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FAMILIES OF EXPLICIT QUASI-HYPERBOLIC AND HYPERBOLIC SURFACES

Natalia Garcia-Fritz [1], Giancarlo Urzúa[2]

[1,2]Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Chile

FAMILIES OF EXPLICIT QUASI-HYPERBOLIC AND HYPERBOLIC SURFACES

NATALIA GARCIA-FRITZ AND GIANCARLO URZÚA

Abstract. We construct explicit families of quasi-hyperbolic and hyperbolic surfaces. This is based on earlier work of Vojta, and the recent expansion and generalization of it by the first author. In this paper we further extend it to the singular case, obtaining results for the surface of cuboids, the generalized surfaces of cuboids, and other families of Diophantine surfaces of general type. In particular, we produce explicit families of smooth complete intersection surfaces of multidegrees $(m_1, \ldots, m_n)$ in $\mathbb{P}^{n+2}$ which are hyperbolic, for any $n \geq 8$ and any degrees $m_i \geq 2$. We also show similar results for complete intersection surfaces in $\mathbb{P}^{n+2}$ for $n = 4, 5, 6, 7$. These families give evidence for [Dem18, Conjecture 0.18] in the case of surfaces.

1. Introduction

The purpose of this paper is to give an explicit method to find low genus curves in a wide range of algebraic surfaces. The method is based on an earlier work of Vojta [V00], which has roots in the seminal work of Bogomolov [B77] (see [D79]), and the recent expansion and generalization of Vojta's method by the first author [GF15]. In this paper we further extend it to the singular case.

In addition, we show that the method allows us to test hyperbolicity on these surfaces. In particular, we show new examples of families of quasi-hyperbolic and hyperbolic surfaces. This part is based on Nevanlinna theory (cf. [V11]). We recall some definitions to be precise. An *entire curve* in a variety X is the image of a nonconstant holomorphic map $\mathbb{C} \to X$. A surface X is said to be *quasi-hyperbolic* if all entire curves are contained in a proper Zariski closed subset of X. A surface X is said to be *hyperbolic* if it has no entire curves. Hence, when X is a smooth projective surface, we have that X is hyperbolic if it is in the sense of Kobayashi or in the sense of Brody; cf. [Kob].

A main motivation for us comes from describing the set of rational points of particular Diophantine varieties under the Bombieri-Lang conjecture. For instance, by finding all curves of geometric genus less than or equal to one, Vojta [V00] shows that the "n squares problem" of Büchi follows from that conjecture, and later the first author shows that the analogous problem for arbitrary k-powers would also be a consequence of it [GF16].

Let us consider one example, which will be used in Section 2 to develop the ideas and computations around the method. In this example the singularities are rational double points of type A.

Let $n \geq 3$, $m \geq 2$ be integers. Let $\{F_1, \ldots, F_{nm}\}$ and $\{G_1, \ldots, G_{nm}\}$ be collections of distinct nm vertical and horizontal fibres of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively.

Let us denote the elements of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$ by (a, b). Then we have

$$F_{km+1} + \ldots + F_{(k+1)m} + G_{km+1} + \ldots + G_{(k+1)m} = (m, m)$$

in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ for $0 \leq k \leq n-1$. These equations define a tower of n cyclic covers of degree m

$$X_n \to X_{n-1} \to \ldots \to X_1 \to \mathbb{P}^1 \times \mathbb{P}^1.$$

All X_k are normal projective surfaces with km^{k+1} singularities of type

$$A_{m-1}\colon\ (0,0) \in (z^m - xy) \subset \mathbb{C}^3.$$

The surface X_n is simply connected. It also has ample canonical class, and so it is of general type.

Let $n = 3$. We can take as a model of $\mathbb{P}^1 \times \mathbb{P}^1$ the quadric

$$(z_0 z_3 - z_1 z_2) \subset \mathbb{P}^3,$$

and so the surface X_3 can be presented as

$$\prod_{i=1}^{m}(z_0 - a_i z_1 - b_i z_2 + a_i b_i z_3) = z_4^m, \quad \prod_{i=1}^{m}(z_0 - c_i z_1 - d_i z_2 + c_i d_i z_3) = z_5^m,$$

$$\prod_{i=1}^{m}(z_0 - e_i z_1 - f_i z_2 + e_i f_i z_3) = z_6^m, \qquad z_0 z_3 - z_1 z_2 = 0$$

in $\mathbb{P}^6$ for distinct $a_i, c_i, e_i \in \mathbb{C}$ and distinct $b_i, d_i, f_i \in \mathbb{C}$. For $m = 2$ and a specific choice of $a_i, c_i, e_i, b_i, d_i, f_i$, the surface X_3 is isomorphic to the surface of cuboids S defined by

$$x_0^2 + x_1^2 + x_2^2 = x_3^2, \quad x_0^2 + x_1^2 = x_4^2, \quad x_0^2 + x_2^2 = x_5^2, \quad x_1^2 + x_2^2 = x_6^2$$

in $\mathbb{P}^6$ (see Section 3). A positive rational point in S would realize a perfect cuboid. It is unknown if there are any such points. This famous old problem goes back to Euler; cf. [ST10], [vLu00], [B13], [FS16]. According to the *Bombieri-Lang conjecture*, outside of a certain finite set of curves of geometric genus at most one, there can only be a finite number of solutions. Hence it is of interest to find all such curves. We prove the following (see Sections 2 and 6).

Theorem 1.1. *The surfaces X_n are hyperbolic for any $m > 2$.*

Recall that the surface of cuboids S is the surface X_n with $m = 2$, $n = 3$ and a particular choice of vertical and horizontal fibres. For S, the results are less strong and involve further adaptations of the method. Details are worked out in Section 3.

Theorem 1.2. *Let S be the surface of cuboids. Let $S' \to S$ be its minimal resolution, and let E be the sum of the 48 exceptional curves. We have:*

(a) *Every curve of geometric genus 0 or 1 must contain at least 2 of the 48 singularities of S.*
(b) *If $C \subset S$ is a curve which is smooth at the singular points of S, then $deg(C) \leq 4g(C) + 44$.*
(c) *If $C \subset S'$ is a rational curve which is neither exceptional nor contained in $x_0 x_1 x_2 x_3 = 0$, then $C \cdot E \geq 8$.*

In Section 4 we develop the method for arbitrary cyclic quotient singularities, which is captured in the following particular example. Let us consider the lines $L_{t,u} = (t^2x + tuy + u^2z) \subset \mathbb{P}^2$ for $[t,u] \in \mathbb{P}^1$. They are precisely the tangent lines to the conic $(y^2 - 4xz) \subset \mathbb{P}^2$. Let $\{L_1, \ldots, L_d\}$ be distinct lines such that $L_i = L_{t_i,u_i}$ for some $[t_i, u_i] \in \mathbb{P}^1$. Let us take positive integers $a_1, \ldots, a_d$ such that $\sum_{i=1}^d a_i = mR$ for some integers $m, R > 0$. Assume that $a_i < m$ and $\gcd(a_i, m) = 1$ for all i, and that $a_i + a_j$ is not divisible by m for all $i \neq j$. The method we develop in Section 4 allows us to prove the following.

Theorem 1.3. *If $4m < d$, then the surface*

$$(t_1^2x + t_1u_1y + u_1^2z)^{a_1} \cdots (t_d^2x + t_du_dy + u_d^2z)^{a_d} = w^m$$

in $\mathbb{P}(1,1,1,R)$ contains no curves with geometric genus ≤ 1 apart from the $\mathbb{P}^1$'s defined by $t_i^2x + t_iu_iy + u_i^2z = 0$. Its normalization is a simply connected normal projective surface with ample canonical class.

The Diophantine hypersurface $(\prod_{i=1}^{15}(i^2x + iy + z) = w^3) \subset \mathbb{P}(1,1,1,5)$ works as an example for Theorem 1.3.

Let X be a smooth projective surface, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega_X^1)$ for some line bundle $\mathcal{L}$, and some integer $r > 0$. Key in the above results is the notion of ω-integral curves; see Definition 2.1. We construct surfaces via a composition of cyclic covers of the fixed surface X, which are branched along ω-integral curves, and for which we know all ω-integral curves on X. In Section 5 we prove the following general theorem.

Theorem 1.4. *Assume we have the relations $m_i\mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j}D_{i,j}$ in Pic(X), for some line bundles $\mathcal{M}_i$, with $0 < a_{i,j} < m_i$ and gcd$(a_{i,j}, m_i) = 1$ for all i,j. Assume that the divisor $\sum_{i=1}^n \sum_{j=1}^{s_i} D_{i,j}$ has simple normal crossings with $D_{i,j}$ ω-integral curves. Then a tower of n cyclic covers of degree m*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 := X$$

is defined, where all X_k are normal projective surfaces with only cyclic quotient singularities. If

$$\sum_{i=1}^n \frac{1}{m_i}\Big(\sum_{j=1}^{s_i} D_{i,j}\Big) - \mathcal{L} \ \textit{is } \mathbb{Q}\textit{-ample, and} \ \ a_{i,j} \not\equiv -a_{i,j'} (\mathrm{mod}\, m_i) \ \ \textit{for all } i, \ j \neq j',$$

then X_n can have curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in X.

From this theorem, and via a particular result of Vojta [V00b] in Nevanlinna theory for algebraic varieties, in Section 6 we obtain the following.

Theorem 1.5. *Let us consider the hypothesis and notation as in Theorem 1.4. In addition, assume that all solutions to the differential equation given by $\omega = 0$ on X are ω-integral curves. Then an entire curve in X_n must be contained in the set of preimages of ω-integral curves in X. In particular, if the set of preimages of ω-integral curves in X does not contain curves of geometric genus 0 or 1, then X_n is hyperbolic.*

The following application of the method and its adaptations produces explicit families of smooth complete intersection surfaces which are hyperbolic and have arbitrary multidegrees, in particular, low multidegrees. We note that it is key for this application to work with singularities.

