a$ in Ω, he showed that u has to be radially symmetric and the domain Ω – a standard annulus. Several years later, Sirakov [Sir01] removed the extra assumption and proved a more general rigidity theorem, allowing for separate constant

Dirichlet conditions $u|_{\Gamma_i} = a_i > 0$ and separate constant Neumann conditions $u_\nu|_{\Gamma_i} = c_i \le 0$ to be imposed on each connected component Γ_i of the inner boundary $\partial\Omega_1$, $i = 1, \dots, k$. The assumption of non-positivity of each Neumann condition $u_\nu|_{\Gamma_i} = c_i \le 0$ is crucial for the moving planes method to run and yield the radial symmetry and monotonicity of solutions. Furthemore, the rigidity theorems by Reichel and Sirakov apply to a more general class of second-order elliptic equations in which the non-linearity f is allowed to depend also in $|\nabla u|$. In $n = 2$ dimensions, a symmetry result for (1.1) in the case $f \equiv 1$ was obtained earlier by Willms, Gladwell and Siegel [WGS94] under some additional boundary curvature assumptions. See also [KSV05] for a complex analytic approach to (1.1) when $n = 2$ and $f \equiv 1$.

Constructions of non-trivial solutions to overdetermined elliptic problems have been prominent in the literature in recent years. Many of them have been driven by a famous conjecture of Berestycki, Caffarelli and Nirenberg [BCN97], according to which, if f is a Lipschitz function and $\Omega \subset \mathbb{R}^n$ is an unbounded smooth domain, such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected, then the overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0, \quad u > 0 & \text{in} \quad \Omega, \\ u = 0, \quad u_\nu = \text{const} & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

admits a positive bounded solution if and only if Ω is a half space, a cylinder, the complement of a cylinder or the complement of a ball. Many of this constructions have been done by means of Bifurcation Theory, and the solutions constructed are close, but different, from known, trivial solutions. First, Sicbaldi [Sic10] constructed domain counterexamples Ω to the conjecture when $n \ge 3$ and $f(u) = \lambda u$, which bifurcate from cylinders $B^{n-1} \times \mathbb{R}$ for appropriate $\lambda > 0$. Then Ros, Ruiz and Sicbaldi [RRS19] constructed a different set of counterexamples for all dimensions $n \ge 2$ and $f(u) = u^p - u$, $p > 1$, that bifurcate from the complement of a ball $\mathbb{R}^n \setminus \overline{B^n_\lambda}$. The main tool behind the two results is a bifurcation theorem by Krasnoselskii that is based on topological degree theory and which yields a sequence of domains Ω, rather than a smooth branch. Schlenk and Sicbaldi [SS12] managed to strengthen the construction in [Sic10] through the use of the Crandall-Rabinowitz Bifurcation Theorem to obtain a smooth branch of perturbed cylinders Ω. A similar approach leading to perturbed generalized cylinders was pursued by Fall, Minlend and Weth [FMW17] in the case $f \equiv 1$. The bifurcation method has also been successful in finding nontrivial solutions in versions of (1.2) set in Riemannian manifolds [MS16, FMW18]. Other methods for the construction of solutions in overdetermined elliptic problems have also been developed, for instance, in [DS09, HHP11, Kam13, KLT13, Tra14, Sic14, DPPW15, DS15, FM15, LWW17, JP18] and references therein.

In the present work, we focus on the case in which the Neumann data on the *inner* boundary is positive:

$$u|_{\partial\Omega_0} = 0, \quad u|_{\partial\Omega_1} = a \ge 0 \quad \text{and} \quad u_\nu|_{\partial\Omega_0} = c_0 > 0, \quad u_\nu|_{\partial\Omega_1} = c_1 > 0. \tag{1.3}$$

We make the following crucial observation: over annuli Ω, problem (1.1) now possesses radial solutions that are *not monotone* along the radius – see Lemma 3.3 in Chapter 3 and Proposition 4.2 in Chapter 4 for the existence of one-parameter families of such examples. Their presence hints that proving radial symmetry for solutions of (1.1) would be out of the scope of the moving planes method. Thus, one is led to conjecture that, under (1.3), *radial rigidity does not hold* for solutions of (1.1).

Our main results confirms that this is indeed the case for some non-linearities $f(u)$.

Theorem 1.1. *Let $f \equiv 1$. Then there exist bounded, real analytic annular domains of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1} \subset \mathbb{R}^n$, which are different from standard annuli, such that the overdetermined problem (1.1) admits a solution $u \in C^\infty(\overline{\Omega})$ satisfying (1.3) for some positive constants a and $c_0 = c_1$.*

Theorem 1.1 was shown by the author and his advisor in [KS20], and its proof is the content of Chapter 3. We largely follow the exposition in [KS20], but we provide some additional details and references to help readers who are not experts in the field.

Our second result treats a family of non-linearities of power type, and can be seen as the analogous result to that of Ros, Ruiz and Sicbaldi in [RRS19] in the setting of bounded annular domains.

Theorem 1.2. *Let $f(u) = u^p - \kappa u$, where $\kappa \geq 0$ and $1 < p < \frac{n+2}{n-2}$, when $n \geq 3$, and $p > 1$, when $n = 2$. Then there exist bounded, C^2 annular domains of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1} \subset \mathbb{R}^n$, which are different from standard annuli, such that the overdetermined problem (1.1) admits a solution $u \in C^2(\overline{\Omega})$ satisfying (1.3) for $a = 0$ and some positive constants $c_0 \neq c_1$.*

The proof of Theorem 1.2 is the content of Chapter 4.

This thesis is organized as follows. In Chapter 2 we review some of the results in the theory of overdetermined elliptic problems, focussing on the rigidity results of Serrin and Sirakov proved by means of the moving planes method (Section 2.2), and the construction of non-trivial solutions to some overdetermined problems by means of Bifurcation Theory (Section 2.3). In Chapters 3 and 4 we give more refined, quantitative versions of Theorems 1.1 and 1.2, and finally provide their proofs, which are based on bifurcation arguments. The theorems of Bifurcation theory we apply in these constructions are stated in the Appendix A.

Although the results in Chapter 3 and Chapter 4 are independent, the basic strategy and layout are very similar, and thus some computations in the first stages of the proof of Theorem 1.2 are only sketched. For this reason, we recommend that the reader goes over Sections 3.1 and 3.2 in Chapter 3 before starting Chapter 4. As the computations are more explicit, and the techniques more elementary in Chapter 3 than those in Chapter 4, useful intuition can be gained there first.

The purpose of this work is two-fold. On the one hand, its aim is to communicate the contribution of the two main results given by Theorems 1.1 and 1.2. On the other, it is conceived to serve as an introduction to the field for those who are taking their first steps in this direction, as the author did a couple of years ago.

Quick overview of overdetermined elliptic problems

The purpose of this chapter is to give a basic account of the theory of overdetermined elliptic problems. Since a complete review could become technical and extensive very quickly, we limit ourselves to cite some of the classical results and the recent contributions relevant for our construction, explaining briefly how they were proved, providing the appropriate references. In Section 2.1 we give some examples in which overdetermined problems arise. In Section 2.2 we state some of the rigidity results, most notably the theorem of Serrin (Theorem 2.2) and the theorem of Sirakov (Theorem 2.7), and in Section 2.3 we review some of the symmetry breaking constructions of solutions which have been obtained by means of Bifurcation theory (see Appendix A) in recent years.

2.1 Motivation

To provide some motivation to the study of overdetermined problems, we will describe three situations in which this type of problem arises. The first two are problems which come from the modelling of physical situations, and the third is a problem in spectral geometry.

Physics is one of the sources for overdetermined elliptic problems. Such problems appear, for instance, in fluid dynamics, the theory of elasticity, and electrostatics. The survey [Sir02] contains many physical applications, and in this section we will describe two of the examples given there.

The first problem was considered by Serrin in [Ser71]. Suppose that a viscous incompressible fluid is moving in a straight pipe with a given cross section. We can fix Cartesian coordinates (x, y, z) in such a way that the z-axis is directed along the pipe, and so the cross section of the pipe can be represented by a domain Ω in the (x, y)-plane. The differential equation that models this situation is

$$-\Delta u = A \quad \text{in} \quad \Omega, \tag{2.1}$$

where u is the flow velocity of the fluid, and A is a positive constant depending on the length of the pipe, the dynamic viscosity of the fluid η, and the change of pressure between the two ends of the pipe. The adherence condition on the wall of the pipe is then expressed as the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

The tangential stress of the fluid on the wall $\partial\Omega$ of the pipe is computed as $\eta u_\nu|_{\partial\Omega}$. In applications, the determination of the points on $\partial\Omega$ where the stress is maximal is an important problem, and therefore one can ask the following question: when is the tangential stress the same at each point of the wall $\partial\Omega$? Thus, we are asking for a solution u to the Dirichlet problem (2.1)-(2.2) to satisfy, in addition, a constant Neumann condition

$$\frac{\partial u}{\partial \nu} = \text{const} \quad \text{on} \quad \partial\Omega,$$

which makes the problem for u overdetermined.

Another overdetermined problem arises in electrostatics. Consider a smooth conducting body $\Omega \subset \mathbb{R}^3$, with a charge distribution $\rho \in C(\partial\Omega)$ on its boundary. The charge distribution ρ is said to be at equilibrium if the *single-layer potential* induced by ρ,

$$\Psi(x) := -\frac{1}{4\pi} \int_{\partial\Omega} \frac{\rho(y)}{|x-y|}\, dS(y),$$

is constant in Ω. Note that Ψ is harmonic in $\mathbb{R}^3 \setminus \partial\Omega$. The question we want to answer is: what conducting bodies admit a constant charge distribution which is also at equilibrium? Suppose ρ is a constant charge distribution which is also at equilibrium. Then the single-layer potential Ψ is constant on $\partial\Omega$, and satisfies

$$\frac{\partial \Psi}{\partial \nu} = -\rho = \text{const} \quad \text{on} \quad \partial\Omega,$$

where ν is the *inner* unit normal with respect to $\mathbb{R}^3 \setminus \overline{\Omega}$. Thus, Ψ satisfies the overdetermined problem.

$$\begin{cases} \Delta\Psi = 0 & \text{in} \quad \mathbb{R}^3 \setminus \Omega, \\ \Psi = \text{const} & \text{on} \quad \partial\Omega, \\ \Psi_\nu = \text{const} & \text{on} \quad \partial\Omega. \end{cases}$$

To finish this section, we describe a spectral-geometric problem. Let M be a Riemannian manifold, and let $0 < \kappa < \text{vol}(M)$. Let $\Omega \subset M$ be a domain with smooth boundary, and let $\lambda_1 = \lambda_1(\Omega)$ be the first eigenvalue of the Laplace-Beltrami operator in $\Omega \subset M$, taken with a zero Dirichlet boundary condition. The question we want to answer is: for which domains of fixed finite volume κ does the first eigenvalue $\lambda_1(\Omega)$ take an extremum value? In other words, we are seeking for critical points of the functional $\Omega \mapsto \lambda_1(\Omega)$ under the constraint $\text{vol}(\Omega) = \kappa$. Such domains are called *extremal domains*, and are characterized by the following property: there exists a positive solution to the overdetermined problem

$$\begin{cases} \Delta_M u + \lambda_1 u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u_\nu = \text{const} & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

In other words, the first Dirichlet eigenfunction for Δ_M in Ω also satisfies a constant Neumann condition (see [SS12]).

In Sections 2.2 and 2.3 we will see how this problems are treated, and what type of results have been obtained in this direction.

2.2 Rigidity results

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain, $n \geq 2$. Consider the model overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u_\nu = \text{const} & \text{on} \quad \partial\Omega, \end{cases} \tag{2.4}$$

and let $u \in C^2(\overline{\Omega})$ be a solution to (2.4). Note that by the maximum principle $u > 0$ in Ω.

Problem (2.4) was first treated by Serrin in his seminal paper [Ser71]. A first look at the problem reveals that when $\Omega = B_R(x_0)$ is a ball, it possess a unique solution given by

$$u(x) = \frac{R^2 - |x - x_0|^2}{2n}. \tag{2.5}$$

A natural question to ask is whether there exists a domain Ω different from a ball which supports a solution to (2.4). The answer to that question turns out to be negative, as was shown by Serrin. We thus say that problem (2.4) is *rigid*: the only domains for which the problem are solvable are balls and the corresponding solution u is monotonically decreasing along the radius.

Serrin had a very clever insight on the geometry of problem (2.4), showing rigidity by means of the *moving planes method*. The method was first introduced by Alexandrov in [Ale62] to prove that *the only closed constant mean curvature hypersurfaces isometrically embedded in Euclidean space are the spheres*, and Serrin's result can be seen as an analogue to Alexandrov's. The method consists in the following. Suppose $T_0 \subset \mathbb{R}^n$ is an hyperplane such that $T_0 \cap \Omega = \varnothing$. Then T_0 can be continuously moved in direction normal to itself towards the domain Ω until it touches the boundary $\partial\Omega$ at some point for the first time. From that moment onwards the resulting hyperplane T will cut a portion of Ω, which we call $\Sigma = \Sigma(T)$. Let $\Sigma' = \Sigma'(T)$ be the set resulting after the reflection of Σ with respect to T. We see that $\Sigma' \subset \Omega$ at the beginning of the process until at least one of the two occurs:

- Σ' becomes internally tangent to the boundary $\partial\Omega$ at some point not belonging to T;
- T becomes orthogonal to $\partial\Omega$ at some point.

Denote the hyperplane T at which one of these positions occur for the first time by T'. The goal now is to show that Ω must be symmetric with respect to T'. If this can be shown for any hyperplane T' whose normal is any given direction in $\mathbb{R}^n$, then Ω must be radially symmetric with respect to some point. Moreover, Ω cannot have n-dimensional holes; otherwise there would be a T' at which Σ' is internally tangent to $\partial\Omega$ at some point, and Ω would not be symmetric with respect to T', contradicting our assumption. It follows then that Ω is a ball, since they are the only radially symmetric domains satisfying this property.

We now sketch the proof given by Serrin. Introduce the function $v \in C^2(\Sigma')$ defined by

$$v(x) := u(x'), \quad x \in \Sigma',$$

where x' denotes the reflection of x with respect to T'. Note that v satisfies

$$v = u \quad \text{on} \quad \partial\Sigma' \cap T', \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Sigma' \setminus T'.$$

Then, as $\Delta v(x) = \Delta u(x')$ in Σ', we have that $w := u - v$ is a solution to

$$\begin{cases} \Delta w = 0 & \text{in} \quad \Sigma', \\ w = 0 & \text{on} \quad \partial\Sigma' \cap T', \\ w \geq 0 & \text{on} \quad \partial\Sigma' \setminus T'. \end{cases}$$

The strong maximum principle then implies that

$$w > 0 \quad \text{in} \quad \Sigma' \tag{2.6}$$

or else $w \equiv 0$ in Σ'. In the later case, we have that $u(x') = v(x)$ for all $x \in \Sigma'$, which means that u is symmetric with respect to T', and since $u > 0$ in Ω, this means that $\partial\Sigma' \setminus T'$ must coincide with that portion of $\partial\Omega$ contained in the same side of T'. As in the construction $\Sigma' \subset \Omega$, we the have that Ω is symmetric with respect to T'. That Ω is a ball follows from the discussion above. Thus, we have to rule out the case (2.6). In the case that Σ' is internally tangent at some point $x \in \partial\Omega$, we have at this point that

$$\frac{\partial w}{\partial \nu}(x) = \frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(x') = 0,$$

but the Hopf lemma implies $w_\nu(x) > 0$, which is a contradiction. The case in which T' is normal to $\partial\Omega$ at some point x is a bit more delicate, but can be ruled out in a similar fashion by a version of the Hopf lemma in boundary corners given by Serrin, namely:

Lemma 2.1 (Serrin). *Let $D' \subset \mathbb{R}^n$ be a C^2-domain, and let T be a hyperplane containing the normal to $\partial D'$ at some point $x \in D' \cap T$. Let D denote the portion of D lying on some particular side of T.*

Suppose that $w \in C^2(\overline{D})$ and satisfies

$$w \geq 0, \quad \Delta w \leq 0 \quad \textit{in} \quad D, \quad \textit{and} \quad w(x) = 0.$$

Let s be any direction at x which enters D non-tangentially. Then either

$$\frac{\partial w}{\partial s}(x) > 0 \quad \textit{or} \quad \frac{\partial^2 w}{\partial s^2} > 0$$

unless $w \equiv 0$.

The proof finishes after applying Lemma 2.1 to $D = \Sigma'$ and the function $w = u - v$, and showing that the first and second derivatives of u and v must coincide in the point x at which T' is normal to $\partial\Omega$, which is clearly a contradiction.

A careful analysis shows that what is needed in the above proof is the following:

- the differential equation is invariant under the substitution $x \mapsto x'$;
- the differential equation satisfies the strong maximum principle, the Hopf lemma and its boundary corner version in Lemma 2.1.

The above, as noted by Serrin, is satisfied for any C^1 non-linearity of the form $f(u, |\nabla u|)$ when the solution u is positive. Then holds the following generalization of the previous result, also proved in [Ser71].

Theorem 2.2 (Serrin). *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain, and let $f(u, s)$ be a C^1 function. Let $u \in C^2(\overline{\Omega})$ be a positive solution to the overdetermined problem*

$$\begin{cases} \Delta u + f(u, |\nabla u|) = 0, \quad u > 0 & \textit{in} \quad \Omega, \\ \qquad\qquad u = 0, \quad u_\nu = \textit{const} & \textit{on} \quad \partial\Omega. \end{cases} \tag{2.7}$$

Then Ω must be a ball and u is radially symmetric and monotone decreasing along the radius.

The moving planes method was later greatly developed and exploited in the analysis of the symmetries of solutions of elliptic differential equations. In their classical work [GNN79], Gidas, Ni and Nirenberg employed this powerful method in proving the following.

Theorem 2.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a ball, f is a Lipschitz non-linearity, and let $u \in C^2(\overline{\Omega})$ be a positive solution to the Dirichlet problem*

$$\begin{cases} \Delta u + f(u) = 0 & \textit{in} \quad \Omega, \\ \qquad u = 0 & \textit{on} \quad \partial\Omega. \end{cases} \tag{2.8}$$

Then u is radially symmetric with respect to the center of the ball Ω and decreasing along the radius.

Note that in the above theorem it is assumed *a priori* that Ω is a ball, in contrast to Theorem 2.2, where it is derived as a consequence. What makes this conclusion possible it is exactly the extra Neumann condition present in problem (2.7).

The proof of Theorem 2.3 is very similar to that of Theorem 2.2: one reflects the domain Ω with respect to an hyperplane T at some critical position, and then, applying the various forms of the maximum principle, shows the solution u to equation (2.8), and hence the domain Ω, has to be symmetric with respect to T. We will explain how the moving planes method works in this case, and in a somewhat more general context, in some more detail later on.

The method, through the use of the maximum principles, has the drawback that it relies on the smoothness of the boundary of the domain Ω . For example, the method fails in proving symmetry for solutions with respect to each axis when Ω is a cube. However, Berestycki and Nirenberg in [BN91] generalized the method removing the assumptions on smoothness. In fact, they require no smoothness at all. We cite the following theorem, which is a generalization of Theorem 2.3.

Theorem 2.4 (Berestycki-Nirenberg)**.** *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain which is convex in the x_1 direction and symmetric with respect to the hyperplane $x_1 = 0$. Let f be Lipschitz function, and let $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ be a positive solution to (2.8). Then u is symmetric with respect to x_1 and $u_{x_1} < 0$ for $x_1 > 0$ in Ω.*

The proof again relies on the moving planes method, though the classical maximum principles of Hopf are now replaced by a more general notion and a generalization of the maximum principle in *narrow domains*. We make this more precise in the following.

Consider a second order uniformly elliptic operator L in a bounded domain $\Omega \subset \mathbb{R}^n$

$$L = \sum_{j,k=1}^{n} a_{jk}\partial_{jk} + \sum_{j=1}^{n} b_j \partial_j + c$$

with bounded measurable coefficients a_{jk}, b_j and c. We say that the maximum principle holds for L in Ω if

$$Lz \geq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \limsup_{x \to \partial\Omega} z(x) \leq 0,$$

implies $z \leq 0$ in Ω. In [BN91] is proved:

Lemma 2.5. *Assume* $\operatorname{diam} \Omega \leq d$. *There exists $\delta > 0$ depending only on the dimension n, d, the ellipticity constant of L and a bound for its coefficients, such that the maximum principle holds for L in Ω provided its measure $|\Omega| < \delta$.*

Thus, Lemma 2.5 asserts that for a given linear elliptic operator with bounded coefficients L, the maximum principle holds for a domain Ω with bounded diameter if it has small enough measure, in other words it is *narrow*. We will give a sketch of the proof of Theorem 2.4 contained in [BN91].

Let Ω be as in Theorem 2.4, and let $a := \inf_{x\in\Omega} x_1$, and let T_λ be the hyperplane $x_1 = \lambda$ for $\lambda \in (-a, 0)$. Denote

$$\Sigma_\lambda = \{x \in \Omega : x_1 < \lambda\},$$

and define in Σ_λ the functions

$$v(x_1, x') := u(2\lambda - x_1, x'), \quad w_\lambda(x) := v(x) - u(x).$$

Note that $(2\lambda - x_1, x')$ is the reflection of the point (x_1, x') trough the hyperplane T_λ. Now, we have

$$\Delta w_\lambda = \Delta v - \Delta u = f(v) - f(u) = f'(\xi)(v - u)$$

for some ξ between u and v. Since f is Lipschitz, then f' is bounded. Therefore w_λ is a solution to

$$\begin{cases} \Delta w + c_\lambda w = 0 & \text{in} \quad \Sigma_\lambda, \\ w \geq 0 & \text{on} \quad \partial\Sigma_\lambda, \end{cases}$$

for a function c_λ uniformly bounded in λ. We wish to prove

$$w_\lambda > 0 \quad \text{in} \quad \Sigma_\lambda.$$

If $\lambda + a > 0$ is small enough, then $|\Sigma_\lambda|$ is also small, and by Lemma 2.5 the maximum principle holds in Σ_λ and we can deduce $w > 0$ in Σ_λ. Take then $\mu \leq 0$ to be the supremum over all λ such that the above holds. If $\mu = 0$ the proof is finished. So suppose $\mu < 0$ and look for a contradiction. By continuity $w_\mu(x) \geq 0$ in Σ_μ. Since $u > 0$ in Ω, we have $w \not\equiv 0$ on $\partial\Sigma_\mu$, and by the usual maximum principle $w > 0$ in Σ_μ. The idea is now to show that $w_{\mu+\varepsilon} > 0$ in $\Sigma_{\mu+\varepsilon}$ for $\varepsilon > 0$ small, which would contradict the definition of μ. For this, fix $\delta > 0$ small such that Lemma 2.5 holds, an approximate Σ_μ by a compact subset K such that $|\Sigma_\mu \setminus K| \leq \delta/2$. Then $w_\mu > 0$ in K by compactness, and by continuity $w_{\mu+\varepsilon} > 0$ in K for all small $\varepsilon > 0$ such that $|\Sigma_{\mu+\varepsilon} \setminus K| \leq \delta$. We then apply the maximum principle of Lemma 2.5 to $w_{\mu+\varepsilon}$ and show that $w_{\mu+\varepsilon} > 0$ in $\Sigma_{\mu+\varepsilon} \setminus K$. Therefore, $w_{\mu+\varepsilon} > 0$ in $\Sigma_{\mu+\varepsilon}$, which contradicts the maximality of μ.

This refined moving plane method proved very successful and simplified the proofs of the symmetry results contained in [GNN79]. It also served in generalizing Serrin's result to other homogeneous manifolds than $\mathbb{R}^n$. Using the machinery developed by Berestycki and Nirenberg, Kumaresan and Prajapat proved in [KP98] an analogous result to Theorem 2.2, now set in the sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$. In this context, the moving planes is replaced by moving closed, totally geodesic hypersurfaces $\Gamma \subset M$.

Theorem 2.6 (Kumaresan-Prajapat)**.** *Let M be the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$. Let $\Omega \subset M$ be a bounded C^1-domain, contained in a hemisphere when $M = \mathbb{S}^n$, and let f be a C^1*

function such that the overdetermined problem

$$\begin{cases} \Delta_M u + f(u) = 0, \quad u > 0 & \textit{in} \quad \Omega, \\ u = 0, \quad u_\nu = const & \textit{on} \quad \partial\Omega. \end{cases} \tag{2.9}$$

possesses a positive solution $u \in C^2(\overline{\Omega})$*, where* Δ_M *denotes the Laplace-Beltrami operator in* M*. Then* Ω *is a geodesic ball and* u *is radially symmetric.*

A natural way to proceed from the above mentioned rigidity theorems is to consider domains that have holes *a priori*. More explicitly, consider a C^2-domain $\Omega_0 \subset \mathbb{R}^n$, and a finite union of C^2-domains $\Omega_1 \subset \mathbb{R}^n$, where Ω_1 is bounded and $\Omega_1 \Subset \Omega_0$. Let us write the *inner* boundary $\partial\Omega_1$ as

$$\partial\Omega_1 = \bigcup_{i=1}^{k} \Gamma_i,$$

where Γ_i are the distinct connected components of $\partial\Omega_1$. Let $\Omega := \Omega_0 \setminus \overline{\Omega_1}$ be connected, and consider the following overdetermined problem

$$\begin{cases} \Delta u + f(u, |\nabla u|) = 0, \quad u > 0 & \text{in} \quad \Omega, \\ u = 0, \quad u_\nu = c_0 & \text{on} \quad \partial\Omega_0, \\ u = a_i, \quad u_\nu = c_i & \text{on} \quad \Gamma_i, \quad i = 1, \dots, k, \end{cases} \tag{2.10}$$

for some real constants c_0, c_i and $a_i \geq 0$, for $i = 1, \dots, k$. Note that the method of Berestycki and Nirenberg cannot be applied to (2.10) since the underlying domain Ω is not convex in any direction. In this setting, Reichel [Rei95] studied problem 2.10 and adapted the moving plane method of Serrin to work under suitable hypotheses over the non-linear term $f(u, |\nabla u|)$. He concluded that with the additional hypotheses over the boundary conditions and the solution u

$$a_i = a > 0, \quad c_i = c \leq 0, \quad i = 1, \dots, k, \quad \text{and} \quad u < a \quad \text{in} \quad \Omega, \tag{2.11}$$

rigidity holds in the following sense: the domains Ω_0 and Ω_1 must be concentric balls, in particular $k = 1$ and Ω is a standard annulus, and the solution u is radially symmetric and decreasing along the radial direction.

Reichel also considered the case when $\Omega_0 = \mathbb{R}^n$ and the boundary condition over $\partial\Omega_0$ in (2.10) is replaced by the natural

$$\lim_{|x| \to \infty} u = 0, \qquad \lim_{|x| \to \infty} \nabla u = 0. \tag{2.12}$$

Again, using his adaptation of the moving planes method he showed in [Rei97], under the boundary conditions (2.11)-(2.12), that for (2.10) to be solvable Ω_1 must be a ball, and the solution u radially symmetric and decreasing along the radius.

Later, it was the work of Sirakov in [Sir01] which unified all previous results. He removed the assumptions (2.11) for the boundary conditions and treated the more general case in (2.10) where he allowed distinct positive constants $a_i > 0$ and distinct non-positive constants $c_i \leq 0$. Thus, Sirakov proved:

Theorem 2.7 (Sirakov)**.** *Let $\Omega \subset \mathbb{R}^n$ be a C^2-domain of the form $\Omega_0 \setminus \overline{\Omega_1}$, where $\Omega_1 \Subset \Omega_0$, and let $f(u,s)$ be a locally Lipschitz function.*

Suppose first that $\Omega_0 \subset \mathbb{R}^n$ is bounded, and let $u \in C^2(\overline{\Omega})$ be a positive solution to the overdetermined problem (2.10) for some positive constants $a_i > 0$ and some non-negative constants $c_i \leq 0$. Then Ω_0 and Ω_1 are two concentric balls, and u is radially symmetric and decreasing along the radial direction.

Suppose now that $\Omega_0 = \mathbb{R}^n$, and further suppose that $f(u,s)$ is non-increasing for small values of u. Let $u \in C^2(\overline{\Omega})$ be a positive solution to the overdetermined problem (2.10) for some positive constants $a_i > 0$ and some non-negative constants $c_i \leq 0$, where the boundary condition over $\partial\Omega_0$ is replaced by (2.12). Then Ω_1 is a ball, and u is radially symmetric and decreasing along the radial direction.

We want to point out that the hypothesis of non-positivity of the Neumann condition $u_\nu|_{\partial\Omega_1}$ is crucial for the moving planes method to run in the proof of Theorem 2.7. Indeed, when this condition is removed, the the overdetermined problem (2.10) can have a *non-monotone* positive solution u. Thus, if this hypothesis is removed, rigidity in the sense of Theorem 2.7 may fail to hold. The work present in this thesis aims to show that the condition on non-positivity of the Neumann condition is actually sharp in the case of bounded annular domains, in the sense that when $u_\nu|_{\partial\Omega_1} > 0$ is allowed there exist non radially symmetric positive solutions to the overdetermined problem (2.10).

Of the many symmetry results for overdetermined problems set in unbounded domains, a particularly influential one was obtained by Berestycki, Caffarelli and Nirenberg in [BCN97]. Their work was motivated by the study of free boundary problems, which through the application of a blow-up technique for studying the regularity of the free boundary, led to the study of overdetermined problems now set in Lipschitz epigraphs. In this setting, they were able to show rigidity for such overdetermined problems under growth conditions on the boundary of the epigraph. Their result was the following:

Theorem 2.8 (Berestycki-Caffarelli-Nirenberg)**.** *Let $\varphi \in C^2(\mathbb{R}^{n-1})$ be a Lipschitz function such that*

$$\lim_{|x|\to\infty} \varphi(x+\tau) - \varphi(x) = 0 \quad \text{uniformly in} \quad \tau \in \mathbb{R}^{n-1},$$

and let $\Omega \subset \mathbb{R}^n$ be the epigraph of φ,

$$\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > \varphi(x_1, \dots, x_n)\}.$$

Let $f(s)$ be a Lipschitz function, and suppose there exist $0 < s_0 < s_1 < \mu$ and $\delta > 0$ such that

- *$f(s) > 0$ for $0 < s < \mu$ and $f(s) \leq 0$ for $s \geq \mu$;*
- *$f(s) \geq \delta s$ for $0 \leq s \leq s_0$;*
- *f is non-increasing in (s_1, μ)*

If there exists a positive bounded solution $u \in C^2(\Omega) \cap L^\infty(\Omega)$ to the problem

$$\begin{cases} \Delta u + f(u) = 0, \quad u > 0 & \text{in} \quad \Omega, \\ u = 0, \quad u_\nu = const & \text{on} \quad \partial\Omega, \end{cases} \tag{2.13}$$

then φ is constant and u is monotone along the x_n variable.

Motivated by this result, Berestycki et al. proposed the following conjecture: if Ω is a smooth domain with $\mathbb{R}^n \setminus \overline{\Omega}$ connected, and there exists a Lipschitz function f such that (2.13) has a positive bounded solution, then Ω must be a ball, a cylinder, a half-space or the complement of one of them [BCN97]. The Berestycki-Caffarelli-Nirenberg Conjecture would be ultimately proven false in dimensions $n \geq 3$ in [Sic10], and later in all dimensions $n \geq 2$ in [RRS19]. Other set of counterexamples can be found in [DPPW15]. We will discuss these results in Section 2.3.

2.3 Symmetry breaking results

The results described in the previous section make us wonder if there are non-trivial domains which support a positive solution to an overdetermined problem. The answer to that question is affirmative, and in this section we review some of the constructions prominent in the literature of the past decade.

We return to the problem of finding *extremal domains* in a given Riemannian manifold, described in Section 2.1. That is, the problem of finding a positive solution to

$$\begin{cases} \Delta_M u + \lambda_1 u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u_\nu = \text{const} & \text{on} \quad \partial\Omega, \end{cases} \tag{2.14}$$

where $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue for the Laplace-Beltrami operator Δ_M in $\Omega \subset M$, and the domain Ω has a fixed finite volume. When $M = \mathbb{R}^n$, by Serrin's Theorem 2.2 extremal domains can only be balls. In fact this is an old result known as the *Faber-Krahn inequality*: the balls minimize the first Dirichlet eigenvalue for the Laplacian among all domains with the same volume. When we remove the volume constraint, the problem for minimizing or maximizing $\lambda_1(\Omega)$ loses its meaning, but the problem (2.14) still makes sense. It was in this setting that Sicbaldi [Sic10] constructed a family of non-trivial cylinder-type domains which constitute counterexamples to the Berestycki-Caffarelli-Nirenberg Conjecture.

We describe now the construction in [Sic10]. Consider the unit ball $B_1 \subset \mathbb{R}^{n-1}$ centred at the origin, the first Dirichlet eigenvalue $\lambda_1 > 0$ for the Laplacian in B_1, which is the square of the first positive zero of the Bessel function $J_{n/2-1}$, and the first eigenfunction $\tilde{\phi}$ associated to λ_1 normalized in the L^2-norm. So, $\tilde{\phi}$ is a solution to

$$\begin{cases} \Delta\tilde{\phi} + \lambda_1\tilde{\phi} = 0 & \text{in} \quad B_1, \\ \tilde{\phi} = 0 & \text{on} \quad \partial B_1. \end{cases}$$

Also, $\tilde{\phi}$ is smooth and can be chosen positive in B_1 (see [Eva10]), and by the symmetry theorem of Gidas-Ni-Nirenberg, Theorem 2.8, $\tilde{\phi}$ is also radially symmetric. This means $\tilde{\phi}$ also satisfies a constant Neumann condition along ∂B_1. Therefore, the function $\phi \in C^\infty(\overline{B_1 \times \mathbb{R}})$ defined by

$$\phi(x,t) := \tilde{\phi}(x), \quad x \in B_1, \quad t \in \mathbb{R},$$

is clearly a solution to

$$\begin{cases} \Delta\phi + \lambda_1\phi = 0 & \text{in} \quad B_1 \times \mathbb{R}, \\ \phi = 0 & \text{on} \quad \partial B_1 \times \mathbb{R}, \\ \phi_\nu = \text{const} & \text{on} \quad \partial B_1 \times \mathbb{R}. \end{cases} \tag{2.15}$$

Now, define the perturbed periodic cylinder

$$C_v^T := \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 \le |x| \le 1 + v(t)\}.$$

for $v \in C^{2,\alpha}(\mathbb{R})$ T-periodic and such that $v(t) > -1$ for all $t \in \mathbb{R}$. Then there exists a constant $\lambda = \lambda_{v,T}$ and a unique L^2-normalized positive function $\phi = \phi_{v,T} \in C^{2,\alpha}(C_v^T)$ (see [GT15]) which is a solution to the Dirichlet problem

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in} \quad C_v^T, \\ \phi = 0 & \text{on} \quad \partial C_v^T, \end{cases} \tag{2.16}$$

Uniqueness of the solution implies that $\phi_{v,T}$ is T-periodic in the t variable, that is

$$\phi(x, t+T) = \phi(x,t) \quad \text{for all} \quad (x,t) \in C_v^T.$$

After identifying ∂C_v^T with $\partial B_1 \times \mathbb{R}$, we can define the operator

$$N(v,T) = \frac{\partial \phi_{v,T}}{\partial \nu} - \fint_{\partial B_1 \times \mathbb{R}} \frac{\partial \phi_{v,T}}{\partial \nu}\, dS,$$

where by the symbol $\fint_{\partial\Omega}$ we represent the mean value over $\partial\Omega$. Now, by the Schauder regularity theory and the T-periodicity, $N(\cdot,T)$ defines an operator between $U \subset C^{2,\alpha}(\mathbb{R}/T\mathbb{Z})$ and $C^{1,\alpha}(\mathbb{R}/T\mathbb{Z})$, where U is a neighbourhood of the zero function $v = 0$. After a change of scales we may consider $N(\cdot,T) : U \subset C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \to C^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$. Then, we see that the solution $\phi_{v,T}$ to (2.16) solves the overdetermined problem (2.14) for $\Omega = C_v^T$ if and only if

$$N(v,T) = 0. \tag{2.17}$$

Note that when $v = 0$, then $\phi_{0,T}$ is the unique solution to (2.15) for all $T > 0$. Therefore, there exists a trivial branch of solutions $(0, T)$ to equation (2.17). The equation above can be seen as a family of operator equations depending on a parameter T, and thus the methods of Bifurcation theory can be applied to it (see Appendix A). The idea can be roughly explained as follows. By the Implicit Function Theorem, if the linearization $dN(0, T_0) : C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \to C^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ is an isomorphism of Banach spaces, then in a small neighbourhood of $(0, T_0) \in C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \times (0, \infty)$ the only solutions to (2.17) are the trivial $(0, T)$. Therefore, a necessary condition to find non-trivial solutions to (2.17) near $(0, T_0)$ is that $dN(0, T_0)$ is *degenerate*, in the sense that its kernel $\ker dN(0, T_0)$ is non-trivial. In this way, a crucial step in the construction is to show that such T_0 exists, and the next step is to show that the hypotheses of a bifurcation theorem are fulfilled. Following this reasoning, Sicbaldi concluded:

Theorem 2.9 (Sicbaldi). *Let $n \geq 2$ and $\alpha \in (0, 1)$. There exists a positive number T_*, a sequence of positive numbers $T_j \to T_*$, and a sequence of non-zero T_j-periodic functions $v_j \in C^{2,\alpha}(\mathbb{R})$ converging to 0 such that the domains*

$$\Omega_j := C_{v_j}^{T_j} = \left\{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < 1 + v_j(t)\right\}$$

support a positive solution $u_j \in C^{2,\alpha}(\Omega_j)$ to (2.14).

