Ruled strips with asymptotically diverging twisting

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Abstract

We consider the Dirichlet Laplacian in a two-dimensional strip composed of segments translated along a straight line with respect to a rotation angle with velocity diverging at infinity. We show that this model exhibits a "raise of dimension" at infinity leading to an essential spectrum determined by an asymptotic three-dimensional tube of annular cross-section. If the cross-section of the asymptotic tube is a disk, we also prove the existence of discrete eigenvalues below the essential spectrum.

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1 Introduction

1.1 Motivation and context

The Laplacian arises in many areas of natural sciences as the leading contribution in the stationary representation of the partial differential equations determining the time evolution. The solution of the time-dependent problem can thus be inferred from the spectral analysis of the Laplacian, but the latter is also important on its own for spectral quantities have typically direct physical interpretations. Moreover, the structure of the underlying manifold leads to an interesting interaction of analysis and geometry.

An enormous amount of work has been conducted over the last century to understand spectral-geometric properties of the Laplacian on Riemannian manifolds. The situation is probably best understood for the two extreme cases of complete non-compact and compact manifolds, while non-complete non-compact problems, where the Laplacian may possess both the essential and discrete spectra, are typically much less studied. A prominent exception is provided by *tubes*, *i.e.* tubular neighbourhoods of submanifolds, whose spectrum is relatively well understood due to an intensive study of various quantum models motivated by the advent of nanotechnology. For unbounded tubes embedded in the Euclidean space and Dirichlet boundary conditions, typical spectral properties of the Laplacian can be roughly summarised as follows:

1. If the principal curvatures of a complete non-compact submanifold $\Sigma \subset \mathbb{R}^{\mathsf{dim}+\mathsf{codim}}$ of dimension dim vanish at infinity and the transport of the tube cross-section $\omega \subset \mathbb{R}^{\mathsf{codim}}$ along Σ is asymptotically parallel, then the essential spectrum of the Dirichlet Laplacian in the tube coincides with the half-line $[E_1, \infty)$, where E_1 is the lowest Dirichlet eigenvalue of ω: see [16].

2. In the precedent situation, if the transport of ω along Σ is parallel, then the Dirichlet Laplacian in the tube always possesses discrete eigenvalues below E_1 whenever Σ is parabolic (typically $\mathsf{dim} \leq 2$) and non-trivially curved: see [4, 2] for $\mathsf{dim} = 1$ and [5, 1, 20, 21, 22] for $\mathsf{codim} = 1$.

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3. On the other hand, if Σ is totally geodesic and the transport of ω along Σ is not parallel (in which case we say that the tube is *twisted*), then the spectrum of the Dirichlet Laplacian is purely essential and Hardy-type inequalities hold: see [6, 17] for $\mathsf{dim} = 1$ and $\mathsf{codim} = 2$.

Analogous results are also known for tubes embedded in a Riemannian manifold $\mathcal{A}$ instead of the Euclidean space: see [11, 12, 19, 10, 8, 18]. In the simplest non-trivial situation where Σ is a curve in a two-dimensional surface $\mathcal{A}$ and the cross-section ω is a symmetric interval, any non-trivial curvature of Σ and/or non-negative Gauss curvature of $\mathcal{A}$, both vanishing at infinity in an appropriate sense, lead to the existence of discrete spectra (see [11]), while Hardy-type inequalities hold if Σ is a geodesic and $\mathcal{A}$ is non-positively curved (see [12, 10]). Hence the negative curvature of $\mathcal{A}$ acts as twisting, and in fact the prominent example of "strips on ruled surfaces" from [12] can be re-considered as "twisted strips" in $\mathbb{R}^3$. This latter geometric setting will be our primary concern in this paper.

In general, it is also known that the essential spectrum may change if the curvatures of Σ or $\mathcal{A}$ do not vanish at infinity or the transport of the cross-section ω along Σ is not asymptotically parallel. In summary, the curvature of Σ has the tendency to lower the spectrum, while twisting has the opposite effect: see [13]. A striking illustration of the latter is given in the recent paper [14] of the first author in which it is shown that the spectrum of the Dirichlet Laplacian in a three-dimensional tube in $\mathcal{A} = \mathbb{R}^3$ about one-dimensional flat submanifold $\Sigma = \mathbb{R}$ can even become purely discrete if the velocity of rotation of a two-dimensional non-symmetric cross-section ω along Σ diverges at infinity. This fact can be seen as an analogue for three-dimensional tubes of Donnelly's result for complete surfaces [3] which says that the spectrum of the Laplacian is purely discrete if the Gauss curvature diverges to minus infinity at infinity.

The present paper was initiated as an attempt to generalise the Euclidean model of [14] to a manifold setting. However, we soon realised that the features of this generalisation are actually quite different and lead to new phenomena which we believe are of interest for the spectral-geometric community. Indeed, the model that we consider exhibits a sort of *raise of dimension* at infinity, which is responsible not only for a shift of the essential spectrum but also for the existence of discrete eigenvalues.

1.2 Model and main results

To keep the model as simple as possible, we restrict to the one-dimensional base manifold $\Sigma := \{(s,0,0) : s \in \mathbb{R}\}$ that we regard as a submanifold of a ruled surface $\mathcal{A}$ in $\mathbb{R}^3$ parameterised by the mapping

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3 : \big\{(s,t) \mapsto \big(s, t\cos\theta(s), t\sin\theta(s)\big)\big\}\,, \tag{1}$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a (locally) Lipschitz continuous function. The Gauss curvature of $\mathcal{A}$ is given by

$$K(s,t) = -\frac{\theta'(s)^2}{\big[1+\theta'(s)^2\,t^2\big]^2}\,,$$

and in agreement with the general result [9, Prop. 3.7.5] we observe that it is everywhere non-positive. For the tube cross-section, we set $\omega := (a_1, a_2)$, where $a_1 < a_2$ are two real numbers. In the terminology of the previous subsection: $\mathsf{dim} = \mathsf{codim} = 1$; Σ is parabolic and totally geodesic; and ω is symmetric or non-symmetric. The tube is defined as the image

$$\Omega := \mathcal{L}\big(\mathbb{R} \times (a_1, a_2)\big)\,,$$

which has a geometrical meaning of a *twisted strip* composed of segments (a_1, a_2) translated along the straight line Σ in $\mathbb{R}^3$ with respect to the rotation angle θ: see Figure 1. It is a special case of strips on surfaces studied in [11, 12, 10].