Theorem 1.6. *There are explicit families of smooth complete intersections in $\mathbb{P}^{n+2}$ which are hyperbolic for multidegrees*

(a) $(m_1, \ldots, m_n)$ *when* $n \geq 8$ *and* $m_i \geq 2$.
(b) $(m_1, \ldots, m_n)$ *when* $n \geq 5$ *and* $m_i \geq 3$.
(c) $(2, m_1, \ldots, m_{n-1})$ *when* $n \geq 6$, $m_i \geq 2$ *for all* $i < n-1$, $m_{n-1} \geq 3$.
(d) $(2, m_1, \ldots, m_{n-1})$ *when* $n \geq 4$ *and* $m_i \geq 3$.

The explicit families are shown at the end of Section 6. This theorem gives evidence for [Dem18, Conjecture 0.18] in the case of surfaces. Smooth complete intersection surfaces $X_{n,k} \subset \mathbb{P}^n$ of multidegree $(k, \ldots, k)$ and hyperbolic have been constructed in [GF16] for $k = 3$, $n \geq 6$; $k = 4, 5$, $n \geq 5$; $k \geq 6$, $n \geq 4$. We also point out that hyperbolic complete intersections of high multidegree have been constructed by Brotbek [Br14] (see also [X15]).

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2. The generalized surfaces of cuboids

Let $n \geq 3$, $m \geq 2$ be integers. Let $\{F_1, \ldots, F_{nm}\}$ and $\{G_1, \ldots, G_{nm}\}$ be collections of distinct nm vertical and horizontal fibres of $\mathbb{P}^1 \times \mathbb{P}^1$. Let us denote the elements of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$ by (a, b). Then we have

$$F_{km+1} + \ldots + F_{(k+1)m} + G_{km+1} + \ldots + G_{(k+1)m} = (m, m)$$

for $0 \leq k \leq n-1$. These expressions define a tower of n cyclic covers of degree m

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 := \mathbb{P}^1 \times \mathbb{P}^1$$

inductively as follows: Let $f_{k+1} \colon X_{k+1} \to X_k$ be the cyclic cover defined by the equation of line bundles

$$g_k^*\big(F_{km+1} + \ldots + F_{(k+1)m} + G_{km+1} + \ldots + G_{(k+1)m}\big) \simeq g_k^*(1,1)^{\otimes m},$$

where $g_k := f_1 \circ \cdots \circ f_k$. To be more precise, the surface X_{k+1} is defined as $X_{k+1} := \mathrm{Spec}_{X_k} \bigoplus_{j=0}^{m-1} g_k^*(1,1)^{-j}$, and so the finite morphism f_{k+1} satisfies $f_{k+1*}\mathcal{O}_{X_{k+1}} = \bigoplus_{j=0}^{m-1} g_k^*(1,1)^{-j}$; for details see e.g. [U10, Section 1]. All X_k are normal projective surfaces with km^{k+1} singularities of type

$$A_{m-1}\colon \ (0,0) \in (z^m - xy) \subset \mathbb{C}^3.$$

As shown in the introduction for $n = 3$, the surfaces X_k are complete intersections in $\mathbb{P}^{k+3}$, and so they are simply connected.

Let $D_k \subset X_{k-1}$ be the branch divisor of $f_k \colon X_k \to X_{k-1}$, this is

$$D_k := g_{k-1}^*\big(F_{km+1} + \ldots + F_{(k+1)m} + G_{km+1} + \ldots + G_{(k+1)m}\big).$$

Hence it is a collection of $2m$ smooth curves which form a simple normal crossings divisor with m^{k+1} nodes. By the Riemann-Hurwitz formula, each smooth curve Γ in D_k satisfies

$$2g(\Gamma) - 2 = m^{k-1}((m-1)(k-1) - 2),$$

and $\Gamma^2 = 0$, where $g(\Gamma)$ is the genus of Γ. We have $D_k^2 = 2m^{k+1}$, and $D_k \cdot K_{X_{k-1}} = 2m(2g(\Gamma) - 2)$. We also have the formulas

$$\chi(\mathcal{O}_{X_k}) = m\chi(\mathcal{O}_{X_{k-1}}) + \frac{(m-1)(2m-1)}{12m} D_k^2 + \frac{(m-1)}{4} D_k \cdot K_{X_{k-1}},$$

and $K_{X_k}^2 = mK_{X_{k-1}}^2 + \frac{(m-1)^2}{m} D_k^2 + 2(m-1) D_k \cdot K_{X_{k-1}}$, and so

$$K_{X_k}^2 - 8\chi(\mathcal{O}_{X_k}) = -\frac{2k}{3} m^k (m^2 - 1) < 0.$$

In fact $\frac{K_{X_k}^2}{\chi(\mathcal{O}_{X_k})}$ approaches 8 as $k >> 0$. We also have

$$K_{X_k} \sim g_k^*\big((-2,-2) + \frac{(m-1)}{m}(m,m)k)\big) = (k(m-1) - 2) g_k^*(1,1),$$

and so X_k has ample canonical class if and only if $k(m-1) > 2$. Hence for $k = n$ the surface X_n is of general type.

Let $\sigma_n \colon X'_n \to X_n$ be the minimal resolution of the singularities in X_n, and let $g'_n = \sigma_n \circ g_n$. Then

$${g'_n}^*(mn, mn) = mR + mE$$

where R is the strict transform by σ_n of the branch divisor (in X_0) of g_n, and E is the (reduced) sum of the exceptional curves of σ_n. That is a simple local toric computation; see e.g. [HTU17, 2.1]. Therefore, we have

$${g'_n}^*(n, n) = R + E. \tag{2.1}$$

We now recall the key general notion of ω-integral curve (see [V00, Definition 2.4], [GF15, Definition 3.2]).

Definition 2.1. Let X be a smooth surface. Let $r \geq 1$ be an integer, let $\mathcal{L}$ be an invertible sheaf on X, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega_X^1)$. Let C be an irreducible curve on X, and let $\varphi_C \colon \tilde{C} \to X$ be the normalization of $C \subset X$. The curve C is said to be *ω-integral* if $\varphi_C^*\omega \in H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^r\Omega_{\tilde{C}}^1)$ is zero.

Let $\omega \in H^0(X_0, (2,2) \otimes S^2\Omega_{X_0}^1)$ be the global section $z_3^2 dz_1 dz_2$. Consider the isomorphism $h \colon \mathbb{P}^1 \times \mathbb{P}^1 \to X_0$ given by $h([x,y] \times [w,z]) = [xw, xz, yw, yz]$. The section ω corresponds to the section $y^2 z^2 dx dw$ under h. Therefore, horizontal and vertical fibres are ω-integral, and this is the complete set of ω-integral curves by [GF15, 6.6]. In particular the branch loci of $g_n \colon X_n \to X_0$ is formed by ω-integral curves.

Notation 2.2. Let $X, r, \mathcal{L}$, and ω be as in Definition 2.1. Let Y be a smooth surface, and $\pi \colon Y \to X$ be a dominant morphism. We denote by $\pi^\bullet\omega$ the image of ω under the natural pull-back morphism $H^0(X, \mathcal{L} \otimes S^r\Omega_X^1) \to H^0(Y, \pi^*(\mathcal{L}) \otimes S^r\Omega_Y^1)$; see [GF15].

As in [GF15, Theorem 3.87], we have that there exists a section ω' in $H^0(X'_n, \mathcal{O}_{X'_n}(-(m-1)R) \otimes g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n})$ whose image under the natural morphism

$$H^0\big(X'_n, \mathcal{O}_{X'_n}(-(m-1)R) \otimes g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n}\big) \to H^0\big(X'_n, g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n}\big)$$

is precisely $g_n'^{\bullet}\omega$. We now show the existence of a section ω'' in a "more negative" sheaf, which also has $g_n'^{\bullet}\omega$ as image.

Lemma 2.3. *Let $m > 2$. Then there is ω'' in $H^0(X'_n, \mathcal{O}_{X'_n}(-(m-1)R - E) \otimes g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n})$ whose image under the natural morphism*

$$H^0\big(X'_n, \mathcal{O}_{X'_n}(-(m-1)R-E) \otimes g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n}\big) \to H^0\big(X'_n, g_n'^*(2,2) \otimes S^2\Omega^1_{X'_n}\big)$$

is $g_n'^{\bullet}\omega$.

Proof. We only need to prove that $g_n'^{\bullet}\omega$ vanishes along the divisor E. This is a general toric local computation with differentials. So let us say that $\omega \in H^0(X_0, \mathcal{L} \otimes S^r\Omega^1_{X_0})$, where in our case $\mathcal{L} = (2,2)$ and $r = 2$.

Let us consider one node P of the branch divisor of g_k for some $0 < k < n$. Take one preimage Q of this node P by the morphism g_n. At Q we have a rational double point of type A_{m-1}, which is in particular a cyclic quotient singularity. By [BHPV04, III, Theorem 5.1], the singularity and the map $g_n : X_n \to X_0$ is locally analytically isomorphic to $U = (z^m - xy) \subset \mathbb{C}^3$ and the projection $g(x,y,z) = (x,y)$ respectively. Thus to compute the pull-back of ω under g_n, we use this local model. Moreover, this model has a toric description as follows. See e.g [R03].