Note that when the dimension $n \geq 3$, the sets $\mathbb{R}^n \setminus \overline{\Omega_j}$ are connected, and thus the sequence Ω_j is a family of counterexamples to the Berestycki-Caffarelli-Nirenberg Conjecture. Note, however, that when the dimension $n = 2$, the sets $\mathbb{R}^n \setminus \overline{\Omega_j}$ have two connected components, and thus do not contradict the conjecture.

The domains in Theorem 2.9 were constructed by means of the Krasnoselskii Bifurcation Theorem A.2, and a careful study of the linearized operator $dN(0, T)$ and how its kernel varied with respect to varying the bifurcation parameter T was needed. Later, Schlenk and Sicbaldi [SS12] managed to strengthen the proofs in the construction and showed that the domains Ω_j in fact belong to a smooth family of bifurcating domains to equation (2.14). The latter construction is based on the Crandall-Rabinowitz Bifurcation Theorem A.1, which yileds a smooth curve of solutions rather than a sequence converging to the trivial branch at the bifurcation point. We will make use of each of these two bifurcation methods later in Chapters 3 and 4, and show how they work in detail.

Fall, Minlend and Weth [FMW17] considered a related overdetermined problem in generalized cylinder-type domains, obtaining a similar result.

Theorem 2.10 (Fall-Minlend-Weth). *Let $m, n \geq 1$, and $\alpha \in (0, 1)$. Then there exists $\lambda_* = \lambda_*(n) > 0$ and a smooth curve*

$$s \in (-\varepsilon, \varepsilon) \mapsto (\lambda_s, \varphi_s) \in (0, \infty) \times C^{2,\alpha}(\mathbb{R}^m)$$

with $\varphi_0 = 0$, $\lambda_0 = \lambda_*$, and φ_s a non-constant $\mathbb{R}^m/2\pi\mathbb{Z}^m$-periodic function for every $s \neq 0$, such that there exists a solution $u \in C^{2,\alpha}(\overline{\Omega_s})$ to the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \textit{in} \quad \Omega_s, \\ u = 0 & \textit{on} \quad \partial\Omega_s, \\ u_\nu = \lambda_s/n & \textit{on} \quad \partial\Omega_s, \end{cases} \tag{2.18}$$

in the domains

$$\Omega_s := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| < \lambda_s + \varphi_s(y)\} \subset \mathbb{R}^{n+m}.$$

The method of construction follows the same scheme as in [Sic10, SS12]. Note that in the cylinders

$$\Omega_\lambda := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| < \lambda\}$$

there exists a solution to the overdetermined problem (2.18) given by

$$u_\lambda(x,y) := \frac{\lambda^2 - |x|^2}{2n}.$$

Consider the perturbed cylinders

$$\Omega_\lambda^\varphi := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| < \lambda + \varphi(y)\},$$

where the perturbations are functions $\varphi \in C^{2,\alpha}(\mathbb{R}^m)$ which are $\mathbb{R}^m/2\pi\mathbb{Z}^m$-periodic. Then, consider the operator

$$F(\varphi,\lambda) := \frac{\partial u_{\varphi,\lambda}}{\partial \nu} - \frac{\lambda}{n},$$

where $u = u_{\varphi,\lambda} \in C^{2,\alpha}(\overline{\Omega_\lambda^\varphi})$ is a solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega_\lambda^\varphi, \\ u = 0 & \text{on} \quad \partial\Omega_\lambda^\varphi. \end{cases} \tag{2.19}$$

Then, the solution $u_{\lambda,\varphi}$ to the problem (2.19) satisfies a constant Neumann condition if

$$F(\varphi,\lambda) = 0. \tag{2.20}$$

Note that for $\varphi = 0$, the solution $u_{0,\lambda} = u_\lambda$, and so there is a trivial branch of solutions $(0,\lambda)$ to (2.20). The goal then, to prove Theorem 2.10, is to verify the hypotheses of the Crandall-Rabinowitz Theorem A.1 in equation (2.20) and conclude that bifurcations of non-trivial solutions occur at a critical value of the parameter λ.

Fall, Minlend and Weth also considered the above overdetermined problem in the sphere $\mathbb{S}^n$ [FMW18]. Recall that by Theorem 2.6, if the domain $\Omega \subset \mathbb{S}^n$ is contained in a hemisphere, then it must be a geodesic ball to support a solution to 2.9. The observation of Fall et al. is that the domains

$$D_\lambda := \{((\cos\theta)\sigma, \sin\theta) \in \mathbb{S}^n : \sigma \in \mathbb{S}^{n-1},\ |\theta| < \lambda\}, \quad \lambda \in (0, \pi/2),$$

also support a solution to

$$\begin{cases} -\Delta_{\mathbb{S}^n} u = 1 & \text{in} \quad D, \\ u = 0 & \text{on} \quad \partial D, \\ u_\nu = \text{const} & \text{on} \quad \partial D, \end{cases} \tag{2.21}$$

The domains D_λ are tubular neighbourhoods of the equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$, and thus are not contained in a hemisphere of $\mathbb{S}^n$. Also, the solution to (2.21) is not monotone along the θ-variable. Therefore, there is the possibility of obtaining solutions to (2.21) in domains D which are small perturbations of the domains D_λ. The result in [FMW18] is obtained in this line of reasoning.

Theorem 2.11 (Fall-Minlend-Weth)**.** *Let $n \geq 2$, and $\alpha \in (0,1)$. Then there exists a strictly decreasing sequence $\lambda_j \to 0$ with the following property: for each j there exists $e_j > 0$ and a smooth curve*

$$s \in (-\varepsilon_j, \varepsilon_j) \mapsto (\lambda_j(s), \varphi_j(s)) \in (0, \pi/2) \times C^{2,\alpha}(\mathbb{S}^n)$$

with $\varphi_j(0) = 0$, $\lambda_j(0) = \lambda_j$, and $\varphi_j(s)$ a non-constant function for every $s \neq 0$, such that there exists a solution $u \in C^{2,\alpha}(\overline{D_j(s)})$ to the overdetermined problem (2.21) in the domains

$$D_j(s) := \{((\cos\theta)\sigma, \sin\theta) \in \mathbb{S}^n : \sigma \in \mathbb{S}^{n-1},\ |\theta| < \lambda_j(s) + \varphi_j(s)\}.$$

Note that the bifurcating domains $D_j(s)$ have two connected boundary components, in contrast to the domains in Theorems 2.9 and 2.10, which may appear as an extra difficulty. However, the symmetry of the sphere $\mathbb{S}^n$ with respect to its equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ allows a symmetric perturbation of the domains D_λ with respect to $\mathbb{S}^{n-1}$. Thus, in the construction in Theorem 2.11, the treatment of the disconnected boundary can be reduced to the treatment of *one* connected component, which by the symmetry of the domain will imply the desired properties. In Chapters 3 and 4 we will treat overdetermined problems in which the underlying domains have two *non symmetric* connected boundary components, which does not allow the treatment of the construction in [FMW18].

The last result we want to discuss is the one due to Ros, Ruiz and Sicbaldi [RRS19]. Their construction gives, finally, a negative answer to the Berestycki-Caffarelli-Nirenberg Conjecture in dimension $n = 2$, the only dimension remaining.

Theorem 2.12 (Ros-Ruiz-Sicbaldi)**.** *Let $n \geq 2$, $1 < p < \frac{p+2}{p-2}$, when $n \geq 3$, and $p > 1$, when $n = 2$. Then there exist exterior C^2-domains $\Omega \subset \mathbb{R}^n$ different from the complement of a ball, such that the overdetermined problem*

$$\begin{cases} -\Delta u + u - u^p = 0, \quad u > 0 & \text{in} \quad \Omega, \\ u = 0, \quad u_\nu = const & \text{on} \quad \partial\Omega, \end{cases} \tag{2.22}$$

admits a bounded solution.

The construction of the domains in Theorem 2.12 is aligned with that of Theorem 2.9. The authors notice that there exists a positive, radially symmetric solution to (2.22), which is also bounded, wheren the domain Ω is the complement of a ball $B_R \subset \mathbb{R}^n$. Under some symmetry conditions, Ros, Ruiz and Sicbaldi show that for radii R less than a fixed raidius R_0, the Dirichlet problem in equation (2.22) admits a unique (in the appropriate function space) positive solution $u = u_{v,R}$ in the perturbed domains

$$B^c_{R,v} := \{x \in \mathbb{R}^n : |x| > R + v\},$$

where $v \in C^{2,\alpha}(\mathbb{S}^{n-1})$ is small, but not 0. Then, the authors rephrase the problem as an operator equation $F(v, R) = 0$, where

$$F(v, R) := \frac{\partial u_{v,R}}{\partial \nu} - \fint_{\partial B_R} \frac{\partial u_{v,R}}{\partial \nu}\, dS,$$

and then derive the required properties to apply the Krasnoselskii Bifurcation Theorem A.2, and show bifurcation of non-trivial solutions to $F(v, R)$ at a some critical value R_*.

A point worth of noting is that equation (2.22) is non-linear, in contrast to the constructions described above. The non-linear term puts an extra difficulty, since the linearized equation may fail to be invertible. Ros et al. overcome this difficulty restricting to function spaces which satisfy symmetry conditions, and which guarantee the existence of the interval $(0, R_0)$ in which the positive bounded solutions $u_{v,R}$ can be shown to exist and be unique. Then, the main difficulty in the construction is to show that bifurcations occur precisely in the interval $(0, R_0)$.

In Chapter 3 and Chapter 4 we give new constructions of non-trivial solutions to overdetermined problems in annular domains by means of Bifurcation theoretic methods. Our constructions are done in the spirit of the constructions described above, but bear new difficulties not present before. The novelty of these constructions is that they deal, unlike previous ones, with domains whose boundaries are not connected, and are not symmetric in an obvious geometric way. This is a new feature, and as far as we are aware, our treatment is the first in this line of results that deals with it.

First result

In the following chapter we show the first of the two main results obtained. The exposition here is largely contained in [KS20] . However, that paper is aimed to an expert audience and thus some details in the proofs are omitted. Since it is the intention of this work to serve as an introduction to the field, those details are provided here or else the appropriate reference for their discussion.

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1}$, where $\Omega_1 \Subset \Omega_0$. We focus on the problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega_0, \\ u = a & \text{on} \quad \partial\Omega_1, \\ u_\nu = c & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

where a is a positive constant. Note that by the maximum principle any solution u to (3.1) is automatically positive in Ω.

The main result of this Chapter is the following.

Theorem 3.1. *There exist bounded, real analytic annular domains of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1} \subset \mathbb{R}^n$, which are different from standard annuli, such that the overdetermined problem (3.1) admits a solution $u \in C^\infty(\overline{\Omega})$ for some positive constants a and c.*

We construct these nontrivial annular domains and their corresponding solutions by the means of the Crandall-Rabinowitz Bifurcation Theorem [CR71] (see Theorem A.1 in the Appendix A). In this manner, we actually obtain a smooth branch of domains and solutions (in fact, a whole sequence of distinct branches) bifurcating from the trivial branch of standard annuli admitting the radial, non-monotone solutions of (3.1) given by Lemma 3.3 in Section 3.1. This is more precisely stated in the body of Theorem 3.5 in Section 3.1, of which Theorem 3.1 is an immediate corollary.

The overdetermined problem (3.1) has a relation to the so-called *Cheeger problem*: given a bounded domain $\Omega \subset \mathbb{R}^n$, find

$$h_1(\Omega) := \inf \left\{ \frac{P(E)}{|E|} : E \subset \Omega \text{ Lebesgue measurable}, |E| > 0 \right\} \tag{3.2}$$

where $|E|$ is the Lebesgue measure of E and $P(E)$ denotes the perimeter functional ([Giu84]). The constant $h_1(\Omega)$ is known as the *Cheeger constant* for the domain Ω, and a subset $E \subset \Omega$, for which the infimum in (3.2) is attained, is called a *Cheeger set* for Ω. See the surveys [Par11] and [Leo15] for an overview of the Cheeger problem and a discussion of applications. It turns out that the domains Ω constructed in Theorem 3.1 are precisely their own Cheeger sets. Such domains are called *self-Cheeger.*

Corollary 3.2. *Let Ω be any one of the real analytic annular domains in Theorem 3.1 that admits a solution $u \in C^\infty(\overline{\Omega})$ of (3.1). Then*

$$h_1(\Omega) = \frac{P(\Omega)}{|\Omega|} = \frac{1}{c}$$

and Ω is the unique (up to sets of zero measure) minimizer of the Cheeger problem (3.2).

In this manner, Corollary 3.2 establishes the existence of non-radially symmetric domains that are self-Cheeger.

Our approach to Theorem 3.1 is aligned with that of Schlenk and Sicbaldi in [SS12] and Fall, Minlend and Weth in [FMW17, FMW18]: we translate the problem to a non-linear, non-local operator equation in appropriate function spaces and we derive the necessary spectral and Fredholm tranversality properties of the linearized operators in order to implement the Crandall-Rabinowitz Bifurcation Theorem. However, unlike the quoted results above where symmetry considerations allow the authors to perturb all connected boundary components of the underlying domains in the same symmetric fashion, we are bound by the geometry of the standard annulus $\Omega_\lambda = \{x \in \mathbb{R}^n : \lambda < |x| < 1\}$ to deform its two *non-symmetric* boundary components differently. Thus, the function spaces that we work in are necessarily product spaces of two factors that correspond to the two separate connected components of $\partial\Omega_\lambda$.

The chapter is organized as follows. In the next section we outline the strategy of the construction leading up to the statement of Theorem 3.5, which refines Theorem 3.1, and we show how the latter follows from the former. In Section 3.2 we set up the problem as an operator equation $F_\lambda(\mathbf{v}) = \mathbf{0}$, where $F_\lambda : U \subset (C^{2,\alpha}(\mathbb{S}^{n-1}))^2 \to (C^{1,\alpha}(\mathbb{S}^{n-1}))^2$, and compute a formula for its linearization $L_\lambda := dF_\lambda|_{\mathbf{v}=\mathbf{0}}$ in terms of the Dirichlet-to-Neumann operator for the Laplacian in Ω_λ (Proposition 3.6). In Section 3.3 we study the spectrum of L_λ: we find that for each $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, L_λ has *two* different eigenvalue branches $\mu_{k,1}(\lambda) < \mu_{k,2}(\lambda)$ with associated eigenvectors in the subspaces $\mathbb{R}\mathcal{Y}_k \oplus \mathbb{R}\mathcal{Y}_k$, where $\mathcal{Y}_k$ is any spherical harmonic of degree k. In Lemmas 3.9 and 3.10 we establish key monotonicity properties for $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ in both λ and k, from which we infer that for $k \geq 2$ the first eigenvalue branch $\mu_{k,1}(\lambda)$ is strictly decreasing in $\lambda \in (0, 1)$, changing sign once, while the second $\mu_{k,2}(\lambda) > 0$ is always positive. In Section 3.4 we restrict the operators to pairs of functions on the sphere which are invariant under the action of an appropriate group of isometries G so as to ensure that, whenever 0 is an eigenvalue of the restricted L_λ, it is simple. We then verify the relevant Fredholm mapping properties (Proposition

3.14), necessary to apply the Crandall-Rabinowitz Bifurcation Theorem and complete the proof of Theorem 3.5. Finally, in Section 3.5 we provide the proof of Corollary 3.2.

3.1 Outline of strategy and refinement of the main theorem

Let us first introduce some notation. For any $\lambda \in (0,1)$ we denote the standard annulus of inner radius λ and outer radius 1 by

$$\Omega_\lambda := \{x \in \mathbb{R}^n : \lambda < |x| < 1\}$$

and let its two boundary components be

$$\Gamma_1 := \{x \in \mathbb{R}^n : |x| = 1\} = \mathbb{S}^{n-1},$$
$$\Gamma_\lambda := \{x \in \mathbb{R}^n : |x| = \lambda\} = \lambda\mathbb{S}^{n-1},$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, centered at the origin.

We will construct the nontrivial solutions u and domains Ω solving (3.1) by bifurcating away, at certain critical values of the bifurcation parameter λ, from the branch of non-monotone radial solutions u_λ of (3.1) defined on the annuli Ω_λ, for which $\partial_\nu u_\lambda = c_\lambda$ is the same constant on all of $\partial\Omega_\lambda$. We describe this branch of solutions explicitly in the lemma below.

Lemma 3.3. *For each $\lambda \in (0,1)$, there exist unique positive values a_λ and c_λ given by*

$$a_\lambda = \begin{cases} \frac{1}{2}\lambda \log \lambda + \frac{1}{4}\left(1-\lambda^2\right) & \text{if } n=2, \\ \frac{1}{n}\lambda \frac{\lambda^{n-2}-1}{n-2}\frac{1+\lambda}{1+\lambda^{n-1}} + \frac{1}{2n}\left(1-\lambda^2\right) & \text{if } n \geq 3, \end{cases} \tag{3.3}$$

$$c_\lambda = \frac{1}{n}\frac{1-\lambda^n}{1+\lambda^{n-1}} \tag{3.4}$$

such that for $\Omega = \Omega_\lambda$ the problem (3.1) has a unique positive solution $u = u_\lambda$ with boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_1, \qquad u = a_\lambda \quad \text{on} \quad \Gamma_\lambda, \qquad u_\nu = c_\lambda \quad \text{on} \quad \partial\Omega_\lambda.$$

The solution is radially symmetric and given by

$$u_\lambda(x) = \begin{cases} \frac{1}{2}\lambda \log|x| + \frac{1}{4}\left(1-|x|^2\right), & \text{if} \quad n=2, \\ \frac{\lambda^{n-1}}{n(n-2)}\frac{1+\lambda}{1+\lambda^{n-1}}\left(1-|x|^{2-n}\right) + \frac{1}{2n}\left(1-|x|^2\right) & \text{if} \quad n \geq 3. \end{cases} \tag{3.5}$$

Proof. If $u = u(r)$, where $r = |x|$, is a radially symmetric solution to (3.1), then u satisfies the ODE

$$u'' + \frac{n-1}{r}u' = -1,$$

where the prime denotes differentiation with respect to r. Then simple integration yields

$$u'(r) = \frac{C}{r^{n-1}} - \frac{r}{n}. \tag{3.6}$$

Note that $u_\nu|_{\Gamma_\lambda} = u'(\lambda)$ and $u_\nu|_{\Gamma_1} = -u'(1)$. Therefore, solving $-u'(1) = u'(\lambda)$ for C, we obtain

$$C = \frac{1+\lambda}{n(1+\lambda^{1-n})}.$$

The formulas (3.3)-(3.5) for a_λ, c_λ and u_λ follow by integrating (3.6) once again and setting $u(1) = 0$. It remains to confirm that $a_\lambda > 0$ when $\lambda \in (0,1)$. When $n = 2$, this follows from the observation that

$$\lim_{\lambda\uparrow 1} \frac{da_\lambda}{d\lambda} = 0 = \lim_{\lambda\uparrow 1} a_\lambda \quad \text{and} \quad \frac{d^2 a_\lambda}{d\lambda^2} = (\lambda^{-1} - 1)/2 > 0 \quad \text{for} \quad \lambda \in (0,1).$$

For $n \geq 3$, we can rewrite the expression (3.3) for a_λ as

$$a_\lambda = \frac{1+\lambda}{2n(n-2)(1+\lambda^{n-1})} g(\lambda) \quad \text{where} \quad g(\lambda) = (n-2) - n\lambda - (n-2)\lambda^n + n\lambda^{n-1}.$$

Then the positivity of $g(\lambda)$, and thus of a_λ, over $\lambda \in (0,1)$, follows from the fact that

$$\lim_{\lambda\uparrow 1} g'(\lambda) = 0 = \lim_{\lambda\uparrow 1} g(\lambda) \quad \text{and} \quad \frac{d^2 g(\lambda)}{d\lambda^2} = n(n-1)(n-2)\lambda^{n-3}(1-\lambda) > 0 \quad \text{for} \quad \lambda \in (0,1).$$

□

We will be perturbing the boundary of each annulus Ω_λ in the direction of the *inner* unit normal to $\partial\Omega_\lambda$. For a pair of functions $\mathbf{v} = (v_1, v_2) \in C^{2,\alpha}(\mathbb{S}^{n-1}) \times C^{2,\alpha}(\mathbb{S}^{n-1})$ of sufficiently small $C^{2,\alpha}$-norm, $0 < \alpha < 1$, denote the $\mathbf{v}$-deformation of Ω_λ by:

$$\Omega_\lambda^{\mathbf{v}} := \{x \in \mathbb{R}^n : \lambda + v_1(x/|x|) < |x| < 1 - v_2(x/|x|)\},$$

so that its boundary $\partial\Omega_\lambda^{\mathbf{v}} = \Gamma_1^{\mathbf{v}} \cup \Gamma_\lambda^{\mathbf{v}}$, where

$$\Gamma_1^{\mathbf{v}} := \{x \in \mathbb{R}^n : |x| = 1 - v_2(x/|x|)\},$$
$$\Gamma_\lambda^{\mathbf{v}} := \{x \in \mathbb{R}^n : |x| = \lambda + v_1(x/|x|)\}.$$

Our perturbations $\mathbf{v}$ will ultimately be taken to be invariant with respect to the action of a certain subgroup G of the orthogonal group $O(n)$. We call a domain $\Omega \subset \mathbb{R}^n$ G-invariant if it is invariant under the action of G, and a function $\psi : \Omega \to \mathbb{R}$, defined on a G-invariant domain Ω, is called G-invariant if

$$\psi = \psi \circ g, \quad \text{for every} \quad g \in G.$$

The notation for the usual Hölder and Sobolev function spaces, restricted to G-invariant functions, will include a subscript-G, as in $C_G^{k,\alpha}$, L_G^2, H_G^k, etc. We point out that these spaces are closed subsets of the corresponding function spaces, and hence Banach and Hilbert spaces on their own.

We know that for each $\lambda \in (0,1)$ and every $\mathbf{v} \in (C^{2,\alpha}(\mathbb{S}^{n-1}))^2$ of appropriately small $C^{2,\alpha}$-norm, the Dirichlet problem in the perturbed annulus $\Omega_\lambda^{\mathbf{v}}$

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega_\lambda^{\mathbf{v}}, \\ u = 0 & \text{on} \quad \Gamma_1^{\mathbf{v}}, \\ u = a_\lambda & \text{on} \quad \Gamma_\lambda^{\mathbf{v}}, \end{cases} \tag{3.7}$$

with a_λ defined as in (3.3), has a unique solution $u = u_\lambda(\mathbf{v}) \in C^{2,\alpha}(\overline{\Omega^{\mathbf{v}}_\lambda})$ [GT15]. Moreover, $u_\lambda(\mathbf{0}) = u_\lambda$, the map $(\mathbf{v}, \lambda) \mapsto u_\lambda(\mathbf{v})$ is smooth by standard regularity theory. Note also that if $\mathbf{v}$ is G-invariant, then the domain $\Omega^{\mathbf{v}}_\lambda$ is also G-invariant, and so are equations (3.7). Therefore, the uniqueness for the solution of the Dirichlet problem implies that $u_\lambda(\mathbf{v})$ and $\partial_\nu u_\lambda(\mathbf{v})$ are G-invariant functions.

We would like to find out when the Dirichlet problem solution $u_\lambda(\mathbf{v})$ also satisfies a constant Neumann condition on $\partial\Omega^{\mathbf{v}}_\lambda$. For the purpose, given $\lambda \in (0,1)$ and $U \subset C^{2,\alpha}(\mathbb{S}^{n-1}) \times C^{2,\alpha}(\mathbb{S}^{n-1})$ – a sufficiently small neighbourhood of $\mathbf{0}$, we define the operator

$$F_\lambda : U \to C^{1,\alpha}(\mathbb{S}^{n-1}) \times C^{1,\alpha}(\mathbb{S}^{n-1}),$$
$$F_\lambda(\mathbf{v}) := \frac{1}{c_\lambda}\left(\frac{\partial u_\lambda(\mathbf{v})}{\partial \nu}\bigg|_{\Gamma^{\mathbf{v}}_\lambda} - c_\lambda,\ \frac{\partial u_\lambda(\mathbf{v})}{\partial \nu}\bigg|_{\Gamma^{\mathbf{v}}_1} - c_\lambda \right), \tag{3.8}$$

where we identify functions on $\partial\Omega^{\mathbf{v}}_\lambda$ with pairs of functions on $\mathbb{S}^{n-1}$ in a way that is going to be precise later. The Schauder regularity theory [GT15, Gri11] implies that (3.8) is a good definition, and the factor of $1/c_\lambda$ provides a convenient normalization. Now, the solution $u_\lambda(\mathbf{v})$ of the Dirichlet problem (3.7) in $\Omega^{\mathbf{v}}_\lambda$ solves the full overdetermined problem (3.1) if

$$F_\lambda(\mathbf{v}) = \mathbf{0}. \tag{3.9}$$

Note that $F_\lambda(\mathbf{0}) = \mathbf{0}$ for every $\lambda \in (0,1)$. Our goal is to find a branch of solutions $(\mathbf{v}, \lambda)$ of (3.9), bifurcating from this trivial branch $(\mathbf{0}, \lambda)$. To achieve it, we will need to understand how the kernel of the linearization $L_\lambda := dF_\lambda|_{\mathbf{v}=\mathbf{0}}$ depends on λ. In Proposition 3.6 of the next section we will derive a working formula for L_λ:

$$L_\lambda(w_1, w_2) = \left(-\frac{\partial \phi_{\mathbf{w}}}{\partial \nu}\bigg|_{\Gamma_\lambda} + \frac{w_1}{c_\lambda}\frac{\partial^2 u_\lambda}{\partial r^2}\bigg|_{\Gamma_\lambda},\ -\frac{\partial \phi_{\mathbf{w}}}{\partial \nu}\bigg|_{\Gamma_1} + \frac{w_2}{c_\lambda}\frac{\partial^2 u_\lambda}{\partial r^2}\bigg|_{\Gamma_1} \right),$$

where $\phi_{\mathbf{w}}$ is the harmonic function ϕ on Ω_λ with boundary values $\phi|_{\Gamma_\lambda}(x) = w_1(x/|x|)$ and $\phi|_{\Gamma_1}(x) = w_2(x/|x|)$.

In order to study the kernel of L_λ, we will look more generally at the eigenvalue problem

$$L_\lambda(\mathbf{w}) = \mu(\lambda)\mathbf{w}, \qquad \mathbf{w} \in (C^{2,\alpha}(\mathbb{S}^{n-1}))^2.$$

For each spherical harmonic $\mathcal{Y}_k$ of degree $k \in \mathbb{N}_0$, the subspace

$$W_k = \mathrm{Span}\{(\mathcal{Y}_k, 0), (0, \mathcal{Y}_k)\}$$

is invariant under L_λ and decomposes into eigenspaces for $L_\lambda|_{W_k}$, associated with two distinct eigenvalues $\mu_{k,1}(\lambda) < \mu_{k,2}(\lambda)$, for which we will calculate explicit formulas in Section 3.3. We will study the dependence of these eigenvalues on both $\lambda \in (0,1)$ and $k \in \mathbb{N}_0$, focussing on whether they cross 0 as λ varies in $(0,1)$. It turns out that when $k = 1$, $\mu_{1,1}(\lambda) < 0$ while $\mu_{1,2}(\lambda) = 0$ for all λ, i.e.

the eigenspace correponding to $\mu_{1,2}(\lambda)$ is always in the kernel of L_λ (see Remark 3.3). This part of ker L_λ comes from the deformations of Ω_λ, generated by translations. What we find for $k \geq 2$ is that the first eigenvalue branch $\mu_{k,1}(\lambda)$ is *strictly decreasing* in λ and that it crosses 0 at a unique $\lambda_k^* \in (0,1)$. This is done in Lemma 3.9. Moreover, we establish in Lemma 3.10 that both $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ increase *strictly* in $k \in \mathbb{N}_0$ for fixed λ. This means that for $k \geq 2$, the second eigenvalue branch $\mu_{k,2}(\lambda) > 0$ never crosses 0, while the values of λ_k^*, at which $\mu_{k,1}(\lambda)$ is zero, form a strictly increasing k-sequence; in addition, the eigenvalues for $k = 0$, $\mu_{0,1}(\lambda) < \mu_{0,2}(\lambda) < 0$ (Proposition 3.12). Therefore, at the critical values $\lambda = \lambda_k^*$, $k \geq 2$, the kernel of $L_{\lambda_k^*}$ over $(C^{2,\alpha}(\mathbb{S}^{n-1}))^2$ consists of the $\mu_{k,1}(\lambda_k^*)$-eigenspace plus the always present component of the $\mu_{1,2} \equiv 0$ eigenspace.

We prove the λ-monotonicity of the first branch $\mu_{k,1}(\lambda)$, $k \geq 2$, by analyzing the explicit formula for $\mu_{k,1}(\lambda)$ directly and using some delicate estimates, based on hyperbolic trigonometric identities (see the proof of Lemma 3.9). Unfortunately, this approach does not extend to the second eigenvalue branch $\mu_{k,2}(\lambda)$, $k \geq 2$, which we also believe to be decreasing in λ, based on numerics. Showing the latter is ultimately not necessary, since we prove that $\mu_{k,2}(\lambda) > 0$ never contributes to the kernel of L_λ. Neither do the eigenvalue branches for $k = 0$, $\mu_{0,j}(\lambda)$, $j = 1, 2$, which are shown to be strictly negative. In order to establish the k-monotonicity of $\mu_{k,j}(\lambda)$, for fixed λ, $j = 1, 2$, we treat k as a continuous positive variable and extend the $\mu_{k,j}(\lambda)$ to be smooth functions in $(k, \lambda) \in (0, \infty) \times (0, 1)$, continuous up to $k = 0$ (see Remark 3.2). Even so, trying to prove $\partial_k \mu_{k,j}(\lambda) > 0$ directly from the formula for the eigenvalue turns out to be unyielding, and we accomplish it instead by looking at the matrix representation $M_{\lambda,k}$ of $L_\lambda|_{W_k}$ and showing that $\partial_k M_{\lambda,k}$ is positive definite (see the proof of Lemma 3.10).

The Crandall-Rabinowitz Bifurcation Theorem requires that the linearized operator has a one-dimensional kernel at bifurcation values. To achieve this, in Section 3.4 we will restrict the operators F_λ and L_λ to G-invariant functions in $(C^{2,\alpha}(\mathbb{S}^{n-1}))^2$. The symmetry group G will be chosen so as to completely eliminate the eigenspaces of L_λ corresponding to $k = 1$ (which contain the $\mu_{1,2} = 0$ component of ker L_λ), and ensure that, whenever $\mu_{k,1}(\lambda)$ is an eigenvalue of the restricted L_λ for some $k \geq 2$, it is of multiplicity 1. Additionally, we will choose the group G in a way that will guarantee that the constructed G-invariant domains $\Omega_\lambda^{\mathbf{v}}$ are not merely translations of the standard annulus.

More precisely, let $-\Delta_{\mathbb{S}^{n-1}}$ be the Laplace-Beltrami operator on the sphere $\mathbb{S}^{n-1}$ and let $\{\sigma_i\}_{i=0}^{\infty}$ be the sequence of its eigenvalues, i.e. $\sigma_i = i(i + n - 2)$. We shall fix any group $G \leqslant O(n)$ that has the following two properties:

(P1) If T is a translation of $\mathbb{R}^n$ and $T(\mathbb{S}^{n-1})$ is a G-invariant set, then T is trivial.

(P2) If $\{\sigma_{i_k}\}_{k=0}^{\infty}$ are the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ when restricted to the G-invariant functions, then σ_{i_k} has multiplicity equal to 1, i.e. there exists a unique (up to normalization) G-invariant spherical harmonic of degree i_k, $k \in \mathbb{N}_0$.

Note that $i_0 = 0$ and that because of (P1), spherical harmonics of degree 1 are not G-invariant, i.e. $i_1 \geq 2$. An example of such a group G can be established as a consequence of the following lemma and the fact that spherical harmonics are precisely given by the restrictions to $\mathbb{S}^{n-1}$ of homogeneous harmonic polynomials in n variables.

Lemma 3.4. *Let $G = O(n-1) \times \mathbb{Z}_2$ act on $\mathbb{R}^n$ as orthogonal transformations on the first $n-1$ variables and reflections with respect to the hyperplane $x_n = 0$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous G-invariant harmonic polynomial. Then the degree of f is even. Moreover, for each $k \in \mathbb{N}_0$ the vector space of homogeneous G-invariant harmonic polynomials of degree $2k$ is one-dimensional.*

Proof. Clearly if f is constant, then it is G-invariant. If f is a nonconstant homogeneous G-invariant polynomial, then

$$f(x_1, \ldots, x_n) = f(\pm|x'|, 0, \ldots, 0, x_n) = f(x', \pm x_n), \quad \text{for every} \quad (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Thus, the monomials comprising f have to be products of even powers of $|x'|$ and x_n, implying that $\deg(f) = 2k$, for some $k \in \mathbb{N}$. Since f is homogeneous, it takes the form

$$f(x', x_n) = \sum_{j=0}^{k} a_j x_n^{2j} |x'|^{2(k-j)}, \quad (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

for some coefficients $a_j \in \mathbb{R}$, $j = 0, \ldots, k$. Computing the Laplacian of f,

$$\Delta f(x', x_n) = 2 \sum_{j=0}^{k-1} \{a_j (k-j)[n-1+2(k-j-1)] + a_{j+1}(j+1)(2j+1)\} x_n^{2j} |x'|^{2(k-j-1)},$$

so that f is harmonic if and only if its coefficients satisfy the homogeneous linear system

$$a_j(k-j)[n-1+2(k-j-1)] + a_{j+1}(j+1)(2j+1) = 0, \quad \text{for} \quad j = 0, \ldots, k-1.$$

We easily see that fixing $a_0 = 1$ determines uniquely the remaining coefficients. $\square$

The group $G = O(n-1) \times \mathbb{Z}_2$ clearly satisfies (P1), and because of Lemma 3.4, it also satisfies (P2) with $i_k = 2k$ and $\sigma_{i_k} = 2k(2k+n-2)$, $k \in \mathbb{N}_0$.

Let us denote $Y_k := \mathcal{Y}_{i_k}$, where $\mathcal{Y}_{i_k}$ is the unique G-invariant spherical harmonic of degree $i_k \in \mathbb{N}_0$, normalized in the L^2-norm, that is,

$$\Delta_{\mathbb{S}^{n-1}} Y_k + \sigma_{i_k} Y_k = 0, \qquad \int_{\mathbb{S}^{n-1}} |Y_k|^2 \, dS = 1, \qquad k \in \mathbb{N}_0.$$

Finally, denote by

$$\langle \mathbf{w}, \mathbf{z} \rangle_\lambda := \lambda^{n-1} \int_{\mathbb{S}^{n-1}} w_1 z_1 \, dS + \int_{\mathbb{S}^{n-1}} w_2 z_2 \, dS, \qquad \mathbf{w}, \mathbf{z} \in \left(L^2(\mathbb{S}^{n-1})\right)^2, \tag{3.10}$$

the inner product on $L^2(\mathbb{S}^{n-1}) \times L^2(\mathbb{S}^{n-1})$ induced by the standard inner product on $L^2(\partial\Omega_\lambda)$ under the natural identification

$$\begin{array}{rcl} \mathbf{w} & \leftrightarrow & \psi \\ (w_1(x), w_2(x)) & = & (\psi(\lambda x), \psi(x)) \quad \text{for all } x \in \mathbb{S}^{n-1}, \end{array} \tag{3.11}$$

and let

$$\|\mathbf{w}\|_\lambda := \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle_\lambda}$$

be the induced L^2-norm. We point out that L_λ is formally self-adjoint with respect to $\langle \cdot, \cdot \rangle_\lambda$ (see Remark 3.1).

As a result of restricting to pairs of G-invariant functions in $C^{2,\alpha}(\mathbb{S}^{n-1})$, the linearized operator $L_\lambda : (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2 \to (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$ will now possess a one-dimensional kernel at each critical value $\lambda_k := \lambda^*_{i_k}$, $k \in \mathbb{N}$ – spanned by an element of the form $\mathbf{z}_k = (a_k Y_k, b_k Y_k)$ – and its image will be the closed subspace of co-dimension 1 orthogonal to $\mathbf{z}_k$ with respect to the inner product (3.10). Moreover, because of the strict λ-monotonicity of $\mu_{i_k,1}(\lambda)$, the tranversality condition

$$\partial_\lambda L_\lambda|_{\lambda=\lambda_k}(\mathbf{z}_k) = \mu'_{i_k,1}(\lambda_k)\mathbf{z}_k - L_\lambda(\mathbf{z}'_k) \notin \text{im } L_{\lambda_k}$$

is going to hold. Invoking the Crandall-Rabinowitz Bifurcation Theorem A.1, we reach at the statement of the refinement of Theorem 3.1.