We are interested in the operator $-\Delta_D^\Omega$ in $L^2(\Omega)$ that acts as the Laplace-Beltrami operator in Ω and satisfies Dirichlet boundary conditions on $\partial\Omega$; it is defined in a standard way as the Friedrichs extension of the differential expression initially defined on compactly supported smooth functions in Ω, see Section 2.1 below. Our primary interest is to investigate spectral properties of $-\Delta_D^\Omega$ in the regime of diverging twisting, *i.e.*,

$$\lim_{|s|\to\infty} |\theta'(s)| = \infty\,. \tag{2}$$

In this case, the Gauss curvature $K(s,t)$ tends to zero as $|s| \to \infty$ for every non-zero t, but the fact that the transport of ω along Σ (when the latter is regarded as embedded in $\mathbb{R}^3$) is far from being parallel (in

which case $\theta' = 0$) leads to peculiar spectral properties. In fact, looking at Figure 1, one can convince oneself that the two-dimensional strip Ω actually looks at infinity like a three-dimensional tube of annular cross-section

$$A_{r_1,r_2} := \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}\,, \tag{3}$$

where

$$r_1 := \begin{cases} \min\{|a_1|,|a_2|\} & \text{if} \quad a_1 a_2 \geq 0\,, \\ 0 & \text{if} \quad a_1 a_2 \leq 0\,, \end{cases} \qquad r_2 := \max\{|a_1|,|a_2|\}\,. \tag{4}$$

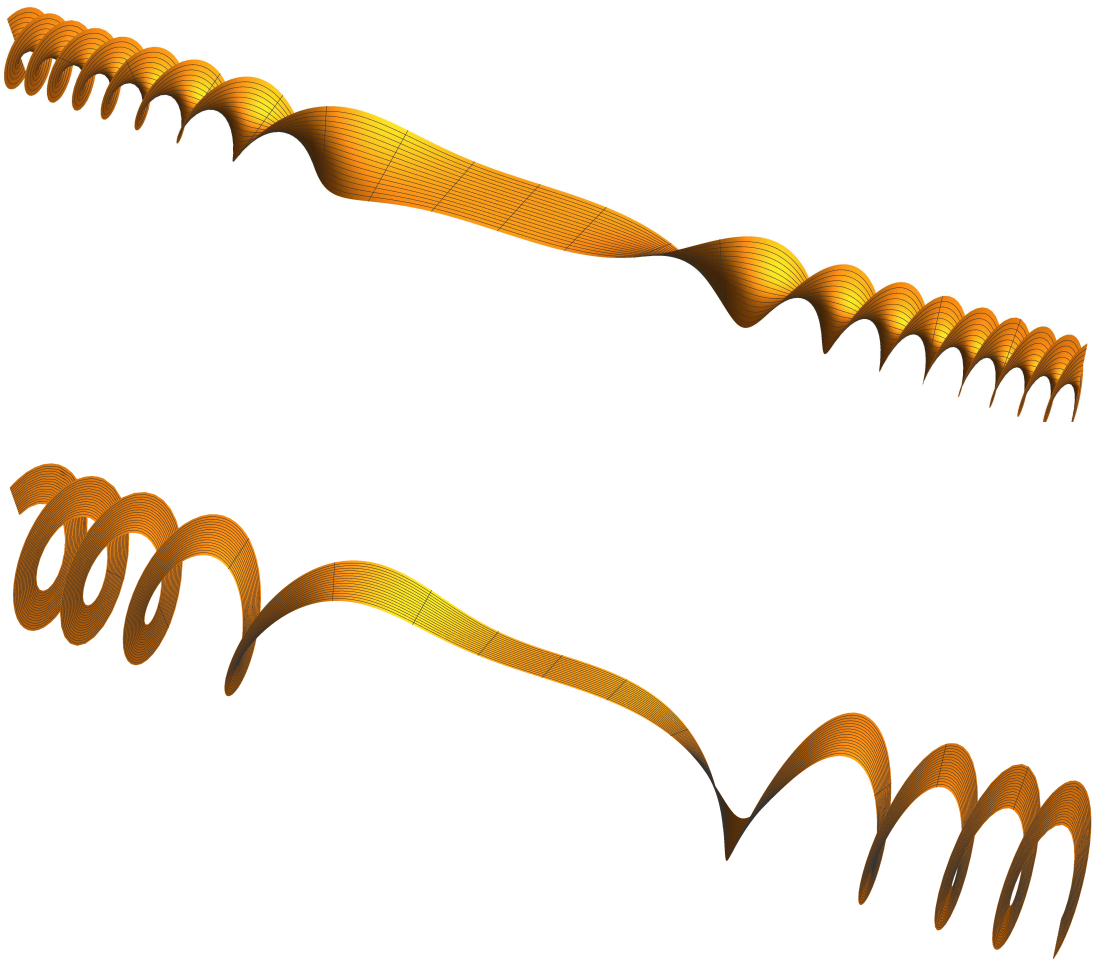


Figure 1: Ruled strips with asymptotically diverging twisting (case $a_1 a_2 \leq 0$ above, and case $a_1 a_2 > 0$ below).

Our first main result about the essential spectrum shows that this intuition is correct:

Theorem 1. *If* (2) *holds, then*

$$\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [\lambda_1, \infty)\,, \tag{5}$$

where λ_1 *denotes the lowest eigenvalue of the Dirichlet Laplacian in* A_{r_1,r_2}.

This theorem puts into evidence a difference with respect to twisted strips with asymptotically vanishing twisting, *i.e.* $\theta'(s) \to 0$ as $|s| \to \infty$, in which case it is known that

$$\inf \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,,$$

where $E_1 := \pi^2/(a_2 - a_1)^2$ is the lowest eigenvalue of the Dirichlet Laplacian in the cross-section $\omega = (a_1, a_2)$. Depending on values of a_1 and a_2, the number λ_1 may be either below or above E_1 (and

by continuity there are also a_1 and a_2 for which it equals E_1), but it is always bounded from above by $E_2 := (2\pi)^2/(a_2-a_1)^2$, the second eigenvalue of the Dirichlet Laplacian in ω. Hence the present situation substantially differs from the case of three-dimensional tubes with asymptotically diverging twisting studied in [14]; in view of (5), there is always some essential spectrum in twisted strips, while it is shown in [14] that the spectrum is purely discrete for tubes satisfying (2) in a regime analogous to $a_1 a_2 > 0$. Also, technically the proof of Theorem 1 differs from the standard proofs for asymptotically flat geometries as well as from the demonstration given in [14].

Our second main result shows that there is always spectrum below λ_1 provided that the cross-section ω is twisted with respect to a point *inside* the interval:

Theorem 2. *If* (2) *and* $a_1 a_2 \leq 0$ *hold, then*

$$\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \cap (0, \lambda_1) \neq \varnothing \,.$$

To show that discrete eigenvalues of $-\Delta_D^\Omega$ also exist in some cases $a_1 a_2 > 0$ remains an open problem. On the other hand, we conjecture that the discrete spectrum is empty whenever $a_1 a_2$ is large enough.

Finally, we leave as an open problem the study of the nature of the essential spectrum located in Theorem 1 (possible existence and location or absence of embedded eigenvalues and the absence of singular continuous spectrum). In the case of twisting vanishing at infinity, this study was performed in [15] with help of Mourre theory. In the present regime (2), however, the choice of the conjugate operator is not clear.

The paper is organised as follows: Section 2 is devoted to necessary prerequisites, in particular the definition of the Laplacian $-\Delta_D^\Omega$, and Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

2 Preliminaries

Let us start with general preliminaries valid for an arbitrary twisting angle θ.