Let $\sigma' : V \to U$ be the minimal resolution, and $g' : V \to \mathbb{C}^2$ the composition of σ' with g. Let E_0 be the (reduced) preimage under g' of $x = 0$, and let E_m be the (reduced) preimage under g' of $y = 0$. Let $E_1, E_2, \ldots, E_{m-1}$ be the chain of $\mathbb{P}^1$'s in V which corresponds to the exceptional divisor of σ'. Hence $E_0, E_1, \ldots, E_{m-1}, E_m$ also form a chain. Let $u_i = 0$ be the local coordinate defining E_i, so that at each node of $E_0, E_1, \ldots, E_{m-1}, E_m$ we have local coordinates u_i, u_{i+1} for V. Then (see [R03, Example 3.1]) we have that locally g' is given by

$$g'(u_i, u_{i+1}) = (u_i^{m-i} u_{i+1}^{m-i-1}, u_i^i u_{i+1}^{i+1}).$$

Therefore we have the pull-back relations

$$dx = (m-i)u_i^{m-i-1}u_{i+1}^{m-i-1}du_i + (m-i-1)u_i^{m-i}u_{i+1}^{m-i-2}du_{i+1}$$

and

$$dy = iu_i^{i-1}u_{i+1}^{i+1}du_i + (i+1)u_i^i u_{i+1}^i du_{i+1}.$$

Via a local trivialization of $\mathcal{L}$, we can identify ω with a section of $S^r\Omega^1_{\mathbb{C}^2}$ around $(0,0)$ as

$$\omega = a_0 dx^{\otimes r} + a_1 dx^{\otimes(r-1)} \otimes dy + \ldots + a_r dy^{\otimes r},$$

where the a_i are holomorphic around $(0,0)$. Since $x = 0$ and $y = 0$ are ω-integral curves, then as in [GF15, Theorem 3.87], we have that $a_0 = ya'_0$ and $a_r = xa'_r$. Therefore the pull-back of $a_0 dx^{\otimes r}$ and $a_r dy^{\otimes r}$ vanish along the divisor $E_1 + \ldots + E_{m-1}$. On the other hand, the pull-back of $dx^{\otimes(r-k)} \otimes dy^{\otimes k}$ for $0 < k < r$ vanishes on $u_i = 0$ if and only if $i - 1 > 0$ or $m - i - 1 > 0$, and both inequalities hold because $m > 2$. Therefore the pull-back of ω vanishes along $E_0 + E_1 + \ldots + E_{m-1} + E_m$. Moreover, by [GF15, Theorem 3.87], it vanishes of order $m-1$ along E_0 and E_m. $\square$

Remark 2.4. The previous lemma is valid for any tower of cyclic morphisms of order $m > 2$ branched along a simple normal crossings divisor which is formed by ω-integral curves, so that the singularities are rational double points of type A_{m-1}.

Theorem 2.5. *Let $m > 2$, and let $C \subset X'_n$ be a curve of geometric genus g which is not an exceptional curve of σ_n. If*

$$\frac{4g-4}{n-2} < g'_n(C) \cdot (1,1),$$

then $g'_n(C)$ is an ω-integral curve.

Proof. By Lemma 2.3, we have

$$\omega'' \in H^0(X'_n, \mathcal{O}_{X'_n}(-(m-1)R - E) \otimes g'^*_n(2,2) \otimes S^2\Omega^1_{X'_n})$$

whose image under the natural morphism

$$H^0\big(X'_n, \mathcal{O}_{X'_n}(-(m-1)R-E) \otimes g'^*_n(2,2) \otimes S^2\Omega^1_{X'_n}\big) \to H^0\big(X'_n, g'^*_n(2,2) \otimes S^2\Omega^1_{X'_n}\big)$$

is $g'^{\bullet}_n\omega$. Let $\varphi_C \colon \tilde{C} \to X'_n$ be the normalization of $C \subset X'_n$. By Equality (2.1), we obtain that

$$g'^*_n(-n,-n) - (m-2)R = -(m-1)R - E$$

in $\mathrm{Pic}(X'_n)$. In this way

$$\mathcal{O}_{X'_n}(-(m-1)R - E) \otimes g'^*_n(2,2) \simeq g'^*_n(2-n, 2-n) \otimes \mathcal{O}_{X'_n}(-(m-2)R) =: \mathcal{L},$$

and so $\deg_{\tilde{C}}\left(\varphi_C^*\mathcal{L} \otimes S^2\Omega^1_{\tilde{C}}\right) \leq (2-n, 2-n) \cdot g'_n(C) + 2(2g-2) < 0$ by the projection formula and the hypothesis. Therefore $H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^2\Omega^1_{\tilde{C}}) = 0$, and C is a ω''-integral curve. As in [GF15, Proposition 3.88], the ω''-integral curves in X'_n are also $g'^{\bullet}_n\omega$-integral curves. On the other hand, by [GF15, Theorem 3.35] the $g'^{\bullet}_n\omega$-integral curves in X'_n are either the exceptional divisors of σ_n or curves $C \subset X'_n$ such that $g'_n(C)$ is ω-integral in X_0. $\square$

Corollary 2.6. *There are no curves of geometric genus ≤ 1 in X_n for any $m > 2$.*

Proof. By Theorem 2.5, if C is a curve in X'_n of geometric genus 0 or 1 which is not an exceptional curve of $\sigma_n \colon X'_n \to X_n$, then $g'_n(C)$ is ω-integral. All ω-integral curves are fibres of $\mathbb{P}^1 \times \mathbb{P}^1$. But, since $n > 2$ and $m > 2$, we have that the preimage of a fibre in $\mathbb{P}^1 \times \mathbb{P}^1$ has geometric genus bigger than 1. Therefore the only curves of geometric genus 0 or 1 in X'_n are the exceptional curves. As σ_n contracts them, X_n has no such curves. $\square$

Remark 2.7. Theorem 2.5 is not true for $m = 2$, since otherwise we would obtain all curves with geometric genus ≤ 1 from fibres, and that is not the case (see Section 3). The problem is that Lemma 2.3 does not work for $m = 2$, showing optimality in that sense. We will revisit this issue in Section 4.

Example 2.8. Under the Bombieri-Lang conjecture, Corollary 2.6 says that, for example, the complete intersection surface

$$\prod_{i=1}^{m}(x_0 - ix_1 + i^2x_2) = x_4^m \qquad \prod_{i=1}^{m}(x_0 - (i+m)x_1 + (i+m)^2x_2) = x_5^m$$

$$\prod_{i=1}^{m}(x_0 - (i-m)x_1 + (i-m)^2x_2) = x_6^m \qquad x_0x_3 + x_2^2 = x_1x_2$$

in $\mathbb{P}^6$ can only have a finite number of points in $\mathbb{P}^6(\mathbb{Q})$. Here $n = 3$, and we have chosen a specific model and set of parameters in $\mathbb{Z}$.

3. The surface of cuboids

We know that Theorem 2.5 does not work for the surface of cuboids

$$z_0z_3 = z_4^2 \qquad (z_0 - z_3)^2 + (z_1 + z_2)^2 = z_5^2$$

$$(z_0 + z_3)^2 - (z_1 + z_2)^2 = z_6^2 \qquad z_0z_3 = z_1z_2$$

in $\mathbb{P}^6$, since here $m = 2$. The purpose of this section is to adapt the method to get some results on rational curves of this surface. We follow the notation of Section 2 for this particular example, and so the surface of cuboids is denoted by X_3. We recall that X_3 is isomorphic to the original surface of cuboids:

$$x_0^2 + x_1^2 + x_2^2 = x_3^2, \quad x_0^2 + x_1^2 = x_4^2, \quad x_0^2 + x_2^2 = x_5^2, \quad x_1^2 + x_2^2 = x_6^2$$

in $\mathbb{P}^6$, by the isomorphism $z_0 = x_0 - ix_1$, $z_1 = x_3 - x_2$, $z_2 = x_3 + x_2$, $z_3 = x_0 + ix_1$, $z_4 = x_4$, $z_5 = 2x_5$, $z_6 = 2ix_6$, where $i = \sqrt{-1}$. This surface has been extensively studied, because it is related to the Perfect Cuboid Problem of Euler. Some references are [ST10], [vLu00], [B13], [FS16].

There are at least 92 curves of geometric genus zero or one in X_3. They are all smooth (see [vLu00]):

- The irreducible components of $x_0x_1x_2x_3 = 0$, which are 32 rational curves;
- The irreducible components of $x_4x_5x_6 = 0$, which are 12 elliptic curves, corresponding to the pull-backs of the 6 horizontal fibres $\{F_1, \ldots, F_6\}$, and the 6 vertical fibres $\{G_1, \ldots, G_6\}$;
- The curve defined by the equations

  $$x_0 = x_1, \quad x_4 = x_5, \quad \sqrt{2}x_0 = x_6, \quad x_2^2 + x_6^2 = x_3^2, \quad 2x_5^2 + x_6^2 = 2x_3^2,$$

  and the curves obtained as orbits by applying the automorphisms of X_3. This gives us 48 elliptic curves.

It was proved by Stoll and Testa [ST10] that every curve of geometric genus ≤ 1 in X_3 of degree less than or equal to 4 belongs to this list, and they conjectured that these are all the curves with geometric genus ≤ 1 in this surface.

Using the global section (appearing in Section 2)

$$\omega \in H^0(X_0, (2,2) \otimes S^2\Omega^1_{X_0}),$$

from a weaker version of Lemma 2.3 that works for the case $m = 2$, it is obtained the following (cf. Corollary 6.43 and Theorem 6.48 in [GF15]):

Theorem 3.1. *Let C be an irreducible curve on X_3 with strict transform $C' \subseteq X'_3$. If*

$$\deg_{\tilde{C}}(\varphi_C^*(\mathcal{O}_{X_3}(-R+E)\otimes g_3'^*(2,2))\otimes S^2\Omega^1_{\tilde{C}}) = -\deg(C)+(E.C')+4g(C)-4$$

is negative, then $g'_n(C)$ is an ω-integral curve.