Theorem 3.5. *Let $n \geq 2$, $\alpha \in (0,1)$, and let G and Y_k, $k \in \mathbb{N}$, be as above. There is a strictly increasing sequence $\{\lambda_k\}_{k=1}^\infty$ of positive real numbers with $\lim_{k\to\infty} \lambda_k = 1$ and a sequence $\{\mathbf{z}_k\}_{k=1}^\infty$ of non-zero elements of the form $\mathbf{z}_k = (a_k Y_k, b_k Y_k)$ with $\|\mathbf{z}_k\|_{\lambda_k} = 1$, satisfying the following: for each $k \in \mathbb{N}$, there exists $\varepsilon > 0$ and a smooth curve*

$$\begin{array}{rcl} (-\varepsilon, \varepsilon) & \to & \left(C^{2,\alpha}(\mathbb{S}^{n-1})\right)^2 \times (0,1) \\ s & \mapsto & (\mathbf{w}(s), \lambda(s)) \end{array}$$

satisfying $\mathbf{w}(0) = 0$, $\lambda(0) = \lambda_k$, such that for $\mathbf{v}(s) \in C^{2,\alpha}(\mathbb{S}^{n-1}) \times C^{2,\alpha}(\mathbb{S}^{n-1})$ defined by

$$\mathbf{v}(s) = s(\mathbf{z}_k + \mathbf{w}(s)),$$

the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \textit{in} \quad \Omega^{\mathbf{v}(s)}_{\lambda(s)}, \\ \quad u = 0 & \textit{on} \quad \Gamma^{\mathbf{v}(s)}_1, \\ \quad u = \textit{const} > 0 & \textit{on} \quad \Gamma^{\mathbf{v}(s)}_{\lambda(s)}, \\ \quad u_\nu = \textit{const} > 0 & \textit{on} \quad \partial\Omega^{\mathbf{v}(s)}_{\lambda(s)}, \end{cases} \tag{3.12}$$

admits a positive solution $u \in C^{2,\alpha}(\overline{\Omega}^{\mathbf{v}(s)}_{\lambda(s)})$. Moreover, for every $s \in (-\varepsilon, \varepsilon)$ the two components of $\mathbf{w}(s) = (w_1(s), w_2(s))$ are G-invariant functions that satisfy

$$\langle \mathbf{w}(s), \mathbf{z}_k \rangle_{\lambda_k} = 0. \tag{3.13}$$

Let us show how Theorem 3.5 entails Theorem 3.1.

Proof of Theorem 3.1. Fix any $k \in \mathbb{N}$ and $\alpha \in (0,1)$. We need only explain why for $s \neq 0$ the $C^{2,\alpha}$ domains $\Omega_{\lambda(s)}^{\mathbf{v}(s)}$, constructed in Theorem 3.5, are different from a standard annulus, and why their boundaries are actually real analytic.

Since the functions $v_1(s)$, $v_2(s)$, are G-invariant, the corresponding domains $\Omega_{\lambda(s)}^{\mathbf{v}(s)}$ and solutions u of (3.12) are also G-invariant. We point out that the orthogonality condition (3.13) implies that for $s \neq 0$, the non-zero $\mathbf{v}(s)$ is also non-constant since $\mathbf{z}_k$ is non-constant and

$$\langle \mathbf{v}(s), \mathbf{z}_k \rangle_{\lambda_k} = s \neq 0,$$

which means that at least one of the boundary components $\Gamma_r^{\mathbf{v}(s)}$, $r = \lambda(s), 1$, is different from a central dilation of $\mathbb{S}^{n-1}$ with respect to the origin. In addition, Property (P1) of G prevents $\Omega_{\lambda(s)}^{\mathbf{v}(s)}$ from being an affine transformation of the annulus Ω_λ that involves a non-trivial translation. All this guarantees the nontriviality of $\Omega_{\lambda(s)}^{\mathbf{v}(s)}$.

The domains $\Omega = \Omega_{\lambda(s)}^{\mathbf{v}(s)}$ are constructed to be of class $C^{2,\alpha}$, but by the classical regularity result of Kinderlehrer and Nirenberg [KN77], the boundary of a $C^{2,\alpha}$-domain Ω, admitting a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to (3.1), gets upgraded to a real analytic one. The solution u itself is real analytic up to the boundary. □

3.2 Reformulating the problem and deriving its linearization

Let us first recast the operator F_λ, defined in (3.8), by pulling back the Dirichlet problem (3.7) from $\Omega_\lambda^{\mathbf{v}}$ to the annulus Ω_λ, where we shall use polar coordinates

$$(0,\infty) \times \mathbb{S}^{n-1} \cong \mathbb{R}^n \setminus \{0\} \quad \text{under} \quad (r,\theta) \mapsto x = r\theta$$

to describe the geometry. In this way, $\Omega_\lambda \cong (\lambda, 1) \times \mathbb{S}^{n-1}$, its boundary components $\Gamma_\lambda \cong \{\lambda\} \times \mathbb{S}^{n-1}$, $\Gamma_1 \cong \{1\} \times \mathbb{S}^{n-1}$ are two copies of $\mathbb{S}^{n-1}$, and we naturally get the identification of functions (3.11).

For any $\mathbf{v} = (v_1, v_2) \in U \subset (C^{2,\alpha}(\mathbb{S}^{n-1}))^2$ of sufficiently small norm, we consider the diffeomorphism $\Phi : \Omega_\lambda \to \Omega_\lambda^{\mathbf{v}}$ defined in polar coordinates by

$$\Phi(r,\theta) = \big(\big(1 + \eta_1(r)v_1(\theta) + \eta_2(r)v_2(\theta)\big)r, \theta\big)\,, \tag{3.14}$$

where η_1, η_2 are smooth functions satisfying

$$\eta_1(r) = \begin{cases} 1/\lambda & \text{if} \quad r \leq \lambda + \delta, \\ 0 & \text{if} \quad r \geq \lambda + 2\delta, \end{cases} \qquad \eta_2(r) = \begin{cases} -1 & \text{if} \quad r \geq 1 - \delta, \\ 0 & \text{if} \quad r \leq 1 - 2\delta, \end{cases} \tag{3.15}$$

for some small enough $\delta > 0$. We set on Ω_λ the pull-back metric $g = \Phi^* g_0$, where g_0 is the Euclidean metric on $\Omega_\lambda^{\mathbf{v}}$. That is, whenever X, Y are tangent vectors to Ω_λ, we have

$$g(X, Y) = g_0(\Phi_* X, \Phi_* Y). \tag{3.16}$$

We now compute the metric g in a neighbourhood of the boundary $\partial\Omega_\lambda$. Since near $\partial\Omega_\lambda$ η_1 and η_2 are locally constant function, from (3.14) we see

$$\Phi_* \partial_r = (1 + \eta_j v_j)\partial_r, \quad \Phi_* \nabla_\theta = r\eta_j \nabla_\theta v_j + (1 + \eta_j v_j) r \nabla_\theta \theta.$$

Now, in polar coordinates the euclidean metric $g_0 = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$, where $g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$. Then, by the relation (3.16), we see the metric g equals

$$g = (1 + \eta_j v_j)^2 dr^2 + 2r\eta_j \,(1 + \eta_j v_j) dr dv_j + r^2 \eta_j^2 dv_j^2 + r^2 (1 + \eta_j v_j)^2 g_{\mathbb{S}^{n-1}}, \tag{3.17}$$

in a neighbourhood of $\partial\Omega_\lambda$. Since Φ is an isometry between (Ω_λ, g) and $(\Omega_\lambda^{\mathbf{v}}, g_0)$, $u_\lambda(\mathbf{v})$ is the solution of the Dirichlet problem (3.7) in $\Omega_\lambda^{\mathbf{v}}$ if and only if $u_\lambda^*(\mathbf{v}) := \Phi^* u_\lambda(\mathbf{v})$ is the solution of

$$\begin{cases} -\Delta_g u = 1 & \text{in} \quad \Omega_\lambda, \\ \qquad u = 0 & \text{on} \quad \Gamma_1, \\ \qquad u = a_\lambda & \text{on} \quad \Gamma_\lambda, \end{cases} \tag{3.18}$$

where $-\Delta_g$ is the Laplace-Beltrami operator in (Ω_λ, g). Note that from formula (3.17) follows that the metric g is a smooth function of $\mathbf{v}$, and when expressed in this coordinates, the operator $-\Delta_g$ has the form

$$\Delta_g = \frac{1}{\sqrt{\det g}} \partial_j \left(\sqrt{\det g} g^{ij} \partial_j \right),$$

where g^{ij} are the coefficients of the inverse metric g^{-1}. Thus, by the Schauder regularity theory [GT15], $u_\lambda^*(\mathbf{v}) \in C^{2,\alpha}(\overline{\Omega_\lambda})$ and it depends smoothly on $\mathbf{v}$.

Let ν^* denote the *inner* unit normal field to $\partial\Omega_\lambda$ with respect to the metric g. We have $\Phi_* \nu^* = \nu$, and to find the expression for $\partial_\nu u_\lambda(\mathbf{v})$ in these coordinates, we need to compute $\partial_{\nu^*} u_\lambda^*(\mathbf{v}) = g(\nabla_g u_\lambda^*(\mathbf{v}), \nu^*)$. In the new coordinates, the operator $F_\lambda : U \to (C^{2,\alpha}(\mathbb{S}^{n-1})^2$ thus becomes

$$F_\lambda(\mathbf{v}) := \frac{1}{c_\lambda} \left(\frac{\partial u_\lambda^*(\mathbf{v})}{\partial \nu^*} \bigg|_{\Gamma_\lambda} - c_\lambda, \, \frac{\partial u_\lambda^*(\mathbf{v})}{\partial \nu^*} \bigg|_{\Gamma_1} - c_\lambda \right). \tag{3.19}$$

Then $F_\lambda(\mathbf{0}) = \mathbf{0}$ for all $\lambda \in (0,1)$, since $u_\lambda^*(\mathbf{0}) = u_\lambda$, and $F_\lambda(\mathbf{v}) = \mathbf{0}$ if and only if $\partial_{\nu^*} u_\lambda^*(\mathbf{v}) = c_\lambda$ is constant on $\partial\Omega_\lambda$. The latter implies that $u_\lambda(\mathbf{v})$ solves the overdetermined problem (3.1) in $\Omega_\lambda^{\mathbf{v}}$.

In the following proposition we will compute the linearization at $\mathbf{v} = \mathbf{0}$ of the operator $F_\lambda(\mathbf{v})$, reformulated as in (3.19). Recall that we use the identification (3.11) of functions on $\partial\Omega_\lambda$ with a pair of functions on $\mathbb{S}^{n-1}$.

Proposition 3.6. *The smooth operator F_λ, defined in (3.19), has a linearization at $\mathbf{v} = \mathbf{0}$,*

$$L_\lambda := dF_\lambda|_{\mathbf{v}=\mathbf{0}} : (C^{2,\alpha}(\mathbb{S}^{n-1}))^2 \to (C^{1,\alpha}(\mathbb{S}^{n-1}))^2,$$

given by

$$L_\lambda(w_1, w_2) = \left(- \left.\frac{\partial \phi_{\mathbf{w}}}{\partial \nu}\right|_{\Gamma_\lambda} + \frac{w_1}{c_\lambda} \left.\frac{\partial^2 u_\lambda}{\partial r^2}\right|_{\Gamma_\lambda}, - \left.\frac{\partial \phi_{\mathbf{w}}}{\partial \nu}\right|_{\Gamma_1} + \frac{w_2}{c_\lambda} \left.\frac{\partial^2 u_\lambda}{\partial r^2}\right|_{\Gamma_1} \right), \tag{3.20}$$

where for $\mathbf{w} = (w_1, w_2) \in (C^{2,\alpha}(\mathbb{S}^{n-1}))^2$, $\phi_{\mathbf{w}}$ denotes the harmonic function $\phi \in C^{2,\alpha}(\overline{\Omega}_\lambda)$ with boundary values $\phi|_{\Gamma_\lambda}(x) = w_1(x/|x|)$ and $\phi|_{\Gamma_1}(x) = w_2(x/|x|)$.

Proof. As F_λ is a smooth operator, its linearization at $\mathbf{0}$ is given by the directional derivative

$$L_\lambda(\mathbf{w}) = \lim_{t\to 0} \frac{F_\lambda(t\mathbf{w})}{t}.$$

Write $\mathbf{v} = (v_1, v_2) = t(w_1, w_2)$ for small t and consider the diffeomorphism $\Phi = \Phi_t$ defined in (3.14) and the induced metric $g = g_t$ on Ω_λ. Note that when $t = 0$, the diffeomorphism Φ_0 is the identity in Ω_λ and g_0 is the euclidean metric. Let $u_\lambda^*(\mathbf{v}) = u_t$ be the solution of the Dirichlet problem (3.18) in Ω_λ, which smoothly depends on the parameter t. Since u_λ is a radial function and can be extended by (3.5) to solve $-\Delta_{g_0} u_\lambda = 1$ in the whole of $\mathbb{R}^n \setminus \{0\}$, we have that $u_\lambda^* := \Phi_t^* u_\lambda$ is well defined and solves

$$-\Delta_{g_t} u = 1 \quad \text{in} \quad \Omega_\lambda.$$

Expanding $u_\lambda^* = u_\lambda^*(r,\theta)$ in a neighbourhood of $\partial\Omega_\lambda$ to first order in t, we obtain

$$u_\lambda^*(r,\theta) = u_\lambda(r + tr\eta_j(r)w_j(\theta)) = u_\lambda(r) + tr\eta_j(r)w_j(\theta)\frac{\partial u_\lambda}{\partial r} + O(t^2),$$

where η_j, $j = 1, 2$, are the functions defined in (3.15).

Let $\psi_t := u_t - u_\lambda^*$. Then $\psi_t \in C^{2,\alpha}(\overline{\Omega_\lambda})$ is a solution of

$$\begin{cases} \Delta_{g_t} \psi_t = 0 & \text{in} \quad \Omega_\lambda, \\ \quad \psi_t = -u_\lambda^* & \text{on} \quad \Gamma_1, \\ \quad \psi_t = a_\lambda - u_\lambda^* & \text{on} \quad \Gamma_\lambda, \end{cases} \tag{3.21}$$

which depends smoothly on t, with $\psi_0 = 0$. Setting $\dot\psi := \frac{d}{dt}\psi_t\big|_{t=0}$, we can differentiate (3.21) at $t = 0$ to obtain

$$\left.\frac{d}{dt}(\Delta_{g_t}\psi_t)\right|_{t=0} = \frac{d}{dt}\Delta_{g_t}\Big|_{t=0} \psi_0 + \Delta_{g_0}\dot\psi = \Delta\dot\psi,$$

and therefore $\dot\psi$ is a solution to

$$\begin{cases} \Delta\dot\psi = 0 & \text{in} \quad \Omega_\lambda, \\ \quad \dot\psi = (\partial_r u_\lambda) w_2 = -c_\lambda w_2 & \text{on} \quad \Gamma_1, \\ \quad \dot\psi = -(\partial_r u_\lambda) w_1 = -c_\lambda w_1 & \text{on} \quad \Gamma_\lambda, \end{cases}$$

so that

$$\dot{\psi} = -c_\lambda \phi_{\mathbf{w}}. \tag{3.22}$$

Now, given that $\psi_t = t\dot{\psi} + O(t^2)$, we have in a neighbourhood of $\partial\Omega_\lambda$

$$u_t = u_\lambda + t\left(\dot{\psi} + r\eta_j w_j \frac{\partial u_\lambda}{\partial r}\right) + O(t^2). \tag{3.23}$$

Recall that $\nu^* = \nu_t$ denotes the *inner* unit normal field to $\partial\Omega_\lambda$ with respect to the metric g_t. Below we will compute $\partial_{\nu_t} u_t$ to first order in t. As u_t is constant on each boundary component, it follows that

$$\partial_{\nu_t} u_t = |\nabla_{g_t} u_t| = \sqrt{g_t^{rr}}|\partial_r u_t| \quad \text{on} \quad \partial\Omega_\lambda. \tag{3.24}$$

Using formula (3.17) for the metric g_t to calculate its inverse g_t^{-1} to first order in t, we see that the component $g_t^{rr} = 1 - 2t\eta_j w_j + O(t^2)$, so that near $\partial\Omega_\lambda$,

$$\sqrt{g_t^{rr}} = 1 - t\eta_j w_j + O(t^2). \tag{3.25}$$

Taking into account that η_j is constant in a neighbourhood of Γ_1 and Γ_λ, we differentiate (3.23) with respect to r to find that near $\partial\Omega_\lambda$,

$$\frac{\partial u_t}{\partial r} = \frac{\partial u_\lambda}{\partial r} + t\left(\frac{\partial \dot{\psi}}{\partial r} + \eta_j w_j \frac{\partial u_\lambda}{\partial r} + r\eta_j w_j \frac{\partial^2 u_\lambda}{\partial r^2}\right) + O(t^2). \tag{3.26}$$

From (3.24)-(3.26) and the fact that $|z + tA| = |z| + t\,\mathrm{sgn}(z)A$ for $|tA| < |z|$, we compute

$$\begin{aligned} \frac{\partial u_t}{\partial \nu_t} = \sqrt{g_t^{rr}}\left|\frac{\partial u_t}{\partial r}\right| &= \left|\frac{\partial u_\lambda}{\partial r} + t\left(\frac{\partial \dot{\psi}}{\partial r} + r\eta_j w_j \frac{\partial^2 u_\lambda}{\partial r^2}\right) + O(t^2)\right| \\ &= \left|\frac{\partial u_\lambda}{\partial r}\right| + t\,\mathrm{sgn}(\partial_r u_\lambda)\left(\frac{\partial \dot{\psi}}{\partial r} + r\eta_j w_j \frac{\partial^2 u_\lambda}{\partial r^2}\right) + O(t^2) \quad \text{on} \quad \partial\Omega_\lambda. \end{aligned} \tag{3.27}$$

Now, as $|\partial_r u_\lambda| = \partial_\nu u_\lambda$, $\mathrm{sgn}(\partial_r u_\lambda)\partial_r = \partial_\nu$ and $\mathrm{sgn}(\partial_r u_\lambda) r\eta_j = 1$ on $\partial\Omega_\lambda$, (3.27) yields

$$\frac{\partial u_t}{\partial \nu_t} = \frac{\partial u_\lambda}{\partial \nu} + t\left(\frac{\partial \dot{\psi}}{\partial \nu} + w_j \frac{\partial^2 u_\lambda}{\partial r^2}\right) + O(t^2) \quad \text{on} \quad \partial\Omega_\lambda. \tag{3.28}$$

The formula (3.20) for L_λ hence follows from (3.28) and (3.22). □

3.3 Spectrum of the linearized operator

In this section we give an account of the spectral properties of the linearized operator L_λ, which we derived in Proposition 3.6.

Recall that a function $\mathcal{Y} \in C^\infty(\mathbb{S}^{n-1})$ is a spherical harmonic of degree $k \in \mathbb{N}_0$ if it is an eigenfunction of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$, that is,

$$\Delta_{\mathbb{S}^{n-1}}\mathcal{Y} + \sigma_k \mathcal{Y} = 0,$$

where $\sigma_k := k(k+n-2)$ is the corresponding eigenvalue. We first observe that the subspace W generated by $\{(\mathcal{Y},0),(0,\mathcal{Y})\}$ is invariant under L_λ and we shall derive a matrix representation of $L_\lambda|_W$ with respect to a certain convenient basis.

Lemma 3.7. *Let $\mathcal{Y} \in C^\infty(\mathbb{S}^{n-1})$ be a spherical harmonic of degree $k \in \mathbb{N}_0$ and unit $L^2(\mathbb{S}^{n-1})$ norm, and let W be the subspace of $(C^\infty(\mathbb{S}^{n-1}))^2$ spanned by $\{(\mathcal{Y},0),(0,\mathcal{Y})\}$. Then W is invariant under the action of L_λ. Moreover, if*

$$\mathbf{e}_1 := (\lambda^{\frac{1-n}{2}}\mathcal{Y},0), \quad \mathbf{e}_2 := (0,\mathcal{Y}), \tag{3.29}$$

then $\mathcal{B} = \{\mathbf{e}_1,\mathbf{e}_2\}$ is an orthonormal basis for W with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$ on $L^2(\mathbb{S}^{n-1}) \times L^2(\mathbb{S}^{n-1})$, defined in (3.10), and the matrix representing the restriction $L_\lambda|_W$ with respect to the basis $\mathcal{B}$ is given by

$$M_{\lambda,k} = \begin{pmatrix} \frac{1}{\lambda}\frac{(k+n-2)\lambda^{2-n-k}+k\lambda^k}{\lambda^{2-n-k}-\lambda^k} - \frac{n-1}{\lambda} & \lambda^{\frac{1-n}{2}}\frac{2-n-2k}{\lambda^{2-n-k}-\lambda^k} \\ \lambda^{\frac{1-n}{2}}\frac{2-n-2k}{\lambda^{2-n-k}-\lambda^k} & \frac{k\lambda^{2-n-k}+(k+n-2)\lambda^k}{\lambda^{2-n-k}-\lambda^k} + (n-1) \end{pmatrix} - \frac{1}{c_\lambda}\,\mathrm{id}, \tag{3.30}$$

for $k \geq 1$, while for $k = 0$,

$$M_{\lambda,0} = \begin{pmatrix} \frac{1}{\lambda}\frac{(n-2)\lambda^{2-n}}{\lambda^{2-n}-1} - \frac{n-1}{\lambda} & \lambda^{\frac{1-n}{2}}\frac{2-n}{\lambda^{2-n}-1} \\ \lambda^{\frac{1-n}{2}}\frac{2-n}{\lambda^{2-n}-1} & \frac{n-2}{\lambda^{2-n}-1} + (n-1) \end{pmatrix} - \frac{1}{c_\lambda}\,\mathrm{id} \quad \textit{when} \quad n \geq 3, \tag{3.31}$$

$$M_{\lambda,0} = \begin{pmatrix} -\frac{1}{\lambda}\frac{1}{\log\lambda} - \frac{1}{\lambda} & \lambda^{-\frac{1}{2}}\frac{1}{\log\lambda} \\ \lambda^{-\frac{1}{2}}\frac{1}{\log\lambda} & -\frac{1}{\log\lambda} + 1 \end{pmatrix} - \frac{1}{c_\lambda}\,\mathrm{id} \quad \textit{when} \quad n = 2. \tag{3.32}$$

Proof. It is easy to verify that $\mathcal{B}$ is an orthonormal basis for W with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$. Let $\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2$. Recall that in polar coordinates the Laplacian can be expressed as

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}.$$

Then, for $n \geq 3$ and $k \in \mathbb{N}_0$, as well as for $n = 2$, $k \in \mathbb{N}$, the function $\phi_{\mathbf{w}} \in C^\infty(\overline{\Omega_\lambda})$, defined in polar coordinates $r \in [\lambda,1]$, $\theta \in \mathbb{S}^{n-1}$ by

$$\phi_{\mathbf{w}}(r,\theta) = (aA(r) + bB(r))\mathcal{Y}(\theta),$$

where

$$A(r) = \lambda^{\frac{1-n}{2}}\frac{r^{2-n-k} - r^k}{\lambda^{2-n-k} - \lambda^k}, \qquad B(r) = \frac{\lambda^{2-n-k}r^k - \lambda^k r^{2-n-k}}{\lambda^{2-n-k} - \lambda^k}, \tag{3.33}$$

can be checked to be harmonic in Ω_λ, and satisfies

$$\phi_{\mathbf{w}}(\lambda,\theta) = a\lambda^{\frac{1-n}{2}}\mathcal{Y}(\theta), \quad \phi_{\mathbf{w}}(1,\theta) = b\mathcal{Y}(\theta).$$

Formula (3.20) for L_λ hence gives

$$\begin{aligned} L_\lambda(a\mathbf{e}_1 + b\mathbf{e}_2) &= \begin{pmatrix} -(aA'(\lambda) + bB'(\lambda))\mathcal{Y} + (\partial_r^2 u_\lambda(\lambda)/c_\lambda)a\lambda^{\frac{1-n}{2}}\mathcal{Y} \\ (aA'(1) + bB'(1))\mathcal{Y} + (\partial_r^2 u_\lambda(1)/c_\lambda)b\mathcal{Y} \end{pmatrix} \\ &= \begin{pmatrix} -(aA'(\lambda) + bB'(\lambda))\mathcal{Y} - \frac{n-1}{\lambda}a\lambda^{\frac{1-n}{2}}\mathcal{Y} \\ (aA'(1) + bB'(1))\mathcal{Y} + (n-1)b\mathcal{Y} \end{pmatrix} - \frac{1}{c_\lambda}(a\mathbf{e}_1 + b\mathbf{e}_2), \end{aligned} \tag{3.34}$$

where in the last equality we used the fact that $\partial_r^2 u_\lambda = -1 - \frac{n-1}{r}\partial_r u_\lambda$. Therefore, W is an invariant subspace of L_λ. Plugging in (3.33) in (3.34), we easily derive formulas (3.30) and (3.31) for the matrix representation $M_{\lambda,k}$ of $L_\lambda|_W$ with respect to the basis $\mathcal{B}$.

When $n = 2$, and $k = 0$, the substitute for (3.33) is

$$A(r) = \lambda^{-1/2}\frac{\log r}{\log \lambda}, \qquad B(r) = -\frac{\log r - \log \lambda}{\log \lambda} \tag{3.35}$$

and we easily check again the invariance of W under L_λ and derive (3.32). □

Remark 3.1. *Note that the matrix $M_{\lambda,k}$ in Lemma 3.7 is symmetric. This is not surprising taking into account the fact that the operator L_λ is formally self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$ over the space $V := (C^{2,\alpha}(\mathbb{S}^{n-1}))^2$. Indeed, for $\mathbf{w}_1, \mathbf{w}_2 \in V$, identified as functions on $\partial\Omega_\lambda$ under (3.11), let $\phi_{\mathbf{w}_1}, \phi_{\mathbf{w}_2}$ denote their corresponding harmonic extensions to $\overline{\Omega_\lambda}$. Using formula (3.20) for L_λ and the definition (3.10) of the inner product $\langle\cdot,\cdot\rangle_\lambda$, an application of Green's formula and harmonicity yield*

$$\begin{aligned}\langle L_\lambda \mathbf{w}_1, \mathbf{w}_2\rangle_\lambda &= \int_{\partial\Omega_\lambda}\left(-\frac{\partial\phi_{\mathbf{w}_1}}{\partial\nu} + \frac{\partial^2 u_\lambda}{\partial r^2}\frac{\phi_{\mathbf{w}_1}}{c_\lambda}\right)\phi_{\mathbf{w}_2}\, dS \\ &= \int_{\partial\Omega_\lambda}\phi_{\mathbf{w}_1}\left(-\frac{\partial\phi_{\mathbf{w}_2}}{\partial\nu} + \frac{\partial^2 u_\lambda}{\partial r^2}\frac{\phi_{\mathbf{w}_2}}{c_\lambda}\right) dS = \langle \mathbf{w}_1, L_\lambda \mathbf{w}_2\rangle_\lambda.\end{aligned}$$

Remark 3.2. *Note that the matrices $M_{\lambda,k}$, $\lambda \in (0,1)$, $k \in \mathbb{N}_0$, derived in Lemma 3.7, fit in a two-parameter family of symmetric matrices*

$$\mathcal{M} = \{M_{\lambda,k} : (\lambda,k) \in (0,1)\times[0,\infty)\}. \tag{3.36}$$

where we define $M_{\lambda,k}$ for non-integral $k \in [0,\infty)$ by the analytic formula in equation (3.30). In that way, the family $\mathcal{M}$ is analytic in both $(\lambda,k) \in (0,1)\times(0,\infty)$. Moreover, we can see that $\mathcal{M}$ is continuous up to $(\lambda,k) \in (0,1)\times\{0\}$ after easily checking

$$\lim_{k\downarrow 0} M_{\lambda,k} = M_{\lambda,0},$$

where $M_{\lambda,0}$ is given by (3.31) when $n \geq 3$ and by (3.32) when $n = 2$.

The symmetric matrices of $\mathcal{M}$ are never multiples of the identity (the off-diagonal entries are non-zero), whence every $M_{\lambda,k} \in \mathcal{M}$ has two distinct real eigenvalues

$$\mu_{k,1}(\lambda) < \mu_{k,2}(\lambda)$$

and each is a smooth function of $(\lambda,k) \in (0,1)\times(0,\infty)$, continuous up to $(\lambda,k) \in (0,1)\times\{0\}$. Furthermore, since any eigenvector of $M_{\lambda,k} \in \mathcal{M}$ has two non-zero entries, we can define $v_j(\lambda,k) \in \mathbb{R}^2$ to be the unique eigenvector of $M_{\lambda,k}$, associated with $\mu_{k,j}(\lambda)$, $j = 1,2$, that has unit Euclidean norm and positive first entry. Clearly, the eigenvector $v_j(\lambda,k)$, $j = 1,2$, depends smoothly in $(\lambda,k) \in (0,1)\times(0,\infty)$ and is continuous up to $(\lambda,k) \in (0,1)\times\{0\}$, as well.

Before we continue, it will be convenient to recast the matrices $M_{\lambda,k} \in \mathcal{M}$ in new notation that will greatly facilitate the computations when we analyze the behaviour of its eigenvalues $\mu_{k,j}(\lambda)$, $j = 1, 2$. For that purpose, first define the matrix

$$\tilde{M}_{\lambda,k} := M_{\lambda,k} + \frac{1}{c_\lambda}\,\mathrm{id}, \qquad M_{\lambda,k} \in \mathcal{M} \tag{3.37}$$

whose eigenspaces are the same as those of $M_{\lambda,k}$ and whose eigenvalues $\tilde{\mu}_{k,j}(\lambda)$ are shifts of $\mu_{k,j}(\lambda)$ by $1/c_\lambda$:

$$\mu_{k,j}(\lambda) = \tilde{\mu}_{k,j}(\lambda) - \frac{1}{c_\lambda}, \qquad j = 1, 2. \tag{3.38}$$

For a given $k \in (0, \infty)$ we shall denote

$$\alpha := \frac{n}{2} + k - 1, \quad \alpha \in (0,\infty) \quad \text{and} \quad e^\omega := \lambda^{-\alpha}, \quad \omega \in (0, \infty). \tag{3.39}$$

Lemma 3.8. *Let $k \in (0,\infty)$ be given and let α, ω be as in (3.39). The matrix $\tilde{M}_{\lambda,k}$ defined in (3.37) takes the form*

$$\tilde{M}_{\lambda,k} = \begin{pmatrix} \frac{1}{\lambda}(\alpha \coth\omega - \frac{n}{2}) & -\frac{\alpha}{\sqrt{\lambda}}\frac{1}{\sinh\omega} \\ -\frac{\alpha}{\sqrt{\lambda}}\frac{1}{\sinh\omega} & \alpha\coth\omega + \frac{n}{2} \end{pmatrix} \tag{3.40}$$

and its eigenvalues are given by

$$\tilde{\mu}_{k,j}(\lambda) = \frac{C \mp \sqrt{C^2 - 4\lambda D}}{2\lambda}, \qquad j = 1, 2, \tag{3.41}$$

where

$$C = \alpha(\lambda+1)\coth\omega + \frac{n}{2}(\lambda - 1), \qquad D = \alpha^2 - \frac{n^2}{4} = (n+k-1)(k-1). \tag{3.42}$$

Proof. These are straightforward computations. First note that

$$\begin{aligned}(k+n-2)\lambda^{1-n/2-k} + k\lambda^{n/2+k-1} &= (2\alpha - k)\lambda^{-\alpha} + k\lambda^{\alpha} \\ &= \left(\frac{n}{2} - 1\right)(\lambda^{-\alpha} - \lambda^{\alpha}) + \alpha(\lambda^{-\alpha} + \lambda^{\alpha})\end{aligned}$$

and, using the expresion in (3.30), we find the $(1,1)$-entry of $\tilde{M}_{\lambda,k}$ to be

$$\begin{aligned}\frac{1}{\lambda}\left(\frac{(k+n-2)\lambda^{1-n/2-k} + k\lambda^{n/2+k-1}}{\lambda^{1-n/2-k} - \lambda^{n/2+k-1}} - (n-1)\right) &= \frac{1}{\lambda}\left(\left(\frac{n}{2} - 1\right) + \alpha\frac{\lambda^{-\alpha} + \lambda^{\alpha}}{\lambda^{-\alpha} - \lambda^{\alpha}} - (n-1)\right) \\ &= \frac{1}{\lambda}\left(\alpha\coth\omega - \frac{n}{2}\right).\end{aligned}$$

In a similar fashion we compute the $(2,2)$-entry of $\tilde{M}_{\lambda,k}$. Also

$$\lambda^{\frac{1-n}{2}}\frac{2-n-2k}{\lambda^{2-n-k} - \lambda^k} = -\frac{2\alpha}{\sqrt{\lambda}}\frac{1}{\lambda^{-\alpha} - \lambda^{\alpha}} = \frac{-\alpha}{\sqrt{\lambda}}\frac{1}{\sinh\omega}.$$

This establishes equation (3.40). The characteristic equation for $\tilde{M}_{\lambda,k}$ then computes to

$$\lambda\tilde{\mu}^2 - \left\{\alpha(\lambda+1)\coth\omega + \frac{n}{2}(\lambda-1)\right\}\tilde{\mu} + \left\{\alpha^2 - \frac{n^2}{4}\right\} = 0,$$

from which we derive formulas (3.41)-(3.42) for its eigenvalues. ∎

Remark 3.3. *Note that when $k = 1$, we have $\alpha = n/2$ and $e^{\omega} = \lambda^{-n/2}$, so that C and D in (3.42) evaluate to*

$$C = \frac{n}{2}\left((\lambda+1)\frac{\lambda^{-n/2}+\lambda^{n/2}}{\lambda^{-n/2}-\lambda^{n/2}} + \lambda - 1\right) = n\frac{\lambda+\lambda^n}{1-\lambda^n} = \frac{\lambda}{c_\lambda}, \qquad D = 0.$$

Hence, (3.41) gives us $\tilde{\mu}_{1,1}(\lambda) = 0$ and $\tilde{\mu}_{1,2}(\lambda) = 1/c_\lambda$, and the eigenvalues of the matrix $M_{\lambda,1}$ are then given by

$$\mu_{1,1}(\lambda) = -\frac{1}{c_\lambda}, \qquad \mu_{1,2}(\lambda) = 0. \tag{3.43}$$

We observe that, over subspaces $W = Span\{(\mathcal{Y}, 0), (0, \mathcal{Y})\}$, where $\mathcal{Y}$ is a spherical harmonic of degree 1, the linearization $L_\lambda|_W$ has a kernel of dimension 1 for every $\lambda \in (0, 1)$. As spherical harmonics of degree 1 are the restriction to $\mathbb{S}^{n-1}$ of degree 1 homogeneous harmonic polynomials in $\mathbb{R}^n$, this kernel precisely corresponds to deformations of the standard annulus Ω_λ generated by translations.

In the following key sequence of lemmas we will examine the behaviour of the eigenvalues $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ in both $k \in \mathbb{N}_0$ and $\lambda \in (0, 1)$. See Figure 3.1 below for a plot of these branches for $k = 0, 1, 2, 3$, in dimension $n = 3$.

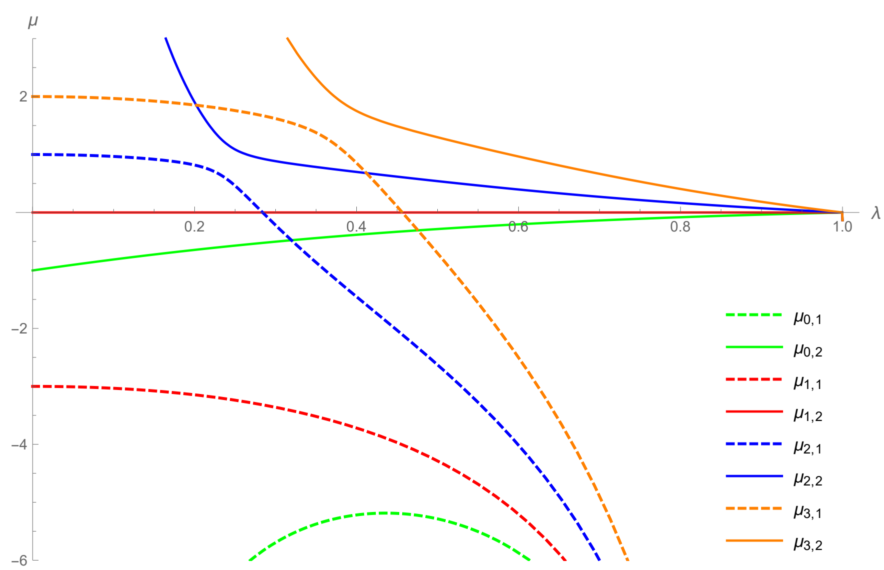

Figure 3.1: Mathematica plot of the eigenvalues $\mu_{k,j}$, $k = 0, 1, 2, 3$ and $j = 1, 2$, as a function of $\lambda \in (0, 1)$ for $n = 3$.

First, we will establish the first branch $\mu_{k,1}(\lambda)$ is strictly decreasing in $\lambda \in (0, 1)$ for any given $k \geq 2$. The proof of the next lemma is based on a delicate use of hyperbolic trigonometric identities.

Lemma 3.9. *For $k \in \mathbb{N}$, let $M_{\lambda,k}$ be the matrix defined in (3.30) and let $\mu_{k,1} : (0, 1) \to \mathbb{R}$ denote its first eigenvalue. For every $k \geq 2$ the following are satisfied:*

- $\lim_{\lambda \downarrow 0} \mu_{k,1}(\lambda) = k-1$, $\lim_{\lambda \uparrow 1} \mu_{k,1}(\lambda) = -\infty$;
- $\mu_{k,1}'(\lambda) < 0$, *and so* $\mu_{k,1}(\lambda)$ *is strictly decreasing in* λ.