2.1 Fermi coordinates

Using the (Fermi) coordinates (s,t) defined by (1), Ω can be identified with the Riemannian manifold $\big(\mathbb{R}\times(a_1,a_2), G\big)$, where G is the metric induced by $\mathcal{L}$,

$$G := \nabla\mathcal{L}\cdot(\nabla\mathcal{L})^\top = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad f(s,t) := \sqrt{1+\theta'(s)^2\,t^2}\,. \tag{6}$$

The function f is the Jacobian of $\mathcal{L}$. In these coordinates, $-\Delta_D^\Omega$ can be identified with the self-adjoint operator H in the Hilbert space

$$\mathcal{H} := L^2\big(\mathbb{R}\times(a_1,a_2), f(s,t)\,\mathrm{d}s\,\mathrm{d}t\big)$$

associated with the closure h of the form

$$\dot{h}[\Psi] := \big(\partial_i\Psi, G^{ij}\partial_j\Psi\big)_{\mathcal{H}} = \int f^{-1}\,|\partial_s\Psi|^2 + \int f\,|\partial_t\Psi|^2\,, \qquad \mathsf{D}(\dot{h}) := C_0^\infty\big(\mathbb{R}\times(a_1,a_2)\big)\,.$$

Here $\int$ denotes the integration over $\mathbb{R}\times(a_1,a_2)$ and the arguments of the functions are suppressed for brevity. Furthermore, we adopt the standard notation G^{ij} for the coefficients of the inverse metric G^{-1} and the Einstein summation convention is used with the range of indices being $1,2$. Throughout the paper, the first and second variables are consistently denoted by s and t, respectively.

By definition, we have

$$\mathsf{D}(h) = \overline{C_0^\infty\big(\mathbb{R}\times(a_1,a_2)\big)}^{\|\cdot\|_{\mathfrak{H}_1}} =: W^{1,2}\big(\mathbb{R}\times(a_1,a_2), G\big)$$

with the Sobolev-type norm $\|\cdot\|_{\mathfrak{H}_1}$ given by

$$\|\Psi\|_{\mathfrak{H}_1} := \sqrt{\dot{h}[\Psi] + \|\Psi\|_{\mathcal{H}}^2} = \sqrt{\int f^{-1}\,|\partial_s\Psi|^2 + \int f\,|\partial_t\Psi|^2 + \int f\,|\Psi|^2}\,. \tag{7}$$

In a distributional sense, H acts as

$$H = -|G|^{-1/2}\partial_i|G|^{1/2}G^{ij}\partial_j = -f^{-1}\partial_s f^{-1}\partial_s - f^{-1}\partial_t f\partial_t \qquad \text{with} \qquad |G| := \det(G) = f^2. \tag{8}$$

Remark 1. For the purpose of this remark only, let us assume for a moment extra hypotheses about the differentiability of θ, say

$$\theta'',\, \theta''' \in L^\infty_{\mathrm{loc}}(\mathbb{R})\,. \tag{9}$$

Then H is unitarily equivalent to a Schrödinger-type operator (see [15, Sec. 3.2] or [7, Sec. 4.4])

$$\hat{H} = -\partial_s f^{-2} \partial_s - \partial_t^2 + V \qquad \text{in} \qquad L^2\big(\mathbb{R}\times(a_1,a_2)\big)\,,$$

with potential

$$V := V_1 + V_2\,, \qquad V_1 := -\frac{5}{4}\frac{(\partial_s f)^2}{f^4} + \frac{1}{2}\frac{\partial_s^2 f}{f^3}\,, \qquad V_2 := -\frac{1}{4}\frac{(\partial_t f)^2}{f^2} + \frac{1}{2}\frac{\partial_t^2 f}{f}\,.$$

More specifically, $\hat{H}$ is the operator associated with the closure $\hat{h}$ of the form

$$\dot{\hat{h}}[\Psi] := \int f^{-2}\,|\partial_s\Psi|^2 + \int |\partial_t\Psi|^2 + \int V\,|\Psi|^2\,, \qquad \mathsf{D}(\dot{\hat{h}}) := C_0^\infty\big(\mathbb{R}\times(a_1,a_2)\big)\,.$$

Using the special form of f given in (6), we obtain

$$V_1(s,t) = -\frac{7}{4}\frac{\theta'(s)^2\,\theta''(s)^2\,t^4}{\big[1+\theta'(s)^2\,t^2\big]^3} + \frac{1}{2}\frac{\theta''(s)^2\,t^2}{\big[1+\theta'(s)^2\,t^2\big]^2} + \frac{1}{2}\frac{\theta'(s)\,\theta'''(s)\,t^3}{\big[1+\theta'(s)^2\,t^2\big]^2}\,, \tag{10}$$

$$V_2(s,t) = \frac{\theta'(s)^2\,\big[2-\theta'(s)^2\,t^2\big]}{4\,\big[1+\theta'(s)^2\,t^2\big]^2}\,. \tag{11}$$

Note that (11) makes sense under the present minimal (Lipschitz) regularity assumptions on θ.

In the present work, in order to avoid the additional regularity hypotheses, we do not use the unitarily equivalent operator $\hat{H}$ to establish spectral properties of H. However, $\hat{H}$ will be occasionally recalled to get some insights into the problem.

2.2 A lower bound

We follow the method of [12] (see also [10, Sec. 7.2]) to get a convenient lower bound to H. In the sense of quadratic forms in $\mathcal{H}$, we have

$$H \geq -f^{-1}\partial_s f^{-1}\partial_s + \lambda\,, \tag{12}$$

where, for almost every fixed $s\in\mathbb{R}$, $\lambda(s)$ denotes the lowest eigenvalue of the "transverse" operator

$$L_s := -f(s,t)^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}\,f(s,t)\,\frac{\mathrm{d}}{\mathrm{d}t} \qquad \text{in} \qquad \mathcal{H}_s := L^2\big((a_1,a_2), f(s,t)\,\mathrm{d}t\big)\,, \tag{13}$$

subject to Dirichlet boundary conditions at a_1 and a_2. (With an abuse of notation, we denote by the same symbol λ both the function on $\mathbb{R}$ and $\lambda\otimes 1$ on $\mathbb{R}\times(a_1,a_2)$.) More precisely, L_s is the operator associated with the closure ℓ_s of the form

$$\dot{\ell}_s[\psi] := \int_{a_1}^{a_2} |\psi'(t)|^2\,f(s,t)\,\mathrm{d}t\,, \qquad \mathsf{D}(\dot{\ell}_s) := C_0^\infty\big((a_1,a_2)\big)\,,$$

in the Hilbert space $\mathcal{H}_s$. We have

$$\lambda(s) = \inf_{\substack{\psi\in C_0^\infty((a_1,a_2))\\ \psi\neq 0}} \frac{\displaystyle\int_{a_1}^{a_2} |\psi'(t)|^2\,f(s,t)\,\mathrm{d}t}{\displaystyle\int_{a_1}^{a_2} |\psi(t)|^2\,f(s,t)\,\mathrm{d}t}\,. \tag{14}$$

Remark 2. The operator L_s is unitarily equivalent to

$$\hat{L}_s := -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V_2(s,t) \qquad \text{in} \qquad L^2\big((a_1,a_2)\big)\,, \tag{15}$$

where V_2 is defined in (11). More precisely, $\hat{L}_s$ is the operator associated with the closure $\hat{\ell}_s$ of the form

$$\dot{\hat{\ell}}_s[\psi] := \int_{a_1}^{a_2} |\psi'(t)|^2 \, \mathrm{d}t + \int_{a_1}^{a_2} V_2(s,t) \, |\psi(t)|^2 \, \mathrm{d}t \,, \qquad \mathsf{D}(\dot{\hat{\ell}}_s) := C_0^\infty\big((a_1,a_2)\big) \,.$$

Consequently, we also have

$$\lambda(s) = \inf_{\substack{\psi \in C_0^\infty((a_1,a_2)) \\ \psi \neq 0}} \frac{\displaystyle\int_{a_1}^{a_2} |\psi'(t)|^2 \, \mathrm{d}t + \int_{a_1}^{a_2} V_2(s,t) \, |\psi(t)|^2 \, \mathrm{d}t}{\displaystyle\int_{a_1}^{a_2} |\psi(t)|^2 \, \mathrm{d}t} \,. \tag{16}$$

