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Spectral Asymptotics for Compactly Supported Electric Perturbations of the Landau Hamiltonian

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Abstract

We consider compactly supported electric perturbations of the Landau Hamiltonian, and study the accumulation of the discrete spectrum to the so called Landau levels. We review a result from [10], and then study an improved version from [5] describing the accumulation in terms of the logarithmic capacity of the support of the perturbation. We make the observation that the latter one requires the perturbation to be discontinuous at the boundary of its support, and use it as a motivation to give a result containing sufficient conditions for the asymptotics in [5] to hold true if the perturbation is continuous, under some regularity assumptions on its support.

Resumen

En este trabajo consideramos perturbaciones del Hamiltoniano de Landau por potenciales eléctricos con soporte compacto, y estudiamos la acumulación del espectro discreto hacia los llamados niveles de Landau. Primero revisamos un resultado de [10], y luego estudiamos una versión mejorada de [5] que describe la acumulación en términos de la capacidad logarítmica del soporte de la perturbación. Observamos que la segunda versión requiere que la perturbación sea discontinua en la frontera de su soporte, y usamos esto como motivación para enunciar un resultado que entrega condiciones suficientes para la validez de las relaciones asintóticas de [5] en el caso en que la perturbación es continua, bajo algunas hipótesis de regularidad sobre su soporte.

1 Introduction

Our main object of study is the Landau Hamiltonian, which is the 2D Schrödinger operator describing the behavior of a non-relativistic, spinless quantum particle in the plane, subject to a constant magnetic field. The starting point is the fact that the spectrum of this operator, which consists of an arithmetic progression of positive eigenvalues of infinite multiplicity called the Landau levels, remains to be the essential spectrum under the effect of electric perturbations which are relatively compact in the sense of quadratic forms. However, with such perturbations, discrete eigenvalues may appear, and it is of interest to study the rate at which they shall accumulate to the Landau levels.

Recent works by Raikov-Warzel [10] and Filonov-Pushnitski [5] have addressed the issue of accumulation in the case of compactly supported electric perturbations with fixed sign. While [10] gives asymptotics for the case of bounded perturbations, [5] allows for perturbations in $L^p(\mathbb{R}^2)$ classes with $p > 1$. More remarkable is the fact that [5] refines [10], in the sense that it contributes with an aditional asymptotic term that exhibits in a explicit way the dependence on the perturbation via the logarithmic capacity of its support. However, the result in [5] includes an assumption that can only hold if the perturbation is discontinuous at the boundary of its support, so the question arises of whether the asymptotics in [5] still hold true if the perturbation is allowed to be continuous. In this work, we present a result which guarantees that this is indeed possible under some regularity assumptions on the exterior boundary of the support of the perturbation.

This work is organized as follows. We begin by describing the main features of the Landau Hamiltonian, and then discuss the general problem of perturbations by relatively compact electric potentials. Next, we review the mentioned results of [10] and [5], and conclude by studying the case of continuous perturbations. Throughout this work, except for Section 3.1, we will only consider perturbations with fixed sign.

2 The Landau Hamiltonian

Let $b > 0$ be a constant magnetic field in $\mathbb{R}^2$, and consider the magnetic potential $A(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x})) = \frac{b}{2}(-x_2, x_1)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, associated with this field, in the sense that $b = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_1}$. Next, consider the symmetric operators in $L^2(\mathbb{R}^2)$ given by

$$\Pi_j = -i\frac{\partial}{\partial x_j} - A_j, \quad j \in \{1, 2\}.$$

The *Landau Hamiltonian* on $\mathbb{R}^2$ is then defined as the self-adjoint operator

$$\begin{aligned} H_0 &= \Pi_1^2 + \Pi_2^2 \\ &= (-i\nabla - A)^2 \end{aligned} \tag{1}$$

in $L^2(\mathbb{R}^2)$, generated by the closure of the quadratic form

$$\|u\|_A^2 = \int_{\mathbb{R}^2} |i\nabla u + Au|^2 d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^2),$$

where $d\mathbf{x}$ denotes the Lebesgue measure in $\mathbb{R}^2$.

In order to describe the spectral properties of $H_0(b)$, we introduce the *magnetic annihilation* and *creation operators* given, respectively, by

$$a = \Pi_1 + i\Pi_2, \quad a^* = \Pi_1 - i\Pi_2. \tag{2}$$

The operators a and a^* are mutually adjoint in $L^2(\mathbb{R}^2)$, and admit the following representation:

$$a = -2i\mathrm{e}^{-\varphi}\overline{\partial}\mathrm{e}^{\varphi}, \quad a^* = -2i\mathrm{e}^{\varphi}\partial\mathrm{e}^{-\varphi}, \tag{3}$$

where $\varphi(\mathbf{x}) = \frac{b|\mathbf{x}|^2}{4}$, $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$ and $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$, for $\mathbf{x} \in \mathbb{R}^2$ as above. Moreover, the operators a and a^* satisfy the relations

$$[a, a^*] = 2b, \tag{4}$$

and

$$H_0 = a^*a + b = aa^* - b, \tag{5}$$

We describe $\sigma(H_0)$ with the help of a convenient unitary equivalence. Consider the linear mapping κ_b on $\mathbb{R}^4$ defined by

$$\kappa_b(\mathbf{x}, \boldsymbol{\xi}) = \left(\frac{1}{\sqrt{b}}(x_1 - \xi_2), \frac{1}{\sqrt{b}}(\xi_1 - x_2), \frac{\sqrt{b}}{2}(\xi_1 + x_2), -\frac{\sqrt{b}}{2}(\xi_2 + x_1)\right),$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$. The mapping κ_b is symplectic in the sense that

$$\boldsymbol{\sigma}(\kappa_b\mathbf{v}, \kappa_b\mathbf{v}') = \boldsymbol{\sigma}(\mathbf{v}, \mathbf{v}'), \quad \mathbf{v}, \mathbf{v}' \in \mathbb{R}^4, \tag{6}$$

where $\boldsymbol{\sigma}$ is the symplectic form on $\mathbb{R}^4$ defined by

$$\boldsymbol{\sigma}((\mathbf{x},\boldsymbol{\xi}),(\mathbf{x}',\boldsymbol{\xi}')) = \boldsymbol{\xi}\cdot\mathbf{x}' - \mathbf{x}\cdot\boldsymbol{\xi}',$$

and $\cdot$ denotes the standard scalar product in $\mathbb{R}^2$.

