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Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a singularity at $u = 0$

In a series of papers with Daniela Giachetti (Rome, Italy) and Pedro J. Martínez-Aparicio (Cartagena, Spain), we have considered the following semilinear elliptic equation with a singularity at $u = 0$

$$u \geq 0 \quad \text{in } \Omega,$$

$$-div\, A(x)Du = F(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $F(x, s)$ is a Carathéodory function which satisfies

$$0 \leq F(x, s) \leq \frac{h(x)}{\Gamma(s)} \quad \text{a.e. } x \in \Omega, \quad \forall s > 0,$$

with h in some $L^r(\Omega)$ and Γ a $C^1([0, +\infty[)$ function such that $\Gamma(0) = 0$ and $\Gamma'(s) > 0$ for every $s > 0$.

In the case where the singularity is mild, i.e. when

$$0 \leq F(x, s) \leq h(x)(\frac{1}{s} + 1) \quad \text{a.e. } x \in \Omega, \quad \forall s > 0,$$

we define the solution as a function $u \in H_0^1(\Omega)$ which satisfies the equation for test functions $v \in H_0^1(\Omega)$. We prove the existence of such a solution and its stability with respect to variations of $F(x, s)$; we also prove that such a solution is unique when $F(x, s)$ is nonincreasing in s. A key ingredient in the proof is to prove that the integral $\displaystyle\int_{\{u \leq \delta\}} F(x, u)\, v$ tends to zero in a controlled way as δ tends to zero.

In the case where the singularity is stronger, the solution in general does not belong to $H_0^1(\Omega)$ anymore. We then introduce a notion of solution which is more sophisticated: the solution is required to belong to the class of functions such that $G_k(u) \in H_0^1(\Omega)$ and $\varphi T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where as usual $G_k(s) = (s - k)^+$ and $T_k(s) = \inf \{s, k\}$ for $s > 0$, while the equation has to be satisfied for a non standard class of nonnegative test functions, in the spirit of the notion of solutions defined by transposition. This definition allows us to perform, mutatis mutandis, the various computations that we made in the case of mild singularities, and to prove the existence of such a solution and its stability with respect to variations of $F(x, s)$; we also prove that such a solution is unique when $F(x, s)$ is nonincreasing in s.

More recently, in a paper in collaboration with Juan Casado-Diaz (Sevilla, Spain), we have considered the case where $F(x, s)$ can change sign. In this case we prove that the equation still has a nonnegative solution, and that this solution is unique among the nonnegative solutions when $F(x, s)$ is nonincreasing in s. On the other hand, we prove that when the singularity is not mild, there exist no solution which changes sign. Finally, by means of an explicit example, we prove that the equation can have many solutions which change sign.