Proof. Fix $k \in \mathbb{R}$, $k > 1$. We shall first prove that $\mu_{k,1}'(\lambda) < 0$, where $'$ denotes differentiating with respect to λ. As $\frac{1}{c_\lambda} = n\frac{1+\lambda^{n-1}}{1-\lambda^n}$ is strictly increasing in λ, by (3.38) it suffices to show that $\tilde{\mu}_{k,1}'(\lambda) \leq 0$. Since by (3.41)

$$\tilde{\mu}_{k,1}(\lambda) = \frac{2D}{C + \sqrt{C^2 - 4\lambda D}}$$

and given that $D = (n+k-1)(k-1)$ is positive and constant in λ, we only need to show that

$$\frac{\partial}{\partial \lambda}\left(C + \sqrt{C^2 - 4\lambda D}\right) = \frac{C'(\sqrt{C^2-4\lambda D} + C) - 2D}{\sqrt{C^2 - 4\lambda D}} > 0. \tag{3.44}$$

Using the identity $\coth^2\omega - 1/\sinh^2\omega = 1$ successively, we get

$$\begin{aligned}
C^2 - 4\lambda D &= \left\{\alpha(1+\lambda)\coth\omega - \frac{n}{2}(1-\lambda)\right\}^2 - 4\lambda\left\{\alpha^2 - \frac{n^2}{4}\right\} \\
&= \alpha^2(1+\lambda)^2\left(1 + \frac{1}{\sinh^2\omega}\right) - n\alpha(1+\lambda)(1-\lambda)\coth\omega + \frac{n^2}{4}(1-\lambda)^2 - 4\lambda\left(\alpha^2 - \frac{n^2}{4}\right) \\
&= \alpha^2(1-\lambda)^2 - n(1+\lambda)(1-\lambda)\coth\omega + \frac{n^2}{4}(1+\lambda)^2 + \frac{\alpha^2(1+\lambda)^2}{\sinh^2\omega} \\
&= \alpha^2(1-\lambda)^2\left(1 + \frac{1}{\sinh^2\omega}\right) - n(1+\lambda)(1-\lambda)\coth\omega + \frac{n^2}{4}(1+\lambda)^2 + \frac{4\lambda\alpha^2}{\sinh^2\omega} \\
&= \left\{\alpha(1-\lambda)\coth\omega - \frac{n}{2}(1+\lambda)\right\}^2 + \frac{4\lambda\alpha^2}{\sinh^2\omega},
\end{aligned}$$

which gives the estimate

$$\sqrt{C^2 - 4\lambda D} > \alpha(1-\lambda)\coth\omega - \frac{n}{2}(1+\lambda). \tag{3.45}$$

On the other hand, using the fact that $\omega' = -\alpha/\lambda$,

$$C' = \alpha\coth\omega + \frac{1+\lambda}{\lambda}\frac{\alpha^2}{\sinh^2\omega} + \frac{n}{2} > \alpha\coth\omega + \frac{\alpha^2}{\sinh^2\omega} + \frac{n}{2}, \tag{3.46}$$

so that (3.45) and (3.46) yield

$$\begin{aligned}
C'(\sqrt{C^2-4\lambda D} + C) - 2D &> \left(\alpha\coth\omega + \frac{\alpha^2}{\sinh^2\omega} + \frac{n}{2}\right)(2\alpha\coth\omega - n) - \left(2\alpha^2 - \frac{n^2}{2}\right) \\
&= 2\alpha^2\left(1 + \frac{1}{\sinh^2\omega}\right) + \frac{2\alpha^3\coth\omega}{\sinh^2\omega} - \frac{n\alpha^2}{\sinh^2\omega} - 2\alpha^2 \\
&= \frac{2\alpha^2}{\sinh^2\omega}\left(1 + \alpha\coth\omega - \frac{n}{2}\right) > \frac{2\alpha^2}{\sinh^2\omega}\left(1 + \alpha - \frac{n}{2}\right) = \frac{2\alpha^2 k}{\sinh^2\omega} > 0,
\end{aligned}$$

where in the penultimate inequality we used $\coth\omega > 1$. This confirms (3.44) and completes the proof of the strict monotonicity of $\mu_{k,1}(\lambda)$ in λ.

To derive the limiting behaviour of $\mu_{k,1}(\lambda)$ as $\lambda \downarrow 0$, we first note that from the definition (3.39), we have $\lim_{\lambda\downarrow 0}\omega = \infty$, so that (3.42) gives $\lim_{\lambda\downarrow 0} C = \alpha - n/2 = k-1$, and since $\lim_{\lambda\downarrow 0} 1/c_\lambda = n$, we calculate

$$\lim_{\lambda\downarrow 0}\mu_{k,1}(\lambda) = \lim_{\lambda\downarrow 0}\left(\frac{2D}{C + \sqrt{C^2 - 4\lambda D}} - \frac{1}{c_\lambda}\right) = \frac{D}{k-1} - n = k-1.$$

As to the limiting behaviour of $\mu_{k,1}(\lambda)$ as $\lambda \uparrow 1$, the fact that $\tilde{\mu}_{k,1}(\lambda)$ is decreasing in λ and $\lim_{\lambda \uparrow 1} 1/c_\lambda = \infty$ yields

$$\lim_{\lambda \uparrow 1} \mu_{k,1}(\lambda) = -\infty.$$

□

Next, we will prove that, for fixed λ, both $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ increase with k. As the explicit formulas (3.41)-(3.42) for the eigenvalues turn out to be unyielding, we accomplish this instead by treating k as a continuous variable and showing that $\partial_k M_{\lambda,k}$ is positive definite.

Lemma 3.10. *For fixed $\lambda \in (0,1)$ and $j = 1,2$ the sequence $\{\mu_{k,j}(\lambda)\}_{k=0}^{\infty}$ is strictly increasing.*

Proof. We shall treat $k \in [0,\infty)$ as a continuous variable, following the discussion in Remark 3.2. First, we restrict ourselves to $k > 0$ and fix λ. Recall that in the remark we defined $v = v_j(\lambda,k) \in \mathbb{R}^2$ to be the unique eigenvector of $M_{\lambda,k}$ associated with eigenvalue $\mu_{k,j}(\lambda)$, $j = 1,2$, which has unit Euclidean norm and positive first entry. Then $\partial_k v \in \mathbb{R}^2$ is orthogonal to v and since $M_{\lambda,k}$ is symmetric, we have

$$\langle M_{\lambda,k}(\partial_k v), v\rangle = \langle \partial_k v, M_{\lambda,k} v\rangle = \mu_{k,j}\langle \partial_k v, v\rangle = 0. \tag{3.47}$$

Differentiating the identity $\mu_{k,j} = \langle M_{\lambda,k} v, v\rangle$ with respect to k and using (3.47), we obtain

$$\partial_k \mu_{k,j} = \langle (\partial_k M_{\lambda,k}) v, v\rangle + \langle M_{\lambda,k}(\partial_k v), v\rangle + \langle M_{\lambda,k} v, \partial_k v\rangle = \langle \partial_k M_{\lambda,k} v, v\rangle.$$

Therefore, we will have the desired $\partial_k \mu_{k,j} > 0$ once we show that the symmetric matrix $\partial_k M_{\lambda,k}$ is positive definite. Using $\partial\alpha/\partial k = 1$ and $\partial\omega/\partial k = \omega/\alpha$, we compute from (3.40)

$$\partial_k M_{\lambda,k} = \partial_k \tilde{M}_{\lambda,k} = \begin{pmatrix} \frac{1}{\lambda}\left(\coth\omega - \frac{\omega}{\sinh^2\omega}\right) & \frac{1}{\sqrt{\lambda}}\left(\frac{\omega\cosh\omega}{\sinh^2\omega} - \frac{1}{\sinh\omega}\right) \\ \frac{1}{\sqrt{\lambda}}\left(\frac{\omega\cosh\omega}{\sinh^2\omega} - \frac{1}{\sinh\omega}\right) & \coth\omega - \frac{\omega}{\sinh^2\omega}. \end{pmatrix}.$$

We see that its determinant

$$\begin{aligned} \det(\partial_k M_{\lambda,k}) &= \frac{1}{\lambda}\left\{\left(\coth\omega - \frac{\omega}{\sinh^2\omega}\right)^2 - \left(\frac{\omega\cosh\omega}{\sinh^2\omega} - \frac{1}{\sinh\omega}\right)^2\right\} \\ &= \frac{1}{\lambda\sinh^4\omega}(\sinh^2\omega - \omega^2)(\cosh^2\omega - 1) > 0, \end{aligned}$$

as $\sinh\omega > \omega$ and $\cosh\omega > 1$ for $\omega \in (0,\infty)$. Furthermore, the $(2,2)$-entry of $\partial_k M_{\lambda,k}$ satisfies

$$\coth\omega - \frac{\omega}{\sinh^2\omega} = \frac{\cosh\omega\sinh\omega - \omega}{\sinh^2\omega} > \frac{\sinh\omega - \omega}{\sinh^2\omega} > 0.$$

Therefore, by Sylvester's criterion the matrix $\partial_k M_{\lambda,k}$ is positive definite for $k > 0$. Since according to Remark 3.2, $\mu_{k,j}(\lambda)$ is continuous in $k \in [0,\infty)$ for fixed λ, we can conclude

$$\mu_{k+1,j}(\lambda) > \mu_{k,j}(\lambda) \quad \text{for all} \quad k \in \mathbb{N}_0, \quad j = 1,2.$$

□

In the final lemma of this section we derive the asymptotics of $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ as $k \to \infty$.

Lemma 3.11. *For fixed $\lambda \in (0,1)$ the sequences $\{\mu_{k,j}(\lambda)\}_{k=0}^{\infty}$, $j = 1, 2$, have the asymptotics*

$$\lim_{k\to\infty} \frac{\mu_{k,1}(\lambda)}{k} = 1, \qquad \lim_{k\to\infty} \frac{\mu_{k,2}(\lambda)}{k} = \frac{1}{\lambda}.$$

Proof. From the definition of C and D in (3.42) and the fact that $\lim_{k\to\infty} \coth \omega = 1$, we calculate

$$\lim_{k\to\infty} \frac{C}{k} = 1 + \lambda, \qquad \lim_{k\to\infty} \frac{D}{k^2} = 1.$$

Hence, using equations (3.38) and (3.41), we obtain

$$\begin{aligned}
\lim_{k\to\infty} \frac{\mu_{k,j}(\lambda)}{k} &= \lim_{k\to\infty} \left(\frac{\tilde{\mu}_{k,j}(\lambda)}{k} - \frac{1}{c_\lambda k} \right) = \lim_{k\to\infty} \frac{1}{2\lambda} \left(\frac{C}{k} \mp \sqrt{\frac{C^2}{k^2} - \frac{4\lambda D}{k^2}} \right) \\
&= \frac{(1+\lambda) \mp \sqrt{(1+\lambda)^2 - 4\lambda}}{2\lambda} = \begin{cases} 1 & j = 1 \\ 1/\lambda & j = 2 \end{cases}.
\end{aligned}$$

□

As a corollary to the lemmas above, we state the following proposition.

Proposition 3.12. *Let $k \in \mathbb{N}_0$ and let $\mu_{k,1}(\lambda)$ and $\mu_{k,2}(\lambda)$ be the eigenvalues of the matrix $M_{\lambda,k}$ defined in Lemma 3.7. The following statements are satisfied:*

- *both eigenvalues for $k = 0$ are negative*

$$\mu_{0,1}(\lambda) < \mu_{0,2}(\lambda) < 0 \quad \textit{for all} \quad \lambda \in (0,1); \tag{3.48}$$

- *for $k = 1$, the eigenvalues are equal to*

$$\mu_{1,1}(\lambda) = -1/c_\lambda, \qquad \mu_{1,2}(\lambda) = 0; \tag{3.49}$$

- *for every $k \geq 2$, the second eigenvalue*

$$\mu_{k,2}(\lambda) > 0 \quad \textit{for all} \quad \lambda \in (0,1); \tag{3.50}$$

- *for every $k \geq 2$, there exists a unique value $\lambda_k^* \in (0,1)$ such that the first eigenvalue*

$$\mu_{k,1}(\lambda_k^*) = 0. \tag{3.51}$$

  Moreover, the sequence $\{\lambda_k^*\}_{k=2}^{\infty}$ is strictly increasing with $\lim_{k\to\infty} \lambda_k^* = 1$.

Proof. In (3.43) we calculated that $\mu_{1,2} \equiv 0$, hence by Lemma 3.10 we have that for $k \geq 2$

$$\mu_{k,2} > \mu_{1,2} \equiv 0 > \mu_{0,2} > \mu_{0,1}.$$

so that we show both (3.48) and (3.50). Equation (3.49) is (3.43) reproduced here for the sake of completeness. Only the last bullet point remains to be established.

According to Lemma 3.9, for $k \geq 2$ the first branch $\mu_{k,1}(\lambda)$ is strictly decreasing in λ and

$$\lim_{\lambda \downarrow 0} \mu_{k,1}(\lambda) = k - 1 > 0 \quad \text{while} \quad \lim_{\lambda \uparrow 1} \mu_{k,1}(\lambda) = -\infty.$$

Thus, when $k \geq 2$, $\mu_{k,1}(\lambda)$ has a unique zero $\lambda = \lambda_k^*$ in $(0,1)$, where it changes sign from positive to negative. Since the k-monotonicity Lemma 3.10 implies that

$$\mu_{k+1,1}(\lambda_k^*) > \mu_{k,1}(\lambda_k^*) = 0$$

we must have $\lambda_{k+1}^* > \lambda_k^*$, so that the sequence of zeros $\{\lambda_k^*\}_{k=2}^{\infty}$ is strictly increasing. Denote its limit by $l = \lim_{k\to\infty} \lambda_k^*$. Obviously, $\lambda_k^* \leq l \leq 1$ for all $k \geq 2$. If it were the case that $l < 1$, we would have by the asymptotic behaviour of $\mu_{k,1}(\lambda)$, established in Lemma 3.11, that for any large enough k, $\mu_{k,1}(l)/k > \frac{1}{2}$. But then the zero λ_k^* of $\mu_{k,1}(\lambda)$ would have to be greater than l, which is a contradiction. Hence, $l = 1$. $\square$

3.4 The proof of Theorem 3.5

We now turn to the proof of Theorem 3.5. Following the discussion given in Section 3.1, it will be necessary to specialize to functions that are invariant under the action of a subgroup G of the orthogonal group $O(n)$ satisfying (P1)-(P2), stated in Section 3.1. Recall that $C_G^{k,\alpha}(\mathbb{S}^{n-1})$ denotes the Hölder space of G-invariant functions.

We begin by observing that the operator F_λ defined in (3.19) restricts to the G-invariant function spaces $(C_G^{k,\alpha}(\mathbb{S}^{n-1}))^2$ and, therefore, so does its linearization L_λ.

Lemma 3.13. *The nonlinear operator F_λ defined in (3.19) and its linearization $L_\lambda = dF_\lambda|_{\mathbf{v}=\mathbf{0}}$ have well defined restrictions*

$$\begin{array}{rccc} F_\lambda: & U & \to & \left(C_G^{1,\alpha}(\mathbb{S}^{n-1})\right)^2, \\ L_\lambda: & \left(C_G^{2,\alpha}(\mathbb{S}^{n-1})\right)^2 & \to & \left(C_G^{1,\alpha}(\mathbb{S}^{n-1})\right)^2, \end{array}$$

where $U \subset (C_G^{2,\alpha}(\mathbb{S}^{n-1}))^2$ is a sufficiently small neighbourhood of $\mathbf{0}$.

Proof. We just have to explain why $F_\lambda(\mathbf{v}) \in (C_G^{1,\alpha}(\mathbb{S}^{n-1}))^2$ if $\mathbf{v} \in U \subset (C_G^{2,\alpha}(\mathbb{S}^{n-1}))^2$. Clearly, if $\mathbf{v}$ is G-invariant, then so is the pull-back metric $g = g(\mathbf{v}) = \Phi^* g_0$ on Ω_λ, where Φ is the diffeomorphism defined in (3.14). Hence, by unique solvability, the solution $u_\lambda^*(\mathbf{v}) \in C^{2,\alpha}(\overline{\Omega_\lambda})$ of the Dirichlet problem (3.18) is also G-invariant, and we confirm that $F_\lambda(\mathbf{v})$ belongs to $(C_G^{1,\alpha}(\mathbb{S}^{n-1}))^2$, indeed. $\square$

Recall that properties (P1)-(P2) of G say that the G-invariant spherical harmonics are only the ones of degree $\{i_k\}_{k\in\mathbb{N}_0}$, with $i_0 = 0$ and $i_1 \geq 2$, and for each $k \in \mathbb{N}_0$, they form a one-dimensional subspace – spanned by the unique G-invariant spherical harmonic Y_k of degree i_k and unit $L^2(\mathbb{S}^{n-1})$ norm. For each $k \in \mathbb{N}_0$, let $W_k = \mathrm{Span}\{(Y_k, 0), (0, Y_k)\}$, let $\mathcal{B}_k = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis for W_k, defined in (3.29), and let M_{λ,i_k} be the matrix of $L_\lambda|_{W_k}$ with respect to $\mathcal{B}_k$. Also, recall that in Remark 3.2 we chose the eigenvector

$$v_j(\lambda, i_k) = (a_{k,j}, b_{k,j}), \quad k \in \mathbb{N}_0,\ j = 1,2, \quad \text{where} \quad a_{k,j} > 0 \quad \text{and} \quad a_{k,j}^2 + b_{k,j}^2 = 1,$$

to span the eigenspace of M_{λ,i_k}, associated with $\mu_{i_k,j}(\lambda)$. The corresponding eigenvector of L_λ is

$$\mathbf{z}_{k,j} := a_{k,j}\mathbf{e}_1 + b_{k,j}\mathbf{e}_2, \quad k \in \mathbb{N}_0,\ j = 1,2 \quad \text{and its norm} \quad \|\mathbf{z}_{k,j}\|_\lambda = 1. \tag{3.52}$$

Remark 3.4. *The sequence of eigenvectors $\{\mathbf{z}_{k,j}(\lambda)\}_{k\in\mathbb{N}_0, j=1,2}$ of L_λ forms an orthonormal basis for the Hilbert space $L^2_G(\mathbb{S}^{n-1}) \times L^2_G(\mathbb{S}^{n-1})$, endowed with the inner product $\langle\cdot,\cdot\rangle_\lambda$ defined in (3.10), which is equivalent to the usual one. Indeed, this follows from the fact that $\{Y_k\}_{k\in\mathbb{N}_0}$ is an orthonormal basis for $L^2_G(\mathbb{S}^{n-1})$.*

Since $i_1 \geq 2$, Proposition 3.12 says that the eigenvalues $\mu_{i_k,1}(\lambda)$, $k \in \mathbb{N}$, cross 0 at values $\lambda_k := \lambda^*_{i_k} \in (0,1)$, with $\lambda_k \uparrow 1$, while the eigenvalues $\mu_{i_k,2}(\lambda) > 0$. In addition, the eigenvalues $\mu_{i_0,1}(\lambda)$ and $\mu_{i_0,2}(\lambda)$ are strictly negative for all $\lambda \in (0,1)$. Theorem 3.5 will follow after a direct application of the Crandall-Rabinowitz Theorem (see Theorem A.1) to the smooth family of nonlinear operators $F_\lambda : U \to (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$ from Lemma 3.13, and the following proposition puts us exactly in the framework of that theorem. In order to simplify notation, we will denote

$$\mathbf{z}_k := \mathbf{z}_{k,1} \quad \text{for} \quad k \in \mathbb{N},$$

where $\mathbf{z}_{k,1}$ is defined in (3.52).

Proposition 3.14. *For every $k \in \mathbb{N}$, the linear operator $L_{\lambda_k} : (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2 \to (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$ in Lemma 3.13 has kernel of dimension 1 spanned by $\mathbf{z}_k$, closed image of co-dimension 1 given by*

$$\mathrm{im}\, L_{\lambda_k} = \left\{ \mathbf{w} \in \left(C^{1,\alpha}_G(\mathbb{S}^{n-1})\right)^2 : \langle \mathbf{w}, \mathbf{z}_k\rangle_{\lambda_k} = 0 \right\}, \tag{3.53}$$

and satisfies

$$\partial_\lambda L_\lambda|_{\lambda=\lambda_k}(\mathbf{z}_k) \notin \mathrm{im}\, L_{\lambda_k}. \tag{3.54}$$

Proof. The proof of (3.53) follows closely the one in [FMW18, Proposition 5.1]. Our first observation is that the Sobolev space $H^s(\mathbb{S}^{n-1})$ can be characterized as the subspace of funtions $v \in L^2(\mathbb{S}^{n-1})$ such that

$$\sum_{j=0}^{\infty}(1 + j^2)^s \|P_j(v)\|^2_{L^2} < \infty,$$

where P_j denotes the L^2-orthogonal projection on the subspace generated by the spherical harmonics of degree j. As stated in Remark 3.4, the sequence $\{\mathbf{z}_{k,j}\}$ is an orthonormal basis for $(L^2_G(\mathbb{S}^{n-1}))^2$ with inner product $\langle\cdot,\cdot\rangle_\lambda$, and so we can define the map $(H^2_G(\mathbb{S}^{n-1}))^2 \to (H^1_G(\mathbb{S}^{n-1}))^2$

$$\mathbf{w} = \sum_{\ell=0}^{\infty}(a_{\ell,1}\mathbf{z}_{\ell,1} + a_{\ell,2}\mathbf{z}_{\ell,2}) \mapsto \sum_{\ell=0}^{\infty}(a_{\ell,1}\mu_{i_\ell,1}(\lambda)\mathbf{z}_{\ell,1} + a_{\ell,2}\mu_{i_\ell,2}(\lambda)\mathbf{z}_{\ell,2}). \tag{3.55}$$

Due to the asymptotic behavior of the sequences $\{\mu_{m,j}(\lambda)\}_{m=1}^{\infty}$ proved in Lemma 3.11, we can see that (3.55) defines a continuous mapping. Since it agrees with L_λ on finite linear combinations of $\{\mathbf{z}_{k,j}\}$, which are dense both in $(C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2$ and $(H^2_G(\mathbb{S}^{n-1}))^2$, (3.55) defines an extension of L_λ. Moreover, for $\lambda = \lambda_k$ the map

$$\mathbf{w} = \sum_{\ell=0}^{\infty}(b_{\ell,1}\mathbf{z}_{\ell,1} + b_{\ell,2}\mathbf{z}_{\ell,2}) \mapsto \sum_{\substack{\ell=0\\ \ell\neq k}}^{\infty}\left(\frac{b_{\ell,1}}{\mu_{i_\ell,1}(\lambda_k)}\mathbf{z}_{\ell,1} + \frac{b_{\ell,2}}{\mu_{i_\ell,2}(\lambda_k)}\mathbf{z}_{\ell,2}\right) + \frac{b_{k,2}}{\mu_{i_k,2}(\lambda_k)}\mathbf{z}_{k,2}$$

is a right inverse for L_{λ_k}, which is also continuous by Lemma 3.11. Thus, L_{λ_k} defines an isomorphism between the spaces

$$\mathfrak{X}_k := \left\{\mathbf{v} \in \left(H^2_G(\mathbb{S}^{n-1})\right)^2 : \langle \mathbf{v}, \mathbf{z}_k\rangle_{\lambda_k} = 0\right\},$$
$$\mathfrak{Y}_k := \left\{\mathbf{v} \in \left(H^1_G(\mathbb{S}^{n-1})\right)^2 : \langle \mathbf{v}, \mathbf{z}_k\rangle_{\lambda_k} = 0\right\}.$$

It follows that $L_{\lambda_k} : \mathfrak{X}_k \cap (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2 \to \mathfrak{Y}_k \cap (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$ is a well defined, injective mapping. It only remains to prove its surjectivity.

Let $D : \mathfrak{X}_k \to \mathfrak{Y}_k$ denote the difference

$$D\mathbf{w} = L_{\lambda_k}(\mathbf{w}) - c_{\lambda_k}^{-1}\left(\partial_r^2 u_{\lambda_k}\big|_{\Gamma_{\lambda_k}} w_1,\, \partial_r^2 u_{\lambda_k}\big|_{\Gamma_1} w_2\right),$$

which precisely corresponds to the Dirichlet-to-Neumann operator for the Laplacian. We note that for every $\mathbf{w} \in \mathfrak{X}_k$,

$$\langle D\mathbf{w}, 1\rangle_{\lambda_k} = 0. \tag{3.56}$$

Indeed, since each component of $\mathbf{z}_{\ell,1}$ and $\mathbf{z}_{\ell,2}$ has zero integral over $\mathbb{S}^{n-1}$ for $\ell \geq 1$, it suffices to check that

$$\langle D\mathbf{z}_{0,j}, 1\rangle_{\lambda_k} = 0, \quad j = 1, 2.$$

Denoting by $\phi_{\mathbf{z}_{0,j}}$, $j = 1, 2$, the harmonic function in Ω_{λ_k} whose boundary data on Γ_{λ_k} and Γ_1 is given by the respective components of $\mathbf{z}_{0,j}$ (which are constants), we verify that

$$\langle D\mathbf{z}_{0,j}, 1\rangle_{\lambda_k} = \int_{\partial\Omega_{\lambda_k}} -\frac{\partial \phi_{\mathbf{z}_{0,j}}}{\partial \nu}\, dS = 0, \quad j = 1, 2,$$

due to the harmonicity of $\phi_{\mathbf{z}_{0,j}}$.

With this in mind, let us proceed to establish the stated surjectivity. Because L_{λ_k} is an isomorphism between $\mathfrak{X}_k$ and $\mathfrak{Y}_k$, for any $\mathbf{y} \in \mathfrak{Y}_k \cap (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$, there exists a unique $\mathbf{w} \in \mathfrak{X}_k$

such that $L_{\lambda_k}(\mathbf{w}) = \mathbf{y}$. This means that the solution $\phi \in H^2(\Omega_{\lambda_k})$ of the Dirichlet problem

$$\begin{cases} \Delta\phi = 0 & \text{in} \quad \Omega_{\lambda_k}, \\ \phi = w_1 & \text{on} \quad \Gamma_{\lambda_k}, \\ \phi = w_2 & \text{on} \quad \Gamma_1, \end{cases} \tag{3.57}$$

also satisfies the Neumann problem

$$\begin{cases} \Delta\phi = 0 & \text{in} \quad \Omega_{\lambda_k}, \\ -\phi_\nu = y_1 - \dfrac{w_1}{c_{\lambda_k}}\dfrac{\partial^2 u_{\lambda_k}}{\partial r^2} & \text{on} \quad \Gamma_{\lambda_k}, \\ -\phi_\nu = y_2 - \dfrac{w_2}{c_{\lambda_k}}\dfrac{\partial^2 u_{\lambda_k}}{\partial r^2} & \text{on} \quad \Gamma_1. \end{cases} \tag{3.58}$$

Let us denote by $C^{s,\alpha}_m(\partial\Omega_{\lambda_k}), H^s_m(\partial\Omega_{\lambda_k})$ the usual Hölder and Sobolev spaces over the boundary $\partial\Omega_{\lambda_k}$, restricted to functions that have zero mean over $\partial\Omega_{\lambda_k}$. Because of (3.56), the Neumann condition in (3.58) belongs to $C^{1,\alpha}_m(\partial\Omega_{\lambda_k}) + H^2_m(\partial\Omega_{\lambda_k})$. Therefore, by elliptic regularity for the Neumann problem [Gri11], it must be that

$$\phi \in C^{2,\alpha}(\overline{\Omega_{\lambda_k}}) + H^3(\Omega_{\lambda_k}).$$

Let us argue by induction that

$$\phi \in C^{2,\alpha}(\overline{\Omega_{\lambda_k}}) + H^{s/2}(\Omega_{\lambda_k}) \quad \text{for all} \quad s \in \mathbb{N}, \quad s \geq 6.$$

Indeed, from the inductive assumption we see that the trace

$$\phi|_{\partial\Omega_{\lambda_k}} \in C^{2,\alpha}_m(\partial\Omega_{\lambda_k}) + H^{(s-1)/2}_m(\partial\Omega_{\lambda_k})$$

which, in turn, implies that the Neumann condition in (3.58) is in $C^{1,\alpha}_m(\partial\Omega_{\lambda_k}) + H^{(s-1)/2}_m(\partial\Omega_{\lambda_k})$. Hence, by elliptic regularity for the Neumann problem, $\phi \in C^{2,\alpha}(\overline{\Omega_{\lambda_k}}) + H^{(s+1)/2}(\Omega_{\lambda_k})$, which completes the inductive step. By Sobolev emdedding, we now conclude that $\phi \in C^{2,\alpha}(\overline{\Omega_{\lambda_k}})$, so that its boundary values $\mathbf{w} \in (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2$.

Therefore, $L_{\lambda_k} : \mathfrak{X}_k \cap (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2 \to \mathfrak{Y}_k \cap (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2$ is an isomorphism. This readily implies equality (3.53) and that $\ker L_{\lambda_k}$ is spanned by $\mathbf{z}_k$. The tranversality condition (3.54) follows from the fact that

$$\partial_\lambda L_\lambda|_{\lambda=\lambda_k}(\mathbf{z}_k) = \mu'_{i_k,1}(\lambda_k)\mathbf{z}_k - L_\lambda(\mathbf{z}'_k),$$

which by Lemma 3.9 has a non-trivial component along $\mathbf{z}_k$, and thus cannot belong to $\operatorname{im} L_{\lambda_k}$. $\square$

Proof of Theorem 3.5. Let $I_k \subset (0,1)$ be a small interval around each critical value $\lambda_k \uparrow 1$. Let $U_k \subset (C^{2,\alpha}_G(\mathbb{S}^{n-1}))^2$ be an appropriately small neighbourhood of $\mathbf{0}$, such that for all $\lambda \in I_k$, F_λ is well defined on U_k via (3.19). Then the operator

$$F : U_k \times I_k \to Y := (C^{1,\alpha}_G(\mathbb{S}^{n-1}))^2, \qquad F(\mathbf{v}, \lambda) := F_\lambda(\mathbf{v}),$$

is in $C^\infty(U_k \times I_k, Y)$, and by Proposition 3.14, we can apply the Crandall-Rabinowitz Bifurcation Theorem A.1 to get a smooth curve

$$\begin{array}{ccc} (-\varepsilon, \varepsilon) & \to & \left(C^{2,\alpha}_G(\mathbb{S}^{n-1})\right)^2 \times I_k \\ s & \mapsto & (\mathbf{w}(s), \lambda(s)) \end{array}$$

such that

- $\mathbf{w}(0) = \mathbf{0}$, $\lambda(0) = \lambda_k$, and $\langle \mathbf{w}(s), \mathbf{z}_k \rangle_{\lambda_k} = 0$;
- $F_{\lambda(s)}(\mathbf{v}(s)) = 0$, where $\mathbf{v}(s) = s(\mathbf{z}_k + \mathbf{w}(s))$.

Then, for every $s \in (-\varepsilon, \varepsilon)$, the solution $u_{\lambda(s)}(\mathbf{v}(s)) \in C^{2,\alpha}_G(\overline{\Omega}^{\mathbf{v}(s)}_{\lambda(s)})$ to the Dirichlet problem (3.7) also solves the overdetermined problem (3.12). $\square$

3.5 Proof of Corollary 3.2

The corollary follows directly from the more general [Min17, Theorem 1.2]. For the sake of completeness, we shall provide the proof in our particular setting.

Let Ω be any one of the domains constucted in Theorem 3.1 and let $u \in C^\infty(\overline{\Omega})$ be the solution of the corresponding overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega_0, \\ u = a & \text{on} \quad \partial\Omega_1, \\ u_\nu = c & \text{on} \quad \partial\Omega. \end{cases}$$

for some constants $a > 0$ and $c > 0$.

Proof of Corollary 3.2. First, let us show that

$$|\nabla u| < c \quad \text{in} \quad \Omega. \tag{3.59}$$

Indeed, since $-\Delta u = 1$,

$$\Delta |\nabla u|^2 = 2|D^2 u|^2 + 2\nabla u \cdot \nabla(\Delta u) = 2|D^2 u|^2 > 0,$$

so that the function $|\nabla u|^2$ is subharmonic in Ω and, by the strong maximum principle

$$|\nabla u|^2(x) < \sup_{\partial\Omega} |\nabla u|^2 = c^2 \quad \text{for all} \quad x \in \Omega.$$

Now let $E \subseteq \Omega$ be any subset of finite perimeter and let $\partial^* E \subseteq \partial E$ be its reduced boundary – where one can define a measure-theoretic inner unit normal ν_E. By De Giorgi's theorem on the regularity of sets of finite perimeter (see [Giu84, Chapter 4]), the $(n-1)$-dimensional Haudorff measure $H^{n-1}(\partial^* E) = P(E)$ and we can apply the version of the Divergence Theorem to obtain

$$|E| = \int_E (-\Delta u)\, dx = \int_{\partial^* E} \nabla u \cdot \nu_E \, dH^{n-1} \le \int_{\partial^* E} |\nabla u| \, dH^{n-1} \le cH^{n-1}(\partial^* E) = cP(E), \quad (3.60)$$

where we used (3.59) in the last inequality above. Hence,

$$P(E)/|E| \ge 1/c = P(\Omega)/|\Omega|.$$

and we conclude that Ω is self-Cheeger.

It remains to show that Ω is the unique minimizer, up to a set of zero Lebesgue measure. Let $E \subseteq \Omega$ be another minimizing subset of finite perimeter with $|E| > 0$ and $P(E)/|E| = 1/c$. If equality holds in (3.60), then $|\nabla u| = c \quad H^{n-1}$-a.e. on $\partial^* E$, and (3.59) implies that $H^{n-1}(\partial^* E \cap \Omega) = 0$. Now, according to De Giorgi's theorem,

$$0 = H^{n-1}(\partial^* E \cap \Omega) = \int_\Omega |D\chi_E| := \sup \left\{ \int_\Omega \chi_E \operatorname{div} \phi \, dx : \phi \in C_0^1(\Omega; \mathbb{R}^n),\ |\phi| \le 1 \right\}.$$

Hence, $\chi_E \in L^1(\Omega)$ has a weak derivative $\partial \chi_E = 0$ a.e. in Ω, which implies that $\chi_E = \text{const}$ a.e. in Ω. Since $|E| > 0$, we conclude that $\chi_E = 1$ a.e. in Ω, i.e. $E = \Omega$, up to a set of measure 0. $\square$

Second result

In this Chapter we give another construction of solutions to an overdetermined elliptic problem in annular domains. The result in this case is a consequence of the Krasnoselskii Bifurcation Theorem (see Theorem A.2 in the Appendix A). Thus, we only show that on each neighbourhood of the trivial solution for a certain critical value of a parameter contains a nontrivial solution. Whether or not this solutions belong to a smooth branch is beyond of the scope of this method.

The proofs and methods in this chapter are of a different kind of that in the previous one. In the previous chapter, due to the nature of the differential equation, we could rely on explicit computations to carry out the proofs. Therefore, in the absence of explicit formulas, we will rely on a variational characterizations of the relevant quantities involved, and on the direct methods of the calculus of variations. This sets an extra difficultly in setting up the problem, and we had to give up the approach of the Crandall-Rabinowitz Theorem.

We follow the approach of Ros et al. in [RRS19], and the result presented here can be seen as the analogue of their result in the case of bounded annular domains.

In this chapter we treat the non-linearity $f(u) = u^p - \kappa u$, for $\kappa \geq 0$ and $p > 1$. Thus, we want to see whether there exists a solution to the problem

$$\begin{cases} -\Delta u + \kappa u = u^p, & u > 0 \quad \text{in} \quad \Omega, \\ \qquad u = 0, & u_\nu = c_0 \quad \text{on} \quad \partial\Omega_0, \\ \qquad u = 0, & u_\nu = c_1 \quad \text{on} \quad \partial\Omega_1, \end{cases} \tag{4.1}$$

for some constants $c_0, c_1 > 0$. The main theorem of this chapter gives an affirmative response.

Theorem 4.1. *Let $n \geq 2$, $\kappa \geq 0$ and $p \in (1, \frac{n+2}{n-2})$, when $n \geq 3$, and $p > 1$, when $n = 2$. Then there exist bounded, annular C^2-domains of the form $\Omega = \Omega_0 \setminus \overline{\Omega_1} \subset \mathbb{R}^n$, which are different from standard annuli, such that the overdetermined problem (4.1) admits a positive solution $u \in C^2(\overline{\Omega})$ for some positive constants c_0 and c_1.*

The chapter is organized as follows. In Section 4.1 we review the basic results present in the literature concerning the Dirichlet problem for the equation of our interest and the existence and non-degeneracy of radially symmetric solutions over annuli and balls (Theorem 4.2). In Section 4.2 we outline the strategy we will follow for the construction of non-trivial solutions to (4.1), and state

Theorem 4.6, which is a quantitative version of Theorem 4.1, and show how the latter follows from the former. In Section 4.3 we translate the problem to an operator equation of the form $F_\lambda(v) = 0$ in the appropriate function spaces, and compute a formula for its linearization $L_\lambda := dF|_{v=0}$ (Proposition 4.9). Then, in Section 4.4, we study the first eigenvalue μ_λ of the linearized operator L_λ by means of a variational characterization (Proposition 4.11) and then show the existence of a critical value λ^* such that $\mu_{\lambda^*} = 0$ (Proposition 4.16). In Section 4.5 we reformulate out problem in terms of an operator equation $R_\lambda(v) = 0$, which is derived from the previous operator Φ_λ, so as to meet the requirements for the application of the Krasnoselskii Bifurcation Theorem, and finally prove Theorem 4.6. In Section 4.6 we prove the crucial Propositions 4.4 and 4.14, which were postponed to the end due to the length of their proofs. Finally, we leave the Appendix 4.A to state and prove some technical lemmas needed in Section 4.6.