2.3 An effective operator

Now let us look at the case of asymptotically diverging twisting. Assuming (2), we already saw that the strip Ω looks at infinity like a three-dimensional tube of annular cross-section (3). If r_1 is positive (*i.e.* $a_1 a_2 > 0$), A_{r_1,r_2} is an annulus of radii r_1 and r_2, while in the case $r_1 = 0$ (*i.e.* $a_1 a_2 \leq 0$) A_{0,r_2} can be identified with a disk of radius r_2 (which we understand as a degenerate annulus, to keep the same terminology). By the rotational symmetry, the lowest Dirichlet eigenvalue of A_{r_1,r_2} satisfies

$$\lambda_1 = \inf_{\substack{\psi \in C_0^\infty((r_1,r_2)) \\ \psi \neq 0}} \frac{\displaystyle\int_{r_1}^{r_2} |\psi'(r)|^2 \, r \, \mathrm{d}r}{\displaystyle\int_{r_1}^{r_2} |\psi(r)|^2 \, r \, \mathrm{d}r} \,. \tag{17}$$

To further support the above geometric intuition, let us notice that the Jacobian $f(s,t)$ asymptotically decouples as follows

$$\lim_{|s|\to\infty} \frac{f(s,t)}{f_\infty(s,t)} = 1 \qquad \text{with} \qquad f_\infty(s,t) := |\theta'(s)| \, |t| \tag{18}$$

for every non-zero t (that is, for all $t \in (a_1,a_2)$ if $a_1 a_2 > 0$). It is therefore natural to consider the following asymptotic version of the operator L_s introduced in (13):

$$L_\infty := -|t|^{-1} \, \frac{\mathrm{d}}{\mathrm{d}t} \, |t| \, \frac{\mathrm{d}}{\mathrm{d}t} \qquad \text{in} \qquad \mathcal{H}_\infty := L^2\big((a_1,a_2), |t| \, \mathrm{d}t\big) \,, \tag{19}$$

subject to Dirichlet boundary conditions at a_1 and a_2. More precisely, L_∞ is the operator associated with the closure ℓ_∞ of the form

$$\dot{\ell}_\infty[\psi] := \int_{a_1}^{a_2} |\psi'(t)|^2 \, |t| \, \mathrm{d}t \,, \qquad \mathsf{D}(\dot{\ell}_\infty) := C_0^\infty\big((a_1,a_2)\big) \,,$$

in the Hilbert space $\mathcal{H}_\infty$. It is easy to check that the lowest eigenvalue of L_∞ coincides with (17), *i.e.*

$$\lambda_1 = \inf_{\substack{\psi \in C_0^\infty((a_1,a_2)) \\ \psi \neq 0}} \frac{\displaystyle\int_{a_1}^{a_2} |\psi'(t)|^2 \, |t| \, \mathrm{d}t}{\displaystyle\int_{a_1}^{a_2} |\psi(t)|^2 \, |t| \, \mathrm{d}t} \,. \tag{20}$$

The claim is indeed obvious for $a_1 a_2 \geq 0$.

If $a_1 a_2 \leq 0$, the lowest eigenvalue of L_∞ is actually determined by the minimum between the lowest eigenvalues of the Dirichlet Laplacian in disjoint disks of radii $|a_1|$ and a_2. More specifically, if $a_1 a_2 < 0$, the eigenfunction ψ_1 of L_∞ corresponding to λ_1 satisfies the boundary value problem

$$\begin{cases} -t^{-1}\big(t \, \psi_1'(t)\big)' = \lambda_1 \psi_1(t) \,, \quad t \in (a_1,0) \cup (0,a_2) \,, \\ \psi_1(a_1) = \psi_1(a_2) = 0 \,, \\ \psi_1'(0^-) = \psi_1'(0^+) = 0 \,. \end{cases} \tag{21}$$

Hence ψ_1 is not necessarily continuous at 0 and the problem is decoupled into the disjoint intervals $(a_1, 0)$ and $(0, a_2)$ via the extra Neumann boundary condition.

Standard arguments show that ψ_1 is infinitely smooth in $(a_1, a_2)\setminus\{0\}$ and can be chosen non-negative. In fact, ψ_1 is then positive if $a_1 a_2 \geq 0$. If $a_1 a_2 < 0$, however, ψ_1 is positive only in one of the intervals $(a_1, 0)$ or $(0, a_2)$, while it is necessarily zero in the other (namely in the smaller). The corresponding eigenvalue λ_1 is always simple unless $|a_1| = a_2$, in which case it is doubly degenerate (but it is of course simple as an eigenvalue of the Dirichlet Laplacian in A_{0,r_2}).

Remark 3. In particular, if $r_1 = 0$ (*i.e.* $a_1 a_2 \leq 0$), then

$$\lambda_1 = \left(\frac{j_{0,1}}{r_2}\right)^2, \tag{22}$$

where $j_{0,1} \approx 2.4$ is the first positive zero of the Bessel function J_0. If moreover $r_2 = |a_1|$ (*i.e.* $|a_1| \geq a_2$), then

$$\psi_1(t) = \begin{cases} J_0\big(\sqrt{\lambda_1}\, t\big) & \text{if} \quad t \in (a_1, 0)\,, \\ 0 & \text{if} \quad t \in (0, a_2)\,. \end{cases} \tag{23}$$

Let us denote by $\{\lambda_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ the set of eigenvalues of L_∞ and corresponding eigenfunctions, respectively. We choose ψ_k real-valued and normalised to 1 in $\mathcal{H}_\infty$. The boundary value problem (21) holds for each couple (λ_k, ψ_k) instead of (λ_1, ψ_1). Note that the set $\{\lambda_k\}_{k=1}^\infty$ coincides with the eigenvalues of the Dirichlet Laplacian in the annulus A_{r_1,r_2} associated with radially symmetric eigenfunctions.

3 The essential spectrum

Let

$$E_k := \left(\frac{k\pi}{a_2 - a_1}\right)^2, \qquad k = 1, 2, \dots,$$

denote the eigenvalues of the Dirichlet Laplacian in $L^2\big((a_1, a_2)\big)$. The corresponding normalised eigenfunctions are given by

$$\chi_k(t) := \sqrt{\frac{2}{a_2 - a_1}}\, \sin\big(\sqrt{E_k}\,(t - a_1)\big)\,. \tag{24}$$

It was shown in [10, Thm. 5.1] that

$$\lim_{|s|\to\infty} |\theta'(s)| = 0 \qquad \Longrightarrow \qquad \sigma_{\mathrm{ess}}(H) = [E_1, \infty)\,. \tag{25}$$

This result is expected because under the condition the strip Ω looks at infinity like a straight strip of width $a_2 - a_1$ and the spectrum of the latter coincides with the interval $[E_1, \infty)$.

Our goal is to show that the condition (2) actually shifts the essential spectrum in the sense of Theorem 1. This result is also intuitively clear because, under condition (2), the strip Ω looks at infinity like a three-dimensional tube of cross-section A_{r_1,r_2}, and λ_1 is the lowest eigenvalue of the Dirichlet Laplacian in A_{r_1,r_2}, see Section 2.3. However, giving a rigorous proof of this result requires some work, as we shall see.