This result is not enough to capture all the curves of geometric genus ≤ 1 on X_3, but it allows us to give extra information about these curves, as the following corollaries show (cf. Proposition 1.11 and Corollary 1.13 in [GF15]):

Corollary 3.2. *Every curve of geometric genus zero or one on X_3 contains at least two of the 48 singular points of X_3.*

It was known by work of Freitag and Salvati Manni [FS16] (from [B13]) that a curve of geometric genus zero or one must contain at least one of the 48 singular points of X_3.

Corollary 3.3. *Let C be an irreducible curve in X_3, smooth at the singularities of this surface (C can have singularities outside of the 48 singular points of X_3). Then*

$$\deg(C) \leq 4g(C) + 44.$$

This was also obtained by Kani for smooth curves [Kan14], using different methods, and it improves a result of Freitag and Salvati Manni from [FS16]. In this work we want to improve these results, by using different global twisted differentials at the same time, in order to get better control on the exceptional divisors.

Recall the tower of cyclic covers of degree $m = 2$

$$X_3 \to X_2 \to X_1 \to \mathbb{P}^1 \times \mathbb{P}^1.$$

In this case R consists of the horizontal fibres at the points

$$[1:1],\ [1:-1],\ [1:i],\ [1:-i],\ [1:0],\ [0:1],$$

and the vertical fibres at the same points. We will denote the horizontal fibre at $[a:b]$ by $h_{b/a}$ and the vertical fibre at $[a:b]$ by $v_{b/a}$, with the convention that $1/0 = \infty$.

The image of $x_4 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ consists of $h_1 \cup h_{-1} \cup v_1 \cup v_{-1}$, the image of $x_5 = 0$ consists of $h_i \cup h_{-i} \cup v_i \cup v_{-i}$, and the image of $x_6 = 0$ consists of $h_0 \cup h_\infty \cup v_0 \cup v_\infty$. Thus, each of these consists of two horizontal fibres and two vertical fibres, intersecting at 4 points. Over each of the 12 intersection points, there are 4 out of the 48 singular points of X_3, and every singular points of X_3 maps to one of these intersections.

The image of $x_0 = 0$ under g_3 is the curve $C_0 = \{xz+yw=0\}$. Similarly, the image of $x_1 = 0$ is $C_1 = \{yw - xz = 0\}$, the image of $x_2 = 0$ is $C_2 = \{yz - xw = 0\}$, and the image of $x_3 = 0$ is $C_3 = \{xw + yz = 0\}$.

Consider the following global sections in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, (3,3) \otimes S^2\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1})$:

$$\omega_0 = (xz - yw)y^2z^2dxdw \qquad \omega_1 = (yw - xz)y^2z^2dxdw$$
$$\omega_2 = (yz - xw)y^2z^2dxdw \qquad \omega_3 = (xw + yz)y^2z^2dxdw.$$

For each $0 \leq i \leq 3$, the ω_i-integral curves consist of the horizontal fibres, the vertical fibres, and C_i.

Let E_i be the sum of the exceptional divisors from the 24 singular points of X_3 whose image belongs to C_i, and let E_i' be the sum of the exceptional divisors from the 24 singular points whose image does not belong to C_i. We have the following version of Lemma 2.3 adapted to these new global sections.

Lemma 3.4. *For each $0 \leq i \leq 3$, there is ω_i'' in*

$$H^0(X_3', \mathcal{O}_{X_3'}(-R - E_i) \otimes g_3'^*(3,3) \otimes S^2\Omega^1_{X_3'})$$

whose image under the natural morphism

$$H^0\big(X_3', \mathcal{O}_{X_3'}(-R - E_i) \otimes g_3'^*(3,3) \otimes S^2\Omega^1_{X_3'}\big) \to H^0\big(X_3', g_3'^*(3,3) \otimes S^2\Omega^1_{X_3'}\big)$$

is $g_3'^{\bullet}\omega_i$.

Proof. Fix $0 \leq i \leq 3$. We know that $g_3'^{\bullet}\omega_i$ vanishes along R. We will prove that $g_3'^{\bullet}\omega$ vanishes along the divisor E_i.

We consider a node P of the branch divisor of g_k for some $0 < k < 3$, and such that $P \in g_k'(C_i)$. At the preimage Q of P, we have a singularity of type A_1. This singularity is locally analytically isomorphic to $(z^2 - xy) \subset \mathbb{C}^3$ and $g(x, y, z) = (x, y)$.

Let E be the exceptional divisor at Q, let $u_1 = 0$ be the local coordinate defining it, and, as before, the local coordinates $u_0 = 0$, $u_2 = 0$ define the (reduced) preimages of $x = 0$ and $y = 0$ respectively. Then we have

$$dx = 2u_0u_1du_0 + u_0^2du_1, \qquad dy = u_2^2du_1 + 2u_1u_2du_2.$$

Via a local trivialization of ${g'}_3^*(3,3)$, we can identify ω_0 with a section of $S^2\Omega^1_{\mathbb{C}^2}$ around $(0,0)$ as $\omega_0 = (x-y)\bar{\omega}$ with $\bar{\omega} = a_0dx^{\otimes 2} + a_1dxdy + a_2dy^{\otimes 2}$. Since ${g'}_3^*(x) = u_0^2u_1$, and ${g'}_3^*(y) = u_1u_2^2$, we obtain that the pull-back of ω_0 vanishes along E with order one. Doing this for every singular point contained in C_0, we obtain that ${g'}_3^{\bullet}\omega_0$ vanishes along the divisor E_0. A similar computation shows that for every i, the pull-back of ω_i vanishes along the divisor E_i. $\square$

Theorem 3.5. *Let $C \subset X_3'$ be a curve of geometric genus 0, which is not an exceptional curve of σ_3, and it is not in the pull-back of the C_i's. Then for every $0 \leq i \leq 3$, we have $(C.E_i') \geq 4$.*

Proof. From Proposition 3.88 in [GF15], we have that the ω_i''-integral curves in X_3' are among the pull-back of the horizontal fibres, the pull-back of the vertical fibres and the curve C_i. Let $C \subset X_3'$ be a curve of geometric genus zero. We have

$$\mathcal{O}_{X_3'}(-R - E_i) \otimes g_3'^*(3,3) = \mathcal{O}_{X_3'}(-R - E_i + R + E) = \mathcal{O}_{X_3'}(E_i'),$$

thus if for some $0 \leq i \leq 3$ we have $(C.E_i) < 4$, then we obtain

$$\deg_{\tilde{C}}(\varphi_C^*\mathcal{O}_{X_3'}(-R - E_i) \otimes \varphi_C^* g_3'^*(3,3) \otimes S^2\Omega^1_{X_3'}) = (C.E_i') - 4 < 0,$$

hence C must be an ω_i''-integral curve. $\square$

Corollary 3.6. *Let $C \subset X_3'$ be a curve of geometric genus 0, which is not an exceptional curve of σ_3, and not in the pull-back of the C_i's. Then $(C.E) \geq 8$.*

Proof. We know that $(C.E_i') \geq 4$ for each $0 \leq i \leq 3$. Since $E_0' + E_1' + E_2' + E_3' = 2E$, we obtain $(C.E) \geq 8$. $\square$

4. Low genus curves in cyclic covers

4.1. **Local picture.** We first recall the local picture of a cyclic cover, together with cyclic quotient singularities, and their minimal resolution.

Let $0 < q < m$ be integers with $\gcd(q,m) = 1$. Consider the action of $\tau(x,y) = (\mu x, \mu^q y)$ on $\mathbb{C}^2$, where μ is a primitive m-th root of 1. A *cyclic quotient singularity* $\frac{1}{m}(1,q)$ is a germ at the origin of the quotient of $\mathbb{C}^2$ by $\langle \tau \rangle$; cf. [BHPV04, III §5]. For us it will be useful the following toric description. Consider the inclusions of rings

$$\mathbb{C}[x^m, y^m] \subset \mathbb{C}[x^m, y^m, x^{m-q}y] \subset \mathbb{C}[x,y]^{\langle \tau \rangle} \subset \mathbb{C}[x,y].$$

We note that

$$\mathbb{C}[x^m, y^m, x^{m-q}y] \simeq \mathbb{C}[u,v,w]/(uv^{m-q} - w^m),$$

where $v = x^m$, $u = y^m$, and $w = x^{m-q}y$. The inclusions define morphisms between the corresponding spectrums of the rings, which translates into the maps

$$\mathbb{C}^2 \xrightarrow{q} \mathbb{C}^2/\langle \tau \rangle \xrightarrow{\eta} (uv^{m-q} - w^m) \subset \mathbb{C}^3 \xrightarrow{r} \mathbb{C}^2$$

where $r(u,v,w) = (u,v)$ is the cyclic cover branch along $\{uv^{m-q} = 0\}$ of degree m, η is the normalization map, and q is the quotient map. As in [BHPV04, III §5], around $(0,0) \in \mathbb{C}^2$ the local picture for any cyclic cover of degree m is given by $\{u^a v^b = w^m\} \subset \mathbb{C}^3 \to \mathbb{C}^2$, $(u,v,w) \mapsto (u,v)$, where $\gcd(a,m)$=$\gcd(b,m) = 1$, and q is such that $aq + b \equiv 0$ modulo m.

Let $\sigma \colon \widetilde{Y} \to Y$ be the minimal resolution of $Y := \frac{1}{m}(1,q)$. Figure 1 shows the exceptional curves $E_i = \mathbb{P}^1$ of σ, for $1 \leq i \leq s$, and the strict transforms E_0 and E_{s+1} of $(y = 0)$ and $(x = 0)$ respectively.