In addition, κ_b satisfies

$$(\mathcal{H}_b \circ \kappa_b)(\boldsymbol{\xi},\mathbf{x}) = b(x_1^2+\xi_1^2), \tag{7}$$

where $\mathcal{H}_0(\mathbf{x},\boldsymbol{\xi};b) = (\xi_1+bx_2/2)^2+(\xi_2-bx_1/2)^2$ is the Weyl symbol corresponding to H_0. In this situation, Theorem A.2 in [4], Chapter 7, guarantees the existence of a unitary operator $\mathcal{W}_b$ in $L^2(\mathbb{R}^2)$ satisfying

$$\mathcal{W}_b^* H_0 \mathcal{W}_b = (b\mathfrak{h})\otimes I_y, \tag{8}$$

where I_y is the identity in $L^2(\mathbb{R})$ and

$$\mathfrak{h} = -\frac{d^2}{dx^2} + x^2$$

is the one-dimensional *harmonic oscillator*, self-adjoint in $L^2(\mathbb{R})$ and essentially self-adjoint on $C_0^\infty(\mathbb{R})$. Moreover, the relations

$$\mathcal{W}_b^* a \mathcal{W}_b = (\sqrt{b}\alpha)\otimes I_y, \quad \mathcal{W}_b^* a^* \mathcal{W}_b = (\sqrt{b}\alpha^*)\otimes I_y, \tag{9}$$

hold, α and α^* being the standard annihilation and creation operators, given by

$$\alpha = -i\frac{d}{dx} - ix, \quad \alpha^* = -i\frac{d}{dx} + ix,$$

closed on the domains

$$\mathrm{Dom}(\alpha) = \mathrm{Dom}(\alpha^*) = \mathrm{Dom}(\mathfrak{h}^{1/2}).$$

The well known spectral properties of $\mathfrak{h}$ and relations (8) and (9) imply that

$$\sigma(H_0) = \bigcup_{q\in\mathbb{Z}_+}\{\Lambda_q\}, \quad \Lambda_q := b(2q+1),$$

and

$$\ker(H_0-\Lambda_q) = (a^*)^q \ker a, \quad q\in\mathbb{Z}_+. \tag{10}$$

Now, using (3) we see that

$$\ker a = \{u\in L^2(\mathbb{R}^2) : u = g\mathrm{e}^{-\varphi}, \overline{\partial} g = 0\}.$$

Hence, an orthonormal basis for $\ker a$ is given by the family

$$\varphi_{k,0}(\mathbf{x}) = \sqrt{\frac{b}{2\pi}}\sqrt{\frac{1}{k!}}\left(\sqrt{\frac{b}{2}}z\right)^k \mathrm{e}^{-b|\mathbf{x}|^2/4}, \quad \mathbf{x} \in \mathbb{R}^2, \ k \in \mathbb{Z}_+. \tag{11}$$

Next, we set

$$\tilde{\varphi}_{k,q} = (a^*)^q \varphi_{k,0}, \quad q \in \mathbb{Z}_+,$$

and use (4) to compute

$$\begin{aligned} \langle \tilde{\varphi}_{k,q}, \tilde{\varphi}_{\ell,q} \rangle_{L^2(\mathbb{R}^2)} &= \langle a^q (a^*)^q \varphi_{k,0}, \varphi_{\ell,0} \rangle_{L^2(\mathbb{R}^2)} \\ &= (2b)^q q! \delta_{k\ell}, \qquad\qquad k, \ell \in \mathbb{Z}_+, \end{aligned}$$

which implies, together with (10), that the functions

$$\varphi_{k,q} := \frac{\tilde{\varphi}_{k,q}}{\sqrt{(2b)^q q!}}, \quad k \in \mathbb{Z}_+,$$

constitute an orthonormal basis of $\ker(H_0 - \Lambda_q)$. Note that, in paticular, we have

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0),$$

since $\dim \ker(H_0 - \Lambda_q) = \infty$ for every $q \in \mathbb{Z}_+$.

By using the generalized Laguerre polynomials

$$L_q^{(k-q)}(\xi) = \sum_{\ell=0}^{q} \binom{k}{q-\ell} \frac{(-\xi)^\ell}{\ell!}, \quad \xi \in \mathbb{R},$$

where $\binom{k}{q-\ell} := 0$ if $k < q - \ell$, it is possible to derive a explicit formula for $\varphi_{k,q}$. Namely,

$$\varphi_{k,q}(\mathbf{x}) = \frac{1}{i^q}\sqrt{\frac{b}{2\pi}}\sqrt{\frac{q!}{k!}}\left(\sqrt{\frac{b}{2}}z\right)^{k-q} L_q^{(k-q)}\left(\frac{b|\mathbf{x}|^2}{2}\right) \mathrm{e}^{-b|\mathbf{x}|^2/4}, \quad \mathbf{x} \in \mathbb{R}^2. \tag{12}$$

It is customary to refer Λ_q as the *q-th Landau level*, and the family $\{\varphi_{k,q}\}_{k\in\mathbb{Z}_+}$ as the *canonical basis* of $\ker(H_0 - \Lambda_q)$. For $q \in \mathbb{Z}_+$, we are going to denote the spectral projection of H_0 onto $\ker(H_0 - \Lambda_q)$ by P_q.