Using χ_1 with $a_1 := r_1$ and $a_2 := r_2$ as a test function in (17), we obtain the upper bound

$$\lambda_1 < \left(\frac{\pi}{r_2 - r_1}\right)^2 \tag{26}$$

(the strict inequality follows from the fact that the chosen test function differs from the eigenfunction ψ_1). If $a_1 a_2 \geq 0$, then $r_2 - r_1 = a_2 - a_1$, so (26) shows that $\lambda_1 < E_1$ in this case. On the other hand, if the interval (a_1, a_2) is symmetric, *i.e.* $a_2 = -a_1$ (a particular case of $a_1 a_2 \leq 0$), it follows from (22) that $E_1 < \lambda_1 < E_2$. If $a_1 a_2 \leq 0$ without any further restriction on a_1 and a_2, we have $r_2 - r_1 = r_2 \geq (a_2 - a_1)/2$, so (26) together with this crude bound shows only that $\lambda_1 < E_2$. Summing up, we see that, depending on values of a_1 and a_2, λ_1 may be either below or above E_1 (and by continuity there are also a_1 and a_2 for which it equals E_1), but it is always bounded from above by E_2.

Theorem 1 is established in two steps. First, we establish a lower bound to the threshold of the essential spectrum:

Lemma 1. *If* (2) *holds, then* $\inf\sigma_{\rm ess}(H)\geq\lambda_1$.

Proof. By imposing an extra Neumann condition at the segments $\{s=\pm s_0\}$ with $s_0>0$, we obtain

$$H\geq H^N_{\rm int}\oplus H^N_{\rm ext}\,,$$

where $H^N_{\rm int}$ is the self-adjoint operator in $\mathcal{H}_{\rm int}:=L^2\big((-s_0,s_0)\times(a_1,a_2),f(s,t)\,{\rm d}s\,{\rm d}t\big)$ associated with the closure $h^N_{\rm int}$ of the form

$$\dot h^N_{\rm int}[\Psi]:=\big(\partial_i\Psi,G^{ij}\partial_j\Psi\big)_{\mathcal{H}_{\rm int}}\,,\qquad \mathsf{D}(\dot h^N_{\rm int}):=C^\infty_0\big(\mathbb{R}\times(a_1,a_2)\big)\restriction(-s_0,s_0)\times(a_1,a_2)\,,$$

and $H^N_{\rm ext}$ is an analogously defined operator in $\mathcal{H}_{\rm ext}:=L^2\big(\mathbb{R}\setminus[-s_0,s_0]\times(a_1,a_2),f(s,t)\,{\rm d}s\,{\rm d}t\big)$. Since the spectrum of $H^N_{\rm int}$ is purely discrete, the minimax principle yields

$$\inf\sigma_{\rm ess}(H)\geq\inf\sigma_{\rm ess}(H^N_{\rm ext})\geq\inf\sigma(H^N_{\rm ext})\,. \tag{27}$$

A lower bound analogous to (12) holds for $H^N_{\rm ext}$ as well. Consequently, neglecting the differential part in (12), we obtain

$$\inf\sigma(H^N_{\rm ext})\geq\inf_{|s|\geq s_0}\lambda(s)\,. \tag{28}$$

This estimate together with (27) yields

$$\inf\sigma(H)\geq\inf_{|s|\geq s_0}\lambda(s)\,.$$

Since $s_0>0$ is arbitrary, the left-hand side is independent of s_0 and the essential spectrum is a closed set, it follows that it is enough to show that (recall (14) and (20))

$$\liminf_{|s|\to\infty}\lambda(s)\geq\lambda_1 \tag{29}$$

in order to conclude with the proof of the lemma.

To establish (29), since $\lambda(s)$ (respectively, λ_1) is the lowest eigenvalue of the operator L_s (respectively, L_∞), one is lead to establishing a sort of convergence of L_s to L_∞ as $|s|\to\infty$. For this, let ψ_s denote an eigenfunction of L_s corresponding to $\lambda(s)=:\lambda_s$, with ψ_s normalised to 1 in $\mathcal{H}_s$, and let us write the eigenvalue equation $L_s\psi_s=\lambda_s\psi_s$ in the weak form

$$\ell_s(\varphi,\psi_s)=\lambda_s\,(\varphi,\psi_s)_{\mathcal{H}_s}\,,\qquad\forall\varphi\in\mathsf{D}(\ell_s)\,. \tag{30}$$

First of all, we remark that λ_s is uniformly bounded in $s\in\mathbb{R}$ due to (26). This can be verified by using a test function in (14) supported away from zero (it is only needed if $a_1a_2\leq0$) and estimating $f(s,t)\geq|\theta'(s)|\,|t|$ in the denominator and $f(s,t)\leq2|\theta'(s)|\,|t|$ in the numerator (the second bound is valid for all s sufficiently large due to (2)), so that the s-dependent terms $|\theta'(s)|$ cancel in the Rayleigh quotient. Consequently, putting $\varphi:=\psi_s$ in (30), we obtain

$$\|\psi_s'\|^2_{\mathcal{H}_s}\leq C\,,$$

where C is a constant independent of s. Recalling the definition of f in (6), we deduce

$$\int_{a_1}^{a_2}|\psi_s'(t)|^2\,{\rm d}t\leq C\qquad\text{and}\qquad|\theta'(s)|\int_{a_1}^{a_2}|\psi_s'(t)|^2\,|t|\,{\rm d}t\leq C\,. \tag{31}$$

Similarly, from the normalisation of ψ_s, we deduce

$$\int_{a_1}^{a_2}|\psi_s(t)|^2\,{\rm d}t\leq1\qquad\text{and}\qquad|\theta'(s)|\int_{a_1}^{a_2}|\psi_s(t)|^2\,|t|\,{\rm d}t\leq1\,. \tag{32}$$

We may conclude from the latter inequalities in (31) and (32) that $\{\phi_s\}_{s\in\mathbb{R}}$ with $\phi_s(t):=|\theta'(s)|^{1/2}\psi_s(t)$ is a bounded family of functions in the weighted Sobolev space $\mathsf{D}(\ell_\infty)$, while $\{\psi_s\}_{s\in\mathbb{R}}$ converges to zero in $\mathcal{H}_\infty$ as $|s|\to\infty$ due to (2). At the same time, from the former inequalities in (31) and (32) we deduce that $\{\psi_s\}_{s\in\mathbb{R}}$ is a bounded family of functions in the standard Sobolev space $W^{1,2}\big((a_1,a_2)\big)$.