4.1 Existence, uniqueness, and Morse index of radial solutions

Recall that for any $\lambda \in (0,1)$ we defined in the previous chapter the annulus

$$\Omega_\lambda := \{x \in \mathbb{R}^n : \lambda < |x| < 1\}$$

and let its boundary components be

$$\Gamma_r := r\mathbb{S}^{n-1}, \quad r = \lambda, 1.$$

We will denote by Ω_0 or B the unit ball centred at the origin:

$$\Omega_0 = B := \{x \in \mathbb{R}^n : |x| < 1\}.$$

The basic approach to the Theorem 4.1 follows very close the one in the previous chapter: we will construct a non-trivial solution u and domains Ω bifurcating away from the branch of non-monotone radially symmetric solutions defined on the standard annuli. In the absence of explicit formulas analogous to the ones derived in Lemma 3.3 in the previous chapter, we need to gather certain known facts present in the literature about the radially symmetric solutions to (4.1), such as existence, uniqueness, and non-degeneracy.

We look at the Dirichlet problem

$$\begin{cases} -\Delta u + \kappa u = u^p & \text{in} \quad \Omega_\lambda, \\ u = 0 & \text{on} \quad \partial\Omega_\lambda, \end{cases} \tag{4.2}$$

for $\kappa \geq 0$ and $p > 1$. Note that by the Hopf Lemma, any positive solution to (4.2) satisfies $u_\nu|_{\partial\Omega_\lambda} > 0$. Furthermore, if u is radially symmetric then $u_\nu = \text{const}$ over each boundary component Γ_1 and

Γ_λ. When $1 < p < \frac{n+2}{n-2}$, with the convention that $\frac{n+2}{n-2} = \infty$ when $n = 2$, it is shown in [AR73] with a variational approach that there is a positive radially symmetric solution $u_\lambda \in C^\infty(\overline{\Omega_\lambda})$ to (4.2). The solution u_λ is constructed using the powerful machinery of Critical Point Theory as a critical point for the functional

$$J_\lambda(u) := \frac{1}{2}\int_{\Omega_\lambda} |\nabla u|^2 + \kappa|u|^2 \, dx - \frac{1}{p+1}\int_{\Omega_\lambda} |u|^{p+1} \, dx, \quad u \in H^1_0(\Omega_\lambda), \tag{4.3}$$

and satisfies

$$J_\lambda(u_\lambda) = \inf_{\substack{u \in H^1_{0,\rho}(\Omega_\lambda) \\ u \neq 0}} \max_{t>0} J_\lambda(tu),$$

where $H^1_{0,\rho}(\Omega_\lambda)$ denotes the Sobolev space of radially symmetric functions (see [SW10]). The proof is based on the *Mountain Pass Theorem* and relies on the compactness of the Sobolev embedding $H^1 \hookrightarrow L^{p+1}$ given by the Rellich-Kondrachov Theorem.

Since u_λ is radially symmetric, it can also be thought of as a function of the variable $r := |x|$. Thus, in this framework, $u_\lambda = u_\lambda(r)$ is a solution the ODE boundary value problem

$$u'' + \frac{n-1}{r}u' + u^p - \kappa u = 0, \qquad u(\lambda) = 0 = u(1). \tag{4.4}$$

The *Morse index* of u_λ in the space $H^1_{0,\rho}(\Omega_\lambda)$ is defined as the dimension of the maximal subspace $V \subset H^1_{0,\rho}(\Omega_\lambda)$ such that the second variation

$$d^2 J_\lambda|_{u=u_\lambda}(v, v) < 0 \quad \text{for all} \quad v \in V \setminus \{0\},$$

or equivalently, is the number of negative eigenvalues of the linearized operator

$$-\Delta + \kappa - pu_\lambda^{p-1} : H^1_{0,\rho}(\Omega_\lambda) \to H^{-1}_\rho(\Omega_\lambda), \tag{4.5}$$

where $H^{-1}_\rho(\Omega_\lambda)$ denotes the dual space to $H^1_\rho(\Omega_\lambda)$.

We say that u_λ is *non-degenerate* in $H^1_{0,\rho}(\Omega_\lambda)$ if 0 is *not* an eigenvalue for (4.5), or equivalently, the Dirichlet problem

$$\begin{cases} -\Delta\psi + \kappa\psi - pu_\lambda^{p-1}\psi = 0 & \text{in} \quad \Omega_\lambda, \\ \psi = 0 & \text{on} \quad \partial\Omega_\lambda, \end{cases}$$

only has the trivial solution $\psi = 0$ in $H^1_{0,\rho}(\Omega_\lambda)$.

We summarize all the properties of u_λ that we will need in the following proposition.

Proposition 4.2. *Let $\lambda \in [0, 1)$, $n \geq 2$, $\kappa \geq 0$ and $p \in (1, \frac{n+2}{n-2})$. Then there exists a unique positive radially symmetric solution $u \in C^\infty(\overline{\Omega_\lambda})$ to the problem (4.2). We denote this solution by u_λ.*

Moreover, the solution u_λ is non-degenerate in the Sobolev space of radially symmetric functions $H^1_{0,\rho}(\Omega_\lambda)$ and its Morse index in this space is equal to 1.

Proof. The existence of u_λ follows from the discussion above. Furthermore, since the solutions u_λ have the mountain pass structure, by the result in [Hof84] their Morse index in the space $H^1_{0,\rho}(\Omega_\lambda)$ is at most 1. Now, multiplying the the equation by u_λ (4.2) and integrating by parts we arrive to the identity

$$\int_{\Omega_\lambda} |\nabla u_\lambda|^2 + \kappa u_\lambda^2 \, dx = \int_{\Omega_\lambda} u_\lambda^{p+1} \, dx, \tag{4.6}$$

and thus, because $p > 1$,

$$d^2 J_\lambda|_{u=u_\lambda}(u_\lambda, u_\lambda) = \int_{\Omega_\lambda} |\nabla u_\lambda|^2 + \kappa u_\lambda^2 - p u_\lambda^{p+1} \, dx = (1-p) \int_{\Omega_\lambda} u_\lambda^{p+1} \, dx < 0.$$

This implies that the Morse index of u_λ is at least, and hence equal to, 1.

Uniqueness in the case $\lambda = 0$ was proved by Gidas, Ni and Nirenberg [GNN79], and a proof of uniqueness and non-degeneracy can be found in [Sri93, DGP99]. Uniqueness in the annulus $\lambda > 0$, in case $\kappa > 0$, was shown in [Tan03] for dimensions $n \geq 3$ and later in [FMT08] for all dimensions $n \geq 2$ along with the non-degeneracy of u_λ in the space $H^1_{0,\rho}(\Omega_\lambda)$. Uniqueness in the case $\kappa = 0$ was shown, for instance, in [Ni83]. It only remains to prove non-degeneracy in the case $\kappa = 0$. We give here a simple argument based on Sturm-Liouville Theory.

Suppose that the solution u_λ is degenerate. This means that there exists a $\psi \in H^1_{0,\rho}(\Omega_\lambda)$ which is a solution to the ODE

$$\psi'' + \frac{n-1}{r}\psi' + p u_\lambda^{p-1}\psi = 0, \quad \text{in} \quad (\lambda, 1), \quad \psi(\lambda) = 0 = \psi(1), \tag{4.7}$$

where $r := |x|$, $x \in \Omega_\lambda$. If we multiply the above equation by u_λ, multiply the equation (4.4) by ψ, and then integrate by parts we arrive at

$$\int_\lambda^1 (\psi' u_\lambda' - p u_\lambda^p \psi) r^{n-1} \, dr = 0, \qquad \int_\lambda^1 (\psi' u_\lambda' - u_\lambda^p \psi) r^{n-1} \, dr = 0.$$

Subtracting the above identities we see that

$$(p-1) \int_\lambda^1 u_\lambda^p \psi r^{n-1} \, dr = 0.$$

Now, let $v := u_\lambda'$. Then differentiating equation (4.4) we see v satisfies

$$v'' + \frac{n-1}{r} v' + \left(p u_\lambda^{p-1} - \frac{n-1}{r^2} \right) v = 0 \quad \text{in} \quad (\lambda, 1). \tag{4.8}$$

Consider the function $\phi(r) := r v(r)$, then

$$\phi' = r v' + v, \quad \phi'' = r v'' + 2v'.$$

Putting this into equation (4.8) we thus find that ϕ solves de ODE

$$\phi'' + \frac{n-1}{r}\phi' + p u_\lambda^{p-1}\phi = 2\left(v' + \frac{n-1}{r} v \right) = -2u_\lambda^p.$$

If we multiply the above equation by ψ and integrate by parts we arrive to the identity

$$\int_\lambda^1 (\phi'\psi' - pu_\lambda^{p-1}\phi\psi)r^{n-1}\,dr = 2\int_\lambda^1 u_\lambda^p\psi r^{n-1}\,dr = 0,$$

and therefore, if we multiply (4.7) by ϕ and integrate by parts once again,

$$\phi(1)\psi'(1) - \phi(\lambda)\psi'(\lambda)\lambda^{n-1} - \int_\lambda^1 (\phi'\psi' - pu_\lambda^{p-1}\phi\psi)r^{n-1}\,dr = \phi(1)\psi'(1) - \phi(\lambda)\psi'(\lambda)\lambda^{n-1} = 0. \quad (4.9)$$

Now, note that because u_λ is positive in $(\lambda, 1)$ we have $u_\lambda'(\lambda) > 0$ and $u_\lambda'(1) < 0$, which implies $\phi(\lambda) > 0$ and $\phi(1) < 0$. Furthermore, because the Morse index of u_λ is 1, then 0 is the second eigenvalue of the linearised operator, and thus by the Sturm Comparison Theorem ψ has exactly one zero in $(\lambda, 1)$. This implies that $\psi'(\lambda)$ and $\psi'(1)$ have the same sign. But then

$$\phi(1)\psi'(1) - \phi(\lambda)\psi'(\lambda)\lambda^{n-1} \neq 0,$$

which contradicts (4.9). Therefore, ψ must be identically zero or, in other words, u_λ is non-degenerate. □

For $\lambda \in [0, 1)$, we denote by z_λ the radially symmetric function with negative eigenvalue $\tau_\lambda < 0$ normalized in the H^1-norm, that is $z_\lambda \in C^\infty(\Omega_\lambda) \cap H^1_{0,\rho}(\Omega_\lambda)$ solves

$$\begin{cases} -\Delta z + \kappa z - pu_\lambda^{p-1} z = \tau_\lambda z & \text{in} \quad \Omega_\lambda, \\ z = 0 & \text{on} \quad \partial\Omega_\lambda, \end{cases} \quad (4.10)$$

From now on we fix $\kappa \geq 0$ and $p \in (1, \frac{n+2}{n-2})$.

Let us define the quadratic form $\tilde{Q}_\lambda : H_0^1(\Omega_\lambda) \to \mathbb{R}$ associated to the operator $-\Delta + \kappa - pu_\lambda^{p-1}$,

$$\tilde{Q}_\lambda(\psi) := \int_{\Omega_\lambda} |\nabla\psi|^2 + \kappa|\psi|^2 - pu_\lambda^{p-1}|\psi|^2\,dx,$$

and define the space

$$H^{1,*}(\Omega_\lambda) := \left\{\psi \in H^1(\Omega_\lambda) : \int_{\Omega_\lambda} \psi z_\lambda\,dx = 0\right\}.$$

Recall that for a subgroup G of the orthogonal group $O(n)$ we denoted the spaces of G-invariant functions with a subscript-G, and set $H^{1,*}_{0,G}(\Omega_\lambda) := H^1_{0,G}(\Omega_\lambda) \cap H^{1,*}(\Omega_\lambda)$. Note that Proposition 4.2 implies that for every $\lambda \in (0, 1)$, $\tilde{Q}_\lambda(\psi) > 0$ for $\psi \in H^{1,*}_{0,\rho}(\Omega_\lambda) \setminus \{0\}$. However, this is not true for some λ if ψ is not radially symmetric, as states the following lemma.

Lemma 4.3. *There exists $\delta \in (0, 1)$ such that*

$$\tilde{\chi}_\lambda := \inf_{\substack{\psi \in H^{1,*}_{0,G}(\Omega_\lambda) \\ \|\psi\|_{L^2(\Omega_\lambda)} = 1}} \tilde{Q}_\lambda(\psi) < 0 \quad (4.11)$$

for all $\lambda \in (\delta, 1)$.

Proof. Let Y be a non-zero G-invariant spherical harmonic normalized in the L^2-norm and with corresponding eigenvalue $\sigma > 0$. Let us define in polar coordinates the function

$$\psi(r,\theta) := u_\lambda(r)Y(\theta), \quad (r,\theta) \in (\lambda, 1) \times \mathbb{S}^{n-1}.$$

Then $\psi \in H^{1,*}_{0,G}(\Omega_\lambda)$. Moreover, because

$$|\nabla\psi(r,\theta)|^2 = |u'_\lambda(r)|^2|Y(\theta)|^2 + \frac{1}{r^2}|u_\lambda(r)|^2|\nabla_\theta Y(\theta)|^2,$$

we find that

$$\tilde{Q}_\lambda(\psi) = \int_\lambda^1 \left(|u'_\lambda(r)|^2 + \frac{\sigma}{r^2}|u_\lambda(r)|^2 + \kappa|u_\lambda(r)|^2 - p|u_\lambda(r)|^{p+1}\right) r^{n-1}\, dr.$$

Now, recall that by the identity (4.6)

$$\int_\lambda^1 \left(|u'_\lambda(r)|^2 + \kappa|u_\lambda(r)|^2\right) r^{n-1}\, dr = \int_\lambda^1 |u_\lambda(r)|^{p+1} r^{n-1}\, dr$$

and therefore

$$\tilde{Q}_\lambda(\psi) = \int_\lambda^1 \left\{(1-p)\left(|u'_\lambda(r)|^2 + \kappa|u_\lambda(r)|^2\right) + \frac{\sigma}{r^2}|u_\lambda(r)|^2\right\} r^{n-1}\, dr \tag{4.12}$$

Moreover, by the Rayleigh-Ritz formula for the first eigenvalue of the Laplacian we have the inequality

$$\int_\lambda^1 |u'_\lambda(r)|^2 r^{n-1}\, dr \geq \kappa_\lambda \int_\lambda^1 |u_\lambda(r)|^2 r^{n-1}\, dr, \tag{4.13}$$

where κ_λ is the first Dirichlet eigenvalue for the Laplacian in Ω_λ. Hence, plugging (4.13) in (4.12) yields the inequality

$$\tilde{Q}_\lambda(\psi) \leq \int_\lambda^1 \left\{(1-p)(\kappa_\lambda + \kappa) + \frac{\sigma}{\lambda^2}\right\} |u_\lambda(r)|^2 r^{n-1}\, dr \tag{4.14}$$

Now, we claim that

$$\lim_{\lambda \uparrow 1} \kappa_\lambda = +\infty.$$

Indeed, by the Cheeger inequality (see [Leo15]), there holds

$$\kappa_\lambda \geq \frac{h(\Omega_\lambda)}{4},$$

where $h(\Omega_\lambda)$ is the Cheeger constant. As was shown in Corollary 3.2 in Chapter 3, the Cheeger constant for the annulus Ω_λ is given by

$$h(\Omega_\lambda) = \frac{|\partial\Omega_\lambda|}{|\Omega_\lambda|} = n\frac{1+\lambda^{n-1}}{1-\lambda^n}$$

which goes to ∞ as $\lambda \uparrow 1$. This sets the claim. Thus, from inequality (4.14) we have $\tilde{Q}_\lambda(\psi) < 0$ for λ sufficiently close to 1. □

Remark 4.1. *As a corollary of the proof of Lemma 4.3 we have that the Morse index of u_λ in the Sobolev space $H^1_{0,G}(\Omega_\lambda)$ diverges to ∞ as $\lambda \uparrow 1$. Thus, one is led to think that there might exist infinitely many branches of non radially symmetric solutions to the problem (4.2) bifurcating away from u_λ at some critical values of λ. This question has been studied in [Lin90, Lin93] for a general class of non-linearities which include $f(u) = u^p - \kappa u$.*

Remark 4.2. *We note that $\tilde{\chi}_\lambda$ is nothing but the second eigenvalue of $-\Delta + \kappa - p u_\lambda^{p-1}$ in the space $H^1_{0,G}(\Omega_\lambda)$, and therefore there exists a non-zero function $\psi \in H^{1,*}_{0,G}(\Omega_\lambda)$ which is a weak solution to*

$$\begin{cases} -\Delta\psi + \kappa\psi - p u_\lambda^{p-1}\psi = \tilde{\chi}_\lambda \psi & \text{in} \quad \Omega_\lambda, \\ \psi = 0 & \text{on} \quad \partial\Omega_\lambda. \end{cases}$$

More generally, we will call u_λ *non-degenerate* in the space $H^1_{0,G}(\Omega_\lambda)$ if 0 is *not* an eigenvalue of the linearized operator

$$-\Delta + \kappa - p u_\lambda^{p-1} : H^1_{0,G}(\Omega_\lambda) \to H^{-1}_G(\Omega_\lambda). \tag{4.15}$$

With the insight of Lemma 4.3, we see that the solution u_λ might be degenerate in the space $H^1_{0,G}(\Omega_\lambda)$ for some values of $\lambda \in (0,1)$. We point out that the non-degeneracy of u_λ is crucial for the construction, since it guarantees that a solution to (4.2) in a perturbed G-invariant annulus is unique and also G-invariant. We did not encounter this issue in the previous chapter since the Laplacian $\Delta : H^1_0(\Omega_\lambda) \to H^{-1}(\Omega_\lambda)$ is an isomorphism *for all* $\lambda \in (0,1)$. However, as a consequence of the next proposition, we conclude that the degeneracy of u_λ can be ruled out if λ is sufficiently small.

Proposition 4.4. *There exists $\varepsilon \in (0,1)$ such that $\tilde{\chi}_\lambda > 0$ for all $\lambda \in (0,\varepsilon)$.*

The proof of Proposition 4.4 is quite technical and is postponed to the end of the Chapter. We also note that G-invariance is needed in this proof, and that is why we work from the very start in spaces of G-invariant functions, contrary to what has been done in the previous chapter.

To simplify notation, we use $f(u) = u^p - \kappa u$ and $f'(u) = p u^{p-1} - \kappa$ in the following. The enthusiastic reader can check that much of the computations in this chapter hold for a general non-linearity $f(u)$ that may not be of the specified form above. Therefore, one can hope for a generalization of Theorem 4.1 which covers a much more general class of non-linear equations. However, some of the results, like uniqueness of solutions, have not been proved in such a general case and so the arguments used here do not carry out to those cases.

Proposition 4.4 implies the existence of a subinterval $(0,\lambda_0) \subset (0,1)$ where the linearized operator (4.15) has a trivial kernel. By the Fredholm alternative (see [Eva10]), (4.15) is in fact an isomorphism. This is enough to conclude that for each $w \in H^{1/2}_G(\partial\Omega_\lambda)$ there exist a unique

H^1_G-weak solution to the problem

$$\begin{cases} \Delta\psi + f'(u_\lambda)\psi = 0 & \text{in} \quad \Omega_\lambda, \\ \psi = w & \text{on} \quad \partial\Omega_\lambda. \end{cases}$$

in the sense that $\psi|_{\partial\Omega_\lambda} = w$ as a trace and

$$\int_{\Omega_\lambda} \nabla\psi\cdot\nabla\zeta - f'(u_\lambda)\psi\zeta \, dx = 0, \quad \text{for all} \quad \zeta \in H^1_{0,G}(\Omega_\lambda). \tag{4.16}$$

We shall work in the maximal interval such that the above holds. For the purpose, let us call

$$\Lambda := \inf\left\{\lambda \in (0,1) : \tilde{\chi}_\lambda \le 0\right\}, \tag{4.17}$$

where $\tilde{\chi}_\lambda$ is defined in (4.11). By Lemma 4.3 Λ is well defined, and by Proposition 4.4 $\Lambda > 0$.

We summarize the above discussion in the following lemma.

Lemma 4.5. *Let $\lambda \in (0,1)$. Suppose the linearized operator*

$$\Delta + f'(u_\lambda) : H^1_{0,G}(\Omega_\lambda) \to H^{-1}_G(\Omega_\lambda)$$

has a trivial kernel. Then it is an isomorphism, and for each $w \in H^{1/2}_G(\partial\Omega_\lambda)$ there exists a unique weak solution $\psi_w \in H^1_G(\Omega_\lambda)$ to the Dirichlet problem

$$\begin{cases} \Delta\psi + f'(u_\lambda)\psi = 0 & \text{in} \quad \Omega_\lambda, \\ \psi = w & \text{on} \quad \partial\Omega_\lambda. \end{cases} \tag{4.18}$$

Moreover, for every $0 < \alpha < \min\{1, p-1\}$, $\Delta + f'(u_\lambda) : C^{2;\alpha}_{0,G}(\Omega_\lambda) \to C^{0;\alpha}_G(\overline{\Omega_\lambda})$ is also an isomorphism, and if $w \in C^{2,\alpha}_G(\partial\Omega_\lambda)$ then the weak solution ψ_w belongs to $C^{2,\alpha}_G(\overline{\Omega_\lambda})$ and is, in fact, a classical solution.

The above holds, in particular, for all $\lambda \in (0,\Lambda)$.

Proof. For the first statement, we only need to explain why the weak solution $\psi = \psi_w$ in H^1_G-sense is in fact a weak solution in the classical sense, that is, (4.16) holds for every $\zeta \in H^1_0(\Omega_\lambda)$ regardless of G-invariance. We decompose ψ into its Fourier series

$$\psi(r,\theta) = \sum_{k=0}^{\infty} \varphi_k(r) Y_k(\theta),$$

where $\{Y_k\}_{k=0}^{\infty}$ denotes a sequence of G-invariant spherical harmonics which span $L^2_G(\mathbb{S}^{n-1})$. We claim the identity (4.16) in fact holds for every $\zeta \in H^1(\Omega_\lambda)$. Indeed, we decompose a general $\zeta \in H^1(\Omega_\lambda)$ into its Fourier series

$$\zeta(r,\theta) = \sum_{k=0}^{\infty}\sum_{j=0}^{m_k} \xi_{k,j}(r) Y_{k,j}(\theta).$$

where $Y_{k,j}$, $j = 1, \ldots, m_k$, are the L^2-normalized spherical harmonics of degree k. Then, by linearity, it is enough to cheek that (4.16) holds for a function $\zeta \in H^1(\Omega_\lambda)$ of the form

$$\zeta(r, \theta) = \xi(r) Y(\theta),$$

where Y is any non-zero spherical harmonic. Since (4.16) holds for such ζ if Y is G-invariant, we may suppose Y is not G-invariant. We note that for each $k \geq 0$

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \nabla(\varphi_k(r) Y_k) \cdot \nabla(\xi(r) Y)\, d\theta &= \varphi_k'(r)\xi'(r) \int_{\mathbb{S}^{n-1}} Y_k Y\, d\theta + \frac{1}{r^2}\varphi_k(r)\xi(r) \int_{\mathbb{S}^{n-1}} \nabla_\theta Y_k \cdot \nabla_\theta Y\, d\theta \\ &= \left(\varphi_k'(r)\xi'(r) + \frac{\sigma_k}{r^2}\varphi_k(r)\xi(r)\right) \int_{\mathbb{S}^{n-1}} Y_k(\theta) Y(\theta)\, d\theta = 0,\end{aligned}$$

since Y_k and Y are L^2-othogonal for every $k \geq 0$. Multiplying the above identity by r^{n-1}, then integrating with respect to r between λ and 1, and summing over $k \geq 0$ hence yields

$$\int_{\Omega_\lambda} \nabla\psi \cdot \nabla\zeta\, dx = 0.$$

An analogous argument shows that

$$\int_{\Omega_\lambda} f'(u_\lambda)\psi\zeta\, dx = 0.$$

Therefore, (4.16) holds for each $\zeta \in H^1(\Omega_\lambda)$.

To establish the Hölder regularity, we note that because $u_\lambda \in C^\infty(\overline{\Omega_\lambda})$ and $p > 1$, then $u_\lambda^{p-1} \in C^{0,\alpha}(\overline{\Omega_\lambda})$ for all $0 < \alpha < \min\{1, p-1\}$. Thus, the coefficients in (4.18) are in $C^{0,\alpha}(\overline{\Omega_\lambda})$. The second statement of the lemma is therefore a consequence of the Schauder regularity theory (see [GT15, Gri11]).

Finally, note that by the definition of Λ, we have $\tilde{\chi}_\lambda > 0$ for all $\lambda \in (0, \Lambda)$, and then, by Proposition 4.2, the operator $\Delta + f'(u_\lambda) : H^1_{0,G}(\Omega_\lambda) \to H^{-1}_G(\Omega_\lambda)$ has a trivial kernel. Thus, the above results apply for all $\lambda \in (0, \Lambda)$. $\square$

4.2 Outline of strategy and refinement of Theorem 4.1

In this construction we shall work from the very start on spaces of G-invariant functions. From now on, we shall fix a subgroup G of the orthogonal group $O(n)$ satisfying the properties

(P1') If T is a translation of $\mathbb{R}^n$ and $T(\mathbb{S}^{n-1})$ is a G-invariant set, then T is trivial.

(P2') If $\{\sigma_{i_k}\}_{k=0}^\infty$ are the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ when restricted to the G-invariant functions, then σ_{i_1} has multiplicity equal to 1, i.e. there exists a unique (up to normalization) G-invariant spherical harmonic of degree i_1.

Note that property (P1') is the same as property (P1), but (P2') is weaker that (P2). The explanation for this is that we will construct bifurcating solutions to (4.23) which cluster at a *single* critical value of the parameter $\lambda \in (0,1)$. Clearly the example $G = O(n-1) \times \mathbb{Z}_2$ given in Lemma 3.4 satisfies property (P2'). Another more general example is given by $G = O(m) \times O(n-m)$ for $1 \leq m \leq n-1$, which acts on $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. In this case, the first non-zero G-invarinat spherical harmonic, up to normalization, is the restriction to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomial of degree 2

$$f(x_1, \ldots, x_n) = (n-m)(x_1^2 + \cdots + x_m^2) - m(x_{m-1}^2 + \cdots + x_n^2).$$

More examples are given in [RRS19].

Because there is no ambiguity, we relabel the sequence of restricted eigenvalues $\{\sigma_{i_k}\}_{k=0}^{\infty}$ to $\{\sigma_k\}_{k=0}^{\infty}$. Thus

$$\sigma_k := i_k(i_k + n - 2), \quad k \in \mathbb{N}_0. \tag{4.19}$$

Note that $\sigma_0 = 0$ and by property (P1') $\sigma_1 \geq 2n$. For a given $k \in \mathbb{N}_0$, let us denote by m_k the multiplicity of σ_k and by

$$Y_{k,j}, \quad j = 1, \ldots, m_k,$$

the G-invariant spherical harmonics of degree i_k, normalized in the L^2-norm. Therefore

$$\Delta_{\mathbb{S}^{n-1}} Y_{k,j} + \sigma_k Y_{k,j} = 0, \quad \int_{\mathbb{S}^{n-1}} |Y_{k,j}|^2 \, dS, \quad j = 1, \ldots, m_k.$$

By property (P2'), we have $m_1 = 1$, and we denote the unique G-invariant and L^2-normalized spherical harmonic of degree i_1 by Y_1.

Since by Serrin's result (Theorem 2.2 in Chapter 2) a solution to the overdetermined problem (4.1) cannot satisfy $u_\nu|_{\partial\Omega} = c$ with the same constant c over each boundary component of $\partial\Omega$. Because of this lack of symmetry it will be convenient, for reasons that will become apparent latter in Section 4.3, to re-define the perturbed annulus as

$$\Omega_\lambda^v := \left\{ x \in \mathbb{R}^n : \lambda + \frac{v(\lambda x/|x|)}{u_\lambda'(\lambda)} < |x| < 1 + \frac{v(x/|x|)}{u_\lambda'(1)} \right\},$$

where u_λ is the radially symmetric solution to (4.4) given by Proposition 4.2, and so its boundary components now are

$$\Gamma_1^v := \left\{ x \in \mathbb{R}^n : |x| = 1 + \frac{v(x/|x|)}{u_\lambda'(1)} \right\},$$
$$\Gamma_\lambda^v := \left\{ x \in \mathbb{R}^n : |x| = \lambda + \frac{v(\lambda x/|x|)}{u_\lambda'(\lambda)} \right\}.$$

We want to see whether a positive solution to the Dirichlet problem in the perturbed annulus

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega_\lambda^v, \\ u = 0 & \text{on} \quad \partial\Omega_\lambda^v. \end{cases} \tag{4.20}$$

also satisfies a constant Neumann condition over each boundary component for some non-zero and non-constant $v \in C^{2,\alpha}(\partial\Omega_\lambda)$. We point out that in the present case, the Krasnoselskii Bifurcation

Theorem does not provide the orthogonality condition (3.13) which ensures that that the bifurcating perturbations v are non-trivial, and this forces us to work orthogonal to the functions which are locally constant on $\partial\Omega_\lambda$ from the start. We denote the function spaces that satisfy this property by a subscript-m. For instance,

$$C^{k,\alpha}_{G,m}(\partial\Omega_\lambda) := \left\{ v \in C^{2,\alpha}_G(\partial\Omega_\lambda) : \int_{\Gamma_r} v\, dS = 0, \quad r = 1, \lambda \right\},$$

and the spaces $C^{k,\alpha}_{G,m}$ and $H^k_{G,m}$ over Ω_λ and $\partial\Omega_\lambda$ are analogously defined. We also set $H^{1,*}_{G,m} := H^1_{G,m} \cap H^{1,*}$.

Suppose the Dirichlet problem (4.20) has a unique G-invariant positive solution $u_\lambda(v)$ for $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ with sufficiently small norm, and that $u_\lambda(0) = u_\lambda$. Then, for $U \subset C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ a sufficiently small neighbourhood of the zero function, we define the operator $F_\lambda : U \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, by the component-wise formula

$$F_\lambda(v)|_{\Gamma_r} := \left.\frac{\partial u_\lambda(v)}{\partial \nu}\right|_{\Gamma_r} - \fint_{\Gamma_r} \frac{\partial u_\lambda(v)}{\partial \nu}\, dS, \quad r = 1, \lambda, \tag{4.21}$$

where we are identifying $\partial\Omega^v_\lambda$ with $\partial\Omega_\lambda$, $\fint_\Gamma$ denotes the mean value over Γ. The Schauder regularity theory [GT15, Gri11] implies F_λ is well defined for appropriate $\alpha \in (0,1)$. Therefore, the solution $u_\lambda(v)$ for the Dirichlet problem (4.20) solves the overdetermined problem (4.1) if and only if

$$F_\lambda(v) = 0. \tag{4.22}$$

As before, we want to find non-trivial solutions (λ, v) to the equation (4.22) bifurcating from the trivial branch $(\lambda, 0)$. Thus, we are let to study the linearizarion $L_\lambda := dF_\lambda|_{v=0}$ and how its kernel depends on the parameter λ. In Proposition 4.9 we compute

$$L_\lambda(w) = -\frac{\partial \psi_w}{\partial \nu} + H_\lambda w, \quad w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda),$$

where $\psi_w \in C^{2,\alpha}_G(\overline{\Omega_\lambda})$ is the solution to (4.18) given by Lemma 4.5, and H_λ denotes the mean curvature of $\partial\Omega_\lambda$ with respect to the *inner* normal, that is

$$H_\lambda(x) = \begin{cases} n-1, & |x| = 1, \\ -\frac{n-1}{\lambda}, & |x| = \lambda. \end{cases}$$

Since we lack the explicit formulas as we had in Section 3.3 in Chapter 3, we instead focus on the *first* eigenvalue of L_λ, which we call μ_λ, and study it through an associated quadratic form $Q_\lambda : H^{1,*}_{G,m}(\Omega_\lambda) \to \mathbb{R}$, given by

$$Q_\lambda(\psi) := \int_{\Omega_\lambda} |\nabla\psi|^2 - f'(u_\lambda)\psi^2\, dx + \int_{\partial\Omega_\lambda} H_\lambda \psi^2\, dS.$$

In Proposition 4.11 we derive a variational characterization for μ_λ in terms of Q_λ, which allows us to prove in Proposition 4.16 that there is a critical value $\lambda^* \in (0, \Lambda)$ at which μ_λ crosses 0.

Since the operator F_λ defined in (4.21) turns out to be inappropriate to apply the Krasnoselskii Bifurcation Theorem to it, in Section 4.5 we define a new operator $R_\lambda : U \subset C^{1,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$ which has the form

$$R_\lambda = \mathrm{id} - K_\lambda,$$

where K_λ is a compact operator, and such that $R_\lambda(v) = 0$ if and only if $F_\lambda(v) = 0$. Thus, $u_\lambda(v)$ solves the overdetermined problem (4.1) if and only if $R_\lambda(v) = 0$. For this reformulation, we study the invertibility of the family of operators $L_\lambda + \mu\,\mathrm{id}$, where $\mu > -\mu_\lambda$ is a parameter (Lemma 4.17). In Lemma 4.19 we show that the eigenvalues of the linearization $dR_\lambda|_{v=0}$ can be computed in terms of the eigenvalues of L_λ, and in particular that the number of negative eigenvalues of both operators is the same, and in Lemma 4.20 we show the existence of a subinterval $[\Lambda_1, \Lambda_2] \subset (0, \Lambda)$ that contains the critical value λ^*, such that $dR_{\Lambda_1}|_{v=0}$ is an isomorphism with no negative eigenvalues, and $dR_{\Lambda_2}|_{v=0}$ is an isomorphism with exactly one negative eigenvalue. Therefore, we can apply the Krasnoselskii Bifurcation Theorem A.2 to the family of operators R_λ, $\lambda \in [\Lambda_1, \Lambda_2]$, and show the refined version of Theorem 4.1 below.

Theorem 4.6. *Let $n \geq 2$, $\kappa \geq 0$, $1 < p < \frac{n+2}{n-2}$ and $0 < \alpha < \min\{1, p-1\}$. Let $G \leqslant O(n)$ satisfy (P1')-(P2'). Then there exists a subinterval $[\Lambda_1, \Lambda_2] \subset (0, \Lambda)$ depending on p, κ, and i_1 with the following property: for every $\varepsilon > 0$ there exists $\lambda \in (\Lambda_1, \Lambda_2)$ and a non-zero $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ with $\|v\|_{C^{2,\alpha}(\partial\Omega_\lambda)} < \varepsilon$, such that the overdetermined problem*

$$\begin{cases} -\Delta u + \kappa u = u^p, \quad u > 0 & \text{in} \quad \Omega^v_\lambda, \\ u = 0, \quad u_\nu = \mathit{const} & \text{on} \quad \Gamma^v_1, \\ u = 0, \quad u_\nu = \mathit{const} & \text{on} \quad \Gamma^v_\lambda, \end{cases} \tag{4.23}$$

admits a positive solution $u \in C^{2,\alpha}_G(\overline{\Omega_\lambda})$.

Let us show how to deduce Theorem 4.1 from the above Theorem 4.6.

Proof of Theorem 4.1. Fix $\kappa \geq 0$, $1 < p < \frac{n+2}{n-2}$ and $0 < \alpha < \min\{1, p-1\}$. We only need to explain why the domains Ω^v_λ given by Theorem 4.6 are different from standard annuli.