3 Electric perturbations

3.1 Relative compactness

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function, playing the role of an electric potential, and consider the perturbed Hamiltonian

$$H_V := H_0 + V.$$

Moreover, assume that V is suitable enough so that H_V is self-adjoint in $L^2(\mathbb{R}^2)$. It is of interest to find conditions on V under which relevant information about $\sigma(H_V)$ can be obtained. This can be done, for example, by taking V to be H_0-form compact, meaning that

$$|V|^{1/2} H_0^{-1/2} \in S_\infty(L^2(\mathbb{R}^2)), \tag{13}$$

where $S_\infty(L^2(\mathbb{R}^2))$ denotes the class of compact operators in $L^2(\mathbb{R}^2)$. Under this assumption, we define H_V as the self-adjoint operator in $L^2(\mathbb{R}^2)$ generated by the closed lower bounded quadratic form

$$\int_{\mathbb{R}^2} \left(|i\nabla u + Au|^2 + V|u|^2 \right) \mathrm{d}\mathbf{x}, \quad u \in \mathrm{Dom}(H_0^{1/2}).$$

Weyl theorem on relatively compact perturbations implies that

$$\sigma_{\mathrm{ess}}(H_V) = \sigma_{\mathrm{ess}}(H_0).$$

In particular, it follows that the elements of $\sigma(H_V) \backslash \sigma(H_0)$ are isolated eigenvalues of finite multiplicity.

A wide variety of perturbations V satisfying (13) is provided by the class $\mathcal{L}^p(\mathbb{R}^2)$ of measurable functions $V : \mathbb{R}^2 \to \mathbb{R}$ such that, for every $\varepsilon > 0$, there exist functions $V_1, V_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$V = V_1 + V_2, \quad V_1 \in L^p(\mathbb{R}^2), \quad \sup_{\mathbf{x} \in \mathbb{R}^2} |V_2(\mathbf{x})| \leq \varepsilon.$$

Indeed, it is known that for $p > 1$, any element $V \in \mathcal{L}^p(\mathbb{R}^2)$ satisfies

$$|V|^{1/2}(-\Delta + 1)^{-1/2} \in S_\infty(L^2(\mathbb{R}^2)). \tag{14}$$

Now, the diamagnetic inequality implies that

$$|V|^{1/2}(H_0 + 1)^{-1/2} \mathrel{\overset{\leq}{\cdot}} |V|^{1/2}(-\Delta + 1)^{-1/2}, \tag{15}$$

where, for any pair of bounded operators T, S on $L^2(\mathbb{R}^2)$, the relation $T \overset{\leq}{\cdot} S$ means that

$$|(Tu)(\mathbf{x})| \leq (S|u|)(\mathbf{x}), \quad u \in L^2(\mathbb{R}^2),$$

for almost every $\mathbf{x} \in \mathbb{R}^2$. Using (14), (15) and Theorem 2.2 from [1], we deduce that

$$|V|^{1/2}(H_0+1)^{-1/2} \in S_\infty(L^2(\mathbb{R}^2)). \tag{16}$$

Thus, writing

$$|V|^{1/2}H_0^{-1/2} = |V|^{1/2}(H_0+1)^{-1/2}(H_0+1)^{1/2}H_0^{-1/2}$$

and noting that $(H_0+1)^{1/2}H_0^{-1/2}$ is bounded, it becomes clear from (16) that (13) holds.

3.2 Signed perturbations

Let $V : \mathbb{R}^2 \to [0, \infty)$ be a measurable function such that (13) holds, and consider the problem of studying the spectrum of the perturbed operators

$$H_{\pm V} := H_0 \pm V.$$

According to the discussion in the previous section, the spectrum of $H_{\pm V}$ outside $\sigma(H_0)$ is discrete, and thus it can only accumulate to the Landau levels. Moreover, since $V \geq 0$, the elements in $\sigma_{\rm d}(H_V)$ can only accumulate to the Landau levels from above, while those in $\sigma_{\rm d}(H_{-V})$ can only do so from below.

In order to study this accumulations, we introduce the intervals

$$\mathrm{I}_0^- := (-\infty, \Lambda_0), \quad \mathrm{I}_{q+1}^- := (\Lambda_q, \Lambda_{q+1}), \quad \mathrm{I}_q^+ := (\Lambda_q, \Lambda_{q+1}), \quad q \in \mathbb{Z}_+, \tag{17}$$

and

$$\mathrm{I}_q^\pm(\lambda) := \{s \in \mathrm{I}_q^\pm : |s - \Lambda_q| > \lambda\}, \quad q \in \mathbb{Z}_+, \ \lambda \in (0, 2b), \tag{18}$$

along with the counting functions

$$\mathcal{N}_q^\pm(\lambda) = \mathrm{rank}\ \mathbb{1}_{\mathrm{I}_q^\pm(\lambda)}(H_{\pm V}), \quad q \in \mathbb{Z}_+, \ \lambda \in (0, 2b). \tag{19}$$

In addition, for any compact self-adjoint operator T we define

$$n_\pm(\lambda; T) := \mathrm{rank}\ \mathbb{1}_{(\lambda,\infty)}(\pm T), \quad \lambda > 0.$$

Here and in the sequel, $\mathbb{1}_S$ denotes the characteristic function of the set S. If T is a self-adjoint operator in a Hilbert space and $S \subset \mathbb{R}$ is a Borel set, then $\mathbb{1}_S(T)$ denotes the spectral projection of T associated with S.