Consequently, the sets $\{\psi_s\}_{s\in\mathbb{R}}$ and $\{\phi_s\}_{s\in\mathbb{R}}$ are precompact in the weak topology of the spaces $W^{1,2}\big((a_1,a_2)\big)$ and $\mathsf{D}(\ell_\infty)$, respectively. Let ψ_∞ and ϕ_∞ denote weak limit points, *i.e.* there exists a real sequence $\{s_j\}_{j=1}^\infty$ such that $|s_j|\to\infty$ as $j\to\infty$ for which

$$\begin{aligned} &\psi_{s_j} \xrightarrow[j\to\infty]{w} \psi_\infty \quad \text{in} \quad W^{1,2}\big((a_1,a_2)\big)\,, \qquad && \phi_{s_j} \xrightarrow[j\to\infty]{w} \phi_\infty \quad \text{in} \quad \mathsf{D}(\ell_\infty)\,, \\ &\psi_{s_j} \xrightarrow[j\to\infty]{s} \psi_\infty \quad \text{in} \quad L^2\big((a_1,a_2)\big)\,, && \phi_{s_j} \xrightarrow[j\to\infty]{s} \phi_\infty \quad \text{in} \quad \mathcal{H}_\infty\,. \end{aligned} \tag{33}$$

Here the strong limits follow from the compactness of the embeddings $W^{1,2}\big((a_1,a_2)\big)\hookrightarrow L^2\big((a_1,a_2)\big)$ and $\mathsf{D}(\ell_\infty)\hookrightarrow\mathcal{H}_\infty$. We claim that

$$\psi_\infty = 0 \qquad \text{in} \qquad L^2\big((a_1,a_2)\big).$$

By virtue of the latter inequality in (32), it is clear for the case $a_1a_2>0$ when we can bound $|t|\geq r_1>0$. If $a_1a_2\leq 0$, we deduce the claim as follows. On the one hand, for every sequence of positive numbers $\{\varepsilon_j\}_{j=1}^\infty$ such that $\varepsilon_j\to 0$ as $j\to\infty$, we have the convergence result

$$\int_{(a_1,a_2)\setminus[-\varepsilon_j,\varepsilon_j]} |\psi_{s_j}(t)|^2\,\mathrm{d}t \xrightarrow[j\to\infty]{} \int_{(a_1,a_2)} |\psi_\infty(t)|^2\,\mathrm{d}t$$

by the monotone convergence theorem. On the other hand, we have

$$\int_{(a_1,a_2)\setminus[-\varepsilon_j,\varepsilon_j]} |\psi_{s_j}(t)|^2\,\mathrm{d}t \leq \frac{1}{\varepsilon_j}\int_{(a_1,a_2)\setminus[-\varepsilon_j,\varepsilon_j]} |\psi_{s_j}(t)|^2\,|t|\,\mathrm{d}t \leq \frac{1}{\varepsilon_j\,|\theta'(s_j)|}\int_{(a_1,a_2)} |\phi_{s_j}(t)|^2\,|t|\,\mathrm{d}t\,,$$

where the right-hand side tends to zero as $j\to\infty$ provided that we choose for instance $\varepsilon_j := |\theta'(s_j)|^{-1/2}$. Here we use the hypothesis (2). The fact that $\psi_\infty=0$ implies $\phi_\infty\neq 0$. Indeed, recalling the definition of f in (6), we have

$$1 = \|\psi_{s_j}\|^2_{\mathcal{H}_{s_j}} \leq \int_{a_1}^{a_2} |\psi_{s_j}(t)|^2\,\mathrm{d}t + \int_{a_1}^{a_2} |\phi_{s_j}(t)|^2\,|t|\,\mathrm{d}t \xrightarrow[j\to\infty]{} \int_{a_1}^{a_2} |\phi_\infty(t)|^2\,|t|\,\mathrm{d}t\,.$$

In the eigenvalue equation (30), let us take $\varphi\in C_0^\infty\big((a_1,a_2)\big)$, the core of both ℓ_s and ℓ_∞, write s_j instead of s, multiply by $|\theta'(s_j)|^{1/2}$ and take the limit $j\to\infty$. Using (33), we obtain

$$\ell_\infty(\varphi,\phi_\infty) = \lambda_\infty\,(\varphi,\phi_\infty)_{\mathcal{H}_\infty} \qquad \text{with} \qquad \lambda_\infty := \lim_{j\to\infty}\lambda_{s_j}\,. \tag{34}$$

Since ϕ_∞ is non-zero, it enables us to conclude that the eigenvalue λ_{s_j} of L_{s_j} tends to an eigenvalue of L_∞ as $j\to\infty$ with eigenfunction ϕ_∞. Consequently, $\lambda_\infty\geq\lambda_1$. Since the same result (34) is obtained for *any* weak limit point of $\{\phi_s\}_{s\in\mathbb{R}}$, we arrive at the desired bound (29).

Summing up, combining (29) with (27) and (28) concludes the proof of the lemma. □

Now we turn to the opposite inclusion $\sigma_{\mathrm{ess}}(H)\supset[\lambda_1,\infty)$. Employing (18) in (8), it is expected that the essential spectrum of H will be determined by the asymptotic operator

$$H_\infty := -\frac{|\theta'(s)|^{-1}\partial_s|\theta'(s)|^{-1}\partial_s}{t^2} - |t|^{-1}\partial_t|t|\partial_t \qquad \text{in} \qquad L^2\big(\mathbb{R}\times(a_1,a_2), |\theta'(s)|\,|t|\,\mathrm{d}s\,\mathrm{d}t\big)\,, \tag{35}$$

for which we can assume that $\theta'(s)\neq 0$ for all $s\in\mathbb{R}$ by using (2) and possibly re-defining θ' on a compact set (the latter does not influence the essential spectrum). More specifically, H_∞ is the operator associated with the closure h_∞ of the form

$$\dot{h}_\infty[\Psi] := \int f_\infty^{-1}\,|\partial_s\Psi|^2 + \int f_\infty\,|\partial_t\Psi|^2\,, \qquad \mathsf{D}(\dot{h}_\infty) := C_0^\infty\big(\mathbb{R}\times[(a_1,a_2)\setminus\{0\}]\big)\,,$$

with the function f_∞ defined in (18). Notice that the one-dimensional operator $-|\theta'(s)|^{-1}\partial_s|\theta'(s)|^{-1}\partial_s$ in $L^2\big(\mathbb{R},|\theta'(s)|\,\mathrm{d}s\big)$, understood as the Friedrichs extension of the operator initially defined on the domain $C_0^\infty(\mathbb{R})$, is unitarily equivalent to the standard Laplacian $-\partial_u^2$ in $L^2(\mathbb{R})$ with the usual domain $W^{2,2}(\mathbb{R})$; indeed, the unitarily equivalence is accomplished by the change of variables $u=\int_0^s|\theta'(\sigma)|\,\mathrm{d}\sigma$. Since

the operator $-\partial_u^2$ can be decomposed using the Fourier transform, H_∞ is unitarily equivalent to the direct-integral operator

$$H_\infty \cong \int_{\mathbb{R}}^{\oplus} L_\infty^m \, \mathrm{d}m \qquad \text{with} \qquad L_\infty^m := -\frac{1}{|t|}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|t|\,\frac{\mathrm{d}}{\mathrm{d}t} + \frac{m^2}{t^2}\,. \tag{36}$$

More specifically, L_∞^m is the operator in $L^2\big((a_1,a_2),|t|\,\mathrm{d}t\big)$ associated with the closure ℓ_∞^m of the form

$$\dot\ell_\infty^m[\psi] := \int_{a_1}^{a_2} |\psi'(t)|^2\,|t|\,\mathrm{d}t\,, \qquad \mathsf{D}\big(\dot\ell_\infty^m\big) := C_0^\infty\big((a_1,a_2)\setminus\{0\}\big)\,.$$

Let us denote by $\{\lambda_k^m\}_{k=1}^\infty$ and $\{\psi_k^m\}_{k=1}^\infty$ the set of eigenvalues of L_∞^m and corresponding eigenfunctions, respectively, with ψ_k^m normalised to 1 in $L^2\big((a_1,a_2),|t|\,\mathrm{d}t\big)$. Recalling (19), notice that $\{\lambda_k^0\}_{k=1}^\infty = \{\lambda_k\}_{k=1}^\infty$ and that ψ_k^0 can be chosen in such a way that $\{\psi_k^0\}_{k=1}^\infty = \{\psi_k\}_{k=1}^\infty$. It follows from (36) that

$$\sigma(H_\infty) = \sigma_{\mathrm{ess}}(H_\infty) = [\lambda_1,\infty)\,.$$

This is an *a posteriori* evidence supporting (5). Here we use H_∞ as an inspiration to choose an appropriate singular sequence to prove the following lemma with help of a Weyl-type criterion:

Lemma 2. *If* (2) *holds, then* $\sigma_{\mathrm{ess}}(H)\supset[\lambda_1,\infty)$.