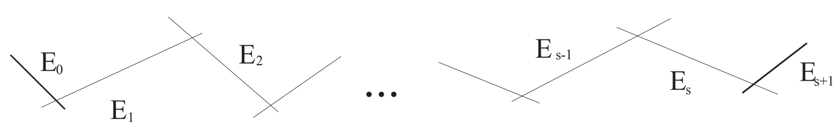

Figure 1. Exceptional divisors over $\frac{1}{m}(1,q)$, E_0 and E_{s+1}

The numbers $E_i^2 = -b_i$ are computed using the *Hirzebruch-Jung continued fraction*

$$\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}} =: [b_1, \ldots, b_s].$$

The continued fraction $[b_1, \ldots, b_s]$ defines the sequence of integers

$$0 = \beta_{s+1} < 1 = \beta_s < \ldots < q = \beta_1 < m = \beta_0$$

where $\beta_{i+1} = b_i \beta_i - \beta_{i-1}$. In this way, $\frac{\beta_{i-1}}{\beta_i} = [b_i, \ldots, b_s]$. Partial fractions $\frac{\alpha_i}{\gamma_i} = [b_1, \ldots, b_{i-1}]$ are computed through the sequences

$$0 = \alpha_0 < 1 = \alpha_1 < \ldots < q^{-1} = \alpha_s < m = \alpha_{s+1},$$

where $\alpha_{i+1} = b_i\alpha_i - \alpha_{i-1}$ (q^{-1} is the integer such that $0 < q^{-1} < m$ and $qq^{-1} \equiv 1 (\mathrm{mod}\ m)$), and $\gamma_0 = -1$, $\gamma_1 = 0$, $\gamma_{i+1} = b_i\gamma_i - \gamma_{i-1}$. We have $\alpha_{i+1}\gamma_i - \alpha_i\gamma_{i+1} = -1$, $\beta_i = q\alpha_i - m\gamma_i$, and $\frac{m}{q^{-1}} = [b_s, \ldots, b_1]$. These numbers appear in the pull-back formulas

$$g'^*\big((u=0)\big) = \sum_{i=0}^{s+1} \beta_i E_i, \qquad g'^*\big((v=0)\big) = \sum_{i=0}^{s+1} \alpha_i E_i,$$

where $g' := \sigma \circ \eta \circ r$, and $K_{\widetilde{Y}} \equiv \sigma^*(K_Y) + \sum_{i=1}^{s}(-1 + \frac{\beta_i + \alpha_i}{m})E_i$. The numbers $d_i := -1 + \frac{\beta_i + \alpha_i}{m}$ are the *discrepancies* of E_i. Let u_i be a local coordinate defining E_i, so that at each node of $E_0, E_1, \ldots, E_s, E_{s+1}$ we have local coordinates u_i, u_{i+1} for $\widetilde{Y}$. Then (see [R03]) we have that locally g' is given by

$$g'(u_i, u_{i+1}) = (u_i^{\beta_i} u_{i+1}^{\beta_{i+1}}, u_i^{\alpha_i} u_{i+1}^{\alpha_{i+1}}).$$

Therefore we have the pull-back relations

$$du = \beta_i u_i^{\beta_i - 1} u_{i+1}^{\beta_{i+1}} du_i + \beta_{i+1} u_i^{\beta_i} u_{i+1}^{\beta_{i+1}-1} du_{i+1}$$

and

$$dv = \alpha_i u_i^{\alpha_i - 1} u_{i+1}^{\alpha_{i+1}} du_i + \alpha_{i+1} u_i^{\alpha_i} u_{i+1}^{\alpha_{i+1}-1} du_{i+1}.$$

4.2. **Global picture.** The following is taken from [U10, Section 1]. Let X be a smooth projective surface over $\mathbb{C}$, and let $\sum_{j=1}^{d} D_j$ be a simple normal crossings divisor in X, that is, the irreducible curves D_j are all smooth, and the singularities of the divisor are at most nodes. Let us assume the existence of a line bundle $\mathcal{M}$ on X such that

$$\mathcal{O}_X\big(a_1D_1 + a_2D_2 + \ldots + a_dD_d\big) \simeq \mathcal{M}^{\otimes m}$$

for some integers $0 < a_j < m$ such that $\gcd(a_j, m) = 1$. With this data, one constructs a smooth projective surface Y' which represents the "m-th root of $D := \sum_{j=1}^{d} a_jD_j$" as follows. Let $s \in H^0(X, \mathcal{O}_X(D))$ be a section whose zero locus is D. This section defines a structure of $\mathcal{O}_X$-algebra on $\bigoplus_{j=0}^{m-1} \mathcal{M}^{-j}$ by means of the induced injection $\mathcal{M}^{-m} \simeq \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$. Then we have the affine morphism $f_0 \colon Y_0 \to X$, where $Y_0 := \mathrm{Spec}_X\left(\bigoplus_{i=0}^{m-1} \mathcal{M}^{-i}\right)$. The variety Y_0 might not be normal. To normalize it, we define the line bundles

$$\mathcal{M}^{(i)} := \mathcal{M}^i \otimes \mathcal{O}_X\Big(-\sum_{j=1}^{d}\Big[\frac{a_j i}{m}\Big]D_j\Big)$$

on X for $0 \leq i < m$. Then $\eta \colon Y := \mathrm{Spec}_X\left(\bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)}\right) \to Y_0$ is the normalization of Y_0. Hence if $f \colon Y \to X$ is the composition of η with f_0, then $f_*\mathcal{O}_Y = \bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)}$. We note that Y may have only cyclic quotient singularities over the nodes of $\sum_{j=1}^{d} D_j$. More precisely, given a node in $D_i \cap D_j$, we have one singularity in Y over that node (since $\gcd(a_j, m) = 1$ for all j), and it is of type $\frac{1}{m}(1, q)$ where $a_iq + a_j \equiv 0$ modulo m. Locally around that singularity, the map $f \colon Y \to X$ is isomorphic to the local picture described in Subsection 4.1, i.e. it is $\mathbb{C}^2/\langle\tau\rangle \to \mathbb{C}^2$ where $\{u = 0\} = D_i$ and $\{v = 0\} = D_j$.

Let $\sigma\colon Y' \to Y$ be the minimal resolution of the singularities in Y. The surface Y' is a smooth (irreducible) projective surface. Let $f'\colon Y' \to X$ be the composition of σ with f. Then $f'_* \mathcal{O}_{Y'} = \bigoplus_{i=0}^{m-1} \mathcal{M}^{-(i)}$. Again, the local picture of f' over a node of $D_i \cap D_j$ is as in Subsection 4.1, and so f is locally isomorphic to $\sigma \circ \eta \circ r$.

Now let us consider $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_X)$ for some line bundle $\mathcal{L}$ on X, and some integer $r > 0$.

Theorem 4.1. *Assume that D_j is ω-integral (Definition 2.1) for all j. If $\sum_{j=1}^d D_j - m\mathcal{L}$ is ample and $a_j \not\equiv -a_{j'}$ modulo m for all $j \neq j'$, then Y can have curves curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in X.*

Proof. The proof follows the strategy of Section 2: Lemma 2.3, Theorem 2.5, and Corollary 2.6. As in Lemma 2.3, let us prove the existence of ω'' in $H^0(Y', \mathcal{O}_{Y'}(-(m-1)R - E) \otimes f'^*\mathcal{L} \otimes S^r\Omega^1_{Y'})$, where R is the sum of the strict transforms of the D_j, and $E = \sum_k E_k$ is the sum of all exceptional curves of σ. This is a local computation, and so let $D_i = \{u = 0\}$ and $D_j = \{v = 0\}$ at a node of $D_i \cap D_j$ in X. We assume that the cyclic quotient singularity is $\frac{1}{m}(1, q)$ with continued fraction of length s. Following the proof of Lemma 2.3, we only need to check that the pull-back of $du^{\otimes r-k} \otimes dv^{\otimes k}$ by f' vanishes on E_l, where $0 < k < r$ and $0 \leq l \leq s+1$. According to the local computation in Subsection 4.1, this happens if and only if $\alpha_l > 1$ or $\beta_l > 1$. So assume that $\alpha_l \leq 1$ and $\beta_l \leq 1$. If $\alpha_l = 0$, then $l = 0$ and so $\beta_l = \beta_0 = m > 1$ a contradiction. The same for $\beta_l = 0$. If $\alpha_l = 1$, then $l = 1$ and $\beta_l = \beta_1 = q \geq 1$. Hence $q = 1$, but this singularity is $\frac{1}{m}(1, 1)$ and so the multiplicities a_i, a_j of D_i, D_j respectively must satisfy $a_i + a_j \equiv 0$ modulo m. But this is contrary to our assumptions. Same for $\beta_l = 1$. Therefore for any l we have $\alpha_l > 1$ or $\beta_l > 1$.

On the other hand, we have the numerical equivalence

$$f'^*\Big(\sum_{j=1}^d D_j\Big) \equiv mR + m\sum_k (1 + d_k)E_k,$$

where d_k is the discrepancy associated to E_k (see the end of Subsection 4.1). Hence we obtain

$$\frac{1}{m} f'^*\Big(-\sum_{j=1}^d D_j\Big) - (m-2)R + \sum_k d_k E_k + f'^*\mathcal{L} \equiv -(m-1)R - E + f'^*\mathcal{L}.$$

We recall that $-1 < d_k < 0$ for all k.

Let $\mathcal{N} := -(m-1)R - E + f'^*\mathcal{L}$, let $C \subset Y'$ be a curve of geometric genus g, and not exceptional for σ. Let $\varphi_C\colon \tilde{C} \to Y'$ be the normalization of $C \subset Y'$. Then

$$\deg_{\tilde{C}}\left(\varphi_C^*\mathcal{N} \otimes S^r\Omega^1_{\tilde{C}}\right) \leq \frac{1}{m}\Big(-\sum_{j=1}^d D_j + m\mathcal{L}\Big) \cdot f'(C) + r(2g-2)$$

by the projection formula. By our hypothesis we have

$$\Big(-\sum_{j=1}^{d} D_j + m\mathcal{L}\Big)\cdot f'(C) < 0,$$

and so if $g \leq 1$, then $\deg_{\tilde{C}}\left(\varphi_C^*\mathcal{N}\otimes S^r\Omega^1_{\tilde{C}}\right) < 0$, and so the curve C is ω''-integral. As in Lemma 2.3, we conclude that $f'(C)$ must be an ω-integral curve in X. Since $\sigma\colon Y' \to Y$ is a birational morphism contracting E, we obtain that Y can have curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in X.