In this language, our interest is on studying the asymptotic behavior of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$. The key ingredient is going to be the fact that this asymptotic behavior can be derived from that of $n_+(\lambda; P_q V P_q)$ as $\lambda \downarrow 0$, which can actually be described in a fairly precise way for certain classes of perturbations V. This reduction to the study of n_+ is encoded in Proposition 1 below, which in turn is based on a suitable version of Weyl inequalities for the functions $n_\pm$ of a sum of two compact operators, and on a generalization of the Birman-Schwinger principle. We give statements for this two results, respectively, without proof:

Lemma 1 ([3], Theorem 9, Section 9.2)**.** *Let T_1, T_2 be compact self-adjoint operators on a Hilbert space. Then, for each $s > 0$ and $\varepsilon \in (0,1)$ we have*

$$\begin{aligned} n_\pm(s(1+\varepsilon); T_1) - n_\mp(s\varepsilon; T_2) &\le n_\pm(s; T_1 + T_2) \\ &\le n_\pm(s(1-\varepsilon); T_1) + n_\pm(s\varepsilon; T_2). \end{aligned} \tag{20}$$

Lemma 2 ([2], Proposition 1.5)**.** *Let $V : \mathbb{R}^2 \to [0,\infty)$ be a measurable function such that* (13) *holds. Then, for each $E \in \rho(H_0)$ we have*

$$\begin{aligned} \nu_\pm(E; V) :&= \sum_{0<g<1} \dim\ker(H_0 \pm gV - E) \\ &= n_\pm(1; V^{1/2}(E - H_0)^{-1} V^{1/2}), \end{aligned} \tag{21}$$

where the sum is taken over the finite set of numbers $g \in (0,1)$ for which $\dim\ker(H_0 \pm gV - E) \neq 0$.

Proposition 1 ([10], Proposition 4.1)**.** *Let $V : \mathbb{R}^2 \to [0,\infty)$ be a measurable function such that* (13) *holds and let $q \in \mathbb{Z}_+$. Then, for every $\varepsilon \in (0,1)$ and small $\lambda > 0$ we have*

$$\begin{aligned} n_+(\lambda; (1-\varepsilon) P_q V P_q) + O(1) &\le \mathcal{N}_q^{\pm}(\lambda) \\ &\le n_+(\lambda; (1+\varepsilon) P_q V P_q) + O(1), \quad \lambda \downarrow 0. \end{aligned} \tag{22}$$

Proof. Let $q \in \mathbb{Z}_+$. If $q \geq 1$, pick $\lambda' \in (0, 2b)$ so that $\{\Lambda_q \pm \lambda'\} \cup \mathrm{I}_q^{\pm}(\lambda') \subset \rho(H_{\pm V})$, following the notation introduced (18). Else, if $q = 0$, take $\lambda' \in (0, 2b)$ so that $\{\Lambda_q + \lambda'\} \cup \mathrm{I}_q^{+}(\lambda') \subset \rho(H_V)$. In any case, denote $\Lambda'_\pm = \Lambda_q \pm \lambda'$ and observe that for every $\lambda \in (0, \lambda')$ we have, for $q \geq 1$,

$$\mathcal{N}_q^{\pm}(\lambda) = \nu_\pm(\Lambda_q \pm \lambda; V) - \nu_\pm(\Lambda'_\pm; V), \tag{23}$$

while for $q = 0$ we have

$$\mathcal{N}_0^{-}(\lambda) = \nu_-(\Lambda_0 - \lambda; V), \quad \mathcal{N}_0^{+}(\lambda) = \nu_+(\Lambda_q + \lambda; V) - \nu_+(\Lambda'_+; V). \tag{24}$$

Thus, using (21) and noting that

$$n_\pm(1; V^{1/2}(\Lambda'_\pm - H_0)^{-1}V^{1/2}) = O(1), \quad \lambda \downarrow 0,$$

which follows from the fact that $V^{1/2}H_0^{-1/2}$ is compact, we deduce from (23) and (24) that

$$\mathcal{N}_q^\pm(\lambda) = n_\pm(1; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}V^{1/2}) + O(1), \quad \lambda \downarrow 0. \tag{25}$$

We are going to estimate the first term at the r.h.s. of (25). To do this, fix $\varepsilon \in (0,1)$ and set $Q_q = I - P_q$, I being the identity map in $L^2(\mathbb{R}^2)$. Using (20) with $T_1 := V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}P_qV^{1/2}$ and $T_2 := V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}Q_qV^{1/2}$, we obtain

$$\begin{aligned} &n_\pm(1; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}V^{1/2}) \\ &\qquad \geq n_\pm\left(\frac{1}{1-\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}P_qV^{1/2}\right) \\ &\qquad - n_\mp\left(\frac{\varepsilon}{1-\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}Q_qV^{1/2}\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} &n_\pm(1; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}V^{1/2}) \\ &\qquad \leq n_\pm\left(\frac{1}{1+\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}P_qV^{1/2}\right) \\ &\qquad + n_\pm\left(\frac{\varepsilon}{1+\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}Q_qV^{1/2}\right). \end{aligned} \tag{27}$$

Now we deal with the r.h.s. of equations (26) and (27). On one hand, using the fact that the non-zero singular numbers of $P_qV^{1/2}$ and $V^{1/2}P_q$ coincide, we see that for $\rho \in \{1-\varepsilon, 1+\varepsilon\}$ we have

$$\begin{aligned} n_\pm\left(\frac{1}{\rho}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1}P_qV^{1/2}\right) &= n_\pm(\lambda; \pm\rho V^{1/2}P_qV^{1/2}) \\ &= n_+(\lambda; \rho P_qVP_q). \end{aligned} \tag{28}$$

On the other hand, picking constants $C_{q,\pm} > 0$ for which the operator inequalities

$$\begin{aligned} |\Lambda_q \pm \lambda - H_0|^{-1}Q_q &= \sum_{\substack{\ell \in \mathbb{Z}_+ \\ \ell \neq q}} |\Lambda_q \pm \lambda - \Lambda_\ell|^{-1}P_\ell \\ &\leq C_{q,\pm} \sum_{\ell \in \mathbb{Z}_+} \Lambda_\ell^{-1}P_\ell = C_{q,\pm}H_0^{-1} \end{aligned}$$

hold uniformly for $\lambda \in (0, \lambda')$, we deduce from the fact that $V^{1/2} H_0^{-1/2}$ is compact that

$$n_{\mp}\left(\frac{\varepsilon}{1-\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1} Q_q V^{1/2}\right) = O(1), \quad \lambda \downarrow 0, \tag{29}$$

and

$$n_{\pm}\left(\frac{\varepsilon}{1+\varepsilon}; V^{1/2}(\Lambda_q \pm \lambda - H_0)^{-1} Q_q V^{1/2}\right) = O(1), \quad \lambda \downarrow 0. \tag{30}$$

Thus, putting together (25), (26), (27), (28), (29) and (30) we obtain (22). $\square$

4 Rough eigenvalue asymptotics for bounded compactly supported perturbations

We now turn to study explicit descriptions of the accumulation of $\sigma_{\rm d}(H_{\pm V})$ to the Landau levels for some specific classes of perturbations $V : \mathbb{R}^2 \to [0, \infty)$. In this section, we consider one of the main results in [10], concerning the case in which V is taken to be bounded and compactly supported.