Note that the G-invariance of the functions v imply the G-invariance of the domains Ω^v_λ. Also, since a non-zero $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ has zero mean over each boundary component Γ_1, Γ_λ, then v must be non-constant, and thus at least one of the boundary components Γ^v_1 or Γ^v_λ is different from a dilation of the sphere $\mathbb{S}^{n-1}$ with respect to the origin. In addition, property (P1) of the group G rules out that Ω^v_λ is a mere translation of the annulus Ω_λ. All this together implies the non-triviality of Ω^v_λ. $\square$

4.3 Reformulating the problem and deriving its linearization

Fix $0 < \alpha < \min\{1, p-1\}$. As before, we pullback the problem (4.20) to the fixed domain Ω_λ. For $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ consider the diffeomorphism $\Phi = \Phi_\lambda(v) : \Omega_\lambda \to \Omega^v_\lambda$ as defined in (3.14) in polar coordinates

$$\Phi(r,\theta) = \big(\big(1 + \eta_1(r)v(\lambda\theta) + \eta_2(r)v(\theta)\big)r, \theta\big)\,, \tag{4.24}$$

with the slight modification that now η_1, η_2 are smooth functions satisfying

$$\eta_1(r) = \begin{cases} (\lambda u'_\lambda(\lambda))^{-1} & \text{if} \quad r \le \lambda + \delta, \\ 0 & \text{if} \quad r \ge \lambda + 2\delta, \end{cases} \qquad \eta_2(r) = \begin{cases} u'_\lambda(1)^{-1} & \text{if} \quad r \ge 1 - \delta, \\ 0 & \text{if} \quad r \le 1 - 2\delta, \end{cases} \tag{4.25}$$

for some small enough $\delta > 0$. We set on Ω_λ the pull-back metric $g = g_\lambda(v) := \Phi^* g_0$ of the Euclidean metric g_0 on Ω^v_λ. Recall that near $\partial\Omega_\lambda$, the metric g equals

$$g = (1 + \eta_j v)^2 dr^2 + 2r\eta_j\,(1 + \eta_j v) dr dv + r^2 \eta_j^2 dv^2 + r^2 (1 + \eta_j v)^2 g_{\mathbb{S}^{n-1}}, \tag{4.26}$$

Now consider the problem

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{in} \quad \Omega_\lambda, \\ u = 0 & \text{on} \quad \partial\Omega_\lambda. \end{cases} \tag{4.27}$$

where Δ_g is the Laplace-Beltrami operator for the metric g in Ω_λ. Suppose the Dirichlet problem (4.20) has a unique G-invariant positive solution $u_\lambda(v)$ for $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ with sufficiently small norm. Then, the pullbacked problem (4.27) has a unique G-invariant solution given by $u^*_\lambda(v) := \Phi^* u_\lambda(v)$. For $U \subset C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ a sufficiently small neighbourhood of the zero function, we define the operator $F_\lambda : U \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, by the component-wise formula

$$F_\lambda(v)|_{\Gamma_r} := \left.\frac{\partial u^*_\lambda(v)}{\partial \nu^*}\right|_{\Gamma_r} - \!\!\!\!\!\!\int_{\Gamma_r} \frac{\partial u^*_\lambda(v)}{\partial \nu^*}\, dS, \quad r = 1, \lambda, \tag{4.28}$$

where $\nu^* := \Phi^*\nu$ is the inner unit normal to $\partial\Omega_\lambda$ with respect to the metric g, $\partial_{\nu^*} u^*_\lambda(v) := g(\nabla_g u^*_\lambda(v), \nu^*)$, and dS denotes the *standard* surface measure on $\partial\Omega_\lambda$. Then, as before, $F_\lambda(0) = 0$ for all $\lambda \in (0,1)$ and $F_\lambda(v) = 0$ if and only if $\partial_{\nu^*} u^*_\lambda(v)$ is constant over each boundary component Γ_1, Γ_λ and $u_\lambda(v)$ solves the overdetermined problem (4.1) (cf. Section 3.2 in Chapter 3).

As a consequence of Lemma 4.7 below, F_λ is a well defined C^1 mapping which depends C^1 on the parameter $\lambda \in (0,1)$, and $F_\lambda(0) = 0$ for all $\lambda \in (0, \Lambda)$.

Lemma 4.7. *Let $\lambda \in (0, \Lambda)$. For each $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ of sufficiently small norm there exists a unique positive solution $u_\lambda(v) \in C^{2,\alpha}_G(\overline{\Omega^v_\lambda})$ to the Dirichlet problem (4.20). Moreover, the mapping $(\lambda, v) \mapsto u_\lambda(v)$ is smooth and $u_\lambda(0) = u_\lambda$.*

Proof. Fix $\lambda_0 \in (0,\Lambda)$ and consider the diffeomorphism $\tilde{\Phi} = \tilde{\Phi}_\lambda(v) : \Omega_{\lambda_0} \to \Omega_\lambda^v$ defined in polar coordinates as

$$\tilde{\Phi}(r,\theta) = \Phi\left(\frac{1-\lambda}{1-\lambda_0}r + \frac{\lambda-\lambda_0}{1-\lambda_0}, \theta\right), \quad r \in (\lambda, 1), \quad \theta \in \mathbb{S}^{n-1},$$

where $\Phi : \Omega_\lambda \to \Omega_\lambda^v$ is the diffeomorphism (4.24). Let $\tilde{g} = \tilde{g}_\lambda(v)$ be the pullback of the euclidean metric in Ω_λ^v by $\tilde{\Phi}$. Note that when $\lambda = \lambda_0$, $v = 0$ then $\tilde{\Phi} = \Phi$ is the identity on Ω_λ, and so $\tilde{g}_{\lambda_0}(0)$ is nothing but the euclidean metric in Ω_{λ_0}. Then the problem (4.27) is equivalent to

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{in} \quad \Omega_{\lambda_0}, \\ u = 0 & \text{on} \quad \partial\Omega_{\lambda_0}. \end{cases} \tag{4.29}$$

Now, consider the mapping $S : C^{2,\alpha}_{0,G}(\Omega_{\lambda_0}) \times U \times I \to C^{0,\alpha}_G(\overline{\Omega_{\lambda_0}})$, where $U \times I \subset C^{2,\alpha}_{G,m}(\partial\Omega_{\lambda_0}) \times (0,\Lambda)$ is a small enough neighbourhood of $(0,\lambda_0)$, defined by

$$S(\psi, v, \lambda) := -\Delta_g(u_{\lambda_0} + \psi) + \kappa(u_{\lambda_0} + \psi) - (u_{\lambda_0} + \psi)_+^p.$$

Here $\phi_+ := \max\{\phi, 0\}$ denotes the positive part of ϕ. Then S is a C^1 mapping. We compute the partial differential

$$\partial_\psi S|_{(0,0,\lambda_0)}(\psi_0) = -\Delta\psi_0 + \kappa\psi_0 - p u_{\lambda_0}^{p-1}\psi_0,$$

which by Lemma 4.5 is an isomorphism between the spaces $C^{2,\alpha}_{0,G}(\Omega_{\lambda_0}) \to C^{0,\alpha}_G(\overline{\Omega_{\lambda_0}})$. Therefore, we can apply the Implicit Function Theorem to conclude that for $(v,\lambda) \in C^{2,\alpha}_{G,m}(\partial\Omega_{\lambda_0}) \times (0,\Lambda)$ in a neighbourhood of $(0,\lambda_0)$ that we still denote $U \times I$ there exists a unique $\psi(v,\lambda) \in C^{2,\alpha}_{0,G}(\Omega_{\lambda_0})$ such that

$$S(\psi(v,\lambda), v, \lambda) = 0, \quad \psi(0,\lambda_0) = 0,$$

and the mapping $(v,\lambda) \to \psi(v,\lambda)$ is in C^1. Thus, $u = u_\lambda(v) := (u_{\lambda_0} + \psi(v,\lambda)) \circ \tilde{\Phi}^{-1} \in C^{2,\alpha}_{0,G}(\overline{\Omega_\lambda^v})$ is a solution to

$$-\Delta u + \kappa u - u_+^p = 0 \quad \text{in} \quad \Omega_\lambda^v$$

and since $-\Delta u + \kappa u = u_+^p \geq 0$, by the maximum principle we have either $u > 0$ or $u \equiv 0$ in Ω_λ^v. Since $\psi(0,\lambda_0) = 0$ we have $u_{\lambda_0}(0) = u_{\lambda_0} > 0$, by continuity, possibly after shrinking $U \times I$ a bit we can rule out $u_\lambda(v) \equiv 0$. Only remains to show that $u_\lambda(0) = u_\lambda$. Note that $u_\lambda(0) \circ \tilde{\Phi} = u_{\lambda_0} + \psi(0,\lambda)$ is a solution to problem (4.29), but $u_\lambda \circ \tilde{\Phi}$ is also a solution. By the continuity of $\lambda \mapsto u_\lambda$ we have, maybe after taking a smaller interval I, that

$$u_\lambda \circ \tilde{\Phi}_\lambda(0) - u_{\lambda_0} \in U, \quad \text{for all} \quad \lambda \in I.$$

Hence, by the uniqueness of $\psi(v,\lambda)$ we have $\psi(0,\lambda) = u_\lambda \circ \tilde{\Phi}_\lambda(0) - u_{\lambda_0}$ and thus $u_\lambda(0) = u_\lambda$.

Because the above argument is true for every λ_0, the proposition follows. □

As F_λ is of class C^1, it has a well defined and bounded linearization at $v = 0$,

$$L_\lambda := dF_\lambda|_{v=0} : C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda). \tag{4.30}$$

In what follows, we compute the L_λ. The computation largely follows the one in the previous chapter, so we only point out how to adapt it to this situation. But first, we will need the following lemma.

Lemma 4.8. *Let $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ and $\psi = \psi_w \in C^{2,\alpha}_{G}(\overline{\Omega_\lambda})$ a solution of (4.18). Then*

$$\int_{\Omega_\lambda} \psi z_\lambda \, dx = 0, \qquad \int_{\Gamma_r} \frac{\partial \psi}{\partial \nu} \, dS = 0, \quad r = 1, \lambda.$$

Proof. By Green's formula and equations (4.10) and (4.18) we have

$$\tau_\lambda \int_{\Omega_\lambda} \psi z_\lambda \, dx = \int_{\Omega_\lambda} z_\lambda \Delta\psi - \psi\Delta z_\lambda \, dx = \int_{\partial\Omega_\lambda} \psi \frac{\partial z_\lambda}{\partial\nu} - z_\lambda \frac{\partial\psi}{\partial\nu} \, dS = \int_{\partial\Omega_\lambda} \psi \frac{\partial z_\lambda}{\partial\nu} \, dS,$$

where in the last equality we used $z_\lambda = 0$ on $\partial\Omega_\lambda$. Now, because z_λ is radially symmetric, we have that $\frac{\partial z_\lambda}{\partial\nu}$ is constant over each component of $\partial\Omega_\lambda$, and since ψ has zero mean over each component of $\partial\Omega_\lambda$, the first identity follows.

Note that, by Lemma 4.5 and Proposition 4.2, for all $\lambda \in (0, 1)$ there exist unique radially symmetric solutions $\zeta_1, \zeta_2 \in C^{2,\alpha}(\overline{\Omega_\lambda})$ which are solutions to

$$\Delta\zeta + f'(u_\lambda)\zeta = 0 \quad \text{in} \quad \Omega_\lambda, \tag{4.31}$$

with boundary conditions

$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \in \Gamma_\lambda, \\ 1 & \text{if } x \in \Gamma_1, \end{cases} \qquad \zeta_2(x) = \begin{cases} 1 & \text{if } x \in \Gamma_\lambda, \\ 0 & \text{if } x \in \Gamma_1. \end{cases} \tag{4.32}$$

Again, by the Green identity we have and the fact that $\frac{\partial\zeta_j}{\partial\nu}$ is constant on each component of $\partial\Omega_\lambda$ we see

$$\int_{\partial\Omega_\lambda} \psi \frac{\partial\zeta_j}{\partial\nu} - \zeta_j \frac{\partial\psi}{\partial\nu} \, dS = \int_{\partial\Omega_\lambda} \zeta_j \frac{\partial\psi}{\partial\nu} \, dS = 0, \quad j = 1, 2.$$

The second identity follows. □

Proposition 4.9. *For $\lambda \in (0, \Lambda)$, the linear operator L_λ defined in (4.30) is given by*

$$L_\lambda(w) = -\frac{\partial\psi_w}{\partial\nu} + H_\lambda w, \quad w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda), \tag{4.33}$$

where $\psi_w \in C^{2,\alpha}_{G}(\overline{\Omega_\lambda})$ is the solution to (4.18) given by Lemma 4.5.

Proof. As F_λ is a C^1 operator, L_λ is given by the directional derivative

$$L_\lambda(w) = \lim_{t\to 0} \frac{F_\lambda(tw)}{t}.$$

As before, write $v = tw$ for small t and consider the diffeomorphism $\Phi = \Phi_t$ defined in (4.24) and the induced metric $g = g_t$ on Ω_λ. Let $u^*_\lambda(v) = u_t$ be the solution of the Dirichlet problem (4.27) in Ω_λ, which depends C^1 on the parameter t, and recall that $u_0 = u_\lambda$. Note that u_λ is a can be extended by (4.4) to solve $\Delta_{g_0} u_\lambda + f(u_\lambda) = 0$ in a neighbourhood of Ω_λ. Then $u^*_\lambda := \Phi^*_t u_\lambda$ is well defined for small t and solves the equation

$$\Delta_{g_t} u + f(u) = 0 \quad \text{in} \quad \Omega_\lambda.$$

Expanding $u^*_\lambda = u^*_\lambda(r,\theta)$ in a neighbourhood of $\partial\Omega_\lambda$ to first order in t, we obtain

$$u^*_\lambda(r,\theta) = u_\lambda(r + tr\eta_j w) = u_\lambda(r) + tr\eta_j w \frac{\partial u_\lambda}{\partial r} + O(t^2), \tag{4.34}$$

where η_1, η_2 are the functions defined in (4.25). Let $\psi_t := u_t - u^*_\lambda$. Then $\psi_t \in C^{2,\alpha}_G(\overline{\Omega}_\lambda)$ is a solution of

$$\begin{cases} \Delta_{g_t}\psi - f(u_t) + f(u^*_\lambda) = 0 & \text{in} \quad \Omega_\lambda, \\ \psi = -u^*_\lambda & \text{on} \quad \partial\Omega_\lambda, \end{cases} \tag{4.35}$$

which depends C^1 on t, with $\psi_0 = 0$. Set $\dot\psi := \frac{d}{dt}\psi_t\big|_{t=0}$, then from (4.34) we see

$$\dot\psi = -r\eta_j w \frac{\partial u_\lambda}{\partial r} = -w, \quad \text{on} \quad \partial\Omega_\lambda.$$

Then, taking into account (4.34) we can differentiate (4.35) at $t = 0$ to obtain

$$\begin{cases} \Delta\dot\psi + f'(u_\lambda)\dot\psi = 0, & \text{in} \quad \Omega_\lambda, \\ \dot\psi = -w & \text{on} \quad \partial\Omega_\lambda, \end{cases}$$

and so $\dot\psi = -\psi_w$. Now, given that $\psi_t = t\dot\psi + O(t^2)$, we have in a neighbourhood of $\partial\Omega_\lambda$

$$u_t = u_\lambda + t\left(-\psi_w + r\eta_j w \frac{\partial u_\lambda}{\partial r}\right) + O(t^2).$$

Recall that $\nu^* = \nu_t$ denotes the *inner* unit normal field to $\partial\Omega_\lambda$ with respect to the metric g_t. The computation that follows reproduces verbatim from the same stage in Proposition 3.6 in the previous chapter, and thus we arrive at

$$\frac{\partial u_t}{\partial \nu_t} = \frac{\partial u_\lambda}{\partial \nu} + t\left(-\frac{\partial \psi_w}{\partial \nu} + \operatorname{sgn}(\partial_r u_\lambda) r\eta_j w \frac{\partial^2 u_\lambda}{\partial r^2}\right) + O(t^2) \quad \text{on} \quad \partial\Omega_\lambda. \tag{4.36}$$

Note that $\partial^2_r u_\lambda = -\frac{n-1}{r}\partial_r u_\lambda$ on $\partial\Omega_\lambda$, and therefore $\operatorname{sgn}(\partial_r u_\lambda) r\eta_j \partial^2_r u_\lambda = H_\lambda$ on $\partial\Omega_\lambda$. For the last step, we note that since $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$, from Lemma 4.8 and the fact that $\partial_r u_\lambda$ is constant on Γ_r, $r = 1, \lambda$, follows that

$$⨍_{\Gamma_r} \frac{\partial u_t}{\partial \nu_t}\, dS = ⨍_{\Gamma_r} \frac{\partial u_\lambda}{\partial \nu}\, dS + t ⨍_{\Gamma_r} \left(-\frac{\partial \psi_w}{\partial \nu} + H_\lambda w\right) dS + O(t^2) = \frac{\partial u_\lambda}{\partial \nu} + O(t^2). \tag{4.37}$$

We subtract (4.37) from (4.36) to find

$$F_\lambda(tw) = t\left(-\frac{\partial \psi_w}{\partial \nu} + H_\lambda w\right) + O(t^2).$$

Then, formula (4.33) for $L_\lambda(w)$ follows at once. $\square$

4.4 The linearized operator and its first eigenvalue

In this section we study the linearized operator L_λ. We won't give a detailed account of the spectrum of L_λ as we did in Section 3.3 in Chapter 3, since this is not necessary for our purposes now. Instead, we will focus on the *first* eigenvalue of L_λ and show that it crosses 0 at some critical value $\lambda^* \in (0,1)$. Our methods here are mostly of variational nature, and we will need first to make a couple of definitions before we proceed.

Define the bilinear form associated to the operator L_λ as

$$b_\lambda(w_1,w_2) := \int_{\partial\Omega_\lambda} w_1 L_\lambda(w_2)\, dS = \int_{\partial\Omega_\lambda} -\psi_1 \frac{\partial \psi_2}{\partial \nu} + H_\lambda w_1 w_2 \, dS, \quad w_1,w_2 \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda), \tag{4.38}$$

where $\psi_j = \psi_{w_j}$, $j=1,2$, and its corresponding quadratic form

$$q_\lambda(w) := b_\lambda(w,w), \quad w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda).$$

An integration by parts in (4.38) using equation (4.18) and the observation $\psi_j|_{\partial\Omega_\lambda} = w_j$ shows that

$$b_\lambda(w_1,w_2) = \int_{\Omega_\lambda} \nabla\psi_1 \cdot \nabla\psi_2 - f'(u_\lambda)\psi_1\psi_2 \, dx + \int_{\partial\Omega_\lambda} H_\lambda \psi_1\psi_2 \, dS. \tag{4.39}$$

Therefore b_λ is a symmetric bilinear form. Note that by elliptic estimates (see [Gri11, LM12]) we have

$$|b_\lambda(w_1,w_2)| \le C\|w_1\|_{H^{1/2}(\partial\Omega_\lambda)}\|w_2\|_{H^{1/2}(\partial\Omega_\lambda)}$$

for some constant $C>0$ independent of w_1, w_2. Thus, the bilinear form b_λ lets us extend the operator L_λ to the spaces $H^{1/2}_{G,m}(\partial\Omega_\lambda) \to H^{-1/2}_{G,m}(\partial\Omega_\lambda)$ defining

$$\langle L_\lambda(w_1); w_2\rangle := b_\lambda(w_1,w_2), \quad w_1,w_2 \in H^{1/2}_{G,m}(\partial\Omega_\lambda). \tag{4.40}$$

where $\langle\cdot;\cdot\rangle$ denotes the duality pairing between $H^{1/2}_{G,m}(\partial\Omega_\lambda)$ and $H^{-1/2}_{G,m}(\partial\Omega_\lambda)$. We also define the quadratic form $Q_\lambda : H^1_G(\Omega_\lambda) \to \mathbb{R}$ by

$$Q_\lambda(\psi) := \int_{\Omega_\lambda} |\nabla\psi|^2 - f'(u_\lambda)\psi^2 \, dx + \int_{\partial\Omega_\lambda} H_\lambda \psi^2 \, dS. \tag{4.41}$$

Thus, formula (4.39) implies that

$$q_\lambda(w) = Q_\lambda(\psi_w), \quad w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda). \tag{4.42}$$

Since b_λ is symmetric, the spectrum of $L_\lambda : H^{1/2}_{G,m}(\partial\Omega_\lambda) \to H^{-1/2}_{G,m}(\partial\Omega_\lambda)$ is real. We say that $\mu\in\mathbb{R}$ is an eigenvalue of L_λ if there exists $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ such that

$$L_\lambda w = \mu w.$$

We are interested in establishing whether there exist $\lambda \in (0,\Lambda)$ such that the kernel of L_λ is non-trivial, or equivalently if $\mu = 0$ is an eigenvalue. For this purpose, we first characterize the eigenvalues of L_λ in terms of functions $\psi \in C^{2,\alpha}_G(\overline{\Omega_\lambda})$ rather than functions w defined on the boundary $\partial\Omega_\lambda$.

Lemma 4.10. *Let $\lambda \in (0, \Lambda)$. Then $\mu \in \mathbb{R}$ is an eigenvalue of L_λ if and only if there exists a non-zero solution $\psi \in C^{2,\alpha}_{G,m}(\overline{\Omega_\lambda})$ to the problem*

$$\begin{cases} \Delta\psi + f'(u_\lambda)\psi = 0 & \text{in} \quad \Omega_\lambda, \\ -\psi_\nu + H_\lambda\psi = \mu\psi & \text{on} \quad \partial\Omega_\lambda. \end{cases} \tag{4.43}$$

Proof. If $\mu \in \mathbb{R}$ is an eigenvalue of L_λ, then there exists a non-zero $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ such that $L_\lambda w = \mu w$. Therefore, the function $\psi_w \in C^{2,\alpha}_{G,m}(\overline{\Omega_\lambda})$ in Proposition 4.9 is also a solution of (4.43). On the other hand, let $\psi \in C^{2,\alpha}_{G,m}(\Omega_\lambda)$ be a non-zero solution of (4.43) for some $\mu \in \mathbb{R}$, and set $w := \psi|_{\partial\Omega_\lambda}$. Then $w \neq 0$ as $\lambda < \Lambda$ and so it is an eigenfunction of L_λ. □

In the next proposition we give a variational characterization of the smallest possible eigenvalue μ_λ of L_λ, which we call the *first eigenvalue.*

Proposition 4.11. *Let $\lambda \in (0, \Lambda)$, and define*

$$\mu_\lambda := \inf_{\substack{\psi \in H^{1,*}_{G,m}(\Omega_\lambda) \\ \|\psi\|_{L^2(\partial\Omega_\lambda)}=1}} Q_\lambda(\psi). \tag{4.44}$$

Then μ_λ is finite and achieved at a function $\psi_\lambda \in C^{2,\alpha}_{G,m}(\overline{\Omega_\lambda})$ which is a solution to problem (4.43) with $\mu = \mu_\lambda$. Thus, μ_λ is an eigenvalue of L_λ. Moreover, if $\mu \in \mathbb{R}$ is another eigenvalue of L_λ then $\mu \geq \mu_\lambda$.

Proof. Let ψ_k be a minimizing sequence for μ_λ. If $\{\psi_k\}$ is unbounded in the H^1-norm, then define $\phi_k := \psi_k / \|\psi_k\|_{H^1}$. Thus $\|\phi_k\|_{L^2(\partial\Omega_\lambda)} \to 0$, and so we can assume $\phi_k \rightharpoonup \phi$ weakly in $H^1(\Omega_\lambda)$ to some $\phi \in H^{1,*}_{0,G}(\Omega_\lambda)$. As Q_λ is weakly sequentially lower semi-continuous and $\lambda \in (0, \Lambda)$ we have

$$\liminf_{k\to\infty} Q_\lambda(\phi_k) \geq Q_\lambda(\phi) \geq 0,$$

with $Q_\lambda(\phi) > 0$ if $\phi \neq 0$. Suppose first that $\phi \neq 0$, then

$$\liminf_{k\to\infty} Q_\lambda(\psi_k) = \liminf_{k\to\infty} \|\psi_k\|^2_{H^1(\Omega_\lambda)} Q_\lambda(\phi_k) = +\infty,$$

a contradiction. If $\phi = 0$ then, since $\phi_k \to \phi$ strongly in $L^2(\Omega_\lambda)$, we have

$$\liminf_{k\to\infty} Q_\lambda(\phi_k) \geq \liminf_{k\to\infty} \int_{\Omega_\lambda} |\nabla\phi_k|^2 + |\phi_k|^2 \, dx - \liminf_{k\to\infty} \int_{\Omega_\lambda} (f'(u_\lambda) + 1)|\phi_k|^2 \, dx = 1$$

and therefore

$$\liminf_{k\to\infty} Q_\lambda(\psi_k) = \liminf_{k\to\infty} \|\psi_k\|^2_{H^1} Q_\lambda(\phi_k) = +\infty,$$

which again is impossible. Hence, the sequence ψ_k is bounded in the H^1-norm, and without loss of generality we may assume $\psi \rightharpoonup \psi$ weakly in $H^1(\Omega_\lambda)$. By weak lower semi-continuity of Q_λ there holds

$$\mu_\lambda = \liminf_{k\to\infty} Q_\lambda(\psi_k) \geq Q_\lambda(\psi) \geq \mu_\lambda.$$

Therefore, μ_λ is finite and achieved at ψ. Now, there exist Lagrange multipliers α_j, $j = 1, \dots, 4$ such that for every $\zeta \in H^1_G(\Omega_\lambda)$

$$\int_{\Omega_\lambda} \nabla\psi \cdot \nabla\zeta - f'(u_\lambda)\psi\zeta \, dx + \int_{\partial\Omega_\lambda} H_\lambda \psi\zeta \, dS = \\ = \alpha_1 \int_{\Omega_\lambda} z_\lambda \zeta \, dx + \alpha_2 \int_{\partial\Omega_\lambda} \psi\zeta \, dS + \alpha_3 \int_{\Gamma_1} \zeta \, dS + \alpha_4 \int_{\Gamma_\lambda} \zeta \, dS. \quad (4.45)$$

Plugging in $\zeta = z_\lambda$ we find $\alpha_1 = 0$, plugging $\zeta = \psi$ we see $\alpha_2 = \mu_\lambda$. With the functions $\zeta = \zeta_j$ defined in (4.31)-(4.32) we find $\alpha_3, \alpha_4 = 0$. Separating variables, by the same arguments held in Lemma 4.5 we see the identity (4.45) holds for every $\zeta \in H^1(\Omega_\lambda)$. Thus, ψ is a weak solution to

$$\begin{cases} \Delta\psi + f'(u_\lambda)\psi = 0 & \text{in} \quad \Omega_\lambda, \\ -\psi_\nu + H_\lambda\psi = \mu_\lambda\psi & \text{on} \quad \partial\Omega_\lambda. \end{cases}$$

By regularity theory for the Neumann problem, $\psi \in C^{2,\alpha}_G(\overline{\Omega_\lambda})$. That μ_λ is an eigenvalue of L_λ follows from Lemma 4.11. Finally, if $\mu \in \mathbb{R}$ is another eigenvalue of L_λ, then by (4.38) and (4.42) we have

$$\mu \int_{\partial\Omega_\lambda} w^2 \, dS = b_\lambda(w, w) = Q_\lambda(\psi_w)$$

for some non-zero $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$. Then $\mu \geq \mu_\lambda$ follows from (4.44). $\square$

By the previous proposition, we see that the behaviour of μ_λ, and in particular whether it changes sign at some λ, is completely determined by the quadratic form Q_λ. Since $H^{1,*}_{0,G}(\Omega_\lambda) \subset H^{1,*}_{G,m}(\Omega_\lambda)$ and $Q_\lambda(\psi) = \tilde{Q}_\lambda(\psi)$ for all $\psi \in H^1_{0,G}(\Omega_\lambda)$, by Lemma 4.3 there exist a $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$ such that $Q_\lambda(\psi) < 0$ if λ is sufficiently close to 1. We thus want to show Q_λ is positive in the space $H^{1,*}_{G,m}(\Omega_\lambda)$ for λ close to 0, and therefore conclude $\mu_\lambda = 0$ for some critical value of λ. This will need some preparation, starting with the following lemma.

Lemma 4.12. *Let $\lambda \in (0, 1)$. Then for all $\psi \in H^1_{G,m}(\Omega_\lambda)$ we have the inequality*

$$\frac{1}{\lambda} \int_{\Gamma_\lambda} |\psi|^2 \, dS - \int_{\Gamma_1} |\psi|^2 \, dS \leq \frac{1}{n} \int_{\Omega_\lambda} |\nabla\psi|^2 \, dx \quad (4.46)$$

Proof. Let $\psi \in H^1_{G,m}(\Omega_\lambda)$. We decompose ψ into its Fourier series

$$\psi(r, \theta) = \varphi_0(r) + \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \varphi_{k,j}(r) Y_{k,j}(\theta), \quad (4.47)$$

Since ψ has zero mean over each component of $\partial\Omega_\lambda$ we have $\varphi_0(\lambda) = 0 = \varphi_0(1)$. Thus inequality (4.46) reads

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \lambda^{n-2} \varphi_{k,j}(\lambda)^2 - \varphi_{k,j}(1)^2 \leq \frac{1}{n} \int_\lambda^1 r^{n-1} \varphi_0'(r)^2 \, dr \\ + \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \int_\lambda^1 r^{n-1} \varphi_{k,j}'(r)^2 + \sigma_k r^{n-3} \varphi_{k,j}(r)^2 \, dr.$$

Thus, it is sufficient to prove for each $\sigma = \sigma_k$ and $\varphi = \varphi_{k,j}$, $k \in \mathbb{N}$, the inequality

$$\lambda^{n-2}\varphi(\lambda)^2 - \varphi(1)^2 \leq \frac{1}{n}\int_\lambda^1 r^{n-1}\varphi'(r)^2 + \sigma r^{n-3}\varphi(r)^2\, dr. \tag{4.48}$$

We write the left hand side in the above equation as

$$\begin{aligned}\lambda^{n-2}\varphi(\lambda)^2 - \varphi(1)^2 &= -\int_\lambda^1 (r^{n-2}\varphi(r)^2)'\, dr \\ &= -\int_\lambda^1 2r^{n-2}\varphi(r)\varphi'(r)\, dr + (2-n)\int_\lambda^1 r^{n-3}\varphi(r)^2\, dr. \end{aligned}\tag{4.49}$$

Now, by the Arithmetic mean - Geometric mean inequality the first term in the right hand side of (4.49) can be bounded as

$$\begin{aligned}\int_\lambda^1 2r^{n-2}|\varphi(r)\varphi'(r)|\, dr = \int_\lambda^1 2\sqrt{n}r^{(n-3)/2}|\varphi(r)|\frac{r^{(n-1)/2}}{\sqrt{n}}|\varphi'(r)|\, dr \leq \\ \leq \int_\lambda^1 nr^{n-3}\varphi(r)^2 + \frac{r^{n-1}}{n}\varphi'(r)^2\, dr. \end{aligned}\tag{4.50}$$

Therefore, pluging (4.50) in (4.49) yields

$$\begin{aligned}\lambda^{n-2}\varphi(\lambda)^2 - \varphi(1)^2 \leq \int_\lambda^1 nr^{n-3}\varphi(r)^2 + \frac{r^{n-1}}{n}\varphi'(r)^2\, dr + (2-n)\int_\lambda^1 r^{n-3}\varphi(r)^2\, dr = \\ = \frac{1}{n}\int_\lambda^1 r^{n-1}\varphi'(r)^2 + 2nr^{n-3}\varphi(r)^2\, dr. \end{aligned}\tag{4.51}$$

Taking into account that by G-invariance $\sigma \geq \sigma_1 \geq 2n$, the inequality in (4.48) follows from (4.51). The lemma is now proved. $\square$

To study the sign change of the first eigenvalue μ_λ, we will need to consider a related, more tractable quantity, which we define in the lemma below.

Lemma 4.13. *For every $\lambda \in (0,1)$ the infimum*

$$\chi_\lambda := \inf_{\substack{\psi \in H^{1,*}_{G,m}(\Omega_\lambda) \\ \|\psi\|_{L^2(\Omega_\lambda)}=1}} Q_\lambda(\psi) \tag{4.52}$$

is finite and attained. Moreover, a minimizer for χ_λ is a non-zero function $\psi_\lambda \in H^{1,*}_{G,m}(\Omega_\lambda) \cap C^{2,\alpha}(\overline{\Omega_\lambda})$ which is a solution to

$$\begin{cases} -\Delta\psi + \kappa\psi - pu_\lambda^{p-1}\psi = \chi_\lambda\psi & \text{in} \quad \Omega_\lambda, \\ \qquad -\psi_\nu + H_\lambda\psi = 0 & \text{on} \quad \partial\Omega_\lambda. \end{cases} \tag{4.53}$$

and we have the bound

$$\chi_\lambda \geq \kappa - p\sup_{\Omega_\lambda} u_\lambda^{p+1} \tag{4.54}$$

Proof. First note that by the inequality (4.46) in Lemma 4.12, for every $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$ we have

$$Q_\lambda(\psi) \geq \frac{1}{n}\int_{\Omega_\lambda} |\nabla\psi|^2\,dx + \int_{\Omega_\lambda} \kappa|\psi|^2 - pu_\lambda^{p-1}|\psi|^2\,dx \geq (\kappa - p\sup_{\Omega_\lambda} u_\lambda^{p+1})\int_{\Omega_\lambda} |\psi|^2\,dx,$$

Hence, inequality (4.54) holds, in particular χ_λ is finite. Now, let $\{\psi_k\}$ be a minimizing sequence for χ_λ. If the sequence is unbounded in the H^1-norm, then we define $\phi_k := \psi_k/\|\psi_k\|_{H^1(\Omega_\lambda)}$. Because of $\|\psi_k\|_{L^2(\Omega_\lambda)} = 1$, we have $\|\phi_k\|_{L^2(\Omega_\lambda)} \to 0$, and so without loss of generality we can suppose that $\phi_k \rightharpoonup 0$ weakly in $H^1(\Omega_\lambda)$. By the compactness of the trace operator $H^1(\Omega_\lambda) \to L^2(\partial\Omega_\lambda)$ we have $\|\phi_k\|_{L^2(\partial\Omega_\lambda)} \to 0$, and so

$$\lim_{k\to\infty}\int_{\partial\Omega_\lambda} H_\lambda|\phi_k|^2\,dS = 0.$$

Therefore,

$$\lim_{k\to\infty}\int_{\Omega_\lambda} |\nabla\phi_k|^2\,dx = \lim_{k\to\infty} Q_\lambda(\phi_k) - \chi_\lambda\int_{\Omega_\lambda} |\phi_k|^2\,dx = 0,$$

which contradicts the fact that $\|\phi_k\|_{H^1(\Omega_\lambda)} = 1$. Therefore, the sequence $\{\psi_k\}$ must be bounded in the H^1-norm, in which case, up to subsequence, we can suppose $\psi_k \rightharpoonup \psi$ weakly in $H^1(\Omega_\lambda)$. Since Q is weakly sequentially lower semi-continuous we conclude that χ_λ is attained at ψ. Now, there exist Lagrange multipliers $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ such that

$$\int_{\Omega_\lambda} \nabla\psi\cdot\nabla\zeta + \kappa\psi\zeta - pu_\lambda^{p-1}\psi\zeta\,dx + \int_{\partial\Omega_\lambda} H_\lambda\psi\zeta\,dS = \int_{\Omega_\lambda} \alpha_1\psi\zeta + \alpha_2 z_\lambda\zeta\,dx + \alpha_3\int_{\Gamma_\lambda}\zeta\,dS + \alpha_4\int_{\Gamma_1}\zeta\,dS,$$

for all $\zeta \in H^1_G(\Omega_\lambda)$. Plugging in $\zeta = \psi$, we find $\alpha_1 = \chi_\lambda$. With $\zeta = z_\lambda$, we find $\alpha_2 = 0$, and with $\zeta = \zeta_1, \zeta_2$, where the ζ_j are defined by (4.31)-(4.32), we find $\alpha_3, \alpha_4 = 0$. Thus, for each $\zeta \in H^1_G(\Omega_\lambda)$ holds the identity

$$\int_{\Omega_\lambda} \nabla\psi\cdot\nabla\zeta + \kappa\psi\zeta - pu_\lambda^{p-1}\psi\zeta\,dx + \int_{\partial\Omega_\lambda} H_\lambda\psi\zeta\,dS = \chi_\lambda\int_{\Omega_\lambda}\psi\zeta\,dx. \tag{4.55}$$

By the same argument as in the proof of Lemma 4.5 the identity (4.55) in fact holds for every $\zeta \in H^1(\Omega_\lambda)$, meaning that ψ is a weak solution to the problem (4.53). Finally, Schauder elliptic regularity theory implies $\psi \in C^{2,\alpha}(\overline{\Omega_\lambda})$. □

The last ingredient is

Proposition 4.14. *There exists $\varepsilon \in (0,1)$ such that $\chi_\lambda > 0$ for all $\lambda \in (0,\varepsilon)$.*

We will give a proof of this proposition at the end of the chapter.

Let us define

$$\lambda^* := \inf\{\lambda \in (0,1) : \chi_\lambda < 0\}, \tag{4.56}$$

where χ_λ is defined in (4.52). By Proposition 4.14 we have that $\lambda^* > 0$.

Lemma 4.15. *We have $\lambda^* < \Lambda$.*

Proof. Note that by the definition of λ^* we have $\chi_{\lambda^*} = 0$. Therefore, it is sufficient to show that $\chi_\Lambda < 0$. Since $H^{1,*}_{0,G}(\Omega_\lambda) \subset H^{1,*}_{G,m}(\Omega_\lambda)$, it is clear that $\chi_\Lambda \leq \tilde{\chi}_\Lambda$, and by the definition of Λ in (4.17) $\tilde{\chi}_\Lambda = 0$. We now argue by contradiction. Suppose that $\chi_\Lambda = 0$. Since $\tilde{\chi}_\Lambda = 0$, there exists a non-zero $\psi \in H^{1,*}_{0,G}(\Omega_\Lambda)$ such that $\tilde{Q}_\Lambda(\psi) = 0$, and since $\tilde{Q}_\Lambda(\psi) = Q_\Lambda(\psi)$, by our supposition ψ also minimizes Q_Λ over $H^{1,*}_{G,m}(\Omega_\Lambda)$. Hence, by Lemma 4.13 ψ is a solution of

$$\begin{cases} \Delta\psi + f'(u_\Lambda)\psi = 0 & \text{in} \quad \Omega_\Lambda, \\ -\psi_\nu + H_\Lambda\psi = 0 & \text{on} \quad \partial\Omega_\Lambda. \end{cases}$$

Thus, $\psi, \psi_\nu = 0$ on $\partial\Omega_\Lambda$, but this implies that $\psi = 0$ in Ω_Λ, a contradiction. The lemma is now proved. □

We are ready to state the main proposition of this section.

