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Weyl group actions on Jacobian varieties

Por

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Abstract

In this work we study and characterize the families of curves with the action of a Weyl group, such that the group algebra decomposition for the corresponding Jacobian becomes a product of elliptic curves.

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Contents

List of Symbols

$T = V/L$	:	Complex torus induced by the quotient of the vector space V over the lattice L.
A	:	Abelian variety.
Π	:	Period matrix of a complex torus.
$\mathrm{Hom}\,(T_1, T_2)$	:	The set of morphisms of T_1 into T_2.
$\hat{T} = \overline{\Omega}/\hat{L}$	:	The dual torus.
$H_1\,(X, \mathbb{Z})$	:	First homology group of X.
$T_g G$	:	Tangent space of G at g.
$\mathrm{mult}_p\,(F)$	:	The multiplicity of the holomorphic map F at the point p.
$\deg\,(F)$	:	Degree of the holomorphic map F.
$M\,(n \times k, K)$	:	The space of $n \times k$ matrices over K.
$\mathrm{Sp}\,(2g, \mathbb{Z})$	:	The symplectic group.
$\mathcal{M}_g$	:	The Moduli space of the isomorphism classes of compact Riemann surfaces of genus g.
$\mathcal{A}_g$	:	The Moduli space of the isomorphism classes of principally polarized abelian varieties of dimension g. .
$\lvert G\rvert, \lvert g\rvert$	:	Cardinality of the group G order of the element $g \in G$.
$H \leq G$	:	H is subgroup of G.
$N \trianglelefteq G$	:	N is a normal subgroup of G.
G/N	:	Quotient group of G by the normal subgroup N.
$\lvert G : H\rvert$	:	Index of H in G.
$G_1 \rtimes G_2$	:	semidirect product of G_1 by G_2.
$\langle g \rangle$	:	subgroup generated by g.

$\mathbf{C}_G(H), \mathbf{N}_G(H)$	:	centralizer of H in G normalizer of H in G.
$\mathbf{Z}(G)$	:	center of G.
$[a,b]$	:	commutator of a and b, $[a,b] = a^{-1}b^{-1}ab$.
$[H,K]$	:	commutator subgroup of H and K, $[H,K] = \Big\langle [h,k] : h \in H, k \in K \Big\rangle$.
G'	:	The commutator subgroup of G, $[G,G]$.
ω	:	A meromorphic 1-form.
D	:	A divisor of a Riemann surface.
$\mathrm{Div}(X)$	:	The set of divisors of X.
(f)	:	A canonical divisor of a meromorphic function f.
$\mathrm{PDiv}(X)$	:	The set of principal divisors of X.
$\mathrm{KDiv}(X)$	:	The set of canonical divisors of X.
(ω)	:	A canonical divisor of a meromorphic 1-form.
$\mathcal{M}(X)$	:	The set of meromorphic functions of X.
$\mathcal{M}^{(1)}(X)$	:	The set of meromorphic 1-forms of X.
$\Omega^1(X)$	:	The space of meromorphic 1-forms of X.
$\mathrm{Jac}(X)$	:	The jacobian variety of a Riemann surface.
$\mathrm{End}(V)$	:	The set of endomorphisms of V.
$K[G]$	:	The set group algebra of G over the field K.
$\mathrm{GL}(V)$	:	The set of the linear isomorphisms of the vector space V.
$\mathrm{Irr}_K(G)$	:	A complete set of irreducible representations of a group G over the field K.
$\mathrm{Ind}_H^G \varphi$	:	The induced representation on G.
$\ell_F(\varphi)$	:	The Schur index of φ with respect to F.
$\mathrm{Gal}(K_V/\mathbb{Q})$	:	The Galois group of the field extension $K_v/\mathbb{Q}$.

$\mathbf{N}(G,K)$	:	The sum of the degrees of the irreducible representations of G over the field K.
$Cl(G)$	:	The number of conjugacy classes of a group G.
$H \cong N$	:	H is isomorphic as groups to N.
$S_n = A_{n-1}$	:	The symmetric group over n symbols using Coxeter notation in the right side.
B_n	:	The demihedral group of rank n
C_n	:	The hyper-octahedral group of rank n.
$O(V)$	:	The set of orthogonal transformations of V.
Φ	:	A root system.
$p(n)$	:	The partition function.
$G \curvearrowright X$	:	An action of G on X.
$\sigma = (\gamma; m_1, \ldots, m_r)$	:	Signature of an action.
$\mathbb{H}_g$	:	The siegel space of degree g.
$\operatorname{ord}(G)$	:	the set of nontrivial orders of a group G.
$\mathbb{O}_p^H$	:	Orbit of p by the subgroup H
$\mathbb{C}$	:	The Complex plane
$\mathbb{P}_1$	:	The Riemann sphere
$\mathbb{H}$	:	The upper half plane, $\{z \in \mathbb{C} : \Im z > 0\}$
Δ	:	The unit disc, $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
S, X, Y	:	use to be Riemann surfaces
S/G	:	The Orbit surface of S by the action of G
π_H	:	the intermediate Galois covering, $\pi_H : S \to S/H$, induced by the action of the subgroup $H \leq G$,
π^H	:	the intermediate covering, $\pi^H : S/H \to S/G$, induced by the action of the subgroup

$H \leq G$.

χ_ρ	:	The character of the linear representation ρ.
Ω_{G_p}	:	Left transversal of $\mathbf{N}_G\left(G_p\right)$ in G, $\{l_1, ..., l_s \in G : G = \bigsqcup_{j=1}^{s} l_j \cdot \mathbf{N}_G\left(G_p\right)\}\}$

Chapter 1

Introduction

Ekedahl and Serre [ES93] ask two questions concerning completely decomposable Jacobian varieties over $\mathbb{C}$.

- **Question 1** Is it true that, for all positive numbers g, there exists a curve of genus g whose Jacobian is completely decomposable?
- **Question 2** Are the genera of curves with completely decomposable Jacobians bounded?

In their work, they construct examples of curves of genus up to 1297. However, even for that range there are genera for which there are no examples of curves having completely decomposable Jacobian.

The topic of completely decomposable Jacobian varieties has produced much research. For these reasons, there are many works about it, see for instance [Kan94], [CRR14], [Yam+07],[MSV09], [Nak+07], [Pau13] and [Pau08].

For a group G, let $\{\mathcal{W}_1, \ldots, \mathcal{W}_l\}$ be a complete set of the irreducible $\mathbb{Q}$-representations of G. If G acts on an abelian variety A, then the group algebra decomposition is given by

$$A \sim B_1^{n_1} \times \ldots \times B_l^{n_l}$$

where $n_i = \dim(V_i)/m_{\mathbb{Q}}(V_i)$, V_i is a complex irreducible representation associated to $\mathcal{W}_i$ and $m_{\mathbb{Q}}(V_i)$ is the Schur index of V_i. (See [LR04] for details).

A Riemann surface X is said to satisfy **Hypothesis A** with respect to a finite group G if there exist an action of G on X such that the corresponding Jacobian is completely decomposable. This hypothesis implies that the following holds:

$$g = \sum n_i \cdot \dim B_i \leq \sum n_i \leq |G|$$

where g is the genus of X. Thus, in order to make examples of completely decomposable Jacobians (using the group algebra decomposition) with genera of high order it is necessary to use groups of big order. With this in mind, it is natural to find for a given family of finite groups $\mathcal{G}$, all the Riemann surfaces that satisfy **Hypothesis A** for some element of $\mathcal{G}$. We call this the classification problem for the family $\mathcal{G}$. In general, this problem is difficult to solve. See for instance, [CRR14], where the classification problem is solved for the family of symmetric groups. The methods used for this case, are specific for the symmetric family. In this work, we continued the classification problem for the Weyl groups, this is an important family of finite Coxeter groups that includes the symmetric family. The solution that we present can be used for any group such that its irreducible complex representations are realizable over $\mathbb{Q}$ this property is known as absolutely irreducible.

This work has the following structure; through chapter 1 to 6 we include some preliminares needed in the classification problem for this family of groups. In Chapter 7, we prove necessary conditions for the actions that we are looking for. Finally, in Chapter 8 we solve the classification problem using the tools provided in the other chapters.

Chapter 2

Abelian Varieties

2.1 Basic concepts

In this chapter we introduce abelian varieties and some of its properties that will be used through this text. Most of the theory and definitions that will be used in this chapter are based in the book Complex abelian varieties of Birkhenhake & Lange [BL13] and the notes "Introduccion a las variedades abelianas y grupos Kleinianos" of Rubí Rodríguez and Rubén Hidalgo [RH05].

Definition 2.1. A *complex torus* T of dimension g is the quotient group of a complex vector space V of dimension g and a *lattice* L of V, i.e a discrete subgroup of rank $2g$. Let $\{e_1, \ldots, e_g\}$ be a basis of V and $\{\lambda_1, \ldots, \lambda_{2g}\}$ a basis of L. For each λ_i write $\lambda_j = \sum_i \alpha_{ij} e_i$. The the matrix,

$$\Pi = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,2g} \\ \vdots & & \vdots \\ \alpha_{g,1} & \cdots & \alpha_{g,2g} \end{bmatrix}$$

is the *period matrix* of T.

A matrix $\Pi \in M(g \times 2g, \mathbb{C})$ is the period matrix of some complex torus if and only if the matrix,

$$P = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$$

is non-singular.

Definition 2.2. Let $T_i = V_i/L_i$ be a complex torus of dimension g_i for $i = 1, 2$. A *morphism* of T_1 to T_2 is a holomorphic map $f \colon T_1 \to T_2$ compatible with the group structure.

It is well known that any holomorphic map between complex tori is the composition of a homomorphism with a translation. Furthermore, any homomorphism $f \colon T_1 \to T_2$ is induced by a unique $\mathbb{C}$-linear map $F \colon V_1 \to V_2$ such that $F(L_1) \subset L_2$. We denote by $\mathrm{Hom}(T_1, T_2)$ the set of morphisms from T_1 into T_2. Note that this set endowed with pointwise addition is an abelian group.

The uniqueness of the map F defines a group monomorphism

$$\rho_a \colon \mathrm{Hom}(T_1, T_2) \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$$
$$f \mapsto F.$$

The restriction of F to the lattice L_1 induces a monomorphism,

$$\rho_r\colon \operatorname{Hom}(T_1, T_2) \to \operatorname{Hom}_{\mathbb{Z}}(L_1, L_2)$$
$$f \mapsto F|_{L_1}.$$

Definition 2.3. The maps $\rho_a(f)$ and $\rho_r(f)$ are the *analytic, and rational representations* of f.

Since $\operatorname{Hom}_{\mathbb{Z}}(L_1, L_2) \cong \mathbb{Z}^{4g_1g_2}$, the injectivity of ρ_r implies that

$$\operatorname{Hom}(T_1, T_2) \cong \mathbb{Z}^m$$

for some $m \leq 4g_1g_2$. In particular, if $T_1 = T_2$ then ρ_a, ρ_r are representations of $\operatorname{End}(T), \operatorname{End}_{\mathbb{Q}}(T)$ respectively. These representations are related by

$$\rho_r \otimes \mathbb{C} \cong \rho_a \oplus \overline{\rho_a}.$$

Thus, the complexification of the rational representation is the direct sum of the analytic and the conjugated analytic representation.

Remark 2.4. Let $\Pi_i \in M(g_i \times 2g_i, \mathbb{C})$ denote the period matrix of the torus $T_i = V_i/L_i$. With respect to the basis L_i, there exists matrices $A \in M(g_2 \times g_1, \mathbb{C})$, $R \in M(2g_2 \times 2g_1, \mathbb{C})$ corresponding to the analytical and rational representations respectively. The condition $\rho_a(f)(L_1) \subset L_2$ can be rewritten as,

$$A\Pi_1 = \Pi_2 R.$$

Conversely, given $A \in M(g_2 \times g_1, \mathbb{C}), R \in M(2g_2 \times 2g_1, \mathbb{C})$ satisfying the above equation, we can define a morphism $f\colon T_1 \to T_2$ with corresponding analytical and rational representations given by A and R respectively. In other words, any morphism is determined by matrices satisfying these conditions. In particular, f is an isomorphism if and only if $A \in \operatorname{GL}(2g, \mathbb{Z})$ and $R \in \operatorname{GL}(2g, \mathbb{Z})$

Definition 2.5. Let $T = V/L$ be a complex torus. A set S is said to be a subtorus of T if there exists a subspace $W \leq V$ and a lattice M of W such that $M \subset W \cap L$ and $S = W/M$.

Remark 2.6. If $f\colon T_1 \to T_2$ is a morphism, then $\operatorname{Im}(f)$ and the connected component $(\ker f)_0$ of $\ker f$ containing 0 are examples of subtori of T_1 and T_2 respectively.

Definition 2.7. An *isogeny* of a complex torus T_1 to a complex torus T_2 is a surjective morphism $T_1 \to T_2$ with finite kernel. Note that a morphism $T_1 \to T_2$ is an isogeny if and only if it is surjective and $\dim T_1 = \dim T_2$.

Proposition 2.8. *Let $f\colon T_1 \to T_2$ be an isogeny of complex tori. Then there exists a natural number n and an isogeny $g\colon T_2 \to T_1$ such that $g \circ f = n_1$ and $f \circ g = n_2$, where n_i is the endomorphism $n_i\colon T_i \to T_i$ defined by $n_i([v]) = [nv]$.*

Proof. See [BL13, Ch. 1] □

Remark 2.9. The above result implies that complex tori as objects and isogenies as arrows define a category. Another consequence is that isogenies are units of the ring $\operatorname{End}_{\mathbb{Q}}(T)$.

Let $T = V/L$ be a complex torus of dimension g. Consider the $\mathbb{C}$-vector space

$$\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) := \{h\colon V \to \mathbb{C} |\ h \text{ is } \mathbb{C}\text{-antilinear}\}.$$

The map $h \mapsto \Im h$ defines an $\mathbb{R}$-isomorphism from $\overline{\Omega}$ to $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. It follows that the $\mathbb{R}$-bilinear form,

$$\begin{aligned} \overline{\Omega} \times V &\to \mathbb{R} \\ (h, v) &\mapsto \Im h(v) \end{aligned}$$

is non-degenerate. Hence the set $\hat{L} = \{h \in \overline{\Omega} |\ \mathrm{Im}\, h(L) \subset \mathbb{Z}\}$ is a lattice in $\overline{\Omega}$, which is called the *dual lattice* of L. The quotient

$$\hat{T} = \overline{\Omega}/\hat{L}$$

is a complex torus of dimension g, called the *dual torus*.

2.2 Abelian varieties

Remark 2.10. For a complex vector space V, there exists a correspondence

$$\left\{ \begin{array}{c} \text{Real bilinear anti-symmetric} \\ \text{forms } E \text{ satisfying} \\ E(iv, iu) = E(u, v) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Hermitian forms } H \\ \text{on } V \end{array} \right\}.$$

The natural correspondences are given by $E(v, w) \mapsto E(iv, w) + iE(v, w)$ and $H \mapsto Im(H)$. Because of this, E and H will be used interchangeably.

Definition 2.11. Let H a Hermitian form on V such that $Im(H)(L \times L) \subset \mathbb{Z}$. We define the function

$$\begin{aligned} \phi_H =\colon V &\to \overline{\Omega} \\ v &\mapsto H_v, \end{aligned}$$

where $H_v(w) = H(v, w)$. Note that $\phi_H(L) \subset \hat{L}$. Hence this map induces a morphism

$$\lambda_H = \lambda_E \colon T \to \hat{T}$$

with analytical representation ϕ_H.

Proposition 2.12. *Let $T = V/L$ be a torus and H a Hermitian form on V satisfying $H(L \times L) \subset \mathbb{Z}$. Then the following are equivalent:*

1. *H is non-degenerate.*
2. *ϕ_H is an isomorphism.*
3. *λ_H is an isogeny.*

Proof. See [RH05, Ch. 2]. □

Definition 2.13. A *polarization* on a torus $T = V/L$ is a non-degenerate real alternating form E on V satisfying

1. $E(iu, iv) = E(u, v)$,

2. $E\left(L \times L\right) \subset \mathbb{Z}$.

Definition 2.14. Let $T = V/L$ be a complex torus of dimension g. Then T is said to be an *abelian variety* if there exists a polarization on T. The pair (T, H) is a *polarized abelian variety*.

Remark 2.15. Not every complex torus has a polarization. However, the next example will show that for the 1-dimensional case it is always possible to polarize them.

Example 2.16. Let $\mathbb{H}$ be the upper half complex plane. Given $\tau \in \mathbb{H}$, let $L = \mathbb{Z} + \tau\mathbb{Z}$. We call the torus $T = \mathbb{C}/L$ an *elliptic curve*. The function given by

$$E\left(\alpha_1 + \beta_1\tau, \alpha_2 + \beta_2\tau\right) = \frac{-\Im\left(\left(\alpha_1 + \beta_1\tau\right)\left(\alpha_2 + \beta_2\tau\right)\right)}{\Im(\tau)} = \alpha_1\beta_2 - \alpha_2\beta_1.$$

for $\alpha_i, \beta_i \in \mathbb{R}$ is a polarization of T.

Remark 2.17. If we identify the elements 1 and τ as 1-cycles that generate the lattice $H_1\left(X, \mathbb{Z}\right)$, the above formula gives the geometric intersection number of the 1-cycles $\alpha_i + \beta_i\tau$ for all integral values of α_i, β_i.

Definition 2.18. Given a complex torus T_1, a polarized abelian variety (T_2, H), and a morphism with finite kernel $f\colon T_1 \to T_2$, we define a polarization on T_1 by

$$f^*\left(H\right)\left(v_1, v_2\right) := H\big(\rho_a\left(f\right)\left(v_1\right), \rho_a\left(f\right)\left(v_2\right)\big).$$

This polarization is called the *induced polarization* or the *pullback polarization* of f.

Definition 2.19. If $\left(A_1, H_1\right), \left(A_2, H_2\right)$ are polarized abelian varieties, then $f\colon A_1 \to A_2$ is said to be a *morphism of polarized abelian varieties* if f is a morphism of complex tori and satisfies the relation $f^*\left(H_2\right) = H_1$. An *automorphism* of a polarized abelian variety is a bijective morphism $f\colon A \to A$ such that f and f^{-1} preserve the polarization.

Proposition 2.20. *For every polarized abelian variety (T, E) with $T = V/L$, there exists a basis of L such that E can be written as*

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_g \end{pmatrix}$, *with d_i positive integers such that $d_i | d_{i+1}$. Such basis is called* simplectic. *The sequence $(d_1, \ldots, d_g)$ is called the* type of the polarization. *If D is the identity matrix, then E is a* principal polarization, *and T is a* principally polarized abelian variety *(ppav).*

Proposition 2.21 (Riemann Relations)**.** *Let T be a complex torus and Π a period matrix for T. Then T is an abelian variety if and only if there exists a nondegenerated alternating integral matrix E of size $2g \times 2g$ such that*

1. $\Pi E^{-1} \Pi^t = 0$.

2. $-i \cdot \Pi E^{-1} \overline{\Pi}^t$ *is positive definite.*

Proof. See [BL13, Ch. 4]. □

Corollary 2.22. *If $A = (T, E)$ is a polarized abelian variety of type $(d_1, \ldots, d_g)$, then there exist a basis of V and a symplectic basis of L, such that a period matrix of T has the form:*

$$\Pi = (Z\ D)\,,$$

where D is the diagonal matrix that contains as diagonal the type of the polarization and Z is a complex $g \times g$ symmetric matrix such that $\Im(Z)$ is positive definite.

Definition 2.23. Let A be an abelian variety, and A' a sub-torus. Then A' is said to be a *sub-abelian variety* if the polarization of A' is induced by the inclusion map $i\colon A' \to A$. If there exists a sub-abelian variety B of A such that $A = A' + B$, then we say that A' and B are complementary.

An important property of abelian varieties is that every sub-torus (in particular any abelian subvariety) has a complementary abelian subvariety. This does not hold for arbitrary tori.

Theorem 2.24 (Poncairé's Reducibility Theorem)**.** *Let (A, H) be a polarized abelian variety and S a sub-torus of A. Then there exists a sub-torus R of A such that $A = S + R$ and $S \cap R$ is finite.*

Proof. See [RH05, Ch. 2]. □

Definition 2.25. An abelian variety is called simple if it has no proper abelian sub-varieties.

Proposition 2.26 (Poncairé Decomposition)**.** *Given an abelian variety A, there exist simple abelian varieties A_i not isogenous to each other, positive integers n_i and an isogeny*

$$A \to X_1^{n_1} \times \ldots \times X_r^{n_r}$$

Moreover, the integers and the abelian varieties A_i are unique up to isogenies and permutations. If all the components A_i are elliptic curves, then A is said completely decomposable.

Proof. It is straightforward from the definition of simple abelian variety and Poncaire's Decomposition. □

Remark 2.27. Finding decompositions for arbitrary Abelian Varieties is in general hard. We will see a way to find similar decompositions by using group actions. It is still an open question to determine when they correspond to the Poncairé decomposition.

2.3 Algebraic groups

In this section we consider mostly [SH77] and [Mil86]. Regular maps are the morphisms of the category of algebraic varieties.

Definition 2.28. An *algebraic group* is an algebraic variety G which is at the same time a group, in such a way that the product and inverse map are regular maps. An *algebraic subgroup H* of G is a subgroup that is closed in G. Finally, a *morphism of algebraic groups* is a group homomorphism which is a regular map.

Theorem 2.29. *An algebraic group is a nonsingular variety.*

Proof. Note that for each $h \in G$, the map

$$t_h \colon G \to G$$
$$t_h(g) = hg$$

is an automorphism of the variety. It follows that G is homogeneous. The property of singularity is invariant under isomorphism of varieties. It follows that the set of singular points is empty or G. Finally, recall that the set of singular points of a variety is a proper subset. Hence, the algebraic group is nonsingular. □

The next theorem will be useful to prove that regular maps defined on abelian varieties are very rigid.

Theorem 2.30 (Rigidity Theorem)**.** *Let $f \colon X \times Y \to Z$ be a regular map of irreducible varieties X, Y and Z, with X projective. Suppose that for some point $y_0 \in Y$, the image of $X \times y_0$ under f is a single point in Z. Then, the image of $X \times y$ is a single point for every $y \in Y$.*

Proof. See [SH77, Ch. 4]. □

Theorem 2.31. *Let $\psi \colon G \to H$ be a regular map of an abelian variety G to an algebraic group H, then there exists a group morphism $\varphi \colon G \to H$, such that $\psi(g) = \psi(e)\varphi(g)$, where $e \in G$ is the identity element. In particular, two abelian varieties are isomorphic as varieties if and only if they are isomorphic as algebraic groups.*

Proof. It is enough to show that $g \mapsto \varphi(g) := \psi^{-1}(e)\psi(g)$ is a group homomorphism. To accomplish this, define the auxiliar function $G \times G \to H$ given by $(g, g') \mapsto \varphi(g)\varphi(g')(\varphi(gg'))^{-1}$ and apply the Rigidity Theorem. □

2.3.1 Relation between Abelian varieties and algebraic groups

Proposition 2.32. *Let G be an irreducible and projective algebraic group of dimension g over the complex numbers. Then, there exists a morphism*

$$\exp \colon T_eG \to G,$$

that is surjective and its kernel is a lattice in the tangent space $T_eG \cong \mathbb{C}^g$.

Proof. See [Mil86, Ch. 1, Section 2]. □

Remark 2.33. The above proposition implies that irreducible projective algebraic groups are complex tori. Conversely, a complex torus is *projective* if it can be viewed as an irreducible projective algebraic group.

Theorem 2.34. *A complex torus is projective if and only if it is an abelian variety i.e there exists a polarization on it.*

Proof. It is almost the same as [Mil86, Thm. 2.8]. □

The above theorem allows us to apply the theory of varieties to abelian varieties. The projective irreducible algebraic groups will also be called abelian varieties.

Chapter 3

Riemann Surfaces

3.1 Basic definitions and notation

A *Riemann surface* is a one-dimensional connected complex manifold. Unless otherwise stated we assume that the Riemann surface is also compact. It can be proved that the compact Riemann surfaces are in correspondence with algebraic curves. For this reason, both terms will be used interchangeable.

Definition 3.1. Let U and S be complex manifolds. A *(smooth) covering* of S is a surjective continuous function $f\colon U \to S$ such that for each $s \in S$ there exists an open neighborhood V_s of s in S for which $f^{-1}(V_s)$ consists of a disjoint union of open sets U_i with $f\mid_{U_i}\colon U_i \to V_s$ a homeomorphism.

Definition 3.2. A map $F\colon X \to Y$ between Riemann surfaces is said to be *holormorphic* if it is holomorphic in local coordinates, i.e if for each $x \in X$ there exist charts $\varphi_1\colon U_1 \to V_1$ on X, $\varphi_2\colon U_2 \to V_2$ on Y with $x \in U_1$, $F(x) \in U_2$, such that $\varphi_2 \circ F \circ \varphi_1$ is holomorphic at $\varphi_1(x)$.

Remark 3.3. Using local coordinates we can define *meromorphic functions* $f\colon X \to \mathbb{C}$ and their respective *poles* and *orders*. It can be proved that these concepts are well defined. The set of meromorphic functions on X is denoted by $\mathcal{M}(X)$.

Proposition 3.4 (Local normal form). *Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces and p a point of X. Then there is a unique integer $m \geq 1$ such that F can be written in local coordinates near p as z^m. This number is denoted by* $\operatorname{mult}_p(F)$.

Proof. See [Mir95, Ch. 2, Section 4]. □

Remark 3.5. In the above case, the proposition implies that the set of points satisfying $\operatorname{mult}_p(F) \geq 2$ is discrete and therefore finite.

Definition 3.6. A non-constant holomorphic map $F\colon X \to Y$ is called a *branched covering*. A *ramification point* $P \in X$, is a point such that $\operatorname{mult}_p(F) \geq 2$. The image of a ramification point is a *branch value*.

Remark 3.7. For a branched covering $F\colon X \to Y$, let B be the set of branch values of F. Then the map $X - F^{-1}(B) \to Y - B$ is a covering map. That is the reason of why these maps are called branched coverings.

Proposition 3.8. *Let $F\colon X \to Y$ be a non-constant holomorphic function. Then the number,*

$$\sum_{p \in F^{-1}(y)} \operatorname{mult}_p(F)$$

is independent of $y \in Y$. This value is defined as the degree of F *and is denoted by* $\deg(F)$.

Proof. See [Mir95, Ch. 4, Section 4]. □

Definition 3.9. Let S and M be complex manifolds. A *covering transformation* of a (smooth) *covering* $f : S \to M$ is a homeomorphism of S onto itself which interchanges points having the same projection on M i.e the set of topological automorphisms of S, such that the following diagram commutes,

$$\begin{array}{ccc} S & \xrightarrow{h} & S \\ & \searrow^{f} & \downarrow{\scriptstyle f} \\ & & M \end{array}$$

These automorphisms form a group called *the Galois group of the covering* and is denoted by $Gal(f : S \to M)$.

Remark 3.10. In the situation of $f : S \to M$ being a covering, if M is a Riemann Surface there is a natural analytic structure on S that makes f holomorphic and the covering transformations holomorphic mappings from S onto itself.

Definition 3.11. The Galois group of a covering is called *fiber transitive* if there is a transformation in the group which carries any point P_1 over P into any other prescribed point P_2 over P. In this situation the covering $f : S \to M$ is a *Galois covering*.

3.2 Group actions on Riemann surfaces

Throughout this work G will denote a finite group and $G \curvearrowright X$ an action of G on a Riemann surface X. Actions are considered to be holomorphic and effective, in other words, they can be represented by a monomorphism from G to the analytical automorphism group of X, $\mathrm{Aut}(X)$.

Definition 3.12. Let G be a group acting on a Riemann surface X. For a point p of X and H a subgroup of G, G_p denotes the *stabilizer* of p, $\mathbb{O}_p^H$ the *orbit* of p with respect to H, X/H the *Quotient surface*, and π_H its quotient map.

Proposition 3.13. *Let G be a group acting on a Riemann surface X, and $p \in X$. Then G_p is a cyclic subgroup.*

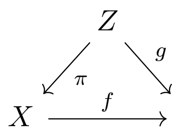
Proof. See [Mir95, Ch. 3, Section 3]. □

Definition 3.14. Let G be a group acting on a Riemann surface X and H a subgroup of G. Then the induced action of H produces two branched coverings,

$$\pi_H : X \to X/H$$
$$\pi^H : X/H \to X/G$$

It can be proved that π_H is always Galois and π^H is Galois if and only if H is a normal subgroup of G. The induced action of a subgroup and its corresponding projection is called the *intermediate quotient*.

Definition 3.15. Let $f\colon X \to Y$ be an holomorphic function between Riemann surfaces. The *Galois closure* of f is a Galois covering $g\colon Z \to Y$ of the lowest possible degree with the property that there is a third morphism $\pi\colon Z \to X$ making the diagram commute



The next result is remarkable because it allows us to make explicit computations of the ramification points of a holomorphic map. It will be used primarily for the quotient map.

Theorem 3.16 (Riemann-Hurwitz's Formula)**.** *Let $F\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(F)\,(2g(Y) - 2) + \sum_{p \in X} (\mathrm{mult}_P(F) - 1).$$

Proof. See [Mir95, Ch. 2, Section 4]. □

Lemma 3.17. *Let G be a group acting on a Riemann surface X. Then the quotient map $\pi\colon X \to X/G$ is holomorphic with degree $|G|$ and $\mathrm{mult}_p(\pi) = |G_p|$ for any $p \in X$.*

Proof. See [Mir95, Ch. 3, Section 3]. □

Definition 3.18. For a branch value $y \in X/G$ we say that y is *marked* with n if the stabilizer of any point in its fiber has order n. Note that the stabilizers corresponding to points of the same orbit are in the same conjugacy class. Hence this number is well defined.

Applying the Hurwitz's Formula and the above lemma we obtain the following result.

Corollary 3.19. *Let G be a group acting on a Riemann surface X and π its quotient map. If $y_i \in Y$ are the branch values of π, marked with r_i, then*

$$2g(X) - 2 = |G| \left(2g(X/G) - 2 + \sum_i 1 - \frac{1}{r_i} \right).$$

Definition 3.20. Let G_i be a nontrivial cyclic subgroup of a group G acting on a Riemann surface X. A branch value $p \in S/G$ is *of type* G_i if the stabilizer of the points in its fiber are the elements of the complete conjugacy class of the group G_i.

3.3 Differential forms

In this section we consider the definitions given in [Mir95].

Definition 3.21. For an open set $V \subset \mathbb{C}$, an expression ω of the form $\omega = f(z)\,dz$ is called a *holomorphic* 1*-form* if the function f is holomorphic on V.

Definition 3.22. Let X be a Riemann surface. A *holomorphic* 1*-form* on X is a collection of holomorphic 1-forms $\{\omega_\varphi = f_\varphi(z_\varphi)\, dz_\varphi\}$, one for each chart $\varphi\colon U \to V$, where ω_φ is defined on the open set V, such that if two charts $\varphi_i\colon U_i \to V_i$ have overlapping domains, then

$$f_{\varphi_2}(w) = f_{\varphi_1}(T(w))\, T'(w)\,.$$

where $z = T(w) = \varphi_1 \circ \varphi_2^{-1}(w)$ is the corresponding change of coordinates from U_2 to U_1. In this case we say that ω_{φ_1} transforms to ω_{φ_2} under T. The space of holomorphic 1-forms on X is denoted by $\Omega^1(X)$

Remark 3.23. In both definitions, if we change the word holomorphic for meromorphic, we obtain the definition of a *meromorphic* 1*-form* over an open set $V \subset \mathbb{C}$ and over a Riemann surface X. Informally, a meromorphic 1-form ω is an expression that can be written in local coordinates as $f(z)dz$, where f is a meromorphic function. The set of meromorphic 1-forms is denoted by $\mathcal{M}^{(1)}$.

Definition 3.24. We say that ω has a pole at $p \in X$ if $f(z)$ has a pole at p, provided that ω can be written as $f(z)dz$ in a neighborhood of the point p. We define the order of ω at p by,

$$\mathrm{ord}_p(\omega) = \mathrm{ord}_p(f)$$

It is easy to prove that both definitions do not depend on the chart.

Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces and ω be a holomorphic 1-form on Y. We define a holomorphic 1-form $F^*\omega$ on X as follows: fix a chart $\varphi\colon U \to V$ on X such that $F(U)$ is contained in the domain of a chart U' of Y. Now we have a local coordinate z on U' and w on U such that $z = h(w)$ for some holomorphic function h. If ω has the form $f(z)\, dz$, we define the *pullback* of ω via F by

$$F^*\omega = f(h(w))\, dw.$$

Note that $F^*\omega$ is a holomorphic 1-form on X.

3.4 Divisors on a Riemann Surface

Definition 3.25. For a Riemann surface X, we define the group of *divisors* of X denoted by $\mathrm{Div}(X)$ as the free abelian group over the set X. Thus, its elements are formal sums with finite support of points on X. The element of $\mathrm{Div}(X)$ are called *divisors*. Given two divisors $D_1 = \sum n_x x$ and $D_2 = \sum m_x x$, we say that $D_1 \geq D_2$ if $n_x \geq m_x$ for all $x \in X$. Given a divisor $D = \sum_{x\in X} n_x x$, the *degree* of D is the number $\deg(D) = \sum_{x\in X} n_x$. Note that $\deg$ is a homomorphism from $\mathrm{Div}(X)$ to the group $\mathbb{Z}$. The kernel of $\deg$ is denoted by $\mathrm{Div}_0(X)$.

Definition 3.26. For a meromorphic function f on X, we define the *principal divisor* of f by

$$(f) := \sum_{x\in X} \mathrm{ord}_x(f)$$

The set of principal divisors of X is a subgroup of $\mathrm{Div}\,(X)$ denoted by $\mathrm{PDiv}\,(X)$. Given two divisors D_1, D_2, we say that they are *linearly equivalent* if $D_1 - D_2 \in \mathrm{PDiv}\,(X)$.

Remark 3.27. Recall that the poles of a meromorphic function form a finite set for compact Riemann surfaces. Hence, a principal divisor is in fact a divisor on X. Furthermore $(f) \in \mathrm{Div}_0\,(X)$ since the sum of the degrees of a meromorphic function is 0 (see [Mir95, Ch. 2, Section 4]). In particular, the degree is stable under linear equivalence.

Definition 3.28. For a meromorphic 1-form ω on X, we define the *canonical divisor* of ω by,

$$(\omega) := \sum_{x \in X} \mathrm{ord}_x\,(\omega) x$$

The set of canonical divisors is denoted by $\mathrm{KDiv}\,(X)$.

Remark 3.29. Again the set of poles of a meromorphic 1-form is a discrete set, thus the canonical divisor is in fact a divisor. The following lemma implies that all the canonical divisors over X have the same degree.

Lemma 3.30. *Let ω_1, ω_2 be meromorphic 1-forms on a Riemann surface X, then there exist a meromorphic function f satisfying $\omega_1 = f\omega_2$. Furthermore, it can be proved that the degree of all the divisors equals $2g - 2$. In particular, all the canonical divisors on X are linearly equivalent.*

Proof. See [Tal09]. □

Definition 3.31. For a divisor D, we define the sets,

$$L\,(D) = \{f \in \mathcal{M}\,(X) : (f) + D \geq 0\} \cup \{0\}$$
$$I\,(D) = \{\omega \in \mathcal{M}^{(1)} : (\omega) \geq D\} \cup \{0\}$$

where $\mathcal{M}\,(X)$ and $\mathcal{M}^{(1)}\,(X)$ denote the sets of meromorphic functions and meromorphic 1-forms respectively. Note that $L\,(D)$ and $I\,(D)$ are vector spaces over $\mathbb{C}$. We define the numbers $l\,(D)$ and $i\,(D)$ to be the dimensions of these vector spaces respectively.

Remark 3.32. If the degree of D is negative, then the space $L\,(D)$ is empty. In fact, if $f \in L\,(D)$ then

$$0 \leq \deg\big((f) + D\big) = \deg\,(f) + \deg D = \deg D.$$

Note that $L\,(0) = \{f \in \mathcal{M}\,(X) : (f) \geq 0\}$ is the space of holomorphic functions on X. Since X is compact, this is the set of constant functions and it is therefore isomorphic to $\mathbb{C}$. It follows that $l\,(0) = 1$.

Lemma 3.33. *Let D be a divisor and K a canonical divisor. Then the spaces $I(K - D)$ and $L(D)$ are isomorphic as complex vector spaces. In particular, the space $I(D)$ is isomorphic to $L(K - D)$ and $L\,(K)$ is isomorphic to $\Omega^1\,(X)$.*

Proof. The divisor K can be written as $K = (\omega)$ for some meromorphic 1-form ω. Note that the equality,

$$(f) + D = (f\omega) - (\omega) + D = (f\omega) - (K - D)$$

implies that a meromorphic function f lies in $L(D)$ if and only if $f\omega \in I(K-D)$. Thus, we have a morphism

$$L(D) \to I(K-D)$$
$$f \mapsto f\omega$$

Lemma 3.30 implies the existence of an inverse map. Taking $K-D$ instead of D we get an isomorphism between $I(D)$ with $L(K-D)$. Finally $K=D$ gives the isomorphism of $L(K)$ with $\Omega^1(X)$. □

Lemma 3.34. *Let D_1, D_2 be linearly equivalent divisors. Then the spaces $I(D_1)$ and $I(D_2)$ are isomorphic. The same holds for $L(D_1)$ and $L(D_2)$.*

Proof. Let f be a meromorphic function, such that $D_1 = D_2 + (f)$. Given an element $\omega \in I(D_1)$, note that the element $f\omega \in I(D_2)$ since

$$(f\omega) + D_2 = (f) + (\omega) + D_2 = (\omega) + D_1 \geq 0.$$

This defines a morphism $\phi\colon L(D_1) \to L(D_2)$ between the $\mathbb{C}$ vector spaces. Since f is a meromorphic function, its reciprocal f^{-1} is meromorphic and defines the inverse of ϕ. The proof for the isomorphism between the spaces $L(D_i)$ is analog. □

Theorem 3.35 (Riemann Roch Theorem)**.** *If D is a divisor on a Riemann surface of genus g. Then the following holds*

$$l(D) - i(D) = \deg D + 1 - g$$

Proof. See [Mir95] and [Tal09]. □

Corollary 3.36. *Let X be a Riemann surface of genus g. Then $\Omega^1(X)$ is a $\mathbb{C}$-vector space of dimension g.*

Proof. It is clear that $\Omega^1(X)$ is a $\mathbb{C}$-vector space so it suffices to prove that it has dimension g. Considering the Riemann Roch theorem for the divisor $D=0$, we get,

$$l(0) - i(0) = \deg(0) + 1 - g$$

From Remark 3.32 we know that $l(0) = 1$ and obviously $\deg(0) = 0$. It follows that $i(0) = g$. The result follows from Lemma 3.33, since this implies that $I(0) \cong \Omega^1(X)$. □

3.5 Integrals and the Jacobian variety.

For this section we consider mostly [Mir95].

Now we define the concept of integral of a 1-form over a path. A *path* on a Riemann surface X is a continuous and piecewise $\mathcal{C}^\infty$ function $\gamma : [a,b] \to X$. The path is *closed* if $\gamma(a) = \gamma(b)$.

Definition 3.37. Let X be a Riemann surface, γ a path on X and ω a holomorphic 1-form on X. Choose a partition $[a_i, a_{i+1}]$ of $[a,b]$ such that each image of γ restricted to this interval is a $\mathcal{C}^\infty$ curve γ_i contained in the domain U_i of a chart φ_i. Then for the open set U_i, ω has the form $\omega = f_i(z)\,dz$

with f_i holomorphic. We define the integral of ω along γ as the complex number

$$\int_\gamma \omega = \sum_i \int_{a_{i-1}}^{a_i} f_i\big(\phi_i \circ \gamma_i(t)\big)dt.$$

It can be proved that the value of the integral is independent of the parametrization and the partition.

Proposition 3.38. *Let γ_1 and γ_2 be homotopic paths on X and ω a 1-form. Then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

The above proposition implies that integrals of holomorphic 1-forms can be defined in homology classes of $\mathrm{H}_1(X,\mathbb{Z})$. This induces the following morphism:

$$\begin{aligned} \mathrm{H}_1(X,\mathbb{Z}) :\ &\to \Omega^1(X)^* \\ [a] &\mapsto \int_{[a]} \end{aligned}$$

Moreover, it can be proved that this morphism is injective. See [Mir95] for references.

Definition 3.39. Let Λ denote the image of the above monomorphism. Then $\mathrm{H}_1(X,\mathbb{Z})$ can be regarded as a subgroup of $\Omega^1(X)^*$. An element $T \in \Lambda$ is called a *period* of X.

We have identifications $\Lambda \cong \mathrm{H}_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ and $\Omega^1(X)^*$ is a vector space of dimension g. Therefore Λ is a lattice of this space.

Definition 3.40. Let X be a Riemann surface of genus g. The *Jacobian* of X, denoted by $\mathrm{Jac}(X)$, is the g-dimensional complex torus

$$\mathrm{Jac}(X) = \Omega^1(X)^* \big/ \Lambda \tag{3.1}$$

Theorem 3.41. *Let X be a Riemann surface of genus g. Then the Jacobian variety $\mathrm{Jac}(X)$ can be regarded as a principally polarized abelian variety.*

Sketch of proof. We know that $\mathrm{Jac}(X)$ is a complex torus of dimension g. The principal polarization is given by the geometric intersection number in $H_1(X,\mathbb{Z})$ extended to $\Omega^1(X)^*$ by the injection as in Remark 2.17. □

3.5.1 Actions on the Jacobian Variety

For this section, we consider [Mir95] and [Wol02].

Remark 3.42. Given a linear transformation $T\colon V \to W$ between vector spaces V, then there exist a dual map $T^*\colon W^* \to V^*$. Consider fixed bases for V and W, and the corresponding dual bases for V^* and W^*. The matrix representation of T and T^* are related by $[T^*] = [T]^t$. If a group G acts on a vector space V, it induces an action of G on V^* given by:

$$\begin{aligned} G \times V^* &\to V^* \\ (g,f) &\mapsto f(g^{-1}) \end{aligned}$$

For each $g \in G$ the associated matrices are related by $[g^*] = [g]^{-t}$. This is the *dual action*.

Given an action of a finite group G on a Riemann surface X, let $\varphi\colon G \to \mathrm{End}\,(X)$ be the associated morphism. Note that the pullback ϕ^* induces an action on $\Omega^1\,(X)$ in the following way,

$$\begin{aligned} \psi\colon G &\to \mathrm{End}\left(\Omega^1\,(X)\right) \\ g &\mapsto \psi_g \end{aligned}$$

where $\psi_g\,(\omega) = \varphi^*_{g^{-1}}\omega$. The dual action induces an action of G on the space $\Omega^1\,(X)^*$. This defines an action of G on the space $\Omega^1\,(X)^*$.

Lemma 3.43. *Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces, $\gamma\colon [a,b] \to X$ a path on X and ω a 1-form on Y, then*

$$\int_{F_*\gamma} \omega = \int_\gamma F^*\omega \tag{3.2}$$

where $F_*\gamma = \gamma \circ F$.

The above lemma implies that the action of G on $\Omega^1\,(X)^*$ takes period elements to period elements. Hence there is an induced action on $\mathrm{Jac}\,(X)$. In conclusion, any action of a group G on a Riemann surface X induces an action on $\mathrm{Jac}\,(X)$. This action is called the *natural action on the Jacobian variety*.

The following theorem relates the automorphism groups associated to a Riemann surface and its corresponding Jacobian variety.

Theorem 3.44. *If a curve X has a non trivial group of automorphisms, then*

$$\mathrm{Aut}\,(X) = \begin{cases} \mathrm{Aut}\,(JX)\,, & \textit{if } X \textit{ is hyperelliptic,} \\ \mathrm{Aut}\,(JX)\,/\{\pm 1\}, & \textit{if not.} \end{cases}$$

Proof. See [Wol02] and the references given in [LR04]. ☐

Chapter 4

Representation theory of finite groups

4.1 Semisimple rings properties

Definition 4.1. A ring R with unit is said to satisfy the *minimum condition* if every nonempty collection of left ideals has a minimal element. Furthermore if R does not have non-trivial two sided ideals then R is called simple.

Remark 4.2. If K is a field and G is a finite group, then the group algebra $K[G]$ can be regarded as a K-vector space. Note that the left ideal of $K[G]$ are vector subspaces. In particular, this algebra satisfies the minimum condition.

Definition 4.3. Let R be a nonzero ring and M a nonzero R-module. The module M is said to be *completely reducible* if it is a direct sum of irreducible R-submodules.

Theorem 4.1 (Weddeburn's Theorem)**.** *Let R be a nonzero ring with unit satisfying the minimum condition. Then the following are equivalent:*

1. *Every R-module is completely reducible.*
2. *The ring R considered as a left R-module is a direct sum*

$$R = L_1 \oplus \ldots \oplus L_n$$

   where each L_i is a simple R-module (i.e a minimal left ideal) with $L_i = Re_i$, for some $e_i \in R$ with e_i idempotent orthogonal such that $1 = e_1 + \ldots + e_n$.
3. *As rings, R is isomorphic to a direct product of matrix rings over division rings, i.e $R \cong R_1 \times \ldots \times R_r$ where R_j is a two-sided ideal of R and is isomorphic to the ring of matrices of size $n_j \times n_j$ with entries in a division ring D_j. The integer r, n_j and the division rings D_j (up to isomorphism) are uniquely determined by R.*

Proof. See [DF04, Section 18.2]. □

Definition 4.4. A ring R satisfying any of the equivalent properties in the above theorem is called *semisimple*.

Theorem 4.5 (Maschke's Theorem)**.** *Let G be a finite group and K a field whose characteristic does not does not divide the order of G. If M is a $K[G]$-module and U is a submodule of M, then there exists a submodule V of M such that $M = U \oplus V$. In other words, every submodule is a direct sum of M.*

Proof. See [DF04, Section 18.1]. □

Corollary 4.6. *If K is a field and G is a finite group whose order is not divisible by $\operatorname{char} K$, then the group algebra $K[G]$ is a semisimple ring.*

Proof. This follows directly from Maschke's Theorem and the first equivalence in Wedderburn's Theorem. □

Proposition 4.7. *Let $\triangle$ be a division ring, $n \in \mathbb{N}$ and R the ring of all $n \times n$ matrices over $\triangle$ with identity I. Then*

1. *R is a simple ring.*
2. *Let e_i be the matrix with a 1 in the position i, i and zero elsewhere. Then e_i are orthogonal primitive idempotents whose sums is the identity.*
3. *Let $L_i = Re_i$. Then L_i is a simple left R-module. Every simple left R-module is isomorphic to L_1. In particular $R = L_1 \oplus \dots L_n \cong L_1^n$.*

Proof. See [DF04, Section 18.2]. □

Lemma 4.8 (Schur's Lemma)**.** *Let R be an arbitrary nonzero ring, M, N simple R-modules and $\varphi\colon M \to N$ a nonzero R-module homomorphism. Then φ is an isomorphism and $\operatorname{Hom}_R(M, M)$ is a division ring.*

Proof. Recall that the image and kernel of a morphism between R-modules are submodules N and M respectively. □

Proposition 4.9. *Let R be a semisimple ring, then every irreducible R-module is isomorphic to some minimal left ideal of R.*

Proof. Weddeburn's Theorem implies that there exist left R-modules such that $R = Re_1 \oplus \dots Re_n$. Let M be a nonzero R-module. Note that M can be decomposed in the following way

$$M = \sum_{m \in M} \sum_{i=1}^{n} Re_i m.$$

Therefore, some factor $Re_k m$ is nonzero. Define the map $\varphi\colon Re_k \to M$ by $re_k \mapsto re_k m$. Note that ϕ is a nonzero R-homomorphism between simple R-modules, so the above lemma implies that φ is an isomorphism. □

Remark 4.10. Note that this proposition implies that the decomposition of R as a sum of minimal left ideals in the second equivalence of Wedderburn's Theorem is unique up to permutations.

Lemma 4.11. *Let R be a semisimple ring and let L_1, L_2 be minimal left ideals of R. Then L_1 and L_2 are isomorphic if and only if there exists some $r \in L_1$ such that $L_1 = L_2 r$. Otherwise, $L_2 r = 0$ for every $r \in L_1$.*

Sketch of proof. The proof consists in defining R-homomorphisms of the form $x \mapsto rx$ and the fact that both ideals are simple R-modules to conclude that the kernel and the image are both trivial. □

Definition 4.12. Let R be a semisimple ring and L a minimal left ideal of R. We define C_L as the sum of all the minimal left ideals of R isomorphic to L. The ideal C_L is called a *simple component* of R.

Theorem 4.13. *Let R be a semisimple ring, L be a minimal left ideal and C_L the simple component associated to L. Then C_L is a simple ring and $R = \bigoplus C_L$ where L runs over a complete set of non isomorphic minimal left ideals of R. Moreover for each two sided ideal I of R, the ideal can be decomposed as a direct sum of simple components of R*

Proof. The proof is straightforward. □

Remark 4.14. It follows from Wedderburn's Theorem and the above result that the simple components of a ring are isomorphic to matrix rings over a division ring. In particular, let R_1, R_2 be different components in the decomposition given by the third equivalence in Weddeburn's Theorem and let L_1, L_2 be minimal left ideals contained in them respectively. Then L_1 and L_2 are not isomorphic.

4.2 Basic concepts of representation theory

In this section G will denote a finite group.

Definition 4.15. Let G be a group and V a vector space over K. A K-*representation* of G is a homomorphism $T\colon G \to \mathrm{GL}\,(V)$. We usually write T_g for $T(g)$. Two representations T, T' are said to be equivalent if there exists a K-isomorphism $S\colon V \to V'$ such that

$$T'_g S = S T_g.$$

The *degree* of T is the dimension of the vector space V over K and is denoted by $\deg T$. For this work, we will consider $K = \mathbb{Q}, \mathbb{C}$.

Remark 4.16. A representation $\varphi\colon G \to \mathrm{GL}\,(V)$ is uniquely determined by giving V a structure of $K[G]$-module. The equivalence of representations is translated into a isomorphism of $K[G]$-modules.

Definition 4.17. Let $T\colon G \to \mathrm{GL}\,(V)$ be a representation. A subspace $W \leq V$ is *G-invariant* if, for all $g \in G$, one has $\phi_g\,(W) \subset W$. A nonzero representation is said to be an *irreducible representation* or *irrep* if the only G-invariant subspaces are $\{0\}$ and V.

We introduce the concept of a character associated to a representation $\varphi\colon G \to \mathrm{GL}\,(V)$.

Definition 4.18. Let $\varphi\colon G \to \mathrm{GL}\,(V)$ be a representation. The *character* associated to the representation φ is the function $\chi_\varphi\colon G \to K$ defined by $\chi_\varphi = \mathrm{Tr}\,(\varphi_g)$. The characters associated to irreps are called *irreducible characters*. We use the symbol $\mathrm{Irr}_K\,(G)$, to denote the complete set of irreducible representations of G over K.

Remark 4.19. Observe that any character is constant on the conjugacy classes of G. A function with this property is called a *class function*.

Theorem 4.20. *For a finite group G, the following holds:*

1. $|\,\mathrm{Irr}_{\mathbb{C}}\,(G)\,| = Cl\,(G)$*, where* $Cl\,(G)$ *denotes the number of conjugacy classes of* G*.*

2. $|G| = \sum_{V_i \in Irr_{\mathbb{C}}(G)} (\dim V_i)^2$.

Proof. See [Ste11, Ch. 4] □

Theorem 4.21. *Let G be a finite group. Then $|\operatorname{Irr}_{\mathbb{Q}}(G)|$ is equal to the number of conjugacy classes of cyclic subgroups of G.*

Proof. See [Ser12, Ch. 13]. □

Given a representation of a subgroup $H \leq G$ we define an induced representation on the group G.

Definition 4.22. Let H be a subgroup of G, $\varphi\colon H \to \mathrm{GL}(V)$ be a representation of H and $g_1, \ldots, g_r$ be a full set of representatives in G of the left cosets of H of degree d. We define the map $\dot{\varphi_x}\colon G \to \mathrm{GL}(V)$ by

$$\dot{\varphi_x} = \begin{cases} \varphi_x, & \text{if } x \in H \\ 0, & \text{if } x \notin H. \end{cases}$$

where 0 is the $d \times d$ matrix with 0 on all its entries. We define the *induced representation* by

$$\operatorname{Ind}_H^G \varphi\colon G \to \mathrm{GL}_{md}(\mathbb{C}).$$
$$\left[\operatorname{Ind}_H^G \varphi\right]_{ij} = \dot{\varphi}_{g_i^{-1} g g_j}.$$

Proposition 4.23. *Let G be a group, $H \leq G$ a subgroup and $\varphi\colon H \to \mathrm{GL}(V)$ a representation of H. Then the degree of the induced representation is given by:*

$$\deg \operatorname{Ind}_H^G \varphi = [G : H] \deg \varphi.$$

Proof. To prove that the induced representation is a representation is straightforward. The degree of this is given by construction. □

Definition 4.24. Let $\varphi\colon G \to \mathrm{GL}_n(K)$ be a representation of G and $F \subset K$ be a subfield of K. We say that φ is *realizable* in F if there exists a representation $\psi\colon G \to \mathrm{GL}_n(F)$ such that φ and ψ are K-equivalent representations. In terms of modules: let M be the $K[G]$-module associated to φ. Then φ is realizable in F if there exists a $F[G]$-module N such that M and $N \otimes_F K$ are isomorphic as left $K[G]$-modules.

Let χ be the character of an irreducible K-representation $\varphi\colon G \to \mathrm{GL}(V)$. For any subfield $F \subset K$ we define the field $F_V = F(\chi) = F(\chi(g) :\ g \in G)$. This field is contained in a cyclotomic extension, since $\chi(g)$ is the sum of roots of unity. In particular, this extension is Galois of finite degree. The Galois group of this representation is defined as the Galois group of the field extension F_V/F, and it is denoted $\mathrm{Gal}(F_V/F)$. Finally, for a subgroup $H \leq G$, $\mathrm{Fix}_H(V)$ denotes the set of fixed points of V under the action of H.

Definition 4.25. Let φ be an irreducible K-representation of a group G with character χ and F be a subfield $F \subset K$. We define the *Schur index* of φ with respect to F as

$$\ell_F(\varphi) = \min\{|L : F(\chi)| \mid \varphi \text{ is realizable over } L\}.$$

Remark 4.26. If the Schur index is taken with respect to $\mathbb{Q}$, then we omit the subscript of the field. In this case the schur index is denoted by ℓ_V, where V is the associated $K[G]$-algebra of the representation.

Definition 4.27. Let $\varphi\colon G \to \mathrm{GL}(V)$ be a $\mathbb{Q}$-irreducible representation of G, we say that φ is *absolutely irreducible* if $\varphi \otimes_{\mathbb{Q}} \mathbb{C}$ is irreducible over $\mathbb{C}$. If all representations of G are absolutely irreducible, then we say that G is absolutely irreducible. An example of absolutely irreducible groups is given by the family of symmetric groups S_n (See [CRR14]) more generally any Weyl group is absolutely irreducible (See Theorem 5.38). Note that the product of absolutely irreducible groups is absolutely irreducible.

Let V be a complex irreducible representation of G, χ its character, ℓ_V its Schur index and K_V the extension field of V. Note that σV is an irreducible representation for each $\sigma \in \mathrm{Gal}(K_V/\mathbb{Q})$. Note that these representations are those associated to the characters $\sigma\chi$, thus they are not isomorphic to each other. We call the set $\{\sigma V : \sigma \in \mathrm{Gal}(K_V/\mathbb{Q})\}$ the *Galois class* of V.

Theorem 4.28. *Let $\{V_1, \ldots, V_r\}$ be a complete set of representatives of Galois classes from the set $\mathrm{Irr}_{\mathbb{C}}(G)$ and let $K_j = K_{V_j}$, $\ell_j = \ell_{V_j}$ and $G_j = \mathrm{Gal}(K_j/\mathbb{Q})$. Then for $\mathcal{W} \in \mathrm{Irr}_{\mathbb{Q}}(G)$ there exists precisely one V_j satisfying*

$$\mathcal{W} \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{i=1}^{\ell_j} \bigoplus_{\sigma \in G_j} \sigma V_j := \bigoplus_{\sigma \in G_j} (\sigma V_j)^{\ell_j}$$

Conversely, the right side of the equality is the complefixication of $\mathcal{W}$.

Proof. See [CR66, Ch. 10, Section 70]. □

Remark 4.29. If $\varphi\colon \to \mathrm{GL}(V)$ is a K-representation which is realizable over a subfield $F \subset K$, then $l_F(\varphi) = 1$. We will use this for absolutely irreducible representations.

Definition 4.30. Let G be a finite group and K a field of characteristic 0. Let $\{U_i\}_{i \in I}$ be an indexation of the set $\mathrm{Irr}_K(G)$. We define the value $\mathbf{N}(G, K) = \sum \dim(U_i)$. Note that if G is absolutely irreducible then $\mathrm{Irr}_{\mathbb{Q}}(G) = \mathrm{Irr}_{\mathbb{C}}(G)$ and $\mathbf{N}(G, \mathbb{Q}) = \mathbf{N}(G, \mathbb{C})$. In this case we simplify the notation by $\mathrm{Irr}(G)$ and $\mathbf{N}(G)$ respectively.

4.3 Representations of a semi direct product with an abelian factor

Let A,H be finite groups with A abelian and let $G = A \rtimes H$ be a semi direct product of them. There is a method to construct all the irreducible representations over $\mathbb{C}$ of the group G, known as *Little Groups Method* which will be explained below

Let Ω_A be a complete set of inequivalent irreducible representations of A over $\mathbb{C}$. Since A is abelian, the elements of Ω_A are homomorphisms $\rho_i\colon A \to \mathbb{C}^*$. There is a natural action of G on the set Ω_A given by:

$$G \times \Omega_A \xrightarrow{\phi} \Omega_A$$
$$(g, \rho) \mapsto \rho^g,$$

where ρ^g is defined by $\rho^g(a) = \rho\left(g^{-1}ag\right)$ and is known as the *conjugate representation*. Denote by $\Omega_A^H = \{\rho_j\}_{j\in J}$ a set of representatives of the orbits by the subgroup H in Ω_A. For each $j \in J$, let $H_j = \mathrm{Stab}_{\rho_j}^H$ and $G_j = \mathrm{Stab}_{\rho_j}^G$. Observe that G_j is given by $G_j = A \rtimes H_j$. We define an extension $\overline{\rho}_j$ of ρ_j on the subgroup G_j by

$$\overline{\rho}_j(ah_j) = \rho_j(a).$$

For each $a \in A$ and each $h_j \in H_j$, it is easy to show that this is an irreducible representation of G_j of degree 1. Let $\{\sigma_{ij}\}_{i\in I}$ be a complete set of the complex inequivalent irreps of H_j. In a similar way as before, extend these representations to obtain representations $\{\overline{\sigma}_{ij}\}_{i\in I}$ of G_j. Taking tensor product defines a representation $\overline{\rho}_j \otimes \overline{\sigma}_{ij}$ of G_j. Let θ_{ij} be the induced representation on G of the latter representation; i.e θ_{ij} is defined as

$$\theta_{ij} = \mathrm{Ind}_{G_j}^G\left(\overline{\rho}_j \otimes \overline{\sigma}_{ij}\right).$$

Proposition 4.31. *Let $G = A \rtimes H$ be a semi direct product with A an abelian group and let $\{\theta_{ij}\}$ be the set of representations of G constructed above. Then the following holds:*

- *Each θ_{ij} is irreducible.*
- *The irreps θ_{ij} and $\theta_{i'j'}$ are equivalent if and only if $i = i'$ and $j = j'$.*
- *Every irreducible representation of G is isomorphic to one of the θ_{ij}.*

Proof. See [Ser12, Ch. 8]. □

4.4 The Isotypical decomposition

Let A be an abelian variety, G a finite group acting on A and $\mathbb{Q}[G]$ the group algebra of G over the field $\mathbb{Q}$. This action induces a $\mathbb{Q}$-algebra homomorphism $\psi\colon \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$.

Any element $t \in \mathbb{Q}[G]$ defines an abelian subvariety $t(A) := \mathrm{Im}(\psi(mt))$ where $m \in \mathbb{Z}$ is an integer such that $nt \in \mathbb{Z}[G]$. From this, it follows that $\psi(mt) \in \mathrm{End}(A)$. Note that this definition does not depend on the chosen integer m since any abelian variety is a divisible group. We choose special elements to induce a decomposition of A.

For this purpose, use Wedderburn's Theorem to decompose the group algebra $\mathbb{Q}[G]$ as

$$\mathbb{Q}[G] = Q_1 \times \ldots \times Q_l.$$

with Q_i a simple $\mathbb{Q}$-algebra. Let e_i be the unit element of the algebra Q_i. Considering e_i as elements of $\mathbb{Q}[G]$, they form a set of orthogonal idempotents contained in the center of $\mathbb{Q}[G]$ that satisfy the following equation

$$1 = e_1 + \ldots + e_l.$$

Proposition 4.32. *If A_i denotes the abelian variety $A_i = e_i(A)$ for $i = 1, \ldots, l$, then*

1. *A_i is a G-stable abelian subvariety of A with $\mathrm{Hom}_G(A_i, A_j) = 0$ for $i \neq j$.*

2. *The addition map induces an isogeny*

$$\varphi \colon A_1 \times \ldots \times A_l \to A.$$

This decomposition is unique up to permutations and is called the isotypical decomposition of A. *The* A_i *are called the* isotypical components.

Proof. See [LR04]. □

Now we decompose the isotypical components into a product of pairwise isogenous abelian subvarieties.

Theorem 4.2 (Isogeny decomposition). *Let* $A_i \neq 0$ *be one of the nonzero isotypical components of an abelian variety* A *associated to the irreducible* $\mathbb{Q}$-*representation* $\mathcal{W}_i$, *let* Q_i *be the simple* $\mathbb{Q}$-*algebra associated to it and let* e_i *denote the unit element of* Q_i. *Then there are idempotent elements* $q_{i1}, \ldots, q_{is_i} \in Q_i$ *such that* $e_i = \sum_j q_{ij}$. *Define abelian subvarieties* B_{ij} *by* $B_{ij} := q_{ij}(A)$. *Then the following holds:*

1. *Let* $n_i := \dim_D(\mathcal{W}_i)$ *with* $D := \operatorname{End}_G(\mathcal{W}_i)$ *and let* V_i *a complex representation associated to* $\mathcal{W}_i$. *Then* $n_i = \dim(V_i)/m_{\mathbb{Q}}(V_i) = s_i$.

2. *There is an isogeny* $\mu_i \colon B_{i1} \times \ldots B_{in_i} \to A_i$.

3. *The abelian subvarieties* B_{ij} *are pairwise isogenous. Therefore any* A_i *is isogenous to a power of an abelian variety* B_i

$$A_i \sim B_i^{n_i}$$

In conclusion, there exist abelian subvarieties $B_1, \ldots, B_l$ *of* A *and an isogeny*

$$A \sim B_1^{n_1} \times \ldots \times B_l^{n_l}.$$

Proof. See [LR04] and [CR06]. □

Remark 4.33. The factors B_i of the isogeny decomposition are called *primitive factors*. Moreover, note that if $\mathcal{W}_i$ is an absolutely irreducible representation, then $n_i = \dim(V_i) = \dim(\mathcal{W}_i)$.

Corollary 4.34. *Suppose that* G *is an absolutely irreducible group acting on a Riemann surface* X. *Let* $\{V_i\}_{i=1}^{l}$ *denote the set* $\operatorname{Irr}(G)$. *Consider the induced action on* $\operatorname{Jac}(X)$ *and apply the isogeny decomposition to this abelian variety. This implies that there exist abelian subvarieties* $B_1, \ldots, B_l$ *of* $\operatorname{Jac}(X)$ *such that:*

$$\operatorname{Jac}(X) \sim B_1^{\dim V_1} \times \ldots \times B_l^{\dim V_l}.$$

We always assume that V_1 *is the trivial representation. In this case, it can be proved that the component* B_1 *is isogenous to the Jacobian variety of the curve* X/G, *from which follows that* $\dim(X/G) = \dim B_1$.

Definition 4.35. Suppose that a finite group G is acting on a Riemann surface X. Furthermore, suppose that the dimension of the factors B_i associated to the decomposition of $\operatorname{Jac}(X)$ is at most 1. Then we say that the pair (G, X) (or simply G if there is no ambiguity) satisfies the **Hypothesis A** . In particular, $\operatorname{Jac}(X)$ is a completely decomposable abelian variety.

Chapter 5

Coxeter Groups and Weyl Groups

Throughout this chapter, V will denote a real Euclidian space. Most of the material of this chapter is based on [Hum92] and [BB06].

5.1 Finite reflection groups

Definition 5.1. Let V be a real Euclidian space endowed with a positive definite symmetric bilinear form $\langle , \rangle$. Let $\alpha \in V$ be an element of the space and H_α the hyperplane orthogonal to the line L_α that passes through α. A *reflection* with respect to α is a linear operator $T = T_\alpha$ on V such that:

$$T(\alpha) = -\alpha$$
$$T|_{H_\alpha} = id_{H_\alpha}$$

A finite group generated by reflections is called a *finite reflection group*. From now on we denote by W a finite reflection group acting on the Euclidian space V. We say that a reflection group W is *essentially relative* to V if there are no nonzero fixed points.

Remark 5.2. The operator T_α is given by:

$$T_\alpha v = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

It follows that T_α is an orthogonal transformation of order 2 in the group of all orthogonal transformations on V, $O(V)$.

Example 5.3. We introduce three of the most important families of finite reflection groups.

- $(S_n,\ n \geq 2)$ The symmetric group S_n can be thought of as a subgroup of $O(n,\mathbb{R})$ as follows: a permutation $\sigma \in S_n$ acts on $\mathbb{R}^n$ by permuting the standard basis. Note that the transposition (ij) corresponds to the reflection $T_{e_i - e_j}$.
- $(B_n,\ n \geq 2)$. For $V = \mathbb{R}^n$, note that the group generated by the reflections $T_{e_1}, \ldots T_{e_n}$ (called *the group of sign changes*) is isomorphic to $\mathbb{Z}_2^n$, which intersects S_n trivially and is also normalized by S_n. Thus the semidirect product of S_n and the group of sign changes yields a reflection group of order $2^n \cdot n!$. This group is called the *Hyper-octahedral group* of rank n.

- (D_n, $n \geq 3$). Again consider the space $V = \mathbb{R}^n$. Consider H the subgroup of the sign changes generated by the reflections $T_{e_i+e_j}$ (for $i \neq j$). Note that H is a subgroup of index 2 in the group of sign changes. The semidirect product of S_n and the group of sign changes yields another reflection group of order $2^{n-1} \cdot n!$ which is called the *Demihedral group* of rank n.

Remark 5.4. It can be proved that the groups S_4 and D_3 are isomorphic.

The goal of the following results is to understand the internal structure of a finite reflection group. We start by studying properties of their action over Euclidian spaces.

Proposition 5.5. *Let $F \in O(V)$ and $\alpha \in V$ be a nonzero vector, then $FT_\alpha F^{-1} = T_{F\alpha}$. In particular $T_\alpha \in W$ if and only if $T_{w\alpha} \in W$ (assuming $w \in W$).*

Proof. It is clear that $FT_\alpha F^{-1}$ maps $t\alpha$ to its negative. Now we will prove that $FT_\alpha F^{-1}$ fixes the plane $H_{F\alpha}$. Recall that since F preserves the dot product, $v \in H_\alpha$ if and only if $Fv \in H_{F\alpha}$. Hence, whenever v lies in H_α,

$$FT_\alpha F^{-1}(Fv) = FT_\alpha v = Fv,$$

and the proposition is proved. □

Definition 5.6. We say that $\Phi \subset V$ is a *root system* if,

1. Φ is finite.
2. $\Phi \cap L_\alpha = \{\pm\alpha\}$ for every $\alpha \in \Phi$.
3. Φ is stable under Φ-reflections, i.e, $T_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

We say that the group W is associated to Φ if it is generated by the elements T_α where α ranges over Φ. Elements of Φ are called *roots*, W a Φ-group, and $|\Phi|$ is the rank of the group.

Proposition 5.7. *Let W be a finite reflection group, then W can be realized as a Φ-group for some root system Φ. Conversely, any Φ-group is finite.*

Proof. Suppose that W is a finite reflection group, let $\{T_{\alpha_i}\}$ be a finite set of reflections generating W with no elements proportional to each other. Define Φ as the set of unit vectors lying on the lines L_{α_i}. Then Proposition 5.5 implies that Φ satisfies the third hypothesis in order to be a root system. First and second hypotheses are satisfied by construction.

Conversely, suppose that W is a Φ-group. Note that T_α fixes $\langle\Phi\rangle^\perp$ pointwise for each $\alpha \in \Phi$. It follows that a monomorphism $W\colon \to S_{|\Phi|}$ exists, hence W is finite. □

Remark 5.8. A finite reflection group is henceforth to be studied with a corresponding root system.

Definition 5.9. Let $<$ be a total order on the euclidian space V that respects the vector space structure. A set is *positive* if every element is greater than 0. Negative subsets are defined in the same way.

Remark 5.10. From now on, a total order on a euclidian space is assumed to respect the algebraic structure. Every Euclidian space has a total order, since the lexicographic order on V provides an example.

Definition 5.11. Let Φ be a root system Φ. Then $\Delta \subset \Phi$ is called a *simple system* if it is a basis for $\langle \Phi \rangle$ and each root is a linear combination of elements of Δ with coefficients of the same sign.

Theorem 5.1. *Let Δ be a simple system in the root system Φ, then there exists a unique positive system containing Δ, satisfying $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in Δ. Moreover, every positive system in Φ contains a unique simple system.*

Proof. See [Hum92, Ch. 1, Section 3]. □

Theorem 5.2. *Let W be a finite reflection group with a root system Φ and a simple system Δ in Φ. Then W is characterized by the presentation,*

$$W = \left\langle T_\alpha, \alpha \in \Delta \middle| (T_\alpha T_\beta)^{m(\alpha,\beta)} = e \right\rangle.$$

where $m(\alpha, \beta)$ denotes the order of $T_\alpha T_\beta$ in W.

Proof. See [Hum92, Ch. 1, Section 9]. □

Definition 5.12. We say that a root system $\Phi \subset V$ is *crystallographic* if,

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z} \quad \forall \alpha, \beta \in \Phi,$$

These values are called *Cartan integers*. In this case, the Φ-group is a *Weyl group*.

Remark 5.13. The crystallographic condition implies that Φ is contained in the $\mathbb{Z}$-span of Δ. Thus if W is also assumed to be essential then the $\mathbb{Z}$-span of Δ is a lattice stable under the action of W.

Proposition 5.14. *The Weyl groups are the finite reflection groups for which $m(s, s') \in \{2, 3, 4, 6\}$ for all $s \neq s'$ in W.*

Proof. See [Hum92, Ch. 2]. □

Remark 5.15. It can be proved that for an irreducible root system there can be at most two values for the length $(\alpha, \alpha)^{1/2}$. These are called the *short and long roots* of the system. If all roots have the same length they are taken to be long by definition.

Example 5.16. The following are examples of crystallographic root systems and their respective simple system Δ.

1. $(S_n, n \geq 2)$ Let $H \subset \mathbb{R}^n$ be the orthogonal space to the element $\sum e_i$ and the *standard lattice* $L = \mathbb{Z}_1 e_1 + \ldots + \mathbb{Z}_n e_n$. The root system is given by
$$\Phi = \{v \in H \cap L : |v|^2 = 2\}$$
Then Φ is given by the elements
$$e_i - e_j \quad (1 \leq i \neq j \leq n + 1)$$
And Δ is given by
$$\Delta = \{e_i - e_{i+1} : i = 1 \ldots n\}.$$

2. $(B_n, n \geq 2)$. Let $V = \mathbb{R}^n$ and L the standard lattice. Then Φ is defined by
$$\Phi = \{v \in L : |v|^2 \in \{1, 2\}\}.$$
Note that the short roots of Φ are given by $\pm e_i$ and the long roots are $\pm e_i \pm e_j \, (i < j)$. Thus Δ is given by $v_1 = e_1 - e_2, v_2 = e_2 - e_3, \ldots, v_{n-1} = e_{n-1} - e_n, v_n = e_n$.

5.2 Coxeter groups

Theorem 5.2 motivates the following definition:

Definition 5.17. Given a set S and a function $m\colon S \times S \to \{1, 2, \ldots, \infty\}$ satisfying

$$m\left(s, s'\right) = m\left(s', s\right)$$
$$m\left(s, s'\right) = 1 \Leftrightarrow s = s',$$

let W be the quotient group F/N, where F is the free group on the set S and N is the normal subgroup generated by the elements $(ss')^{m(s,s')}$. Then the pair (W, S) is called a *Coxeter system*, W a *Coxeter group*, m the Coxeter function, S the set of *Coxeter generators*, and $|S|$ its *rank*. Note that finite reflection groups are examples of Coxeter groups.

Remark 5.18. We make the convention that $m\left(s, s'\right) = \infty$ means that there are no relations between the elements s, s' in the group W.

Definition 5.19. Given a Coxeter system (W, S), the associated *Coxeter graph* is the undirected graph $(S, \{(s, s') \mid m\left(s, s'\right) \geq 3\})$ where arcs are labeled by m. It is customary to omit the label 3 as it occurs frequently. A Coxeter system is said to be *irreducible* if the corresponding graph is connected.

Example 5.20. The usual notation for the presentation of the dihedral group D_6 of order 12 is given as follows

$$\left\langle r, s : r^6 = s^2, srs = r^{-1}\right\rangle$$

Following this notation, consider the following subsets of D_6.

$$S = \{r, r^5 s\}$$
$$S' = \{s, r^3, rs\}$$

Note that both sets define Coxeter generators thus Coxeter systems, and in fact they are irreducible. Moreover, the rank of these systems are 2 and 3 respectively. Despite that, the associated groups of both systems are isomorphic. In conclusion, the Coxeter system is not determined by the isomorphism class of its Coxeter group.

For a subset $J \subset S$, the subgroup of W generated by J is the *parabolic subgroup* of J and is denoted by W_J. The following result implies that every finite reflection group can be decomposed into the product of irreducible groups and parabolic subgroups.

Proposition 5.21. *Let (W, S) be a finite Coxeter system with associated graph Γ, Γ_i a connected component, and S_i the corresponding subset of S. Then W is isomorphic to $\prod_i W_{S_i}$.*

Remark 5.22. The definition of a Coxeter system leaves some subtleties that need to be considered. A priori, it may happen that some element of S could be trivial in W. For example: it can be shown that the presentation

$$\langle x, y : x^4 = y^3, xy = y^2x^2 \rangle$$

induces the trivial group. The following result implies that this does not happen for Coxeter groups.

Proposition 5.23. *For a Coxeter group W there is a unique morphism $f: W \to \{-1, 1\}$ sending each $s \in S$ to -1. In particular, each s is non trivial in W and W has a subgroup of index 2.*

Proof. Let F be the free group over the set S. We define the function $f: S \to \{-1, 1\}$ sending each s into -1. Note that the elements $(ss')^{m(s,s')}$ lie in the kernel, it follows that f can be extended to the desired morphism. □

5.2.1 Geometric representation

Given a Coxeter system (W, S), we define a faithful representation $\sigma: W \to \mathrm{GL}(V)$ for some real vector space V with a basis indexed by S. The idea is to send each $s \in W$ to a quasi-reflection in V. In this context, a *quasi-reflection* is a linear operator $T: V \to V$ which fixes some hyperplane and sends some vector to its negative. Moreover, we define an associated bilinear form B related to the Coxeter function $m(s, s')$ which will determine a geometry in the space V.

Definition 5.24. Let V be a finite dimensional real vector space and B a bilinear form on V. A vector $v \in V$ is said to be *isotropic* with respect to B if $B(v, v) = 0$. The bilinear form B is *non-isotropic* if it has no nonzero isotropic vectors. Furthermore, if B is non-degenerate and $W \leq V$, we define the *perp space*

$$W_B^{\perp} = \{v \in V | B(w, v) = 0 \text{ for all } w \in W\}.$$

Lemma 5.25. *Let B be a non-degenerate bilinear form on V and $W \leq V$ a subspace. Then $\dim W_B^{\perp} = \dim V - \dim W$. In particular, if B is non isotropic, then $V = W_B^{\perp} \oplus W$.*

Proof. The proof is straightforward. □

Definition 5.26. Given a Coxeter system (W, S), and a real vector space V having a basis indexed by $\{a_s | s \in S\}$. We define a symmetric bilinear form B on V by

$$B(a_s, a_{s'}) = -\cos \frac{\pi}{m(s, s')}$$

If $m(s, s') = \infty$ then we interpret this expression as $B(a_s, a_{s'}) = -1$. For each $s \in S$ we define the operator $T_s: V \to V$,

$$T_s v = v - 2B(a_s, v)\, a_s$$

Remark 5.27. Since $T_s(a_s) = -a_s$ and fixes $(Ra_s)^{\perp}_{B}$ pointwise, the operator T_s is a quasi-reflection. Note that T_s belongs to $\mathrm{GL}(V)$ for each $s \in S$.

Proposition 5.28. *The map $s \mapsto T_s$ can be extended uniquely to a representation $\sigma \colon W \to \mathrm{GL}(V)$. Moreover, for each pair $(s, s') \in S$, the order of ss' in W is precisely $m(s, s')$. This representation is called the* geometric representation *of the Coxeter system and is in fact faithful.*

Proof. See [Hum92, Ch. 5, Section 4]. □

Theorem 5.3. *Let (W, S) be a Coxeter system. Then the following statements are equivalent,*

1. *W is finite.*
2. *The bilinear form B is positive definite.*
3. *W is a finite reflection group.*

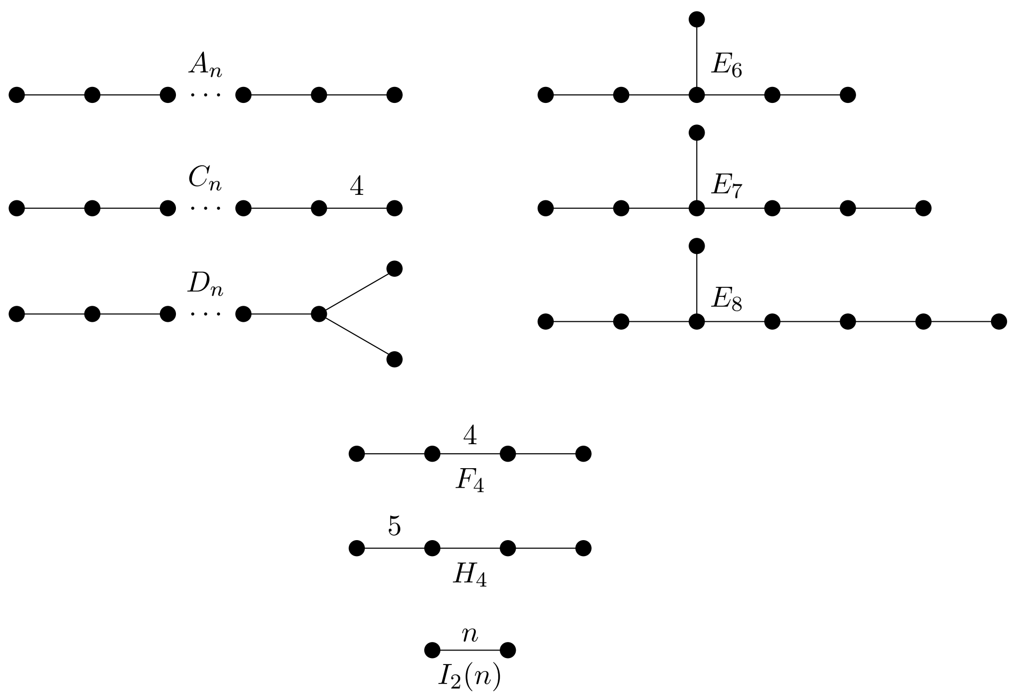
In other words, finite reflection groups and finite Coxeter groups are the same.

Proof. See [Hum92, Ch. 6, Section 4] for details. □

5.2.2 Classification of finite Coxeter groups.

Recall that a Coxeter group can be described by its Coxeter graph.

Theorem 5.4. *Let G be an irreducible finite Coxeter group. Then G is isomorphic to one of the following groups,*



Moreover $I_2(6) \cong G_2$, $I_2(4) \cong B_2$, $I_2(3) \cong A_2 = S_3$. And the other groups are pairwise non isomorphic to each other.

Proof. See [Hum92] or [BB06]. □

Remark 5.29. From now on, we denote the groups $I_2(6)$ and S_{n+1} by G_2 and A_n respectively. The subscripts indicate the number of vertices in each graph. Proposition 5.14 implies that the irreducible finite Weyl groups are given by: $A_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. In particular, with the exception of G_2 none of these groups is isomorphic to a dihedral group.

5.3 Further properties of Coxeter groups

In this section we will study properties of the groups C_n and D_n. We start with the representation theory of these groups using the Little Group Method, see [Ser12, Ch. 8, Section 2] for details. Recall that C_n is isomorphic to $\mathbb{Z}_2^n \rtimes S_n$ and has D_n as a subgroup of index 2.

5.3.1 Representations of C_n

We follow the same notation used in the Little Group Method. We find an isomorphism with the set Ω_A. Let Ω_A be indexed by

$$\Omega_A = \{\rho_0, \ldots, \rho_{2^n-1}\}$$

Furthermore, we will find explicitly the groups G_k.

Proposition 5.30. *There is an isomorphism of groups between Ω_A and $\mathbb{Z}_2^n$, such that the action ϕ of S_n in Ω_A is equivalent to the standard action (permutation of coordinates).*

Sketch of proof. Let x_j be the canonical vector of $\mathbb{Z}_2^n$ and let $f\colon \{-1, 1\} \to \mathbb{Z}_2$ be the isomorphism between these two groups. We define the map $\psi\colon \Omega_A \to \mathbb{Z}_2^n$ by the rule

$$\psi(\rho) = (f \circ \rho(x_1), \ldots, f \circ \rho(x_n))$$

This is the required isomorphism. □

Remark 5.31. The above proposition allows us to change the problem under the identification: by abuse of notation we do not distinguish between the elements ρ in Ω_A and their image $\psi(\rho)$.

The orbits under the standard action of S_n consist of the n-tuples having the same number of zero entries. For each $i = 0, \ldots, n$, let 0_i be the set of n-tuples having exactly i entries with one. The cardinality of O_i is $\binom{n}{i}$. As representatives for the orbits O_i we take the elements,

$$a_i = \begin{cases} (0, \ldots, 0, 1, \ldots, 1), & \text{if } i < n - i \\ (1, \ldots, 1, 0, \ldots, 0), & \text{if } i > n - i \end{cases}.$$

Therefore the set $\Omega_A^{S_n}$ is given by $\Omega_A^{S_n} = \{a_0, \ldots, a_n\}$. For a fixed element $x \in \mathbb{Z}_2^n$, it follows that $\sigma \in \mathrm{Stab}_x^{S_n}$ if and only if there exist disjoint permutations σ_0, σ_1 such that $\sigma = \sigma_1\sigma_0$ and σ_i stabilizes the coordinates of x with value i. In particular,

$$H_j = \mathrm{Stab}_{\rho_j}^{H} \cong S_j \times S_{n-j}.$$

Remark 5.32. By the Little Group Method, the set $\mathrm{Irr}_{\mathbb{C}}(C_n)$ is given by the representations $\theta_{ij} = \mathrm{Ind}_{G_j}^{G}\left(\overline{\rho_j} \otimes \overline{\sigma_{ij}}\right)$. Hence, in order to construct the representations, it suffices to find the representations of the groups $S_j \times S_{n-j}$ para $j \in \{0, \ldots, n\}$. We will analyze each case:

1. If $G_0 = G$, then $\theta_{i0} = \overline{\rho}_0 \otimes \overline{\sigma}_{i0}$ are representations of G, of the type σ_{i0} for S_n and the trivial action for $\mathbb{Z}_2^n$.
2. If $G_n = G$, then $\theta_{in} = \overline{\rho}_n \otimes \overline{\sigma}_{in}$ are representations of G, in the way σ_{in} for S_n and the generators of $\mathbb{Z}_2^n$ acts as $-Id$.
3. For the other cases $G_k \cong \mathbb{Z}_2^n \rtimes (S_k \times S_{n-k})$. In this case, the degree of the representation θ_{ik} will be divisible by $[G : G_k]$. In particular, none of them will have degree 1.

Using the fact that S_n has only 2 representations of dimension 1, we see that C_n has exactly 4 of them. Moreover, we have that C_n has exactly $\sum_{j=0}^{n} p(j) \cdot p(n-j)$ inequivalent irreps over $\mathbb{C}$. Hence, the following holds:

Corollary 5.33. *The number of irreducible representations of the group C_n is given by*

$$Cl(C_n) = \sum_{j=0}^{n} p(j) \cdot p(n-j).$$

Corollary 5.34. *The commutator subgroup of C_n has index 4.*

5.3.2 Properties of Weyl groups

The following results will be used to deduce necessary properties that signatures must satisfy.

Lemma 5.35. *Suppose that G is a finite group and H a normal subgroup of G. Then,*

$$Cl(H) \leq [G : H] \cdot Cl(G).$$

In particular $Cl(D_n) \leq 2 \cdot Cl(C_n)$

Sketch of proof. Consider a set $\{g_1, \ldots, g_r\}$ of representatives of the cosets of G/H where $r = [G : H]$. For each $[h]_H$ conjugacy class in $Cl(H)$. We define for each g_i the class $[g_i^{-1} h g_i]$ in the set $Cl(G)$. □

Lemma 5.36. *For each positive integer n, the partition function satisfies*

$$p(n+2) \leq p(n+1) + p(n)$$

In particular, the inequality $p(n+1) < 2 \cdot p(n)$ holds.

Proof. For a natural number $n \in \mathbb{N}$ consider the set P_n of all its partitions. Its elements are tuples of positive numbers that sum n. For each integer $a \in \{1, \ldots, n\}$ consider the subset $P_n(a)$ of tuples of P_n that contains a as en element. It easily follows that

$$p(n) = |P_n|$$
$$|P_n(a)| = |P_{n-a}|$$

From this it follows that

$$|P_{n+2}| = |P_{n+2}(1)| + |P_{n+2}(1)^c| = |P_{n+1}| + |P_{n+2}(1)^c|$$

since $P_n = P_{n+2}(2)$. It is enough to prove that $|P_{n+2}(1)^c| \leq P_{n+2}(2)$. Let $x = (n_1, \ldots, n_r)$ be an element of $P_{n+2}(1)^c$. Recall that $n_i > n_{i+1}$ and $\sum n_i = n+2$. Thus the element $n_r \geq 2$ (otherwise x would be an element of $P_{n+2}(1)$. Define the element $z = (n_1, \ldots, n_{r-1}, 2, n_r - 2)$. If n_r=0 then ignore the 0 given at the end. This defines an injection from $P_{n+2}(1)^c$ into $P_{n+2}(2)$. □

Definition 5.37. For a finite group G let $\mathrm{ord}(G)$ be the set of orders of non-trivial elements of G. It is called the *order set* of G. Moreover, for each $n \in \mathrm{ord}(G)$ let $\mathrm{ord}(G)_n$ denote the set of elements of G of order n.

Theorem 5.38. *The complex representations of Weyl groups are realizable over $\mathbb{Q}$. In other words, Weyl groups are absolutely irreducible.*

Proof. References are given in [Hum92, Ch. 8, Section 10]. □

Remark 5.39. The irreducible Weyl groups of the above list are not abelian groups.

The following table contains values associated to some Weyl groups. These values will be useful for studying signatures. Recall that $\mathbf{N}(G)$ is defined in Definition 4.30 and $Cl(G)$ denotes the number of conjugacy classes of G.

TABLE 5.1: Values of Weyl groups

Group	$\|G\|$	$\mathbf{N}(G)$	$Cl(G)$	$\mathrm{ord}(G)$
C_3	48	20	10	$\{2,3,4,6\}$
C_4	384	76	20	$\{2,3,4,6,8\}$
C_5	3840	312	36	$\{2,3,4,5,6,8,10,12\}$
C_6	46080	1384	65	$\{2,3,4,5,6,8,9,10,12,16,18\}$
D_3	24	10	5	$\{2,3,4\}$
D_4	192	44	13	$\{2,3,4,6\}$
D_5	1920	156	18	$\{2,3,4,5,6,8,12\}$
D_6	23040	752	37	$\{2,3,4,5,6,8,10,12\}$
D_7	322560	3256	55	$\{2,3,4,5,6,7,8,10,12,20,24\}$
S_3	6	4	3	$\{2,3\}$
S_5	120	26	7	$\{2,3,4,5,6\}$
S_6	720	76	11	$\{2,3,4,5,6,8,9\}$
S_7	5040	232	15	$\{2,3,4,5,6,7,8,9,10,12\}$
F_4	1152	140	25	$\{2,3,4,6,8,12\}$
G_2	12	8	6	$\{2,3,6\}$
E_6	51840	892	25	$\{2,3,4,5,6,8,9,10,12\}$
E_7	2903040	10208	60	Not necessary
E_8	696729600	199952	112	

Proposition 5.40. *Let D_{2n} denote the dihedral group of order $2n$, then the commutator subgroup D'_{2n} is given by $\langle r^2 \rangle$. In particular, the order set of G'_2 is given by* $\operatorname{ord}(G'_2) = \{3\}$.

Proof. Note that $\langle r^2 \rangle$ is a normal subgroup of D_{2n} and $D_{2n}/\langle r^2 \rangle$ is an abelian group, since the images of r and s commute in the quotient. Thus, the commutator subgroup is contained in $\langle r^2 \rangle$. It follows that $\langle r^2 \rangle = D'_{2n}$ since the only subgroup of $\langle r^2 \rangle$ is not normal. □

Proposition 5.41. *Let G_2 be the dihedral group Weyl group. Then*

- $\operatorname{ord}(G_2)_2 = \{r^k s : k \in \{0, \ldots, 5\}\} \cup \{r^3\}$
- $\operatorname{ord}(G_2)_3 = \{r^2, r^4\}$
- $\operatorname{ord}(G_2)_6 = \{r, r^5\}$.

Proof. It is straightforward. □

Remark 5.42. The group C_3 is isomorphic to $S_4 \times S_4$. We know that its commutator subgroup has index 4, under this identification it easily follows that the commutator subgroup is given by $H \times \{1\}$ where H denotes the alternating group of S_4. In particular, $\operatorname{ord}(C'_3) = \{2, 3\}$.

Definition 5.43. For a finite group G, the minimum number of generators of G, denoted by $d(G)$ is defined as the minimum cardinality of all the generating sets of the group G. Note that if $\varphi\colon G \to H$ is a surjective homomorphism, then $d(G) \geq d(H)$.

Chapter 6

Further properties of actions over Riemann Surfaces

6.1 Signature of an action

We define some invariants that encode the geometric information associated to actions of groups on Riemann surfaces. The symbol G will always denote a finite group acting on a Riemann surface X and γ the genus of X/G. The symbol π will be the associated quotient map $\pi\colon X \to X/G$. Note that the action of G on X defines an equivalence relation on X given by the orbits. Considering the above equivalence, a set is called a *complete branch set* if it is a maximal set of non-equivalent ramification points with respect to the action of G. From now on $\{p_1, \ldots, p_r\}$ denotes a complete branch set.

Definition 6.1. Let G acting on the Riemann surface X. The *signature* of the action $G \curvearrowright X$, is the tuple $\sigma = (\gamma; m_1, \ldots, m_r)$, where γ equals the genus of X/G and m_i is the order of the stabilizer subgroup of p_i. In some cases we reduce the notation of the signature, if a_j denotes the number of orbits of ramification points where the quotient map is j to 1. We define the *ordered signature* σ^* of the action as the tuple $\sigma^* = (\gamma; j^{a_j}, \ldots, t^{a_t})$. By simplicity, we abbreviate the ordered signature σ^* by signature. To avoid confusions, the symbols will be preserved (σ^* and σ respectively). Note that if $a_i \neq 0$ then $i \in \operatorname{ord}(G)$, the set of orders on non trivial elements of G.

Theorem 6.2 (Riemann existence theorem). *A group G acts on a curve X of genus $\boldsymbol{g}$ with ordered signature $\sigma^* = (\gamma; 2^{a_2}, \ldots, t^{a_t})$ if and only if the* Riemann-Hurwitz equation

$$\boldsymbol{g} = |G|\,(\gamma - 1) + 1 + \frac{|G|}{2} \sum_{i=2}^{t} a_i \left(1 - \frac{1}{i}\right).$$

is satisfied, and there is a tuple

$$\mathcal{V} = (\alpha_1, \ldots, \alpha_\gamma, \beta_1, \ldots, \beta_\gamma, g_{2,1}, \ldots, g_{2,a_2}, \ldots, g_{t,a_t})\,.$$

of elements of G that generate G, with $g_{i,k}$ of order i, that satisfies:

$$\prod_{j=1}^{\gamma}[\alpha_j, \beta_j] \prod_{i=1}^{t} \prod_{k=1}^{a_j} g_{i,k} = e. \tag{6.1}$$

Remark 6.3. Given a signature σ^*, the tuple $\mathcal{V}$ is called a *generating vector* for the signature. From now on the symbol **S** will denote the sum $\sum a_i \left(1 - \frac{1}{i}\right)$

(if there is ambiguity we use the notation $\mathbf{S}(G)$ or even $\mathbf{S}(G,\sigma)$ to specify the dependence of this number on G and σ). Note that the generating vector has length $2\gamma+\sum a_i$ and that $\mathbf{S} \geq \frac{1}{2}\sum a_i$. From this and the fact that $a_i \geq 0$, the following holds

- $\sum a_i = 0$ if and only if $0 \leq \mathbf{S} < 1/2$.
- $\sum a_i = 1$ if and only if $1/2 \leq \mathbf{S} < 1$.

Theorem 6.4 (Hurwitz's Theorem)**.** *Let G be a finite group acting on a Riemann surface X of genus $\boldsymbol{g} \geq 2$. Then*

$$|G| \leq 84\,(\boldsymbol{g}-1)\,.$$

If the equality occurs then the group is said to be a Hurwitz group*.*

Proof. See [Mir95, Ch. 3]. □

Definition 6.5 (Geometric signature)**.** Let G be a finite group acting on a Riemann surface X and $\{p_1,\ldots,p_r\}$ a complete branch set. For each p_j, we consider its stabilizer G_j. We define the *geometric signature* of $G \curvearrowright X$ as the tuple:

$$(\gamma;[m_1,C_1],\ldots,[m_r,C_r])$$

where γ is the genus of the quotient curve X/G, m_j the order of the group G_j and C_j the conjugacy class of G_j.

Remark 6.6. The geometric signature consists of a signature $(\gamma;m_1,\ldots,m_r)$ and a specification of the type of the branch values. Note that the geometric signature does not depend on the set $\{p_1,\ldots,p_r\}$ but on the action itself. In other words, if another complete branch set is taken, the geometric signature is the same. However in order to make computations, we choose a set of representatives of those conjugacy classes; the results will be the same.

Theorem 6.1. *Let G be a finite group acting on a Riemann surface X with geometric signature $(\gamma;[m_1,C_1],\ldots,[m_r,C_r])$. Let $\mathcal{W}_i$ be an irreducible and nontrivial rational representation of G and B_i the associated subvariety of the isogeny decomposition of* $\mathrm{Jac}\,(X)$*. Then the dimension of B_i is given by*

$$\dim B_i = k_i\left(\dim V_i\,(\gamma-1)+\frac{1}{2}\sum_{k=1}^{r}\left(\dim V_i - \dim \mathrm{Fix}_{G_k} V_i\right)\right)$$

where G_k is a representative of the conjugacy class C_k, V_i is a complex irreducible representation associated to $\mathcal{W}_i$, and $k_i = l_i \cdot |\,\mathrm{Gal}\,(K_i/\mathbb{Q})\,|$.

Proof. See [Roj02]. □

6.2 Signature of the restricted action

For a subgroup H of G, we are interested in the induced action on this subgroup. Recall that the quotient map π_H is Galois, thus its structure is described by its signature. We consider [Roj02] for this section.

Definition 6.7. Given an action of a group G on a Riemann surface X and a point $p \in X/G$ for a fixed fiber $\pi^{-1}(p)$, a *package* is the set of points in $\pi^{-1}(p)$ with the same stabilizer. We also define the set

$$\Omega_{G_p} = \{ \text{ left transversal of } N_G(G_p) \text{ in } G\}$$

Remark 6.8. Given two points x_1, x_2 in the fiber $\pi^{-1}(p)$, let g_1, g_2 be elements of the group such that $x_i = h_i(p)$ for $i = 1, 2$. The relation in their stabilizers is given by

$$\begin{aligned} G_{x_1} = G_{x_2} &\Leftrightarrow h_1 G_p h_1^{-1} = h_2 G_p h_2^{-1} \\ &\Leftrightarrow h_2^{-1} h_1 G_p (h_2^{-1} h_1)^{-1} = G_p \\ &\Leftrightarrow h_2^{-1} h_1 G_p (h_2^{-1} h_1)^{-1} = G_p \end{aligned}$$

hence, the number of packages is $[G : N_G(G_p)]$. Moreover each of them has the same number of elements.

6.2.1 Genus of X/H

Let $\{\ell_j\}$ be an indexation of the set Ω_p. Each ℓ_j determines one package with stabilizer $G_p^{\ell_j^{-1}}$. We follow this notation for the rest of the section.

Proposition 6.9. *Let S be a curve with action of G with geometric signature $\Gamma = (\gamma; [m_1, C_1], ..., [m_s, C_s])$; for each $H \leq G$, we will denote by S/H the quotient curve of S by the action of H; then the Riemann-Hurwitz formula for the covering $\pi_H : X \to X/H$ can be expressed as follows*

$$g_X = |H| \left(g_{X/H} - 1\right) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} \sum_{l_j \in \Omega_{G_i}} [N_G(G_i) : G_i] \cdot \left(|G_i^{\ell_j^{-1}} \cap H| - 1\right)$$

where G_i is a representative for the conjugacy class C_i, and Ω_{G_i} is a left transversal of the normalizer in G of G_i.

Proof. See [Roj02]. □

Corollary 6.10. *Applying the above proposition to $H = G$, the above equation can be simplified to*

$$g_X =| G | (g_{X/G} - 1) + 1 + \frac{1}{2} \sum_{G_i \in \Gamma} [G : G_i] \cdot \left(| G_i | -1\right)$$

Thus, the genus of X/H can be completely described in terms of the genus of X and the subgroups G_i.

A closed formula for the genus of X/G is given in the next proposition.

Proposition 6.11. *Let X be a curve admitting G action with geometric signature*

$$\Gamma := (\gamma; [m_1, C_1], ..., [m_r, C_r]).$$

For each H subgroup of G, the genus of the quotient curve X/H is given by

$$g_{X/H} = [G : H]\left(g_{S/G} - 1\right) + 1 + \frac{1}{2}\sum_{G_i \in \Gamma}\sum_{l_j \in \Omega_{G_i}} \frac{[N_G(G_i) : G_i] \mid G_i^{l_j^{-1}} \cap H \mid}{\mid H \mid}\left(\frac{\mid G_i^{l_j^{-1}} \mid}{\mid G_i^{l_j^{-1}} \cap H \mid} - 1\right),$$

where G_i is a representative for the conjugacy class C_i, and Ω_{G_i} is a left transversal of the normalizer in G of G_i for all i.

Proof. See [Roj02]. □

6.2.2 Marked points for the restricted action

Since we know the value of the genus of X/H, to describe completely the restricted action, it remains to study the marked points with respect to H. Recall that the covering π_H is Galois, thus the points of the fiber of every branch value of S/H have stabilizer of the same order, moreover they are conjugate in H. Let $(\gamma; [m_1, C_1], ..., [m_r, C_r])$ be the geometric signature of the covering map $\pi\colon X \to X/G$. For each $i = 1, \ldots, r$ consider the algorithm:

- Choose an element $l_1 \in \Omega_{G_i}$ and define the set

$$L_1^i := \{l_j \in \Omega_{G_i} : |G_i^{l_j^{-1}} \cap H| = |G_i^{l_1^{-1}} \cap H|\}$$

- Now, choose another element $l_2 \in \Omega_{G_i} \setminus L_1^i$, $|G_i^{l_2^{-1}} \cap H|$ and define the corresponding set L_2^i in the same way as before, and so on. The set Ω_{G_i} is finite, thus the algorithm ends.

Remark 6.12. We denote by ν_i the number of steps of this algorithm. The number ν_i is the same as the number of sets L_k^i that we used to partition Ω_{G_i}. Thus, it follows that

$$\sum_1^{\nu_i} |L_k^i| = |\Omega_{G_i}| = |G : N_G(G_i)|$$

The following result easily follows from this partition of Ω_{G_i}.

Proposition 6.13. *Let X be a curve with action of G, with geometric signature*

$$\Gamma = (\gamma; [m_1, C_1], ..., [m_s, C_s])$$

then, for each $C_i \in \Gamma$ there are

$$|L_k^i|\left(\frac{|N_G(G_i) : G_i||G_i^{l_k^{-1}} \cap H|}{|H|}\right)$$

points marked with the number $|G_i^{l_k^{-1}} \cap H|$ on S/H for the action of $H \leq G$, $k = 1, \ldots, \nu_i$.

Remark 6.14. Thus, given an action of a finite group G over a Riemann surface X, we can obtain all the data of the quotient map π_H i.e the geometric signature of this map. Note that some of these numbers may be trivial. To write the corresponding geometric signature, do not consider the cases when $|G_i^{l_k^{-1}} \cap H|$ is trivial. However, these points are still important for the covering map $\pi^H : X/H \to X/G$. See [Roj02] for more details.

6.3 Moduli spaces

Informally, by a moduli space we mean a space with some geometric structure whose points represent objects or isomorphism classes of such objects. In this section we construct the moduli space of principally polarized abelian varieties of dimension g and also we show some properties of the moduli space of Riemann surfaces of genus g with k marked points. For this part we use mostly [Ara12, Ch. 4] and [Rod14].

6.3.1 Moduli space of ppav of dimension g

To construct the moduli space of the isomorphism classes of ppav of dimension g we parametrize the set of period matrix corresponding to these abelian varieties. Then we identify the elements that produce the same isomorphism class to produce a quotient space. Recall that a $g \times 2g$ matrix Π is the period matrix of some abelian variety if and only if it satisfies the Riemann relations. Moreover Corollary 2.22 applied to ppavs, implies that there exists a basis such that the period matrix can be written as $\Pi = (\Omega \; I_g)$ where I_g denotes the the identity $g \times g$ matrix and Ω is a complex $g \times g$ symmetric matrix such that $\Im(Z)$ is positive definite. With this in mind we define the following set.

Definition 6.15. The *Siegel space* of degree g is defines as follows

$$\mathbb{H}_g = \{\Omega \in M(n \times n, \mathbb{C}) : \Omega = \Omega^t \text{ and } \Im(\Omega) \text{ is positive definite}\}$$

Remark 6.16. The siegel space $\mathbb{H}_g$ is an open set of the space of $g \times g$ symmetric matrices over $\mathbb{C}$. Thus its dimension is given by $g(g+1)/2$.

Proposition 6.17. *Let Π be a $2g \times g$ matrix written in the form $(I_g \; \Omega)$ with I, Ω $g \times g$ matrices. Then Π is a period matrix for some ppav A if and only if $\Omega \in \mathbb{H}_g$.*

Proof. It follows directly from the Riemann relations. □

This suggests that we can parametrize the isomorphism classes of ppavs of dimension g using the elements of $\mathbb{H}_g$, under the rule that sends each $\Omega \in \mathbb{H}_g$ to the isomorphism class associated to the period matrix $\Pi_\Omega = (\Omega \; I_g)$. This application covers all the isomorphism classes of ppavs of dimension g since given any ppav (A, H) of dimension g there exists a basis such that the period matrix of A is written in the form $\Pi = (Z \; I_g)$ with $Z \in \mathbb{H}_g$.

Remark 6.18. Now we identify the elements $\Omega_1, \Omega_2 \in \mathbb{H}_g$ that produce the same isomorphism class. To accomplish this we introduce the following group.

Definition 6.19. For a positive number g, we define the *symplectic group*

$$\mathrm{Sp}\,(2g,\mathbb{Z}) = \{R \in \mathrm{GL}\,(2g,\mathbb{Z}) : R^t E R = E\}$$

where

$$E = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

It is an easy exercise to prove that this is indeed a group with the standard product of matrices.

Lemma 6.20. *Let R be a matrix of size $2g \times 2g$ written in the form*

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for A, B, C, D $g \times g$ matrices. Then $R \in \mathrm{Sp}\,(2g,\mathbb{Z})$ if and only if the following holds

- *AB^t is symmetric.*
- *CD^t is symmetric.*
- *$AD^t - BC^t = I_g$.*

Theorem 6.21. *The group $\mathrm{Sp}\,(2g,\mathbb{Z})$ acts on the Siegel space $\mathbb{H}_g$ by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} * \Omega = (A + \Omega \cdot C)^{-1} (B + \Omega \cdot D)\,.$$

Proof. For more details see [Ara12]. □

Proposition 6.22. *Given elements Ω_1 and Ω_2 in the Siegel space $\mathbb{H}_g$, then their associated isomorphism classes of ppavs of dimension g are the same if and only if there exists a matrix $R \in \mathrm{Sp}\,(2g,\mathbb{Z})$ such that $\Omega_1 = M * \Omega_2$.*

Sketch of proof. Let $A_i = \mathbb{C}^g/\Lambda_i$ be the abelian variety associated to the isomorphism class given by Ω_i and Π_i its matrix. Choose bases of $\mathbb{C}^g$ and the associated lattice to write the matrix Π_i as $(I\ \Omega_i)$. Suppose that $f\colon A_1 \to A_2$ is an isomorphism of ppavs. From Remark 2.4 there exist invertible matrices A, R satisfying

$$A \cdot [I_g\ \Omega_1] = [I_g\ \Omega_2] \cdot R$$

Recall that the pullback polarization by f on A_1 is represented by the matrix $R^t E R$. From this, it follows that f preserves the polarization if and only if the matrix R belongs to $\mathrm{Sp}\,(2g,\mathbb{Z})$. Moreover, if that is the case a simple computation shows that the matrix Ω_1 is given by

$$\Omega_1 = (A + \Omega_2 \cdot C)^{-1} (B + \Omega_2 \cdot D)\,.$$

i.e Ω_1 and Ω_2 belong to the same orbit by the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$. The reciprocal is a direct consequence of the above computations. □

Remark 6.23. In conclusion, the quotient space $\mathbb{H}_g/\,\mathrm{Sp}\,(2g,\mathbb{Z})$ induced by the action can be identified with the moduli space $\mathcal{A}_g$ of all ppavs of dimension g.

Proposition 6.24. *The action of* $\mathrm{Sp}(2g, \mathbb{Z})$ *in* $\mathbb{H}_g$ *is properly discontinuous. Therefore the quotient is a Hausdorff space.*

Proof. See [Ara12] for details. □

Several deep questions regarding the geometry of $\mathcal{A}_g$ remain unknown. Coleman's conjecture states that given $g \geq 4$ there are only finitely many Riemann surfaces, up to isomorphism, of genus g and Jacobian of CM type. This conjecture is known to be false for $g \leq 7$, but it is still open for $g \geq 8$. See [MO11] for more details.

6.3.2 Torelli Theorem and moduli space of Riemann surfaces

Remark 6.25. We will not go into depth about the structure of the geometry of the moduli space $\mathcal{M}_g$ of the isomorphism classes of compact Riemann surfaces of genus g. The only aspect that we are concerned with is the existence of the space and the dimension it has.

Theorem 6.26. *The moduli space of compact Riemann surfaces of genus* g*,* $\mathcal{M}_g$ *has a natural structure of a quasi projective variety of dimension* $3g - 3$*.*

Proof. See [Ric10] and the references given in [Rod14]. □

For a Riemann surface X we denote by θ_X the polarization induced on $\mathrm{Jac}(X)$ by the geometric intersection.

Theorem 6.27 (Torelli Theorem)**.** *Let* X_1, X_2 *be two Riemann surfaces of the same genus such that* $(\mathrm{Jac}(X_1), \theta_1)$ *and* $(\mathrm{Jac}(X_2), \theta_2)$ *are isomorphic as polarized abelian varieties. Then* X_1 *and* X_2 *are isomorphic as Riemann surfaces.*

Proof. See [Pet14]. □

Definition 6.28. Let $g > 1$ be a positive integer. The *period map* with respect to g is defined as follows

$$\begin{aligned} p\colon \mathcal{M}_g &\to \mathcal{A}_g \\ [X] &\mapsto [\mathrm{Jac}(X)] \end{aligned}$$

i.e carries isomorphism classes of Riemann surfaces into the isomorphism classes of the respectively Jacobian in the category of abelian varieties. The image of the period map is called the *Jacobian locus*.