Theorem 1 ([10], Theorem 2.2)**.** *Let $V : \mathbb{R}^2 \to [0, \infty)$ be a bounded measurable function. Assume that the support of V is compact, and there exists a constant $C_- > 0$ such that $V \geq C_-$ on a non-empty open subset of $\mathbb{R}^2$. Then, for each $q \in \mathbb{Z}_+$ we have*

$$\lim_{\lambda \downarrow 0} \frac{\mathcal{N}_q^{\pm}(\lambda)}{(\log|\log \lambda|)^{-1}|\log \lambda|} = 1, \tag{31}$$

$\mathcal{N}_q^{\pm}$ being the counting functions defined in (19)*.*

Remark 1. From the proof below we can see that if $V : \mathbb{R}^2 \to [0, \infty)$ is any measurable function satisfying (13) and the condition $V \geq C_-$ in Theorem 1, then the operator H_V (resp., H_{-V}) has an infinite number of eigenvalues, counting multiplicities, in each interval I_q^+ (resp., I_q^-), $q \in \mathbb{Z}_+$.

In virtue of Remark 1, for any such perturbation V and $q \in \mathbb{Z}_+$ we introduce the non-increasing sequence of eigenvalues of H_V in the interval I_q^+ and the non-decreasing sequence of eigenvalues of H_{-V} in the interval I_q^-, and denote them, respectively, by

$$\{\lambda_{q,k}^+\}_{k \in \mathbb{Z}_+} \quad \text{and} \quad \{\lambda_{q,k}^-\}_{k \in \mathbb{Z}_+}. \tag{32}$$

Here and in the sequel, we enumerate eigenvalues counting multiplicities. Thus, in particular, our discussion at the beginning of section 3.2 implies that

$$\lambda_{q,k}^+ \downarrow \Lambda_q \quad \text{and} \quad \lambda_{q,k}^- \uparrow \Lambda_q \quad \text{as} \quad k \to \infty, \quad q \in \mathbb{Z}_+. \tag{33}$$

The proof of Theorem 1 given in [10] has two main ingredients:

1. The reduction given by Proposition 1;

2. A study of the spectral asymptotics for the operators $P_q V P_q$, $q \in \mathbb{Z}_+$.

We shall now review this proof as an instructive example of the benefits of having the explicit representation for the canonical basis of $\ker(H_0 - \Lambda_q)$ given in (12). For $\mathbf{x} \in \mathbb{R}^2$ and $r > 0$, set

$$B_r(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^2 : |\mathbf{z} - \mathbf{x}| < r\}.$$

Proof. Let $q \in \mathbb{Z}_+$. In short, we are going to use Proposition 1 to reduce our problem to that of studying the operator $P_q V P_q$, and then we are going to compare V with characteristic functions of some balls, for which radial symmetry will allow to obtain the desired asymptotics. We begin by noting that if $u \in L^2(\mathbb{R}^2)$ is radially symmetric, then the eigenvalues of the compact operator $P_q u P_q$, are given by

$$s_{q,k}(u) := \langle u\varphi_{q,k}, \varphi_{q,k}\rangle_{L^2(\mathbb{R}^2)}, \quad k \in \mathbb{Z}_+.$$

This is an immediate consequence of the fact that the relation

$$\langle u\varphi_{q,k}, \varphi_{q,\ell}\rangle_{L^2(\mathbb{R}^2)} = \left\{\int_0^\infty f_{q,k,\ell}(r)\mathrm{d}r\right\}\left\{\int_0^{2\pi} e^{i(k-\ell)\theta}\mathrm{d}\theta\right\}, \quad k,\ell \in \mathbb{Z}_+,$$

holds for some function $f_{q,k,\ell} : [0,\infty) \to \mathbb{R}$, which follows from the symmetry of u.

In particular, if $u = \mathbb{1}_{B_r(0)}$, $r > 0$, one can use the explicit formulas (12) to deduce (see [10], Proposition 3.2) that

$$\lim_{k\to\infty} \frac{\log s_{q,k}(u)}{k\log k} = -1. \tag{34}$$

Next, using the hypotheses on V, pick constants $C_\pm > 0$, radii $r_\pm > 0$ and points $\mathbf{x}^\pm \in \mathbb{R}^2$ such that

$$C_- \mathbb{1}_{B_{r_-}(\mathbf{x}^-)}(\mathbf{x}) \le V(\mathbf{x}) \le C_+ \mathbb{1}_{B_{r_+}(\mathbf{x}^+)}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{35}$$

Then, estimates (22) and the min-max principle imply that

$$\mathcal{N}_q^\pm(\lambda) \ge n_+(\lambda; (1-\varepsilon)C_- P_q \mathbb{1}_{B_{r_-}(\mathbf{x}^-)} P_q) + O(1), \quad \lambda \downarrow 0, \tag{36}$$

and

$$\mathcal{N}_q^\pm(\lambda) \le n_+(\lambda; (1+\varepsilon)C_+ P_q \mathbb{1}_{B_{r_+}(\mathbf{x}^+)} P_q) + O(1), \quad \lambda \downarrow 0. \tag{37}$$