Proposition 4.16. *Let λ^* be defined as in (4.56). The following is satisfied:*

- *there exists $\Lambda_1 \in (0, \lambda^*)$ such that $\mu_\lambda > 0$ for all $\lambda \in (0, \Lambda_1]$;*
- *$\mu_{\lambda^*} = 0$;*
- *for each $\varepsilon > 0$ such that $\lambda^* + \varepsilon < \Lambda$ there exist a $\lambda_\varepsilon \in (\lambda^*, \lambda^* + \varepsilon)$ such that $\mu_{\lambda_\varepsilon} < 0$.*

Proof. From Proposition 4.14, for λ close to 0 the quadratic form Q_λ is strictly positive, and then from Proposition 4.11 follows the first point.

For the second point, note that from the definition of λ^* in (4.56), $Q_{\lambda^*} \geq 0$ in $H^{1,*}_{G,m}(\Omega_{\lambda^*})$, and then by Proposition 4.11 we have $\mu_{\lambda^*} \geq 0$. Also, since $\chi_{\lambda^*} = 0$, by Lemma 4.13 there exists a non-zero solution $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$ to

$$\begin{cases} \Delta\psi + f'(u_{\lambda^*})\psi = 0 & \text{in} \quad \Omega_{\lambda^*}, \\ -\psi_\nu + H_{\lambda^*}\psi = 0 & \text{on} \quad \partial\Omega_{\lambda^*}. \end{cases}$$

Thus, by Lemma 4.10 $\mu = 0$ is an eigenvalue of L_{λ^*} and $\mu_{\lambda^*} \leq 0$. We then conclude that $\mu_{\lambda^*} = 0$.

Finally, for the last point, note that by the definition of λ^*, for each $\varepsilon > 0$ such that $\lambda^* + \varepsilon < \Lambda$ there exist a $\lambda_\varepsilon \in (\lambda^*, \lambda^* + \varepsilon)$ such that $\chi_{\lambda_\varepsilon} < 0$. Let ψ_ε be a minimizer for $\chi_{\lambda_\varepsilon}$. Then by the same argument as in Lemma 4.15 we have $\psi_\varepsilon \neq 0$ on $\partial\Omega_{\lambda_\varepsilon}$, and therefore

$$\mu_{\lambda_\varepsilon} \leq \frac{Q_{\lambda_\varepsilon}(\psi_\varepsilon)}{\|\psi_\varepsilon\|_{L^2(\partial\Omega_{\lambda_\varepsilon})}} < 0.$$

□

4.5 The proof of Theorem 4.1

In this section we shall re-write our problem in a more convenient way so as to apply Krasnoselskii's Bifurcation Theorem. For this, the following is needed.

Lemma 4.17. *Let $\lambda \in (0, \Lambda)$. Then for each $\mu > -\mu_\lambda$ the operator*

$$L_\lambda + \mu \,\mathrm{id} : C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$$

is an isomorphism.

Proof. Let us first consider the bilinear form associated with the operator $L_\lambda + \mu\,\mathrm{id}$

$$b_{\lambda,\mu}(w_1, w_2) := b_\lambda(w_1, w_2) + \mu \int_{\partial\Omega_\lambda} w_1 w_2 \, dS, \quad w_1, w_2 \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda).$$

where b_λ is the bilinear form associated to L_λ defined in (4.38). By means of $b_{\lambda,\mu}$, the operator $L_\lambda + \mu\,\mathrm{id}$ can be extended to an operator $L_\lambda + \mu\,\mathrm{id} : H^{1/2}_{G,m}(\partial\Omega_\lambda) \to H^{-1/2}_{G,m}(\partial\Omega_\lambda)$ as in (4.40). Note that by (4.42), the quadratic form $q_{\lambda,\mu}$ associated to $b_{\lambda,\mu}$ is given by

$$q_{\lambda,\mu}(w) = Q_\lambda(\psi_w) + \mu \int_{\partial\Omega_\lambda} \psi_w^2 \, dS,$$

We will show that $b_{\lambda,\mu} : H^{1/2}_{G,m}(\partial\Omega_\lambda) \times H^{1/2}_{G,m}(\partial\Omega_\lambda) \to \mathbb{R}$ is a coercive bilinear form.

Define

$$\gamma := \inf_{\substack{\psi \in H^{1,*}_{G,m}(\Omega_\lambda) \\ \|\psi\|_{H^1(\Omega_\lambda)} = 1}} Q_\lambda(\psi) + \mu \int_{\partial\Omega_\lambda} \psi^2 \, dS.$$

and let $\{\psi_k\}$ be a minimizing sequence for γ weakly converging in $H^1(\Omega_\lambda)$ to some ψ. Note that as $H^{1,*}_{G,m}(\Omega_\lambda)$ is a closed subspace of $H^1(\Omega_\lambda)$, we have $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$. Then by the weak lower semicontinuity of Q_λ

$$\gamma = \liminf_{k\to\infty} Q_\lambda(\psi_k) + \mu \int_{\partial\Omega_\lambda} \psi_k^2 \, dS \geq Q_\lambda(\psi) + \mu \int_{\partial\Omega_\lambda} \psi^2 \, dS \geq (\mu_\lambda + \mu) \int_{\partial\Omega_\lambda} \psi^2 \, dS \geq 0,$$

where in the second inequality we used Proposition 4.11. If ψ_k fails to converge strongly in $H^1(\Omega_\lambda)$ to ψ, the the first inequality above is strict, and so $\gamma > 0$. If $\psi_k \to \psi$ strongly, then $\|\psi\|_{H^1} = 1$, in particular $\psi \neq 0$. If $\psi = 0$ on $\partial\Omega_\lambda$, then $\gamma \geq Q_\lambda(\psi) > 0$ since $\lambda < \Lambda$, and if $\psi \neq 0$ on $\partial\Omega_\lambda$ the last inequality above is strict and also $\gamma > 0$.

Now, by the boundedness of the trace operator we have

$$Q_\lambda(\psi) + \mu \int_{\partial\Omega_\lambda} \psi^2 \, dS \geq \gamma \|\psi\|_{H^1(\Omega_\lambda)} \geq \gamma' \|\psi\|_{H^{1/2}(\partial\Omega_\lambda)}$$

for some $\gamma' > 0$. Thus, $b_{\lambda,\mu}$ is coercive, and by the Lax-Milgram theorem the operator $L_\lambda + \mu\,\mathrm{id} : H^{1/2}_{G,m}(\partial\Omega_\lambda) \to H^{-1/2}_{G,m}(\partial\Omega_\lambda)$ is an isomorphism.

With that in mind, we now prove that the operator, when restricted to the Hölder spaces $C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, is also an isomorphism. Clearly, it is injective since it has a trivial kernel. Let $y \in C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, then there exists $w \in H^{1/2}_{G,m}(\partial\Omega_\lambda)$ such that $L_\lambda w = y - \mu w$. This implies that the weak solution $\psi_w \in H^{1,*}_{G,m}(\Omega_\lambda)$ to the Dirichlet problem (4.18) also satisfies the Neumann condition

$$-\frac{\partial \psi_w}{\partial \nu} = y - (H_\lambda + \mu) w \quad \text{on} \quad \partial\Omega_\lambda.$$

Then, by elliptic regularity for the Neumann problem, $\psi_w \in C^{2,\alpha}_G(\overline{\Omega_\lambda}) + H^{3/2}_G(\Omega_\lambda)$, which in turn implies that the trace $w = \psi_w|_{\partial\Omega_\lambda} \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) + H^1_{G,m}(\partial\Omega_\lambda)$. An inductive argument analogous to that in the proof of Propostion 3.14 in Chapter 3 shows

$$w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) + H^{s/2}_{G,m}(\partial\Omega_\lambda) \quad \text{for all} \quad s \geq 1.$$

By Sobolev embedding, we conclude that $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$. Thus, L_λ is also surjective, and hence an isomorphism. $\square$

Now we state the lemma we need to rewrite our problem in a convenient way, so as to apply the Krasnoselskii Bifurcation Theorem. It is a direct corollary to Lemma 4.17.

Lemma 4.18. *There exists $\varepsilon > 0$ such that for each $\lambda \in (0, \lambda^* + \varepsilon)$ we have $\mu_\lambda > -1$ and the operator*

$$L_\lambda + \mathrm{id} : C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$$

is an isomorphism.

Proof. Since $\mu_{\lambda^*} = 0$ by Proposition 4.16, by the continuity of $\lambda \mapsto \mu_\lambda$ there exists $\varepsilon > 0$ such that $\mu_\lambda > -1$ for every $\lambda < \lambda^* + \varepsilon$. The result follows after Lemma 4.17. $\square$

According to Lemma 4.18 and the Inverse Function Theorem, we can chose $\Lambda_2 \in (\lambda^*, \Lambda)$ and a neighbourhood $V \subset C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$ of 0 such that the operator $F_\lambda + \mathrm{id} : U \to V$ is a diffeomorphism for every $\lambda \in (0, \Lambda_2]$. Then, we can define the inverse operator $K_\lambda := (F_\lambda + \mathrm{id})^{-1} : V \to U$. Since the embedding $C^{2,\alpha}_{G,m}(\partial\Omega_\lambda) \hookrightarrow C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$ is compact, we have that K_λ is a compact operator when we look U as a subset of $C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, as we shall do from now on. We then define the operator $R_\lambda : V \to U$ as

$$R_\lambda(v) = v - K_\lambda(v). \tag{4.57}$$

We note that $v \in C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$ and $R_\lambda(v) = 0$ if and only if $v \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ and $F_\lambda(v) = 0$.

Lemma 4.19. *Let $\lambda \in (0, \Lambda_2]$ be as above and let R_λ be the operator defined in (4.57). Then the number of negative eigenvalues of its linearization at $v = 0$, $dR_\lambda|_{v=0} : C^{1,\alpha}_{G,m}(\partial\Omega_\lambda) \to C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$, is the same as the number of negative eigenvalues of L_λ. Moreover, $dR_\lambda|_{v=0}$ is an isomorphism if and only if L_λ is an isomorphism.*

Proof. Let $w \in C^{1,\alpha}_{G,m}(\partial\Omega_\lambda)$ be an eigenfunction for $dR_\lambda|_{v=0}$ associated with the eigenvalue μ. Note that $dR_\lambda|_{v=0} = \mathrm{id} - dK_\lambda|_{v=0}$, and so

$$dK_\lambda|_{v=0}(w) = (1 - \mu)w.$$

As the image $\operatorname{im} dK_\lambda|_{v=0} \subset C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$ and $\ker dK_\lambda|_{v=0} = \{0\}$, then $1 - \mu \neq 0$ and $w \in C^{2,\alpha}_{G,m}(\partial\Omega_\lambda)$. Now, because $dK_\lambda|_{v=0}^{-1} = \mathrm{id} + L_\lambda$, we see that

$$dR_\lambda|_{v=0}(w) = \mu w \quad \text{if and only if} \quad L_\lambda(w) = \frac{\mu}{1 - \mu} w. \tag{4.58}$$

Therefore, as $\lambda < \Lambda_2$, by Lemma 4.17

$$\frac{\mu}{1-\mu} = -1 + \frac{1}{1-\mu} > -1,$$

which implies $\mu < 1$. Then, from (4.58) hence follows $\mu < 0$ if and only if $\mu/(1-\mu)$ is a negative eigenvalue of L_λ.

Now, note that as $dR_\lambda|_{v=0}$ has the form of identity plus a compact operator, it is an isomorphism if and only if it has a trivial kernel. From the proof of Lemma 4.17 we see that also L_λ is an isomorphism if and only if it has a trivial kernel. Then the last claim of the lemma follows from (4.58). □

The only thing remaining to show is

Lemma 4.20. *We can choose $\Lambda_2 \in (\lambda^*, \Lambda)$ such that $dR_{\Lambda_2}|_{v=0}$ is an isomorphism with an odd number of negative eigenvalues, counted with algebraic multiplicity.*

Proof. In view of Lemma 4.19, we just need to show the statement replacing $dR_\lambda|_{v=0}$ by L_λ, for $\lambda = \Lambda_2$. Consider the quadratic form Q_λ defined in (4.41), and let $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$. We decompose ψ into its Fourier series

$$\psi(r,\theta) = \varphi_0(r) + \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} \varphi_{k,j}(r) Y_{k,j}(\theta).$$

Then we have

$$Q_\lambda(\psi) = Q_{\lambda,0}(\varphi_0) + \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} Q_{\lambda,k}(\varphi_{k,j}), \tag{4.59}$$

where for $\varphi \in H^1_\rho(\Omega_\lambda)$ we define

$$Q_{\lambda,k}(\varphi) := \int_\lambda^1 r^{n-1}\left(\varphi'(r)^2 - f'(u_\lambda)\varphi(r)^2\right) + \sigma_k r^{n-3}\varphi(r)^2\, dr + (n-1)\left(\varphi(1)^2 - \lambda^{n-2}\varphi(\lambda)^2\right).$$

Note that as σ_k is increasing with k, we have

$$Q_{\lambda,k+1}(\varphi) > Q_{\lambda,k}(\varphi), \quad \text{for all} \quad \varphi \in H^1_\rho(\Omega_\lambda)\setminus\{0\}, \quad k \in \mathbb{N}. \tag{4.60}$$

We see $Q_{\lambda,0}(\psi) = Q_\lambda(\psi)$ whenever $\psi \in H^1_\rho(\Omega_\lambda)$. Note also that $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$ implies $\varphi_0(\lambda) = 0 = \varphi_0(1)$ and

$$\int_{\Omega_\lambda} \psi z_\lambda \, dx = 0.$$

Therefore, as u_λ is non-degenerate in the space $H^1_{0,\rho}(\Omega_\lambda)$ and its Morse index is 1, we must have $Q_{\lambda,0}(\varphi_0) > 0$ whenever $\varphi_0 \neq 0$.

Now, we look at the case $\lambda = \lambda^*$. As in this case $Q_\lambda \geq 0$ in $H^{1,*}_{G,m}(\Omega_\lambda)$, then by (4.59) $Q_{\lambda,k} \geq 0$ in $H^1_\rho(\Omega_\lambda)$ for all $k \geq 1$. Recall that by Lemma 4.13 there exists a non-zero $\psi_0 \in H^{1,*}_{G,m}(\Omega_\lambda)$ such that $Q_\lambda(\psi_0) = 0$. We claim that for this ψ holds

$$\varphi_{k,j} = 0 \quad \text{for all} \quad k \geq 2, \quad j = 1, \ldots, m_k.$$

Indeed, if $\varphi_{k,j} \neq 0$ for some $k \geq 2$, then by (4.60) we would have

$$Q_{\lambda,k-1}(\varphi_{k,j}) < Q_{\lambda,k}(\varphi_{k,j}) = 0,$$

which contradicts that $Q_{\lambda,k-1} \geq 0$. Hence, $\varphi_{k,j} = 0$ for all $k \geq 2$. Then

$$Q_\lambda(\psi_0) = Q_{\lambda,0}(\varphi_0) + Q_{\lambda,1}(\varphi_1) = 0,$$

and since $\varphi_0 \neq 0$ implies $Q_{\lambda,0}(\varphi_0) > 0$, we must have $\varphi_0 = 0$. Therefore, $\varphi_1 \neq 0$ and $Q_{\lambda,1}(\varphi_1) = 0$. This implies that any non-zero $\psi \in H^{1,*}_{G,m}(\Omega_\lambda)$ such that $Q_\lambda(\psi) = 0$ must be a multiple of $\psi_0 = \varphi_1(r)Y_1(\theta)$ and thus, by Proposition 4.11 the kernel of L_λ is one-dimensional.

We conclude the proof of the lemma noting that by Proposition 4.16 we can choose $\Lambda_2 \in (\lambda^*, \Lambda)$ such that $\mu_{\Lambda_2} < 0$, and by continuity if Λ_2 is close enough to λ^*, then

$$Q_{\Lambda_2,k}(\varphi) > 0 \quad \text{for all} \quad \varphi \in H^1_\rho(\Omega_{\Lambda_2}) \setminus \{0\}, \quad k \geq 2,$$

so that 0 is not an eigenvalue of L_λ. By the same arguments as above, μ_{Λ_2} is simple. This concludes the proof. □

We now are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\Lambda_1 \in (0, \lambda^*)$ given by Propostion 4.16 such that $\mu_{\Lambda_1} > 0$, and let $\Lambda_2 \in (\lambda^*, \Lambda)$, given by Lemmas 4.19 and 4.20, such that $\mu_{\Lambda_2} < 0$ is the only negative eigenvalue of L_{Λ_2}. We now make the identification of functions $v \in C^{k,\alpha}(\partial\Omega_\lambda)$ with a pair of functions $\mathbf{v} \in (C^{k,\alpha}(\mathbb{S}^{n-1}))^2$ as in (3.11), and let the operators F_λ and R_λ, defined in (4.28) and (4.57) respectively, act on these spaces. Let $U \subset (C^{1,\alpha}_{G,m}(\mathbb{S}^{n-1}))^2$ be a sufficiently small neighbourhood of $\mathbf{0}$ such that we can define the operator

$$R : U \times [\Lambda_1, \Lambda_2] \to (C^{1,\alpha}_{G,m}(\mathbb{S}^{n-1}))^2, \qquad R(\mathbf{v}, \lambda) = R_\lambda(\mathbf{v}).$$

Then, we can apply the Krasnoselskii Bifurcation Theorem A.2 to conclude that each neighbourhood of $V \subset U \times [\Lambda_1, \Lambda_2]$ of $\{\mathbf{0}\} \times [\Lambda_1, \Lambda_2]$ contains a solution to

$$R(\mathbf{v}, \lambda) = 0, \quad \lambda \in (\Lambda_1, \Lambda_2), \quad \mathbf{v} \neq 0.$$

Since $R(\mathbf{v}, \lambda) = \mathbf{0}$ if and only if $F_\lambda(\mathbf{v}) = \mathbf{0}$, this concludes the proof. □

4.6 Proof of Propositions 4.4 and 4.14

First note that for every $\lambda \in (0, 1)$, we have $H^1_{0,G}(\Omega_\lambda) \subset H^1_{G,m}(\Omega_\lambda)$ and $Q_\lambda(\psi) = \tilde{Q}_\lambda(\psi)$ for every $\psi \in H^1_{0,G}(\Omega_\lambda)$. Hence, $\tilde{\chi}_\lambda \geq \chi_\lambda$, and so Proposition 4.4 follows from Proposition 4.14. Therefore, we only need to show that Q_λ is positive on $H^{1,*}_{G,m}(\Omega_\lambda)$ for all λ is sufficiently small. This proof is somewhat delicate and will need some preparation. First, we state the following lemma, which is rather technical and its proof is given in Subsection 4.A.1 in the Appendix.

Lemma 4.21. *There exists a family of extension operators $E_\lambda : H^1_{G,m}(\Omega_\lambda) \to H^1_{G,m}(B)$ such that for each λ_0, there exists a constant $C = C(\lambda_0, n)$ for which*

$$\|E_\lambda \psi\|_{H^1(B)} \leq C\|\psi\|_{H^1(\Omega_\lambda)},$$

for all $\psi \in H^1_{G,m}(\Omega_\lambda)$ and all $\lambda \in (0, \lambda_0)$.

We now turn to the proof Proposition 4.14, which we argue by contradiction. Let us suppose that there is a sequence of positive $\lambda_k \downarrow 0$ such that $\chi_{\lambda_k} \leq 0$. For the sake of notational simplicity we replace every subindex λ_k by k. The behaviour of the functions u_k and z_k will be needed.

Lemma 4.22. *Consider the functions u_k extended to $B_k = B \setminus \Omega_k$ by 0. Then there exists a uniform bound*

$$\|u_k\|_{L^\infty(B)} \leq C \quad \textit{for all} \quad k \geq 1,$$

and, up to subsequence, we have that $u_k \to u_0$ strongly in $H^1(B)$.

Proof. The uniform boundedness was established in [GPY03] for a general non-linearity $f(u)$ of sub-critical growth of which $\kappa u - u^p$ is a particular case. In the same paper is shown that, up to subsequence $u_k \rightharpoonup u_0$ weakly in $H^1(B)$. We give here an alternative proof of this last fact that fits in our special case. Recall that u_k satisfies the min-max characterization (see [SW10])

$$J_k(u_k) = \inf_{\substack{u \in H^1_{0,\rho}(\Omega_k) \\ u \neq 0}} \max_{t>0} J_k(tu), \tag{4.61}$$

where J_k is the energy functional

$$J_k(u) = \frac{1}{2}\int_{\Omega_k} |\nabla u|^2 + \kappa u^2 \, dx - \frac{1}{p+1}\int_{\Omega_k} |u|^{p+1} \, dx,$$

and that u_k satisfies the integral identity (4.6) which we reproduce here

$$\int_B |\nabla u_k|^2 + \kappa |u_k|^2 \, dx = \int_B |u_k|^{p+1} \, dx.$$

Since $H^1_{0,\rho}(\Omega_k) \subset H^1_{0,\rho}(\Omega_{k+1}) \subset H^1_{0,\rho}(B)$ up to extension by 0, from (4.61) we see $J_k(u_k)$ is decreasing in k, and so in particular it is bounded. Also, if we subtract (4.6) from $(p+1)J_k(u_k)$ we find

$$(p+1)J_k(u_k) = \frac{p-1}{2}\int_{\Omega_k} |\nabla u_k|^2 + \kappa u_k^2 \, dx.$$

Thus, from the boundedness of $J_k(u_k)$ we see that $\|u_k\|_{H^1(B)}$ is bounded. Then, up to a subsequence, we can assume $u_k \rightharpoonup u$ weakly in $H^1(B)$ to some $u \in H^1_{0,\rho}(B)$. We point out that because of the compactness of the embeddings $H^1_0(B) \hookrightarrow L^2(B), L^{p+1}(B)$ we have $u_k \to u$ strongly in $L^{p+1}(B)$ and $L^2(B)$. Then, from identity (4.6) we can conclude $\|u_k\|_{H^1(B)} \to \|u\|_{H^1(B)}$, which together with weak convergence imply that $u_k \to u$ strongly in $H^1(B)$. Now, let $\zeta \in C_0^\infty(B \setminus \{0\})$, then $\operatorname{supp} \zeta \subset \Omega_k$ for all k large enough. Multiplying (4.2) by ζ and integrating by parts we see

$$\int_B \nabla u_k \cdot \nabla \zeta + \kappa u_k \zeta \, dx = \int_B u_k^p \zeta \, dx$$

and letting $k \to \infty$ in the above equation we find u is a radially symmetric weak solution to the problem

$$\begin{cases} -\Delta u + \kappa u = u^p & \text{in} \quad B \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

Because $u \in H^1(B)$, the singularity is removable (see Subsection 4.A.2 in the Appendix). Then, u is a weak solution to the problem above in the full ball B. By uniqueness of the radially symmetric (see Proposition 4.2) solution we must have either $u = u_0$ or $u \equiv 0$. We rule out the second possibility noting that by the monotonicity of $J_k(u_k)$ we have

$$\frac{p-1}{2} \int_B |\nabla u|^2 + \kappa u^2 \, dx. = (p+1) \lim_{k\to\infty} J_k(u_k) \geq (p+1) J_0(u_0) > 0.$$

This concludes the proof. □

An analogous result is satisfied by the sequence of eigenfunctions z_k.

Lemma 4.23. *Consider the functions z_k extended to $B_k = B \setminus \Omega_k$ by 0. Then, up to subsequence, we have that $z_k \to z_0$ strongly in $H^1(B)$.*

Proof. As $\|z_k\|_{H^1(B)} = 1$, without loss of generality we can suppose that $z_k \rightharpoonup z$ weakly in $H^1(B)$. Then, as $u_k \to u_0$ and $z_k \to z$ strongly in $L^{p+1}(B)$, we have $u_k^{p-1} \to u_0^{p-1}$ strongly in $L^{\frac{p+1}{p-1}}(B)$ and $z_k^2 \to z^2$ strongly in $L^{\frac{p+1}{2}}(B)$. Therefore, since $\frac{p+1}{p-1}$ and $\frac{p+1}{2}$ are dual exponents,

$$\lim_{k\to\infty} \int_B u_k^{p-1} z_k^2 \, dx = \int_B u_0^{p-1} z^2 \, dx. \tag{4.62}$$

We next show that $z \neq 0$. Indeed, if $z = 0$ then from the strong convergence $z_k \to z$ in $L^2(B)$ and the identity (4.62) follows that

$$\lim_{k\to\infty} \int_B \kappa z_k^2 - p u_k^{p-1} z_k^2 = 0.$$

Now, recall that z_k is a solution to (4.10),

$$-\Delta z_k + \kappa z_k - p u_k^{p-1} z_k = \tau_k z_k \quad \text{in} \quad \Omega_k.$$

Multiplying the above equation by z_k and integrating by parts we arrive to the identity

$$\int_B |\nabla z_k|^2 + \kappa z_k^2 - p u_k^{p-1} z_k^2 \, dx = \tau_k \int_B z_k^2 \, dx, \tag{4.63}$$

from where we can conclude

$$0 \leq \limsup_{k\to\infty} \int_B |\nabla z_k|^2 \, dx \leq \limsup_{k\to\infty} \tau_k \int_B z_k^2 \, dx \leq 0,$$

which contradicts the fact that $\|z_k\|_{H^1(B)} = 1$ for all $k \geq 1$. Hence, $z \neq 0$.

Now, from identity (4.63) the same argument which proves (4.54) in Lemma 4.13 shows that

$$\tau_k \geq \kappa - p \sup_{\Omega_k} u_k^{p-1},$$

which together with the uniform bound on u_k implies that $\{\tau_k\}$ is bounded. So, up to subsequence, we can suppose that $\tau_k \to \tau \le 0$.

Let $\zeta \in C_0^\infty(B \setminus \{0\})$. By the Hölder inequality

$$\int_B |u_k^{p-1} z_k \zeta - u_0^{p-1} z\zeta|\,dx \le \int_B |u_k^{p-1} - u_0^{p-1}||z_k\zeta| + |u_0^{p-1}\zeta||z_k - z|\,dx \le$$
$$\le \|u_k^{p-1} - u_0^{p-1}\|_{L^{\frac{p+1}{p-1}}} \|z_k\|_{L^{p+1}} \|\zeta\|_{L^{p+1}} + \|u_0\|_{L^{p+1}} \|\zeta\|_{L^{p+1}} \|z_k - z\|_{L^{p+1}} \to 0. \quad (4.64)$$

Then weak convergence implies

$$\lim_{k\to\infty} \int_B \nabla z_k \cdot \nabla\zeta + \kappa z\zeta - p u_k^{p-1} z\zeta - \tau_k z_k \zeta\,dx = \int_B \nabla z \cdot \nabla\zeta + \kappa z\zeta - p u_0^{p-1} z\zeta - \tau z\zeta\,dx = 0$$

Therefore, z is a nonzero weak solution to

$$\begin{cases} -\Delta z + \kappa z - p u_0^{p-1} z = \tau z & \text{in} \quad B \setminus \{0\}, \\ z = 0 & \text{on} \quad \partial B. \end{cases}$$

Since $z \in H^1(B)$, the singularity at 0 is removable. Because z_k is radially symmetric for all $k \ge 1$, so is z. Thus, we can conclude that $\tau = \tau_0$ and $z = a z_0$ for some $a \ne 0$. Finally, from (4.63) and the strong convergence $z_k \to z$ in $L^2(B)$ we see that $1 = \|z_k\|_{H^1(B)} \to \|z\|_{H^1(B)}$, and so $a = 1$ and the convergence $z_k \to z_0$ is strong in $H^1(B)$. □

Proof of Proposition 4.14. Let us denote by ψ_k a minimizer for $\chi_k \le 0$, extended to $B_k = B \setminus \Omega_k$ as in Lemma 4.21, and normalized in the $H^1(\Omega_k)$-norm. By the uniform bound $\|u_k\|_{L^\infty} \le C$ and Lemma 4.13, we have that the sequence $\{\chi_k\}$ is bounded below. Therefore, without loss of generality we assume that $\chi_k \to \chi_0 \le 0$. Also, by Lemma 4.21 the sequence $\{\psi_k\}$ is bounded in the $H^1(B)$-norm, and so we can assume that there exists $\psi_0 \in H^1(B)$ such that $\psi_k \rightharpoonup \psi_0$ weakly in $H^1(B)$, and thus $\psi_k \rightharpoonup \psi_0$ weakly in $H^1(\Omega_\lambda)$ for every $\lambda \in (0,1)$. Recall that each ψ_k is a weak solution to

$$\begin{cases} -\Delta\psi + \kappa\psi - p u_k^{p-1}\psi = \chi_k \psi & \text{in} \quad \Omega_k, \\ -\psi_\nu + H\psi = 0 & \text{on} \quad \partial\Omega_k. \end{cases}$$

We will first show that under these hypotheses the weak limit ψ_0 is nonzero. First note that

$$\lim_{k\to\infty} \int_{\Omega_k} |\psi_k|^2\,dx = \int_B |\psi_0|^2\,dx. \quad (4.65)$$

Indeed, let $\varepsilon \in (0,1)$. We have $\psi_k \to \psi_0$ strongly in $L^2(B_\varepsilon)$, and $B_k \subset B_\varepsilon$ for k large enough. Thus

$$\limsup_{k\to\infty} \int_{B_k} |\psi_k|^2\,dx \le \lim_{k\to\infty} \int_{B_\varepsilon} |\psi_k|^2\,dx = \int_{B_\varepsilon} |\psi_0|^2\,dx,$$

which goes to 0 as $\varepsilon \to 0$. The equality (4.65) follows at once.

Using the identity

$$Q_k(\psi_k) = \chi_k \int_{\Omega_k} |\psi_k|^2\,dx$$

and the inequality (4.46) in Lema 4.12 we can conclude that

$$\frac{1}{n}\int_{\Omega_k} |\nabla\psi_k|^2\,dx + \int_{\Omega_k} \kappa|\psi_k|^2 - pu_k^{p-1}|\psi_k|^2\,dx \le \chi_k \int_{\Omega_k} |\psi_k|^2\,dx \le 0. \tag{4.66}$$

If $\psi_0 = 0$, then by (4.65) and the uniform bound $\|u_k\|_{L^\infty} \le C$ follows that

$$\lim_{k\to\infty} \int_{\Omega_k} \kappa|\psi_k|^2 - pu_k^{p-1}|\psi_k|^2\,dx = 0,$$

and therefore the inequality (4.66) implies

$$\limsup_{k\to\infty} \frac{1}{n}\int_{\Omega_k} |\nabla\psi_k|^2\,dx \le 0,$$

which contradicts our assumption $\|\psi_k\|_{H^1(\Omega_k)} = 1$. We then conclude that $\psi_0 \neq 0$.

Now, let $\zeta \in C^\infty(\overline{B})$ be such that $0 \notin \operatorname{supp}\zeta$. Then $\operatorname{supp}\zeta \subset \Omega_\varepsilon$ for some $\varepsilon \in (0,1)$, and so $\zeta|_{\Gamma_k} = 0$ for k sufficiently large. Then by the compactness of the trace operator

$$\lim_{k\to\infty} \int_{\partial\Omega_k} H\psi_k\zeta\,dS = (n-1)\int_{\partial B} \psi_0\zeta\,dS.$$

Also, by the same arguments which prove (4.64) in Lemma 4.23 we find

$$\lim_{k\to\infty} \int_{\Omega_k} u_k^{p-1}\psi_k\zeta\,dx = \int_B u_0^{p-1}\psi_0\zeta\,dx.$$

This together with weak convergence imply that

$$\begin{aligned}\lim_{k\to\infty} \int_{\Omega_k} \nabla\psi_k\cdot\nabla\zeta + \kappa\psi_k\zeta - pu_k^{p-1}\psi_k\zeta - \chi_k\psi_k\zeta\,dx + \int_{\partial\Omega_k} H\psi_k\zeta\,dS =\\ = \int_B \nabla\psi_0\cdot\nabla\zeta + \kappa\psi_0\zeta - pu_0^{p-1}\psi_0\zeta - \chi_0\psi_0\zeta\,dx + (n-1)\int_{\partial B}\psi_0\zeta\,dS = 0.\end{aligned} \tag{4.67}$$

Therefore, ψ_0 is a weak solution to

$$\begin{cases} -\Delta\psi + \kappa\psi_0 - pu_0^{p-1}\psi = \chi_0\psi & \text{in} \quad B\setminus\{0\},\\ \quad -\psi_\nu + (n-1)\psi = 0 & \text{on} \quad \partial B. \end{cases} \tag{4.68}$$

Because $\psi_0 \in H^1(B)$, the singularity at 0 is removable, and ψ_0 is a solution to (4.68) in the full ball B. We also note that, since $z_k \to z_0$ strongly in $L^2(B)$, we have

$$0 = \lim_{k\to\infty} \int_{\Omega_k} \psi_k z_k\,dx = \int_B \psi_0 z_0\,dx = 0,$$

and since $H^1_{G,m}(B) \subset H^1(B)$ is closed, we find $\psi_0 \in H^{1,*}_{G,m}(B)$.

We use separation of variables on ψ_0 and write

$$\psi_0(r,\theta) = \psi(r,\theta) = \varphi_0(r) + \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} \varphi_{k,j}(r)Y_{k,j}(\theta),$$

where, recall, $Y_{k,j}$ denote the G-invariant spherical harmonics of degree k and m_k the multiplicity of the corresponding eigenvalue σ_k. In this coordinates, the Laplacian is written as

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}},$$

where ∂_r acts only on the r variable and $\Delta_{\mathbb{S}^{n-1}}$ acts only on the θ variable. Recalling that

$$\Delta_{\mathbb{S}^{n-1}} Y_{k,j} + \sigma_k Y_{k,j} = 0, \quad k \in \mathbb{N}_0, \quad j = 1, \dots, m_k,$$

we see that solving the problem (4.68) for ψ_0 is equivalent to solving the boundary value problems of the family of ordinary differential equations

$$\begin{cases} \varphi_{k,j}'' + \dfrac{n-1}{r}\varphi_{k,j}' + \left(pu_0^{p-1} - \kappa - \dfrac{\sigma_k}{r^2}\right)\varphi_{k,j} + \chi_0 \varphi_{k,j} = 0 \quad \text{in} \quad (0,1), \\ \varphi_{k,j}'(1) + (n-1)\varphi_{k,j}(1) = 0, \\ \varphi_{k,j}'(0) = 0, \end{cases} \tag{4.69}$$

for $k \in \mathbb{N}_0$, $j = 1, \dots, m_k$, and with the convention $\varphi_{0,1} = \varphi_0$. As $\psi \in H^{1,*}_{G,m}(B)$, for $k = 0$ we have $\varphi_0(1) = 0 = \varphi_0'(1)$, and so by (4.69) $\varphi_0 \equiv 0$. But since $\psi_0 \neq 0$, then $\varphi_{k,j} \neq 0$ at least for some k, j. We will show that for each k, j, or equivalently, for each $\sigma_k \geq n-1$, there does not exist a non-zero solution for (4.69) when $\chi_0 \leq 0$, reaching the desired contradiction.

