

# Simultaneous Blow-up for two species Patlack-Keller-Segel System in $\mathbb{R}^2$ .

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## Abstract

The collective synchronised movement of unicellular organisms attracting or repelling by chemical signals are defined as chemotaxis. Evelyn Fox Keller and Lee Segel in [4] proposed one Parabolic-Parabolic system (PKS) to model chemotaxis. The general PKS system is the parabolic-parabolic system

$$\left. \begin{aligned} c_t &= \nabla \cdot (k_1(c, v)\nabla c - k_2(c, v)c\nabla v) + k_3(c, v), \\ \epsilon v_t &= D_v \Delta v + k_4(c, v) - k_5(c, v)v, \end{aligned} \right\} \quad (1)$$

where  $x \in \Omega \subset \mathbb{R}^n$ ,  $t \in \mathbb{R}_0^+$  and the biological meaning of variables are:  $c$  organism density,  $v$  chemical concentration,  $k_1$  organism kinetic movement,  $k_2$  chemical sensibility,  $k_3$  organism fitness function,  $D_v$  chemical diffusion,  $k_4$  chemical production,  $k_5$  chemical degradation, subject to Neumann or Dirichlet boundary conditions and smooth boundary  $\partial\Omega$ , positive initial data are considered  $u(x, 0) = u_0$  and  $v(x, 0) = v_0$ . It was conjectured by S. Childress & J.K. Percus in [1], that in atwo-dimensional domain there exists a critical number  $C$  such that if  $\int u_0(x)dx < C$  then the solution exists globally in time and if  $\int u_0(x)dx > C$  blow-up happens. To different versions of the Keller-Segel model the conjecture has been essentially proved, finding the critical value  $C = 8\pi/\chi$ .

In the case of several chemotactic species a new question arise with the objective to extend the S. Childress & J.K. Percus's conjeture,

***Is there a critical curve in the plane of initial masses  $\theta_1\theta_2$  delimiting on one side global existence and blow-up on the other side?***

Let consider the general PKS parabolic-elliptic system,

$$\left. \begin{aligned} \partial_t u_1 &= \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v_1) - \chi_2 \nabla \cdot (u_1 \nabla v_2) \\ \partial_t u_2 &= \Delta u_2 - \chi_3 \nabla \cdot (u_2 \nabla v_1) - \chi_4 \nabla \cdot (u_2 \nabla v_2) \\ -\Delta v_1 &= \alpha_1 u_1 + \alpha_2 u_2 \\ -\Delta v_2 &= \alpha_3 u_1 + \alpha_4 u_2 \end{aligned} \right\} \quad (2)$$

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where  $x \in \mathbb{R}^2$ ,  $t > 0$ ,  $u_1(x, 0), u_2(x, 0) \in L^1(\mathbb{R}^2, (1 + |x|^2)dx)$ . The biological meaning of variables are:  $u_i$  is the population density,  $v_i$  represent the chemo-attractant concentration,  $\chi_i$  population sensibility at the chemical substance. The first optimal response to this question for one simplified version of (2) it was given in [2, 3] and consider two positive chemotactic populations producing the same chemo-attractant and each population have one different sensibility to the chemical signal. The sharp curves funded in [2, 3] are defined by

$$8\pi \left( \frac{\theta_1}{\chi_1} \mu + \frac{\theta_2}{\chi_2} \right) - (\theta_1 + \theta_2)^2 = 0, \quad \theta_1 = \frac{8\pi}{\chi_1}, \quad \theta_2 = \frac{8\pi}{\chi_2}. \quad (3)$$

Arising a new open question proposed in [2, 3]

*Is the Blow-up for  $u_i$  and  $u_2$  be simultaneous?*

The simultaneous Blow-up occurs when  $u_1$  and  $u_2$  have the same existence time. The method applied to prove the simultaneous Blow-up consist in the following step:

1. The entropy comparison criterion which implies that if one is bounded then another entropy too.
2. The Vallée Poussin criterion to prove uniform integrable or equi-integrable of solutions.
3. Equi-integrability property permits to obtain  $L^p$  bounds for all  $p \in (1, +\infty)$ .
4. The Moser-Trudinger and Zigmud-Calderón inequalities to obtain  $L^\infty$  bound.
5. The Alikatos-Moser iteration presented in [5] over non bounded Domain.

The main result is: **if one solution of (2) have bounded entropy, then the two solutions are bounded in  $L^\infty(0, T)$  or equivalently the Blow-up should be simultaneous.**

## References

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