Now, the *magnetic translations* $\mathcal{T}_{\mathbf{x}'}$ on $\mathbb{R}^2$, defined for $\mathbf{x}' = (x_1', x_2') \in \mathbb{R}^2$ by

$$(\mathcal{T}_{\mathbf{x}'}u)(\mathbf{x}) := \exp\left\{i\frac{b}{2}(x_1' x_2 - x_1 x_2')\right\} u(\mathbf{x}-\mathbf{x}'), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2,$$

commute with H_0, and hence with P_q. This commutation relation implies that

$$P_q \mathbb{1}_{B_{r_\pm}(\mathbf{x}^\pm)} P_q = \mathcal{T}_{\mathbf{x}'}^* P_q \mathbb{1}_{B_{r_\pm}(0)} P_q \mathcal{T}_{\mathbf{x}'},$$

which shows that the operators $P_q \mathbb{1}_{B_{r_\pm}(\mathbf{x}^\pm)} P_q$ and $P_q \mathbb{1}_{B_{r_\pm}(0)} P_q$ are unitarily equivalent. Hence, denoting by $\{s_{q,k,\pm}\}_k$ the non-increasing sequences of eigenvalues of the operators $P_q \mathbb{1}_{B_{r_\pm}(0)} P_q$, $k \in \mathbb{N}$, we see that

$$\begin{aligned}
n_+(E; (1\pm\varepsilon)&C_\pm P_q \mathbb{1}_{B_{r_\pm}(\mathbf{x}^\pm)} P_q) \\
&= n_+(E; (1\pm\varepsilon)C_\pm P_q \mathbb{1}_{B_{r_\pm}(0)} P_q) \\
&= \#\{k \in \mathbb{Z}_+ : (1\pm\varepsilon)C_\pm s_{q,k,\pm} > E\} \\
&= \#\{k \in \mathbb{Z}_+ : \log s_{q,k,\pm} + \log((1\pm\varepsilon)C_\pm) > \log E\}.
\end{aligned} \tag{38}$$

Thus, using (34), we deduce from (38) that

$$\lim_{E \downarrow 0} \frac{n_+(E; (1 \pm \varepsilon) C_\pm P_q \mathbb{1}_{B_{r_\pm}(\mathbf{x}^\pm)} P_q)}{(\log |\log E|)^{-1} |\log E|} = -1,$$

which together with (36) and (37) entails (31). □

We conclude this section by noting that relations (34), (36) and (37) imply that the sequences $\{\lambda_{q,k}^\pm\}_k$ defined in (32), with V as in the statement of Theorem 1, satisfy

$$\log(\pm(\lambda_{q,k}^\pm - \Lambda_q)) = -k \log k (1 + o(1)), \quad k \to \infty, \tag{39}$$

which, as we are going to see in the following section, can be refined.

5 Precise eigenvalue asymptotics for discontinuous compactly supported perturbations

We are now going to study a result from [5] that can be considered as a refinement of Theorem 1, as it gives a more precise description of the rate of the convergence in (33) in terms of the *logarithmic capacity* of the support of the perturbation. Here, the perturbation $V : \mathbb{R}^2 \to [0, \infty)$ will still be compactly supported, but we will also require that it satisfies a positivity condition that can only hold if V is discontinuous at the boundary of its support.

Recall that, for a subset $E \subset \mathbb{R}^2$, the logarithmic capacity of E is defined by

$$\mathrm{Cap}(E) := e^{-\mathcal{V}(E)},$$

where $\mathcal{V}(E)$ is the infimum of

$$\int_E \int_E \log \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathrm{d}\mu(\mathbf{x}) \mathrm{d}\mu(\mathbf{y}),$$

taken over all probability measures μ whose support is a compact subset of E. Here, we adopt the convention that $e^{-\infty} = 0$.

The following are some relevant properties of the logarithmic capacity (see, e.g. [11], [14], [6] or[8]):

- If $E_1 \subset E_2$, then $\mathrm{Cap}(E_1) \leq \mathrm{Cap}(E_2)$;
- $\mathrm{Cap}(\mathbf{x}_0 + rE) = r\mathrm{Cap}(E)$, for every $\mathbf{x}_0 \in \mathbb{R}^2$ and $r > 0$;
- if K is compact, then $\mathrm{Cap}(K)$ coincides with the logarithmic capacity of the boundary of the unbounded component of $\mathbb{R}^2 \backslash K$;
- if $B_1 \subset B_2 \subset \ldots$ are Borel sets and $B = \bigcup_k B_k$, then

$$\mathrm{Cap}(B) = \lim_{k \to \infty} \mathrm{Cap}(B_k).$$

The type of boundary considered in the third point is going to play a crucial role in the proof of the main result in Section 6, so we recall the following definition:

Definition 1. *Let $K \subset \mathbb{R}^2$ be compact. The exterior boundary $\partial_e K$ of K is defined to be the boundary of the unbounded connected component of $\mathbb{R}^2 \backslash K$.*

Remark 2. If two compact subsets of $\mathbb{R}^2$ share exterior boundary, then their logarithmic capacities are equal.

We now state the main result of this section:

Theorem 2 ([5], Theorem 2)**.** *Let $K \subset \mathbb{R}^2$ be a compact set with Lipschitz boundary, and let $V \in L^p(\mathbb{R}^2)$, $p > 1$, be such that $V(\mathbf{x}) = 0$ for $\mathbf{x} \notin K$, and $V(\mathbf{x}) \geq c$ for $\mathbf{x} \in K$ and a constant $c > 0$. Then, for $q \in \mathbb{Z}_+$ and $\{\lambda^{\pm}_{q,k}\}_k$ as defined in* (32)*, we have*

$$\lim_{k\to\infty} (\pm k!(\lambda^{\pm}_{q,k} - \Lambda_q))^{1/k} = \frac{b}{2}(\mathrm{Cap}(K))^2. \tag{40}$$

In a similar fashion to [10], the proof of Theorem 2 given in [5] relies on the reduction given by Proposition 1 and in the study of spectral asymptotics for $P_q V P_q$. This time, we shall briefly discuss a modified version of the reduction argument, while we are just going to give the statement of a result that addresses the issue of spectral asymptotics for $P_q V P_q$ without proof (Lemma 3 below).