□

Remark 4.2. The assumption $a_i + a_j \not\equiv 0$ modulo m is to avoid the situation of cyclic quotient singularities of type $\frac{1}{m}(1,1)$. As we saw in the proof and in Sections 2 and 3 for the A_1 rational double points, the singularities $\frac{1}{m}(1,1)$ do not work for the existence of ω''.

We finish this section with an explicit example, where $X = \mathbb{P}^2$ and D_j are lines. Let us consider the lines $L_{t,u} = (t^2x + tuy + u^2z) \subset \mathbb{P}^2$ for $[t,u] \in \mathbb{P}^1$. They are precisely the tangent lines to the conic $(y^2 - 4xz) \subset \mathbb{P}^2$. Let $\{L_1, \ldots, L_d\}$ be distinct lines such that $L_i = L_{t_i,u_i}$ for some $[t_i, u_i] \in \mathbb{P}^1$. Let us take positive integers $a_1, \ldots, a_d$ such that $\sum_{i=1}^d a_i = mR$ for some integers $m, R > 0$. Assume that $a_i < m$ and $\gcd(a_i, m) = 1$ for all i, and that $a_i + a_j$ is not divisible by m for all $i \neq j$.

Corollary 4.3. *If $4m < d$, then the surface*

$$(t_1^2x + t_1u_1y + u_1^2z)^{a_1}\cdots(t_d^2x + t_du_dy + u_d^2z)^{a_d} = w^m$$

in $\mathbb{P}(1,1,1,R)$ contains no curves of geometric genus ≤ 1 apart from the $\mathbb{P}^1$'s defined by $t_i^2x + t_iu_iy + u_i^2z = 0$. Its normalization is a simply connected normal projective surface with ample canonical class.

Proof. First we need to indicate $\mathcal{L}$, r and ω. By taking the differential of $t_1^2x + t_1u_1y + u_1^2z$ for $u = 1$ and $z = 1$, we obtain the differential

$$\omega = dx^2 - ydxdy + xdy^2$$

which is a global section of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)\otimes S^2\Omega^1_{\mathbb{P}^2})$, and so $\mathcal{L} := \mathcal{O}(4)$ and $r = 2$. Since we are considering this particular ω, we know that the lines $(t_1^2x + t_1u_1y + u_1^2z)$ are all ω-integral. By essentially [GF15, Theorem 3.76], these lines are all the ω-integral curves together with the discriminant curve $(y^2 - 4xz)$.

For the cyclic cover, we are considering $D_j := L_j$ for all j, and

$$\mathcal{O}_{\mathbb{P}^2}\Big(\sum_{j=1}^{d} a_jD_j\Big) \simeq \mathcal{O}_{\mathbb{P}^2}(R)^{\otimes m},$$

and so $\mathcal{M} := \mathcal{O}_{\mathbb{P}^2}(R)$. Note that the variety Y_0 can be considered as

$$Y_0 = \{(t_1^2x + t_1u_1y + u_1^2z)^{a_1}\cdots(t_d^2x + t_du_dy + u_d^2z)^{a_d} = w^m\} \subset \mathbb{P}(1,1,1,R).$$

We also have that $\sum_{j=1}^d D_j - m\mathcal{L} = \mathcal{O}(d - 4m)$, and so it is ample by assumption. Then we can apply Theorem 4.1, and we obtain that Y has curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in $\mathbb{P}^2$. A simple calculation with Riemann-Hurwitz says that the only preimages of ω-integral curves which give a curve of geometric genus 0 or 1 are the ramification curves, with genus 0 indeed. With the normalization map $\eta\colon Y \to Y_0$, we have no modifications on geometric genus of curves, so the same statement holds for Y_0. The claim on simply connectedness and ampleness of canonical class for Y follows from [U10, Theorem 8.5] and the Canonical class formula [U10, Proposition 1.4], which is generalized Riemann-Hurwitz.

□

5. Low genus curves in towers of cyclic covers

In this section we put all together to give the construction of a wide range of algebraic surfaces in which we can control curves of geometric genus ≤ 1.

Theorem 5.1. *Let X be a smooth projective surface, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_X)$. Assume we have the relations $m_i\mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j}D_{i,j}$ in $Pic(X)$, for some line bundles $\mathcal{M}_i$, with $0 < a_{i,j} < m_i$ and $gcd(a_{i,j}, m_i) = 1$ for all i, j. Assume also that the divisor $\sum_{i=1}^{n}\sum_{j=1}^{s_i} D_{i,j}$ has simple normal crossings, and $D_{i,j}$ are ω-integral curves. Then a tower of n cyclic covers of degree m*

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 := X$$

is defined, where all X_k are normal projective surfaces with only cyclic quotient singularities.

If

$$\sum_{i=1}^{n} \frac{1}{m_i}\Big(\sum_{j=1}^{s_i} D_{i,j}\Big) - \mathcal{L} \text{ is } \mathbb{Q}\text{-ample, and } a_{i,j} \not\equiv -a_{i,j'} (\mathrm{mod}\, m_i) \text{ for all } i,\ j \neq j',$$

then X_n can have curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in X.

Proof. The expressions $m_i\mathcal{M}_i = \sum_{j=1}^{s_i} a_{i,j}D_{i,j}$ in $\mathrm{Pic}(X)$ define a tower of n cyclic covers of degree m_i

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 := X$$

inductively as follows: Let $f_{k+1}\colon X_{k+1} \to X_k$ be the cyclic cover defined by

$$g_k^*\mathcal{O}_X\big(\sum_{j=1}^{s_{k+1}} a_{k+1,j}D_{k+1,j}\big) \simeq g_k^*\mathcal{M}_i^{\otimes m_i},$$

where $g_k := f_1 \circ \cdots \circ f_k$ as done in Subsection 4.2. We note that if $D'_{k+1,j}$ is the strict transform of $D_{k+1,j}$ under g_k, then $g_k^*\mathcal{O}_X\big(\sum_{j=1}^{s_{k+1}} a_{k+1,j}D_{k+1,j}\big) = \mathcal{O}_{X_k}\big(\sum_{j=1}^{s_{k+1}} a_{k+1,j}D'_{k+1,j}\big)$, and $\sum_{j=1}^{s_{k+1}} D'_{k+1,j}$ is a simple normal crossings divisor. All X_k are normal projective surfaces with cyclic quotient singularities. Let $\sigma_k\colon X'_k \to X_k$ be the minimal resolution of all singularities in X_k. The surface X'_k is a smooth (irreducible) projective surface. Let $g'_k\colon X'_k \to X$ be the composition of σ_k with g_k.

The proof follows again the strategy of Section 2: Lemma 2.3, Theorem 2.5, and Corollary 2.6. As in Theorem 4.1, one proves the existence of ω'' in $H^0(X'_n, \mathcal{O}_{X'_n}(-R-E) \otimes {g'}_n^{*}\mathcal{L} \otimes S^r\Omega^1_{X'_n})$, where $R = \sum_{i=1}^{n}(m_i - 1)R_i$ and

R_i is the sum of the strict transforms of the $\sum_{j=1}^{s_i} D_{i,j}$, and $E = \sum_k E_k$ is the sum of all exceptional curves of σ_n. For that it is key the hypothesis $a_{i,j} \not\equiv -a_{i,j'} (\text{mod } m_i)$ for all $i, j \neq j'$.

On the other hand, we have the numerical equivalence

$$g'^*_n\Big(\sum_{j=1}^{s_i} D_{i,j}\Big) \equiv m_i R_i + m_i \sum_{k_i} (1 + d_{k_i}) E_{k_i},$$

where the sum of exceptional curves runs over the singularities due to $\sum_{j=1}^{s_i} D_{i,j}$, and d_{k_i} is the discrepancy associated to E_{k_i} (see the end of Subsection 4.1).

Hence we obtain that

$$-g'^*_n\Big(\sum_{i=1}^{n} \frac{1}{m_i}\Big(\sum_{j=1}^{s_i} D_{i,j}\Big)\Big) - \sum_{i=1}^{n} (m_i - 2) R_i + \sum_k d_k E_k + g'^*_n \mathcal{L}$$

is numerically equivalent to $\mathcal{N} := -R - E + g'^*_n \mathcal{L}$. We recall that $-1 < d_k < 0$ for all k.

Let $C \subset X'_n$ be a curve of geometric genus g, and not exceptional for σ_n. Let $\varphi_C \colon \tilde{C} \to X'_n$ be the normalization of $C \subset X'_n$. Then

$$\deg_{\tilde{C}}\left(\varphi_C^* \mathcal{N} \otimes S^r \Omega^1_{\tilde{C}}\right) \leq \Big(-\sum_{i=1}^{n} \frac{1}{m_i}\Big(\sum_{j=1}^{s_i} D_{i,j}\Big) + \mathcal{L}\Big) \cdot g'_n(C) + r(2g-2)$$

by the projection formula. By our hypothesis we have

$$\Big(-\sum_{i=1}^{n} \frac{1}{m_i}\Big(\sum_{j=1}^{s_i} D_{i,j}\Big) + \mathcal{L}\Big) \cdot g'_n(C) < 0,$$

and so if $g \leq 1$, then $\deg_{\tilde{C}}\left(\varphi_C^* \mathcal{N} \otimes S^r \Omega^1_{\tilde{C}}\right) < 0$, and so the curve C is ω''-integral. As in Lemma 2.3, we conclude that $g'_n(C)$ must be an ω-integral curve in X. Since $\sigma_n \colon X'_n \to X_n$ is a birational morphism contracting E, we obtain that X_n can have curves of geometric genus ≤ 1 only in the set of preimages of ω-integral curves in X. $\square$

6. Hyperbolicity

By using certain result in Nevanlinna theory for complex varieties (cf. [V00, V00b]), we will prove that if the normal projective surface X_n in Theorem 5.1 contains no curves of geometric genus ≤ 1 and $\omega = 0$ has only algebraic solutions, then in fact X_n is hyperbolic, that is, the only holomorphic maps $f \colon \mathbb{C} \to X_n$ are constant. These in practice produce many families of smooth projective surfaces of general type which are hyperbolic. For example, at the end of this section we prove existence of hyperbolic complete intersection surfaces of low degrees.