Proof. Let $\varphi_0\in C_0^\infty(\mathbb{R})$ be a real-valued function normalised to 1 in $L^2(\mathbb{R})$ with $\operatorname{supp}\varphi_0\subset[0,1]$. For every $n>0$, we define

$$\varphi_n(s) := n^{-1/2}\,\varphi_0(n^{-1}s-n)\,,$$

so that

$$\operatorname{supp}\varphi_n = n^2 + n\operatorname{supp}\varphi_0 \subset [n^2,n^2+n]\subset[n^2,(n+1)^2] \tag{37}$$

is "localised at infinity" for large n. The normalisation factor is chosen in such a way that also φ_n is normalised to 1 in $L^2(\mathbb{R})$.

For every $m\in\mathbb{R}$, we set

$$\Psi_n^m(s,t) := \varphi_n(s)\,\omega^m(s)\,\psi_1^m(t) \qquad \text{with} \qquad \omega^m(s) := e^{im\int_0^s|\theta'(\sigma)|\,\mathrm{d}\sigma}\,. \tag{38}$$

(For simplicity, below we shall denote by the same symbols φ_n, ω^m and ψ_1^m functions $\varphi_n\otimes 1$, $\omega^m\otimes 1$ and $1\otimes\psi_1^m$ on $\mathbb{R}\times(a_1,a_2)$, respectively.) For every $n>0$ and $m\in\mathbb{R}$, we have $\Psi_n^m\in\mathsf{D}(h)$. Using the pointwise inequality $f\geq 1$ and the normalisation of φ_n, we get

$$\|\Psi_n^m\|_{\mathcal{H}} \geq \|\varphi_n\|_{L^2(\mathbb{R})}\,\|\psi_1^m\|_{L^2((a_1,a_2))} = \|\psi_1^m\|_{L^2((a_1,a_2))} > 0\,. \tag{39}$$

Thus we have a positive lower bound to the norm of Ψ_n^m which is independent of n. Our goal is to show that, for every $m\in\mathbb{R}$,

$$\big\|(H-\lambda_1^m)\Psi_n^m\big\|_{\mathcal{H}_1^*} := \sup_{\substack{\phi\in C_0^\infty(\mathbb{R}\times(a_1,a_2))\\ \phi\neq 0}} \frac{|h(\phi,\Psi_n^m)-\lambda_1^m\,(\phi,\Psi_n^m)_{\mathcal{H}}|}{\|\phi\|_{\mathcal{H}_1}} \;\xrightarrow[n\to\infty]{}\; 0\,, \tag{40}$$

where $\|\cdot\|_{\mathcal{H}_1}$ is the norm introduced in (7). Since $\mathbb{R}\ni m\mapsto\lambda_1^m\in[\lambda_1,\infty)$ is a continuous function with range $[\lambda_1,\infty)$, property (40) ensures that $[\lambda_1,\infty)\subset\sigma_{\mathrm{ess}}(H)$ with help of the Weyl criterion adapted to quadratic forms (*cf.* [16, Thm. 5]). Our inspiration for the choice (38) is the *formal* identity

$$(H_\infty-\lambda_1^m)\,\omega^m\,\psi_1^m = 0\,,$$

which is behind the fact that an analogue of (40) holds for H_∞ instead of H.

Since $\mathbb{R}\ni m\mapsto\lambda_1^m\in[\lambda_1,\infty)$ is even and the essential spectrum is a closed set, it is enough to show (40) for $m>0$. Note that $[0,\infty)\ni m\mapsto\lambda_1^m\in[\lambda_1,\infty)$ is strictly increasing. Let us henceforth assume $m>0$ and write

$$h(\phi,\Psi_n^m)-\lambda_1^m\,(\phi,\Psi_n^m)_{\mathcal{H}} = I_1+I_2\,, \tag{41}$$

where

$$I_1 := \int f_\infty^{-1}\,\overline{\partial_s\phi}\,\partial_s\Psi_n^m + \int f_\infty\,\overline{\partial_t\phi}\,\partial_t\Psi_n^m - \lambda_1^m\int f_\infty\,\overline{\phi}\,\Psi_n^m\,,$$
$$I_2 := \int (f^{-1}-f_\infty^{-1})\,\overline{\partial_s\phi}\,\partial_s\Psi_n^m + \int (f-f_\infty)\,\overline{\partial_t\phi}\,\partial_t\Psi_n^m - \lambda_1^m\int (f-f_\infty)\,\overline{\phi}\,\Psi_n^m\,.$$

If $a_1a_2\leq 0$, notice that $\psi_k^m(0)=0$ whenever $m\neq 0$. Consequently,

$$C_m := \sup_{t\in(a_1,a_2)}\frac{|\psi_1^m(t)|}{|t|} \tag{42}$$

is a finite constant and the integrals above containing f_∞^{-1} are well defined.

Integrating by parts and using the eigenvalue equation that ψ_1^m satisfies, it is easy to check that

$$I_1 = \int f_\infty^{-1}\,\overline{\partial_s\phi}\,\varphi_n'\,\omega^m\,\psi_1^m - im\int |\theta'|\,f_\infty^{-1}\,\overline{\phi}\,\varphi_n'\,\omega^m\,\psi_1^m\,.$$

Consequently,

$$\begin{aligned}
|I_1| &\leq \sqrt{\int f^{-1}\,|\partial_s\phi|^2}\sqrt{\int f\,f_\infty^{-2}\,|\varphi_n'|^2|\psi_1^m|^2} + |m|\sqrt{\int f\,|\phi|^2}\sqrt{\int |\theta'|^2\,f^{-1}\,f_\infty^{-2}\,|\varphi_n'|^2|\psi_1^m|^2}\\
&\leq \|\phi\|_{\mathcal{H}_1}\,C_m\,\left(\sqrt{\int f\,|\theta'|^{-2}\,|\varphi_n'|^2} + |m|\sqrt{\int f^{-1}\,|\varphi_n'|^2}\right)\\
&\leq \|\phi\|_{\mathcal{H}_1}\,C_m\,(1+|m|)\,\|\varphi_n'\|_{L^2(\mathbb{R})}\,,
\end{aligned}$$

where the first estimate follows by the Schwarz inequality, the second bound is due to (42) and the last inequality employs (2) and the fact that φ_n is "localised at infinity". Since

$$\|\varphi_n'\|_{L^2(\mathbb{R})} = n^{-1}\,\|\varphi_0'\|_{L^2(\mathbb{R})}\,,$$

we infer that the I_1-part of (41) verifies (40).