Take V as in the statement of the Theorem 2, and denote by

$$s_{q,0} \geq s_{q,1} \geq \cdots \tag{41}$$

the non-increasing sequence of eigenvalues of $P_q V P_q$, $q \in \mathbb{Z}_+$. The key point is to observe that Proposition 1 implies the existence of an integer $k_0 \in \mathbb{N}$ and constants $C_{q,1}, C_{q,2} > 0$ such that, for all $k \in \mathbb{Z}_+$ sufficiently large, we have

$$C_{q,1} s_{q,k+k_0} \leq \pm(\lambda^{\pm}_{q,k} - \Lambda_q) \leq C_{q,2} s_{q,k-k_0}. \tag{42}$$

Note that if the limit $\lim_{k\to\infty}(k! a_k)^{1/k}$ exists for a given sequence $\{a_k\}_k \subset \mathbb{R}$, then it remains unchanged if $\{a_k\}_k$ is replaced by $\{C a_k\}_k$, $C > 0$ being any constant. Hence, as a consequence of (42), the validity of Theorem 2 reduces to the validity of the following result, for which a proof can be find in [5].

Lemma 3 ([5], Lemma 2)**.** *Let $K \subset \mathbb{R}^2$ be a compact set with Lipschitz boundary, and let $V \in L^p(\mathbb{R}^2)$, $p > 1$, be such that $V(\mathbf{x}) = 0$ for $\mathbf{x} \notin K$, and $V(\mathbf{x}) \geq c > 0$ for $\mathbf{x} \in K$ and some constant c. Then, for $q \in \mathbb{Z}_+$ and $\{s_{q,k}\}_k$ as defined in* (41)*, we have*

$$\lim_{k\to\infty} (k! s_{q,k})^{1/k} = \frac{b}{2}(\mathrm{Cap}(K))^2. \tag{43}$$

Remark 3. Using Stirling's formula, which tells that

$$\log(k!) = k \log k - k + O(\log k), \quad k \to \infty,$$

we obtain from equation (40) the asymptotic relation

$$\log(\pm(\lambda^{\pm}_{q,k} - \Lambda_q)) = -k \log k + k \left\{ 1 + \log\left(\frac{b}{2}(\mathrm{Cap}(K))^2\right) + o(1) \right\}, \quad k \to \infty. \tag{44}$$

In particular, this shows that Theorem 2 yields the asymptotic relation (39). Now, the novelty is that relation (44) allows us to see in a explicit way the dependence of the accumulation on the perturbation itself, via the term involving $\mathrm{Cap}(K)$. It is in this sense that Theorem 2 refines Theorem 1.

6 Precise eigenvalue asymptotics for continuous compactly supported perturbations

Here we prove that asymptotics in (40) remain valid when the perturbation $V : \mathbb{R}^2 \to [0, \infty)$ is taken to be continuous and compactly supported, under appropriate regularity assumptions on its support. In particular, we relax the requirement that V satisfies a condition of the type $V(\mathbf{x}) \geq c > 0$ on its support, condition that is required in Theorem 2 and can only hold if V is discontinuous at the boundary of its support. Here and in the sequel, we will say that a subset $D \subset \mathbb{R}^2$ is a *domain* if it is open, connected and non-empty.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, let $K = \overline{\Omega}$, and let $V : \mathbb{R}^2 \to [0, \infty)$ be a continuous function supported in K. Then V satisfies the hypotheses of Proposition 1, and we can again consider, for each $q \in \mathbb{Z}_+$, the sequences $\{\lambda^{\pm}_{q,k}\}_k$ defined in (32). Applying Proposition 1 as before, we deduce that the asymptotic relations (40) will hold if we verify that (43) holds when we take

$$s_{q,0} \geq s_{q,1} \geq \cdots \tag{45}$$

to be the non-increasing sequence of eigenvalues of the compact operator $P_q V P_q$.

Our proof is based on the following result, which follows closely the lines of [12].

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a family of domains $\Omega_j \subset \mathbb{R}^2$, $j \in \mathbb{N}$, such that $\partial\Omega_j$ is Lipschitz, $\overline{\Omega_j} \subset \Omega$ and*

$$\lim_{j\to\infty} \mathrm{Cap}(\overline{\Omega_j}) = \mathrm{Cap}(\overline{\Omega}). \tag{46}$$

Proof. Let $j \in \mathbb{N}$, $j \geq 2$, be arbitrary, and for any compact set $K \subset \mathbb{R}^2$ with two or more points define

$$\Delta_j(K) = \max_{w_1,\ldots,w_j \in K} \prod_{\substack{k,\ell \\ k\neq\ell}} |w_k - w_\ell|.$$

It is known (see, e.g. [7]) that if K is connected, then

$$j^j \mathrm{Cap}(K)^{j(j-1)} \leq \Delta_j(K) \leq (4e^{-1} \log j + 4)^j j^j \mathrm{Cap}(K)^{j(j-1)}. \tag{47}$$

Set $K = \overline{\Omega}$. By the left-hand inequality in (47), there exist points $w_1, \ldots, w_j \in \overline{\Omega}$ such that

$$\prod_{\substack{k,\ell \\ k\neq\ell}} |w_k - w_\ell| \geq j^j \mathrm{Cap}(\overline{\Omega})^{j(j-1)}.$$

In particular, we can choose $w'_k \in \Omega$ sufficiently close to w_k, $k \in \{1 \ldots, j\}$, so that

$$\prod_{\substack{k,\ell \\ k \neq \ell}} |w'_k - w'_\ell| \geq \mathrm{Cap}(\overline{\Omega})^{j(j-1)}.$$