Corollary 6.29. *The period map* $p\colon \mathcal{M}_g \to \mathcal{A}_g$ *is injective.*

Remark 6.30. The injectivity of the period map and some geometric aspects implies the the following inequality if we compare dimensions

$$3g - 3 \leq \frac{g(g+1)}{2}.$$

From this, it follows that if $g \geq 4$, then the period map is not surjective.

Theorem 6.31. *The dimension of the moduli space of the isomorphism class of curves of a fixed genus g where a finite group G acts with a fixed signature is given by*

$$3\gamma - 3 + k$$

where γ is the genus of the quotient curve by G and k is the number of ramification points for the action.

Proof. See the reference given in [CRR14]. □

Chapter 7

Signature properties of Weyl groups

In this chapter, we prove some necessary conditions that a signature must satisfy in order to be associated with an action that satisfies **Hypothesis A** . Some of these conditions are specifically for absolutely irreducible groups or other specific group properties that a Weyl group satisfies.

7.1 Signature properties

In this section we assume that G denotes a group that is acting on a curve X of genus $\boldsymbol{g}$. We denote by γ the genus of the quotient curve X/G, by $\sigma = (0; j^{a_j}, \ldots, t^{a_t})$ the ordered signature and $\mathcal{V}$ the generating vector associated to this action.

Lemma 7.1. *Suppose that G is acting on X with $\gamma = 0$. Let $\sigma = (0; j^{a_j}, \ldots, t^{a_t})$ be the signature and $\mathcal{V} = (g_{j,1}, \ldots, g_{t,a_t})$ the generating vector. Then the following holds:*

- *If G is non cyclic then $\sum a_i \geq 3$.*
- *If $\sigma^* = \big(0; 2^2, m\big)$ for some $m \geq 2$ then G is isomorphic to the dihedral group of order $2m$. We allow the case $m = 2$ by defining $\big(0; 2^2, 2\big) = \big(0; 2^3\big)$.*

Sketch of proof. The Riemann Hurwitz equation implies that the elements of the generating vector satisfy the equation $\prod g_{i,j} = 1$. It follows from this and the fact that these elements generate the whole group that if we remove any element of the generating vector then the resulting set still generates the group. Combining this with the fact that the length of the generating vector is $\sum a_i$ implies the first part of the lemma. The second result follows directly since if $\sigma^* = \big(0; 2^2, m\big)$ then G is generated by two elements x, y of order 2 such that xy has order m. This implies that G is isomorphic to the dihedral of order $2m$. □

Lemma 7.2. *Let G be a finite group acting on a curve X with $\gamma = 0$. If G has a subgroup H of index 2, then $\sum a_{2i} \geq 2$. In other words, the number of orbits of ramification points having stabilizer of even order is at least 2. In particular, this holds for any finite Coxeter group.*

Proof. Note that the group H contains all the elements of odd order since $x^2 \in H$ and $x^{2j+1} = e$ implies that $x \in H$. This observation implies that $\mathcal{V}$ contains at least one element of even order. Suppose that $\mathcal{V}$ contains only one element $g_{a,b}$ of even order. The Riemann-Hurwitz equation for this case is given by $\prod g_{i,j} = 1$, it follows that $g_{a,b}$ is generated by elements of odd order, in particular $g_{a,b} \in H$. Then $\mathcal{V}$ is not a generating set since the generated group is contained in H. In conclusion, there exist at least two elements on $\mathcal{V}$ of even order. □

Lemma 7.3. *Let G be a group acting on a curve X such that $\gamma = 1$. Let $\sigma = (1; j^{a_j}, \dots, t^{a_t})$ be the corresponding signature, $\mathcal{V} = (\alpha_1, \beta_1, g_{j,1}, \dots, g_{t,a_t})$ the generating vector and **S** the sum $\sum a_i \left(1 - \frac{1}{i}\right)$. Then the following holds:*

- *If $\sum a_i = 0$ (equivalently $0 \le$ **S** $< 1/2$) then G is abelian.*
- *If $\sum a_i = 1$ (equivalently $1/2 \le$ **S** < 1) then the non-trivial element $g_{i,j}$ belongs to G'. In particular, the corresponding $a_{i,j} \in \operatorname{ord}(G')$.*

Sketch of proof. If $\gamma = 1$ then the Riemann Hurwitz equation implies that the elements of $\mathcal{V}$ satisfy the following equation $[\alpha_1, \beta_1] \prod g_{i,j} = 1$. To prove the first assertion, suppose that $\sum a_i = 0$, then the generating vector has the form $\mathcal{V} = (\alpha_1, \beta_1)$, it follows that the equation simplifies to $[\alpha_1, \beta_1] = 1$, in other words the group is abelian. To prove the second part, suppose that $\sum a_i = 1$, it follows that the generating vector has the form $\mathcal{V} = (\alpha_1, \beta_1, g_{j,1})$ for some j. In this case, the elements satisfy the equation $[\alpha_1, \beta_1] \cdot g_{j,1} = 1$. From this, it follows that the element $g_{j,1}$ belongs to the commutator subgroup G'.

□

Remark 7.4. Let G be a finite group and G' its commutator subgroup. Recall that $d(G)$ denotes the minimum number of generators of G (Definition 5.43). For each n, there exist a surjective homomorphism $\varphi \colon G^n \to (G/G')^n$. It follows that $d(G^n) \ge d((G/G')^n)$. Suppose that G is an irreducible Weyl group.

- If $G = C_3, G_2$ then $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case the minimum number of generators of $(G/G')^n$ is $2n$. It follows that $d(G^n) \ge 2n$.
- If $G = S_3, D_3, G_2$ then $G/G' \cong \mathbb{Z}_2$. In this case the minimum number of generators of $(G/G')^n$ is $2n$. It follows that $d(G^n) \ge 2n$.

In particular, the following corollary holds.

Corollary 7.5. *Let G be a finite irreducible Weyl group and suppose that for some integer, the group G^n is acting on a curve X with $\gamma = 0$ and signature $\sigma = (0; j^{a_j}, \dots, t^{a_t})$. Then the following holds*

- *If G is isomorphic to C_3 or $G = G_2$ then $\sum a_i \ge 2n + 1$.*
- *If G is isomorphic to S_3, D_3, D_4 or G_2 then $\sum a_i \ge n + 1$.*

7.2 Bounds for S (G) and N (G).

Recall the definitions of **N** (G) (see Definition 4.30) and **S** (G) (see Remark 6.3). In this section we find some restrictions involving the values **S** (G) and **N** (G). These results will be useful in order to prove that some signatures cannot occur in certain cases.

Lemma 7.6. *Let G be an absolutely irreducible finite group acting on a Riemann surface X of genus $\boldsymbol{g}$ satisfying **Hypothesis A** . Then the following holds:*

1. $$\boldsymbol{g} \leq \boldsymbol{N}(G) + \gamma - 1.$$

2. $$\boldsymbol{S} \leq 2\left(\frac{\boldsymbol{N}(G) - 2 + \gamma}{|G|} + 1 - \gamma\right)$$

Proof. Let $\{V_1, \ldots, V_r\}$ be the set $\mathrm{Irr}(G)$. We are under the assumption that $\mathcal{W}_1$ is the trivial representation, therefore $\gamma = \dim B_1 \leq 1$.

Now, from Theorem 3.41 the dimension of $\mathrm{Jac}(X)$ is the genus of X. From this observation, if we compare dimensions on the isogeny decomposition of $\mathrm{Jac}(X)$ it follows that

$$\boldsymbol{g} = \sum_{i=1}^{r} \dim(V_i) \dim(B_i) \leq \sum_{i=1}^{r} \dim(V_i) = \mathbf{N}(G).$$

where the inequality follows from **Hypothesis A** . The first part of the lemma is obtained by refining this bound. Suppose that $\gamma = 0$ then $\dim B_1 = 0$. In this case the above inequality is strict. It follows that:

$$\boldsymbol{g} \leq \mathbf{N}(G) + \gamma - 1.$$

To obtain the second inequality, consider the covering $\pi\colon X \to X/G$. Applying the Riemann Hurwitz formula and the latter inequality we obtain

$$\boldsymbol{g} = |G|(\gamma - 1) + 1 + \frac{|G|}{2} \cdot \mathbf{S} \leq \mathbf{N}(G) + \gamma - 1$$

the second result follows directly. □

Remark 7.7. In general there are no explicit formulas for the value of $\mathbf{N}(G)$ and not even good bounds for it, since this requires a lot of knowledge about the representation theory of G. See for instance [CRR14, Lemma 2.1]. Fortunately, there is a very simple bound that works on every absolutely irreducible group which involves only the number of conjugacy classes.

Proposition 7.8. *Let G be an absolutely irreducible finite group. Then the following inequality holds*

$$\boldsymbol{N}(G) \leq |G|^{1/2} Cl(G)^{1/2}. \tag{7.1}$$

Proof. From Theorem 4.20 and Cauchy–Schwarz inequality, we have:

$$\mathbf{N}(G)^2 = \sum_{V_i \in \mathrm{Irr}(G)} \dim V_i \cdot 1 \leq |\,\mathrm{Irr}(G)\,| \cdot \sum_{V_i \in \mathrm{Irr}(G)} \dim^2 V_i = Cl(G)\,|G|.$$

□

Remark 7.9. If $G = S_n$ denotes the symmetric group on n letters then it is known that the number of conjugacy classes is the number of partitions of n i.e $Cl(S_n) = p(n)$. Note that the corresponding bound for $\mathbf{S}$ is smaller than the bound given in [CRR14, Lemma 2.1].

Corollary 7.10. *Suppose that an absolutely irreducible group acts G on X satisfying* ***Hypothesis A*** *. Then the following holds:*

$$\boldsymbol{S} \leq 2\left(\sqrt{\frac{Cl\,(G)}{|G|}} + 1 - \gamma\right)$$

Proof. It follows directly from the above result and Lemma 7.6. □

Corollary 7.11. *Let G and X satisfy* ***Hypothesis A*** *with $\gamma = 0$ and G absolutely irreducible. Then the Riemann Hurwitz equation and the inequalities of this section imply the following:*

1. $\frac{85}{42} \leq \boldsymbol{S}$.
2. $|G| \leq 84 \cdot (\boldsymbol{N}\,(G) - 2)$.
3. $|G| < 84^2 \cdot Cl\,(G)$.

Proof. Denote by $\boldsymbol{g}$ the genus of the surface X. We separate the proof of each item by the same indexation.

1. Applying the Hurwitz formula with $\gamma = 0$, we have that

$$\boldsymbol{g} - 1 = |G|\left(\frac{\mathbf{S}}{2} - 1\right)$$

   Recall that $|G| \leq 84\,(\boldsymbol{g} - 1)$, from this the first inequality follows. Note that the equality holds if and only if G is a Hurwitz group.

2. The second inequality is obtained by simplifying the following inequality:

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\,(G) - 2}{|G|} + 1\right)$$

3. By the above, we have that $|G| < 84 \cdot \mathbf{N}\,(G)$. The result follows from applying Proposition 7.8.

□

Finally, we consider an inequality for the number of conjugacy classes of the Weyl group C_n.

Corollary 7.12. *Let G and X satisfying* ***Hypothesis A*** *with $\gamma = 0$. Furthermore, suppose that G is an absolutely irreducible group and n is a positive integer such that the group G^n satisfies* ***Hypothesis A*** *on a curve with genus of the quotient surface 0. Then*

$$n < \frac{2 \cdot \ln(84)}{\ln|G| - \ln Cl\,(G)}$$

Proof. The results follow directly from Corollary 7.11 and the fact the the cardinality and the number of conjugacy classes of a group defines a multiplicative function over cartesian products. □

Remark 7.13. The latter corollaries are useful because they allow us to prove that some groups cannot satisfy **Hypothesis A** with the internal structure of the group and no external theory about signatures or other facts.

Remark 7.14. The representations of a cartesian product of finite groups are given by the tensor product of the representations. This implies that $\mathbf{N}(G)$ is a multiplicative function in cartesian products, more specifically

$$\mathbf{N}(G)\, G \times H = \mathbf{N}(G)\, \mathbf{N}(H)$$

for any pair of finite groups G, H.

Lemma 7.15. *Let $n \geq 2$, then the sequence $Cl(C_n)$ satisfy the following*

$$Cl(C_{n+1}) \leq 4 \cdot Cl(C_n)$$

In particular, if a_n is a positive sequence satisfying $a_{n+1}/a_n > 4$, then the sequence $Cl(C_n)/a_n$ is strictly decreasing.

Sketch of proof. Split the sum and then apply Lemma 5.36 to obtain the following

$$\begin{aligned} Cl(C_{n+1}) &= \sum_{j=0}^{n+1} p(j) \cdot p(n+1-j) \\ &= \sum_{j=0}^{n} \big(p(j) \cdot p(n+1-j)\big) + p(n+1) \cdot p(0) \\ &\leq 2 \cdot \sum_{j=0}^{n} p(j) \cdot p(n-j) + 2 \cdot p(n) \leq 4 \cdot Cl(C_n)\,. \end{aligned}$$

The final step in the inequality follows from $p(n) \leq Cl(C_n)$. □

Remark 7.16. The above lemma will be useful in order to bound the term $\mathbf{N}(G)/|G|$ appearing in some inequalities involving signatures of the groups C_n. Furthermore, this lemma and Lemma 5.35 will be used to bound this term for the family of groups D_n.

Chapter 8

The Classification Problem

The classification problem for a given group G consists in finding all Riemann surfaces X and their respective actions satisfying **Hypothesis A** . We solve this problem for the irreducible Weyl groups. The strategy we follow is given in three steps:

1. In the first place, we propose necessary conditions that the signature must satisfy (the cases $\gamma = 0$ and $\gamma = 1$ will be treated separately). The first condition is the bound given by Lemma 7.6. Note that here we use **Hypothesis A** only to deduce the inequality for **S**. Secondly, if the hypothesis of Section 7.1 are satisfied, we consider these extra conditions. The set of signatures satisfying all these conditions is called *possible data*.

2. In second place, we define the set *real data* as the subset of the possible data for which there exists a generating vector associated to these signatures. Recall that the existence of such a vector is equivalent to the existence of an action.

3. Finally, we define the set *final data* as the subset of the real data satisfying **Hypothesis A** . In order to accomplish this, we calculate the dimensions of the factors B_i associated to the decomposition of $\mathrm{Jac}\,(X)$, using the formula in Theorem 6.1.

Remark 8.1. Note that the conditions formulated in the first step are just necessary conditions but not sufficient to satisfy **Hypothesis A** . That is the reason why we include **Hypothesis A** as an extra condition in the last step. In order to check **Hypothesis A** , we need the second step to ensure the existence of an action.

To find the set *possible data*, we used Haskell (see Appendix A). To find the generating vectors and calculate the dimension of the factors B_i, we used Magma (see [BCP97] and [PR16]).

8.1 The classification problem for the Hyper-octahedral family C_n

In this section we solve the problem for the family of Hyper-octahedral groups C_n, for $n \geq 3$. Suppose that C_n satisfies **Hypothesis A** for some curve X, then the following holds,

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\,(C_n) - 2 + \gamma}{2^n \cdot n!} + 1 - \gamma\right) \leq 2\left(\sqrt{\frac{Cl\,(C_n)}{2^n \cdot n!}} + 1 - \gamma\right) \tag{8.1}$$

Remark 8.2. According to Lemma 7.15, the right hand side of the inequality is a decreasing sequence. We use this fact to bound **S** when we study infinite cases of the form $n \geq a$. Hence, in these cases we use the following inequality:

$$\mathbf{S} \leq 2\left(\sqrt{\frac{Cl\,(C_n)}{2^n \cdot n!}} + 1 - \gamma\right) \leq 2\left(\sqrt{\frac{Cl\,(C_a)}{2^a \cdot a!}} + 1 - \gamma\right).$$

for each $n \geq a$.

8.1.1 Curves with $\gamma = 1$

In this case we solve the problem assuming that the genus of the curve X/G is 1.

- If $n \geq 4$ then:

$$\mathbf{S} \leq 2\left(\sqrt{\frac{Cl\,(C_n)}{2^n \cdot n!}}\right) \leq 2\left(\sqrt{\frac{Cl\,(C_4)}{2^4 \cdot 4!}}\right) = 2 \cdot \sqrt{\frac{20}{2^4 \cdot 4!}} < \frac{1}{2}$$

  If the action exists, then Lemma 7.3 would imply that C_n is an abelian group, which is a contradiction. Therefore, for $n \geq 4$ the group C_n does not satisfy **Hypothesis A** if $\gamma = 1$.

- If $n = 3$ then:

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\,(C_3) - 1}{|C_3|}\right) = \frac{19}{24}$$

  Recall that $\operatorname{ord}(C_3) = \{2, 3, 4, 6\}$, therefore the following holds:

$$\frac{a_2}{2} + \frac{2a_3}{3} + \frac{3a_4}{4} + \frac{5a_6}{6} \leq \frac{19}{24}$$

  The solutions to this inequality are the possible signatures. Recall that $\operatorname{ord}(C_3') = \{2, 3\}$ (See Remark 5.42). We analyze each case.

  1. (1). This signature is not realizable since C_3 is not abelian.
  2. $(1; 4)$. This signature is not realizable in C_3 since $4 \notin C_3'$.
  3. $(1; 2)$. We consider the identification $C_3 \cong S_4 \times \mathbb{Z}_2$. Under this identification, suppose that

$$\mathcal{V} = \left(\left(\alpha, \alpha'\right), \left(\beta, \beta'\right), \left(x, x'\right)\right)$$

     is a generating vector associated to the signature $(1; 2)$. The Riemann Hurwitz equation implies that the first coordinate satisfies the equation $[\alpha, \beta]x = id_{S_4}$. Moreover, this set of elements generates S_4. Hence, the set (α, β, x) is a generating vector of S_4 associated to the signature $(1; |x|)$. Moreover, $|x| \neq 1$ otherwise S_4 would be generated by two commuting elements. We prove in the section corresponding to the dihedral groups that this signature is not realizable in S_4 (recall that $S_4 \cong D_3$). In conclusion, the signature $(1; 2)$ is not realizable in C_3.

4. $(1;3)$. This signature is realizable, we present the generating vector under the identification $C_3 \cong S_4 \times \mathbb{Z}_2$. The vector is given by

$$\left(\alpha,\alpha'\right)=\left((12)\,,\overline{1}\right),\left(\beta,\beta'\right)=\left((1234)\,,\overline{1}\right),\left(x,x'\right)=\left((132)\,,\overline{0}\right).$$

Thus, the real data is given by $(1;3)$. Moreover this is the final data since the decomposition of the curve is given by $E \times E \times E \times E \times E \times E \times E$.

In conclusion, in the case $\gamma = 1$ for the family of groups C_n. The group C_3 is the only one that satisfies **Hypothesis A** .

8.1.2 Case $\gamma = 0$

Suppose that the group C_n satisfies **Hypothesis A** for some curve X and $\gamma = 0$. Then Corollary 7.11 implies that n satisfy the following condition:

$$2^n \cdot n! \leq 84^2 \cdot Cl\left(C_n\right)$$

Lemma 7.15 implies that this inequality does not hold for $n > 6$. Hence (for $\gamma = 0$) the group C_n does not satisfies **Hypothesis A** if $n \geq 7$.

Note that the group C_n satisfies the hypothesis of Corollary 7.11 and lemmas 7.1, 7.2 and 7.6, from this it follows that **S** satisfy the following conditions:

- $\sum a_i \geq 3$.
- $\sum a_{2i} \geq 2$.
- $\sigma^* \neq \left(0; 2^2, m\right)$ for each $m \geq 2$.
- $\dfrac{85}{42} \leq \mathbf{S} \leq 2\left(\dfrac{\mathbf{N}\left(C_n\right)-2}{2^5 \cdot 5!}+1\right)$

We find all the possible signatures satisfying the above conditions. Recall that these are necessary conditions for a signature in order to satisfy **Hypothesis A** for the group. As stated above, the signatures satisfying these conditions will be the possible data.

- If $n = 3$ then

(8.2)
$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(C_3\right)-2}{2^3 \cdot 3!}+1\right) < \frac{11}{4}$$

Recall that $\operatorname{ord}\left(C_3\right) = \{2,3,4,6\}$, therefore the following holds:

(8.3)
$$\frac{a_2}{2}+\frac{2a_3}{3}+\frac{3a_4}{4}+\frac{5a_6}{6} < \frac{11}{4}$$

TABLE 8.1: Data for C_3 with $\gamma = 0$.

Possible data
$(0; 6^2, m)$ for $m = 2, 3, 4, 6.$
$(0; 4^2, m)$ for $m = 3, 4, 6.$
$(0; m, 4, 6)$ for $m = 2, 3.$
$(0; 2, 3^2, m)$ for $m = 4, 6.$
$(0; 2^3, m)$ for $m = 3, 4, 6.$
$(0; 2^2, 6, m)$ for $m = 3, 4, 6.$
$(0; 2^2, 3, m)$ for $m = 3, 4.$
$(0; 2^4, m)$ for $m = 2, 3, 4.$
$(0; 2, 4^2, m)$ for $m = 2, 3, 4.$
$(0; 2, 3, 4, 6)$

Real data
$(0; 4, 4, 6)$
$(0; 2, 4, 6)$
$(0; 2, 3, 4, 4)$
$(0; 2, 2, 6, 6)$
$(0; 2, 2, 4, 6)$
$(0; 2, 2, 4, 4)$
$(0; 2, 2, 3, 6)$
$(0; 2, 2, 3, 4)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$
$(0; 2, 4, 4, 4)$
$(0; 2, 3, 4, 6)$
$(0; 2, 2, 2, 2, 4)$

The **final data** is given by:

1. $(0; 2, 4, 6)$ with $\boldsymbol{g} = 3$
2. $(0; 2^3, 3)$ with $\boldsymbol{g} = 5$
3. $(0; 2^3, 4)$ with $\boldsymbol{g} = 7$
4. $(0; 2^3, 6)$ with $\boldsymbol{g} = 9$
5. $(0; 4^2, 6)$ with $\boldsymbol{g} = 9$
6. $(0; 2^2, 3, 4)$ with $\boldsymbol{g} = 11$
7. $(0; 2^2, 4^2)$ with $\boldsymbol{g} = 13$
8. $(0; 2^2, 3, 6)$ with $\boldsymbol{g} = 13$
9. $(0; 2^2, 4, 6)$ with $\boldsymbol{g} = 15$

- If $n = 4$ then

(8.4) $$\mathbf{S} \leq 2\left(\frac{\mathbf{N}(C_4) - 2}{2^4 \cdot 4!} + 1\right) < 2.39.$$

The data is given by:

TABLE 8.2: Data for C_4 with $\gamma = 0$.

Possible data
$(0; 4^2, m)$ for $m = 4, 6, 8$.
$(0; 3, 6, m)$ for $m = 6, 8$.
$(0; 3, 4, m)$ for $m = 4, 6, 8$.
$(0; 2, 8, m)$ for $m = 6, 8$.
$(0; 2, 4, m)$ for $m = 6, 8$.
$(0; 2^3, m)$ for $m = 3, 4, 6, 8$.
$(0; 2^2, 3^2)$
$(0; 2, 6^2)$
$(0; 2, 3, 8)$

Real data
$(0; 4, 4, 6)$.
$(0; 2, 6, 8)$.
$(0; 2, 4, 6)$.
$(0; 2, 2, 2, 8)$.
$(0; 2, 2, 2, 6)$.
$(0; 2, 2, 2, 4)$.

The final data is given by the signature $(2, 4, 6)$ and it has genus $\boldsymbol{g} = 17$.

- If $n = 5$ then

(8.5)
$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(C_5\right) - 2}{2^5 \cdot 5!} + 1\right) < 2.17$$

TABLE 8.3: Data for C_5 with $\gamma = 0$.

Possible data
$(0; 2, 4, m)$ for $m = 5, 6, 8, 10, 12$.
$(0; 2, 3, m)$ for $m = 8, 10, 12$.
$(0; 2^3, 3)$
$(0; 2, 6^2)$
$(0; 2, 5, 6)$
$(0; 3, 4^2)$

Real data
$(0; 2, 4, 12)$
$(0; 2, 4, 10)$

In this case, the final data is empty.

- Finally if $n = 6$ the possible data is given by the signatures $(2, 4, 5)$ and $(2, 3, 8)$. The real and final data are empty sets.

8.2 The classification problem for the Demihedral family D_n

In this section we solve the problem for the family of Demihedral groups D_n, for $n \geq 3$. Suppose that D_n satisfies **Hypothesis A** for some curve X. Lemma 5.35 implies that $Cl\left(D_n\right) \leq 2 \cdot Cl\left(C_n\right)$. If we apply this in the inequality given by Corollary 7.10, we obtain that

$$\begin{aligned}\mathbf{S} &\leq 2\cdot\sqrt{\frac{Cl\left(D_n\right)}{2^{n-1}\cdot n!}}+2\cdot(1-\gamma)\\ &\leq 2\cdot\sqrt{\frac{2\cdot Cl\left(C_n\right)}{2^{n-1}\cdot n!}}+2\cdot(1-\gamma)\\ &=\sqrt{\frac{\cdot Cl\left(C_n\right)}{2^{n-4}\cdot n!}}+2\cdot(1-\gamma)\end{aligned}$$

Remark 8.3. For the same reason as before, Lemma 7.15 implies that the right hand side of the inequality is a decreasing sequence. We use this fact to bound **S** when we study infinite cases of the form $n \geq a$.

8.2.1 Curves with $\gamma = 1$

In this case we solve the problem assuming that the genus of the curve X/G is 1.

- $(n \geq 5)$. Since the sequence $\sqrt{\frac{Cl(C_n)}{2^{n-4}\cdot n!}}$ is decreasing, it follows that:

$$\mathbf{S} \leq \sqrt{\frac{Cl\left(C_n\right)}{2^{n-4}\cdot n!}} \leq \sqrt{\frac{Cl\left(C_5\right)}{2^{5-4}\cdot 5!}} = \sqrt{\frac{36}{2\cdot 5!}} < \frac{1}{2}.$$

  If the action exists then Lemma 7.3 would imply that D_n is an abelian group, which is a contradiction. Therefore for $n \geq 5$ and $\gamma = 1$ the group D_n does not satisfy **Hypothesis A** .

- If $n = 4$

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(D_4\right)-1}{\left|D_4\right|}\right) = \frac{43}{96} \leq \frac{1}{2}$$

  Hence the group D_4 does not satisfy **Hypothesis A** if $\gamma = 1$.

- If $n = 3$

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(D_3\right)-1}{\left|D_3\right|}\right) = \frac{3}{4}$$

The solutions to this inequality are the possible signatures. Recall that $D_3 \cong S_4$ and S_4' is the alternating group of rank 4 (order 12), thus $\operatorname{ord}\left(D_3'\right) = \{2, 3\}$. We use this identification for the analysis. We analyze each case.

1. (1). This signature is not realizable since G_2 is not abelian.
2. $(1;4)$. This signature is not realizable in D_3 since $4 \notin D_3'$.
3. $(1;2)$ This signature is not realizable in D_3 (See [CRR14]).
4. $(1;3)$. This signature is realizable, it has the following generating vector

$$\alpha_1 = (12)\,, \beta_1 = (1234)\,, g_1 = (132)$$

For $\gamma = 1$ the only action of D_n satisfying **Hypothesis A** is the action of D_3 associated to the signature $(1;3)$. We used [CRR14] for this case.

8.2.2 Case $\gamma = 0$

Suppose that D_n acts on a curve X satisfying the **Hypothesis A** with $\gamma = 0$. Then Corollary 7.11 and Lemma 5.35 implies that the following inequality holds:

$$|D_n| \leq 84^2 \cdot Cl\,(D_n) \leq 2 \cdot 84^2 \cdot Cl\,(C_n)\,.$$

Use Lemma 7.15 to prove that the solution of this inequality is $n \leq 7$. Hence the group D_n does not satisfy **Hypothesis A** for $n \geq 8$. Now we study the remaining cases.

Note that the group D_n satisfies the hypothesis of Corollary 7.11 and lemmas 7.1, 7.2 and 7.6. From this it follows that **S** satisfies the following conditions:

- $\sum a_i \geq 3$.
- $\sum a_{2i} \geq 2$.
- $\sigma^* \neq \left(0; 2^2, m\right)$ for each $m \geq 2$.
- $\dfrac{85}{42} \leq \mathbf{S} < 2\left(\dfrac{\mathbf{N}\,(D_n) - 2}{|D_n|} + 1\right)$

We follow the same strategy used in the classification for C_n. In this case we summarize the steps and write directly the data.

- If $n = 3$ then $\mathbf{S} \leq 8/3$.

TABLE 8.4: Data for D_3 with $\gamma = 0$.

Possible data
$(0; 4, 4, 4)$
$(0; 3, 4, 4)$
$(0; 2, 3, 3, 4)$
$(0; 2, 2, 4, 4)$
$(0; 2, 2, 3, 4)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

Real data
$(0; 3, 4, 4)$
$(0; 2, 3, 3, 4)$
$(0; 2, 2, 4, 4)$
$(0; 2, 2, 3, 4)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

The final data is:

1. $(0; 3, 4, 4)$ with $\boldsymbol{g} = 3$.
2. $(0; 2, 3, 3, 4)$ with $\boldsymbol{g} = 8$.
3. $(0; 2, 2, 4, 4)$ with $\boldsymbol{g} = 7$.
4. $(0; 2, 2, 3, 4)$ with $\boldsymbol{g} = 6$.
5. $(0; 2, 2, 3, 3)$ with $\boldsymbol{g} = 5$.
6. $(0; 2, 2, 2, 4)$ with $\boldsymbol{g} = 4$.

7. $(0; 2, 2, 2, 3)$ with $\boldsymbol{g} = 3$.

- If $n = 4$ then $\mathbf{S} \leq 39/16$.

TABLE 8.5: Data for D_4 with $\gamma = 0$.

Possible data
$(0; 4, 6, 6)$
$(0; 4, 4, 6)$
$(0; 4, 4, 4)$
$(0; 3, 6, 6)$
$(0; 3, 4, 6)$
$(0; 3, 4, 4)$
$(0; 2, 6, 6)$
$(0; 2, 4, 6)$
$(0; 2, 2, 3, 4)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$

Real data
$(0; 4, 4, 6)$
$(0; 3, 4, 4)$
$(0; 2, 2, 3, 4)$
$(0; 2, 2, 3, 3)$

The final data is the signature $(0; 3, 4, 4)$ with $\boldsymbol{g} = 17$.

- If $n = 5$ then $\mathbf{S} \leq 1037/480$.

TABLE 8.6: Data for D_5 with $\gamma = 0$.

Possible data
$(0; 2, 5, 6)$
$(0; 2, 4, 8)$
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$
$(0; 2, 3, 12)$
$(0; 2, 3, 8)$

Real data
$(0; 2, 5, 6)$
$(0; 2, 4, 5)$

The final data is $(0; 2, 4, 5)$ with $\boldsymbol{g} = 49$.

- If $n = 6$ then $\mathbf{S} \leq 793/384$.

  The possible data is given by $(0; 2, 4, 5)$ and $(0; 2, 3, 8)$. However, the real and final data are empty sets.

- If $n = 7$ then $\mathbf{S} < 2.03$.

  In this case, all the data is empty.

8.3 The classification problem for the Symmetric family S_n

Now we solve the problem for the family S_n for $n \geq 3$. Corollary 7.10 implies that

$$\mathbf{S} \leq 2 \cdot \sqrt{\frac{p(n)}{n!}} + 2 \cdot (1 - \gamma)$$

Lemma 5.36 implies that the right hand side of the inequality is a decreasing sequence.

8.3.1 Curves with $\gamma = 1$

In this case we solve the problem assuming that the genus of the curve X/G is 1.

- $(n \geq 5)$. Since the sequence $\sqrt{\frac{p(n)}{n!}}$ is decreasing, it follows that

$$\mathbf{S} \leq 2 \cdot \sqrt{\frac{p(n)}{n!}} \leq 2 \cdot \sqrt{\frac{p(5)}{5!}} \leq 2 \cdot \sqrt{\frac{7}{5!}}.$$

  If the action exists then Lemma 7.3 would imply that S_n is an abelian group, which is a contradiction. Therefore for $n \geq 5$ and $\gamma = 1$ the group S_n does not satisfy **Hypothesis A** .

- $(n = 4)$. Recall that the group S_4 is isomorphic to D_3, thus this case is omitted.

- $(n = 3)$.

$$\mathbf{S} \leq 2 \left(\frac{\mathbf{N}(S_3) - 1}{|S_3|} \right) = 1.$$

The set $\text{ord}(S_3)$ is given by $\{2, 3\}$. Thus the possible signatures satisfy the inequality:

$$\frac{a_2}{2} + \frac{2a_3}{3} \leq 1$$

The solutions to this inequality are the possible signatures. We analyze each case.

1. (1). It is not realizable, otherwise the group S_3 would be abelian.

2. $(1; 2)$. It is not realizable, since the commutator subgroup of S_3 does not have elements of order 2.

3. $(1; 3)$. This signature can be realized with two different generating vectors.

   (a) $\alpha_1 = (1, 2)$, $\beta_1 = (1, 2, 3)$, $g_1 = (1, 2, 3)$

   (b) $\alpha_1 = (1, 2)$, $\beta_1 = (1, 3)$, $g_1 = (1, 2, 3)$

4. $(1; 2, 2)$. This signature can be realized with two different generating vectors:

   (a) $\alpha_1 = (1,2,3)$, $\beta_1 = (1,2,3)$, $g_1 = (1,2)$, $g_2 = (1,2)$
   (b) $\alpha_1 = (1,2)$, $\beta_1 = (1,2,3)$, $g_1 = (1,2)$, $g_2 = (1,3)$

Moreover, these two actions satisfy **Hypothesis A** . For this case we used [CRR14].

8.3.2 Case $\gamma = 0$

Suppose that the group S_n satisfies the **Hypothesis A** for some curve X and $\gamma = 0$. Then Corollary 7.11 implies that n satisfies the following condition:

$$n! \leq 84^2 \cdot p(n)$$

Lemma 5.36 implies that this inequality does not hold for $n \geq 8$. Hence for $\gamma = 0$ the group C_n does not satisfy **Hypothesis A** if $n \geq 8$. We prove the remaining cases. Again, the groups S_n satisfy the hypothesis of Corollary 7.11 and Lemmas 7.1, 7.2 and 7.6. From this it follows that **S** satisfies the following conditions:

- $\sum a_i \geq 3.$
- $\sum a_{2i} \geq 2.$
- $\frac{85}{42} \leq \mathbf{S} < 2\left(\frac{\mathbf{N}(S_n) - 2}{|S_n|} + 1\right)$

For the same reason as before, we omit the case $n = 4$, because $S_4 \cong D_3$ and this case was done.

1. If $n = 3$ then $\mathbf{S} \leq 8/3$.

TABLE 8.7: Data for S_3 with $\gamma = 0$.

Possible data
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

Real data
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 2, 3)$

The **final data** is given by:

   (a) $(0; 2, 2, 3, 3)$ with $\boldsymbol{g} = 2$
   (b) $(0; 2, 2, 2, 2, 3)$ with $\boldsymbol{g} = 3$

2. If $n = 5$ then $\mathbf{S} < 12/5$.

TABLE 8.8: Data for S_5 with $\gamma = 0$.

Possible data
$(0; 4, 5, 6)$
$(0; 4, 4, 6)$
$(0; 4, 4, 5)$
$(0; 4, 4, 4)$
$(0; 3, 6, 6)$
$(0; 3, 4, 6)$
$(0; 3, 4, 4)$
$(0; 2, 6, 6)$
$(0; 2, 5, 6)$
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 5)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$

Real data
$(0; 4, 5, 6)$
$(0; 4, 4, 5)$
$(0; 3, 4, 6)$
$(0; 3, 4, 4)$
$(0; 2, 6, 6)$
$(0; 2, 5, 6)$
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$
$(0; 3, 6, 6)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 5)$
$(0; 2, 2, 2, 4)$

The **final data** is given by:

(a) $(0; 4, 5, 6)$ with $\boldsymbol{g} = 24$
(b) $(0; 4, 4, 5)$ with $\boldsymbol{g} = 19$
(c) $(0; 3, 4, 6)$ with $\boldsymbol{g} = 16$
(d) $(0; 3, 4, 4)$ with $\boldsymbol{g} = 11$
(e) $(0; 2, 6, 6)$ with $\boldsymbol{g} = 11$
(f) $(0; 2, 5, 6)$ with $\boldsymbol{g} = 9$
(g) $(0; 2, 4, 6)$ with $\boldsymbol{g} = 6$
(h) $(0; 2, 4, 5)$ with $\boldsymbol{g} = 4$

3. If $n = 6$ then $\mathbf{S} \leq 397/180$.

TABLE 8.9: Data for S_6 with $\gamma = 0$.

Possible data
$(0; 3, 4, 4)$
$(0; 2, 6, 6)$
$(0; 2, 5, 6)$
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$
$(0; 2, 2, 2, 3)$

Real data
$(0; 2, 6, 6)$
$(0; 2, 5, 6)$

In this case the final data is an empty set.

4. If $n = 7$ then $\mathbf{S} < 527/252$.

TABLE 8.10: Data for S_7 with $\gamma = 0$.

Possible data
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$

Real data
$(0; 2, 4, 6)$

Again, in this case the final data is an empty set.

8.4 The classification problem for the excepcional Weyl groups

In this section we finish the classification problem for the Weyl groups. We follow the same methodology used before. The exceptional Weyl groups are given by E_6, E_7, E_8, F_4, G_2.

8.4.1 Curves with $\gamma = 1$

None of these groups are abelian (Remark 5.39), hence, by Lemma 7.3, we may discard the cases for which $\mathbf{S} < 1/2$. Using Lemma 7.6 we obtain the following bounds for $\mathbf{S}$:

- For E_6
$$\mathbf{S} \leq 2 \cdot \frac{\mathbf{N}(E_6) - 1}{|E_6|} < \frac{1}{2}.$$
- For E_7
$$\mathbf{S} \leq 2 \cdot \frac{\mathbf{N}(E_7) - 1}{|E_7|} < \frac{1}{2}.$$
- For E_8
$$\mathbf{S} \leq 2 \cdot \frac{\mathbf{N}(E_8) - 1}{|E_8|} < \frac{1}{2}.$$
- For F_4
$$\mathbf{S} \leq 2 \cdot \frac{\mathbf{N}(F_4) - 1}{|F_4|} < \frac{1}{2}.$$
- For G_2
$$\mathbf{S} \leq 2 \cdot \frac{\mathbf{N}(G_2) - 1}{|G_2|} = \frac{7}{6}.$$

Therefore, the only group that can satisfy **Hypothesis A** is G_2. Now we study the problem for this group.

From Table 5.1 we know that $\operatorname{ord}(G_2) = \{2, 3, 6\}$ therefore the following holds:
$$\frac{a_2}{2} + \frac{2a_3}{3} + \frac{5a_6}{6} \leq \frac{7}{6}$$

The solutions to this inequality are the possible signatures. From Proposition 5.40 we know that $\operatorname{ord}(G_2') = \{3\}$. We analyze each case.

1. (1). This signature is not realizable since G_2 is not abelian.

2. $(1;6)$. This signature is not realizable since $6 \notin \text{ord}\,(G_2')$.
3. $(1;2)$ is not realizable since $2 \notin \text{ord}\,(G_2')$.
4. $(1;3)$. This signature is realizable since it has the following generating vector $\left(r, s, r^2\right)$.
5. $(1;2,3)$. It is not realizable, since if we suppose that $(\alpha_1, \beta_1, g_2, g_3)$ is a generating vector for this signature with $|g_i| = i$, then the element g_3 can be written as $g_3 = r^{2j}$ with $j \in \{1,2\}$ and g_2 belongs to the set $\text{ord}\,(G_2)_2 = \{r^k s : k \in \{0,\ldots,5\}\} \cup \{r^3\}$. The Riemann Hurwitz equation implies that the elements satisfy the condition $g_2 g_3 \in G_2' = \{r^2, r^4\}$. In particular, this implies that $g_2 \neq r^3$. Otherwise, the product $g_2 g_3$ would have order 6, thus $g_2 g_3 \notin G_2'$. Thus, if we suppose that g_2 can be written in the form $r^k s$ for some k, then the product $g_2 g_3 = r^k s r^{2j} = r^{k-2j} s$. However, this element does not belong to G_2'. In conclusion, this signature is not realizable.
6. $(1;2,2)$. This signature is realizable with generating vector (s, s, rs, rs).

Thus the real data is given by:

- $(1;3)$ with generating vector $\left(r, s, r^2\right)$.
- $(1;2,3)$ with generating vector (s, s, rs, rs).

Moreover, both vectors satisfy **Hypothesis A** . Thus, the final data is the same set.

8.4.2 Curves with $\gamma = 0$

Suppose that G satisfies **Hypothesis A** over a curve X for which $\gamma = 1$. Then, Corollary 7.11 implies that the following condition must hold:

$$|G| \leq 84 \cdot (\mathbf{N}\,(G) - 2)$$

In our family of groups, we checked that E_7, E_8 do not satisfy this condition. Hence, the only remaining candidates are the groups E_6, F_4, G_2. From these groups, by Remark 5.39 we know they are not abelian and all of them have a subgroup of index 2. Therefore, in order to exist, an action satisfying **Hypothesis A** must also satisfy the following conditions:

- $\sum a_i \geq 3$.
- $\sum a_{2i} \geq 2$.
- $\sigma^* \neq \left(0; 2^2, m\right)$ for each $m \geq 2$. (with the exception of the group G_2).
- $\dfrac{85}{42} \leq \mathbf{S} \leq 2\left(\dfrac{\mathbf{N}\,(G) - 2}{|G|} + 1\right)$

We follow the same methodology used before to construct the final data. The conditions given above define the possible data.

- For the group G_2, the following holds:

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}(G_2)-2}{|G_2|}+1\right) = 3.$$

TABLE 8.11: Data for G_2 with $\gamma = 0$.

Possible data
$(0;6,6,6)$
$(0;3,6,6)$
$(0;2,6,6)$
$(0;2,3,6,6)$
$(0;2,3,3,6)$
$(0;2,2,6,6)$
$(0;2,2,3,6)$
$(0;2,2,3,3)$
$(0;2,2,2,6)$
$(0;2,2,2,3)$
$(0;2,2,2,3,3)$
$(0;2,2,2,2,6)$
$(0;2,2,2,2,3)$
$(0;2,2,2,2,2)$

Real data
$(0;2,2,6,6)$
$(0;2,2,3,6)$
$(0;2,2,2,6)$
$(0;2,2,2,3)$
$(0;2,2,2,3,3)$
$(0;2,2,2,2,6)$
$(0;2,2,2,2,3)$
$(0;2,2,2,2,2)$

And the final data is given by:

1. $(0;2,2,2,3)$ with $g=2$
2. $(0;2,2,2,6)$ with $g=3$
3. $(0;2,2,3,6)$ with $g=4$
4. $(0;2,2,2,2,2)$ with $g=4$
5. $(0;2,2,2,2,3)$ with $g=5$
6. $(0;2,2,6,6)$ with $g=5$
7. $(0;2,2,2,2,6)$ with $g=6$

- For the group F_4 The following holds:

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}(F_4)-2}{|F_4|}+1\right) = \frac{215}{96}.$$

TABLE 8.12: Data for F_4 with $\gamma = 0$.

Possible data
$(0; 3, 4, 4)$
$(0; 2, 6, 8)$
$(0; 2, 6, 6)$
$(0; 2, 4, 12)$
$(0; 2, 4, 8)$
$(0; 2, 4, 6)$
$(0; 2, 3, 12)$
$(0; 2, 3, 8)$
$(0; 2, 2, 2, 3)$

The real data is only the signature $(0; 2, 6, 6)$ and the final data is empty.

- The group E_6

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(E_6\right) - 2}{|E_6|} + 1\right) < 2.03.$$

In this case, all the data is empty.

8.5 The classification problem for powers of Weyl Groups

In this section we solve the problem for powers of Weyl groups for the case $\gamma = 0$. We put bounds on the possible power that a Weyl group can satisfy **Hypothesis A** for $\gamma = 0$. Using Corollary 7.12, we know that this power n is bounded by the integer part of the following value

$$\frac{2 \cdot \ln\left(84\right)}{\ln |G| - \ln Cl\left(G\right)}$$

Remark 8.4. For a group G we denote by $\mathrm{Pow}\left(G\right)$ the integer part of the bound given above. The next table contains the values of $\mathrm{Pow}\left(G\right)$ for each irreducible Weyl group that satisfies **Hypothesis A** .

TABLE 8.13: Values of powers of Weyl groups

Group G	Pow (G)
C_3	5
C_4	2
D_3	5
D_4	3
D_5	1
S_3	12
S_5	3
G_2	12

Recall that cartesian products of absolutely irreducible groups are absolutely irreducible, thus it is possible to apply the same method to this family of groups.

8.5.1 Possible data for the power of Weyl groups

We solve the classification problem for the power of Weyl groups for the case $\gamma = 0$. First we bound the integer for which G^n satisfies **Hypothesis A** . This number is given in the table of Remark 8.4. Now we define the possible data for each case. In this case, the conditions to define the possible data are slightly different since we add an extra condition given by Remark 7.4 and Corollary 7.5.

In summary we suppose that G is a finite irreducible group, such that a power G^n is acting on a curve X with $\gamma = 0$ and satisfying **Hypothesis A** . If $\sigma = (0; j^{a_j}, \ldots, t^{a_t})$ denotes the signature associated to this action. Then the following holds

- $\sum a_i \geq 3.$
- $\sum a_{2i} \geq 2.$
- $\sum a_i \geq d\left((G/G')^n\right)$
- $\sigma \neq \left(0; 2^2, m\right)$ for each $m \geq 2.$
- $\dfrac{85}{42} \leq \mathbf{S} \leq 2\left(\dfrac{\mathbf{N}\,(G)^n - 2}{|G|^n} + 1\right)$

Remark 8.5. The value of $d\left((G/G')^n\right)$ given in the third condition appears in Corollary 7.5 and is explained in Remark 7.4.

8.5.2 Powers of C_3

Suppose that C_3^n is satisfying **Hypothesis A** for $n \in \{2, 3, 4, 5\}$. Then the following holds

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}(C_3)^n - 2}{|C_3|^n} + 1\right) \leq 2\left(\frac{\mathbf{N}(C_3)^2 - 2}{|C_3|^2} + 1\right)$$
$$= 2\left(\frac{20^2 - 2}{48^2} + 1\right) = \frac{1351}{576} < \frac{5}{2}$$

On the other hand, Corollary 7.5 implies that

$$\sum a_i \geq 2n + 1 \geq 5$$

Thus

$$\mathbf{S} \geq \frac{1}{2}\sum a_i \geq \frac{5}{2}$$

It follows that the possible data is empty for the group C_3^n if $n \geq 2$.

8.5.3 Powers of C_4

For this group, the table given in Remark 8.4, implies that if C_4^n satisfies **Hypothesis A** , then $n \leq 2$. Therefore, it is enough to study the case $n = 2$. We consider the inequality

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}(C_4)^2 - 2}{|C_4|^2} + 1\right) = 2\left(\frac{76^2 - 2}{384^2} + 1\right) < 2.08$$

If we consider this condition and the ones given above. Using Magma we see that the possible data is an empty set.

8.5.4 Powers of D_3

- $(n = 2)$

  We consider the inequality

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(D_3^2\right) - 2}{|D_3|^2} + 1\right) = 2\left(\frac{10^2 - 2}{24^2} + 1\right) = \frac{337}{144}$$

TABLE 8.14: Data for $(D_3)^2$ with $\gamma = 0$

Possible data
$(0; 4, 4, 6)$
$(0; 4, 4, 4)$
$(0; 3, 6, 6)$
$(0; 3, 4, 12)$
$(0; 3, 4, 6)$
$(0; 3, 4, 4)$
$(0; 2, 12, 12)$
$(0; 2, 6, 12)$
$(0; 2, 6, 6)$
$(0; 2, 4, 12)$
$(0; 2, 4, 6)$
$(0; 2, 3, 12)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 4)$
$(0; 2, 2, 2, 3)$

Real data
$(0; 2, 12, 12)$
$(0; 2, 2, 2, 3)$

The final data is the signature $(0; 2, 2, 2, 3)$ with $g = 49$. The decomposition is given by

$$E^2 \times E^2 \times E^3 \times E^3 \times E^6 \times E^6 \times E^9 \times E^9 \times E^9.$$

- $(n = 3)$ In this case, Corollary 7.5 implies that

$$\sum a_i \geq 4$$

Moreover consider the inequality

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\,(D_3^3) - 2}{|D_3|^3} + 1\right) = 2\left(\frac{10^3 - 2}{24^3} + 1\right) = \frac{7411}{3456} < 2.15.$$

Using Magma with these conditions, we see that the possible data is an empty set.

- $(n = 4, 5)$

Suppose that D_3^n is satisfying **Hypothesis A** for $n \in \{4, 5\}$. The following holds

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\,(D_3)^n - 2}{|D_3|^n} + 1\right) \leq 2\left(\frac{\mathbf{N}\,(D_3)^4 - 2}{|D_3|^4} + 1\right)$$
$$= 2\left(\frac{10^4 - 2}{24^4} + 1\right) = \frac{1351}{576} < \frac{5}{2}$$

On the other hand, Corollary 7.5 implies that

$$\sum a_i \geq n+1 \geq 5$$

Thus

$$\mathbf{S} \geq \frac{1}{2}\sum a_i \geq \frac{5}{2}$$

It follows that the possible data is empty for the group D_3^n if $n \geq 3$

8.5.5 Powers of S_3

- $(n = 2)$

  We consider the inequality

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(S_3^2\right)-2}{|S_3|^2}+1\right) = 2\left(\frac{4^2-2}{6^2}+1\right) = \frac{25}{9}$$

TABLE 8.15: Data for $(S_3)^2$ with $\gamma = 0$

Possible data
$(0;6,6,6)$
$(0;3,6,6)$
$(0;2,6,6)$
$(0;2,3,3,6)$
$(0;2,2,6,6)$
$(0;2,2,3,6)$
$(0;2,2,3,3)$
$(0;2,2,2,6)$
$(0;2,2,2,3)$
$(0;2,2,2,2,3)$
$(0;2,2,2,2,2)$

Real data
$(0;2,6,6)$
$(0;2,2,6,6)$
$(0;2,2,3,6)$
$(0;2,2,2,6)$
$(0;2,2,2,3)$
$(0;2,2,2,2,3)$
$(0;2,2,2,2,2)$

The final data is

1. $(0;2,2,3,6)$ with $g = 10$, the decomposition is given by

$$E^2 \times E^2 \times E^2 \times E^4$$

2. $(0;2,2,2,6)$ with $g = 7$, the decomposition is given by

$$E \times E^2 \times E^4$$

3. $(0;2,2,2,3)$ with $g = 4$, the decomposition is given by

$$E^2 \times E^2$$

4. $(0; 2, 2, 2, 2, 2)$ with $g = 10$, the decomposition is given by

$$E \times E \times E^2 \times E^2 \times E^4$$

- $(n = 3)$

  Corollary 7.5 implies that

$$\sum a_i \geq 4$$

  Moreover the following holds

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(S_3\right)^3 - 2}{|S_3|^3} + 1\right) = 2\left(\frac{4^3 - 2}{6^3} + 1\right) = \frac{139}{54}$$

  Using Magma, we see that the data is given as follows

TABLE 8.16: Data for $(S_3)^3$ with $\gamma = 0$

Possible data
$(0; 2, 2, 3, 6)$
$(0; 2, 2, 3, 3)$
$(0; 2, 2, 2, 6)$
$(0; 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

Real data
$(0; 2, 2, 2, 6)$ $(0; 2, 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

  The final data is the signature $(0; 2, 2, 2, 6)$ with $g = 37$ and decomposition given by

$$E \times E^2 \times E^2 \times E^2 \times E^2 \times E^4 \times E^4 \times E^4 \times E^4 \times E^4 \times E^8$$

- $(4 \leq n \leq 12)$

  Suppose that S_3^n is satisfying **Hypothesis A** for $n \in \{4, \ldots, 12\}$. Then the following holds

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(S_3\right)^n - 2}{|S_3|^n} + 1\right) \leq 2\left(\frac{\mathbf{N}\left(S_3\right)^4 - 2}{|S_3|^4} + 1\right)$$
$$= 2\left(\frac{4^4 - 2}{6^4} + 1\right) < \frac{5}{2}$$

  On the other hand, Corollary 7.5 implies that

$$\sum a_i \geq n + 1 \geq 5$$

  Thus, $\mathbf{S} \geq \frac{1}{2}\sum a_i \geq \frac{5}{2}$. It follows that the possible data is empty for the group S_3^n if $n \geq 4$.

8.5.6 Powers of S_5

- $(n = 2)$

  We consider the inequality

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(S_5\right)^2 - 2}{|S_5|^2} + 1\right) = 2\left(\frac{26^2 - 2}{120^2} + 1\right) < 2.10$$

  Using Magma we see that the possible data is given by

TABLE 8.17: Data for $(S_5)^2$ with $\gamma = 0$

Possible data
$(0; 2, 4, 6)$
$(0; 2, 4, 5)$
$(0; 2, 3, 12)$
$(0; 2, 3, 10)$

  The real and final data are empty sets.

- $(n = 3)$.

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(S_5\right)^3 - 2}{|S_5|^3} + 1\right) = 2\left(\frac{26^3 - 2}{120^3} + 1\right) < 2.03$$

  Corollary 7.5 implies that

$$\sum a_i \geq 4$$

  Using Magma we see that the possible data is an empty set.

8.5.7 Powers of G_2

For this group, the table given in Remark 8.4, implies that if G_2^n satisfies **Hypothesis A** , then $n \leq 12$. We study each case.

- $(n = 2)$

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(G_2\right)^2 - 2}{|G_2|^2} + 1\right) = 2\left(\frac{8^2 - 2}{12^2} + 1\right) = \frac{103}{36}.$$

  Moreover Corollary 7.5 implies that

$$\sum a_i \geq 2n + 1 \geq 5$$

  Using Magma we see that the possible data is given by

TABLE 8.18: Data for $(G_2)^2$ with $\gamma = 0$

Possible data
$(0; 2, 2, 2, 3, 3)$
$(0; 2, 2, 2, 2, 6)$
$(0; 2, 2, 2, 2, 3)$
$(0; 2, 2, 2, 2, 2)$

In this case, the real data and the final data are the signature $(0; 2, 2, 2, 2, 2)$ with $g = 37$ and decomposition given by

$$E \times E \times E \times E \times E \times E^2 \times E^2 \times E^2 \times E^2 \times E^2 \times E^2 \times E^2 \times E^2 \times E^4 \times E^4 \times E^4 \times E^4$$

- $(3 \leq n \leq 12)$
  The following holds

$$\mathbf{S} \leq 2\left(\frac{\mathbf{N}(G_2)^n - 2}{|G_2|^n} + 1\right) \leq 2\left(\frac{\mathbf{N}(G_2)^3 - 2}{|G_2|^3} + 1\right) = \left(\frac{8^3 - 2}{12^3} + 1\right) < 3$$

On the other hand, Corollary 7.5 implies that

$$\sum a_i \geq 7$$

Thus

$$\mathbf{S} \geq \frac{1}{2}\sum a_i \geq \frac{7}{2}$$

It follows that the possible data is empty for the group G_2^n if $n \geq 3$.

8.5.8 Powers of D_4

- $(n = 2)$

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}(D_4)^2 - 2}{|D_4|^2} + 1\right) = 2\left(\frac{44^2 - 2}{192^2} + 1\right) < 2.11$$

Using Magma, we see that the possible data is given by

TABLE 8.19: Data for $(D_4)^2$ with $\gamma = 0$

Possible data
$(0; 2, 4, 6)$
$(0; 2, 3, 12)$

In this case, the real and final data are empty sets.

- $(n = 3)$

$$\frac{85}{42} \leq \mathbf{S} \leq 2\left(\frac{\mathbf{N}\left(D_4^3\right) - 2}{|D_4|^3} + 1\right) = 2\left(\frac{44^3 - 2}{192^3} + 1\right) < 2.03$$

Corollary 7.5 implies that

$$\sum a_i \geq n + 1 \geq 4$$

We use Magma, to see that the possible data is an empty set in this case.

The following two theorem summarizes the classification problem for irreducible Weyl groups and powers of Weyl groups.

Theorem 8.6 (Classification problem for irreducible Weyl groups). *Let X be a curve of genus $\mathbf{g}$ with action of a finite irreducible Weyl Group G. Then the only cases are the following:*

TABLE 8.20: Actions where the group satisfies **Hypothesis A** .

Group	*Genus*	*Signature*	*Dimension*	*Decomposition*
C_3	3	$(0; 2, 4, 6)$	*0*	E^3
	5	$(0; 2, 2, 2, 3)$	*1*	$E^2 \times E^3$
	7	$(0; 2, 2, 2, 4)$	*1*	$E \times E^3 \times E^3$
	9	$(0; 2, 2, 2, 6)$	*1*	$E^3 \times E^3 \times E^3$
	9	$(0; 4, 4, 6)$	*0*	$E^3 \times E^3 \times E^3$
	11	$(0; 2, 2, 3, 4)$	*1*	$E^2 \times E^3 \times E^3 \times E^3$
	13	$(0; 2, 2, 4, 4)$	*1*	$E \times E^3 \times E^3 \times E^3 \times E^3$
	13	$(0; 2, 2, 3, 6)$	*1*	$E^2 \times E^2 \times E^3 \times E^3 \times E^3$
	15	$(0; 2, 2, 4, 6)$	*1*	$E \times E^2 \times E^3 \times E^3 \times E^3 \times E^3$
	17	$(1; 3)$	*1*	$E \times E^2 \times E^2 \times E^3 \times E^3 \times E^3 \times E^3$
C_4	17	$(0; 2, 4, 6)$	*0*	$E^3 \times E^6 \times E^8$
$D_3 \cong S_4$	3	$(0; 3, 4, 4)$	*0*	E^3
	3	$(0; 2, 2, 2, 3)$	*1*	E^3
	4	$(0; 2, 2, 2, 4)$	*1*	$E \times E^3$

Continued on next page

Table 8.20 – continued from previous page

Group	Genus	Signature	Dimension	Decomposition
	5	$(0; 2, 2, 3, 3)$	1	$E^2 \times E^3$
	6	$(0; 2, 2, 3, 4)$	1	$E^3 \times E^3$
	7	$(0; 2, 2, 4, 4)$	1	$E \times E^3 \times E^3$
	8	$(0; 2, 3, 3, 4)$	1	$E^2 \times E^3 \times E^3$
	9	$(1; 3)$	1	$E \times E^2 \times E^3 \times E^3$
D_4	17	$(0; 3, 4, 4)$	0	$E^3 \times E^6 \times E^8$
D_5	49	$(0; 2, 4, 5)$	0	$E^4 \times E^{10} \times E^{15} \times E^{20}$
S_3	2	$(0; 2, 2, 3, 3)$	1	E^2
	3	(0;2,2,2,2,3)	2	$E \times E^2$
	3	(1;3)	1	$E \times E^2$
	4	(1;2,2)	2	$E \times E \times E^2$
S_5	4	(0;2,4,5)	0	E^4
	6	(0;2,4,6)	0	E^6
	9	(0;2,5,6)	0	$E^4 \times E^5$
	11	(0;2,6,6)	0	$E^5 \times E^6$
	11	(0;3,4,4)	0	$E^5 \times E^6$
	16	(0;3,4,6)	0	$E^5 \times E^5 \times E^6$
	19	(0;4,4,5)	0	$E^4 \times E^4 \times E^5 \times E^6$
	24	(0;4,5,6)	0	$E^4 \times E^4 \times E^5 \times E^5 \times E^6$
G_2	2	$(0; 2, 2, 2, 3)$	1	E^2
	3	$(0; 2, 2, 2, 6)$	1	$E \times E^2$
	4	$(0; 2, 2, 3, 6)$	1	$E^2 \times E^2$
	4	$(0; 2, 2, 2, 2, 2)$	2	$E \times E \times E^2$
	5	$(0; 2, 2, 2, 2, 3)$	2	$E \times E^2 \times E^2$
	5	$(0; 2, 2, 6, 6)$	1	$E \times E^2 \times E^2$

Continued on next page

Table 8.20 – continued from previous page

Group	Genus	Signature	Dimension	Decomposition
	6	$(0;2,2,2,2,6)$	2	$E \times E \times E^2 \times E^2$
	5	$(1;3)$	1	$E \times E^2 \times E^2$
	7	$(1;2,2)$	2	$E \times E \times E \times E^2 \times E^2$

Remark 8.7. Each term E^n in the decomposition column denotes an isotypical factor, therefore each of these factors should be indexed by a rational representation of the corresponding groups. However we omitted indexes since some of these groups have many of them. We emphasize that each E^n denotes a different isotypical factor and therefore they are not necessarily isogenous to each other.

Theorem 8.8 (Classification problem for powers of Weyl groups)**.** *Let X be a curve of genus **g** with action of a power of a finite irreducible Weyl Group. Then the only cases are the following:*

1. *For the group D_3^2, the final data is given by the signature $(0;2,2,2,3)$ with $g = 49$ and decomposition given by*

$$E^2 \times E^2 \times E^3 \times E^3 \times E^6 \times E^6 \times E^9 \times E^9 \times E^9$$

2. *For the group S_3^3, the final data is given by the signature $(0;2,2,2,6)$ with $g = 37$ and decomposition given by*

$$E \times E^2 \times E^2 \times E^2 \times E^2 \times E^4 \times E^4 \times E^4 \times E^4 \times E^4 \times E^8$$

3. *For the group G_2^2, the final data is given by the signature $(0;2,2,2,2,2)$ with $g = 37$ and decomposition given by*

$$E \times E \times E \times E \times E \times E^2 \times E^2 \times E^2 \times E^2 \times E^2 \times E^2$$
$$\times E^2 \times E^2 \times E^4 \times E^4 \times E^4 \times E^4$$

4. *For the group S_3^2, the final data is given in the following table*

Group	Genus	Signature	Decomposition
S_3^2	4	$(0;2,2,2,3)$	$E^2 \times E^2$
	7	$(0;2,2,2,6)$	$E \times E^2 \times E^4$
	10	$(0;2,2,2,2,2)$	$E^2 \times E^2 \times E^2 \times E^4$
	10	$(0;2,2,3,6)$	$E \times E \times E^2 \times E^2 \times E^4$

Appendix A

Haskell codes

```
import Data.List

quicksort :: (Ord a) => [a] -> [a]
quicksort [] = []
quicksort (x:xs) =
    let smallerSorted = quicksort [a | a <- xs, a <= x]
        biggerSorted = quicksort [a | a <- xs, a > x]
    in  smallerSorted ++ [x] ++ biggerSorted


partitions' k n
   | k>n = 0
   | k==n = 1
   | otherwise = partitions' (k+1) n + partitions' k (n-k)

temporal = partitions' 1


partitions n = if n == 0 then 0 else temporal n


p n = partitions n


_partitions' k n
   | k>n = []
   | k==n = [[n]]
   | otherwise = [k:i | i<-(_partitions' k (n-k))] ++
   _partitions' (k+1) n_partitions = _partitions' 1


_p n = _partitions n


factorial n = product [1..n]


someDivs' k n
   | kk == n = [k]
   | kk < n && n`mod`k==0 = k:n`quot`k:(someDivs' (k+1) n)
   | kk > n = []
   | otherwise = someDivs' (k+1) n
   where kk = k*k

someDivs = someDivs' 1
```

```
conjweyl n = sum [(p i) * (p (n-i)) | i<-[0..n]]+ 2*(p n)


removeDuplicates :: Eq a => [a] -> [a]
removeDuplicates = rdHelper []
    where rdHelper seen [] = seen
          rdHelper seen (x:xs)
              | x `elem` seen = rdHelper seen xs
              | otherwise = rdHelper (seen ++ [x]) xs


ordersS n = quicksort (removeDuplicates [ product x | x <- _p n,
product x >1])


ordersC n = quicksort (removeDuplicates([ product x | x <- _p n,
product x >1] ++ [ 2*product x | x <- _p n, product x >1]))


orderstC n = map (truncate) (ordersC n)


orderstS n = map (truncate) (ordersS n)


facts n = take n (zipWith (/) [1..] [2..])


crossPow x n = sequence (replicate n x)


dotProd a b = sum (zipWith (*) a b)


vectsh n = map (\x -> (x-1)/x) (ordersC n)


sh x n = dotProd (x) (vectsh n)


par x = if odd x == True then 0 else 1


pard x = if odd x == True then 0.0 else 1.0


paresC n = map (pard) (orderstC n)


filterpro x = if ((dotProd x [1])==2 && (sum x) == 3) then
True else False


limit :: (Integral a) => a -> [a]
```

```
limit 3 = [0..6]
limit 4 = [0..6]
limit 5 = [0..6]
limit x = [0..2]


limitv :: (Integral a) => a -> a
limitv 3 = 6
limitv 4 = 6
limitv 5 = 6
limitv x = 6


bound :: (Integral a) => a -> Double
bound 3 = 2.75
bound 4 = 2.39
bound 5 = 2.17
bound x = 2.06


inequality n = 2*84* 84 * (conjweyl n) - (2^(n-1)* factorial n)


proj i x = head (drop (i-1) x)


dimlim x = truncate (2*x)


pares x = map (pard) x


duplic x y = take x (repeat y)


gen x y = concat(zipWith (duplic) (map (truncate) (x))
(map (truncate) (y)))


tester (a,b,i,d) = [ gen x a | x <-crossPow [0..i] (length a) ,
(dotProd (x) (d)) <b, (dotProd (x) (d))>85/42, (sum x) > 2,
(dotProd x (pares a))>1]
```

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