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MASTER THESIS

Pseudo orbits and topological pressure

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Abstract

In this thesis new, equivalent, definitions of topological pressure of a continuous map defined on a compact metric space with respect to a continuous potential are given. Our definitions make use of the notion of pseudo-orbit. Among other things, we prove that the topological pressure is the exponential growth rate of weighted periodic pseudo-orbits. This result generalizes previous work of M. Barge and R. Swanson on topological entropy.

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Introduction

The realization that complicated behavior exhibited by certain dynamical systems can be studied using methods from measure theory and probability theory was one of the most important breakthroughs in theory of dynamical systems. It was Poincaré who realized that the sole existence of a finite invariant measure yield non trivial information on the orbit structure of the system. *Thermodynamic formalism* is a sub-area of ergodic theory which addresses the problem of choosing relevant invariant measures among the, sometimes very large, set of invariant probabilities. This theory was brought from statistical mechanics into dynamics in the early seventies by Bowen, Ruelle, Sinai and Walters among others. The powerful formalism developed to study equilibrium of systems consisting of a large number of particles (e.g. gases) has been surprisingly efficient to describe certain dynamical systems that exhibit complicated behavior. One of the main objects of the theory is a functional defined on the space of continuous functions called *topological pressure*. We note that the topological entropy coincides with the pressure at the constant function equal to zero. In this thesis we provide a new point of view on this functional. Indeed, our main result is a new definition of the pressure based on work by M. Barge and R. Swanson [2] on topological entropy. We stress that the definition we propose coincides with the classical one. We define the pressure counting the exponential growth of periodic pseudo-orbits. Classical results allow for a definition of pressure, in the context of uniformly expanding (or uniformly hyperbolic) systems, counting the exponential growth of periodic orbits [11]. Interestingly, our definition provides an analogous view point that works even for minimal systems, where no periodic orbit exists.

In 1965 R.L. Adler, A.G. Konheim, and M.H. McAndrew [1] first defined the *topological entropy* of a a continuous map $f : X \to X$ defined on a compact space. Their definition was purely topological and based on open covers. It somehow mimics the definition of entropy of a measure (as defined by Kolomogorov and Sinai in 1959). Later, between 1970 and 1975, E. Dinaburg [7] and R. Bowen [4],[5] introduced equivalent definitions on compact metric spaces. These definitions made a strong use of the metric in the space. It turns out that equivalent metrics yield the same value for the entropy. Therefore, the object it still purely topological, but this metric approach clarified the meaning of the entropy. It was readily observed that for some important classes of systems, namely uniformly expanding/hyperbolic systems, the topological entropy measured the exponential growth of periodic orbits. It was, therefore, possible to define it and compute just by means of the the periodic data of the system.

During the 1970s other major breakthroughs in the theory of dynamical systems occurred. For example, C. Conley [6] introduced several ideas and techniques from algebraic topology in the study of dynamics. For our purposes, one of the main contributions was the definition of pseudo-orbit. While in the classical study of differential equations and dynamical systems we were only interested in orbits, Conley suggested that studying *approximate orbits* could be of great significance. He was indeed right. Ever since, pseudo-orbits have played a major role in dynamics. In the context of thermodynamic formalism, Misiurewicz [9] proved that the topological entropy of a continuous transformation defined on a compact metric space can be computed by means of pseudo-orbits. Years later, M. Barge and R. Swanson [2] showed that the topological entropy is equal to the exponential growth rate of the number of periodic pseudo-orbits. Since there exists minimal systems of positive entropy [12], where there are no periodic orbits and therefore we cannot obtain results similar to those of hyperbolic systems. However, as we can always construct periodic pseudo orbits, it is interesting to wonder if analogously we can calculate the topological entropy not of periodic orbits but with periodic pseudo orbits. The Barge and Swanson's result allows for the recovery of the interpretation of topological entropy we have in the setting of uniformly expanding systems. Namely, the topological entropy measures the exponential growth of pseudo-periodic orbits.

The topological pressure is a generalization of the notion of topological entropy in which points in the space are weighted with a continuous function sometimes called potential. For a continuous map $f : X \to X$ defined on a compact metric space X and a continuous function $\phi : X \to \mathbb{R}$, the pressure $P(\phi, f)$, was defined in 1973 by Ruelle [14] and studied more generally by Walters [16]. It turns out that the topological pressure captures a great amount of the dynamical information of the system. For example, it determines the space of invariant probability measures. It satisfies the so called, variational principle, which establishes a relation between the measurable and the topological dynamics of the system. The pressure has been used to compute the Hausdorff dimension of attractors and to establish some prime number theorems on the periodic orbits of certain systems. It is the fundamental object of thermodynamic formalism. Following the line of thought of Misiurewicz and also that of Barge and Swanson we studied the topological pressure from the pseudo-orbit point of view. In our first main result we provide a characterization of the topological pressure in terms of pseudo-orbits. Details can be found in Chapter 5.

Theorem A. Let f be a continuous transformation in a compact metric space (X, d). Let $\varphi \in C(X)$. Then the topological pressure of φ with respect to f is equal to the pseudo-pressure of φ with respect to f, i.e.,

$$P(\varphi, f) = P_{\psi}(\varphi, f).$$

Our second main results generalizes the fact that for uniformly hyperbolic systems the topological pressure can he computed using the periodic data.

Theorem B. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then the topological pressure of φ with respect to f is equal to $Pe_{\psi}(\varphi, f)$, i.e.,

$$P(\varphi, f) = Pe_{\psi}(\varphi, f),$$

where $Pe_{\psi}(\varphi, f)$ is a functional that depends only on periodic pseudo-orbits of period n as n tends to infinity.

As a consequence of this result. Let f be an expansive homeomorphism on a compact metric space (X,d). Let $\varphi \in C(X)$. If f has the shadowing property. Then

$$P(\varphi, f) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{F\subseteq \mathrm{Fix}(f^n)} \left\{ \sum_{x\in F} \exp\left\{ \sum_{i=1}^{n-1} \varphi(x_i) \right\} \right\}.$$

This thesis is made up of 5 chapters. In the first we have the preliminaries where we present the definitions of topological entropy and pressure, via open coverings, spanning sets, and (n,ε)-separated sets. Some examples, including an overview of the subshift of finite type case, are considered. The second chapter is dedicated to some consequences of the variational principle, for this, we introduce basic concepts of the required ergodic theory. In the third chapter, we present the concept of pseudo-orbit together with some of its properties. Finally, we define the concepts of pseudo-entropy, the exponential growth of periodic pseudo-orbits, pseudo pressure, and the functional Pe_ψ. The fourth chapter is devoted to explaining in detail the theorems of Misiurewiczs, M. Barge, and R. Swanson. In the last chapter, we will show Theorem A, Theorem B and some consequences.

Chapter 1

Preliminaries

This chapter summarizes notions and results that will be used throughout the thesis. The assumed background consists of a basic course on topology and measure theory. We will also introduce elementary notions of topological dynamics and ergodic theory. We refer the reader to [17].

1.1 Topological Entropy

The topological entropy of a dynamical system (X, f), where X is compact, quantifies the complexity of the system. Let $n \in \mathbb{N}$, in the course of the chapter we will see that, as a consequence of Bowen's definition, topological entropy measures the growth rate of the number of different orbits of length n as n tend to infinity. A precise meaning to this claim will be provided in this chapter. In particular, the meaning of different orbits will be explained.

1.1.1 Definition with open covers

Definition 1.1.1. Let X be a topological space. An open **cover of** X is a collection $\mathcal{A}$ of open subsets of X whose union is X. A **subcover of** $\mathcal{A}$ is a subcollection of $\mathcal{A}$ that is still a cover.

Definition 1.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be open covers of a topological space X. We will say that $\mathcal{B}$ is a **refinement** of $\mathcal{A}$, if each element of $\mathcal{B}$ is a subset of some member of $\mathcal{A}$. In this case, we will denote $\mathcal{A} \prec \mathcal{B}$ to express this relation.

Definition 1.1.3. Let $f : X \to X$ be a continuous transformation on a topological space X. Let $x \in X$. For each $n \in \mathbb{N}$, we define **the n-th iteration of x under f** as the n-th composition of f evaluated at x, i.e., $f^n(x) = f \circ f^{n-1}(x)$. We define the **orbit of x under f** as the set which contains all iterations $f^n(x)$ and denote it by $\mathcal{O}_x$, i.e., $\mathcal{O}_x := \{f^n(x) | n \in \mathbb{N}\}$. If f is a homemorphism we can take $n \in \mathbb{Z}$.

Definition 1.1.4. Let f be a continuous transformation on a compact metric space (X, d). For each $n \in \mathbb{N}$, we define the **set of fixed points of f^n** by

$$\mathrm{Fix}(f^n) := \{x \in X \quad | \qquad f^n(x) = x\}.$$

In the definitions that follows f is a continuous transformation in a compact space X. The next definition allows us to construct refinements of a given open cover.

Definition 1.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be open covers of X. We define their **join**, as the collection of all sets of the form $U \cap V$, where $U \in \mathcal{A}$ and $V \in \mathcal{B}$, and denoted by $\mathcal{A} \vee \mathcal{B}$. Note that this join is a refinement of both covers. This allows us to construct refinements of a single open cover $\mathcal{A}$. For each $n \in \mathbb{N}$, we define

$$\mathcal{A}^n := \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A}),$$

where $f^{-i}(\mathcal{A}) := \{f^{-i}(U) : U \in \mathcal{A}\}$.

This open cover $\mathcal{A}^n$, generated by $\mathcal{A}$, allows us to define the first notion of entropy. In order to do that we need the following result [17, Theorem 7.1].

Proposition 1.1.1. Let $\mathcal{A}$ be an open cover of X. We denote by $N(\mathcal{A})$ the number of elements that have the finite open subcovers of $\mathcal{A}$ with the smallest cardinality. Then, the limit $\lim_{n\to\infty} \frac{1}{n} \log(N(\mathcal{A}^n))$ exists and is equal to $\frac{1}{n} \inf_n \log(N(\mathcal{A}^n))$.

Definition 1.1.6. Let $\mathcal{A}$ be an open cover of X. We define **the entropy of f relative to the cover $\mathcal{A}$** by

$$h(f, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log(N(\mathcal{A}^n)).$$

Example 1.1.1. Consider the space S^1. Recall that we can identify this space as the unit circle of the complex plane [15, p34]. Let f be the rotation map by $\frac{\pi}{2}$, i.e., for every $0 \le \theta < 2\pi$, it maps $\exp(i\theta) \mapsto \exp(i(\theta + \frac{\pi}{2}))$. Let γ the finite open cover described by

$$\gamma = \left\{ \left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{3}, \frac{5\pi}{6}\right), \left(\frac{2\pi}{3}, \frac{7\pi}{6}\right), \left(\pi, \frac{3\pi}{2}\right), \left(\frac{4\pi}{3}, \frac{11\pi}{6}\right), \left(\frac{5\pi}{3}, \frac{13\pi}{2}\right) \right\}.$$

First, we note that f is a homeomorphism with period 4. Therefore $\gamma = f^{-4}(\gamma)$ and for every integer $k \ge 4$ we have $\gamma^k = \gamma^4$ as in Definition 1.1.5. If we denote by $\partial\gamma$ the set of extreme points of the intervals in γ, we notice that for every integer $n \ge 4$,

$$\#\gamma^n = \#\gamma^4 \le \#\partial\gamma^4 \le 44$$

Thus, we can estimate the topological entropy of f with respect to the cover γ by

$$\begin{aligned} h(f, \gamma) &= \lim_{n\to\infty} \frac{1}{n} \log(N(\gamma^n)) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(N(\gamma^4)) \\ &\le \lim_{n\to\infty} \frac{1}{n} \log(44)) = 0. \end{aligned}$$

By definition $N(\gamma) \ge 0$, thus we conclude that $h(f, \gamma) = 0$.

Definition 1.1.7. Let f be a continuous transformation in a compact space X. We define the **topological entropy of f** as the supremum among the values $h(f, \mathcal{A})$ overall open covers $\mathcal{A}$ of X, i.e., is given by

$$h(f) := \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is a open cover of X}\}.$$

Remark. Due to the compactness of the space, it is enough to take the supremum over finite open covers.

Proposition 1.1.2. (**Properties of topological entropy**,[17, p165-167])
Let f be a continuous transformation on a compact space X. Let $\mathcal{A}$ and $\mathcal{B}$ be open covers of X.

i If $\mathcal{A} \prec \mathcal{B}$, then $N(\mathcal{A}) \leq N(\mathcal{B})$.

ii If $\mathcal{A} \prec \mathcal{B}$, then $h(f, \mathcal{A}) \leq h(f, \mathcal{B})$.

iii $h(f)$ is non-negative.

iv If $Y \subset X$ a closed subset and invariant under f, then $h(f|_Y) \leq h(f)$.

v Let X_1 and X_2 be compact spaces with continuous maps $f_i : X_i \to X_i$ for $i = 1, 2$. If $\phi : X_1 \to X_2$ is a continuous function such that $\phi X_1 = X_2$ and $\phi f_1 = f_2 \phi$, then $h(f_1) \geq h(f_2)$. The equality holds if ϕ is a homeomorphism.

Remark. This last property says that the topological entropy is an invariant of topological conjugacy [17, Theorem 7.2].

If the space X is endowed with a metric, we present another way of calculating the topological entropy without taking into account all the open covers. To achieve that, we define the diameter of a open cover $\mathcal{A}$ by $\text{diam}(\mathcal{A}) = \sup_{B \in \mathcal{A}} \text{diam}(B)$.

Proposition 1.1.3. ([17, Theorem 7.6])
Let $\{\mathcal{A}_k\}_k$ be any family of open covers such that $diam(\mathcal{A}_k) \to 0$ when $k \to \infty$. Then the topological entropy of f is equal to $\lim_{k \to \infty} h(f, \mathcal{A}_k)$.

Example 1.1.2. Consider S^1 as in Example 1.1.1. Let f be a rotation by any angle $\alpha > 0$. We proceed in the same way for an arbitrary open cover. Let $\mathcal{B}$ be a finite open cover of S^1. Without loss of generality, assume that $\mathcal{B}$ is formed by open intervals. We denote by $\partial \mathcal{B}$ the set of extreme points of the intervals in $\mathcal{B}$. Since f is a homeomorphism, then we note that

$$\partial \mathcal{B}^n = \partial \mathcal{B} \cup f^{-1}(\partial \mathcal{B}) \cup ... \cup f^{-(n-1)}(\partial \mathcal{B}).$$

This allows us to deduce for every $n \in \mathbb{N}$,

$$\# \mathcal{B}^n \leq \# \partial \mathcal{B}^n \leq n \# \partial \mathcal{B}.$$

Therefore we can estimate the topological entropy of f with respect to the open cover $\mathcal{B}$ by

$$\begin{aligned} h(f, \mathcal{B}) &= \lim_{n \to \infty} \frac{1}{n} \log(N(\mathcal{B}^n)) \\ &\leq \lim_{n \to \infty} \frac{1}{n} \log(n N(\mathcal{B})) \\ &\leq \lim_{n \to \infty} \frac{1}{n} (\log(n) + N(\mathcal{B})) \\ &= \lim_{n \to \infty} \frac{1}{n} \log(n) = 0. \end{aligned}$$

Since the topological entropy is non-negative and $\mathcal{B}$ was taken arbitrarily, we conclude that $h(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is a open cover of X}\} = 0$.

Example 1.1.3. Set $\Sigma = \{1, 2, ..., k\}^{\mathbb{N}}$ endowed with a metric d defined by

$$d((x_n), (y)_n) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n},$$

where $(x_n), (y_n) \in \Sigma$. Let $\mathcal{A}$ be the open cover defined by the cylinders $[0; i] := \{(x_n) \in \Sigma : x_0 = i\}$ with $1 \leq i \leq k$. Consider the shift map $\sigma : \Sigma \to \Sigma$ defined by $(x_i)_{i\geq 0} \mapsto (x_i)_{i\geq 1}$. Note that for each $i = 1, 2, ...n$, the preimage of the cylinder $[0; i]$ is

$$\sigma^{-1}([0; i]) = \{(x)_n \in \Sigma : x_1 = i\} = [1; i].$$

Therefore the join of $\mathcal{A}$ with $\sigma^{-1}(\mathcal{A})$ is $\{[0, 1; i, j] : 1 \leq i, j \leq k\}$, i.e., the cylinders of length two centered on the first term. Let $m \in \mathbb{N}$. Generalizing we have that

$$\mathcal{A}^m = \{[0, 1, ..., n-1; i_1, ...i_m] \quad : \quad 1 \leq i_j \leq k \quad \text{for every} \quad 1 \leq j \leq m\}.$$

This is, the cylinders of length m centered on the first term. Note that the elements of $\mathcal{A}^m$ are pairwise disjoint. This allows us to deduce that $N(\mathcal{A}^m) = k^m$, which implies

$$h(\sigma, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log(k^n) = \lim_{n\to\infty} \log(k) = \log(k).$$

On the other hand, the diameter of an element B in $\mathcal{A}^n$ satisfies that

$$\text{diam}(B) \leq \sum_{i=0}^{\infty} \frac{k-1}{2^n} = (k-1) \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{k-1}{2^n} 2 \to 0 \quad \text{when} \quad n \to \infty.$$

Then the diameter of $\mathcal{A}^n$ approaches zero as n goes to infinity. Considering the succession of open coverings $\{\mathcal{A}^n\}_n$, it satisfies the hypothesis of Proposition 1.1.3. Then the topological entropy of Σ for σ is equal to $\lim_{n\to\infty} h(\sigma, \mathcal{A}^n) = \log(k)$.

Remark. The same result holds for the two-side shift σ on the space $\Sigma = \{1, ..., k\}^{\mathbb{Z}}$.

1.1.2 Bowen's definition

A few years after R.L. Adler, A.G. Konheim and M.H. McAndrew defined topological entropy, R. Bowen [4][5] proposed an equivalent definition that holds for metric spaces (not necessarily compact). His definition is based on the notions of spanning and generated sets. In this sub-section we will assume f to be a continuous transformation defined on the metric space (X, d).

Definition 1.1.8. Let $K \subseteq X$ be a compact subset. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset $F \subset K$ is said to (n, ε)**-span** K respect to the transformation f, if for every $x \in K$ there is $y \in F$ such that for every $1 \leq k \leq n-1$, it holds $d(f^k(x), f^k(y)) \leq \varepsilon$.

Definition 1.1.9. Let $K \subseteq X$ be a compact susbset. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Denote by $r_n(\varepsilon, K)$ the smallest cadinality between (n, ε)-spanning sets of K.

Definition 1.1.10. Let $K \subseteq X$ be a compact subset. Let $\varepsilon > 0$. We define $r(\varepsilon, f, K) = \limsup_{n\to\infty} \log(r_n(\varepsilon, K))$

Remark. Because of the compactness of K, there is a finite open cover of K with balls of radius ε, say $\gamma = \{B_1, B_2, ...B_m\}$. Taking a representative for each element of γ^n we obtain a finite (n, ε)-spanning set and therefore $r_n(\varepsilon, K) < \infty$.