Fix $\sigma = \sigma_k \geq n-1$ and suppose $\varphi = \varphi_{k,j}$ is a non-zero solution to (4.69). We note that $\varphi(1) \neq 0$, and that the function $v = u_0'$ is negative and solves

$$v'' + \frac{n-1}{r}v' + \left(pu_0^{p-1} - \kappa - \frac{n-1}{r^2}\right)v = 0 \quad \text{in} \quad (0,1). \tag{4.70}$$

Define

$$w(r) := \frac{\varphi(r)}{v(r)}, \quad r \in (0,1). \tag{4.71}$$

Then a computation shows that $\varphi' = v'w + vw'$ and $\varphi'' = v''w + 2v'w' + vw''$, and plugging this into the equation for φ we find

$$\begin{aligned} vw'' + \left(2v' + \frac{n-1}{r}v\right)w' + \left(v'' + \frac{n-1}{r}v' + \left(pu_0^{p-1} - \kappa - \frac{\sigma}{r^2} + \chi_0\right)v\right)w = \\ = vw'' + \left(2v' + \frac{n-1}{r}v\right)w' + \left(\frac{n-1-\sigma}{r^2} + \chi_0\right)vw = 0. \end{aligned}$$

Also, by the facts $v(1), \varphi(1) \neq 0$ and $v'(1) = -(n-1)v(1)$, $\varphi'(1) = -(n-1)\varphi(1)$ we see that

$$w(1) \neq 0, \quad w'(1) = 0.$$

Therefore, to summarize, w satisfies

$$\begin{cases} w'' + \left(\dfrac{n-1}{r} + \dfrac{2v'}{v}\right)w' + \left(\dfrac{n-1-\sigma}{r^2} + \chi_0\right)w = 0 \quad \text{in} \quad (0,1), \\ w(1) \neq 0, \\ w'(1) = 0. \end{cases} \tag{4.72}$$

Without loss of generality assume $w(1) > 0$. Because $\frac{n-1-\sigma}{r^2} + \chi_0 \leq 0$, and $w'(1) = 0$, by the maximum principle we have

$$w > 0, \quad w' < 0, \quad \text{in} \quad (0,1).$$

Therefore, we may define

$$\xi(r) := r\frac{w'(r)}{w(r)} = r\left(\frac{\varphi'(r)}{\varphi(r)} - \frac{v'(r)}{v(r)}\right), \quad r \in (0,1). \tag{4.73}$$

Then, we have $w'\xi + w\xi' = rw'' + w'$, and so

$$r\frac{w''}{w} = (\xi - 1)\frac{w'}{w} + \xi' = (\xi - 1)\frac{\xi}{r} + \xi'.$$

Dividing (4.72) by w and replacing the above identity shows ξ satisfies the differential equation

$$r\xi' + \left(n - 2 + r\frac{2v'}{v}\right)\xi + \xi^2 + (n - 1 - \sigma + \chi_0 r^2) = 0 \quad \text{in} \quad (0,1). \tag{4.74}$$

We now derive the asymptotic behaviour of $\xi(r)$ as $r \downarrow 0$. For this purpose, we first note that as $v(0) = 0$,

$$\lim_{r\downarrow 0} \frac{v(r)}{r} = v'(0), \tag{4.75}$$

and from the relation $v = u_0'$ and (4.4)

$$v' + \frac{n-1}{r}v = \kappa u_0 - u_0^p, \quad \text{in} \quad (0,1). \tag{4.76}$$

Hence, taking the limit $r \downarrow 0$ in (4.76) and using the identity in (4.75) we find

$$\lim_{r\downarrow 0} v'(r) = \frac{\kappa u_0(0) - u_0(0)^p}{n}, \qquad \lim_{r\downarrow 0} r\frac{v'(r)}{v(r)} = 1. \tag{4.77}$$

First, we show that $v'(0) \neq 0$. Consider the function

$$e(r) := \frac{1}{2}\left(v(r)^2 - \kappa u_0(r)^2\right) + \frac{u_0(r)^{p+1}}{p+1}, \quad r \in [0,1].$$

Differentiating the above expression with respect to r we find

$$e'(r) = v(r)\left(v'(r) - \kappa u_0(r) + u_0(r)^p\right) = -\frac{n-1}{r}v(r)^2 \leq 0, \quad r \in (0,1),$$

where in the last equality we used the relation (4.76). Hence, e is decreasing in $[0,1]$, and therefore

$$u_0(0)\left(\frac{u_0(0)^p}{2} - \kappa\frac{u_0(0)}{2}\right) \geq u_0(0)\left(\frac{u_0(0)^p}{p+1} - \kappa\frac{u_0(0)}{2}\right) = e(0) \geq e(1) = \frac{1}{2}v(1)^2 > 0.$$

Plugging this into (4.77), we see that $v'(0) < 0$. We now claim that $\varphi(0) \neq 0$. Otherwise we would have $\varphi(0) = 0 = \varphi'(0)$, and then by L'Hôpital's rule and the monotonicity of w

$$w(1) \leq \lim_{r\downarrow 0} w(r) = \lim_{r\downarrow 0} \frac{\varphi'(r)}{v'(r)} = 0,$$

which is a contradiction to our assumption $w(1) > 0$. Therefore, $\varphi(0) \neq 0$. With this in mind, from the formulas (4.73) and (4.77) we find the limit

$$\lim_{r\downarrow 0} \xi(r) = \lim_{r\downarrow 0} r\left(\frac{\varphi'(r)}{\varphi(r)} - \frac{v'(r)}{v(r)}\right) = -1, \tag{4.78}$$

and from (4.77)-(4.78) and the differential equation (4.74)

$$\lim_{r\to 0} r\xi'(r) = \sigma.$$

Then, by L'Hôpital's rule

$$1 = \lim_{r\downarrow 0} \frac{\xi(r)/r}{-1/r} = \lim_{r\downarrow 0} \frac{\xi'(r)/r - \xi(r)/r^2}{1/r^2} = \sigma + 1,$$

which implies $\sigma = 0 \geq n-1$. Since $n \geq 2$, this gives the desired contradiction. $\square$

4.A Appendix

4.A.1 Proof of Lemma 4.21

We construct a family of extension operators $E_\lambda : H^1_G(\Omega_\lambda) \to H^1_G(B)$ as follows. For $\psi \in H^1_G(\Omega_\lambda)$, let $\phi \in H^1(B_\lambda)$ be the weak solution to the Dirichlet problem

$$\begin{cases} \Delta\phi = 0 & \text{in} \quad B_\lambda, \\ \phi = \psi & \text{on} \quad \Gamma_\lambda. \end{cases} \tag{4.79}$$

The condition $\phi|_{\Gamma_\lambda} = \psi|_{\Gamma_\lambda}$ is understood in the sense of traces. Since equation (4.79) is invariant under the action of G (Δ is invariant under orthogonal transformations and ψ is G-invariant), the uniqueness of ϕ implies that ϕ is also G-invariant. Thus, $\phi \in H^1_G(B_\lambda)$. Then we define $E_\lambda\psi$ as

$$E_\lambda\psi := \begin{cases} \phi, & \text{if} \quad x \in B_\lambda, \\ \psi, & \text{if} \quad x \in \Omega_\lambda. \end{cases} \tag{4.80}$$

It can be easily checked that $E_\lambda\psi \in H^1_G(B)$ with

$$\nabla(E_\lambda\psi) = \begin{cases} \nabla\phi, & \text{if} \quad x \in B_\lambda, \\ \nabla\psi, & \text{if} \quad x \in \Omega_\lambda. \end{cases}$$

Therefore, (4.80) defines an operator $E_\lambda : H^1_G(\Omega_\lambda) \to H^1_G(B)$. Lemma 4.21 will follow after Lemmas 4.24 and 4.25 below.

Lemma 4.24. *Let $\lambda \in (0,1)$. Let $\phi \in H^1(B_\lambda)$ be a harmonic function and let $\xi \in H^1(\Omega_\lambda)$ be a weak solution to*

$$\begin{cases} \Delta\xi = 0 & \textit{in} \quad \Omega_\lambda, \\ \xi = \phi & \textit{on} \quad \Gamma_\lambda, \\ \xi_\nu = 0 & \textit{on} \quad \Gamma_1 \end{cases} \tag{4.81}$$

in the sense that the trace $\xi|_{\Gamma_\lambda} = \phi|_{\Gamma_\lambda}$ and

$$\int_{\Omega_\lambda} \nabla\xi\cdot\nabla\zeta\, dx = 0, \quad \textit{for all} \quad \zeta \in H^1(\Omega_\lambda), \quad \zeta|_{\Gamma_\lambda} = 0. \tag{4.82}$$

Then we have the estimate

$$\int_{B_\lambda} |\nabla\phi|^2\, dx \le \frac{1+\lambda^n}{1-\lambda^n}\int_{\Omega_\lambda} |\nabla\xi|^2\, dx. \tag{4.83}$$

Proof. Decomposing into Fourier series, it is sufficient to consider functions of the form

$$\phi(r,\theta) = \frac{r^k}{\lambda^k}Y(\theta), \qquad \xi(r,\theta) = \frac{k^2 r^{-\sigma/k} + \sigma r^k}{k^2\lambda^{-\sigma/k} + \sigma\lambda^k}Y(\theta), \qquad \sigma = k(k+n-2), \quad k \in \mathbb{N}, \tag{4.84}$$

where Y is a spherical harmonic associated with the eigenvalue σ. A direct calculation shows ϕ and ξ are harmonic and $\xi|_{\Gamma_\lambda} = \phi|_{\Gamma_\lambda}$. Note that when $k=0$, ϕ is a constant function, and so (4.83) holds trivially. Recall that for a harmonic function h in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ we have

$$\int_\Omega |\nabla h|^2\, dx = \int_{\partial\Omega} -h\frac{\partial h}{\partial\nu}\, dS,$$

where ν is the *inner* normal to $\partial\Omega$. Then, using (4.84) we compute

$$\int_{B_\lambda} |\nabla\phi|^2\, dx = \lambda^{n-1}\int_{\mathbb{S}^{n-1}} Y(\theta)\frac{\partial\phi}{\partial r}(\lambda,\theta)\, d\theta = k\lambda^{n-2}\int_{\mathbb{S}^{n-1}} |Y(\theta)|^2\, d\theta,$$

and

$$\int_{\Omega_\lambda} |\nabla\xi|^2\, dx = \lambda^{n-1}\int_{\mathbb{S}^{n-1}} -Y(\theta)\frac{\partial\xi}{\partial r}(\lambda,\theta)\, d\theta = \lambda^{n-2}k\sigma\frac{\lambda^{-\sigma/k}-\lambda^k}{k^2\lambda^{-\sigma/k}+\sigma\lambda^k}\int_{\mathbb{S}^{n-1}} |Y(\theta)|^2\, d\theta.$$

Therefore, we have

$$\int_{B_\lambda} |\nabla\phi|^2\, dx = \frac{k^2\lambda^{-\sigma/k}+\sigma\lambda^k}{\sigma(\lambda^{-\sigma/k}-\lambda^k)}\int_{\Omega_\lambda} |\nabla\xi|^2\, dx.$$

The result follows by the observation

$$\frac{k^2\lambda^{-\sigma/k}+\sigma\lambda^k}{\sigma(\lambda^{-\sigma/k}-\lambda^k)} \le \frac{\lambda^{-\sigma/k}+\lambda^k}{\lambda^{-\sigma/k}-\lambda^k} \le \frac{1+\lambda^n}{1-\lambda^n}, \quad \text{for all} \quad k \in \mathbb{N}.$$

□

Remark 4.3. *Note that the weak solution $\xi \in H^1(\Omega_\lambda)$ to (4.81) minimizes the Dirichlet energy $\|\nabla\zeta\|_{L^2(\Omega_\lambda)}$ among all functions $\zeta \in H^1(\Omega_\lambda)$ such that $\zeta|_{\Gamma_\lambda} = \phi|_{\Gamma_\lambda}$. Indeed, let ζ satisfy $\zeta|_{\Gamma_\lambda} = \phi|_{\Gamma_\lambda}$, then $(\xi-\zeta)|_{\Gamma_\lambda} = 0$ and by (4.82) we have*

$$\int_{\Omega_\lambda} |\nabla\xi|^2\, dx = \int_{\Omega_\lambda} \nabla\xi\cdot\nabla\zeta\, dx \le \frac{1}{2}\int_{\Omega_\lambda} |\nabla\xi|^2\, dx + \frac{1}{2}\int_{\Omega_\lambda} |\nabla\zeta|^2\, dx,$$

which yields $\|\nabla\xi\|_{L^2(\Omega_\lambda)} \le \|\nabla\zeta\|_{L^2(\Omega_\lambda)}$.

Lemma 4.25. *Let $\lambda_0 \in (0,1)$. Then there exists a constant $C = C(\lambda_0, n)$ such that for each $\psi \in H^1_{G,m}(\Omega_\lambda)$*

$$\|E_\lambda\psi\|_{H^1(B)} \le C\|\psi\|_{H^1(\Omega_\lambda)} \tag{4.85}$$

for all $\lambda \in (0,\lambda_0)$.

Proof. Let $\psi \in H^1_{G,m}(\Omega_\lambda)$ and denote $\phi := (E_\lambda \psi)|_{B_\lambda}$. Let $\xi \in H^1(\Omega_\lambda)$ be weak solution to (4.81). Then $\xi|_{\Gamma_\lambda} = \psi|_{\Gamma_\lambda}$, and by Remark 4.3 $\|\nabla \xi\|_{L^2(\Omega_\lambda)} \leq \|\nabla \psi\|_{L^2(\Omega_\lambda)}$. Then by Lemma 4.24 we have

$$\|\nabla \phi\|_{L^2(B_\lambda)} \leq C(\lambda_0, n)\|\nabla \psi\|_{L^2(\Omega_\lambda)}. \tag{4.86}$$

We note that $\phi \in H^1_{G,m}(B_\lambda)$ has zero mean over B_λ, and so by the Poincaré inequality

$$\|\phi\|^2_{H^1(B_\lambda)} \leq (1 + C\lambda^2)\|\nabla \phi\|^2_{L^2(B_\lambda)}, \tag{4.87}$$

where C is a constant which depends only on the dimension n. The result follows after noting

$$\|E_\lambda \psi\|^2_{H^1(B)} = \|\phi\|^2_{H^1(B_\lambda)} + \|\psi\|^2_{H^1(\Omega_\lambda)}. \tag{4.88}$$

and plugging (4.86) and (4.87) in (4.88). □

4.A.2 Removal of singularities

In this subsection we show that closed subsets which are small in a certain sense can be removed from differential equations. In particular, we show that point singularities can be removed in dimensions $n \geq 2$. A more careful study can be seen, for instance, in [HP70], but we won't need to work in such a generality.

Fix a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, and let $K \subset \Omega$ be a closed set. We define the *capacity* of K as the quantity

$$\operatorname{cap}(K) := \inf\left\{\|u\|_{H^1(\Omega)} : u \in C_0^\infty(\Omega),\ u \geq 1 \text{ in } K\right\}.$$

We say that the closed set K has *zero capacity* if $\operatorname{cap}(K) = 0$.

Lemma 4.26. *Let $K \subset \Omega$ be a closed set of zero capacity. Then $C_0^\infty(\overline{\Omega} \setminus K)$ is dense in $H^1(\Omega)$, and $C_0^\infty(\Omega \setminus K)$ is dense in $H^1_0(\Omega)$.*

Proof. Let $u \in C_0^\infty(\Omega)$ such that $0 \leq u \leq 1$, $u \geq 1$ in K, and $\|u\|_{H^1(\Omega)} < \varepsilon$. Let $\phi \in C^\infty(\overline{\Omega})$. Then $\psi = (1-u)\phi \in C_0^\infty(\overline{\Omega} \setminus K)$ and

$$\|\phi - \psi\|_{H^1(\Omega)} = \|u\phi\|_{H^1(\Omega)}.$$

As $\nabla(u\phi) = u\nabla\phi + \phi\nabla u$, we see

$$\begin{aligned}
\|u\phi\|^2_{H^1(\Omega)} &= \int_\Omega \phi^2(|\nabla u|^2 + u^2) + u^2|\nabla\phi|^2\,dx + \int_\Omega 2u\phi\nabla u\cdot\nabla\phi\,dx \\
&\leq \int_\Omega \phi^2(|\nabla u|^2 + u^2) + u^2|\nabla\phi|^2\,dx + 2\left(\int_\Omega u^2|\nabla\phi|^2\,dx\right)^{1/2}\left(\int_\Omega \phi^2|\nabla u|^2\,dx\right)^{1/2} \\
&\leq C\varepsilon^2,
\end{aligned}$$

where C is a bound for ϕ^2 and $|\nabla\phi|^2$. Thus, every $\phi \in C^\infty(\overline{\Omega})$ can be approximated by a $\psi \in C_0^\infty(\overline{\Omega} \setminus K)$. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, the lemma follows. Clearly the same argument holds for the density of $C_0^\infty(\Omega \setminus K)$ in the space $H^1_0(\Omega)$. □

The lemma can be stated in a more succinct way as $H^1(\Omega) = H^1(\Omega \setminus K)$, where the equality is understood in the sense that both spaces are isomorphic and isometric. As an immediate corollary we have that singularities in a set of zero capacity are removable.

Proposition 4.27. *Let $K \subset \Omega$ be a set of zero capacity. Let L be a second order partial differential operator in divergence form with $L^\infty(\Omega)$ coefficients, and let $u \in H^1(\Omega)$ be a weak solution to*

$$Lu = 0 \quad \text{in} \quad \Omega \setminus K,$$

Then $Lu = 0$ in the whole domain Ω.

Proof. Let $L = -\partial_j(a_{jk}\partial_k) + b_j\partial_j + c$. Then for every $\zeta \in C_0^\infty(\Omega \setminus K)$ holds the identity

$$\int_\Omega a_{jk} \frac{\partial u}{\partial x_k} \frac{\partial \zeta}{\partial x_j} + b_j \frac{\partial u}{\partial x_j} \zeta + cu\zeta \, dx = 0. \tag{4.89}$$

Since by Lemma 4.26 the set $C_0^\infty(\Omega \setminus K)$ is dense in $H_0^1(\Omega)$, identity (4.89) holds for every $\zeta \in C_0^\infty(\Omega)$. Thus, $Lu = 0$ in Ω in the weak sense. $\square$

We now show the particular case of our interest, namely that a single point has zero capacity in dimension $n \geq 2$.

Proposition 4.28. *Let $n \geq 2$. Then the set $\{x_0\} \subset \Omega$ has zero capacity.*

Proof. Since capacity is local in nature, without loss of generality we assume $\Omega = B$ is the unit ball and $x_0 = 0$. Let $\varepsilon \in (0, 1)$, and define the function

$$u_\varepsilon(x) := \begin{cases} \frac{\log|x|}{\log \varepsilon}, & \text{if} \quad \varepsilon \leq |x| \leq 1, \\ 1, & \text{if} \quad |x| \leq \varepsilon. \end{cases}$$

Then $u_\varepsilon \in H_0^1(B)$ and $u_\varepsilon(0) = 1$ for all ε, and

$$\nabla u_\varepsilon(x) = \begin{cases} \frac{x}{(\log \varepsilon)|x|^2}, & \text{if} \quad \varepsilon \leq |x| \leq 1, \\ 0, & \text{if} \quad |x| \leq \varepsilon. \end{cases}$$

Therefore, using polar coordinates the H^1-norms compute to

$$\begin{aligned} \|u_\varepsilon\|^2_{H^1(B)} = \frac{C}{\log^2 \varepsilon} \left(\int_\varepsilon^1 r^{n-3} \, dr + \int_0^1 r^{n-1} \log^2 r \, dr \right) \leq \\ \leq \frac{C}{\log^2 \varepsilon} \left(\int_\varepsilon^1 r^{n-3} \, dr + \int_0^1 \log^2 r \, dr \right), \end{aligned} \tag{4.90}$$

where C is a constant which depends only in the dimension n. A direct computation shows

$$\int_0^1 \log^2 r \, dr = r(\log^2 r - 2\log r + 2)\big|_0^1 = 2.$$

Inequality (4.90) hence yields, when $n \geq 3$,

$$\|u_\varepsilon\|^2_{H^1(B)} \leq \frac{C}{\log^2 \varepsilon} \left(\frac{1 - \varepsilon^{n-2}}{n-2} + 2 \right),$$

which goes to 0 as $\varepsilon \to 0$, and when $n = 2$,

$$\|u_\varepsilon\|^2_{H^1(B)} \leq \frac{C}{\log^2 \varepsilon} \left(-\log \varepsilon + 2 \right),$$

which also converges to 0 as $\varepsilon \to 0$. Thus, $\operatorname{cap}(0) = 0$. $\square$

Remark 4.4. *The condition on the dimension $n \geq 2$ is necessary. Indeed, since in dimension $n = 1$ the Sobolev space H^1 embeds continuously into the space of bounded functions L^∞, for a function $u \in H^1$ such that $u(x_0) \geq 1$ for some point $x_0 \in \mathbb{R}$, we have*

$$\|u\|_{H^1} \geq c\|u\|_{L^\infty} \geq c,$$

for some constant $c > 0$ independent of u. Therefore, $\operatorname{cap}(x_0) \geq c > 0$.

As an immediate corollary to Propositions 4.27 and 4.28, we have that point singularities for H^1 weak solutions to a second order equation are removable in dimensions $n \geq 2$.

Appendix

A.1 Bifurcation theory

This appendix is dedicated to a brief description of the two theorems of Bifurcation theory needed in the proofs of Theorem 3.5 in Chapter 3 and of Theorem 4.6 in Chapter 4, namely, the Crandall-Rabinowitz Bifurcation Theorem and the Krasnoselskii Bifurcation Theorem.

Broadly speaking, Bifurcation theory looks at non-linear equations which depend on a parameter

$$F(x, \lambda) = 0, \tag{A.1}$$

and how solutions to the above equation behave as the parameter varies. Usually, the is a *nice* family of *trivial*, known solutions $\{x_\lambda\}$, but at some critical values of the parameter λ this family may *split* or *bifurcate* into different branches of solutions; thus the name *bifurcation*.

Suppose now that in equation (A.1), $F(\cdot, \lambda)$ is a smooth family of non-linear operators between two Banach spaces X and Y, which smoothly depends on the parameter λ belonging to other Banach space. Also, let $\{x_\lambda\}$ be a family of solutions, smoothly depending on the parameter λ, to (A.1). By the Implicit Function Theorem [Nir74], whenever the linearization of the operator F,

$$\partial_x F(x_\lambda, \lambda) : X \to Y \tag{A.2}$$

is an isomorphism of Banach spaces, that is, it is a bounded linear operator with bounded inverse, then in a neighbourhood of the family $\{x_\lambda\}$ there do not exist other solutions to (A.1). Therefore, a necessary hypothesis for bifurcations to occur is that the linearized operator (A.2) fails to be invertible, or is *degenerate*, at some critical λ_0. Note that we can replace the operator in (A.1) by

$$F(x - x_\lambda, \lambda),$$

and so there is no loss of generality if we further suppose the family $\{x_\lambda\}$ consist only on the zero element $0 \in X$. Typically, we suppose the family of operators $\{F(\cdot, \lambda)\}$ satisfy:

(F1) There exist λ_0 such that the linearized operator (A.2) is Fredholm, that is, its kernel $\ker \partial_x F(0, \lambda_0)$ is finite dimensional and its image $\operatorname{im} \partial_x F(0, \lambda_0)$ is a closed subspace of finite co-dimension.

Property (F1) allows us to decompose the Banach spaces X and Y into a direct sum

$$X = \ker \partial_x F(0,\lambda_0) \oplus X_1, \qquad Y = \operatorname{im} \partial_x F(0,\lambda_0) \oplus Y_1, \tag{A.3}$$

meaning that every element $x \in X$ and $y \in Y$ can be expressed in a unique way as the sum

$$x = x_0 + x_1, \qquad y = y_0 + y_1, \tag{A.4}$$

where $x_0 \in \ker \partial_x F(0,\lambda)$, $y_0 \in \operatorname{im} \partial_x F(0,\lambda)$, $x_1 \in X_1$ and $y_1 \in Y_1$. We now explain a procedure by which the possibly infinite dimensional equation (A.1) can be reduced to a finite set of equations in a finite number of unknowns. Let $Q : Y \to Y_1$ be the projection operator onto the subspace Y_1, and so $\operatorname{id} - Q : Y \to \ker \partial_x F(0,\lambda_0)$ is the projection onto $\ker \partial_x F(0,\lambda_0)$. Then equation (A.1) is equivalent to the system

$$QF(x_0 + x_1, \lambda) = 0, \qquad (\operatorname{id} - Q)F(x_0 + x_1, \lambda) = 0,$$

where we write $x = x_0 + x_1$ according to the decomposition in (A.3)-(A.4). Since the restriction $\partial_x F(0,\lambda_0) : X_1 \to \operatorname{im} \partial_x F(0,\lambda_0)$ is now an isomorphism of Banach spaces, by the Implicit Function Theorem for any λ and x_0 in an appropriately small neighbourhood of λ_0 and $0 \in \ker \partial_x F(0,\lambda_0)$, there exists a unique $x_1(x_0,\lambda)$ in a neighbourhood of $0 \in X_1$ such that

$$QF(x_0 + x_1(x_0,\lambda), \lambda) = 0.$$

Then, substituting $x_1(x_0,\lambda)$ into equation $(\operatorname{id} - Q)F(x_0 + x_1, \lambda) = 0$, we see we only need to solve for x_0 and the parameter λ in

$$(\operatorname{id} - Q)F(x_0 + x_1(x_0,\lambda), \lambda) = 0. \tag{A.5}$$

The element $x_1 \in \ker \partial_x F(0,\lambda_0)$ belongs to a finite dimensional space and the image $\operatorname{im}(\operatorname{id} - Q)$ is also a finite dimensional space. Thus, equation (A.5) is a finite set of (non-linear) equations in a finite number of unknowns depending on a parameter. This is the *Lyapunov-Schmidt reduction*, and (A.5) is sometimes called the *bifurcation equation*.

Lyapunov-Schmidt reduction can be used to prove the Crandall-Rabinowitz Bifurcation Theorem, which is the main tool that allowed us to prove Theorem 3.5 in Chapter 3. We give here a version of the Crandall-Rabinowitz Theorem, equivalent to the one stated in the original exposition [CR71]. A proof of the theorem and more of its applications can be found in [CR71, Kie11].

Theorem A.1 (Crandall-Rabinowitz)**.** *Let X and Y be Banach spaces, and let $U \subset X$ and $I \subset \mathbb{R}$ be open sets, such that $0 \in U$. Let $F \in C^2(U \times I, Y)$ and assume*

- *$F(0,\lambda) = 0$ for all $\lambda \in I$;*
- *$\ker \partial_x F(0,\lambda_0)$ is a dimension 1 subspace and $\operatorname{im} \partial_x F(0,\lambda_0)$ is a closed co-dimension 1 subspace for some $\lambda_0 \in I$;*

- *$\partial_\lambda \partial_x F(0, \lambda_0)(x_0) \notin \operatorname{im} \partial_x F(0, \lambda_0)$, where $x_0 \in X$ spans $\ker \partial_x F(0, \lambda_0)$.*

Write $X = \hat{X} \oplus \mathbb{R}x_0$. Then there exists a C^1 curve

$$(-\varepsilon, \varepsilon) \to \hat{X} \times \mathbb{R}\,, \qquad s \mapsto (x(s), \lambda(s))$$

such that

- *$x(0) = 0$ and $\lambda(0) = \lambda_0$;*
- *$s(x_0 + x(s)) \in U$ and $\lambda(s) \in I$;*
- *$F(s(x_0 + x(s)), \lambda(s)) = 0$.*

Moreover, there is a neighbourhood of $(0, \lambda_0)$ such that $\{(s(x_0 + x(s)), \lambda(s)) : s \in (-\varepsilon, \varepsilon)\}$ are the only solutions bifurcating from $\{(0, \lambda) : \lambda \in I\}$.

Note the above theorem gives an almost complete characterization of the solutions to (A.1) in a neighbourhood of the trivial branch $(0, \lambda)$.

The other Bifurcation theory result we employed is the Krasnoselskii Bifurcation Theorem. The proof of this theorem is based on Topological degree theory and is a consequence of a sign change in the Leray-Schauder degree, or *index*, of the zero solution to (A.1). The natural assumption in this case is that the family of operators in (A.1) now maps into the same Banach space $F(\cdot, \lambda) : X \to X$, and is of the form

$$F(x, \lambda) = x - K(x, \lambda),$$

where K is a compact operator. Thus, the linearization of $F(\cdot, \lambda)$ at 0 has the form

$$\partial_x F(0, \lambda) = \operatorname{id} - \partial_x K(0, \lambda),$$

where $\partial_x K(0, \lambda)$ is a compact linear operator. In other words, $\partial_x F(0, \lambda)$ is a compact perturbation of the identity. Thus, the Riesz-Schauder theory implies it makes sense to assume

(F2) There exist λ_0 such that 0 is an isolated eigenvalue of $\partial_x F(0, \lambda_0)$ of finite algebraic multiplicity.

Suppose now that $\partial_x F(0, \lambda)$ is an isomorphism, which by the Riesz-Schauder theory is equivalent to 0 not being an eigenvalue of $\partial_x F(0, \lambda)$. Then, the Leray-Schauder formula implies the index of the 0 solution to (A.1) can be calculated as

$$i(F(\cdot, \lambda), 0) = (-1)^{m_1 + \cdots + m_k}, \tag{A.6}$$

where $m_1, \ldots, m_k$ are the algebraic multiplicities of the negative eigenvalues $\mu_1, \ldots, \mu_k$ of $\partial_x F(0, \lambda)$. If the index of the 0 solution to (A.1) changes sign as λ crosses λ_0, then bifurcations are expected to occur at λ_0. By formula (A.6), the sign change in the index is equivalent to an odd number of eigenvalues of $\partial_x F(0,\lambda)$ changing sign from negative to positive as λ crosses λ_0. A detailed and more precise exposition can be found in [Nir74] and [Kie11].

We now state a version of the Krasnoselskii Bifurcation Theorem slightly weaker than the classical one. Its proof follows directly from the one given in [Kie11].

Theorem A.2 (Krasnoselskii)**.** *Let X be a Banach space, and let $U \subset X$ be an open set such that $0 \in U$. Let $F \in C^1(U \times [a,b], X)$ and assume*

- *$F(x,\lambda) = x - K(x,\lambda)$, where $x \mapsto K(x,\lambda)$ is a compact mapping for all $\lambda \in (a,b)$;*
- *$\partial_x F(0,a)$ and $\partial_x F(0,b)$ are isomorphisms;*
- *the number of negative eigenvalues of $\partial_x F(0,a)$ minus the number of negative eigenvalues of $\partial_x F(0,b)$ (counted with algebraic multiplicity) is an odd integer.*

Then every neighbourhood of $\{0\} \times [a,b]$ contains a solution of $F(x,\lambda) = 0$, with $\lambda \in (a,b)$ and $x \neq 0$.

A drawback of the above, weaker version is that we cannot identify the bifurcation point λ_0 as in the classical version or as in the Crandall-Rabinowitz Theorem.

Bibliography

[Ale62] AD Alexandrov. A characteristic property of spheres. *Annali di Matematica Pura ed Applicata*, 58(1):303–315, 1962.

[AR73] Antonio Ambrosetti and Paul H Rabinowitz. Dual variational methods in critical point theory and applications. *Journal of functional Analysis*, 14(4):349–381, 1973.

[BCN97] Henri Berestycki, Luis Caffarelli, and Louis Nirenberg. Monotonicity for elliptic equations in unbounded Lipschitz domains. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 50(11):1089–1111, 1997.

[BN91] Henri Berestycki and Louis Nirenberg. On the method of moving planes and the sliding method. *Boletim da Sociedade Brasileira de Matemática-Bulletin of the Brazilian Mathematical Society*, 22(1):1–37, 1991.

[CR71] Michael G Crandall and Paul H Rabinowitz. Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8(2):321–340, 1971.

[DGP99] Lucio Damascelli, Massimo Grossi, and Filomena Pacella. Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 16, pages 631–652. Elsevier, 1999.

[DPPW15] Manuel Del Pino, Frank Pacard, and Juncheng Wei. Serrin's overdetermined problem and constant mean curvature surfaces. *Duke Mathematical Journal*, 164(14):2643–2722, 2015.

[DS09] Daniela De Silva. Existence and regularity of monotone solutions to a free boundary problem. *Amer. J. Math.*, 131(2):351–378, 2009.

[DS15] Erwann Delay and Pieralberto Sicbaldi. Extremal domains for the first eigenvalue in a general compact Riemannian manifold. *Discrete and Continuous Dynamical Systems - A*, 35(12):5799–5825, 2015.

[Eva10] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.

[FM15] Mouhamed Moustapha Fall and Ignace Aristide Minlend. Serrin's over-determined problem on Riemannian manifolds. *Advances in Calculus of Variations*, 8(4):371–400, 2015.

[FMT08] Patricio Felmer, Salomé Martínez, and Kazunaga Tanaka. Uniqueness of radially symmetric positive solutions for $-\Delta u + u = u^p$ in an annulus. *Journal of Differential Equations*, 245(5):1198–1209, 2008.

[FMW17] Mouhamed Moustapha Fall, Ignace Aristide Minlend, and Tobias Weth. Unbounded periodic solutions to Serrin's overdetermined boundary value problem. *Archive for Rational Mechanics and Analysis*, 223(2):737–759, 2017.

[FMW18] Mouhamed Moustapha Fall, Ignace Aristide Minlend, and Tobias Weth. Serrin's overdetermined problem on the sphere. *Calculus of Variations and Partial Differential Equations*, 57(1):3, 2018.

[Giu84] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 2. Springer, 1984.

[GNN79] Basilis Gidas, Wei-Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics*, 68(3):209–243, 1979.

[GPY03] Massimo Grossi, Filomena Pacella, and SL Yadava. Symmetry results for perturbed problems and related questions. *Topological Methods in Nonlinear Analysis*, 21(2):211–226, 2003.

[Gri11] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. SIAM, 2011.

[GT15] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[HHP11] Laurent Hauswirth, Frédéric Hélein, and Frank Pacard. On an overdetermined elliptic problem. *Pacific J. Math.*, 250(2):319–334, 2011.

[Hof84] Helmut Hofer. A note on the topological degree at a critical point of mountainpass-type. *Proceedings of the American Mathematical Society*, 90(2):309–315, 1984.

[HP70] Reese Harvey and John Polking. Removable singularities of solutions of linear partial differential equations. *Acta Mathematica*, 125:39–56, 1970.

[JP18] David Jerison and Kanishka Perera. Higher critical points in an elliptic free boundary problem. *J. Geom Anal.*, 28(2):1258–1294, 2018.

[Kam13] Nikola Kamburov. A free boundary problem inspired by a conjecture of De Giorgi. *Communications in Partial Differential Equations*, 38(3):477–528, 2013.

[Kie11] Hansjörg Kielhöfer. *Bifurcation Theory: An introduction with applications to partial differential equations*, volume 156. Springer Science & Business Media, 2011.

[KLT13] Dmitry Khavinson, Erik Lundberg, and Razvan Teodorescu. An overdetermined problem in potential theory. *Pacific J. Math.*, 265(1):85–111, 2013.

[KN77] David Kinderlehrer and Louis Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[KP98] S. Kumaresan and Jyotshana Prajapat. Serrin's result for hyperbolic space and sphere. *Duke Mathematical Journal*, 91(1):17–28, 1998.

[KS20] Nikola Kamburov and Luciano Sciaraffia. Nontrivial solutions to Serrin's problem in annular domains. *Annales de l'Institut Henri Poincaré (C), Analyse non linéaire*, 2020.

[KSV05] Dmitry Khavinson, Alexander Yu Solynin, and Dimiter Vassilev. Overdetermined boundary value problems, quadrature domains and applications. *Computational Methods and Function Theory*, 5(1):19–48, 2005.

[Leo15] Gian Paolo Leonardi. An overview on the Cheeger problem. In *New trends in shape optimization*, pages 117–139. Springer, 2015.

[Lin90] Song-Sun Lin. Positive radial solutions and non-radial bifurcation for semilinear elliptic equations in annular domains. *Journal of Differential Equations*, 86(2):367–391, 1990.

[Lin93] Song-Sun Lin. Existence of many positive nonradial solutions for nonlinear elliptic-equations on an annulus. *Journal of differential equations*, 103(2):338–349, 1993.

[LM12] Jacques Louis Lions and Enrico Magenes. *Non-homogeneous boundary value problems and applications*, volume 1. Springer Science & Business Media, 2012.

[LWW17] Yong Liu, Kelei Wang, and Juncheng Wei. On one phase free boundary problem in $\mathbb{R}^n$. *arXiv preprint arXiv:1705.07345*, 2017.

[Min17] Ignace Aristide Minlend. Existence of self-Cheeger sets on Riemannian manifolds. *Archiv der Mathematik*, 109(4):393–400, 2017.

[MS16] Filippo Morabito and Pieralberto Sicbaldi. Delaunay type domains for an overdetermined elliptic problem in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. *ESAIM: Control, Optimisation and Calculus of Variations*, 22(1):1–28, 2016.

[Ni83] Wei-Ming Ni. Uniqueness of solutions of nonlinear dirichlet problems. *Journal of Differential equations*, 50(2):289–304, 1983.

[Nir74] Louis Nirenberg. *Topics in nonlinear functional analysis*, volume 6. American Mathematical Soc., 1974.

[Par11] Enea Parini. An introduction to the Cheeger problem. *Surv. Math. Appl.*, 6:9–21, 2011.

[Rei95] Wolfgang Reichel. Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains. *Pittman Research Notes in Mathematics*, pages 164–182, 1995.

[Rei97] Wolfgang Reichel. Radial symmetry for elliptic boundary-value problems on exterior domains. *Archive for Rational Mechanics and Analysis*, 137(4):381–394, 1997.

[RRS19] Antonio Ros, David Ruiz, and Pieralberto Sicbaldi. Solutions to overdetermined elliptic problems in nontrivial exterior domains. *Journal of the European Mathematical Society*, 2019.

[Ser71] James Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, 43(4):304–318, 1971.

[Sic10] Pieralberto Sicbaldi. New extremal domains for the first eigenvalue of the Laplacian in flat tori. *Calculus of Variations and Partial Differential Equations*, 37(3-4):329–344, 2010.

[Sic14] Pieralberto Sicbaldi. Extremal domains of big volume for the first eigenvalue of the Laplace–Beltrami operator in a compact manifold. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 31, pages 1231–1265. Elsevier, 2014.

[Sir01] Boyan Sirakov. Symmetry for exterior elliptic problems and two conjectures in potential theory. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 18(2):135–156, 2001.

[Sir02] Boyan Sirakov. Overdetermined elliptic problems in physics. In *Nonlinear PDE's in Condensed Matter and Reactive Flows*, pages 273–295. Springer, 2002.

[Sri93] PN Srikanth. Uniqueness of solutions of nonlinear dirichlet problems. *Differential and Integral Equations*, 6(3):663–670, 1993.

[SS12] Felix Schlenk and Pieralberto Sicbaldi. Bifurcating extremal domains for the first eigenvalue of the Laplacian. *Advances in Mathematics*, 229(1):602–632, 2012.

[SW10] Andrzej Szulkin and Tobias Weth. The method of nehari manifold. *Handbook of nonconvex analysis and applications*, pages 597–632, 2010.

[Tan03] Moxun Tang. Uniqueness of positive radial solutions for $\Delta u - u + u^p = 0$ on an annulus. *Journal of Differential Equations*, 189(1):148–160, 2003.

[Tra14] Martin Traizet. Classification of the solutions to an overdetermined elliptic problem in the plane. *Geom. Funct. Anal.*, 24(2):690–720, 2014.

[WGS94] N.B. Willms, G.M.L. Gladwell, and D. Siegel. Symmetry theorems for some overdetermined boundary value problems on ring domains. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 45(4):556–579, 1994.