We recall that an *entire curve* in a variety X is a nonconstant holomorphic map $\mathbb{C} \to X$. We begin with some definitions in Nevanlinna theory (cf. [V11, Section 11]). Let X be a smooth projective surface, let $f \colon \mathbb{C} \to X$ be an entire curve, and let D be a divisor on X whose support does not contain the image of f. We define the *counting function* of D in X to be

$$N_f(D,R) = \sum_{0<|z|<R} \mathrm{ord}_z f^*D \cdot \log\Big(\frac{R}{|z|}\Big) + \mathrm{ord}_0 f^*D \cdot \log(R).$$

Let λ be a Weil function for D. We define the *proximity function* for f relative to D to be

$$m_f(D,R) = \int_0^{2\pi} \lambda(f(Re^{i\theta}))\frac{d\theta}{2\pi}.$$

It is defined up to $O(1)$. We can now define the *height* of f relative to D by

$$T_{D,f}(R) = m_f(D,R) + N_f(D,R).$$

The *height* of f relative to an invertible sheaf $\mathcal{L}$ on X is $T_{\mathcal{L},f}(R) = T_{D,f}(R) + O(1)$, where D is any divisor such that $\mathcal{L} \simeq \mathcal{O}_X(D)$.

The following result is [V00b, Corollary 5.2] for $D = 0$ (see also [V00, Proposition 6.1.1]). We use the notation $\log^+(a) = \max\{0, \log(a)\}$.

Theorem 6.1. *Let X be a smooth complex projective variety, let $f\colon \mathbb{C} \to X$ be an entire curve, let r be a positive integer, let $\mathcal{L}$ be a line bundle on X, let ω be a global section of $\mathcal{L}^\vee \otimes S^r\Omega^1_{X/\mathbb{C}}$, and let $\mathcal{A}$ be a line bundle which is big on the Zariski closure of $f(\mathbb{C})$. If $f^*\omega \neq 0$, then*

$$T_{\mathcal{L},f}(R) \leq_{exc} O(\log^+ T_{\mathcal{A},f}(R)) + o\big(\log(R)\big),$$

where the notation $\leq_{exc}$ means that the inequality holds for all $R > 0$ outside of a set of finite Lebesgue measure.

The following will be the main tool to prove hyperbolicity for surfaces.

Corollary 6.2. *Let X be a smooth projective surface, let $\mathcal{L}$ be a big line bundle on X, let $r > 0$ be an integer, and let $\omega \in H^0(X, \mathcal{L}^\vee \otimes S^r\Omega^1_X)$. Assume that $f\colon \mathbb{C} \to X$ is an entire curve whose image is Zariski dense. Then $f^*\omega = 0$.*

Proof. By [V11, Prop. 11.11], we have that $T_{\mathcal{O}(1),f}(R) \leq CT_{\mathcal{L},f}(R) + O(1)$ for all $R > 0$, and a constant $C > 0$ depending on $\mathcal{O}(1)$ and $\mathcal{L}$. Here $\mathcal{O}(1)$ is the hyperplane line bundle given by some embedding of X in a projective space. On the other hand, we have that there are constants $M > 0$ and N such that $M\log(R) + N \leq T_{\mathcal{O}(1),f}(R)$ for all $R > 0$, and so there are constants $M' > 0$ and N' such that $M'\log(R) + N \leq T_{\mathcal{L},f}(R)$.

Let $f^*\omega \neq 0$. From Theorem 6.1 we have that

$$T_{\mathcal{L},f}(R) \leq_{exc} S\log(T_{\mathcal{L},f}(R)) + \epsilon\log(R)$$

for a constant $S > 0$, a given $0 < \epsilon < M'/4$, and $R >> 0$ out of a set of finite Lebesgue measure. But $\log(T_{\mathcal{L},f}(R)) < T_{\mathcal{L},f}(R)/2S$ for $r >> 0$, and so by the inequality above we get $T_{\mathcal{L},f}(R) <_{exc} 2\epsilon\log(R) < M'/2\log(R)$, for certain $R >> 0$. But this contradicts $M'\log(R) + N \leq T_{\mathcal{L},f}(R)$. Therefore $f^*\omega = 0$. $\square$

The following theorem gives a criteria for hyperbolicity of the singular surfaces X_n in Theorem 5.1.

Theorem 6.3. *Let us consider the hypothesis and the notation in Theorem 5.1. In addition, assume that all solutions to the differential equation given by $\omega = 0$ on X are ω-integral curves. Then an entire curve in X_n must be contained in the set of preimages of ω-integral curves in X.*

In particular, if the set of preimages of ω-integral curves in X does not contain curves of geometric genus 0 or 1, then X_n is hyperbolic.

Proof. Let $\sigma_n \colon X'_n \to X_n$ be the minimal resolution of singularities of X_n. Consider an entire curve $f \colon \mathbb{C} \to X_n$. Then it has a lifting $f' \colon \mathbb{C} \to X'_n$. Assume that $f'(\mathbb{C})$ is Zariski dense in X'_n. We have a section

$$\omega'' \in H^0(X'_n, \mathcal{O}_{X'_n}(-R-E) \otimes {g'}^*_n \mathcal{L} \otimes S^r \Omega^1_{X'_n}),$$

and a line bundle $\mathcal{N} := \mathcal{O}_{X'_n}(-R-E) \otimes {g'}^*_n \mathcal{L}$ which is numerically

$$\mathcal{N}^\vee \equiv {g'}^*_n \Big(-\mathcal{L} + \sum_{i=1}^{n} \frac{1}{m_i} \Big(\sum_{j=1}^{s_i} D_{i,j} \Big) \Big) + \sum_{i=1}^{n} (m_i - 2) R_i - \sum_k d_k E_k,$$

where $-1 < d_k < 0$ are the discrepancies of E_k, and $-\mathcal{L} + \sum_{i=1}^n \frac{1}{m_i} \Big(\sum_{j=1}^{s_i} D_{i,j} \Big)$ is an ample divisor in X by hypothesis. Therefore $\mathcal{N}^\vee$ is numerically the sum of the pull-back of an ample divisor plus an effective divisor. It is easy to see that $\mathcal{N}^\vee$ is then big by e.g. [Laz, Corollary 2.2.7]. Therefore by Corollary 6.2 we have that $f^*\omega'' = 0$. But then, locally analytical f satisfies the differential equation given by ${g'}^*_n \omega$, and then this gives a solution to the differential equation given by ω. By hypothesis we know that all solutions are given by ω-integral curves in X, and so this contradicts the Zariski density of $f(\mathbb{C})$.

Therefore $f(\mathbb{C})$ must be contained in an irreducible algebraic curve. But by Liouville's theorem, the geometric genus of this irreducible curve must be less than or equal to 1. Moreover, by Theorem 5.1, all curves of geometric genus ≤ 1 are in the set of preimages of ω-integral curves in X. □

The following corollaries give a proof of Theorem 1.6.

Corollary 6.4. *Let $n \geq 3$, and let $m_i \geq 3$ be n integers. Let $\{a_{i,j}\}$ and $\{b_{i,j}\}$ be two collections of distinct $\sum_{i=1}^n m_i$ complex numbers. Let $\{G_i = G_i(z_0, z_1, z_2, z_3)\}_{i=1}^n$ be a collection of n homogeneous polynomials of degree m_i, such that $z_0 - a_{i,j} z_1 - b_{i,j} z_2 + a_{i,j} b_{i,j} z_3$ does not divide G_i. Then the complete intersection*

$$\prod_{j=1}^{m_i} (z_0 - a_{i,j} z_1 - b_{i,j} z_2 + a_{i,j} b_{i,j} z_3) + t_i G_i = z_{3+i}^{m_i}, \quad z_0 z_3 - z_1 z_2 = 0$$

for $i = 1, \ldots, n$ in $\mathbb{P}^{n+3}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$.

Proof. Let us evaluate the complete intersection

$$\prod_{j=1}^{m_i} (z_0 - a_{i,j} z_1 - b_{i,j} z_2 + a_{i,j} b_{i,j} z_3) + t_i G_i = z_{3+i}^{m_i}, \quad z_0 z_3 - z_1 z_2 = 0$$

for $i = 1, \ldots, n$ in $\mathbb{P}^{n+3}$ in $t_i = 0$ for all i. We denote this surface by X_n, which is as in the construction of the generalized surfaces of cuboids but for not necessarily equal degrees $m_i > 2$ (see Section 2). As in Corollary 2.6, this surface X_n has no curves of geometric genus ≤ 1. In fact, the construction satisfies the hypothesis in Theorem 6.3, and so X_n has no entire curves. We now consider an small deformation of the branched divisor. It gives a smooth branch divisor, and so a smooth surface $X_n(t_1,\dots,t_n)$. At the same time, this gives a small perturbation of the hyperbolic surface X_n, and it is known that hyperbolicity is preserved by small deformations (see [Kob, p.148 Theorem (3.11.1)]). This is a proof of part (d) of Theorem 1.6. $\square$

Next corollary is proved as the previous one, but we need to take care of the degrees equal to 2, which produce A_1 singularities.