Definition 1.1.11. Let $K \subseteq X$ be a compact subset. Let $h_d(f, K) = \lim_{\varepsilon \to 0} r(\varepsilon, f, K)$. We define the **topological entropy of** f as the supremum among the values $h_d(f, K)$, where the supremum is taken over the collection of compact subsets of X, i.e.,

$$h(f)_d := \sup\{h_d(f, K) : K \text{ a compact subset of X}\}.$$

If the metric d is understood, we will write $h(f, K)$ and $h(f)$ for short.

We will now present the third definition of topological entropy via separated sets.

Definition 1.1.12. Let $K \subseteq X$ be a compact subset. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset $E \subseteq K$ is said to be (n, ε)**-separated** if for every $x, y \in E$ with $x \neq y$, there is $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \varepsilon$.

Therefore, a (n, ε)-separated set is a set in which the orbits of length n of each of its elements can be differentiated by an ε-error.

Definition 1.1.13. Let $K \subseteq X$ be a compact subset. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let us denote by $s_n(\varepsilon, K)$ the largest cardinality between (n, ε)-separated subsets of K.

We will see in Proposition 1.1.4 that $s_n(\varepsilon, K) < \infty$.

Definition 1.1.14. Let $K \subseteq X$ be compact. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. We define $s(\varepsilon, f, K)$= $\limsup_{n\to\infty} \frac{1}{n} \log(s_n(\varepsilon, K))$.

The following proposition is presented in [17, p169].

Proposition 1.1.4. Let f be a continuous transformation on a metric space (X, d). Let $K \subseteq X$ be a compact subset. Let $\varepsilon > 0$. Then

$$h_d(f, K) = \lim_{\varepsilon \to 0} s(\varepsilon, K, f).$$

And,

$$h_d(f) = \sup_K \{\lim_{\varepsilon \to 0} s(\varepsilon, K, f) \quad | \quad K \subseteq X \quad \text{compact}\}.$$

Proof. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $K \subseteq X$ a compact subset. Let E be an (n, ε)-separated set of K such that $\#E = s_n(\varepsilon, K, f)$. Let us see that E is a (n, ε)-spanning set. Suppose by contradiction that there exists $x \in K$ such that for every $y \in E$ and every $0 \leq i \leq n-1$, it hols that $d(f^i(x), f^i(y)) > \varepsilon$. Then $E \cup \{x\}$ would be an (n, ε)-separated set, but this cannot be since by definition of $s_n(\varepsilon, K)$, E has maximal cardinality among (n, ε)-separated sets. Therefore E is a (n, ε)-spanning set and thus $r_n(\varepsilon, K) \leq s_n(\varepsilon, K)$.

Let F be a $(n, \frac{\varepsilon}{2})$-spanning set of K. We define the map $\psi : E \to F$ such that it sends each element of E to the closest element of F. Suppose $x, y \in E$ with $x \neq y$ such that $\psi(x) = \psi(y)$, then $d(x, y) \leq d(x, \psi(x)) + d(x, \psi(y)) = \varepsilon$, which contradicts the assumption that E is (n, ε)-separated. Then ψ is injective. So $\#E \leq \#F$, which implies $s_n(\varepsilon, K) \leq r_n(\frac{\varepsilon}{2}, K)$. We conclude that $r_n(\varepsilon, K) \leq s_n(\varepsilon, K) \leq r_n(\frac{\varepsilon}{2}, K)$. Since n was chosen arbitrarily,

$$r(\varepsilon, K) \leq s(\varepsilon, K) \leq r(\frac{\varepsilon}{2}, K)$$

Also ε was arbitrarily chosen, then $h(f) = \lim_{\varepsilon \to 0} r(\varepsilon, K) = \lim_{\varepsilon \to 0} s(\varepsilon, K)$. $\square$

As we said before, the definition of topological entropy is equivalent to the one we gave via open covers, of course, provided that the space is metric and compact. The reader can find a proof in [17, Theorem 7.8]. Now we present an example of a non-compact metric space with infinite topological entropy.

Example 1.1.4. Let $X = \mathbb{R}$ endowed with the Euclidean metric and the continuous transformation f defined by $f(x) = x^2$. Let $K = [3, 4] \subset \mathbb{R}$. We will calculate the entropy of f for this compact subset. Let $x, y \in K$. As $x, y \geq 3$, then $d(f(x), f(y)) = |x^2 - y^2| = d(x, y)|x + y| \leq 6d(x, y)$. So if $n \in \mathbb{N}$, then

$$\begin{aligned} d(f^{n-1}(x), f^{n-1}(y)) &= |x^{2^{n-2}} + y^{2^{n-2}}| d(f^{n-2}(x), f^{n-2}(y)) \\ &\geq 2 * 3^{2^{n-2}} d(f^{n-2}(x), f^{n-2}(y)) \\ &\vdots \\ &\geq 2^{n-1} * 3^{\sum_{i=0}^{n-2} 2^i} |x - y|. \end{aligned}$$

Let $\varepsilon > 0$. Therefore if two points $x, y \in K$ satisfy that

$$|x - y| > \varepsilon / (2^{n-1} * 3^{\sum_{i=0}^{n-2} 2^i} |x - y|),$$

then they are (n, ε)-separated. So in K every (n, ε)-separated set has at most $\lfloor 2^{n-1} 3^{\sum_{i=0}^{n-2} 2^i} |x - y| / \varepsilon \rfloor$ points. This implies that $S_n(\varepsilon, f, K) \geq 2^{n-1} * 3^{\sum_{i=0}^{n-2} 2^i} |x - y| / \varepsilon$. We obtain

$$\begin{aligned} S(\varepsilon, f, K) &= \limsup_{n \to \infty} \frac{1}{n} \log(S_n(\varepsilon, K)) \\ &\geq \limsup_{n \to \infty} \frac{1}{n} \left((n-1) \log(2) + \sum_{i=0}^{n-2} 2^i \log(3) - \log(\varepsilon) \right) \\ &= \limsup_{n \to \infty} \left(\log(2) + \frac{\sum_{i=0}^{n-2} 2^i}{n} \log(3)) \right) = \infty. \end{aligned}$$

Since ε was chosen arbitrarily, we have that $h(f, K) = \infty$. Thus, by Definition 1.1.11 the topological entropy of f is infinite.

Now we cite an example in which the topological entropy can be calculated from periodic points [16, Theorem 7.3 and Theorem 8.17].

Example 1.1.5. Let A be an irreducible $k \times k$ matrix whose entries belong to $\{0, 1\}$. Let $X = \prod_{-\infty}^{\infty} \{0, 1, ...k - 1\}$. Denote

$$X_A := \{(y_i)_{i \in \mathbb{Z}} \quad | \quad y_i \in \{0, 1, ...k - 1\} \quad \text{and} \quad a_{x_i, x_{i+1}} = 1\},$$

where $a_{x_i,x_{i+1}}$ is the $x_i \times x_{i+1}$ entry of A. Let σ_A be the shift map on X_A. The dynamical system (X_A, σ_A) is called subshift of finite type. Let λ be the largest among the eigenvalues of A. It can be shown that

$$h(\sigma_A) = \log \lambda. \tag{1.1}$$

Let $n \in \mathbb{N}$. If $(x_i) \in X_A$, then $(x_i) \in Fix(\sigma_A^n)$ if and only if for all $i \in \mathbb{Z}$, $x_r = x_{r+n}$. That is, the $x_r \times x_r$ entry of the matrix A^n is positive. If we denote $a_{i,j}$ to the entry $i \times j$ of the matrix A, then

$$\begin{aligned}
\#Fix(\sigma_A^n) &= \sum_{i_0,\dots i_{n-1}=0}^{k-1} a_{i_0,i_1} a_{i_1,i_2} \dots a_{i_{n-1},i_0} \\
&= \text{trace of } A^n \\
&= \sum_{i=1}^{k} \lambda_i^n,
\end{aligned}$$

where $\lambda_1, \lambda_2, ..., \lambda_k$ are the eigenvalues of A. So if we divide by λ, we get that

$$\lim_{n\to\infty} \frac{\#Fix(\sigma_A^n)}{\lambda^n} = \lim_{n\to\infty} \frac{\sum_{i=1}^k \lambda_i^n}{\lambda^n} = 1.$$

Which implies that from what was said in Equation (1.1),

$$\lim_{n\to\infty} \frac{1}{n} \log \#Fix(\sigma_A^n) = \log \lambda = h(\sigma_A).$$

1.2 Pressure

Let f be a continuous transformation in a compact metric space (X, d). Let $C(X)$ be the Banach space of continuous real-valued functions endowed with the supremum norm. In this subsection, we will define topological pressure (or pressure for short) as a map $P : C(X) \to \mathbb{R} \cup \{\infty\}$. We will see that pressure will be an extension of topological entropy. Later in Chapter 2, we will show the close relationship between pressure and the set of f-invariant probability measures in X. As we did with topological entropy, we are going to present equivalent definitions, using open covers, separated sets, and spanning sets.

1.2.1 Definition with open covers

Let (X, d) be a compact metric space. We will call the elements of $C(X)$ by "potentials". If $\varphi \in C(X)$, then we will talk about the topological pressure of the potential φ with respect to f. Let $n \in \mathbb{N}$. We denote $\varphi^n(x) = \sum_{i=0}^{n-1} \varphi(f^i(x))$ to the n-th Birkoff sum evaluated at a point $x \in X$ for the potential φ.

Definition 1.2.1. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$, $n \in \mathbb{N}$ and let $\mathcal{A}$ be an open cover of X. We denote

$$P_n(\varphi, f, \mathcal{A}) := \inf \left\{ \sum_{U\in\gamma} \sup_{x\in U} \exp\left(\varphi^n(x)\right) \;\middle|\; \gamma \text{ a finite subcover of } \mathcal{A}^n \right\}.$$

Since φ is bounded in X by compactness, $P_n(\varphi, f, \mathcal{A})$ is the infimum over a subset of bounded real numbers. Thus $P_n(\varphi, f, \mathcal{A}) < \infty$. We define **the pressure of the potential φ with respect to f and the open cover $\mathcal{A}$** by

$$P(\varphi, f, \mathcal{A}) := \limsup_{n\to\infty} \frac{1}{n} \log P_n(\varphi, f, \mathcal{A}).$$

Similar to topological entropy, we want to calculate the pressure as we set the diameter of the cover $\mathcal{A}$ to zero. The following lemma ensures that this limit exists and does not depend on the choice of covers [15, Lemma 10.3.1].

Lemma 1.2.1. Let $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ be any sequence of open covers of X such that

$$\operatorname{diam}(\mathcal{A}_k) \to 0, \quad \text{when} \quad k \to \infty.$$

Then the limit $\lim_{k\to\infty} P(\varphi, f, \mathcal{A}_k)$ exists in $\mathbb{R}\cup\{\infty\}$ and does not depend on the choice of the sequence.

Definition 1.2.2. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$ and $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ be a sequence of open covers such that $\operatorname{diam}(\mathcal{A}_k) \to 0$ as $k \to \infty$. We define the **topological pressure of the potential φ with respect to f** as

$$P(\varphi, f) = \lim_{k\to\infty} P(\varphi, f, \mathcal{A}_k).$$

1.2.2 Definition via (n, ε)-separated sets and spanning sets

Definition 1.2.3. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Denote

$$S_n(\varphi, f, \varepsilon) := \sup\left\{\sum_{x\in F} \exp\left(\varphi^n(x)\right) \quad : \quad E \text{ is a } (n,\varepsilon)\text{-separated set of X}\right\}.$$

$$G_n(\varphi, f, \varepsilon) := \inf\left\{\sum_{x\in E} \exp\left(\varphi^n(x)\right) \quad : \quad F \text{ is a } (n,\varepsilon)\text{-spanning set of X}\right\}.$$

It can be seen these values are finite due to the compactness of X(see Section 1.1.2). Thus, we define

$$S(\varphi, f, \varepsilon) := \limsup_{n\to\infty} \frac{1}{n} \log S_n(\varphi, f, \varepsilon).$$

$$G(\varphi, f, \varepsilon) := \limsup_{n\to\infty} \frac{1}{n} \log G_n(\varphi, f, \varepsilon).$$

Definition 1.2.4. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. We define $S(\varphi, f) = \lim_{\varepsilon\to 0} S(\varphi, f, \varepsilon)$ and $G(\varphi, f) = \lim_{\varepsilon\to 0} G(\varphi, f, \varepsilon)$.

Remark. Let $\varepsilon_1, \varepsilon_2 > 0$ and $n \in \mathbb{N}$. Following Definition 1.1.8, if $\varepsilon_1 > \varepsilon_2$, then each (n, ε_1)-separated set is a (n, ε_2)-separated set. This implies $S_n(\varphi, f, \varepsilon_1) \leq S_n(\varphi, f, \varepsilon_2)$. So the sequence $\{S(\varphi, f, \mathcal{A})\}_n$ is decreasing and monotone. Therefore the limit $\lim_{\varepsilon\to 0} S(\varphi, f, \varepsilon)$ exists in $\mathbb{R} \cup \{\infty\}$. Analogously it can be seen $\lim_{\varepsilon\to 0} G(\varphi, f, \varepsilon)$ exists.

It can be shown that the values $G(\varphi, f)$ and $S(\varphi, f)$ coincide with $P(\varphi, f)$, and therefore the definitions are equivalent. A proof can be found on [15, Proposition 10.3.4].

Proposition 1.2.1. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then $P(\varphi, f) = S(\varphi, f) = G(\varphi, f)$.

Example 1.2.1. Consider the space $\Sigma = \{0, 1, ..., k\}^{\mathbb{N}}$ with the shift map σ and endowed with the metric d as in Example 1.1.3. Let $\overline{0}$ be the element in Σ that contains all its terms equal to 0. Let $c \in \mathbb{R}$. Consider the open cover $\mathcal{A} = \{[0; i] \quad | \quad 0 \leq i \leq k\}$,and the potentials $\varphi_1(x) = d(\overline{0}, x)$,and the constant potential $\varphi_2 = c$. As in Example 1.1.3, the elements of $\mathcal{A}^n$ generates a partition of Σ. Therefore it is the only open subcover of itself to consider. Notice that the n-th Birkhoff sum φ_2^n is constant and equal to nc. Hence we obtain that:

$$P_n(\varphi_2, \sigma, \mathcal{A}) = \sum_{[0,1,...,n-1:i_0,i_1,...i_{n-1}]} \sup_{x \in [0,1,...,n-1:i_0,i_1,...i_{n-1}]} e^{nc} = k^n e^{nc},$$

where $0 \leq i_m \leq k$, for every $1 \leq m \leq n-1$. Thus

$$P(\varphi_2, \sigma, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log(k^n e^{nc}) = \log(k) + c.$$

Since $\operatorname{diam}(\mathcal{A}_k) \to 0$ when k goes to infinity, we conclude by Lemma 1.2.1 that $P(\varphi_2, \sigma) = \log(k) + c$. We will see in Proposition 1.2.2 that this result of using constant potentials or even translate a potential by a constant can be generalized. For φ_1, see that we can bound the nth Birkoff sum. Let $x \in [0, 1, ..., n-1 : i_0, i_1, ...i_{n-1}]$. Thus

$$\varphi_1(x) = \sum_{m=1}^{\infty} \frac{|x_m|}{2^m} \leq \sum_{m=1}^{\infty} \frac{k}{2^m} = 2k.$$

This implies $\varphi_1^n \leq n2k$. So

$$P_n(\varphi_1, \sigma, \mathcal{A}) \leq \sum_{[0,1,...,n-1:i_0,i_1,...i_{n-1}]} \sup_{x \in [0,1,...,n-1:i_0,i_1,...i_{n-1}]} e^{2kn} = k^n e^{2kn}.$$

We conclude that $P(\varphi_1, \sigma, \mathcal{A}) \leq \limsup_{n \to \infty} \frac{1}{n} \log(k^n e^{2kn}) = \log(k) + 2k < \infty$. Therefore the pressure of σ with respect to the potential φ_1 is finite.

Before finishing this subsection we present some of the properties that topological pressure satisfies and that will be necessary for later chapters [17, Theorems 9.7 and 9.8].

Proposition 1.2.2. (**Properties of topological pressure**)
Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi, \phi \in C(X)$. Then:

i $P(0, f) = h(f)$.

ii If $\varphi \leq \phi$, then $P(\varphi, f) \leq P(\phi, f)$. In particular $h(f) + \inf(f) \leq P(\varphi, f) \leq \sup(f)$.

iii $P(.,f)$ is Lipschitz with constant 1, i.e., if $P(.,f) < \infty$, then $|P(\varphi, f) - P(\phi, f)| \leq ||\varphi - \phi||_\infty$.

iv Let $c \in \mathbb{R}$. Then $P(T, f + c) = P(T, f) + c$.

v Let (X_1, d_1) and (X_2, d_2) compact metric spaces with continuous maps $f_i : X_i \to X_i$ for $i = 1, 2$. If $\phi : X_1 \to X_2$ is a surjective continuous map with $\phi f_1 = f_2 \phi$. Then for every $\varphi \in C(X_2)$ we have $P(\varphi \circ \phi, f_1) \geq P(\varphi, f_2)$. The equality holds if ϕ is a homeomorphism.

Now we cite an example from [11, Proposition 5.1] of how to calculate for subshift of finite type the topological pressure with respect to Hölder continuous functions from the periodic points of the shift.

Example 1.2.2. For this, consider the subshift of finite type (X_A, σ_A) as in Example 1.1.5. Let $(x_i) \in X_A$ and $m \in \mathbb{N}$. We will denote for every $r \in \mathbb{Z}$ the finite sequences $\{(x_r, x_{r+1}, ..., x_{r+m})\}$ by words of length m. If η is a word of length $n \in \mathbb{N}$, we denote by $[\eta]$ to the cylinder

$$\{(x_i) \in X_A \quad | \quad (x_0, x_1, ..., x_n) = \eta\}$$

Now we define a particular set of functions, the Hölder continuous functions. Let $\theta \in (0, 1)$ define a metric on X_A as $d_\theta((x_j), (y_j)) = \theta^N$, where $N \in \mathbb{N}$ is the maximum non-negative integer n such that for every $i \in \mathbb{Z}$ with $|i| < n$, we have $x_i = y_i$. Let $f \in X_A :\to \mathbb{C}$ be a continuous function. We define the norm

$$\|f\|_\theta = |f|_\infty + |f|_\theta.$$

Where

$$|f|_\theta = \sup_{n \geq 0} \frac{Var_n f}{\theta^n},$$

and $Var_n(f) = \sup\{|f((x_j)) - f((y_j))|\}$ over all $(x_i), (y_i) \in X_A$ such that for every $|i| < n$, it holds $x_i = y_i$. We say $f \in C(X)$ is Hölder continuous if $\|f\|_\theta < \infty$ with respect to d_θ.