Let $\mathcal{C} = \{\mathbf{x}(s) \in \mathbb{R}^2 : s \in [0,1]\} \subset \Omega$ be a simple closed curve such that $s \mapsto \mathbf{x}(s)$ is C^2 and regular, with

$$\{w'_1, \ldots, w'_j\} \subset \mathcal{C}.$$

Next, introduce the normal unit vector $\mathbf{n} = (-x'_2, x'_1)/|\mathbf{x}'|$, and pick $\varepsilon_j > 0$ small enough so that the domain

$$\Omega_j := \{\mathbf{x}(s) + t\mathbf{n}(s) : s \in [0,1], t \in (-\varepsilon_j, \varepsilon_j)\}$$

has Lipschitz boundary and satisfies $\overline{\Omega_j} \subset \Omega$. The set $\overline{\Omega_j}$ is compact, connected and $\{w'_1, \ldots, w'_j\} \subset \overline{\Omega_j}$, so we have

$$\Delta_j(\overline{\Omega_j}) \geq \mathrm{Cap}(\overline{\Omega})^{j(j-1)}. \tag{48}$$

On the other hand, the right-hand inequality in (47) implies that

$$\Delta_j(\overline{\Omega_j}) \leq (4e^{-1} \log j + 4)^j j^j \mathrm{Cap}(\overline{\Omega_j})^{j(j-1)}. \tag{49}$$

Using (48) and (49) we obtain

$$\mathrm{Cap}(\overline{\Omega}) \leq (4e^{-1} \log j + 4)^{1/(j-1)} j^{1/(j-1)} \mathrm{Cap}(\overline{\Omega_j}),$$

which implies

$$\liminf_{j \to \infty} \mathrm{Cap}(\overline{\Omega_j}) \geq \mathrm{Cap}(\overline{\Omega}). \tag{50}$$

Now, we have $\mathrm{Cap}(\overline{\Omega_j}) \leq \mathrm{Cap}(\overline{\Omega})$ for each j, so the desired conclusion follows from (50). $\square$

We now state and prove the main result of this section:

Theorem 4. *Let $\Omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary, and let $K = \overline{\Omega}$. Let also $V : \mathbb{R}^2 \to \mathbb{R}$ be a continuous, non-negative function, for which there exists a compact set $Z \subset \Omega$ such that $V(\mathbf{x}) = 0$ for $\mathbf{x} \in Z \cup (\mathbb{R}^2 \backslash \Omega)$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \in \Omega \backslash Z$. Then, for $q \in \mathbb{Z}_+$ and $\{\lambda^\pm_{q,k}\}_k$ as defined in* (32), *we have*

$$\lim_{k \to \infty} (\pm k!(\lambda^\pm_{q,k} - \Lambda_q))^{1/k} = \frac{b}{2}(\mathrm{Cap}(K))^2. \tag{51}$$

Proof. Let $q \in \mathbb{Z}_+$. As we already discussed, if suffices to show that, for the sequence $\{s_{q,k}\}_k$ defined in (45), equation (43) holds.

1. Upper bound. Our assumptions on V imply that there exists a constant $C^+ > 0$ such that

$$V(\mathbf{x}) \leq C^+ \mathbb{1}_K(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{52}$$

Next, note that the operator $C^+ P_q \mathbb{1}_K P_q$ is compact, and denote by

$$s_{q,0}(C^+) \geq s_{q,1}(C^+) \geq \cdots$$

the non-increasing sequence of its eigenvalues. Taking (52) into account, we deduce from the min-max principle that

$$s_{q,k} \leq s_{q,k}(C^+), \quad k \in \mathbb{Z}_+. \tag{53}$$

Now, since ∂K is Lipschitz, we can apply Lemma 3 to $\mathbb{1}_K$, and use (53) to obtain

$$\begin{aligned} \limsup_{k\to\infty} (k! s_{q,k})^{1/k} &\leq \lim_{k\to\infty} (k! s_{q,k}(C^+))^{1/k} \\ &= \frac{b}{2} (\mathrm{Cap}(K))^2. \end{aligned} \tag{54}$$

2. Lower bound. Let Ω' be the set of points in Ω that belong to the unbounded connected component of $\mathbb{R}^2 \backslash Z$. Since $\partial_e \overline{\Omega'} = \partial_e K$, we have

$$\mathrm{Cap}(\overline{\Omega'}) = \mathrm{Cap}(K).$$

Thus, since Ω' is a bounded domain, we can apply Theorem 3 to obtain a family of domains $\Omega_j \subset \Omega'$, $j \in \mathbb{N}$, such that $\partial \Omega_j$ is Lipschitz, $\overline{\Omega_j} \subset \Omega'$ and

$$\lim_{j\to\infty} \mathrm{Cap}(\overline{\Omega_j}) = \mathrm{Cap}(K). \tag{55}$$

Since V is continuous and $\overline{\Omega_j} \subset \Omega \backslash Z$ for each $j \in \mathbb{N}$, there exist constants $C_j^- > 0$ such that

$$V(\mathbf{x}) \geq C_j^- \mathbb{1}_{\overline{\Omega_j}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \ j \in \mathbb{N}. \tag{56}$$

As before, the operator $C_j^- P_q \mathbb{1}_{\overline{\Omega_j}} P_q$ is compact, and we denote by

$$s_{q,0}(C_j^-) \geq s_{q,1}(C_j^-) \geq \cdots$$

the non-increasing sequence of its eigenvalues. According to the min-max principle, we deduce from (56) that

$$s_{q,k} \geq s_{q,k}(C_j^-), \quad k \in \mathbb{Z}_+, \ j \in \mathbb{N}. \tag{57}$$

Now, we have that $\partial\overline{\Omega_j}$ is Lipschitz, so we can apply Lemma 3 to $\mathbb{1}_{\overline{\Omega_j}}$, and use (57) to obtain

$$\begin{aligned}\liminf_{k\to\infty}(k!s_{q,k})^{1/k} &\geq \lim_{k\to\infty}(k!s_{q,k}(C_j^-))^{1/k} \\ &= \frac{b}{2}(\mathrm{Cap}(\overline{\Omega_j}))^2, \quad j\in\mathbb{N}.\end{aligned} \tag{58}$$

Thus, taking (54), (55), and (58) into account, we complete the proof by letting $j\to\infty$. $\square$

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