On the action of the semigroup of non singular integral matrices on $\mathbb{R}^n$

Arnaldo Nogueira

Institut de Mathématiques de Marseille

Abstract. Let Γ be the multiplicative semigroup of all $n \times n$ matrices with integral entries and nonzero determinant. Let $1 \leq p \leq n-1$ and $V = \mathbb{R}^{np} = \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ (p copies). Consider the action of Γ on V, given by the natural action on each component, by matrix multiplication on the left. Then for $\mathbf{x} = (x_1, \ldots, x_p) \in V$, the Γ-orbit is dense in V if and only if there is no linear combination $\sum_{j=1}^{p} \lambda_j x_j$, with $\lambda_j \neq 0$ for some j, which is a rational vector in $\mathbb{R}^n$; in fact the assertion holds also for the orbit of the subgroup $\mathrm{SL}(n, \mathbb{Z})$ that is contained in Γ. When $\mathbf{x}$ is such that the Γ-orbit is dense, given $\mathbf{y} \in \mathbb{R}^{np}$, and $\epsilon > 0$ one may ask for $\gamma \in \Gamma$ such that $\|\gamma\mathbf{x} - \mathbf{y}\| < \epsilon$, with a bound on $\|\gamma\|$ in terms of ϵ. There has been considerable interest in the literature in quantitative results of this kind, for various group actions. In particular it was shown by Laurent-Nogueira, for $n = 2$, that given an irrational vector $\mathbf{x}$ in $\mathbb{R}^2$, any target vector $\mathbf{y} \in \mathbb{R}^2$ and $\rho < \frac{1}{3}$ there exist infinitely many γ in $\mathrm{SL}(2, \mathbb{Z})$ such that $\|\gamma\mathbf{x} - \mathbf{y}\| \leq \|\gamma\|^{-\rho}$. In the talk we will describe some results along this theme for the action of Γ; for the case $n = 2$ the result is stronger in import than what is recalled above for $\mathrm{SL}(2, \mathbb{Z})$, in the sense that the corresponding statement holds for all ρ less than 1, in place of $\frac{1}{3}$ for $\mathrm{SL}(2, \mathbb{Z})$.

The talk is based on a joint work with S.G. Dani.