Now we turn to estimating the individual terms of I_2. Similarly as above, noticing the identity $f-f_\infty = 1/(f+f_\infty)$ and recalling that φ_n is real-valued, we estimate

$$\begin{aligned}
\left|\int (f^{-1}-f_\infty^{-1})\,\overline{\partial_s\phi}\,\partial_s\Psi_n^m\right| &\leq \sqrt{\int f^{-1}\,|\partial_s\phi|^2}\sqrt{\int \frac{(|\varphi_n'|^2+m^2|\theta'|^2|\varphi_n|^2)|\psi_1^m|^2}{f\,f_\infty^2\,(f_\infty+f)^2}}\\
&\leq \|\phi\|_{\mathcal{H}_1}\,C_m\sqrt{\int \frac{(|\varphi_n'|^2+m^2|\theta'|^2|\varphi_n|^2)}{f\,|\theta'|^2\,(f_\infty+f)^2}}\,,
\end{aligned}$$

where the square root on the second line tends to zero as $n\to\infty$, due to (2) with help of the dominated convergence theorem. At the same time,

$$\begin{aligned}
\left|\int (f-f_\infty)\,\overline{\partial_t\phi}\,\partial_t\Psi_n^m\right| &\leq \sqrt{\int f\,|\partial_t\phi|^2}\sqrt{\int \frac{|\varphi_n|^2\,|(\psi_1^m)'|^2}{f\,(f_\infty+f)}}\\
&\leq \|\phi\|_{\mathcal{H}_1}\left(\sup_{t\in(a_1,a_2)}|(\psi_1^m)'(t)|^2\right)\sqrt{\int \frac{|\varphi_n|^2}{f\,(f_\infty+f)}}\,,
\end{aligned}$$

where the square root on the second line again tends to zero as $n\to\infty$. Finally,

$$\begin{aligned}
\left|\int (f-f_\infty)\,\overline{\phi}\,\Psi_n^m\right| &\leq \sqrt{\int f\,|\phi|^2}\sqrt{\int \frac{|\varphi_n|^2\,|\psi_1^m|^2}{f\,(f_\infty+f)}}\\
&\leq \|\phi\|_{\mathcal{H}_1}\left(\sup_{t\in(a_1,a_2)}|\psi_1^m(t)|^2\right)\sqrt{\int \frac{|\varphi_n|^2}{f\,(f_\infty+f)}}\,,
\end{aligned}$$

where the square root on the second line also tends to zero as $n\to\infty$. Summing up, we infer that the I_2-part of (41) also verifies (40). □

Theorem 1 follows as a consequence of Lemmata 1 and 2.

Remark 4. Coming back to the unitarily equivalent operator $\hat{H}$ from Remark 1, let us assume in addition to (2) that

$$\lim_{|s|\to\infty} \frac{\theta''(s)}{\theta'(s)^2} = 0 \qquad \text{and} \qquad \lim_{|s|\to\infty} \frac{\theta'''(s)}{\theta'(s)^3} = 0\,. \tag{43}$$

Then

$$V(s,t) \xrightarrow[|s|\to\infty]{} -\frac{1}{4t^2} \tag{44}$$

for all $t \in (a_1, a_2)\setminus\{0\}$ (that is, for all $t \in (a_1, a_2)$ if $a_1 a_2 > 0$). Hence the natural asymptotic counterpart of $\hat{H}$ reads

$$\hat{H}_\infty := -\frac{\partial_s |\theta'(s)|^{-2} \partial_s}{t^2} - \partial_t^2 - \frac{1}{4t^2} \qquad \text{in} \qquad L^2\big(\mathbb{R}\times(a_1,a_2)\big)\,.$$

Let us also remark that the mean curvature of the tube Ω (when regarded as a submanifold of $\mathbb{R}^3$) is given by

$$M(s,t) = -\frac{\theta''(s)\,t}{\big[1+\theta'(s)^2\,t^2\big]^{3/2}}\,.$$

Hence, assuming (2) and the first condition of (43), both the Gauss and mean curvatures of Ω vanish at infinity for all $t \neq 0$ (and even uniformly in t if $a_1 a_2 > 0$). But despite of this "asymptotic flatness", the essential spectrum substantially differs from the standard situation (25), see Theorem 1.

4 The discrete spectrum

The following result shows that there is always *some* spectrum of H below λ_1 provided that the cross-section (a_1, a_2) is twisted with respect to a point *inside* the interval (irrespectively whether (2) holds or not).

Theorem 3. *If $a_1 a_2 \leq 0$, then*

$$\inf\sigma(H) < \lambda_1\,.$$

Proof. As in the proof of Lemma 2, we consider a family of test functions of the form

$$\Psi_n(s,t) := \varphi_n(s)\,\psi_1(t)\,, \tag{45}$$

where ψ_1 is the eigenfunction of L_∞ corresponding to the lowest eigenvalue λ_1 (case $m = 0$ of (38)), but now the "longitudinal" component φ_n is given by

$$\varphi_n(s) := \begin{cases} 1 & \text{if} \quad |s| < n\,, \\ \dfrac{2n-|s|}{n} & \text{if} \quad n \leq |s| \leq 2n\,, \\ 0 & \text{if} \quad |s| > 2n\,. \end{cases} \tag{46}$$

Notice that φ_n converges pointwise to 1 as $n \to \infty$.

Integrating by parts and using the differential equation that ψ_1 satisfies, one verifies that

$$h[\Psi_n] - \lambda_1 \|\Psi_n\|^2_{\mathfrak{H}} = \left\| \frac{\varphi_n'\psi_1}{f} \right\|^2_{\mathfrak{H}} + \big(\varphi_n\psi_1, W\varphi_n\psi_1'\big)_{\mathfrak{H}}\,, \tag{47}$$

where

$$W(s,t) := \frac{1}{t} - \frac{\partial_t f(s,t)}{f(s,t)} = \frac{1}{t\,\big[1+\theta'(s)^2\,t^2\big]}\,. \tag{48}$$

Note that $W\varphi_n\psi_1'$ belongs to $\mathfrak{H}$. Indeed, the fact that $t^{-1}\psi_1'(t)$ remains bounded as $t \to 0$ follows from the Neumann condition that ψ_1 satisfies at zero, *cf.* (21).

By the variational definition of $\inf\sigma(H)$, it is enough to show that the right-hand side of (47) is negative for some n. Using the pointwise inequality $f \geq 1$, we obtain

$$\left\| \frac{\varphi_n' \psi_1}{f} \right\|_{\mathcal{H}}^2 \leq \|\varphi_n'\|_{L^2(\mathbb{R})}^2 \, \|\psi_1\|_{L^2((a_1,a_2))}^2 = n^{-1} \, \|\psi_1\|_{L^2((a_1,a_2))}^2 \,, \tag{49}$$

so this term goes to zero as $n \to \infty$. To show that the second term on the right-hand side of (47) is negative, we use that ψ_1 is decreasing on $(0, a_2)$ and increasing on $(a_1, 0)$. (In fact, because of the decoupling, ψ_1 is necessarily identically zero in one of these intervals.) At the same time, $t \mapsto W(s,t)$ is positive on $(0, a_2)$ and negative on $(a_1, 0)$. Consequently,

$$c_n := \left(\varphi_n \psi_1, W \varphi_n \psi_1'\right)_{\mathcal{H}} < 0 \,. \tag{50}$$

Moreover, $n \mapsto c_n$ is decreasing due to the definition of φ_n. Hence, the right-hand side of (47) is negative for all sufficiently large n and the desired claim follows. □

While Theorem 3 is valid for any twisting angle θ, it is of particular interest for diverging twisting (2), when λ_1 is the threshold of the essential spectrum of H due to Theorem 1. In this case Theorem 3 implies Theorem 2 as a corollary.

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