Corollary 6.5. *Let $r \geq 1$, $s \geq 1$ be integers such that $r+s \geq 5$, and let $\{m_i \geq 3\}_{i=r+1}^{r+s}$ be integers. Let $\{b_{i,j}, c_{i,j}\}_{i=r+1,\dots,r+s}^{j=1,\dots,m_i}$ be two collections of $\sum_{i=r+1}^{r+s} m_i$ distinct complex numbers. Let a_i be a collection of r distinct complex numbers such that $a_i \neq \pm a_j$, $b_{i,j} \neq \pm a_k$, $c_{i,j} \neq \pm a_k$, for all i,j,k. Let $\{F_i = F_i(z_0,z_1,z_2,z_3)\}_{i=1}^{r}$ be a collection of r homogeneous polynomials of degree 2, such that $z_0 - a_i z_1 - a_i z_2 + a_i^2 z_3$ and $z_0 + a_i z_1 + a_i z_2 + a_i^2 z_3$ do not divide F_i. Let $\{G_i = G_i(z_0,z_1,z_2,z_3)\}_{i=r+1}^{r+s}$ be a collection of s homogeneous polynomials of degree m_i, such that $z_0 - b_{i,j} z_1 - c_{i,j} z_2 + b_{i,j} c_{i,j} z_3$ does not divide G_i.*

Then the complete intersection defined by

$$(z_0 - a_i z_1 - a_i z_2 + a_i^2 z_3)(z_0 + a_i z_1 + a_i z_2 + a_i^2 z_3) + t_i F_i = z_{3+i}^2$$

$$\prod_{j=1}^{m_i}(z_0 - b_{i,j} z_1 - c_{i,j} z_2 + b_{i,j} c_{i,j} z_3) + t_i G_i = z_{3+i}^{m_i}, \quad z_0 z_3 - z_1 z_2 = 0$$

for $i = 1,\dots,r+s$ in $\mathbb{P}^{r+s+3}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$.

Proof. Let us evaluate the complete intersection at $t_i = 0$, and denote this surface by X_{r+s}. Then X_{r+s} is constructed from $X_0 := \mathbb{P}^1 \times \mathbb{P}^1$ as for the generalized cuboids but for distinct degrees 2 and m_i. As before, we consider the section $\omega \in H^0(X_0, (2,2) \otimes S^2\Omega^1_{X_0})$ defined by $z_3^2 dz_1 dz_2$. We recall that given the isomorphism $h\colon \mathbb{P}^1 \times \mathbb{P}^1 \to X_0$, $h([x,y]\times[w,z]) = [xw, xz, yw, yz]$, the section ω corresponds to the section $y^2 z^2 dxdw$ under h. The problem is that over multiplicities equal to 2, we obtain A_1 singularities and we cannot apply our results, like the key Lemma 2.3. Instead we consider another section so that we can use the result in Lemma 3.4. For that, let

$$\omega_0 \in H^0(X_0, (4,4) \otimes S^2\Omega^1_{X_0})$$

be defined by $(x^2 - w^2)dxdw$ for affine coordinates x, w. Then if R is the strict transform of the branch divisor in X'_{r+s}, and E is the exceptional divisor of $X'_{r+s} \to X_{r+s}$, then there is

$$\omega'_0 \in H^0(X'_{r+s}, \mathcal{O}_{X'_{r+s}}(-R-E) \otimes g'^*_{r+s}(4,4) \otimes S^2\Omega^1_{X'_{r+s}})$$

corresponding to $g'^{\bullet}_{r+s}\omega_0$. At the same time we have

$$g'^*_{r+s}(r+s, r+s) \equiv R + E,$$

and so $\mathcal{O}_{X'_{r+s}}(-R-E) \otimes g'^*_{r+s}(4,4) \equiv g'^*_{r+s}(-r-s+4, -r-s+4)$. But $r+s \geq 5$, and so we obtain as in Theorem 5.1 and Theorem 6.3 that the surfaces X_{r+s} are hyperbolic. This is indeed because we know all ω_0-integral curves (fibres and the two $(1,1)$ extra curves), and so we can check all the pre-images. To avoid curves of geometric genus ≤ 1, here we use that $s>0$, $m_i \geq 3$, and $s+r \geq 5$. For example $s=0$ would be a problem with the $(1,1)$ curves. This is a proof of part (c) of Theorem 1.6. □

The next corollary uses the example at the end of Section 4.

Corollary 6.6. *Let $n \geq 5$, and let $m_i \geq 3$ be n integers. Let $[a_i, b_i]$ be a collection of distinct $\sum_{i=1}^n m_i$ points in $\mathbb{P}^1$. Let $\{G_i = G_i(x,y,z)\}_{i=1}^n$ be a collection of n homogeneous polynomials of degree m_i, such that $a_j^2x+a_jb_jy+b_j^2z$ does not divide G_i for $j = m_{i-1}, m_{i-1}+1, \ldots, m_i$ (where $m_0 := 1$). Then the complete intersection*

$$\prod_{j=1}^{m_i}(a_j^2x + a_jb_jy + b_j^2z) + t_iG_i = w_i^{m_i}$$

for $i = 1, \ldots, n$ in $\mathbb{P}^{n+2}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$.

Proof. Let us consider instead the situation in Corollary 4.3, but with $n \geq 5$ equations, $a_i = 1$ for all i, and $m_i \geq 3$ for all $i = 1, \ldots, n$. We take $X_0 := \mathbb{P}^2$, $\omega \in H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes S^2\Omega^1_{\mathbb{P}^2})$, and we construct the surface X_n as in Theorem 5.1 from this data. As in Corollary 6.5, for $n \geq 5$ and $m_i \geq 3$, we construct smooth complete intersections in $\mathbb{P}^{n+2}$ of multidegree $(m_1, \ldots, m_n)$ which are hyperbolic, by Theorem 6.3. The proof follows the same strategy as Corollary 6.5, since we know all ω-integral curves. This is a proof of part (b) of Theorem 1.6, and part (a) when all multiplicities are bigger than or equal to 3. □

Corollary 6.7. *Let $r \geq 1$, $s \geq 0$ be integers such that $r+s \geq 7$, and let $\{m_i \geq 3\}_{i=r+1}^{r+s}$ be integers. Let $\{b_{i,j}, c_{i,j}\}_{i=r+1,\ldots,r+s}^{j=1,\ldots,m_i}$ be two collections of $\sum_{i=r+1}^{r+s} m_i$ distinct complex numbers. Let a_i be a collection of r distinct complex numbers such that $a_i \neq a_j \pm 1$, $b_{i,j} \neq a_k$, $c_{i,j} \neq a_k$, $b_{i,j} \neq a_k \pm 1$, and $c_{i,j} \neq a_k \pm 1$ for all i,j,k. Let $\{F_i = F_i(z_0,z_1,z_2,z_3)\}_{i=1}^r$ be a collection of r homogeneous polynomials of degree 2, such that $z_0 - a_iz_1 - a_iz_2 + a_i^2z_3$ and $z_0 - (a_i-1)z_1 - (a_i+1)z_2 + (a_i^2-1)z_3$ do not divide F_i. Let $\{G_i = G_i(z_0,z_1,z_2,z_3)\}_{i=r+1}^{r+s}$ be a collection of s homogeneous polynomials of degree m_i, such that $z_0 - b_{i,j}z_1 - c_{i,j}z_2 + b_{i,j}c_{i,j}z_3$ does not divide G_i.*

Then the complete intersection defined by

$$(z_0 - a_iz_1 - a_iz_2 + a_i^2z_3)(z_0 - (a_i-1)z_1 - (a_i+1)z_2 + (a_i^2-1)z_3) + t_iF_i = z_{3+i}^2$$

$$\prod_{j=1}^{m_i}(z_0 - b_{i,j}z_1 - c_{i,j}z_2 + b_{i,j}c_{i,j}z_3) + t_iG_i = z_{3+i}^{m_i}, \quad z_0z_3 - z_1z_2 = 0$$

for $i = 1, \ldots, r+s$ in $\mathbb{P}^{r+s+3}$ is hyperbolic for sufficiently small $t_i \in \mathbb{C}$.

Proof. This is as in the proof of Corollary 6.5, but with $\omega_0 \in H^0(X_0, (6,6) \otimes S^2\Omega^1_{X_0})$ defined by

$$(x-w)(x-w+1)(x-w-1)(x-w-2)dxdw$$

for affine coordinates x, w. We note that each of the 4 nodes in the 4 fibres $(z_0 - a_iz_1 - a_iz_2 + a_i^2z_3)(z_0 - (a_i-1)z_1 - (a_i+1)z_2 + (a_i^2-1)z_3) = 0$ in $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ belongs to one of the 4 lines $(x-w)(x-w+1)(x-w-1)(x-w-2) = 0$.

This gives that the pull-back of each of the 4 lines do not contain any curves of geometric genus ≤ 1. This is a proof of part (a) of Theorem 1.6, when some (or all) multiplicities are equal to 2. $\square$

The list of multidegrees not included in the previous corollaries is $(2, \ldots, 2)$ in $\mathbb{P}^9$, $\mathbb{P}^8$, and $\mathbb{P}^7$; (m_1, m_2, m_3, m_4) for $m_i > 2$ and $(2, 2, 2, 2)$ in $\mathbb{P}^6$, for which the surface of cuboids is an example. We do not know existence of hyperbolic surfaces in those cases, except for the ones $(k, k, k, k) \subset \mathbb{P}^6$ for $k \geq 3$ [GF16]. This gives explicit evidence for [Dem18, Conjecture 0.18] in the case of surfaces. We point out that hyperbolic complete intersections of high multidegree have been constructed by Brotbek [Br14] (see also [X15]).

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E-mail address: natalia.garcia@mat.uc.cl

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile.

E-mail address: urzua@mat.uc.cl

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile.