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Pre-Publicación MATUC - 2017 -21

GOE STATISTICS FOR ANDERSON MODELS ON ANTITREES AND THIN BOXES IN Z^3 WITH DEFORMED LAPLACIAN

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GOE STATISTICS FOR ANDERSON MODELS ON ANTITREES AND THIN BOXES IN $\mathbb{Z}^3$ WITH DEFORMED LAPLACIAN

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Abstract. Sequences of certain finite graphs - special types of antitrees - are constructed along which the Anderson model shows GOE statistics, i.e. a re-scaled eigenvalue process converges to the Sine_1 process. The Anderson model on the graph is a random matrix being the sum of the adjacency matrix and a random diagonal matrix with independent identically distributed entries along the diagonal. The strength of the randomness stays fixed, there is no re-scaling with matrix size. These considered random matrices giving GOE statistics can also be viewed as random Schrödinger operators $\mathcal{P}\Delta + \mathcal{V}$ on thin finite boxes in $\mathbb{Z}^3$ where the Laplacian Δ is deformed by a projection $\mathcal{P}$ commuting with Δ.

Contents

1. Introduction 1
1.1. The considered graphs and related random matrices 3
1.2. Main results 5
2. Transfer matrices 7
3. The spectrum of $H^w_{n,r,s}$ 9
4. Effective energy, effective potential and elliptic channels 12
5. The limit of thin boxes with fixed width 14
6. The GOE limit 17
Appendix A. Harmonic means of random variables 19
References 20

1. Introduction

In the theory of randomly disordered systems there are two very important models. One is the Anderson model introduced in [And] and the other one are random matrix ensembles such as the Gaussian Orthogonal Ensemble (GOE) introduced by Wigner [Wig]. The latter one model the observed repulsion between energy levels (eigenvalues) in large nuclei. This is characterized by the local eigenvalue statistics which for the GOE is given by the Sine_1 process in the limit where the matrix size goes to infinity, see e.g. [Meh]. This type of limiting statistics is expected for a wide range of disordered systems of the same symmetry class which is referred to as universal behavior or simply *universality*. The GOE statistics

2010 *Mathematics Subject Classification.* Primary 60B20, Secondary 82B44, 60H25, 15B52.

Key words and phrases. universality, local statistics, Anderson model.

This research has received funding from the Chilean CONICYT under grant Fondecyt Regular 1161651 and from the Iniciativa Científica Milenio NC120062 .

applies to models with time reversal symmetry. Without time reversal symmetry (for instance in presence of magnetic phases) disordered systems are expected to follow the local statistics of the Gaussian Unitary Ensemble (GUE).

Universality has been proved for many random matrix ensembles, e.g. [DG, ESY, TV] and particularly very recent works by Ajanki, Erdös, Krüger and Schröder allow very general profiles of covariance structures and dependencies with slow correlation decay of the random entries [AEK, EKS]. However, these ensembles are still far from ensembles of very sparse matrices (many zero entries) or matrices with randomness only along the diagonal, both of which apply for Anderson models.

The Anderson model is supposed to describe random media like doped semi-conductors. Unlike the random matrix ensembles, here one typically considers operators on an infinite dimensional separable Hilbert space, typically $\ell^2(\mathbb{Z}^d)$ or $\ell^($\mathbb{G}$) for some countable graph $\mathbb{G}$ and the random entries just appear on the diagonal. This means that the model is given by a sum $\Delta + \mathcal{V}$ where $\mathcal{V}$ is a real random diagonal multiplication operator (in the canonical basis) with independent identically distributed entries along the diagonal and Δ a graph-Laplacian or adjacency operator. The physically most important models are given by random potential plus the discrete Laplacian on $\mathbb{Z}^d$, $d = 1, 2, 3$. There are also continuous analogues defining Anderson models on $L^2(\mathbb{R}^d)$ where Δ is the actual Laplacian on $\mathbb{R}^d$ and $\mathcal{V}$ random multiplication operator made out of random bump like potentials centered around lattice points and multiplied by i.i.d. real random variables.

For Anderson models one can also consider eigenvalue statistics if one restricts the model to finite boxes in $\mathbb{Z}^d$ or the considered graph $\mathbb{G}$ (finite sub-graphs) along sequences approaching the infinite graph. Restricting the Anderson model to a finite box gives a random matrix. However, the sequences of such random matrices are very different from the random matrix ensembles mentioned above. The random entries are only on the diagonal and the variances are constant and not re-scaled with size of the matrix. The off-diagonal entries are typically very sparse[1] describing the graph structure (edges, edge-weights).

For one and quasi-one dimensional models, e.g. [GMP, KuS, KLS] and for large disorder or at band edges in any dimension, e.g. [FS, AM, DLS, Klo] the Anderson model localizes. This means one has pure point spectrum and exponentially localized eigenfunctions, a phenomenon called Anderson localization. In regimes of Anderson localization one finds Poisson type statistics [Min, Wan, GK].

While there is a huge literature on Anderson localization, there is still a mayor open problem concerning delocalization. For Anderson models of low disorder on $\mathbb{Z}^d$ for dimension $d \geq 3$ it is expected that some absolutely continuous spectrum persists[2]. Moreover, in these delocalized regimes one also expects some form of universality (GOE statistics) for the eigenvalue statistics along increasing boxes approaching $\mathbb{Z}^d$. However, so far even the existence of this delocalized regime for models on $\mathbb{Z}^d$ is mathematically unproven.

Existence of a delocalized regimes for Anderson models was first shown for infinite dimensional regular trees (Bethe lattice) and then extended to similar tree like structures and tree-strips [Kle, ASW, FHS, KLW, AW, KS, Sa1, Sa2]. Only recently some examples of graphs with finite d-dimensional growth rate ($d > 2$) have been introduced with rigorous

[1] meaning most off diagonal entries are zero, the non-zero entries are sparse

[2] The non-disordered Laplacian has purely absolutely continuous spectrum

proof of absolutely continuous spectrum for Anderson models on them, so called antitrees and similar structures [Sa3, Sa4]. The word antitree describes that these graphs are far from trees as they have some local complete-graph-like structures in them. These can be viewed as local mean-field structures which give a local averaging effect on the random potentials preventing localization.

A connection from Anderson models to GOE statistics has been found by considering long strips within $\mathbb{Z}^2$ and re-scaling the random potential. Originally one also had to modify the Laplacian slightly [VV] which was later resolved in [SV]. In this paper we combine methods from [Sa3] with [VV, SV] to construct examples of sequences of Anderson models on finite graphs with fixed disorder strength that show GOE statistic in the limit. An additional re-scaling of the randomness as in [SV, VV] s not needed. As their construction is very similar to the antitrees in [Sa3] we call them antitrees as well. However, as we use some finite base graph $\mathbb{G} \subset \mathbb{Z}^2$ to construct them we call them generally $\mathbb{G}$-antitrees. The locally averaging graph structure of the antitree part replaces the re-scaling of the randomness in [VV, SV]. In some sense this sequence of considered models lies in between the theory of random band matrix ensembles and the Anderson models on $\mathbb{Z}^d$.

1.1. **The considered graphs and related random matrices.** Let us introduce more precisely the graph structures we will consider.

Definition 1. a) A discrete weighted graph $(\mathbb{G}, W)$ is a countable or finite set $\mathbb{G}$ together with a symmetric, real valued weight function $W : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$. Two distinct points $x \neq y \in \mathbb{G}$ are considered to be connected by an edge if and only if $W(x,y) \neq 0$ in which case $W(x,y) = W(y,x) \in \mathbb{R}$ is the edge weight. The diagonal elements $W(x,x)$ will be referred to as point weights. One may think of W as a real symmetric matrix with indexed by points in $\mathbb{G}$. This is the adjacency matrix of the weighted graph.
b) Given a weighted graph $(\mathbb{G}, W)$ and a function $s : \mathbb{G} \to \mathbb{Z}_+^*$ with values inside the strictly positive integers, we define the $\mathbb{G}$-*antitree* with re-normalized weights as the following weighted graph $(\mathbb{A}_{\mathbb{G}}, \mathcal{A})$

$$\mathbb{A}_{\mathbb{G}} := \big\{(x,j) : x \in \mathbb{G},\, j \in \{1,2,\dots,s(x)\}\big\}, \quad \mathcal{A}\big((x,j),(y,k)\big) := W(x,y)/\sqrt{s(x)\,s(y)}$$

If $s(x) = s \in \mathbb{Z}_+^*$ is constant, then we call it the s-fibered $\mathbb{G}$-antitree.

This defintion is a generalization of the antitrees we worked with in [Sa3], in fact the graphs there would all be $\mathbb{Z}_+$-antitrees where $\mathbb{Z}_+$ is the half line of positive integers with edges only between neighbors.

Let us explain the re-normalization. It is a re-normalization with respect to changing the values $s(x)$ as enlarging $s(x)$ significantly enlarges the number of edges locally. Now, in the Hilbert space $\ell^2(\mathbb{A}_{\mathbb{G}})$, the function

$$1_x(y,k) = \begin{cases} 0 & \text{if } x \neq y \\ 1/\sqrt{s(x)} & \text{if } x = y \end{cases}$$

is a vector of unit length. The δ functions $\delta_x(y) = 0$ for $y \neq x$ and $\delta_x(x) = 1$ are unit vectors in $\ell^2(\mathbb{G})$. Then, with $\langle \cdot, \cdot \rangle$ denoting the standard scalar products, we have

$$\langle 1_x, \mathcal{A}\, 1_y \rangle = W(x,y) = \langle \delta_x, W\, \delta_y \rangle$$

Moreover, it is easy to check that the Hilbert sub-space spanned by the unit vectors 1_x, $x \in \mathbb{G}$ is left invariant by $\mathcal{A}$. Hence, the adjacency matrix $\mathcal{A}$ of $\mathbb{A}_{\mathbb{G}}$ reduces to the adjacency matrix W of $\mathbb{G}$ on this Hilbert sub-space.

We will consider antitrees over (long) two-dimensional strips which can also be obtained through some deformation of the Laplacian of 3-dimensional (thin) boxes. Let $n \geq r$, the $n \times r$ strip $\mathbb{Z}_{n\times r}$ with point weight w is the set $\{1,\dots,n\} \times \{1,\dots,r\} \subset \mathbb{Z}^2$ with weight function

$$W(x,y) = \begin{cases} 0 & \text{if } \|x-y\|_1 > 1 \\ 1 & \text{if } \|x-y\|_1 = 1 \\ w & \text{if } x = y \end{cases}$$

The corresponding s-fibered $\mathbb{Z}_{n\times r}$ antitree with re-normalized weights (with respect to W above) shall be denoted by $\mathbb{A}^w_{n\times r,s}$ and the $nrs \times nrs$ adjacency operator correspondingly by $\mathcal{A}^w_{n\times r,s}$. To represent it in matrix form, we will split the $nrs \times nrs$ matrix $\mathcal{A}^w_{n\times r,s}$ into $mr \times mr$ blocks and each of these blocks is split into $s \times s$ blocks. Identifying the base space with

$$\mathbb{Z}_{n\times r\times s} = \{(x_1,x_2,x_3) \in \mathbb{Z}^3,\ 1 \leq x_1 \leq n,\ 1 \leq x_2 \leq r,\ 1 \leq x_3 \leq s\}\ .$$

$\mathcal{A}^w_{n\times r,s}$ can be considered as an operator on $\ell^2(\mathbb{Z}_{n\times r\times s})$ and we use the canonical orthonormal basis (δ_{x_1,x_2,x_3}) with lexicographical order to represent $\mathcal{A}^w_{n\times r,s}$ as a matrix. We start with the mean-field vector

$$1_s := \frac{1}{\sqrt{s}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^s \quad \text{and let} \quad P_s := 1_s 1_s^\top = \frac{1}{s} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{s\times s} \tag{1.1}$$

be the mean-field projection. Then define the $rs \times rs$ matrices with $s \times s$ blocks

$$P_{r,s} := \mathbb{I}_r \otimes P_s = \begin{pmatrix} P_s & & \\ & \ddots & \\ & & P_s \end{pmatrix} \quad \text{and} \quad A^w_{r,s} := \begin{pmatrix} wP_s & P_s & & \\ P_s & \ddots & \ddots & \\ & \ddots & \ddots & P_s \\ & & P_s & wP_s \end{pmatrix}. \tag{1.2}$$

Here, $\mathbb{I}_r$ is the $r \times r$ identity matrix and $A^w_{r,s}$ is block-tri-diagonal with $s \times s$ blocks. Then, finally we have $\mathcal{A}^w_{n\times r,s}$ as a block-tri-diagonal $nrs \times nrs$ matrix structured in $rs \times rs$ blocks

$$\mathcal{A}^w_{n\times r,s} = \begin{pmatrix} A^w_{r,s} & P_{r,s} & & \\ P_{r,s} & A^w_{r,s} & \ddots & \\ & \ddots & \ddots & P_{r,s} \\ & & P_{r,s} & A^w_{r,s} \end{pmatrix} \tag{1.3}$$

Let be given a probability distribution ν on $\mathbb{R}$. The *Anderson type model* on $\mathcal{A}^w_{m,\times r}$ with single site distribution ν is given by the random real symmetric matrix

$$H^w_{n,r,s} := \mathcal{A}^w_{n\times r,s} + \mathcal{V}_{nrs} \tag{1.4}$$

where $\mathcal{V}_{nrs}$ is a $nrs \times nrs$ real diagonal matrix with independent identically ν-distributed random variables along the diagonal. We will assume that the distribution is compactly supported. This is a family of random band-matrices with randomness only on the diagonal, size $N = nrs$, band-width $2rs$ and sparse structure in the entries, but with some local mean-field setup within groups of $s \times s$ blocks.

1.2. **Main results.**

Assumption. (A1) *We assume that the distribution ν of the single site potential (diagonal entries of $\mathcal{V}_{nrs}$) is compactly supported, say in the interval $[-\sigma,\sigma]$.*
(A2) *Furthermore let us assume that the distribution is centered, $\mathbb{E}(v) := \int v\, d\nu(v) = 0$, and non-trivial $\mathbb{E}(v^2) = \int v^2\, d\nu(v) > 0$.*

We will often need averaged quantities over the distribution ν or products thereof $\nu^{\otimes k}$ as the diagonal entries of $\mathcal{V}_{nrs}$ are all independently ν distributed. When the random variables and their dependence on these entries are clear, we will denote the expectations values by $\mathbb{E}$. Furthermore, in these expressions a variable v will express a ν-distributed independent random variable.

Now let us introduce the harmonic mean

$$h_\lambda := \mathbb{E}\left(\left(\frac{1}{\lambda - v}\right)^{-1}\right) = \int \left(\frac{1}{\lambda - v}\right)^{-1} d\nu(v) \tag{1.5}$$

and define the interval

$$I_{w,\nu} := \{\lambda \in \mathbb{R} : |\lambda| > \sigma \quad \text{and} \quad |h_\lambda - w| < 4\}\,. \tag{1.6}$$

The harmonic-mean to arithmetic-mean inequality gives $|h_\lambda| < |\lambda|$ for $|\lambda| > \sigma$ and we find

$$[-4+w, 4+w] \setminus [-\sigma,\sigma] \subset I_{w,\nu}\,. \tag{1.7}$$

So for small σ or large $|w|$ the set $I_{w,\nu}$ will not be empty and a union of two intervals.

Theorem 1. *Let $H^w_{n,r,s}$ be the Anderson models on the antitree $\mathbb{A}^w_{n\times r,s}$ with single site distribution ν under the assumptions* (A1) *and* (A2)*. For almost any $\lambda \in I_{w,\nu}$ there exist sequences $s_k \gg n_k \gg r_k \to \infty$, and normalization constants $\mathcal{N}_k$ such that*

$$\mathcal{N}_k \operatorname{spec}(H^w_{n_k,r_k,s_k} - \lambda) \Rightarrow \ \mathrm{Sine}_1 \quad \text{for } k \to \infty$$

The growth of $s_k/n_k \to \infty$ and $r_k \to \infty$ can be chosen as slow as one wants, meaning that for any increasing function $f(n)$ growing towards infinity one finds sequences s_k, n_k, r_k satisfying this limit with $s_k/n_k < f(n_k)$ and $r_k < f(n_k)$.

Remark. (i) *The spectrum* $\operatorname{spec}(H^w_{n_k,r_k,s_k} - \lambda)$*, i.e. the eigenvalues of $H^w_{n_k,r_k,s_k} - \lambda$ are considered as a random point process and the convergence holds in the sense of a weak limit of random point processes.*

(ii) *Let $d((x,i),(y,j))$ denote the step distance, i.e. the number of steps needed to go from (x,i) to (y,j) in $\mathbb{A}^w_{n\times r,s}$ along edges, where $(x,y) \in \mathbb{Z}_{n\times r}$, $i,j \in \{1,\dots,s\}$. Then, assuming $w \neq 0$ we obtain $d((x,i),(y,j)) = \|x-y\|_1$ for $x \neq y$ or $i = j$ and $d((x,i),(y,j)) = 1$ for $x = y$ and $i \neq j$. Therefore, the graph-diameter is* $\mathrm{diam} = n+r-1$ *(like on the $n\times r$ grid) in the case of $r > 1$ where as the volume (number of vertices) is* $\mathrm{vol} = nrs$*. We may consider sequences such that $n_k \log n_k > s_k > n_k$ and $r_k < \log n_k$ for which we find a logarithmic volume to diameter ratio of* $\log(\mathrm{vol}_k)/\log(\mathrm{diam}_k) \to 2$ *as for boxes approximating 2-dimensions. However, since* $\mathrm{vol}_k \gtrsim \mathrm{diam}_k^2 \log(\mathrm{diam}_k)$ *the growths in k can be viewed to be slightly bigger than 2-dimensional. For resembling d dimensions you expect the volume (number of points) to grow like* diam^d*.*

Let us go back to blocks in $\mathbb{Z}^3$ and consider the set $\mathbb{Z}_{n\times r\times s}$ as introduced above, a $n \times r \times s$ grid within $\mathbb{Z}^3$. Now let us introduce the discrete Laplacian $\Delta_{n,r,s}$ on $\mathbb{Z}_{n\times r\times s}$ but

with periodic boundary conditions in the last coordinate direction. For the other directions we use Dirichlet boundary conditions. This corresponds to introducing an additional edge from points $(x_1, x_2, 1)$ to (x_1, x_2, s) for any x_1, x_2. All the edges get weight one and we have no point weights. This means the matrix (or weight function) associated to $\Delta_{n,r,s}$ is given by

$$\langle \delta_x, \Delta_{n,r,s}\, \delta_y \rangle = \Delta_{m,n,r}(x,y) = \begin{cases} 1 & \text{if } \|x-y\|_1 = 1 \\ 1 & \text{if } \{x_3, y_3\} = \{1, s\} \text{ and } (x_1, x_2) = (y_1, y_2) \\ 0 & \text{else} \end{cases}$$

Using the same basis structure as before we obtain

$$\Delta_{n,r,s} = \begin{pmatrix} \Delta_{r,s} & \mathbb{I}_{rs} & & \\ \mathbb{I}_{rs} & \Delta_{r,s} & \ddots & \\ & \ddots & \ddots & \mathbb{I}_{rs} \\ & & \mathbb{I}_{rs} & \Delta_{r,s} \end{pmatrix}$$

where in general $\mathbb{I}_m$ will denote the $m \times m$ identity matrix and $\Delta_{r,s}$ is an $rs \times rs$ tri-diagonal block matrix made of $s \times s$ blocks given by

$$\Delta_{r,s} = \begin{pmatrix} \Delta_s^p & \mathbb{I}_s & & \\ \mathbb{I}_s & \Delta_s^p & \ddots & \\ & \ddots & \ddots & \mathbb{I}_{rs} \\ & & \mathbb{I}_s & \Delta_s^p \end{pmatrix} \quad \text{where} \quad \Delta_s^p = \begin{pmatrix} 0 & 1 & 0 & & 1 \\ 1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ 1 & & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{s \times s}\,.$$

Because of the periodic boundary condition in the third coordinate we use the superscript 'p' for Δ_s^p. This periodicity is reflected by the top-right and bottom-left entry 1 in Δ_s^p. This Laplacian commutes with the orthogonal projection $\mathcal{P}$ onto the functions which are constant along the third coordinate direction, i.e.

$$\mathcal{P}\Delta_{n,r,s} = \Delta_{n,r,s}\mathcal{P} \quad \text{where} \quad \mathcal{P}\psi(x_1, x_2, x_3) = \frac{1}{s}\sum_{k=1}^{s} \psi(x_1, x_2, k)\,.$$

In matrix form using $s \times s$ blocks we have the block structure $\mathcal{P} = \mathrm{diag}(P_s, \dots, P_s)$ with nr such blocks. Using $\Delta_s 1_s = 2 \cdot 1_s$ implying $P_s \Delta_s^p = \Delta_s^p P_s = 2P_s$ and $\mathbb{I}_s P_s = P_s \mathbb{I}_s = P_s$ we find

$$\mathcal{P}\Delta_{n,r,s} = \mathcal{P}\Delta_{n,r,s}\mathcal{P} = \mathcal{A}^2_{n\times r,s}$$

the adjacency matrix of the s-fibered $\mathbb{Z}_{n\times r}$-antitree where we take the point weight $w = 2$ on $\mathbb{Z}_{n\times r}$. This leads to the following corollary:

Theorem 2. *For almost all $\lambda \in I_{2,\nu}$, in particular almost all energies $\lambda \in [-2, -\sigma) \cup (\sigma, 6]$, there are sequences $s_k \gg n_k \gg r_k \to \infty$ ($r_k \to \infty$, $s_k/n_k \to \infty$ can grow as slow as wanted in comparison to the growth of n_k) such that with the correct normalization $\mathcal{N}_k$ we find*

$$\mathcal{N}_k\, \mathrm{spec}(\mathcal{P}\Delta_{m_k,n_k,r_k} + \mathcal{V}_{m_k n_k r_k} - \lambda) \Rightarrow \mathrm{Sine}_1\,.$$

Remark. (i) *The Laplacian Δ on $\mathbb{Z}^3$ or $\mathbb{N}^3$ can be seen as some sort of limit of $\Delta_{n,r,s}$ for $n, r, s \to \infty$ and has spectrum $[-6, 6]$. Indeed, for any $n, r, s \in \mathbb{N}$ we have $\mathrm{spec}\,\Delta_{n,r,s} \subset [-6, 6]$. However the projection $\mathcal{P}$ reduces to the subspace with top energy for the Laplacian in the x_3 direction leading to $\mathrm{spec}\,\mathcal{P}\Delta_{n,r,s} \subset [-2, 6]$ and for $n, r, s \to \infty$ one fills this interval.*

(ii) *This result can not be seen as a limiting statistics for boxes on some fixed Anderson*

model on a separable, infinite dimensional Hilbert space because there is no limit of the projections $\mathcal{P} = \mathcal{P}(n,r,s)$ in $\ell^2(\mathbb{Z}^3)$ and there is no operator limit of $\mathcal{P}\Delta_{n,r,s}$ for boxes of size $n \times r \times s$ approaching $\mathbb{Z}^3$.

(iii) *With s_k/n_k and r_k growing very slowly, less than any power of n_k, the corresponding subset $\mathbb{Z}_{n_k,r_k,s_k}$ of $\mathbb{Z}^3$ look like very thin rectangular shaped boxes in $\mathbb{Z}^3$.*

2. Transfer matrices

As P_s has rank one it is easy to see that $P_{r,s}$ has rank r and it is an orthogonal projection matrix. Therefore, using the block structure of $\mathcal{A}^w_{n\times r,s}$ in $rs\times rs$ blocks it can be regarded as an r-propagating-channel operator similar like the one-channel and one-propagating-channel operators defined in [Sa3, Sa4]. In particular we will have some transfer matrix structure like in [Sa3]. Let us identify $\psi \in \mathbb{C}^{nrs} = (\mathbb{C}^{rs})^n$ with $(\psi_i)_{i=1}^n$, $\psi_i \in \mathbb{C}^{rs}$ and let $\psi_0 = \vec{0} = \psi_{n+1}$. Moreover, let us write the random diagonal matrix $\mathcal{V}_{nrs}$ in diagonal block form of n blocks of size rs, $\mathcal{V}_{nrs} = \mathrm{diag}(V_1, V_2, \dots V_n)$. Then, we find

$$\left(H^w_{n,r,s}\psi\right)_i \;=\; P_{r,s}(\psi_{i+1} + \psi_{i-1}) \;+\; (A^w_{r,s} + V_i)\,\psi_i\;.$$

By definition of the random diagonal matrix $\mathcal{V}_{nrs}$, the V_i are i.i.d. random diagonal $rs\times rs$ matrices where each V_i has diagonal entries that are all i.i.d. real random variables and ν-distributed. The projection $P_{r,s}$ can be written as

$$P_{r,s} = \Phi_{r,s}\Phi_{r,s}^\top \quad \text{where} \quad \Phi_{r,s} = \begin{pmatrix} 1_s & & 0 \\ & \ddots & \\ 0 & & 1_s \end{pmatrix} \in \mathbb{R}^{rs\times r} \tag{2.1}$$

is a $rs \times r$ matrix. Recall that $1_s \in \mathbb{C}^s$ is the normalized 'mean-field column vector' $1_s = 1/\sqrt{s}\,(1,1,\dots,1)^\top$. Note that $\Phi_{r,s}^\top\Phi_{r,s} = \mathbb{I}_r$.

For $\psi = (\psi_i)_{i=1}^n \in \mathbb{C}^{nrs}$, $\psi_i \in \mathbb{C}^{rs}$ let us define

$$\vec{u}_i = \vec{u}_i(\psi) := \Phi_{r,s}^\top\psi_i \;\in\; \mathbb{C}^r\;, \tag{2.2}$$

then, the eigenvalue equation $H^w_{n,r,s}\psi = z\psi$ gives

$$(z - A^w_{r,s} - V_i)\psi_i \;=\; \Phi_{r,s}\,(\vec{u}_{i+1} \,+\, \vec{u}_{i-1})\;. \tag{2.3}$$

For $z \notin \mathrm{spec}(A^w_{r,s} - V_i))$ it follows that

$$\vec{u}_i \;=\; \Phi_{r,s}^\top\,(z - A^w_{r,s} - V_i)^{-1}\,\Phi_{r,s}\,(\vec{u}_{i+1} \,+\, \vec{u}_{i-1})\;. \tag{2.4}$$

The matrix $\Phi_{r,s}^\top\,(z - A^w_{r,s} - V_i)^{-1}\,\Phi_{r,s}$ is invertible for $z \notin \mathbb{R}$ by definiteness of the imaginary part. Hence, it is defined and invertible for all but finitely many values of $z \in \mathbb{R}$. If it is invertible we can re-write the eigenvalue equation in the following form,

$$\begin{pmatrix} \vec{u}_{i+1} \\ \vec{u}_i \end{pmatrix} = T^{w,z}_{i;r,s}\begin{pmatrix} \vec{u}_i \\ \vec{u}_{i-1} \end{pmatrix}, \quad T^{w,z}_{i;r,s} := \begin{pmatrix} \left(\Phi_{r,s}^\top\,(z\mathbb{I}_{rs} - A^w_{r,s} - V_i)^{-1}\,\Phi_{r,s}\right)^{-1} & -\mathbb{I}_r \\ \mathbb{I}_r & \mathbf{0} \end{pmatrix}. \tag{2.5}$$

We call $T^{w,z}_{i;r,s}$ the i-th transfer matrix at energy z of $\mathcal{A}^w_{n\times r,s}$. We write the energy or spectral parameter z as an upper index because the dependence on z is somewhat of the same flavor as the one on w.

Now let Q_s be an $s\times(s-1)$ matrix such that $(1_s, Q_s)$ is orthogonal, meaning that

$$1_s 1_s^\top \,+\, Q_s Q_s^\top \,=\, \mathbb{I}_s \qquad 1_s^\top Q_s = \mathbf{0}\;.$$

Then, we let

$$Q_{r,s} := \begin{pmatrix} Q_s & & \\ & \ddots & \\ & & Q_s \end{pmatrix} \quad \text{implying} \quad \Phi_{r,s}\Phi_{r,s}^\top + Q_{r,s}Q_{r,s}^\top = \mathbb{I}_{rs} \, .$$

Hence, $(\Phi_{r,s}, Q_{r,s})$ is orthogonal. If M is an invertible $rs \times rs$ matrix where $\Phi_{r,s}^\top M \Phi_{r,s}$ is also invertible then the Schur complement formula gives that

$$\left(\Phi_{r,s}^\top M^{-1} \Phi_{r,s}\right)^{-1} = \Phi_{r,s}^\top M \, \Phi_{r,s} - \Phi_{r,s}^\top M \, Q_{r,s} \left(Q_{r,s}^\top M \, Q_{r,s}\right)^{-1} Q_{r,s}^\top M \, \Phi_{r,s} \, . \tag{2.6}$$

Applying this to $M = z\mathbb{I}_{rs} - A_{r,s}^w - V_k$ and using $P_s Q_s = \mathbf{0}$, $A_{r,s}^w Q_{r,s} = \mathbf{0}$ we obtain

$$\left(\Phi_{r,s}^\top \left(z\mathbb{I}_{rs} - A_{r,s}^w - V_i\right)^{-1} \Phi_{r,s}\right)^{-1} = -\Phi_{r,s}^\top A_{r,s}^w \Phi_{r,s} + \left(\Phi_{r,s}^\top \left(z\mathbb{I}_{rs} - V_i\right)^{-1} \Phi_{r,s}\right)^{-1} \, . \tag{2.7}$$

Using $1_s^\top P_s 1_s = 1$, (1.2) and (2.1) we get

$$\Phi_{r,s}^\top A_{r,s}^w \Phi_{r,s} = \Delta_r^D + w \, \mathbb{I}_r$$

where Δ_r^D is the Dirichlet Laplacian on the line $\mathbb{Z}_r = \{1, \ldots, r\}$ which is slightly different from the periodic one Δ_s^p used above,

$$\Delta_r^D := \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix} .$$

The diagonal matrix V_k can be further partitioned into $s \times s$ blocks to obtain

$$\left(\Phi_{r,s}^\top (z\mathbb{I}_{rs} - V_i)^{-1} \Phi_{r,s}\right)^{-1} =: V_{i;s}^z = \begin{pmatrix} v_{i,1;s}^z & & \\ & \ddots & \\ & & v_{i,r,s}^z \end{pmatrix} \tag{2.8}$$

where

$$v_{i,j;s}^z := \left(1_s^\top (z\mathbb{I}_s - V_{i,j})^{-1} 1_s\right)^{-1} = \left(\frac{1}{s} \sum_{k=1}^{s} \frac{1}{z - v_{i,j,k}}\right)^{-1} \tag{2.9}$$

with $v_{i,j,k}$ being the random potential at the point (i,j,k) so that

$$V_i = \begin{pmatrix} V_{i,1} & & \\ & \ddots & \\ & & V_{i,r} \end{pmatrix} \in \mathbb{R}^{rs \times rs} \, , \qquad V_{i,j} = \begin{pmatrix} v_{i,j,1} & & \\ & \ddots & \\ & & v_{i,j,s} \end{pmatrix} \in \mathbb{R}^{s \times s} \, . \tag{2.10}$$

Therefore we finally obtain

$$T_{i;r,s}^{w,z} = \begin{pmatrix} V_i^z - w\mathbb{I}_r - \Delta_r^D & -\mathbb{I}_r \\ \mathbb{I}_r & \mathbf{0} \end{pmatrix} . \tag{2.11}$$

For some parameters $z = \lambda \in \mathbb{R}$ some of the inverses in the definition of the transfer matrix (2.5) are not defined. However, when ever possible we define it by analytic extension of the map $z \mapsto T_{i;r,s}^{w,z}$. Note that by definiteness of the imaginary parts in the occurring inverses there is never a problem for non-real parameters $z \notin \mathbb{R}$. For this reason we define:

Definition 2. The value $\lambda \in \mathbb{R}$ (spectral parameter) is called singular for $H_{n,r,s}^w$ at the i-th slice if the map $z \mapsto T_{i;r,s}^{w,z}$ is not defined in λ after analytic extensions. We call $\lambda \in \mathbb{R}$ singular for $H_{n,r,s}^w$ if it is singular at some slice $i = 1, \ldots, n$.

Note that by (2.9) and (2.11) the finite set of singular parameters for $H^w_{n,r,s}$ is contained in the convex hull of the support of ν and hence inside the interval $[-\sigma,\sigma]$.

3. The spectrum of $H^w_{n,r,s}$

For the spectrum and the determination of singular energies we may first split of some (trivial) part of the matrix $H^w_{n,r,s}$. For calculating the appearing Schur complement in the i-th transfer matrix it is sufficient to consider the subspace

$$\mathbb{V}_i := \operatorname{span}\left[\bigcup_{k=0}^{\infty} \operatorname{ran}\left(\left(A^w_{r,s}+V_i\right)^k \Phi_{r,s} \right) \right] \tag{3.1}$$

which is the union of all cyclic spaces of $A^w_{r,s}+V_i$ associated to the column vectors of $\Phi_{r,s}$. It is clear that $A^w_{r,s}+V_i$ leaves the (random) subspace $\mathbb{V}_i$ and its orthogonal complement $\mathbb{V}_i^\perp$ invariant. Now, writing $\psi\in\mathbb{C}^{nrs}$ as $(\psi_i)_{i=1}^n$ with $\psi_i\in\mathbb{C}^{rs}$ we use the fact that $\mathbb{C}^{nrs}\cong\prod_{i=1}^n\mathbb{C}^{rs}$ and with this isomorphy we can identify the product $\mathbb{V}$ of the $\mathbb{V}_i$ as subspace of $\mathbb{C}^{nrs}$ and we also have a natural embedding $\hat{\mathbb{V}}_i^\perp$ of the complements $\mathbb{V}_i^\perp$ into $\mathbb{C}^{nrs}$,

$$\mathbb{V} := \prod_{i=1}^n \mathbb{V}_i \subset \mathbb{C}^{nrs} \quad \text{and} \quad \hat{\mathbb{V}}_i^\perp := \prod_{j=1}^{i-1}\{0\} \times \mathbb{V}_i^\perp \times \prod_{j=i+1}^{n}\{0\}\,. \tag{3.2}$$

This means $\psi\in\mathbb{V} \Leftrightarrow \forall i=1,\dots,n : \psi_i\in\mathbb{V}_i$ and $\psi\in\hat{\mathbb{V}}_i^\perp \Leftrightarrow (\psi_j=0 \text{ for } j\neq i \text{ and } \psi_i\in\mathbb{V}_i^\perp)$. We should mention that it is possible that $\mathbb{V}_i^\perp=\{0\}$ for all i and $\mathbb{V}=\mathbb{C}^{nrs}$ is the full space. In fact, for continuous distributions ν of the single-site potentials this will happen with probability one.

Proposition 3.1. *We find including multiplicities that*

$$\operatorname{spec}(H^w_{n,r,s}) = \operatorname{spec}(H^w_{n,r,s}|\mathbb{V}) \cup \bigcup_{i=1}^n \operatorname{spec}(V_i\,|\,\mathbb{V}_i^\perp)\,.$$

where $\mathbb{V}_i^\perp$ is non-trivial and $\operatorname{spec}(V_i\,|\,\mathbb{V}_i^\perp)$ non-empty if and only if there is $j\in\{1,\dots,r\}$ such that $V_{i,j}$ has a multiple eigenvalue with $V_{i,j}$ as defined in (2.10).

Proof. Since $\operatorname{ran}\Phi_{r,s}\in\mathbb{V}_i$ we see that $H^w_{n,r,s}$ leaves $\mathbb{V}$ for any $i=1,\dots,n$ invariant. Similarly, for any $\psi_i\in\mathbb{V}_i^\perp$ we have $P_{r,s}\psi_i=\Phi_{r,s}\Phi_{r,s}^\top\psi_i=0$ giving that $H^w_{n,r,s}$ also leaves all the spaces $\hat{\mathbb{V}}_i^\perp$ invariant and the restrictions of $H^w_{n,r,s}$ to $\hat{\mathbb{V}}_i^\perp$ is isomorphic to the restrictions of $A^w_{r,s}+V_i$ to $\mathbb{V}_i^\perp$. Now for $\psi_i\in\mathbb{V}_i^\perp\in\mathbb{C}^{rs}$ we can split up ψ_i once more into r-parts $(\psi_{i,j})_{j=1}^r$ by $\mathbb{C}^{rs}=(\mathbb{C}^s)^r$ and use the block structure for $A^w_{r,s}$ as in (1.2) and $\Phi_{r,s}$ as in (2.1). Then

$$0=\Phi_{r,s}^\top\psi_i=\begin{pmatrix}1_s & & \\ & \ddots & \\ & & 1_s\end{pmatrix}^\top\begin{pmatrix}\psi_{i,1}\\ \vdots\\ \psi_{i,r}\end{pmatrix}=\begin{pmatrix}1_s^\top\psi_{i,1}\\ \vdots\\ 1_s^\top\psi_{i,r}\end{pmatrix}$$

implies $1_s^\top\psi_{i,j}=0$ for all $j=1,\dots,r$. This in turn implies $P_s\psi_{i,j}=1_s1_s^\top\psi_{i,j}=0$ and from (1.2) we get $A^w_{r,s}\psi_i=0$. Therefore we have

$$A^w_{r,s}\,|\,\mathbb{V}_i^\perp = \mathbf{0} \quad \text{implying} \quad \left(A^w_{r,s}+V_i\right)|\,\mathbb{V}_i^\perp = V_i\,|\,\mathbb{V}_i^\perp\,.$$

By the considerations above we have

$$H^w_{n,r,s} \cong H^w_{n,r,s}\,|\,\mathbb{V} \oplus \bigoplus_{i=1}^n V_i\,|\,\mathbb{V}_i^\perp$$

in terms of an orthogonal sum of operators (in fact matrices). The spectral decomposition follows.

Moreover, by construction, $\mathbb{V}_i^\perp$ is non-trivial precisely if there is a non-zero eigenvector ψ_i of $A^w_{r,s} + V_i$ which is orthogonal to all column vectors of $\Phi_{r,s}$. By the calculations above this is equivalent to finding j and $\psi_{i,j} \neq 0$ such that $1_s^\top \psi_{i,j} = 0$ and $\psi_{i,j}$ is an eigenvector of $V_{i,j}$ as defined in (2.10). Using the fact that $V_{i,j}$ is diagonal, you can find such an eigenvector precisely if $V_{i,j}$ has an eigenvalue with multiplicity more than one. □

Considering the eigenvalue equation (2.3) and (2.4) a solution $\psi = (\psi_i)_{i=1}^n$ can be obtained from a solution $(\vec{u}_i)_i$ of the transfer matrix equation by taking $\psi_i = \Psi_{z,i}\vec{u}_i$ where $\Psi_{z,i}$ is a $rs \times r$ matrix given by

$$\Psi_{z,i} := (z\mathbb{I}_{rs} - A^w_{r,s} - V_i)^{-1}\,\Phi_{r,s}\left(\Phi_{r,s}^\top\left(z\mathbb{I}_{rs} - A^w_{r,s} - V_i\right)^{-1}\,\Phi_{r,s}\right)^{-1}. \tag{3.3}$$

Lemma 3.2. *For any non-singular energy $\lambda \in \mathbb{R}$ the matrices $\Psi_{\lambda,i}$ are defined or can be defined by analytic extension of $z \mapsto \Psi_{z,i}$ at $z = \lambda$.*

Proof. Let λ be non singular for $H^w_{n,r,s}$. If $\lambda \notin \operatorname{spec}(A^w_{r,s} + V_i)$ then the statement is clear by existence of the first inverse in (3.3) and existence of the last term at least by analytic extension in λ. So let λ be an eigenvalue of $A^w_{r,s}+V_i$. In order to show that $\Psi_{\lambda,i}$ is defined by analytic extension, it is sufficient to show that $\varphi^\top\Psi_{\lambda,i}$ can be defined by analytic extension for any eigenvector φ of the real symmetric matrix $A^w_{r,s}+V_i$ because there is an orthonormal basis of eigenvectors. So let $(A^w_{r,s} + V_i)\varphi = \lambda_0\varphi$. Then, for $\varepsilon \neq 0$, $|\varepsilon|$ small,

$$\varphi^\top\psi_{\lambda+\varepsilon,i} = \frac{\varphi^\top\Phi_{r,s}}{\lambda+\varepsilon-\lambda_0}\left(\Phi_{r,s}^\top\left((\lambda+\varepsilon)\mathbb{I}_{rs} - A^w_{r,s} - V_i\right)^{-1}\,\Phi_{r,s}\right)^{-1}.$$

For $\lambda \neq \lambda_0$ it is clear that the limit $\varepsilon \to 0$ exists as λ is non singular and therefore the limit of the second term exists. Let us now assume $\lambda = \lambda_0$. We need to use some Schur complement formulas. Since $\Phi_{r,s}^\top\Phi_{r,s} = \mathbb{I}_r$ we can choose some orthonormal basis for $\mathbb{C}^{rs}$ and $\mathbb{C}^r$ such that $\Phi_{r,s} \equiv \left(\begin{smallmatrix}\mathbb{I}\\ \mathbf{0}\end{smallmatrix}\right)$. We may work in these basises and give $E\mathbb{I} - A^w_{r,s} - V_i$ and φ the corresponding block structures

$$\lambda\,\mathbb{I}_{rs} - A^w_{r,s} - V_i \equiv \begin{pmatrix} A & B \\ B^\top & D\end{pmatrix} \quad \text{and} \quad \varphi \equiv \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.$$

The symbol $\equiv$ shall remind that this is not how the matrices are defined but their appearance after some basis change putting $\Phi_{r,s}$ in the block structure as indicated. Then the eigenvalue equation for φ transforms to

$$A\varphi_1 + B\varphi_2 = 0, \quad B^\top\varphi_1 + D\varphi_2 = 0 \tag{3.4}$$

and we find using the Schur complement formula

$$\begin{aligned}\left(\Phi_{r,s}^\top((\lambda+\varepsilon)\mathbb{I}_{rs} - A^w_{r,s} - V_i)^{-1}\Phi_{r,s}\right)^{-1}\Phi_{r,s}\varphi &\equiv (A + \varepsilon\mathbb{I} - B(D+\varepsilon\mathbb{I})^{-1}B^\top)\varphi_1 \\ = \varepsilon\varphi_1 + A\varphi_1 + B(D+\varepsilon)^{-1}D\varphi_2 &\overset{\varepsilon\to 0}{\longrightarrow} A\varphi_1 + B\varphi_2 - BP_{ker}\varphi_2 = -BP_{ker}\,\varphi_2\end{aligned}$$

where P_{ker} is the orthogonal projection onto the kernel of D (note that D is self-adjoint). We know that the limit $B(D+\varepsilon\mathbb{I})^{-1}B^\top$ exists as λ is not singular. Hence, for any vector v we have that

$$\lim_{\varepsilon\to 0}(B^\top v)^\top(D+\varepsilon\mathbb{I})^{-1}B^\top v \quad \text{exists which implies} \quad B^\top v \in (\ker D)^\perp.$$

Therefore $\operatorname{ran} B^\top \subset (\ker D)^\perp$ and $BP_{ker} = (P_{ker}B^\top)^\top = \mathbf{0}$. Hence,

$$(A + \varepsilon\mathbb{I} - B(D+\varepsilon\mathbb{I})^{-1}B^\top)\varphi_1 \to 0 \quad \text{for} \quad \varepsilon \to 0\,.$$

This implies

$$\left(\varphi^\top \Psi_{\lambda+\bar\varepsilon,i}\right)^\top = \Psi_{\lambda+\bar\varepsilon,i}^\top\, \varphi \;\equiv\; \frac{1}{\varepsilon}\left(A + \varepsilon\mathbb{I} - B(D+\varepsilon\mathbb{I})^{-1}C\right)\varphi_1 \to$$
$$\frac{d}{dz}\left[\left(\Phi_{r,s}^\top (z\mathbb{I}_{rs} - A_{r,s}^w - V_i)^{-1}\Phi_{r,s}\right)^{-1}\,\Phi_{r,s}\varphi\right]_{z=\lambda}$$

for $\varepsilon \to 0$, which exists because the extension of the Schur complement is analytic in $z = \lambda$. □

Let us now introduce the products of the transfer matrices:

$$\mathbb{X}_{i;r,s}^{w,z} := T_{i;r,s}^{w,z}\, T_{i-1;r,s}^{w,z} \cdots T_{2;r,s}^{w,z}\, T_{1;r,s}^{w,z}\,. \tag{3.5}$$

We finally obtain the key Proposition of this section

Proposition 3.3. *Let $\lambda \in \mathbb{R}$ be non-singular for $H_{n,r,s}^w$. Then, λ is an eigenvalue of $H_{n,r,s}^w$ if and only if we have either*

$$\det\left(\begin{pmatrix}\mathbb{I}_r & \mathbf{0}\end{pmatrix}\, \mathbb{X}_{n;r,s}^{w,\lambda}\, \begin{pmatrix}\mathbb{I}_r \\ \mathbf{0}\end{pmatrix}\right) = 0 \tag{3.6}$$

or λ is an eigenvalue of $V_i\,|\,\mathbb{V}_i^\perp$ for some $i = 1,\dots,n$.
Particularly, if $|\lambda| > \sigma$ then λ is an eigenvalue of $H_{n,r,s}^w$ if and only if (3.6) holds.

Note that the second statement follows immediately as $\|V_i\| \le \sigma \Rightarrow \operatorname{spec}(V_i) \subset [-\sigma,\sigma]$ and all singular energies are also inside the interval $[-\sigma,\sigma]$.

Proof. First let $\lambda \in \operatorname{spec}(H_{n,r,s}^w)$, either $\lambda \in \operatorname{spec}(V_i|\mathbb{V}_i)$ for some $i = 1,\dots,n$ or $\lambda \in \operatorname{spec}(H_{n,r,s}^w|\mathbb{V})$. For the letter case let $\psi = (\psi_i)_{i=1}^n$ be a corresponding non-zero eigenvector.
Claim 1: For some $i = 1,\dots,n$ we have $\vec u_i = \Phi_{r,s}^\top\psi_i \neq \vec 0$. If $\Phi_{r,s}^\top\psi_i = 0$ for all $i = 1,\dots,n$, then $H_{n,r,s}^w\psi = \lambda\psi$ also implies $V_i\psi_i = \lambda\psi_i$ and we have $\psi_i \in \mathbb{V}_i^\perp$ and hence $\psi \in \mathbb{V}^\perp$ implying $\psi = 0$ as $\psi \in \mathbb{V}$ as well.
Claim 2: $(\vec u_i)_i = (\Phi_{r,s}^\top\psi_i)_i$ satisfy the transfer matrix equation (2.5) at $z = \lambda$ with $\vec u_0 = \vec u_{n+1} = \vec 0$. If all appearing inverses in the definition of $T_{i;r,s}^{w,\lambda}$ in (2.5) exist for all $i = 1,\dots,n$ then this is clear so we focus on the case when the transfer matrix is defined only by analytic extension. The eigenvalue equation for λ leads to

$$\psi_i = ((\lambda+\varepsilon)\mathbb{I} - A_{r,s}^w - V_i)^{-1}\,(\Phi_{r,s}(\vec u_{i+1} + \vec u_{i-1}) + \varepsilon\,\psi_i)$$

which after multiplying with $\Phi_{r,s}^\top$ from the left gives

$$\left(\Phi_{r,s}^\top\left((\lambda+\varepsilon)\mathbb{I} - A_{r,s}^w - V_i\right)^{-1}\Phi_{r,s}\right)^{-1}\vec u_i = \vec u_{i+1} + \vec u_{i-1} + \varepsilon\,\Psi_{\lambda+\bar\varepsilon,i}^\top\psi_i$$

In both equations we have to set $\vec u_0 = \vec 0$ for $i = 1$ and $\vec u_{n+1} = \vec 0$ for $i = n$. The limit $\varepsilon \to 0$ shows that $(\vec u_i)_i$ satisfies the transfer matrix equation with the transfer matrices defined by analytic extension to λ.

As not all of the $\vec{u}_i$ are zero, and $\vec{u}_0 = \vec{0}$, we find that $\vec{u}_1 \neq \vec{0}$ and we have

$$\begin{pmatrix} \mathbb{I}_r & \mathbf{0} \end{pmatrix} \mathbb{X}^{w,\lambda}_{n;r,s} \begin{pmatrix} \mathbb{I}_r \\ \mathbf{0} \end{pmatrix} \vec{u}_1 = \vec{0} \,. \tag{3.7}$$

This implies (3.6).

Conversely, assume (3.6), then we find $\vec{u}_1 \neq \vec{0}$ satisfying (3.7). We again focus on the case where one or more of the transfer matrices at $z = \lambda$ are only defined by analytic extension. We let $\vec{u}_i \equiv \vec{u}_i(\lambda + \varepsilon)$ be defined by the transfer matrix equation, i.e. $\left(\begin{smallmatrix} \vec{u}_{i+1} \\ \vec{u}_i \end{smallmatrix}\right) = \mathbb{X}^{w,\lambda+\varepsilon}_{i;r,s} \left(\begin{smallmatrix} \vec{u}_1 \\ \vec{0} \end{smallmatrix}\right)$. Note that we will have $\vec{u}_{n+1}(\lambda) = \vec{0}$ by (3.7). Let us define

$$\psi_i(\lambda + \varepsilon) := \Psi_{\lambda+\varepsilon,i}\, \vec{u}_i(\lambda + \varepsilon) \quad \text{and} \quad \psi(\lambda + \varepsilon) = (\psi_i(\lambda + \varepsilon))_{i=1}^n \,.$$

For $\varepsilon \neq 0$ and small where all inverses in the definition of the transfer matrix exist we obtain

$$\left(H^w_{n,r,s} - (\lambda + \varepsilon)\mathbb{I}_{nrs}\right) \psi(\lambda + \varepsilon) = \varphi(\lambda + \varepsilon) = (\varphi_i(\lambda + \varepsilon))_{i=1}^n$$

where

$$\varphi_i(\lambda + \varepsilon) = \vec{0} \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad \varphi_n(\lambda + \varepsilon) = -\Phi_{r,s}\, \vec{u}_{n+1}(\lambda + \varepsilon) \,.$$

In the limit $\varepsilon \to 0$ with Lemma 3.2 we obtain that $\psi(\lambda)$ is a non-zero eigenvector for the eigenvalue λ. □

4. Effective energy, effective potential and elliptic channels

For $|\lambda| > \sigma$ the random variables $(v^\lambda_{i,j;s})_{i,j}$ are well defined and independent identically distributed, the distribution depends on λ and s. Moreover, the law of large numbers gives for $|\lambda| > \sigma$ and $s \to \infty$ a limit distribution concentrated on the point h_λ. From (2.11) we thus define the effective energy by

$$E = E(\lambda) := h_\lambda - w \,. \tag{4.1}$$

Also note that $h_\lambda^{-1} = \mathbb{E}(1/v^\lambda_{i,j;s}) = \mathbb{E}(1/(\lambda - v))$ where v is a ν-distributed random variable. Another important quantity will be the λ-dependent variance

$$\sigma_\lambda^2 := \int \left((\lambda - v)^{-1} - h_\lambda^{-1}\right)^2 d\nu(v) = \mathbb{E}\left(\frac{1}{\lambda - v} - \frac{1}{h_\lambda}\right)^2 \tag{4.2}$$

Moreover, let us define

$$\frac{1}{s} W_s(\lambda) := \mathbb{E}(v^\lambda_{i,j;s}) - h_\lambda \quad \text{and} \quad \frac{1}{\sqrt{s}}\, Y_{i,j;s}(\lambda) := v^\lambda_{i,j;s} - h_\lambda - \frac{1}{s}\, W_s(\lambda) \tag{4.3}$$

From now on we mostly make considerations for a fixed $\lambda \notin [-\sigma, \sigma]$ and will omit the λ-dependence most of the time. Note that $\mathbb{E}(Y_{i,j;s}) = 0$ and in (i, j) we have a family of real (for λ real), independent identically distributed random variables. Harmonic mean estimates for bounded random variables as in Theorem A.1 give

$$W_s(\lambda) = h_\lambda^3\, \sigma_\lambda^2 + \mathcal{O}(1/s) \tag{4.4}$$

$$\mathbb{E}\left(Y^2_{i,j;s}\right) = h_\lambda^4\, \sigma_\lambda^2 + \mathcal{O}(1/s) \quad \text{and} \quad \sup_{s \in \mathbb{N}} \mathbb{E}\left(Y^{2n}_{i,j;s}\right) \leq C_n \,. \tag{4.5}$$

The error bounds are uniform in λ on compact sets outside $[-\sigma, \sigma]$ (including compact subsets of $\mathbb{C}$).

The upper left $r \times r$ block entry of the transfer matrices are given by

$$E\,\mathbb{I}_r \,+\, \frac{1}{\sqrt{s}} Y_{i;s} \,+\, \frac{1}{s}\, W_s\, \mathbb{I}_r \,-\, \Delta_r^D \tag{4.6}$$

where

$$Y_{i;s} \,=\, \mathrm{diag}(Y_{i,1;s}, \ldots, Y_{i,r;s}) \tag{4.7}$$

is the effective random potential in the i-th slice. In the $s \to \infty$ limit the eigenvalues and eigenvectors of $E\mathbb{I}_r - \Delta_r^D$ will classify some of the asymptotic behavior of the products. Let us note that $\lambda \in I_{w,\nu}$ implies $E(\lambda) \in (-4, 4)$. Moreover, $E(\lambda)$ is a continuous and strictly monotone function of λ in $I_{w,\nu}$. As in [SV] we now separate elliptic and hyperbolic channels and diagonalize Δ_r^D by the orthogonal matrix

$$O_{jk} \,:=\, \sqrt{2/(r+1)}\, \sin(\pi\, jk\,/\,(r+1))\;, \quad j,k = 1, \ldots, r\;.$$

The corresponding j-th eigenvector of Δ_r^D corresponding to the j-th column vector of O is given by

$$a_j \,=\, 2\,\cos(\pi j\,/\,(r+1))\;, \quad j = 1, \ldots, r$$

so that

$$O^\top\, \Delta_r^D\, O \,=\, \mathrm{diag}(a_1, \ldots, a_r)\;.$$

We focus on the case $-4 < E(\lambda) \le 0$, the other case is symmetrical. In this case $E - a_j < 2$. In the notions of [SV] we have a parabolic channel if there exists j such that $|E - a_j| = 2$ which in this case means $E - a_j = -2$. For any given r, there are r such values of E (and of λ). The union over $r \in \mathbb{N}$ gives some countable set of values in E and λ respectively. We will omit these values. Then, if there is no parabolic channel, there is $r_{\mathrm{h}} = r_{\mathrm{h}}(r, E)$ such that

$$\begin{aligned} E - a_j \,&<\, -2 && \text{for} \quad j = 1, \ldots, r_{\mathrm{h}} && \text{(hyperbolic channels)} \\ -2 \,<\, E - a_j \,&<\, 2 && \text{for} \quad j = r_{\mathrm{h}} + 1, \ldots, r && \text{(elliptic channels)} \end{aligned}$$

So we have r_{h} hyperbolic and $r_{\mathrm{e}} := r - r_{\mathrm{h}}$ elliptic channels. Note that for any fixed E and $r \to \infty$, $r_{\mathrm{e}} = r_{\mathrm{e}}(r, E)$ is of the order of r, $r_{\mathrm{e}} \sim cr$ for some $c > 0$. Then we define $\gamma_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$, $|z_j| = 1$ by

$$\begin{aligned} \gamma_j + \gamma_j^{-1} \,&=\, E - a_j\,, && |\gamma_j| < 1\,, && \text{for} \quad j = 1, \ldots, r_{\mathrm{h}} \\ z_j + z_j^{-1} \,&=\, E - a_{j+r_{\mathrm{h}}}\,, && |z_j| = 1\,, \mathrm{Im}(z_j) > 0\,, && \text{for} \quad j = 1, \ldots, r_{\mathrm{e}} \end{aligned}$$

and as in [SV] we define

$$\Gamma \,=\, \mathrm{diag}(\gamma_1, \ldots, \gamma_{r_{\mathrm{h}}})\,, \quad Z = \mathrm{diag}(z_1, \ldots, z_{r_{\mathrm{e}}}) \tag{4.8}$$

as well as

$$U := \begin{pmatrix} Z^* & \\ & Z \end{pmatrix}\,, \quad \tilde{O} := \begin{pmatrix} O & \\ & O \end{pmatrix} \quad \text{and} \quad \mathcal{Q} := \begin{pmatrix} \Gamma & & & \Gamma^{-1} \\ & Z^* & Z & \\ \mathbb{I}_{r_{\mathrm{h}}} & & & \mathbb{I}_{r_{\mathrm{h}}} \\ & \mathbb{I}_{r_{\mathrm{e}}} & \mathbb{I}_{r_{\mathrm{e}}} & \end{pmatrix}\,. \tag{4.9}$$

Here, U is a $2r_{\mathrm{e}} \times 2r_{\mathrm{e}}$ diagonal matrix, $\tilde{O}$ a $2r \times 2r$ orthogonal matrix written in $r \times r$ blocks and $\mathcal{Q}$ a $2r \times 2r$ matrix where the rows are divided in 4 blocks of sizes r_{h}, r_{e}, r_{e}, r_{h} and the columns in 4 blocks of sizes r_{h}, r_{e}, r_{h}, r_{e}. All the non-zero blocks indicated above are diagonal square matrices. These matrices depend on λ. In order to get the eigenvalue

processes we will have to vary the spectral parameter around λ but we will use these fixed $\mathcal{Q}$ $\tilde{O}$ and U to describe our basis change cf. (5.2).

But primarily let us set one more demand on the choice of λ or better $E(\lambda)$, respectively. For fixed r the value r_h changes exactly at the points $E = E(\lambda)$ where we have some parabolic channel. Hence we find some maximal open intervals $I(r_\mathrm{e}) \subset I_{w,\nu}$ such that for λ varying in $I(r_\mathrm{e})$ we have that r_h and r_e are constant[3] and $Z = Z(\lambda)$ is an analytically dependent diagonal $r_\mathrm{e} \times r_\mathrm{e}$ matrix.

Definition 3. We say that the matrix $Z = \mathrm{diag}(z_1, \dots, z_{r_\mathrm{e}})$ with $|z_j| = 1, \mathrm{Im}(z_j) > 0$ is **chaotic**, if all of the following apply for all $i, j, k, l \in \{1, \dots, r_\mathrm{e}\}$

$$\begin{aligned} &z_i z_j z_k z_l \neq 1\,, \quad \bar{z}_i z_j z_k z_l \neq 1\,, \\ &\bar{z}_i \bar{z}_j z_k z_l \neq 1 \quad \text{unless} \quad \{i,j\} = \{k,l\}\,. \end{aligned}$$

Then [SV, Lemma 5.2] gives the following.

Lemma 4.1. *For each $r_\mathrm{e} > 0$ and Lebesgue almost all $\lambda \in I(r_\mathrm{e})$ we find that Z as defined above is chaotic and moreover for any unitary diagonal $r_\mathrm{e} \times r_\mathrm{e}$ matrix Z_* there is an increasing sequence $(n_k)_k$ of integers such that $Z^{n_k+1} \to Z_*$ for $k \to \infty$.*

So we will consider λ and r such that we have elliptic channels[4], there is no parabolic channel and such that Z is chaotic.

5. The limit of thin boxes with fixed width

We will first look at the situation $s = mn$ with m and r constant and consider the eigenvalue process for $n \to \infty$. Furthermore, we scale energy differences to λ by $n(h_\lambda^2 \sigma_\lambda^2 + 1)$ (cf. (A.5) and define

$$\lambda_n^\varepsilon := \lambda + \frac{\varepsilon}{n(h_\lambda^2 \sigma_\lambda^2 + 1)} \quad \text{so that} \quad E\left(\lambda_n^\varepsilon\right) = E(\lambda) + \frac{\varepsilon}{n} + \mathcal{O}\left(\frac{\varepsilon^2}{n^2}\right) \tag{5.1}$$

Here, the error bound is uniform for ε/n varying inside a compact set so that $\lambda_n^\varepsilon \in I_{w,\nu}$.

Let us map out the correspondences between notations here and in [SV] in order to understand the relations of the propositions. In principle s amounts to the disorder strength and $\frac{1}{s}$ corresponds to λ^2 or better to $\sigma^2\lambda^2$ in [SV, Section 5], m amounts to σ^{-2}. Note in particular that the use of λ as in this paper does not correlate to the use of λ in [SV]. But $E(\lambda)$ is the mre important quantity here which corresponds to E in [SV], the use of ε is the same. The size of the transfer matrices r here corresponds to d in [SV], moreover r_h and r_e correspond to d_h and d_e in [SV], respectively.

Since $s = mn$ from now on, we will omit the index s and replace it by m and n. Because of the different roles of m and n we will place the indices differently. This way notations correspond somewhat to the ones used in [SV]. Then using the definitions (4.8) and (4.9) for some fixed λ without parabolic channel such that Z is chaotic we define

$$\mathcal{T}_{i;r,n}^{\varepsilon,m} := \mathcal{Q}^{-1}\, \tilde{O}^\top\, T_{i;r,mn}^{w,\lambda_n^\varepsilon}\, \tilde{O}\, \mathcal{Q}\,. \tag{5.2}$$

[3] there are in fact two such regions, one where $E(\lambda)$ is positive and one where it is negative.

[4] this will be the case for fixed $\lambda \in I_{w,\nu}$ and r big enough

For $\varepsilon = 0$ the limit $s \to \infty$ gives the non-random matrix

$$\mathcal{T}_r = \lim_{s\to\infty} \mathcal{T}_{i;r,s}^{0,m} = \begin{pmatrix} \Gamma & & \\ & U & \\ & & \Gamma^{-1} \end{pmatrix} \tag{5.3}$$

written in blocks of sizes $r_{\mathrm{h}}, 2r_{\mathrm{e}}, r_{\mathrm{h}}$. Note that the upper block has the eigenvalues of size (absolute value) < 1, the middle part the eigenvalues of size 1 and the lower part the eigenvalues of size > 1. Hence, when considering products the upper part is decaying, the lower part growing and the middle part stays of order 1. For the products we will look at the same basis changes and scaling and define

$$\mathcal{X}_{i;r,n}^{\varepsilon,m} := \mathcal{Q}^{-1}\,\tilde{O}^{\top}\,\mathbb{X}_{i;r,mn}^{w,\lambda_n^{\varepsilon}}\,\tilde{O}\,\mathcal{Q} = \mathcal{T}_{i;r,n}^{\varepsilon,m}\,\mathcal{T}_{i-1;r,n}^{\varepsilon,m}\cdots\mathcal{T}_{1;r,n}^{\varepsilon,m} \tag{5.4}$$

We also have to consider the impact of the perturbation in the spectral parameter. From (4.3), (4.6), (5.1) we obtain using $s = mn$ that

$$\mathcal{T}_{i;r,n}^{\varepsilon,m} = \mathcal{T}_r + \frac{1}{\sqrt{m}}\frac{1}{\sqrt{n}}\,\mathcal{Y}_{i;n}^{\varepsilon,m} + \left(\frac{\varepsilon}{n} + \frac{W_n^{\varepsilon,m}}{mn}\right)\mathcal{W}_r + \mathcal{O}\left(\frac{\varepsilon^2}{n^2}\right) \tag{5.5}$$

where

$$\mathcal{Y}_{i;n}^{\varepsilon,m} := \mathcal{Q}^{-1}\,\tilde{O}^{\top}\begin{pmatrix} Y_{i;mn}(\lambda_n^{\varepsilon}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\tilde{O}\,\mathcal{Q},\ W_n^{\varepsilon,m} := W_{mn}(\lambda_n^{\varepsilon}),\ \mathcal{W}_r := \mathcal{Q}^{-1}\begin{pmatrix} \mathbb{I}_r & \\ & \mathbf{0} \end{pmatrix}\mathcal{Q}\,.$$

The error term is non-random and the bound is uniform for ε/n varying in compact sets where λ_n^{ε} stays outside $[-\sigma,\sigma]$. Equation (5.5) is in essence the analogue of [SV, equation (5.5)] where $1/\sqrt{m}$ here plays the role of σ there. Some difference is that here the randomness and the drift-term have some dependence on ε and m, however, this dependence will not matter in the limit. Note that by construction $\mathbb{E}(\mathcal{Y}_{i;n}^{\varepsilon,m}) = \mathbf{0}$. Using (4.4) and (4.5) we find for ε varying inside compact sets that

$$W_n^{\varepsilon,m} = h_\lambda^3\sigma_\lambda^2 + \mathcal{O}\left(\frac{1}{mn},\frac{\varepsilon}{n}\right),\ \mathbb{E}\left(Y_{i,j;mn}^2(\lambda_n^{\varepsilon})\right) = h_\lambda^4\sigma_\lambda^2 + \mathcal{O}\left(\frac{1}{mn},\frac{\varepsilon}{n}\right). \tag{5.6}$$

The error terms mean that the reminder terms are bounded by $C(1/(mn) + |\varepsilon|/n)$ with a uniform C as long as ε stays inside some compact interval and n is large enough (depending on that interval) so that always $\lambda_n^{\varepsilon} \in I_{w,\nu}$. Using these bounds, the moment bound in (4.5) and the independence of the $Y_{i,j;s}$ we see that [SV, Theorem 1.1] is applicable towards an SDE limit for the products $\mathcal{X}_{i;n}^{\varepsilon,m}$ for fixed ε, m with scaling $i \sim n$. Moreover, the covariance structure of the appearing Brownian motion in the limit does not depend on ε and the m dependence is simply determined by the factor $1/\sqrt{m}$ in front of the random term in (5.5). More precisely, from the decomposition of $\mathcal{T}_r$ in (5.3) in analogy to [SV, eq. (5.7)] define

$$\mathcal{P}_{\le 1} = \begin{pmatrix} \mathbb{I}_{r_{\mathrm{h}}+2r_{\mathrm{e}}} \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^{2r\times(r_{\mathrm{h}}+2r_{\mathrm{e}})},$$

the projection on the subspaces generated by eigenvectors of $\mathcal{T}_r$ with eigenvalues of size ≤ 1. Then let

$$X_{i;r,n}^{\varepsilon,m} := \begin{pmatrix} \mathbb{I}_{r_{\mathrm{h}}} & \\ & U^{-i} \end{pmatrix}\left(\mathcal{P}_{\le 1}^{\top}\left[\mathcal{X}_{i;r,n}^{\varepsilon,m}\,\mathcal{X}_0\right]^{-1}\mathcal{P}_{\le 1}\right)^{-1}$$

where $\mathcal{X}_0$ is some adequate $r \times r$ matrix such that the Schur complement

$$X_0 := \left(\mathcal{P}_{\le 1}^{\top}\mathcal{X}_0^{-1}\mathcal{P}_{\le 1}\right)^{-1}$$

exists. Then Theorem 1.1 of [SV] gives as weak limit of stochastic processes

$$X^{\varepsilon,m}_{\lfloor tn\rfloor;r,n} \overset{n\to\infty}{\Longrightarrow} \begin{pmatrix} \mathbf{0} & \\ & \Lambda^{\varepsilon,m}_t \end{pmatrix} X_0 \quad \text{for} \quad t>0 \quad \text{with} \quad \Lambda^{\varepsilon,m}_t \in \mathbb{C}^{2r_\mathrm{e}\times 2r_\mathrm{e}}$$

being some stochastic processes with $\Lambda^{\varepsilon,m}_0 := \mathbb{I}_{2r_\mathrm{e}}$ which for (ε,m) fixed satisfy some SDE (stochastic differential equation) in t. A special choice of $\mathcal{X}_0$ and hence X_0 as in [SV] is needed for proving the limiting eigenvalue statistics mentioned further below. Using the calculations for Proposition 5.3 in [SV] combined with the calculation in [SV, Section 5.3], especially [SV, eq. (5.37)] one obtains the following SDEs:

Proposition 5.1. *Let λ be such that Z is chaotic. The family of processes $\Lambda^{\varepsilon,m}_t$ satisfy SDEs in the evolution in t of the form*

$$d\Lambda^{\varepsilon,m}_t = \left(\varepsilon + \frac{h_\lambda^3\sigma_\lambda^2 - q}{m}\right) \mathcal{S} \begin{pmatrix} \mathbb{I}_{r_\mathrm{e}} & \\ & -\mathbb{I}_{r_\mathrm{e}} \end{pmatrix} \Lambda^{\varepsilon,m}_t\, dt + \frac{1}{\sqrt{m}}\, \mathcal{S} \begin{pmatrix} d\mathcal{A}_t & d\mathcal{B}_t \\ -d\mathcal{B}_t^* & -d\overline{\mathcal{A}}_t \end{pmatrix} \Lambda^{\varepsilon,m}_t$$

where

$$\mathcal{S} = \begin{pmatrix} (\overline{Z}-Z)^{-1} & \\ & (\overline{Z}-Z)^{-1} \end{pmatrix}, \quad q = \frac{h_\lambda^4\,\sigma_\lambda^2}{r+1} \sum_{j=1}^{r_\mathrm{h}} (\gamma_j^{-1} - \gamma_j)^{-1}\,.$$

$\mathcal{A}_t$ and $\mathcal{B}_t$ are independent matrix Brownian motions, $\mathcal{A}_t$ is Hermitian and $\mathcal{B}_t$ complex symmetric, i.e.

$$\mathcal{A}_t^* = \mathcal{A}_t\,, \quad \mathcal{B}_t^\top = \mathcal{B}_t$$

with covariance structure

$$\mathbb{E}|(\mathcal{B}_t)_{ij}|^2 = \mathbb{E}|(\mathcal{A}_t)|_{ij}|^2 = \mathbb{E}\big((\mathcal{A}_t)_{ii}(\mathcal{A}_t)_{jj}\big) = h_\lambda^4\sigma_\lambda^2\, t \cdot \begin{cases} \frac{3}{2} & \text{if } i=j \\ 1 & \text{if } i\neq j \end{cases}$$

All covariances which do not follow are zero.

Note, the occuring factors $h_\lambda^4\sigma_\lambda^2$ come from the variance of $Y_{i,j;s}$ as compared to the variance 1 for the potential used in [SV]. As argued in [SV] using tightness one can obtain analytic versions in $\varepsilon, \frac{1}{\sqrt{m}}$ with uniform convergence for ε varying in a compact set and $m \geq 1$. This will imply convergence of level sets in ε seen as point processes. Note that for $|\varepsilon| < C$ and any C there is $n_0 = n_0(C,\lambda)$ such that $\lambda_n^\varepsilon \in I_{w,\nu} \subset \mathbb{R}\setminus[-\sigma,\sigma]$ for any $n > n_0$ and using Proposition 3.3 we find the analogue to [SV, Theorem 5.4].

Proposition 5.2. *Let $\mathcal{E}_{n,r,s}$ be the process of eigenvalues of $H_{n,r,s} - \lambda\mathbb{I}_{nrs}$ re-scaled by the factor $n(h_\lambda^2\sigma_\lambda^2+1)$, i.e. let*

$$\mathcal{E}_{n,r,s} = n(h_\lambda^2\sigma_\lambda^2+1)\,\mathrm{spec}\,(H_{n,r,s} - \lambda\,\mathbb{I}_{nrs})\,.$$

Fixing r let $\lambda \in I_{w,\nu}$ be such that Z (as defined in (4.8)) is chaotic, let n_k be some strictly increasing sequence such that $Z^{n_k+1} \to Z_*$ for $k\to\infty$. Then, $\mathcal{E}_{n_k,r,mn_k}$ converges to the zero process of the determinant of a $r_\mathrm{e}(\lambda)\times r_\mathrm{e}(\lambda)$ matrix,

$$\mathcal{E}_{n_k,r,mn_k} \quad \Longrightarrow \quad \mathrm{zeros}_\varepsilon \det\left(\begin{pmatrix}\overline{Z}_* & Z_*\end{pmatrix} \Lambda_1^{\varepsilon,m} \begin{pmatrix} \mathbb{I}_{r_\mathrm{h}} \\ -\mathbb{I}_{r_\mathrm{h}} \end{pmatrix}\right) \quad \text{for} \quad k\to\infty$$

The important part here is that the SDEs can be jointly solved in ε with analytic versions in ε, therefore the process of zeros makes sense. For the convergence of the level sets note that like explained in [SV] using the uniform bounds for ε varying inside compact

sets one can obtain analytic versions of these processes over some fixed probability space with uniform convergence on compacts in ε.

The factor $(h_\lambda^2\sigma_\lambda^2+1)$ occurs here because it also occurs in the perturbations λ_n^ε of λ. Note that with fixing r and letting $s\sim n$ going to infinity of the same order we basically look at a sequence of graphs resembling a quasi-two-dimensional limit.

6. The GOE limit

Let us now explain how from Propositions 5.1 and 5.2 one can get to the limiting GOE statistics as in [SV, VV]. Formally, the first step is like a derivative of the SDE in Proposition 5.1 for small $1/\sqrt{m}$ when replacing ε by $\varepsilon/\sqrt{m}$ meaning that we zoom in more locally. Then in the $m\to\infty$ lots of(groups) of eigenvalues of this process will move to infinity and some group is left which spaces like the eigenvalues of a random matrix with Gaussian entries. These random matrices are almost like in the GOE ensemble, there is just a bit of a different covariance structure and some dependence. Afterwards, the $r\to\infty$ limit will finally lead to the Sine_1 process. So all together with the limit in the previous structure, it is a triple limit process leading to the GOE statistics.

Fixing r we look at the process $\sqrt{m}\,(X_{i;r,n}^{\varepsilon/\sqrt{m},m}-X_0)$ in a $m\to\infty$ limit. On the level of the limiting process $\Lambda_t^{\varepsilon,m}$ as in Proposition 5.1 let us note that

$$\widehat{\Lambda}_t^{\varepsilon,m}\coloneqq\sqrt{m}\left(\Lambda_t^{\varepsilon/\sqrt{m},m}-\mathbb{I}_{2r_{\mathrm{e}}}\right)$$

satisfies the SDE

$$\begin{aligned}d\widehat{\Lambda}_t^{\varepsilon,m}&=\left(\varepsilon-\frac{h_\lambda^2\sigma_\lambda^2-q}{\sqrt{m}}\right)\mathcal{S}\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}&\\&-\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}\left(\frac{\widehat{\Lambda}_t^{\varepsilon,m}}{\sqrt{m}}+\mathbb{I}_{2r_{\mathrm{e}}}\right)dt\\&\quad+\mathcal{S}\begin{pmatrix}d\mathcal{A}_t&d\mathcal{B}_t\\-d\mathcal{B}_t^*&-d\overline{\mathcal{A}}_t\end{pmatrix}\left(\frac{\widehat{\Lambda}_t^{\varepsilon,m}}{\sqrt{m}}+\mathbb{I}_{2r_{\mathrm{e}}}\right)\quad\text{with}\quad\widehat{\Lambda}_0^{\varepsilon,m}=\mathbf{0}\,.\end{aligned}$$

In the limit $m\to\infty$ the SDE can be easily solved and one finds as in [SV]

$$\widehat{\Lambda}_t^{\varepsilon,m}\overset{m\to\infty}{\Longrightarrow}\Lambda_t^\varepsilon\coloneqq\varepsilon\,t\,\mathcal{S}\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}&\\&-\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}+\mathcal{S}\begin{pmatrix}\mathcal{A}_t&\mathcal{B}_t\\-\mathcal{B}_t^*&\overline{\mathcal{A}}_t\end{pmatrix}.$$

Now taking λ such that Z is chaotic as in Proposition 5.2 and taking a sequence n_k with $Z^{n_k+1}\to\mathbb{I}_{r_{\mathrm{e}}}$ we find the limiting eigenvalue processes

$$\mathcal{E}_{n_k,r,mn_k}\overset{k\to\infty}{\Longrightarrow}\mathcal{E}_{r,m}\coloneqq\mathrm{zeros}_\varepsilon\det\left(\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}&\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}\Lambda_t^{\varepsilon,m}\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}\\-\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}\right).$$

Then, working with analytic versions in ε and $1/\sqrt{m}$ for this family of processes one finds as in [SV]

$$\sqrt{m}\,\mathcal{E}_{r,m}\overset{m\to\infty}{\Longrightarrow}\mathcal{E}_r\coloneqq\mathrm{zeros}_\varepsilon\det\left(\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}&\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}\Lambda_t^\varepsilon\begin{pmatrix}\mathbb{I}_{r_{\mathrm{e}}}\\-\mathbb{I}_{r_{\mathrm{e}}}\end{pmatrix}\right)=\mathrm{spec}\,\mathfrak{Re}(\mathcal{B}_1-\mathcal{A}_1)$$

Completely analogue to [SV, Lemma 5.5] one can get to this limit with a double sequence $n_k\gg m_k\to\infty$, more precisely:

Proposition 6.1. *Let λ be such that Z is chaotic, let n_k be a strictly increasing sequence of natural numbers such that $Z^{n_k+1}\to\mathbb{I}_{r_{\mathrm{e}}}$ and let $m_k\to\infty$ be some increasing sequence*

towards infinity such that $\sqrt{m_k}\,\|Z^{n_k+1} - \mathbb{I}_{r_e}\| \to 0$. *Then for* $t > 0$, *jointly in* $t \in (0,1]$ *and* ε *varying in any finite subset of* $\mathbb{C}$ *we find*

$$\sqrt{m_k}\left(X^{\frac{\varepsilon}{\sqrt{m_k}},m_k}_{\lfloor tn_k\rfloor;r,n_k} - X_0\right) \quad \overset{k\to\infty}{\Longrightarrow} \quad \begin{pmatrix} \mathbf{0} & \\ & \Lambda^{\varepsilon}_t \end{pmatrix} X_0 \,.$$

Moreover, for the re-scaled eigenvalue process $\mathcal{E}_{n,r,s}$ *as defined above we find*

$$\sqrt{m_k}\,\mathcal{E}_{n_k,r,m_kn_k} \quad \overset{k\to\infty}{\Longrightarrow} \quad \mathcal{E}_r = \operatorname{spec}\mathfrak{Re}(\mathcal{B}_1 - \mathcal{A}_1) \,.$$

Let us note that from the process it is obvious that given any (slowly) towards ∞ increasing function $f(n)$ one can choose to consider only sequences such that $m_k < f(n_k)$.

Proof of Theorem 1. Let b be some standard Gaussian variable and $K = K(r_e)$ be an independent real symmetric $r_e \times r_e$ matrix with Gaussian entries such that $\mathbb{E}((K_{ii})^2) = \frac{5}{4}$ and $\mathbb{E}((K^2_{ij}) = 1$ for $i \neq j$. Then in distribution,

$$\mathfrak{Re}(\mathcal{B}_1 - \mathcal{A}_1) \overset{d}{=} \frac{h_\lambda^2\sigma_\lambda}{\sqrt{r+1}}\,(K + b\,\mathbb{I}_{r_e})$$

As explained in [VV], using methods of [ESYY] the local eigenvalue process converges to the Sine_1 process for $r_e \to \infty$, more precisely,

$$\sqrt{r_e}\,\operatorname{spec}(K(r_e) + b\,\mathbb{I}_{r_e}) \quad \overset{r_e\to\infty}{\Longrightarrow} \quad \mathrm{Sine}_1 \,.$$

Now, for almost all $\lambda \in I_{w,\nu}$ i.e. almost all $E(\lambda) \in (-4,4)$ we find that for all $r \in \mathbb{N}$, Z is chaotic and there is no parabolic channel. Let us fix such a λ. Then for $r \to \infty$ we also find $r_e(r,E) \to \infty$ and hence

$$\frac{\sqrt{(r+1)r_e}}{h_\lambda^2\sigma_\lambda}\,\mathcal{E}_r \quad \Longrightarrow \quad \mathrm{Sine}_1 \,.$$

This convergence and the convergence mentioned in Proposition 6.1 happen in the topology of weak convergence for point processes. Therefore, one can construct some diagonal sequence $m_k, n_k, r_k \to \infty$ such that with $s_k = m_kn_k$ and $r_{e,k} = r_e(r_k,E)$ we find

$$\frac{\sqrt{m_k(r_k+1)\,r_{e,k}}}{h_\lambda^2\sigma_\lambda}\,\mathcal{E}_{n_k,r_k,s_k} \quad \overset{k\to\infty}{\Longrightarrow} \quad \mathrm{Sine}_1 \,.$$

This proves Theorem 1 with the normalization constant

$$\mathcal{N}_k := \frac{(h_\lambda^2\sigma_\lambda^2+1)\,\sqrt{n_ks_k(r_k+1)r_{e,k}}}{h_\lambda^2\sigma_\lambda} \,.$$

Now let $f(n)$ be any (slowly) increasing function with $f(n) \to \infty$ for $n \to \infty$. In Proposition 6.1 one may choose $m_k < f(n_k)$ and start the sequence with $n_k > f(r)$. Therefore, we may choose $m_k < f(n_k)$ and $r_k < f(n_k)$. □

Appendix A. Harmonic means of random variables

In the transfer matrices we see effective potentials that are harmonic means of certain independent identically distributed (iid) random variables. Certain estimates are crucial for the proofs. We therefore consider in this section independent identically distributed random variables $X_k \in [a,b]$, $0 < a < b$, $k \in \mathbb{N}$. These variables correspond to $E - v_{i,j,k}$. We will consider the harmonic means V_s and the harmonic average h defined by

$$V_s := \frac{1}{\frac{1}{s}\sum_{k=1}^{n} \frac{1}{X_k}}, \quad h := \frac{1}{\mathbb{E}(1/X_k)}$$

where $\mathbb{E}$ denotes the expectation value. V_s corresponds to the random variables $v^{\lambda}_{i,j;s}$ as in (2.9) and h corresponds to h_λ. The second and third moment of the centered random variable $1/X_j - 1/h$ will be of some importance, therefore let

$$\sigma_m^m := \mathbb{E}((1/X_k - 1/h)^m) .$$

Note $\sigma_1 = 0$ and σ_2^2 is the variance of $1/X_k$ and corresponds to σ_λ^2 in the application of the following estimates.