Let f be a real-valued Hölder function. Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that if $(x_i), (y_i) \in X_A$ with $d_\theta((x_i), (y_i)) < \delta$, then $|f((x_i)) - f((y_i))| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $\theta^N < \delta$. Since X_A is compact, for every word η of length N there exists $(x_i)_\eta \in [\eta]$ that attains its maximum in $[\eta]$. We define the function g by sending each $(y_i) \in X_A$ with $\eta = \{y_0, y_1, ...y_N\}$ to $f(x_\eta)$. Notice this implies $\|f - g\|_\infty < \varepsilon$. Therefore g depends only on the first N coordinates of each $(x_i) \in X_A$. If we replace all words of length $N - 1$ as new symbols, then g is a function that depends only on the first two coordinates x_0, x_1 of each $(x_i) \in X_A$.

Now we define the matrix A_g whose with entries are $a_{i,j} \exp\{g(i, j)\}$, where $a_{i,j}$ is the $i \times j$ entry of A. This matrix coincides with the transfer operator of the function g and as a consequence of the Ruelle-Perron-Frobenius theorem it can be shown that $\exp\{P(g,\sigma_A)\}$ is one of the eigenvalues of the matrix Ag. In fact, $\exp\{P(g,\sigma_A)\}$ is the eigenvalue with highest norm. Therefore

$$\begin{aligned}\sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\} &= \sum_{x_0,x_1,\ldots,x_{n-1},x_0} \exp\{\sum_{i=0}^{n-1} g(\sigma_A^i(x_i))\}\\ &= \sum_{x_0,x_1,\ldots,x_{n-1},x_0} \exp\{\sum_{i=0}^{n-1} g([x_0,x_1]) + \ldots + g([x_{n-1},x_0])\}\\ &= \text{trace of } A_g^n.\end{aligned}$$

As a trace of a matrix is equal to the sum of the eigenvalues of the matrix, if $\exp P(g,\sigma_A)^n, \lambda_2^n, \ldots, \lambda_k^n$ are the eigenvalues of Ag, then

$$\sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\} = \exp\{P(g,\sigma_A)n\} + \lambda_2^n + \ldots + \lambda_k^n.$$

Since $\exp P(g,\sigma_A)n$ is the eigenvalue with highest norm, then

$$\lim_{n\to\infty} \frac{\sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\}}{\exp P(g,\sigma_A)n} = 1. \tag{1.2}$$

Which implies

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\} = P(g,\sigma_A). \tag{1.3}$$

On the other hand, since $|f-g|_\infty < \varepsilon$, then for every $n \in N$ we obtain

$$\begin{aligned}\sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i)) - n\varepsilon\} &\le \sum_{(x_i)\in Fix(\sigma_A)} \exp\{f^n((x_i))\}\\ &\le \sum_{(x_i)\in Fix(\sigma_A)} \exp\{g^n((x_i)) + n\varepsilon\}.\end{aligned}$$

Since n was taken arbitrarily, we get

$$\begin{aligned}-\varepsilon + \frac{1}{n}\log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\} &\le \frac{1}{n}\log \sum_{(x_i)\in Fix(\sigma_A)} \exp\{f^n((x_i))\}\\ &\le \varepsilon + \frac{1}{n}\log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{g^n((x_i))\}.\end{aligned}$$

Thus by Equation (1.3), we get

$$\begin{aligned}P(g,\sigma_A) - \varepsilon &\le \liminf_{n\to\infty} \frac{1}{n}\log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{f^n((x_i))\}\\ &\le \limsup_{n\to\infty} \frac{1}{n}\log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{f^n((x_i))\}\\ &\le P(g,\sigma_A) + \varepsilon\end{aligned}$$

and since the pressure function is Lipchitz with constant 1, $|P(g,\sigma_A) - P(g,\sigma_A)| < \varepsilon$. This implies

$$P(f,\sigma_A) = \lim_{n\to\infty} \frac{1}{n}\log \sum_{(x_i)\in\mathrm{Fix}(\sigma_A^n)} \exp\{f^n((x_i))\}.$$

Chapter 2

Variational principle

One of the main results in the study of topological pressure (and as a consequence also of topological entropy) is the Variational Principle. It establishes a relation between objects of a topological nature with some measure theoretic ones. The Variational Principle we will present was proved originally by Ruelle [13] for some transformations and then extended by Walters [16] to the general case we present here.

2.1 Invariant measures and entropy.

Before presenting the statement of the Variational Principle we will present some necessary definitions of Ergodic theory. Broadly speaking Ergodic theory studies the dynamical systems through the probability measures and looks for the invariant properties through time.

Definition 2.1.1. Let $(M, \mathcal{B}, \mu)$ be a measurable space and $f : M \to M$ a measurable transformation. We will say that a measure μ **is invariant under** f if for each measurable set $E \in \mathcal{B}$, it holds that $\mu(E) = \mu(f^{-1}(E))$. We will say that μ **is** f**-invariant** to mean the same thing.

Definition 2.1.2. Let $(M, \mathcal{B}, \mu)$ be a measurable space and $f : M \to M$ a measurable transformation. We will denote by $\mathcal{M}_1(M)$ **the set of probability measures on M** and $M_f(M)$ **to the set of** f**-invariant probability measures**.

Endowing $\mathcal{M}_1(M)$ the so-called weak* topology and defining a metric in this space of measures, it can be shown that $\mathcal{M}_f(M)$ is nonempty. The following existence theorem can be found in [15, Theorem 2.1].

Theorem 2.1.1. *Let $f : M \to M$ be a continuous transformation on a compact metric space (X, d). Then there exists some probability measure on M invariant under f.*

Definition 2.1.3. Let $(M, \mathcal{B}, \mu)$ be a measurable space and $f : M \to M$ a measurable transformation. We will say that a measurable function $\varphi : M \to \mathbb{R}$ **is invariant under** f if $\varphi = \varphi \circ f$ at μ-almost every point. In this sense, we will say that **a measurable set B is an** f**-invariant set** if the characteristic function χ_B is an invariant function.

Definition 2.1.4. Let $(M, \mathcal{B}, \mu)$ be a measurable space and $f : M \to M$ a measurable transformation. Let $\mu \in \mathcal{M}_1(M)$. We will say that μ **is ergodic with respect to** f if for every invariant set A, it holds that $\mu(A) = 0$ or $\mu(A) = 1$. We denote by $\mathcal{M}_e(M, f)$ the set of all ergodic probability measures with respect to the transformation f. If the space M is understood we write $\mathcal{M}_e(f)$ for short.

There are several definitions equivalent to the one given to say that a measure μ is ergodic. In the following proposition we present some, to see more about it see [15, Proposition 4.1.3].

Proposition 2.1.1. Let $(M, \mathcal{B}, \mu)$ be a measurable space and $f : M \to M$ a measurable transformation. Let $\mu \in \mathcal{M}_1(M)$. The following conditions are equivalent:

i For every invariant set $A \subseteq M$ we have either $\mu(A) = 0$ or $\mu(A) = 1$.

ii Every f-invariant integrable function $\psi : M \to \mathbb{R}$ is constant at μ-almost every point.

Clearly, $\mathcal{M}_e(f) \subseteq \mathcal{M}_1(M)$. Moreover, ergodic measures play an important role among the invariant probability measures since they turn out to be the "extreme points" of $M_1(f)$. In fact, any invariant probability measure can be decomposed as a convex combination of ergodic measures. Such combination is not necessarily provided by finitely many ergodic measures. This result is known as the ergodic decomposition theorem [15, Theorem 5.1.3]. Before stating it, we clarify the notation a bit.

Definition 2.1.5. Let $(M, \mathcal{B}, \mu)$ be a probability space. Let $\mathcal{P}$ be a collection of elements of $\mathcal{B}$. We say that $\mathcal{P}$ is a partition of M if is a disjoint collection of elements of $\mathcal{B}$ whose union id is M. As we did with the open covers in Definition 1.1.5, for every $n \in \mathbb{N}$ we define

$$\mathcal{P}^n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}).$$

Definition 2.1.6. Let $(M, \mathcal{B}, \mu)$ be a probability space and $\mathcal{P}$ a partition of M. We denote by $\pi : M \to \mathcal{P}$ the canonical projection that assigns each point of M the element of $\mathcal{P}$ to which it belongs. We will say that a subset $Q \subseteq \mathcal{P}$ is measurable if and only if $\pi^{-1}(\mathcal{Q})$ is measurable in M. It can be shown that the set $\widehat{B}$ of measurable sets forms a σ-algebra in $\mathcal{P}$. We define the quotient measure $\widehat{\mu}$ such that for every $Q \in \widehat{B}$

$$\widehat{\mu}(Q) = \mu(\pi^{-1}(Q)).$$

Theorem 2.1.2. Let f be a measurable transformation in a complete separable metric space (X, d) and let $\mu \in M_1(f)$. Then there exists a subset $M_0 \subseteq M$ such that $\mu(M_0) = 1$, a partition $\mathcal{P}$ of M_0 into measurable subsets, and a family of probability measures $\{\mu_P : P \in P\}$ on M that satisfy:

i $\mu_P(P) = 1$ for $\widehat{\mu}$-almost every $P \in P$.

ii For every measurable set $E \subset M$, the map $P \to \mu_P(E)$ is measurable.

iii μ_P is invariant and ergodic for $\widehat{\mu}$-almost every $P \in P$.

iv For every measurable set $E \subseteq M$, we have $\mu(E) = \int \mu_P(E) d\widehat{\mu}(P)$.

We will now present the entropy of a dynamical system. Entropy quantifies the degree of "disorder" of a measurable space and has applications in different fields of science.

Definition 2.1.7. Let $(M, \mathcal{B}, \mu)$ be a probability space and $\mathcal{P}$ a partition of M. We define the entropy of the partition $\mathcal{P}$ by

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log(\mu(P)).$$

It can be shown that given a partition $\mathcal{P}$, the sequence $\{H_\mu(\mathcal{P}^n)\}_n$ is subadditive [15, Lemma 9.17], i.e., for every $m, n \geq 1$ it holds

$$H_\mu(\mathcal{P}^{m+n}) \leq H_\mu(\mathcal{P}^m) + H_\mu(\mathcal{P}^n).$$

It is also known [15, p79] that if a sequence of numbers $\{a_n\}_{n\in\mathbb{N}}$ in $[-\infty, \infty)$ is subadditive, then by Fekete's lemma $\lim\limits_{n\to\infty} \dfrac{1}{n} a_n$ exists and is equal to $\inf_n \frac{a_n}{n}$. Therefore the Fekete's lemma allows us to make the following definition.

Definition 2.1.8. Let $(M, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{P}$ be a partition of M. We define **the entropy of f with respect to $\mathcal{P}$** as the limit $\lim\limits_{n\to\infty} \dfrac{1}{n} H_\mu(\mathcal{P}^n)$.

It should be mentioned that there are examples in which this limit is infinite, which motivates the definition of entropy of the system considering only partitions with finite entropy.

Definition 2.1.9. Let $(M, \mathcal{B}, \mu)$ be a probability space. We define the **entropy of the system** (f, μ) by

$$h_\mu(f) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P})$$

where the supremum is taken over all partitions with finite entropy.

Among the properties of the ergodic decomposition of a probability measure μ is the following theorem given by K. Jacobs [15, Theorem 9.6.2], it states that we can calculate the entropy of μ from the entropy's of the ergodic measures that make up its decomposition.

Theorem 2.1.3. Let $f : M \to M$ be a continuous transformation on a compact metric (X, d). Let μ be an invariant probability measure and $\{\mu_P : P \in \mathcal{P}\}$ be its ergodic decomposition. Then

$$h_\mu(f) = \int h_{\mu_P} d\widehat{\mu}(P).$$

2.2 Variational principle

This section is dedicated to presenting the statement of the variational principle and to studying the consequences that will be significant for the results of this thesis.

Theorem 2.2.1. (Variational Principle) Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then

$$P(\varphi, f) = \sup \left\{ h_\mu(f) + \int \varphi d\mu \quad : \quad \mu \in \mathcal{M}_f(X) \right\}.$$

One of the interesting consequences of this theorem and the ergodic decomposition theorem is that for the calculation of pressure it is enough to consider the ergodic probability measures.

Corollary 2.2.1.1. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then

$$P(\varphi, f) = \sup \left\{ h_\nu(f) + \int \varphi d\nu \;\middle|\; \nu \in \mathcal{M}_e(f) \right\}. \tag{2.1}$$

Proof. Let $\nu \in \mathcal{M}_f(X)$ and let $\{\nu_P : p \in \mathcal{P}\}$ be its ergodic decomposition by Theorem 2.1.2. Then by the ergodic decomposition theorem and by Theorem 2.1.3 we have that

$$\begin{aligned} h_\nu(f) + \int \varphi d\nu &= \int h_{\upsilon_P} d\widehat{\nu}(P) + \int \left(\int \varphi d\nu_P \right) d\widehat{\nu}(P) \\ &= \int \left(h_{\nu_P} + \int \varphi d\nu_P \right) d\widehat{\nu}(P). \end{aligned}$$

Considering the supremum on the left side over all invariant probability measures and the supremum on the right side over all ergodic measures we obtain that

$$\sup \left\{ h_\nu + \int \varphi d\nu \;\middle|\; \nu \in \mathcal{M}_f(X) \right\} \leq \sup \left\{ h_\nu + \int \varphi d\nu \;\middle|\; \nu \in \mathcal{M}_e(f) \right\}.$$

On the other hand, since $\mathcal{M}_e(f) \subseteq \mathcal{M}_f(X)$, we get the reciprocal inequality. Therefore $\sup\{h_\nu + \int \varphi d\nu \;\; | \;\; \nu \in \mathcal{M}_1(f)\} = \sup\{h_\nu + \int \varphi d\nu \;\; | \;\; \nu \in \mathcal{M}_e(f)\}$. Thus, by the variational principle we obtain Equation (4.5). □

This collorary will be important in the proof of Theorem 5.0.2. Now we present a concept of topological dynamics that will be relevant in the main results of the thesis. A good source to learn more about it can be found at [17, 5.3].

Definition 2.2.1. Let f be a continuous transformation on a compact metric space (X, d). A point $x \in X$ is said to be **wandering** if there exists a neighborhood U of x such that the elements of the sequence $\{f^{-n}(U)\}_{n\in\mathbb{N}}$ are pairwise disjoint. We define the **non-wandering set for** f as the set of points that are not wandering, and we denote it by $\Omega(f)$, i.e.,

$$\Omega(f) := \left\{ x \in X \quad : \quad \forall \text{ neighborhood U of x } \exists n \in \mathbb{N} \text{ such that } f^{-n}(U) \cap U \neq \varnothing \right\}.$$

Proposition 2.2.1. *Let f be a continuous transformation on a compact metric space (X, d). Then for every invariant probability measure μ,*

$$\mu(\Omega(f)) = 1.$$

Proof. First, we prove that X is separable and second countable. For this it is enough to take for each $n \in N$ balls of radius $\frac{1}{n}$ and use the compactness of X to take a finite subcover $\{B_{1,n}, B_{2,n}, ...B_{k_n,n}\}$. Then consider the collection of balls $\mathcal{A} = \{B_{1,n}, ..., B_{k_n,n} : n \in \mathbb{N}\}$ and the collection B of centers of balls the in $\mathcal{A}$. It is not difficult to see that $\mathcal{A}$ is a countable basis for X and that B is therefore a countable dense set on X.

On the other hand, if $x \in X$ is wandering and let U be the neighborhood of x such that all its pre-images are pairwise disjoint. Then every point $y \in U$ is wandering since U meets Definition 2.2.1. Therefore $X \setminus \Omega(f)$ is an open set. Let μ be an invariant probability measure. Now let $B_m \subseteq X \setminus \Omega(f)$, for some $m \in \mathbb{N}$. Thus as $\{f^{-1}(B_m), f^{-2}(B_m), ...\}$ is a collection of pairwise disjoint sets, μ is f-invariant and $\bigcup_{n=0}^{\infty} f^{-n}(B_m) \subseteq X$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mu(B_m) &= \sum_{n=0}^{\infty} \mu(f^{-n}(B_m)) \\ &= \mu(\bigcup_{n=0}^{\infty} f^{-n}(B_m)) \\ &\leq \mu(X) = 1.\end{aligned}$$

This implies $\mu(B_m) = 0$. Therefore $X \setminus \Omega(f)$ a countable union of sets with measure zero. Thus, $\mu(X \setminus \Omega(f)) = 0$ and consequently $\mu(\Omega(f)) = 1$. Since μ was chosen arbitrarily, we obtain what we wanted to show. □

Corollary 2.2.1.2. Let $f : X \to X$ be continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then

1. $P(\varphi, f) = P(\varphi|_{\Omega(f)}, f|_{\Omega(f)})$
2. $P(\varphi, f) = P(\varphi|_{\bigcap_{n=0}^{\infty} f^n(X)}, f|_{\bigcap_{n=0}^{\infty} f^n(X)})$.

Proof. To prove item 1, it is enough to note that every invariant probability measure on X defines an invariant probability measure in $\Omega(f)$ and simultaneously every invariant probability measure on $\Omega(f)$ can be extended to an invariant probability measure on X by extending $X \setminus \Omega(f)$ as a set of zero measure. Therefore the calculation of the entropy of f with respect to an invariant probability measure μ on X coincides with the entropy of an invariant probability measure in $\Omega(f)$ since they differ from sets of measure zero. Thus, we obtain

$$\sup\{h_\mu(f) \quad | \quad \mu \in M_1(X)\} = \sup\{h_\nu(f|_{\Omega(f)}) \quad | \quad \nu \in M_1(\Omega(f))\}.$$

Applying the variational principle to f and $f|_{\Omega(f)}$ with respect to the potential φ and $\varphi_{\Omega(f)}$ respectively, we obtain $P(\varphi, f) = P(\varphi|_{\Omega(f)}, f|_{\Omega(f)})$. For item 2, let μ be an f-invariant probability measure. Note that $\mu(f^n(X)) = \mu(f^{-n}f^n(X)) = \mu(X) = 1$. Hence $\mu(\bigcap_{n=0}^{\infty} f^n(X)) = 1$. Just as μ was taken arbitrarily, we can identify the space $\mathcal{M}_f(X)$ with $\mathcal{M}_f(\bigcap_{n=0}^{\infty} f^n(X))$. Thus we conclude as the previous item, by applying the variational principle to f and $f|_{\bigcap_{n=0}^{\infty} f^n(X)}$. Obtaining

$$P(\varphi, f) = P(\varphi|_{\bigcap_{n=0}^{\infty} f^n(X)}, f|_{\bigcap_{n=0}^{\infty} f^n(X)}).$$

□

Remark. Note that by Proposition 1.2.2, if we consider the constant potential 0 we will obtain analogous results for the topological entropy, i.e.,

i $h(f) = \sup\{h_\mu(f) : \quad \mu \in \mathcal{M}_e(f)\}$.

ii $h(f) = h(f|_{\Omega(f)})$.

iii $h(f) = h(f|_{\bigcap_{n=0}^{\infty} f^n(X)})$.

Chapter 3

Pseudo-orbits

Recall that the purpose of this thesis is to present alternative definitions of topological entropy and pressure. Our strategy is to make use of pseudo-orbits. This approach has the advantage that, broadly speaking, pseudo-orbits are simpler to handle than orbits. As we already mentioned, in several examples (see for instance Example 1.2.2) the entropy can be computed using the periodic data of the system. In order for this characterization to hold, several properties on the system are required: expansiveness and the shadowing property. While a large class of interesting systems do satisfy them, there is an even larger class of systems that do not. In order to handle these less regular systems we introduce the notion of periodic pseudo orbit and go on to define entropy and pressure using it.

Definition 3.0.1. Let f be a continuous transformation on a compact metric space (X, d) and let $\alpha > 0$. An α**-pseudo-orbit** is an infinite sequence of points $x_1, x_2, ...$ in X such that for every $i \geq 0$, it holds $d(f(x_i), x_{i+1}) \leq \alpha$.

- If the sequence is finite then we call it by α**-chain**.
- We say that an α-pseudo orbit $(x_i)_{i\in\mathbb{N}}$ is **periodic**, if there is $n \in \mathbb{N}$ such that for each $0 \leq r \leq n-1$ and every $k \geq 0$, we have $x_{kn+r} = x_r$. We understand its **period** like the minor n that holds the above, and denote it by $\tau((x_i))$.

Example 3.0.1. As in Example 1.1.3 consider the shift in two symbols $(\{0,1\}^{\mathbb{N}}, \sigma)$.

- Consider any two elements (x_i) and (y_i) whose first seven terms are $(0, 0, 1, 0, 0, 0, 0)$ and $(0, 1, 0, 1, 0, 1, 0)$ respectively. So $d(\sigma((x_i)), (y_i)) \leq \frac{1}{4}$, which implies the set $\{(x_i), (y_i)\}$ is a $\frac{1}{4}$-chain.
- Let $(x_i) \in \{0,1\}^{\mathbb{N}}$. If we denote $(x_i) = (x_{0,i})$ and construct the sequence $\{(x_{n,i})\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$,

  $$(x_{n+1,i}) = (x_{n,1}, x_{n,2}, x_{n,3}, x_{n,4} + 1, x_{n,5}, ...)$$

  Therefore $d(\sigma((x_{n,i}), (x_{n+1,i})) \leq \frac{1}{4}$. This implies $\{(x_{n,i})\}_{n\in\mathbb{N}}$ is a $\frac{1}{4}$-pseudo-orbit.

We now define a special set in X which, as we shall see, has total measure for every f-invariant measure and will help us to calculate the topological entropy and the pressure.

Definition 3.0.2. Let f be a continuous transformation on a compact metric space (X, d). We define the **chain recurrent set of** f as the set containing all points $x \in X$ such that for any $\alpha > 0$ there is a periodic α-pseudo-orbit containing x, and we denote it by $R(f)$.

Proposition 3.0.1. Let f be a continuous function on a compact metric space (X, d). Then:

i The chain recurrent set $R(f)$ is f-invariant in the sense $f(R(f)) \subseteq R(f)$.

ii The chain recurrent set $R(f)$ contains the non-wandering set $\Omega(f)$.

iii The chain recurrent set $R(f)$ is a closed subset of X.

Proof. Let $\alpha > 0$. Since f is uniformly continuous, then there exists $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$, we have $d(f(x), f(y)) < \alpha$. Let $x \in R(f)$. Then there is a periodic δ-pseudo-orbit that contains x. As for all $i \in N$ we have that $|f(x_i) - x_{i+1}| < \delta$, then by the uniform continuity $|f(f(x_i)) - f(x_{i+1})| < \alpha$. Therefore $(f(x_i))$ is a periodic α-pseudo-orbit that contains $f(x)$. As α was arbitrarily taken, then $f(x) \in R(f)$ showing (i).

Now let $x \in \Omega(f)$ and, $\delta > 0$ and $\alpha > 0$ as above. Let $\delta_1 = \min\{\delta, \alpha\}$. By definition of the non-wandering set, there exists $n \in \mathbb{N}$ such that $f^{-n}(B_{\delta_1}(x)) \cap B_{\delta_1}(x) \neq \varnothing$. Let $x_0 \in f^{-n}(B_{\delta_1}(x)) \cap B_{\delta_1}(x)$. Then $\{x, f(x_0), ..., f^{n-1}(x_0)\}$ is an α-chain that defines a periodic α-pseudo-orbit containing x. Again since α was arbitrarily chosen, then $x \in R(f)$, which proves (ii).

To see that the chain recurrent set $R(f)$ is closed, let us see it contains its limit points. Let $\alpha > 0$. Let z be a limit point of $R(f)$ and by uniform continuity take $\delta > 0$ so that for every $x, y \in X$ with $d(x, y) < \delta$, it holds $d(f(x), f(y)) < \frac{\alpha}{2}$. Without loss of generality assume $\delta \leq \frac{\alpha}{2}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $R(f)$ that converges to z. Take $N \in \mathbb{N}$ such that $d(x_N, z) < \delta$. Since $x_N \in R(f)$, then there exists a periodic δ-pseudo-orbit (y_i) that contains x_N. If $\{x_N, y_2, ..., y_{\tau(y_i)-1}\}$ defines a period of (y_i), then by triangular inequality,

$$d(f(z), y_2) \leq d(f(z), f(x_N)) + d(f(x_N), y_2) \leq \alpha, \quad \text{and}$$
$$d(f(x_{\tau((y_i))-1}), z) \leq d(f(x_{\tau((y_i))-1}), x_N) + d(x_N, z) \leq \alpha.$$

Therefore by our choice of δ, we conclude that $\{z, y_2, ..., y_{\tau(y_i)-1}\}$ is an α-chain which defines a periodic α-pseudo orbit that contains z. Since α was arbitrarily chosen, then $z \in R(f)$, which implies $R(f)$ contains all its limit points as we wanted to show. □

Proposition 3.0.2. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. The topological pressure of φ with respect to f is equal to the topological pressure of φ with respect to f restricted to the chain-recurrent set $R(f)$, i.e.,

$$P(\varphi, f) = P(\varphi, f|_{R(f)}).$$

Proof. By the previous proposition, $\Omega(f) \subseteq R(f)$, and by Corollary 2.2.1.2, we obtain:

$$P(\varphi, f) = P(\varphi|_{\Omega(f)}, f|_{\Omega(f)}) \leq P(\varphi|_{R(f)}, f|_{R(f)}) \leq P(\varphi, f).$$

□

Before presenting the following lemma, which will be used to prove Theorem 4.0.2 and Theorem 5.0.2, we will first need to make use of a special metric that can be defined on compact metric spaces, the Hausdorff metric.

Definition 3.0.3. Let (X, d) be a metric space. If $A \subseteq X$ and $\varepsilon > 0$. We denote by $B_\varepsilon(A) := \{x \in X : d(x, A) \leq \varepsilon\}$ the ε-neighborhood of A. Let $\mathcal{H}$ be the set of all closed (non-empty) and bounded subsets of X. If $A, B \in \mathcal{H}$, we define the function $d_H : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$d_H(A, B) := \inf\{\varepsilon > 0 \quad : \quad A \subset B_\varepsilon(B) \quad \text{and} \quad B \subset B_\varepsilon(A)\}.$$

It can be seen that d_H is a metric. We will call it the **Hausdorff metric**.

Remark. The Hausdorff metric has interesting properties (see [10, p278]). One of these is that if (X, d) is a compact space then $(\mathcal{H}, d_H)$ is also a compact space. This fact will be crucial for the next result.

Proposition 3.0.3. Let f be a continuous transformation on a compact metric space (X, d). The chain recurrent set of $f|_{R(f)}$ is the chain recurrent set of f, i.e.,

$$R(f|_{R(f)}) = R(f).$$

Proof. Note that $R(f|_{R(f)})$ is defined by the set of points in $x \in R(f)$ such that for all $\alpha > 0$ there exists a periodic α-pseudo orbit in $R(f) \subseteq X$ that contains x. Thus $x \in R(f)$. Which implies $R(f|_{R(f)}) \subseteq R(f)$, since x was arbitrarily chosen.

Now we show that $R(f) \subseteq R(f|_{R(f)})$. Let $x \in R(f)$. For each positive integer n take a periodic $\frac{1}{n}$-pseudo orbit that contains x, we will denote it by c_n. If we denote for each n, $c_n = (x_{n,i})$ and C_n to its first period $\{x_{n,1}, x_{n,2}, ..., x_{n,\tau(x_{n,i})-1}\}$. By the above remark, the collection of closed subsets $\mathcal{H}$ is compact with the Hausdorff metric, and therefore the sequence $(C_n)_n$ has a convergent sub-sequence to some closed set C.

Let $y \in C$ and $\alpha > 0$. By the uniform continuity of f there exists $\delta > 0$ such that for every $x, z \in X$ with $d(x, z) < \delta$, we have $d(f(x), f(z)) < \frac{\alpha}{3}$. Without loss of generality, assume that $\delta < \frac{\alpha}{3}$. Now we take $N \in \mathbb{N}$ large enough such that $\frac{1}{N} < \frac{\alpha}{3}$ and $d_H(C_N, C) < \delta$. For each $w_j \in C_N$ choose a $z_j \in C$ such that $d(w_j, z_j) < \delta$. In fact, by our choice of N we can assume for some $0 \leq j \leq \tau(c_N)$, that $z_j = y$. Thus for every $0 \leq j < \tau(c_N)$,

$$d(f(z_j), z_{j+1}) \leq d(f(z_j), f(w_j)) + d(f(w_j), w_{j+1}) + d(w_{j+1}, z_{j+1}) \leq \alpha.$$

Therefore $\{z_1, z_2, ..., z_{\tau(c_N)-1}\}$ is an α-chain in C and defines a periodic α-pseudo-orbit in C containing y with period $\tau(c_N)$. Since α was taken arbitrarily, then $y \in R(f)$. In fact $y \in R(f|_C)$. Analogously, since y was taken arbitrarily in C, then $C \subseteq R(f|_C) \subseteq R(f|_{R(C)})$. Finally $x \in C$, since for every $n \in \mathbb{N}$, we have $x \in c_n$. Then $x \in R(f|_{R(f)})$, which implies $R \subseteq R(f|_{R(f)})$. □

This proposition tells us for all $\alpha > 0$ every element in the chain recurrent set $R(f)$ is contained in periodic α-pseudo-orbits of elements in $R(f)$. We now define subsets in $R(f)$ with interesting properties that will be important in the proof of Theorem 4.0.2 and Theorem 5.0.2.

Definition 3.0.4. Let f be a continuous transformation on a compact metric space (X, d). Let $\alpha > 0$. Two points $x, y \in R(f)$ are said to be on the same **α-chain-transitive component of $R(f)$**, if there exists an α-chain from y to x and an α-chain from x to y.

Proposition 3.0.4. Let f be a continuous transformation on a compact metric space (X, d). Let $\alpha > 0$. Then

i Every α-chain-transitive component of $R(f)$ is open and closed in $R(f)$.

ii The α-chain-transitive components partition $R(f)$.

iii There are finite α-chain-transitive components of $R(f)$.

iv Every α-chain-transitive component of $R(f)$ is f-invariant.

Proof. Let $\alpha > 0$. By the uniform continuity of f there exists $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$, it holds $d(f(x), f(y)) < \frac{\alpha}{2}$. Let $x, y \in R(f)$ be such that $d(x, y) < \delta$. Let (x_i) and (y_i) be periodic $\frac{\alpha}{2}$-pseudo-orbits containing x and y respectively. Assume that the first period of (x_i) is $\{x, x_2, ..x_{\tau(x_i)-1}\}$. Then

$$d(f(y), x_2) \leq d(f(y), f(x)) + d(f(x), x_2) \leq \alpha.$$

Thus $\{y, x_2, ..x_{\tau(x_i)-1}, x\}$ is an α-chain from y to x. Analogously replacing a term in the first period of (y_i), we can obtain an α-chain from x to y. We conclude that x and y are in the same α-chain-transitive component in $R(f)$. This implies that if T is an α-chain-transitive component and $x \in T$, then $B_\delta(x) \cap R(f) \subseteq T$. Therefore T is an open subset in $R(f)$ from the topology inherited of X. On the other hand, if z is a limit point of T. Let (z_n) be a sequence in T that converges to z. Then there exists $N \in \mathbb{N}$ such that for every $m \geq N$, we have $d(x_m, z) < \delta$. Thus, x_m and z are in the same α-chain-transitive component, i.e., $z \in T$. Therefore T is closed. This shows (i).

To prove (ii), suppose by contradiction that T_1 and T_2 are different α-chain-transitive components of $R(f)$ such that $T_1 \cap T_2 \neq \varnothing$. As we just proved T_1 and T_2 are open in $R(f)$, then there exists $0 < \delta$ such that for some $x \in T_1 \cap T_2$ we have $B_\delta(x) \subseteq T_1 \cap T_2$. However, this implies that any two elements in this ball are in the same α-chain-transitive component. Thus $T_1 = T_2$, which contradicts our assumption.

Since $R(f)$ is closed and X is compact, we get that $R(f) \subseteq X$ is a compact subset. So we can cover $R(f)$ by finitely many balls of radius δ, say $B_1, B_2, ...B_N$ for some $N \in \mathbb{N}$. From what we said above we know that for every $1 \leq i \leq N$, each subset $B_i \cap R(f)$ is contained in an α-chain-transitive component of $R(f)$. Therefore the cardinality of the α-chain-transitive components of $R(f)$ is bounded by the cardinality of a finite collection of sets, i.e., there are finite α-chain-transitive components of $R(f)$, proving (iii).

Let $\alpha > 0$. Let T be an α-chain-transitive component of $R(f)$ and let $x \in T$. Let $\delta > 0$ be as above. Clearly, $\{x, f(x)\}$ is an α-chain that goes from x to $f(x)$. Without loss of generality assume that $\delta \leq \frac{\alpha}{2}$. Since $x \in R(f)$, then there exists a periodic $\frac{\alpha}{2}$-pseudo-orbit (x_i) such that $\{x, x_2, ..., x_{\tau(x_i)-1}\}$ defines its first period. Then

$$d(f^2(x), x_3) \leq d(f^2(x), f(x_2)) + d(f(x_2), x_3) \leq \alpha.$$

Hence $\{f(x), x_3, ..., x_{\tau(x_i)-1}, x\}$ is an α-chain from $f(x)$ to x. This implies that $f(T) \subseteq T$. Satisfying Definition 2.1.3.X ☐

3.1 Pseudo-entropy

Inspired by Bowen's definition of topological entropy via separated sets, we will define the pseudo entropy using pseudo-orbits. With the final objective of showing in the next chapter two extra theoretical ways to calculate topological entropy.

Definition 3.1.1. Let f be a continuous transformation on a compact metric space (X, d). Let $\alpha > 0$, $n \in \mathbb{N}$ and $\varepsilon > 0$. A collection E of α-pseudo-orbits is said to be (n, ε)**-separated** if for every $(x_i), (y_i) \in E$, with $(x_i) \neq (y_i)$, there is $0 \leq k < n$ such that $d(x_k, y_k) > \varepsilon$. We denote by $s^\alpha(n, \varepsilon)$ the maximum cardinality among (n, ε)-separated collections of α-pseudo-orbits in X.

Remark. Note therefore that if two pseudo-orbits are (n, ε)-separated, then we are differentiating them by an error of ε and the first n terms of each sequence.

Proposition 3.1.1. Let f a continuous transformation on a compact metric space (X, d). Let $\alpha > 0$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Then $s^\alpha(n, \varepsilon) < \infty$

Proof. Consider the open cover of X given by the collection of balls with radius $\varepsilon/2$. Since X is compact, there exists a finite sub-cover $\{B_1, B_2, ...B_M\}$ of X, with $M \in \mathbb{N}$. Therefore $\{B_1, B_2, ...B_M\}^n$ is a finite open cover of X^n. Suppose that E is a (n, ε)-separated collection of α-pseudo-orbits with cardinality greater than M. Thus, there are $(x_i), (y_i) \in E$ and $B \in \{B_1, B_2, ...B_M\}^n$ such that $(x_i)_{i=0}^n, (y_i)_{i=0}^n \in B$. This implies for every $1 \leq i < n$ that $d(x_i, y_i) < \varepsilon$ which cannot be, since (x_i) and (y_i) are in a (n, ε)-separated set. We conclude that every collection of (n, ε)-separated α-pseudo-orbits has cardinality less than or equal to M^n. Thus $s^\alpha(n, \varepsilon) \leq M^n < \infty$. ☐

Definition 3.1.2. Let f be a continuous transformation on a compact metric space (X, d). Denote

- $h^\alpha_\psi(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log(s^\alpha(n, \varepsilon))$.
- $h_\psi(f, \varepsilon) = \lim_{\alpha\to 0} h^\alpha_\psi(f, \varepsilon)$.

We call **pseudo-entropy of** f to the limit

$$h_\psi(f) = \lim_{\varepsilon\to 0} h_\psi(f, \varepsilon).$$

Proposition 3.1.2. Let f be a continuous transformation on a compact metric space (X,d). Let $n \in \mathbb{N}$, $\alpha > 0$ and $\varepsilon > 0$. Then the limits $h^{\alpha}_{\psi}(f,\varepsilon)$, $h_{\psi}(f,\varepsilon)$ and $h_{\psi}(f)$ exist.

Proof. Let $\mathcal{A}$ be the open cover of X given by the collection of balls of radius $\varepsilon/2$. From the proof of Proposition 3.1.1, we know that $s(n,\varepsilon,\alpha) \leq N(\mathcal{A})^n$. Let (x_i) and (y_i) be two α-pseudo-orbits that are (n,ε)-separated. Then for all $m \geq n$ they are (m,ε)-separated by Definition 3.1.1. Hence for all $m \geq n$ we get $0 \leq s^{\alpha}(n,\varepsilon) \leq s^{\alpha}(m,\varepsilon)$. So

$$\sup_{k\geq n} \frac{1}{k} \log(s^{\alpha}(k,\varepsilon)) \leq \sup_{k\geq m} \frac{1}{k} \log(s^{\alpha}(k,\varepsilon)) \leq N(\mathcal{A}).$$

Thus $\{\sup_k \frac{1}{k} \log(s^{\alpha}(k,\varepsilon))\}_{k\in\mathbb{N}}$ is a monotone bounded sequence in $\mathbb{R}$. Therefore $h^{\alpha}_{\psi}(f,\varepsilon)$ exists. Let $\alpha' < \alpha$ and let (x_i) be an α'-pseudo-orbit. Notice (x_i) is also an α-pseudo-orbit by Definition 3.0.1. Therefore for all $\alpha' < \alpha$ we can conclude $0 \leq s^{\alpha}(n,\varepsilon) \leq s^{\alpha'}(m,\varepsilon)$, which implies $h^{\alpha}_{\psi}(f,\varepsilon) \leq h^{\alpha'}_{\psi}(f,\varepsilon)$. Thus, $\{h^{\frac{1}{n}}_{\psi}(f,\varepsilon)\}_n$ is a monotone and bounded sequence. So $h_{\psi}(f,\varepsilon)$ exists.

Finally, let $\varepsilon > \varepsilon' > 0$. If (x_i) and (y_i) are (n,ε)-separated α-pseudo-orbits, then by Definition 3.1.1 they are also (n,ε')-separated. Thus, for all $\varepsilon > \varepsilon'$ we have $0 \leq s^{\alpha}(n,\varepsilon) \leq s^{\alpha}(n,\varepsilon')$ which implies $h_{\psi}(f,\varepsilon) \leq h_{\psi}(f,\varepsilon')$. Then the sequence $\{h_{\psi}(f,\frac{1}{m})\}_{m\in\mathbb{N}}$ is also bounded and increasing and therefore exists. □

3.2 Growth rate of separated periodic α-pseudo-orbits

It is a well known fact that for expansive systems with the shadowing property the topological entropy can be calculated as the exponential growth rate of the number of periodic orbits. It is natural to ask for a result in the same spirit for more general systems. However, note that there exists examples of dynamical systems with no periodic orbits and positive topological entropy [12]. M. Barge and R. Swanson [2] proposed to study periodic pseudo-orbits and look for a similar result. This approach was indeed fruitful. The desired result was obtained in [2] and it is discussed in next chapter in Theorem 4.0.2.

In this section we provide the definition of entropy in this setting. For this, we will restrict the definitions that were used to define pseudo-entropy, considering only periodic pseudo-orbits.

Definition 3.2.1. Let f a continuous transformation on a compact metric space (X,d). Let $\alpha > 0$, $n \in \mathbb{N}$ and $\varepsilon > 0$. We denote $p^{\alpha}(n,\varepsilon)$ as **the maximum cardinality among (n,ε)-separated collections of periodic α-pseudo-orbits of period n.**

Remark. Notice that $p^{\alpha}(n,\varepsilon) \leq s^{\alpha}(n,\varepsilon) < \infty$ by Proposition 3.1.1.

Definition 3.2.2. Let f a continuous transformation on a compact metric space (X,d). Let $\alpha > 0$, $n \in \mathbb{N}$ and $\varepsilon > 0$. We denote

- $H^{\alpha}_{\psi}(f,\varepsilon) = \limsup_{n\to\infty} \dfrac{1}{n} \log p^{\alpha}(n,\varepsilon)$.

- $H_{\psi}(f,\varepsilon) = \lim_{\alpha\to 0} H^{\alpha}_{\psi}(f,\varepsilon)$.

We define **the growth exponential rate of the number of separated periodic α-pseudo-orbits of period n as n tends to infinity** by

$$H_{\psi}(f) = \lim_{\varepsilon\to 0} H_{\psi}(f,\varepsilon).$$

Remark. The limits $H^{\alpha}_{\psi}(f,\varepsilon), H_{\psi}(f,\varepsilon)$, and $H_{\psi}(f)$ exist since they are bounded by the limits of Definition 3.1.2 and by Proposition 3.1.2.

3.3 Pseudo-Pressure

In the same spirit as in the previous section, we will now emulate the definition of topological pressure for pseudo-orbits and periodic pseudo-orbits. This in order to obtain two additional ways of calculating the topological pressure of a compact metric space. In chapter 5 we will obtain these results.

Definition 3.3.1. Let f be a continuous transformation on a compact metric space (X,d). Let $\varphi \in C(X)$, $n \in \mathbb{N}$ and $\varepsilon > 0$. We denote

$$P^{\alpha}_{n}(\varphi, f, \varepsilon) := \sup_{E} \left\{ \sum_{\mathbf{x}\in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f(x_i)) \right) \right\},$$

where the supremum is taken over the (n,ε)-separated collections of α-pseudo-orbits in X. We denote the limits

- $P^{\alpha}_{\psi}(\varphi,f,\varepsilon) := \limsup_{n\to\infty} \dfrac{1}{n} \log P^{\alpha}_{n}(\varphi,f,\varepsilon)$.

- $P_{\psi}(\varphi,f,\varepsilon) = \lim_{\alpha\to 0} P^{\alpha}(\varphi,f,\varepsilon)$.

- $P_{\psi}(\varphi,f) = \lim_{\varepsilon\to 0} P_{\psi}(\varphi,f,\varepsilon)$.

We will call the **pseudo-pressure** of the potential φ with respect to f to the last limit .

Remark. Since X is compact, then $\varphi(X)$ is bounded in $\mathbb{R}$, i.e., there exists $M \in \mathbb{R}_{>0}$ such that for all $x \in X$, $|\varphi(x)| \leq M$. Thus, since $s^{\alpha}(n,\varepsilon)$ is finite, then $|P^{\alpha}_{n}(\varphi,f,\varepsilon,\alpha)| \leq s^{\alpha}(n,\varepsilon)M$. Therefore $P^{\alpha}_{n}(\varphi,f,\varepsilon,\alpha)$ is finite.

On the other hand, analogously to Proposition 3.1.2, it can be shown that the sequences $\{\sup_n \frac{1}{n} \log P^{\alpha}_{n}(\varphi,f,\varepsilon)\}_{n\in\mathbb{N}}$, $\{P^{\frac{1}{m}}_{\psi}(\varphi,f,\varepsilon)\}_{m\in\mathbb{N}}$, and $\{P_{\psi}(\varphi,f,\frac{1}{m})\}_{m\in\mathbb{N}}$ are bounded and monotone. Then the values $P^{\alpha}_{\psi}(\varphi,f,\varepsilon)$, $P_{\psi}(\varphi,f,\varepsilon)$, and $P_{\psi}(\varphi,f)$ exist.

We now make a definition analogous to section 3.2. We will see in the final theorem of the thesis that topological pressure can be calculated from periodic pseudo orbits. For this, we will restrict the definitions that we just used to define pseudo-pressure, considering only periodic pseudo-orbits.

Definition 3.3.2. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$, $n \in \mathbb{N}$ and $\varepsilon > 0$. We denote

$$Pe_n^\alpha(\varphi, f, \varepsilon) := \sup_E \left\{ \sum_{\mathbf{x} \in E} \exp \left(\sum_{i=0}^{n-1} \varphi(f(x_i)) \right) \right\},$$

where the supreme is taken over the (n, ε)-separated collections of periodic α-pseudo-orbits with period n in X. We denote the limits

- $Pe_\psi^\alpha(\varphi, f, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log Pe_n^\alpha(\varphi, f, \varepsilon)$.
- $Pe_\psi(\varphi, f, \varepsilon) := \lim_{\alpha \to 0} Pe_\psi^\alpha(\varphi, f, \varepsilon)$.
- $Pe_\psi(\varphi, f) := \lim_{\varepsilon \to 0} Pe_\psi(\varphi, f, \varepsilon)$.

Remark. The limits $Pe_\psi^\alpha(\varphi, f, \varepsilon), Pe_\psi(\varphi, f, \varepsilon)$, and $Pe_\psi(\varphi, f)$ exist since they are bounded by the limits of Definition 3.3.1 and by the preceding Remark. We will see in the last theorem of this document that $Pe_\psi(\varphi, f)$ is equal to the pressure of φ with respect to f.

Remark. An interesting point is that the values $P_\psi(\varphi, f)$ and $Pe_\psi(\varphi, f)$ coincide with $h_\psi(f)$ and $H_\psi(f)$ respectively when we consider the constant potential 0.

Chapter 4

Pseudo-entropy and topological entropy

In the previous chapters we have introduced several definitions of topological entropy. While we already established that the original definition using open covers and Bowen's definition using separated sets coincide, it remains to be proven that the other two definitions using pseudo-orbits are also equivalent. This will be done in this chapter. We begin proving a result obtained by Misiurewicz [9] in 1986.

Theorem 4.0.1. Let f be a continuous transformation on a compact metric space (X, d). The topological entropy $h(f)$ is equal to the pseudo entropy $h_\psi(f)$.

Proof. We first show that $h(f) \leq h_\psi(f)$. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Let E be a (n, ε)-separated collection of α-pseudo-orbits. If $x \in E$ and for all $i \in \mathbb{N}$ we denote $f^i(x) = x_i$, then it is satisfied that

$$d(f(x_{i-1}), x_i) = d(f(f^{i-1}(x)), f^i(x)) = 0 < \alpha.$$

This means the orbit $\mathcal{O}_x$ is an α-pseudo-orbit. Which implies that the collection of orbits $\{\mathcal{O}_x : x \in E\}$ is a (n, ε)-separated collection of α-pseudo-orbits. Since E was arbitrarily taken, we conclude $s(n, \varepsilon) \leq s^\alpha(n, \varepsilon)$. In the same way, as n, α, and ε were taken arbitrarily, then for all $\alpha > 0$ and $\varepsilon > 0$ it is satisfied that $s(f, \varepsilon) \leq h_\psi^\alpha(f, \varepsilon)$. Which implies for all $\varepsilon > 0$ that $s(f, \varepsilon) \leq h_\psi(f, \varepsilon)$. Thus,

$$h(f) \leq h_\psi(f). \tag{4.1}$$

Now we prove the reciprocal inequality. Consider the space $X^\mathbb{Z}$ which is compact by the compactness of X and Tychonoff's theorem. Let $\underline{d} : X^\mathbb{Z} \times X^\mathbb{Z} \to \mathbb{R}$ be the metric defined by

$$\underline{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}}.$$

It can be seen that due to the compactness of X, this function $\underline{d}$ defines a metric. For each $\alpha \geq 0$ we define the subset

$$Y_\alpha := \{(x_i)_{i\in\mathbb{Z}} : d(f(x_i), x_{i+1}) \leq \alpha \text{ for every } i \in \mathbb{Z})\} \subseteq X^\mathbb{Z}.$$

Note that for each $(x_i) \in Y_\alpha$ and for each $k \in \mathbb{Z}$ the sequence $(x_i)_{i\geq k}$ is an α-pseudo-orbi, and that for $\alpha = 0$ the set Y_0 is the so-called natural extension.

Lemma 4.0.1. Let $\alpha \geq 0$. Then $Y_\alpha \subseteq X^{\mathbb{Z}}$ is a compact subset.

Proof. By compactness of X^Z, it suffices to show that Y_α is closed. Let $\varepsilon > 0$. By uniform continuity of f there exists $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$, it holds that $d(f(x), f(y)) < \varepsilon$. Without loss of generality suppose $\delta < \varepsilon$. Let (z_i) be a limit point of Y_α. Let $(x_{n,i})_{n\in\mathbb{N}}$ be a sequence in Y_α that converges to (z_i). Let $j \in \mathbb{Z}$. Let $N \in \mathbb{N}$ be such that for all $m \geq N$ we have $\underline{d}((z_i), (x_{m,i})) < \delta/2^{|j|}$. Thus for every $m \geq N$ and by definition of $\underline{d}$ we obtain

$$\frac{d(x_{m,j-1}, z_{j-1})}{2^{|j|-1}} < \frac{\delta}{2^{|j|}}, \qquad \text{and} \qquad \frac{d(x_{n,j}, z_j)}{2^{|j|}} < \frac{\delta}{2^{|j|}}.$$

Therefore by triangular inequality

$$\begin{aligned} d(f(z_{j-1}), z_j) &\leq d(z_j, x_{m,j}) + d(f(x_{m,j-1}), x_{m,j}) + d(f(x_{m,j-1}), f(z_{j-1})) \\ &< 2\varepsilon + \alpha. \end{aligned}$$

Since ε was chosen arbitrarily, we have for all $i \in \mathbb{Z}$ that $d(f(z_{i-1}), z_i) \leq \alpha$. Thus $(z_i) \in Y_\alpha$. We conclude Y_α contains all its limits points, e.i., Y_α is a closed subset of $X^{\mathbb{Z}}$ as we wanted to show. □

For every $\alpha \in [0, 1]$, we denote by σ_α the **shift map** on Y_α, as in Example 1.1.3. For every $k \in \mathbb{Z}$, we consider the projection map $\pi_k : X^{\mathbb{Z}} \to X$ given by $\pi_k((x_i)) = x_k$. It can be seen that $f \circ \pi_k = \pi_k \circ \sigma_0$, i.e., f is semi topologically conjugate to σ_0. Thus, by Proposition 1.1.2 we have

$$h(\sigma_0) \geq h(f). \tag{4.2}$$

On the other hand let $\mathcal{A}$ be a finite open cover of Y_0. Suppose $\mathcal{A} := \{A_1, ..., A_m\}$. By the product topology we know that for each $1 \leq i \leq m$,

$$A_i = \left(\prod_{k\in\mathbb{Z}} C_{i,k}\right) \bigcap Y_0,$$

where $C_{i,k}$ is an open subset of X and $C_{i,k} \neq X$ for finitely many k's. Since there are finitely many $\mathcal{A}_i$'s, there exists $1 \leq j \leq m$ such that for some $\widehat{k} \in \mathbb{Z}$ with $C_{j,\widehat{k}} \neq X$, holds that

$$\widehat{k} = \min\{k : 1 \leq i \leq m \quad \text{and} \quad C_{i,k} \neq X\}.$$

Since $\mathcal{A}$ is an open cover of Y_0, then $\pi_{\widehat{k}}(\mathcal{A}) = \{C_{1,\widehat{k}}, C_{2,\widehat{k}}, ..., C_{m,\widehat{k}}\}$ is an open cover of X. For simplicity, we will denote $\mathcal{B} = \pi_{\widehat{k}}(\mathcal{A})$. Note that any open subcover of $\mathcal{B}$, say $\{B_{p_1,\widehat{k}}, ..., B_{p_N,\widehat{k}}\}$ generates by defintion of Y_0 an open subcover of $\mathcal{A}$, it is $\{B_{p_1}, ..., B_{p_N}\}$, where for every $j \in \{p_1, ..., p_N\}$, B_j is an element in $\mathcal{A}$ such that $\pi_{\widehat{k}}(B_j) = B_j$. Thus, since $\pi_{\widehat{k}}|_{Y_0}$ is surjective, then $N(Y_0, \mathcal{A}) = N(X, \mathcal{B})$. We deduce by Definition 1.1.7

$$h(\sigma_0, \mathcal{A}, Y_0) = h(f, \mathcal{B}, X) \leq h(f).$$

Since $\mathcal{A}$ was taken arbitrarily, we can take the supremum over the open covers of Y_0 and conclude that $h(\sigma_0) \leq h(f)$. Together with Equation (4.2) we get

$$h(\sigma_0) = h(f). \tag{4.3}$$

Without loss of generality by Section 2.2, we can assume f to be surjective.

Lemma 4.0.2. The sequence $\{Y_{\frac{1}{m}}\}_{m\in\mathbb{N}}$ converges to Y_0 in the Hausdorff metric $\underline{d}_H$.

Proof. Since X is a compact, let $M \in \mathbb{R}$ be large enough such that for all $x, y \in X$ we have that $d(x, y) \leq M$. Let $\theta > 0$. Let $N \in \mathbb{N}$ such that $M/2^N < \theta$. Now let $n \in \mathbb{N}$ with $n \geq N$. Thus by the uniform continuity of f there exist $0 < \delta_{2n} \leq \delta_{2n-1} \leq ... \leq \delta_1 < \theta$, such that

- If $x, y \in X$ with $d(x, y) \leq \delta_1$, then $d(f(x), f(y)) \leq \theta$.
- If $x, y \in X$ with $d(x, y) \leq \delta_2$ then $d(f(x), f(y)) \leq \frac{\delta_1}{2}$.

  $\vdots$
- If $x, y \in X$ with $d(x, y) \leq \delta_{2n}$ then $d(f(x), f(y)) \leq \frac{\delta_{2n-1}}{2}$.

Let $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\delta_{2n}}{2}$. Let $(z_i) \in Y_{\frac{1}{m}}$. Since we assumed that f is surjective, let $(y_i) \in Y_0$ such that $y_{-n} = z_n$. Thus by definition of $Y_{\frac{1}{m}}$, $d(f(z_{-n}), z_{-n+1}) < \delta_{2n}$. Therefore by triangular inequality

$$d(f^2(z_{-n}), z_{-n+2}) \leq d(f^2(z_{-n}), f(z_{-n+1})) + d(f(z_{-n+1}), z_{-n+2}) \leq \delta_{2n-1}.$$

Which implies

$$d(f^3(z_{-n}), z_{-n+3}) \leq d(f^3(z_{-n}), f(z_{-n+2})) + d(f(z_{-n+2}), z_{-n+3}) \leq \delta_{2n-2}.$$

So inductively we obtain

$$d(f^{2n}(z_{-n}), z_n) \leq d(f^{2n}(z_{-n}), f(z_{n-1})) + d(f(z_{n-1}), z_n) \leq \delta_1.$$

Since for every $1 \leq i \leq 2n$ we have that $\delta_i < \theta$, then each of the above inequalities is bounded by θ, which implies

$$\begin{aligned}
\underline{d}((z_i), (y_i)) &\leq \sum_{i=-\infty}^{-n-1} \frac{d(z_i, y_i)}{2^{|i|}} + \sum_{i=n+1}^{\infty} \frac{d(z_i, y_i)}{2^{|i|}} + \sum_{i=-n}^{n} \frac{\theta}{2^{|i|}} \\
&\leq \sum_{i=-n}^{n} \frac{\theta}{2^{|i|}} + 2\sum_{i=1}^{\infty} \frac{M}{2^{n+i}} \\
&= \theta \sum_{i=-n}^{n} \frac{1}{2^{|i|}} + \frac{M}{2^n} \sum_{i=0}^{\infty} \frac{1}{2^i} \\
&= \theta \sum_{i=-n}^{n} \frac{1}{2^{|i|}} + \frac{M}{2^{n-1}} \leq 4\theta + \frac{M}{2^{n-1}}.
\end{aligned}$$

This implies $\underline{d}((z_i), Y_0) \leq 4\theta + M/2^{n-1}$. Since $n \geq N$, then for every $(z_i) \in Y_{\frac{1}{m}}$, $\underline{d}((z_i), Y_0) \leq 5\theta$. Let $B_\theta(Y_0)$ denote the θ neighborhood of Y_0, i.e., $B_\theta(Y_0) = \{(x_i) \in X : \underline{d}((x_i), Y_0) \leq \theta\}$. Since $Y_0 \subseteq Y_{\frac{1}{m}}$ and (z_i) was taken arbitrarily, then for every $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\delta_{2n}}{2}$ we get $Y_{\frac{1}{m}} \subseteq B_{5\theta}(Y_0)$, i.e.,

$$\underline{d}_H(Y_{\frac{1}{m}}, Y_0) \leq 5\theta.$$

Since θ was chosen arbitrarily, we conclude the sequence $\{Y_{\frac{1}{m}}\}_{m\in\mathbb{N}}$ converges to Y_0 in the Hausdorff metric $\underline{d}_H$. □

In order to calculate and compare for every $\alpha \in [0,1]$ the topological entropy of σ_α, we construct the following covers. Let $\varepsilon > 0$. By compactness of Y_1, there is a finite open cover $\mathcal{A}_{1,\varepsilon}$ by ε-balls that covers it. For each $\alpha \in [0,1]$, we define the open cover of Y_α by

$$\mathcal{A}_{\alpha,\varepsilon} := \{A \cap Y_\alpha \quad | \qquad A \in \mathcal{A}_{1,\varepsilon}\}.$$

Suppose $\{B_1, B_2, ..., B_m\}$ is a finite subcover of $\mathcal{A}_{1,\varepsilon}$ such that $N = N(\mathcal{A}_{0,\varepsilon})$. Since $Y_0 \subseteq X^{\mathbb{Z}}$ is a closed subset, there exists $\theta > 0$ such that

$$B_\theta(Y_0) \subseteq \bigcup_{k=1}^{m} B_k.$$

By the previous lemma, there exists $N \in \mathbb{N}$ such that for all $m \geq N$ we have that $Y_{\frac{1}{m}} \subseteq B_\theta(Y_0)$. Then for every $m \geq N$ we get $N(\mathcal{A}_{0,\varepsilon}) \geq N(\mathcal{A}_{\frac{1}{m},\varepsilon})$. Since $Y_0 \subseteq Y_{\frac{1}{m}}$, the reciprocal inequality holds. Thus

$$\inf_{0<\alpha\leq 1} N(\mathcal{A}_{\alpha,\varepsilon}) = N(\mathcal{A}_{0,\varepsilon}).$$

Since the sequence $(Y_{\frac{1}{m}})_{m\in\mathbb{N}}$ is decreasing under the inclusion relation and by Proposition 1.1.1, we conclude for all $n \in \mathbb{N}$ that

$$\inf_{0<\alpha\leq 1} h(\sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}) \leq \inf_{0<\alpha\leq 1} \inf_n \frac{1}{n} \log(N(\mathcal{A}^n_{\alpha,\varepsilon})) = \inf_{0<\alpha\leq 1} \inf_n \frac{1}{n} \log(N(\mathcal{A}^n_{0,\varepsilon})).$$

Thus,

$$\inf_{0<\alpha\leq 1} h(\sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}) \leq h(\sigma_0, \mathcal{A}_{0,\varepsilon}). \tag{4.4}$$

Lemma 4.0.3. Let $\alpha > 0$ and $\varepsilon > 0$. Then $h_\psi(f, 2\varepsilon, \alpha) \leq h(\sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon})$.