Theorem A.1. *There exists a continuous function $C = C(a,b,h,\sigma_2,\sigma_3)$ such that uniformly in s,*

$$0 < \mathbb{E}(V_s - h) \le \frac{b\,h^2\,\sigma_2^2}{s}, \qquad \left|\mathbb{E}(V_s - h) - \frac{h^3\sigma_2^2}{s}\right| \le \frac{C}{s^2} . \tag{A.1}$$

$$\frac{a^2\,h^2\,\sigma_2}{s} \le \mathbb{E}((V_s - h)^2) \le \frac{b^2\,h^2\,\sigma_2^2}{s}, \qquad \left|\mathbb{E}((V_s - h)^2) - \frac{h^4\sigma_2^2}{s}\right| \le \frac{C}{s^2} . \tag{A.2}$$

Moreover, for the higher moments we find

$$\left|\mathbb{E}((V_s - h)^3)\right| \le \frac{C}{s^2}, \quad \text{and} \quad \mathbb{E}((V_s - h)^{2m}) \le \frac{(2m)!\,h^{2m}\,b^{2m}}{2^m\,m!\,a^{2m}}\,\frac{1}{s^m} \quad \text{for } m \ge 2 . \tag{A.3}$$

Proof. Using $b > a > 0$ we find $V_s \in [a,b]$ and $V_s \le \frac{1}{s}\sum_{k=1} X_k$ as well as $h < \mathbb{E}(X_k)$ by the arithmetic-harmonic mean inequality. Let $Y = 1/V_s - 1/h = \frac{1}{s}\sum_{k=1}^{s}(\frac{1}{X_k} - 1/h)$, then $\mathbb{E}(Y) = 0$, $\mathbb{E}(Y^2) = \sigma_2^2/s$ and $\mathbb{E}(Y^3) = \sigma_3^3/s^2$. Moreover, with $Y_k := 1/X_k - 1/h$ we find $|Y_k| \le \frac{1}{a} - \frac{1}{b} \le \frac{1}{a}$ and

$$\mathbb{E}(Y^{2m}) = \frac{1}{s^{2m}} \sum_{k_1,\dots,k_{2m}=1}^{s} \mathbb{E}(Y_{k_1}\cdots Y_{i_{km}}) \le \frac{1}{s^{2m}}\frac{(2m)!}{2^m m!} \sum_{k_1,\dots,k_m=1}^{s} \mathbb{E}(Y_{k_1}^2 \dots Y_{k_m}^2))$$

where we used that unpaired indices lead to zero expectation and the fact that $\frac{(2m)!}{2^m m!}$ is the number of pairings of the set $\{1,\dots,2m\}$. Now using that there are s^m m-tuples $(k_1,\dots,k_m)$ and using the bound of Y_k as mentioned above we find

$$\mathbb{E}(Y^{2m}) \le \frac{1}{s^m}\frac{(2m)!}{2^m m!}\,\frac{1}{a^{2m}} .$$

Expanding $V_s = h - hYV_s$ repeatedly we obtain

$$V_s - h = -hYV_s = -h^2Y + h^2Y^2V_s = -h^2Y + h^3Y^2 - h^3Y^3V_s . \tag{A.4}$$

As $V_s \in [a,b]$ we can estimate $\mathbb{E}(h^2Y^2V_s) \le h^2 b\,\sigma_2^2/s$ and

$$\left|\mathbb{E}(Y^3V_s)\right| \le \left|\mathbb{E}(hY^3)\right| + \left|\mathbb{E}(hY^4V_s)\right| \le \frac{h(|\sigma_3^3| + 3ba^{-4})}{s^2}$$

which with (A.4) (using the second-last and last term) gives (A.1). Taking powers of (A.4) and using similar estimates lead to (A.2) and (A.3).

For the general moment bound we use $V_s - h = h^2 Y(YV_s - 1)$ from the expansion above. Since $1 - YV_s = V_s/h \in [a/h, b/h]$ we have $|1 - YV_s| \leq b/h$ and therefore,

$$\mathbb{E}((V_s - h)^{2m}) \leq (b/h)^{2m}\, \mathbb{E}((h^2 Y)^{2m}) = h^{2m}\, b^{2m}\, \mathbb{E}(Y^{2m}) \leq \frac{1}{s^m} \frac{(2m)!}{2^m m!} \frac{h^{2m} b^{2m}}{a^{2m}}$$

□

When varying the spectral parameter we also need to understand how the harmonic average varies for the definition in (5.1). This amounts to replacing X_k by $X_{k,\varepsilon} = X_k + \varepsilon$ and recalculating $h_\varepsilon = 1/\mathbb{E}(X_{k,\varepsilon}^{-1})$. Note by the continuity of $C = C(a, b, h, \sigma_2, \sigma_3)$ for the formulas above the error terms will also be uniform in ε along compact sets $|\varepsilon| \leq c$ in ε as long as $c < a$ because $X_{k,\varepsilon} \in [a - c, b + c]$ under such perturbations. Using

$$\frac{1}{X_{k,\varepsilon}} = \frac{1}{X_k + \varepsilon} = \frac{1}{X_k} - \frac{\varepsilon}{X_k^2} + \frac{\varepsilon^2}{X_k^2 X_{k,\varepsilon}}$$

as well as $\mathbb{E}(1/X_k^2) = \sigma_2^2 + 1/h^2$ and defining $C_\varepsilon := \mathbb{E}(\frac{1}{X_k^2 X_{k,\varepsilon}})$ we find

$$h_\varepsilon = \frac{1}{\frac{1}{h} - \varepsilon[\sigma_2^2 + \frac{1}{h^2} - C_\varepsilon \varepsilon]} = h + \varepsilon\,(\sigma_2^2\, h^2 + 1) + \mathcal{O}(\varepsilon^2) \tag{A.5}$$

where the error bound is uniform on compact sets in $|\varepsilon| \leq c$ where $a - c > 0$.

References

[ASW] M. Aizenman, R. Sims and S. Warzel, *Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs*, Prob. Theor. Rel. Fields (2006) **136**:363–394

[AM] M. Aizenman and S. Molchanov, *Localization at large disorder and extreme energies: an elementary derivation*, Commun. Math. Phys. (1993) **157**:245–278

[AW] M. Aizenman and S. Warzel, *Resonant delocalization for random Schrödinger operators on tree graphs*, preprint J. Eur. Math. Soc. (2013) **15**(4):1167–1222

[AEK] O. Ajanki, L. Erdös and T. Krüger, *Universality for general Wigner-type matrices*, Prob. Th. Rel. Fields (2016) published online DOI: 10.1007/s00440-016-0740-2

[And] P. W. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev. (1958) **109**:1492–1505

[DLS] F. Delyon, Y. Levy and B. Souillard, *Anderson localization for multidimensional systems at large disorder or low energy*, Commun. Math. Phys. (1985) **100**:463-470 (1985)

[DG] P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality*, Courant Lect. Notes Math., vol. 18, American Mathematical Society, Providence, RI, 2009.

[EKS] L. Erdös, T. Krüger and D Schröder, *Random Matrices with Slow Correlation Decay*, preprint (2017) arXiv:1705.10661

[ESY] L. Erdös, B. Schlein and H.-T. Yau, *Universality of random matrices and local relaxation flow.*, Inventiones Math. (2011) **185**:75-119 (2011)

[ESYY] L. Erdös, B. Schlein, H.-T. Yau and J. Yin, *The local relaxation flow approach to universality of the local statistics for random matrices*, An. Inst. Henri Poincare Prob. Stat. (2012) **48**:1–46

[FHH] R. Froese, F. Halasan and D. Hasler, *Absolutely continuous spectrum for the Anderson model on a product of a tree with a finite graph*, J. Funct. Analysis (2012) **262**:1011–1042
J. Funct. Anal. (2006) **230**:184–221

[FHS] R. Froese, D. Hasler and W. Spitzer, *Absolutely continuous spectrum for the Anderson Model on a tree: A geometric proof of Klein's Theorem*, Commun. Math. Phys. (2007) **269**:239–257

[FS] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. (1983) **88**:151–184

[GK] F. Germinet and F. Klopp, *Spectral statistics for random Schrödinger operators in the localized regime*, J. Eur. Math. Soc. (2014) **16**:1967–2031

[GMP] Ya. Gol'dsheid, S. Molchanov and L. Pastur, *Pure point spectrum of stochastic one dimensional Schrödinger operators*, Funct. Anal. Appl. (1977) **11**:1–10

[KLW] M. Keller, D. Lenz and S. Warzel, *Absolutely continuous spectrum for random operators on trees of finite cone type*, J. D' Analyse Math. (2012) **118**:363–396

[KLWo] M. Keller, D. Lenz and R. K. Wojciechowski, *Volume Growth, Spectrum and Stochastic Completeness of Infinite Graphs* Math. Z. (2013) **274**:905–932

[Kle] A. Klein, *Absolutely continuous spectrum in the Anderson model on the Bethe lattice*, Math. Res. Lett. (1994) **1**:399–407

[KLS] A. Klein, J. Lacroix and A. Speis, *Localization for the Anderson model on a strip with singular potentials*, J. Funct. Anal. (1990) **94**:135–155

[KS] A. Klein and C. Sadel, *Absolutely Continuous Spectrum for Random Schrödinger Operators on the Bethe Strip*, Math. Nachr. (2012) **285**:5–26

[Klo] F. Klopp, *Weak disorder localization and Lifshitz tails*, Commun. Math. Phys. (2002) **232**:125–155

[KuS] H. Kunz and B. Souillard, *Sur le spectre des operateurs aux differences finies aleatoires*, Commun. Math. Phys. (1980) **78**: 201–246

[Meh] M. Mehta, *Random matrices*, Elsevier/Academic Press, Amsterdam, 2004

[Min] N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Commun. Math. Phys. (1996) **177**:709–725

[Sa1] C. Sadel, *Absolutely continuous spectrum for random Schrödinger operators on tree-strips of finite cone type*, Annales Henri Poincaré (2013) **14**:737–773

[Sa2] C. Sadel, *Absolutely continuous spectrum for random Schrödinger operators on the Fibbonacci and similar tree-strips*, Math. Phys. Anal. Geom. (2014) **17**:409–440

[Sa3] C. Sadel, *Anderson transition at two-dimensional growth rate on antitrees and spectral theory for Operators with one propagating channel*, Annales Henri Poincre (2016) **17**:1631–1675

[Sa4] C. Sadel, *Spectral Theory of one-channel operators and application to absolutely continuous spectrum for Anderson type models*, preprint (2017) arXiv:1703.08055

[SV] C. Sadel and B. Virag, *A central limit theorem for products of random matrices and GOE statistics for the Anderson model on long boxes*, Commun. Math. Phys. (2016) **343**:881–919

[TV] T. Tao and V. Vu, *Random matrices: Universality of local eigenvalue statistics*, Annals of Prob. (2012) **40**:1285–1315

[VV] B. Valko and B. Virag, *Random Schrödinger Operators on long boxes, noise explosion and the GOE*, Trans. Amer. Math. Soc. (2014) **366**:3709–3728

[Wan] W.-M Wang, *Localization and universality of Poisson statistics for the multidimensional Anderson model at weak disorder*, Invent. Math. (2001) **146**:365–398

[Wig] E. P. Wigner, Gatlinberg Conference on Neutron Physics, Oak Ridge National Laboratory Report, ORNL 2309:59

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