Proof. Let $n \in \mathbb{N}$. Let $(\widehat{x}_i)_{i\geq 0}$ and $(\widehat{y}_i)_{i\geq 0}$ be two α-pseudo-orbits $(n, 2\varepsilon)$-separated for f. Then, there is $1 \leq k < n$ such that $d(x_k, y_k) > 2\varepsilon$. Since we assumed f to be subjective, there are $(x_i)_{i\in\mathbb{Z}}, (y_i)_{i\in\mathbb{Z}} \in Y_\alpha$ such that for every $i \in \mathbb{N}$, $x_i = \widehat{x}_i$ and $y_i = \widehat{y}_i$. Thus

$$\underline{d}(\sigma^k_\alpha(x_i), \sigma^k_\alpha(y_i)) \geq \sum_{i=-\infty}^{\infty} \frac{d(x_{i+k}, y_{i+k})}{2^{|i|}} > 2\varepsilon.$$

We obtain that (x_i) and (y_i) are two $(n, 2\varepsilon)$-separated elements in Y_α with respect to σ_α. This implies that if E is a $(n, 2\varepsilon)$-separated collection of α-pseudo-orbits, then we can construct a $(n, 2\varepsilon)$-separated set of Y_α with respect to σ_α with the same cardinality. Therefore

$$s^\alpha(n, 2\varepsilon, f) \leq s(n, 2\varepsilon, \sigma_\alpha). \tag{4.5}$$

On the other hand, let $\mathcal{B} = \{B_1, B_2, ..., B_N\}$ be a finite open subcover of $\mathcal{A}^n_{\alpha,\varepsilon}$ such that $N = N(\mathcal{A}^n_{\alpha,\varepsilon})$. Suppose (x_i) and (y_i) are in the same element of $\mathcal{B}$. So by Definition 1.1.5, this means for each $0 \leq k < n$ that (x_i) and (y_i) are in the same element of $\sigma^{-k}_\alpha(\mathcal{A}_{\alpha,\varepsilon})$, i.e., that $\sigma^k(x_i)$ and $\sigma^k(y_i)$ are in the same element of $\mathcal{A}^n_{\alpha,\varepsilon}$, i.e., $\underline{d}(\sigma^k_\alpha(x_i), \sigma^k_\alpha(y_i)) < 2\varepsilon$. Therefore any two elements of Y_α in the same element of $\mathcal{B}$ cannot be $(n, 2\varepsilon)$-separated with respect to σ_α. This implies that $S(n, 2\varepsilon, \sigma_\alpha) \leq N(\mathcal{A}^n_{\alpha,\varepsilon})$. Thus, from Equation (4.5) we obtain that

$$s^\alpha(n, 2\varepsilon, f) \leq N(\mathcal{A}^n_{\alpha,\varepsilon}).$$

Since n was taken arbitrarily, then

$$h^\alpha_\psi(f, 2\varepsilon) \leq h(\sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}). \tag{4.6}$$

□

Now we conclude the proof. Let $\alpha \in (0,1]$, $\varepsilon > 0$ and $n \in \mathbb{N}$. As we saw in the proof of Proposition 3.1.2, the sequence $\{h^{\frac{1}{m}}_\psi(f, 2\varepsilon)\}_{m\in\mathbb{N}}$ is monotone and decreasing. Therefore $h_\psi(f, 2\varepsilon) \leq h^\alpha_\psi(f, 2\varepsilon)$. By Equation (4.6), for every $\alpha \in (0,1]$ we obtain

$$h_\psi(f, 2\varepsilon) \leq h(\sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}).$$

Since α is arbitrarily and by Equation (4.4), we get

$$h_\psi(f, 2\varepsilon) \leq \inf_{0<\alpha\leq 1} h(\sigma, \varepsilon) \leq h(\sigma_0, \mathcal{A}_{0,\varepsilon}).$$

As ε was also chosen arbitrarily and by Equation (4.3), we conclude that

$$h_\psi(f) = \lim_{\varepsilon\to 0} h_\psi(f, 2\varepsilon) \leq \lim_{\varepsilon\to 0} h(\sigma_0, \mathcal{A}_{0,\varepsilon}) = h(\sigma_0) = h(f).$$

Together with Equation (4.1), we conclude that $h(f) = h_\psi(f)$. □

Corollary 4.0.1.1. Let $\{f_n\}_n$ be a sequence of continuous transformations on a compact metric space (X, d) converging uniformly to a continuous transformation f. Then

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} h(f_n, \varepsilon) \leq h(f). \tag{4.7}$$

Proof. Let $\varepsilon > 0$, $\alpha > 0$, and $m, n \in \mathbb{N}$. Let E be is a (m, ε)-separated set with respect to f_n. Since for every $i \in \mathbb{N}$ we have $d(f^{i+1}_n(x), f_n(f^i_n(x)) = 0$, then $\{\mathcal{O}_{x,f_n} : x \in E\}$ is a (m, ε)-separated collection of α-pseudo-orbits with respect to f_n, where $\mathcal{O}_{x,f_n} = \{f^i_n(x) : i \in \mathbb{N}\}$. By the uniform convergence hypothesis, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n - f|_\infty < \alpha$. Let $i \in \mathbb{N}$, $x \in X$ and $n \geq N$. Then

$$|f^{i+1}_n(x) - f(f^i_n(x))| < \alpha.$$

This implies $\mathcal{O}_{x,f_n}$ is an α-pseudo-orbit with respect to f. Therefore if E is a (m, ε) separated set for f_n with $n \geq N$, then $\{\mathcal{O}_{x,f_n} : x \in E\}$ is also a (m, ε)-separated collection of α-pseudo-orbits with respect to f. Since E was taken arbitrarily, for all $n \geq N$ we have $s(m, \varepsilon, f_n) \leq s^\alpha(m, \varepsilon, f)$. As m was also chosen arbitrarily, then

$$h(\varepsilon, f_n) \leq h^\alpha_\psi(\varepsilon, f).$$

Since N depends on α, then the above inequality implies that

$$\limsup_{n\to\infty} h(\varepsilon, f_n) \leq h_\psi(\varepsilon, f).$$

Finally, since ε was also chosen arbitrarily, then

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} h(\varepsilon, f_n) \leq h_\psi(f).$$

Applying the previous theorem we obtain Equation (4.7). □

Notice that the argument in this proof uses only topological dynamics arguments. The following theorem gives us a second equivalent definition of topological entropy with pseudo orbits. But in this case, restricting us only to periodic pseudo-orbits.. The proof that we are going to present is due to M. Barge and R. Swanson in 1990 [2], using topological arguments, the Misiurewicz theorem together with a consequence of the variational principle as one of its key points.

Theorem 4.0.2. Let f be a continuous transformation on a compact metric space (X, d). Then the topological entropy of f is equal to the growth exponential rate of separated periodic α-pseudo-orbits of period n as n tends to infinity, i.e.,

$$h(f) = H_\psi(f). \tag{4.8}$$

Proof. First we will prove $h(f) \leq H_\psi(f)$. From Proposition 3.0.3 we have with respect the chain recurrent set that

$$h(f) = h(f|_{R(f)}).$$

Then it suffices to show first $h(f|_{R(f)}) \leq H_\psi(f)$. On one hand, by Section 2.2 we know that

$$h(f|_{R(f)}) = \sup\{h_\mu(f|_{R(f)}) : \mu \in \mathcal{M}_e(f|_{R(f)})\}.$$

Let $\alpha > 0$. By Proposition 3.0.4 we also know that α-chain-transitive components of the chain recurrent set $R(f)$ partition $R(f)$, they are finite and f-invariant subsets of $R(f)$. Therefore every ergodic measure in $R(f)$ is supported in some α-chain-transitive component of $R(f)$. Therefore, if $T_1, ..., T_N$ denote the α-chain-transitive components, we get

$$h(f|_{R(f)}) = \sup\left\{h_\mu(f|_{R(f)}) : \mu \in \bigcup_{i=1}^{N} \mathcal{M}_e(f|_{T_i})\right\} \tag{4.9}$$

$$= \max_{1\leq i\leq N} h(f|_{T_i}). \tag{4.10}$$

In the next two lemmas we will seek to relate these components to the periodic pseudo-orbits.

Lemma 4.0.4. Let $\alpha > 0$. Let T be an α-chain-transitive component of $R(f)$. There is $m(T) \in \mathbb{N}$ such that for every $x, y \in T$, there exists a 2α-chain in T from x to y with length less or equal to $m(T)$.

Proof. By the uniform continuity of f, there is $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$, we have $d(f(w), f(z)) < \alpha$. Without loss of generality suppose that $\delta < \alpha$. Since the space $T \times T$ is compact, then we can cover it by finitely many sets of the form $B_\delta(x_i) \times B_\delta(y_i)$, where $B_\delta(x) := \{y \in X : d(x, y) < \alpha\}$, the ball of radius δ centered at x. Suppose $B_\delta(x_1) \times B_\delta(y_1), B_\delta(x_2) \times B_\delta(y_2), .., B_\delta(x_N) \times B_\delta(y_N)$ cover $T \times T$.

Let $1 \leq i \leq N$. Since $x_i, y_i \in T$, let $\{x_i, z_2, ...z_{k-1}, y_i\}$ be an α-chain from x_i to y_i with length $k(x_i, y_i)$. Let $x \in B_\delta(x_i)$ and $y \in B_\delta(y_i)$. Therefore

$$d(f(x), z_2) \leq d(f(x), f(x_i)) + d(f(x_i), z_2) \leq 2\alpha, \quad \text{and}$$
$$d(f(z_{k-1}), y) \leq d(f(z_{k-1}), y_i) + d(y_i, y) \leq 2\alpha.$$

This implies $\{x, z_2, ..., z_{k-1}, y\}$ is a 2α-chain from x to y with length $k(x_i, y_i)$, i.e., for every $1 \leq i \leq N$, every $x \in B_\delta(x_i)$ and $y \in B_\delta(y_i)$, there exists a 2α-chain from x to y with length $k(x_i, y_i)$. As there are finitely many $k(x_i, y_i)$'s, we define

$$m(T) = \max_{1 \leq i \leq N} \{k(x_i, y_i)\}.$$

□

For the following lemma, we will need to add a bit of notation. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Let T be an α-chain-transitive component of $R(f)$. We denote $p_T^\alpha(n, \varepsilon)$ the maximum cardinality between (n, ε)-separated collections of periodic α-pseudo-orbits with period n in T. Similarly we denote $s_T^\alpha(n, \varepsilon)$ the maximum cardinality over (n, ε)-separated collections of α-pseudo-orbits in T.

Lemma 4.0.5. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Let T be a α-chain-transitive component of $R(f)$. Let $m(T)$ be as in Lemma 4.0.4. Then

$$s_T^\alpha(n, \varepsilon) \leq \sum_{i=1}^{m(T)+n-2} p_T^\alpha(i, \varepsilon) \tag{4.11}$$

Proof. Let E be a (n, ε)-separated collection of α-pseudo-orbits. First we denote $E_1 \subseteq E$ the $(1, \varepsilon)$-separated subset of E, i.e., the collection of α-pseudo-orbits in E that separate in $k = 1$. Let $(x_i) \in E_1$, then by the previous lemma there exists a 2α-chain from x_1 to x_1 with length less or equal to $m(T)$. We define the subsets $E_{1,1}, E_{1,2}, ...E_{1,m(T)} \subseteq E_1$, where $E_{1,i}$ is the subset of α-pseudo-orbits (x_i) of E_1 such that there is a 2α-chain from x_1 to x_1 with length $i \leq m(T)$. Let E_2 be the set of α-pseudo-orbits of E that are separated in $k = 2$. If $(x_i) \in E_2$, then by the previous lemma there exists a 2α-chain from x_2 to x_1 with length less or equal to $m(T)$. We thus define the subsets $E_{2,1}, E_{2,2}, ...E_{2,m(T)} \subseteq E_1$, where $E_{2,i}$ is the set of α-pseudo-orbits (x_i) that are separated in $k = 2$ and such that there is a 2α-chain from x_2 to x_1, with length $i \leq m(T)$.

Successively for every $1 \leq j \leq n$ we define E_j as the set of α-pseudo-orbits that separate in $k = j$, and define $E_{j,i}$ as the set of α-pseudo-orbits (x_i) in E_j such that there exists a 2α-chain from x_j to x_1 with length $1 \leq i \leq m(T)$, say $\{x_j, y_2, ...y_{i-1}, x_1\}$. This implies that $\{x_1, x_2, ..., x_{j-1}, x_j, y_2, ..., y_{i-1}\}$ is a 2α-chain with length $i + j \leq n + m(T) - 2$. Also defines a periodic 2α-pseudo-orbit with period $i + j \le n + m(T) - 2$.

We summarize this construction saying that for every $1 \le k \le n$, and for each pair of α-pseudo-orbits (x_i) and (y_i) which are (k, ε)-separated we can construct two periodic 2α-pseudo-orbits $(k + m(T) - 2, \varepsilon)$-separated with period less or equal to $k + m(T) - 2$. This implies $\# \bigcup_{j+i=k} E_{j,i} \le p_T^{2\alpha}(k, \varepsilon)$. Since $E \subseteq \bigsqcup_{j+i=1}^{m(T)+n-2} E_{j,i}$, we get

$$\#E \le \sum_{i+j=1}^{m(T)+n-2} p_T^{2\alpha}(i + j, \varepsilon).$$

As E was arbitrarily chosen, then we conclude Equation (4.11). □

Now we proceed to prove the theorem. Let $\alpha > 0$. By Equation (4.9) let T be an α-chain-transitive component of $R(f)$ such that $h(f) = h(f|_T)$. Theorem 4.0.1 implies that

$$h_\psi(f) = h(f) = h_\psi(f|_T). \tag{4.12}$$

Recall that by Proposition 3.1.2, the sequences $\{h_\psi^{\frac{1}{m}}(f, \varepsilon)\}_{m\in\mathbb{N}}$ and $\{h_\psi(f, \frac{1}{m})\}_{m\in\mathbb{N}}$ are decreasing and monotone. Therefore for all $\varepsilon > 0$ and $\alpha > 0$, we have

$$h_\psi(f) \le \limsup_{n\to\infty} \frac{1}{n} \log s_T^\alpha(n, \varepsilon).$$

Along with the Lemma 4.0.5,

$$h_\psi(f) \le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{m(T)+n-2} p_T^{2\alpha}(i, \varepsilon).$$

For each $k \ge 0$ we define $1 \le i_k \le k$ such that $p_T^{2\alpha}(i_k, \varepsilon) = \max_{1\le i\le k}\{p_T^{2\alpha}(i, \varepsilon)\}$. Then

$$\begin{aligned}
h_\psi(f) &\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{m(T)+n-2} p_T^{2\alpha}(i, \varepsilon) \\
&= \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} \log \sum_{i=1}^{m(T)+n-2} p_T^{2\alpha}(i, \varepsilon) \\
&\le \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} \log((m(T) + n - 2) p_T^{2\alpha}(i_{m(T)+n-2}, \varepsilon)) \\
&\le \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} [\log(m(T) + n - 2) \\
&+ \log(p_T^{2\alpha}(i_{m(T)+n-2}, \varepsilon))] \\
&= \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} \log(p_T^{2\alpha}(i_{m(T)+n-2}, \varepsilon)) \\
&= H_\psi^{2\alpha}(f|_T, \varepsilon).
\end{aligned}$$

Applying Theorem 4.0.1 this implies that $h(f) \le H_\psi^{2\alpha}(f|_T, \varepsilon)$. Therefore $h(f) \le H_\psi^{2\alpha}(f, \varepsilon)$. Since α and ε were arbitrarily chosen, then $h(f) \le H_\psi(f)$. On the other hand, by Definition 3.2.2, it can be shown that for all $\alpha > 0$ and all $\varepsilon > 0$ we have $s^\alpha(f, \varepsilon) \geq p^\alpha(n, \varepsilon)$. Then $h_\psi(f) \geq H_\psi(f)$ and by the Theorem 4.0.1 we conclude $h(f) \geq H_\psi(f)$. This completes the proof. □

Before presenting one of the consequences of this theorem, it is necessary to make the following two definitions.

Definition 4.0.1. Let $f : X \to X$ be continuous transformation on a metric space (X, d). We will say that f is **expansive** if there exists $\varepsilon > 0$ such that for any $x, y \in X$ there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \varepsilon$.

Definition 4.0.2. Let $f : X \to X$ be a continuous transformation on a metric space (X, d). We will say that f has the **shadowing property** if for each $\beta > 0$ there exists $\alpha > 0$ such that if (x_i) is an α-pseudo-orbit, then there exists $y \in X$ such that for all $n \in \mathbb{N}$, it holds that $d(x_n, f^n(y)) \leq \beta$. We will also say that the orbit O_y "shadows" (x_i).

The next known result [3] can be proven with the previous theorem.

Corollary 4.0.2.1. Let f be an expansive homeomorphism on a compact metric space (X, d). If f has the shadowing property, then

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Fix}(f^n).$$

Proof. Since f is expansive, let $\varepsilon > 0$ be as in Definition 4.0.2 and let $n \in \mathbb{N}$. Let $x, y \in \mathrm{Fix}(f^n)$ with $x \neq y$. Thus, there exists $N \in \mathbb{N}$ such that $d(f^N(x), f^N(y)) > \varepsilon$. Since f is a bijection, we can assume for some $1 \leq k \leq n$ that $d(f^k(x), f^k(y)) > \varepsilon$. Then x and y are (n, ε)-separated. Let $\alpha > 0$. Recall by Definition 3.0.1, that the orbits $\mathcal{O}_x$ and $\mathcal{O}_y$ are periodic α-pseudo-orbits of period n. Thus $\mathcal{O}_x$ and $\mathcal{O}_y$ are also (n, ε)-separated α-pseudo orbits. Which implies $\{\mathcal{O}_x : x \in \mathrm{Fix}(f^n)\}$ is a (n, ε)-separated collection of periodic α-pseudo-orbits of period n. Then

$$\#\mathrm{Fix}(f^n) \leq p^\alpha(n, \varepsilon). \tag{4.13}$$

On the other hand, let $\beta \leq \varepsilon/2$. Since f has the shadowing property, there exists $\alpha > 0$ such that if (x_i) is an α-pseudo-orbit then there exists $y \in X$ such that $\mathcal{O}_y$ shadows (x_i). Now suppose (x_i) is a periodic α-pseudo-orbit with period n. Then there exists $y \in X$ such that for all $m \in \mathbb{N}$, $d(x_m, f^m(y)) < \beta$. Thus by the periodicity, for all $1 \leq r \leq n$ and all $k \in \mathbb{Z}$

$$\begin{aligned} d(f^r(y), f^{r+kn}(y)) &\leq d(f^r(y), x_r) + d(x_r, f^{r+kn}(y)) \\ &\leq \frac{\varepsilon}{2} + d(x_{r+kn}, f^{r+kn}(x)) \\ &\leq \varepsilon. \end{aligned}$$

Now we show $y \in \mathrm{Fix}(f^n)$. Suppose by contradiction and without loss of generality, there exist $k_1, k_2 \in \mathbb{Z}$ such that $f^{nk_1}(y) \neq f^{nk_2}(y)$. Since f is expansive, there exists $N \in \mathbb{N}$ such that $d(f^{N+nk_1)}(y), f^{N+nk_2}(y)) > \varepsilon$. But this contradicts the inequality above. Thus we conclude that $y \in \mathrm{Fix}(f^n)$. This implies that for all periodic α-pseudo-orbit with period n, there exists a periodic orbit with period n that shadows the pseudo-orbit.

Let (x_i) and (y_i) two (n, ε)-separated periodic α-pseudo-orbits with period n. Let $x, y \in X$ be such that $\mathcal{O}_x$ and $\mathcal{O}_y$ and shadows (x_i) and (y_i) respectively. Since f is expansive, it can be shown that $x \neq y$. We conclude that periodic α-pseudo-orbits with period n can be identified injectively with elements in $\mathrm{Fix}(f^n)$. This implies

$$p^\alpha(n, \varepsilon) \leq \#\mathrm{Fix}(f^n).$$

Thus with Equation (4.13), we obtain $p^\alpha(n, \varepsilon) = \#\mathrm{Fix}(f^n)$. Since n was chosen arbitrarily, then

$$H_\psi^\alpha(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Fix}(f^n).$$

Since this holds for all α, then $H_\psi(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Fix}(f^n)$. Also since $\varepsilon > 0$ was taken arbitrarly, then

$$H_\psi(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Fix}(f^n)$$

Finally, Applying Theorem 4.0.2 we obtain

$$h(f) = H_\psi(f) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Fix}(f^n).$$

□

Chapter 5

Pseudo-Pressure and Pressure

This last chapter of the thesis contains our original work. We extend both, the results of Misiurewicz and that of M. Barge and R. Swanson, to the context of topological pressure. We prove that the pseudo-pressure, the functional Pe_ψ and the topological pressure coincide. Our techniques of proof are an adaptation of the methods developed to handle entropy. We would like to point out that, apparently, similar results were obtained by Lian-Fa He in [8]. Unfortunately, the article is not available online and moreover it is in chinese.

Theorem 5.0.1. Let f be a continuous transformation in a compact metric space (X, d). Let $\varphi \in C(X)$. Then the topological pressure of φ with respect to f is equal to the pseudo-pressure of φ with respect to f, i.e.,

$$P(\varphi, f) = P_\psi(\varphi, f).$$

Proof. As in Theorem 4.0.1, let us consider the space $(X^{\mathbb{Z}}, \underline{d})$. For each $\alpha \geq 0$ consider the closed subsets described by

$$Y_\alpha := \{(..., x_{-1}; x_0, x_1, ...) : d(f(x_{i-1}), x_i) \leq \alpha \text{ for every } i \in \mathbb{Z})\}.$$

Let $\varepsilon > 0$. For every $k \in \mathbb{Z}$, consider the projection map $\pi_k : X^{\mathbb{Z}} \to X$. For every $\alpha \in [0, 1]$, let σ_α be the shift map on Y_α.

Lemma 5.0.1. There exists $k \in \mathbb{Z}$ such that the pressure of $\varphi \circ \pi_k$ with respect to σ_0 in Y_0 is equal to the pressure of φ with respect to f in X, i.e.,

$$P(\varphi \circ \pi_k, \sigma_0) = P(\varphi, f).$$

Proof. Let $\mathcal{A}$ be a finite open cover of Y_0. Suppose $\mathcal{A} := \{A_1, ..., A_m\}$. By the product topology we know that for each $1 \leq i \leq m$,

$$A_i = \left(\prod_{k \in \mathbb{Z}} C_{i,k} \right) \bigcap Y_0,$$

where $C_{i,k}$ is an open subset of X and $C_{i,k} \neq X$ for finitely many k's. Since there are finitely many $\mathcal{A}_i$'s, there exists $1 \leq j \leq m$ such that for some $\widehat{k} \in \mathbb{Z}$ with $C_{j,\widehat{k}} \neq X$, holds that

$$\widehat{k} = \min\{k : 1 \leq i \leq m \quad \text{and} \quad C_{i,k} \neq X\}.$$

Since $\mathcal{A}$ is an open cover of Y_0, then $\pi_{\widehat{k}}(\mathcal{A}) = \{C_{1,\widehat{k}}, C_{2,\widehat{k}}, ..., C_{m,\widehat{k}}\}$ is an open cover of X. For simplicity, we will denote $\mathcal{B} = \pi_{\widehat{k}}(\mathcal{A})$. Note that any open subcover of $\mathcal{B}$, say $\gamma = \{B_{p_1,\widehat{k}}, ..., B_{p_N,\widehat{k}}\}$ generates by defintion of Y_0 an open subcover of $\mathcal{A}$, it is $\widehat{\gamma} = \{B_{p_1}, ..., B_{p_N}\}$, where for every $j \in \{p_1, ..., p_N\}$, B_j is an element in $\mathcal{A}$ such that $\pi_{\widehat{k}}(B_j) = B_j$. Let $n \in \mathbb{N}$. Since $f \circ \pi_{\widehat{k}} = \pi_{\widehat{k}} \circ \sigma_0$, if $B_{p_j} \in \widehat{\gamma}$ we get

$$\sup_{(x_j)\in B_{p_j}} \exp\left\{\sum_{i=0}^{n-1} \varphi \circ \pi_{\widehat{k}}(\sigma^i(x_j))\right\} = \sup_{x_1\in B_{p_j,\widehat{k}}} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x_1))\right\}.$$

This implies,

$$\sum_{B_{p_j}\in\widehat{\gamma}^n} \sup_{(x_j)\in B_{p_j}} \exp\left\{\sum_{i=0}^{n-1} \varphi \circ \pi_{\widehat{k}}(\sigma^i(x_j))\right\} = \sum_{B_{p_j,\widehat{k}}\in\gamma^n} \sup_{x\in B_{p_j,\widehat{k}}} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x_1))\right\}.$$

As for the projection map $\pi_{\widehat{k}}$, all open covers of $\mathcal{A}$ generates an open cover of $\mathcal{B}$. We can take the infimum over the finite open sub-covers of either $\mathcal{A}$ or $\mathcal{B}$. Since γ was chosen arbitrarily, then considering the infumum over all finite sub-covers of $\mathcal{B}^n$ on the right side, we obtain

$$P_n(\varphi \circ \pi_{\widehat{k}}, \sigma_0, \mathcal{A}) \leq P_n(\varphi, f, \mathcal{B}).$$

As n was arbitrarily taken, we conclude that

$$P(\varphi \circ \pi_k, \sigma_0, \mathcal{A}) \leq P(\varphi, f, \mathcal{B}) \tag{5.1}$$

Let $m \in \mathbb{N}$. If we intersect $\mathcal{A}$ with balls of radius $\frac{1}{m}$ in $(Y_0, \underline{d})$, we get a open cover A_m such that $\text{diam}(\mathcal{A}_m) \leq \frac{2}{m}$. Thus, we will obtain a sequence $(\mathcal{A}_m)_{m\in\mathbb{N}}$ of open covers such that $\text{diam}(\mathcal{A}_m) \to 0$ when $m \to \infty$. By what we did above, the open covers $\pi_{\widehat{k}}(\mathcal{A}_m) := \mathcal{B}_m$ of X such that the Equation (5.1) holds and $\text{diam}(\mathcal{B})_m \to 0$ when $m \to \infty$. So by Definition 1.2.2, the inequality Equation (5.1) implies $P(\varphi\circ\pi_{\widehat{k}}, \sigma_0) \leq P(\varphi, f)$. On the other hand as $f \circ \pi_{\widehat{k}} = \pi_{\widehat{k}} \circ \sigma_0$ and by Proposition 1.2.2, we have $P(\varphi \circ \pi_{\widehat{k}}, \sigma_0) \geq P(\varphi, f)$. which together with the preceding inequality proves the lemma. □

As in Theorem 4.0.1, let $\mathcal{A}$ be a finite open cover of Y_1 by ε-balls. Thus for every $\alpha \in [0, 1]$, consider the open cover $\mathcal{A}_{\alpha,\varepsilon}$ of Y_α defined by

$$\mathcal{A}_{\alpha,\varepsilon} := \{A \cap Y_\alpha | A \in \mathcal{A}_{1,\varepsilon}\}.$$

Lemma 5.0.2. Let $k \in \mathbb{Z}$ be as in the Lemma 5.0.1. Then

$$\inf_{0<\alpha\leq1} P(\varphi \circ \pi_k, \sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}) \leq P(\varphi \circ \pi_k, \sigma_0, \mathcal{A}_{0,\varepsilon})$$

Proof. Let $\beta > 0$. Notice $\varphi\circ\pi_k$ is uniformly continuous since is a continuous function on a compact metric space. Thus, there exists $\delta > 0$ such that for all $(x_i), (y_i) \in X^{\mathbb{Z}}$ with $\underline{d}((x_i), (y_i)) < \delta$, we have $d(\varphi\circ\pi_k(x_i), \varphi\circ\pi_k(y_i)) < \beta$. Recall from Lemma 4.0.2 that the sequence $(Y_{\frac{1}{m}})_{m\in\mathbb{N}}$ converges to Y_0 in the Hausdorff metric $\underline{d}_H$. Let $M \in \mathbb{N}$ be such that for every $m \geq M$ we get

$$\underline{d}_H(Y_{\frac{1}{m}}, Y_0) < \delta.$$

Let $m \geq M$ and let γ be a finite open subcover of $\mathcal{A}_{\frac{1}{m},\varepsilon}$. First, since γ also covers Y_0, then

$$\sum_{U\in\gamma}\sup_{U\cap Y_0}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_0^i(x_j))\right\} \leq \sum_{U\in\gamma}\sup_{U}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\frac{1}{m}}^i(x_l))\right\}, \tag{5.2}$$

where $(x_j) \in U \cap Y_0$ and $(x_l) \in U$. Thus by the uniform continuity

$$\begin{aligned}\sum_{U\in\gamma}\sup_{U}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\frac{1}{m}}^i(x_l))\right\} &\leq \sum_{U\in\gamma}\sup_{U\cap Y_0}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\frac{1}{m}}^i(x_j))+\beta\right\}\\ &= \exp(n\beta)\sum_{U\in\gamma}\sup_{U\cap Y_0}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\frac{1}{m}}^i(x_j))\right\}.\end{aligned}$$

Let $\mathcal{B} = \{B_1, B_2, ...B_N\}$ be the open cover of Y_0 such that $\mathcal{B} \cap \mathcal{A}_{1,\varepsilon} = \mathcal{A}_{0,\varepsilon}$. As $Y_0 \subseteq Y_{\frac{1}{m}}$ is a closed subset, there exists $\theta > 0$ such that $B_\theta(\mathcal{A}_{0,\varepsilon}) \subseteq \mathcal{B}$, where $B_\theta(\mathcal{A}_{0,\varepsilon}) := \{(x_i) \in X^{\mathbb{Z}} : \underline{d}(x_i, Y_0) < \theta\}$, the θ-neighborhood of $B(\mathcal{A}_{0,\varepsilon})$. Without loss of generality suppose $\delta < \theta$, where δ as above. Then, for every $m \geq M$,

$$\mathcal{A}_{\frac{1}{m},\varepsilon} \subseteq B_\theta(\mathcal{A}_{0,\varepsilon}) \subseteq \mathcal{B}.$$

This implies $\mathcal{B}\cap Y_{\frac{1}{m}} = \mathcal{A}_{\frac{1}{m},\varepsilon}$. Thus, the finite open subcovers of $\mathcal{A}_{\frac{1}{m},\varepsilon}$ are in bijection with the finite subcovers of $\mathcal{A}_{0,\varepsilon}$. If in Equation (5.2) and in the previous inequality we consider the infimum over the finite open sub-covers of $\mathcal{A}_{\frac{1}{m},\varepsilon}$, it is equivalent to consider the infimum over the finite open sub-covers of $\mathcal{A}_{0,\varepsilon}$. Then

$$P_n(\varphi\circ\pi_k, \sigma_0, \mathcal{A}_{0,\varepsilon}) \leq P_n(\varphi\circ\pi_k, \sigma_{\frac{1}{m}}, \mathcal{A}_{\frac{1}{m},\varepsilon}) \leq e^{n\beta}P_n(\varphi\circ\pi_k, \sigma_0, \mathcal{A}_{0,\varepsilon}). \tag{5.3}$$

On the other hand, let $\alpha \geq \alpha'$. Since $Y_{\alpha'} \subseteq Y_\alpha$, then $\mathcal{A}_{\alpha',\varepsilon} \subseteq A_{\alpha,\varepsilon}$. Let $V \in \mathcal{A}_{\alpha,\varepsilon}$ and $n \in \mathbb{N}$, so we have

$$\sup_{(x_j)\in V\cap Y_{\alpha'}}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\alpha'}^i(x_j))\right\} \leq \sup_{(x_j)\in V}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma_{\alpha}^i(x_j))\right\}.$$

This holds for finite open covers of both $\mathcal{A}_{\alpha',\varepsilon}$ and $A_{\alpha,\varepsilon}$, and for every $n \in \mathbb{N}$. Therefore

$$P(\varphi\circ\pi_k, \sigma_{\alpha'}, \mathcal{A}_{\alpha',\varepsilon}) \leq P(\varphi\circ\pi_k, \sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}).$$

Hence $\{P(\varphi\circ\pi_k, \sigma_{\frac{1}{m}}, \mathcal{A}_{\frac{1}{m},\varepsilon})\}_{m\in\mathbb{N}}$ is a monotone decreasing sequence. So

$$\lim_{m\to\infty} P(\varphi\circ\pi_k, \sigma_{\frac{1}{m}}, \mathcal{A}_{\frac{1}{m},\varepsilon}) = \inf_{0<\alpha\leq 1} P(\varphi\circ\pi_k, \sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}). \tag{5.4}$$

Therefore, as β was taken arbitrarily and $M \to \infty$ when $\beta \to 0$, then Equation (5.3) and Equation (5.5) implies

$$\begin{aligned}P_n(\varphi\circ\pi_k, \sigma_0, \mathcal{A}_{0,\varepsilon}) &= \lim_{m\to\infty} P_n(\varphi\circ\pi_k, \sigma_{\frac{1}{m}}, \mathcal{A}_{\frac{1}{m},\varepsilon})\\ &= \inf_{0<\alpha\leq 1} P_n(\varphi\circ\pi_k, \sigma_\alpha, \mathcal{A}_{\alpha,\varepsilon}).\end{aligned}$$

By Definition 1.2.1, we know that $P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}) = \inf_n \frac{1}{n}\log(P_n(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}))$. Let $n\in\mathbb{N}$. Thus, by the preceding equality

$$\begin{aligned}\inf_{0<\alpha\leq 1} P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}) &\leq \inf_{0<\alpha\leq 1}\frac{1}{n}\log(P_n(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}))\\ &\leq \frac{1}{n}\log(P_n(\varphi\circ\pi_k,\sigma_0,\mathcal{A}_{0,\varepsilon})).\end{aligned}$$

Since this holds for every $n\in\mathbb{N}$, we obtain

$$\inf_{0<\alpha\leq 1} P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}) \leq P(\varphi\circ\pi_k,\sigma_0,\mathcal{A}_{0,\varepsilon})$$

□

Lemma 5.0.3. Let $\alpha>0$ and $\varepsilon>0$. Then

$$P^\alpha_\psi(\varphi,f,2\varepsilon)\leq P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}) \tag{5.5}$$

Proof. Let $n\in\mathbb{N}$. Let E be an $(n,2\varepsilon)$-separated set with respect σ_α in Y_α and let γ be a finite open subcover of $\mathcal{A}_{\alpha,\varepsilon}$. From the proof of Lemma 4.0.3, we know that any two $(x_i),(y_i)\in Y_\alpha$ in the same element of $\mathcal{A}^n_{\alpha,\varepsilon}$ cannot be $(n,2\varepsilon)$-separated with respect to σ_α. So each element in E is in a single element of γ. We obtain,

$$\sum_{(x_i)\in E}\exp\left\{\sum_{i=0}^{n-1}\varphi(x_i)\right\}\leq\sum_{U\in\gamma}\sup_{(x_j)\in U}\exp\left\{\sum_{i=0}^{n-1}\varphi\circ\pi_k(\sigma^i_\alpha(x_j))\right\}.$$

Since γ was arbitrarily chosen,

$$\sum_{\mathbf{x}\in E}\exp\left\{\sum_{i=0}^{n-1}\varphi(x_i)\right\}\leq P_n(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}).$$

In the same way, as E was taken arbitrarily, we obtain for all $n\in\mathbb{N}$,

$$P^\alpha_n(\varphi,f,2\varepsilon)\leq P_n(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon}).$$

This implies Equation (5.5). □

Now we will prove Theorem 5.0.1. As in Lemma 5.0.3, we have Equation (5.5) holds for all $\alpha>0$ and all $\varepsilon>0$. Then, by Equation (5.4) and by Lemma 5.0.2 we obtain

$$\begin{aligned}P_\psi(\varphi,f,2\varepsilon)&\leq\lim_{\alpha\to 0}P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon})\\ &=\inf_{0<\alpha\leq 1}P(\varphi\circ\pi_k,\sigma_\alpha,\mathcal{A}_{\alpha,\varepsilon})\\ &\leq P(\varphi\circ\pi_k,\sigma_0,\mathcal{A}_{0,\varepsilon}).\end{aligned}$$

Since $\varepsilon>0$ was arbitrary chosen and applying the Lemma 5.0.1, we obtain

$$P_\psi(\varphi,f)\leq P(\varphi\circ\pi_k,\sigma_0)=P(\varphi,f). \tag{5.6}$$

We will now show the reciprocal inequality. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. As we did at the beginning of the proof of Theorem 4.0.1, it can be shown that if E is a (n, ε)-separated set with respect to f then $\{\mathcal{O}_x : x \in E\}$ is a (n, ε)-separated collection of α-pseudo-orbits. By Definition 1.2.3 of topological pressure via (n, ε)-separated sets and since for every $k \in \mathbb{Z}$ we have $f \circ \pi_k = \pi_k \circ \sigma_\alpha$, we obtain

$$\sum_{x \in E} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} = \sum_{x \in E} \exp\left\{\sum_{i=0}^{n-1} \varphi \circ \pi_k(\sigma_\alpha^i \mathcal{O}_x)\right\} \leq S_n^\alpha(\varphi, f, \varepsilon).$$

Since ε, n, α, and E were taken arbitrarily, respectively we obtain that

$$\begin{aligned} S_n(\varphi, f, \varepsilon) &\leq S_n^\alpha(\varphi, f, \varepsilon), \\ S(\varphi, f, \varepsilon) &\leq P_\psi^\alpha(\varphi, f, \varepsilon), \\ S(\varphi, f, \varepsilon) &\leq P_\psi(\varphi, f, \varepsilon). \end{aligned}$$

As $P(\varphi, f) = S(\varphi, f)$ by Proposition 1.2.1, we get

$$P(\varphi, f) \leq P_\psi(\varphi, f).$$

Together with Equation (5.6) completes the proof. □

We now present a corollary as a generalization of Corollary 4.0.1.1.

Corollary 5.0.1.1. Let $(f_n)_{n \in \mathbb{N}}$ a sequence of continuous transformations in a compact metric space (X, d) that converges uniformly to a continuous transformation f. Let $\varphi \in C(X)$. Then

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} S(\varphi, f_n, \varepsilon) \leq P(\varphi, f) \tag{5.7}$$

Proof. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. By the uniform continuity and as in Corollary 4.0.1.1's proof, there exists $N \in \mathbb{N}$ large enough such that for all $m \geq N$, if E is a (n, ε)-separated set with respect to f, then $\{\mathcal{O}_{x, f_m} : x \in E\}$ is a (n, ε)-separated collection of α-pseudo-orbits with respect to f. Then, by Definition 3.3.1 of pseudo pressure, we get

$$\sum_{x \in E} \exp\left\{\sum_{i=1}^{n-1} \varphi \circ f_m^i(x)\right\} \leq P_n^\alpha(\varphi, f, \varepsilon).$$

As E was taken arbitrarily, then for all $n \in \mathbb{N}$, $S_n(\varphi, f_m, \varepsilon) \leq P_n^\alpha(\varphi, f, \varepsilon)$. This implies that $S(\varphi, f_m, \varepsilon) \leq P_\psi^\alpha(\varphi, f, \varepsilon)$. Just as N depends on α, we get

$$\limsup_{m \to \infty} S(\varphi, f_m, \varepsilon) \leq P_\psi(\varphi, f, \varepsilon).$$

Finally, since $\varepsilon > 0$ is arbitrary and applying Theorem 5.0.1, we conclude that

$$\lim_{\varepsilon \to 0} \limsup_{m \to \infty} S(\varphi, f_m, \varepsilon) \leq P_\psi(\varphi, f) = P(f).$$

□

The following theorem is a generalization of Theorem 4.0.2 proved by M. Barge and R. Swanson, for the computation of topological pressure in a compact metric space with periodic pseudo orbits (recall Definition 3.3.2).

Theorem 5.0.2. Let f be a continuous transformation on a compact metric space (X, d). Let $\varphi \in C(X)$. Then the topological pressure of φ with respect to f is equal to $Pe_\psi(\varphi, f)$, i.e.,

$$P(\varphi, f) = Pe_\psi(\varphi, f) \tag{5.8}$$

Proof. Let $R(f)$ be the chain recurrent set of f (Definition 3.0.2). From Proposition 3.0.3, we know that $P(\varphi, f) = P(\varphi|_{R(f)}, f|_{\varphi|_{R(f)}})$. Then, by Corollary 2.2.1.1 we obtain

$$P(\varphi, f) = \sup\left\{h_\mu(f|_{R(f)}) + \int \varphi d\mu \;\middle|\; \mu \in \mathcal{M}_e(f|_{R(f)})\right\}.$$

Let $\alpha > 0$. Since the α-chain-transitive components of $R(f)$ are f-invariant (Proposition 3.0.4), then every ergodic measure μ of $R(f)$ is supported in some of the finite α-chain-transitive components of $R(f)$, say $T_1, T_2, ...T_N$, where $N \in \mathbb{N}$. Therefore, as $T_1, ..., T_N$ partition $R(f)$, we get that

$$\begin{aligned} P(\varphi, f) &= \sup\left\{h_\mu(f|_{R(f)}) + \int \varphi d\mu \;\middle|\; \mu \in \bigcup_{i=1}^{N} \mathcal{M}_e(f|_{T_i})\right\} & (5.9)\\ &= \max_{1\le i\le N}\left\{P(\varphi|_{T_i}, \varphi|_{T_i})\right\}. & (5.10)\end{aligned}$$

Before presenting the following lemma we introduce new notation. Let $\beta > 0$ and let T be an α-chain-transitive component of $R(f)$. Then, for every $n \in \mathbb{N}$ we define

$$P^\beta_{T,n}(\varphi, f, \varepsilon) := \sup\left\{\sum_{(x_i)\in E} \exp \sum_{i=0}^{n-1} \varphi(x_i)\right\},$$

where the supremum is taken over the collections (n, ε)-separated of β-pseudo-orbits in T. Similarly, for every $n \in \mathbb{N}$ we define

$$Pe^\beta_{T,n}(\varphi, f, \varepsilon) = \sup\left\{\sum_{(x_i)\in F} \exp \sum_{i=0}^{n-1} \varphi(x_i)\right\},$$

where the supremum is considered over the (n, ε)-separated collections of periodic β-pseudo-orbits of period n in T. The following lemma is a generalization of Lemma 4.0.5.

Lemma 5.0.4. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Let T be an α-chain-transitive component of $R(f)$ and let $m(T)$ as in Lemma 4.0.4. Then

$$P^\alpha_{T,n}(\varphi, f, \varepsilon) \le \sum_{i=1}^{m(T)+n-2} Pe^{2\alpha}_{T,i}(\varphi, f, \varepsilon). \tag{5.11}$$

Proof. Let E be a (n,ε)-separated collection of α-pseudo-orbits. As in Lemma 4.0.5, for every $1 \le j \le n$ we define E_j as the set of α-pseudo-orbits that separate in $k = j$, and $E_{j,i}$ as the set of α-pseudo-orbits (x_i) in E_j such that there exists a 2α-chain $\{x_i, y_2, ..y_{i-1}, x_1\}$ with length $1 \le i \le m(T) - 2$. Notice this implies that

$$\{x_1, x_1, ..., x_{j-1}, x_j, y_2, ..., y_{i-1}, x_1\},$$

is a 2α-chain with length $i+j \le n+m(T)-2$. Also it defines a periodic 2α-pseudo-orbit with period $i+j \le n+m(T)-2$. Thus, $\bigcup_{j+i=k} E_{j,i}$ is an $(j+i,\varepsilon)$-separated collection with periodic 2α-pseudo-orbits with period $i+j$. Then, by Definition 3.3.2,

$$\sum_{x_i \in \bigcup_{j+i=k} E_{j,i}} \exp\left\{\sum_{i=0}^{n-1} \varphi(x_i)\right\} \le Pe^{2\alpha}_{T,k}(\varphi, f, \varepsilon).$$

Hence, as $E \subseteq \bigsqcup_{k=1}^{m(T)+n-2} \bigcup_{i+j=k} E_{j,i}$, we get

$$\begin{aligned}
\sum_{(x_i)\in E} \exp\left\{\sum_{i=0}^{n-1} \varphi(x_i)\right\} &\le \sum_{k=1}^{m(T)+n-2} \left(\sum_{(y_m)\in \bigcup_{i+j=k} E_{j,i}} \exp\left\{\sum_{i=0}^{n-1} \varphi(y_i)\right\}\right) \\
&\le \sum_{k=1}^{m(T)+n-2} Pe^{2\alpha}_{T,k}(\varphi, f, \varepsilon).
\end{aligned}$$

Since E is an arbitrary (n,ε)-separated collection of α-pseudo-orbits, we get by Definition 3.3.1 that

$$P^{\alpha}_{T,n}(\varphi, f, \varepsilon) \le \sum_{i=1}^{m(T)+n-2} Pe^{2\alpha}_{T,i}(\varphi, f, \varepsilon).$$

□

Now we will prove the theorem. Let T be an α-chain-transitive component of $R(f)$. Thus by Theorem 5.0.1 and by remark of Definition 3.3.1,

$$\begin{aligned}
P(\varphi|_T, f|_T) &= P_\psi(\varphi|_T, f|_T) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log S^{\alpha}_T(\varphi, n, \varepsilon)
\end{aligned}$$

For each $k \in \mathbb{N}$ let $1 \le i_k \le k$ such that $Pe^{2\alpha}_{T,i_k}(\varphi, f, \varepsilon) = \max_{1\le i\le k} Pe^{2\alpha}_{T,i}(\varphi, f, \varepsilon)$. Then, by Lemma 5.0.4 we get

$$\begin{aligned}
P(\varphi|_T, f|_T) &\le \limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{k=1}^{m(T)+n-2} Pe^{2\alpha}_{T,k}(\varphi, f, \varepsilon)\right) \\
&= \limsup_{n\to\infty} \frac{1}{n+m(T)-2} \log\left(\sum_{k=1}^{n+m(T)-2} Pe^{2\alpha}_{T,k}(\varphi, f, \varepsilon)\right) \\
&\le \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} \log\left([m(T)+n-2] Pe^{2\alpha}_{T,i_{m(T)+n-2}}(\varphi, f, \varepsilon)\right) \\
&\le \limsup_{n\to\infty} \frac{1}{i_{m(T)+n-2}} \log\left(Pe^{2\alpha}_{T,i_{m(T)+n-2}}(\varphi, f, \varepsilon)\right) \\
&= Pe^{2\alpha}_{\psi}(\varphi|_T, f|_T, \varepsilon) \\
&\le Pe2\alpha_{\psi}(\varphi, f, \varepsilon)
\end{aligned}$$

Since $\alpha > 0$ and $\varepsilon > 0$ were arbitrary chosen, we obtain that $P(\varphi|_T, f|_T) \leq Pe_\psi(\varphi, f)$. Also, as T was taken arbitrarily, then by Equation (5.10) we obtain

$$P(\varphi, f) \leq Pe_\psi(\varphi, f). \tag{5.12}$$

We will show the reciprocal inequality. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. If E is an (n, ε)-separated collection of periodic α-pseudo-orbits of period n. By definition, E is also a (n, ε)-separated collection of α-pseudo-orbits, then

$$\sum_{(x_i) \in E} \exp\left\{\sum_{i=0}^{n-1} \varphi(x_i)\right\} \leq P_n^\alpha(\varphi, f, \varepsilon).$$

Since E is arbitrary, we get that

$$Pe_n^\alpha(\varphi, f, \varepsilon) \leq P_n^\alpha(\varphi, f, \varepsilon).$$

In the same way, as α, ε, and n were taken arbitrary, then

$$\begin{aligned} Pe_\psi^\alpha(\varphi, f, \varepsilon) &\leq P_\psi^\alpha(\varphi, f, \varepsilon), \\ Pe_\psi(\varphi, f, \varepsilon) &\leq P_\psi(\varphi, f, \varepsilon), \end{aligned}$$

and applying the Theorem 5.0.1, we obtain

$$Pe_\psi(\varphi, f) \leq P_\psi(\varphi, f) = P(\varphi, f).$$

Together with Equation (5.12) completes the proof. □

Finally, we present a generalization of corollary Corollary 4.0.2.1.

Corollary 5.0.2.1. Let f be an expansive homeomorphism on a compact metric space (X, d). Let $\varphi \in C(X)$. If f has the shadowing property. Then

$$P(\varphi, f) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{F \subseteq \mathrm{Fix}(f^n)} \left\{\sum_{x\in F} \exp\left\{\sum_{i=1}^{n-1} \varphi(x_i)\right\}\right\}. \tag{5.13}$$

Proof. Let $\alpha > 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Since f is expansive and as the proof of Corollary 4.0.2.1, for every $x, y \in \mathrm{Fix}(f^n)$ the orbits $\mathcal{O}_x$ and $\mathcal{O}_y$ are (n, ε)-separated periodic α-pseudo-orbits of period n. Thus, if $F \subseteq \mathrm{Fix}(f^n)$, then the collection of orbits $\{\mathcal{O}_x | x \in F\}$ is an (n, ε)-separated collection of periodic α-pseudo-orbits with period n. This implies that

$$\sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} \leq Pe_n^\alpha(\varphi, f, \varepsilon).$$

Since this holds for every subset of $\mathrm{Fix}(f^n)$, we get

$$\sup_{F \subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} \leq Pe_n^\alpha(\varphi, f, \varepsilon).$$

Since this is true for all $\alpha > 0$, $\varepsilon > 0$, and $n \in N$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} \leq Pe_\psi(\varphi, f). \tag{5.14}$$

We also in saw in the proof of Corollary 4.0.2.1 that, since f has the shadowing property, then for each $\beta > 0$ with $\beta \leq \frac{\varepsilon}{2}$, there exists $\delta > 0$ such that for all periodic α-pseudo-orbit (x_i) with period n we can relate a single periodic point $x \in \mathrm{Fix}(f^n)$ that shadows (x_i), i.e., for every $i \in \mathbb{N}$ we have $d(f^i(x), x_i) < \beta$.

On the other hand, let $\theta > 0$. Since φ is uniformly continuous, then there exists $\beta > 0$ such that for every $x, y \in X$ with $d(x, y) < \beta$, we have $d(\varphi(x), \varphi(y)) < \theta$. Without loss of generality suppose that $\beta < \frac{\varepsilon}{2}$. Let E be a (n, ε)-separated collection of periodic α-pseudo-orbits with period n. Therefore, if $F \subseteq \mathrm{Fix}(f^n)$ is the subset of elements in $\mathrm{Fix}(f^n)$ whose orbits shadows the elements of E, then

$$\sum_{(x_i)\in E} \exp\left\{\sum_{i=0}^{n-1} \varphi(x_i)\right\} \leq \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x)) + \theta\right\}.$$

Since this holds for every E (n, ε)-separated collection of periodic α-pseudo-orbits with period n, we obtain that

$$P_n^\alpha(\varphi, f, \varepsilon) \leq \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x)) + \theta\right\}$$

As n was arbitrarily chosen, we get

$$P_\psi^\alpha(\varphi, f, \varepsilon) \leq \limsup_{n\to\infty} \frac{1}{n} \log e^\theta \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\}$$

Since θ depends on β and β depends on α,

$$\begin{aligned} P_\psi(\varphi, f, \varepsilon) &\leq \lim_{\alpha\to 0} \limsup_{n\to\infty} \frac{1}{n} \log e^\theta \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} \\ &= \lim_{\theta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log e^\theta \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\}. \end{aligned}$$

Also, as this is true for all $\varepsilon > 0$, then

$$P_\psi(\varphi, f) \leq \limsup_{n\to\infty} \frac{1}{n} \log \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\}.$$

Thus, together with Equation (5.14) and applying Theorem 5.0.1, we conclude

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{F\subseteq \mathrm{Fix}(f^n)} \sum_{x\in F} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\} = Pe_\psi(\varphi, f) = P(\varphi, f).$$

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Bibliography

[1] Roy L Adler, Alan G Konheim, and M Harry McAndrew. "Topological entropy". In: *Transactions of the American Mathematical Society* 114.2 (1965), pp. 309–319.

[2] Marcy Barge and Richard Swanson. "Pseudo-orbits and topological entropy". In: *Proceedings of the American Mathematical Society* 109.2 (1990), pp. 559–566.

[3] Rufus Bowen. "Topological entropy and axiom A". In: *Proc. Sympos. Pure Math.* Vol. 14. 1970, pp. 23–41.

[4] Rufus Bowen. "Entropy for group endomorphisms and homogeneous spaces". In: *Transactions of the American Mathematical Society* 153 (1971), pp. 401–414.

[5] Rufus Bowen. "Equilibrium states and the ergodic theory of Anosov diffeomorphisms". In: *Springer Lecture Notes in Math* 470 (1975), pp. 78–104.

[6] Charles Conley. *Isolated invariant sets and the Morse index*. 38. American Mathematical Soc., 1978.

[7] Efim I Dinaburg. "A connection between various entropy characterizations of dynamical systems". In: *Izv. Akad. Nauk SSSR Ser. Mat* 35.324-366 (1971), p. 13.

[8] Lian-fa He. "Pseudo-orbits and topological pressure". In: *Chinese Annals of Mathematics Series A* 17 (1996), pp. 635–642.

[9] Michael Misiurewicz. "Remark on the definition of topological entropy". In: *Dynamical systems and partial differential equations (Caracas, 1984), Univ. Simon Bolivar, Caracas* (1986), pp. 65–67.

[10] James R Munkres. *Topology*. Prentice Hall Upper Saddle River, NJ, 2000.

[11] William Parry and Mark Pollicott. "Zeta functions and the periodic orbit structure of hyperbolic dynamics". In: *Astérisque* 187.188 (1990), pp. 1–268.

[12] Mary Rees. "A Minimal Positive Entropy Homeomorphism of the 2-Torus". In: *Journal of the London Mathematical Society* 2.3 (1981), pp. 537–550.

[13] David Ruelle. "Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification". In: *Bulletin of the American Mathematical Society* 78.6 (1972), pp. 988–991.

[14] David Ruelle. "Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification". In: *Transactions of the American Mathematical Society* 185 (1973), pp. 237–251.

[15] Marcelo Viana and Krerley Oliveira. *Foundations of ergodic theory*. Vol. 151. Cambridge University Press, 2016.

[16] Peter Walters. "A variational principle for the pressure of continuous transformations". In: *American Journal of Mathematics* 97.4 (1975), pp. 937–971.

[17] Peter Walters. *An introduction to ergodic theory*. Vol. 79. Springer Science & Business Media, 2000.