Dependence on the domain geometry of the Hölder estimates for the Neumann problem, with an application to void coalescence in solids

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Abstract

The problem of the sudden growth and coalescence of voids in elastic media is considered. The Dirichlet energy is minimized among incompressible and invertible Sobolev deformations of a two-dimensional domain having n microvoids of radius ε. The constraint is added that the cavities should reach at least certain minimum areas $v_1, \dots, v_n$ after the deformation takes place. They can be thought of as the current areas of the cavities during a quasistatic loading, the variational problem being the way to determine the state to be attained by the elastic body in the next time step. It is proved that if each v_i is smaller than the area of a disk having a certain well defined radius, which is comparable to the distance, in the reference configuration, to either the boundary of the domain or the nearest cavity (whichever is closer), then there exists a range of external loads for which the cavities opened in the body tend to be circular in the limit as $\varepsilon \to 0$. In light of the results by Sivalonagathan & Spector and Henao & Serfaty that cavities always prefer to have a circular shape (unless prevented to do so by the constraint of incompressibility), our theorem suggests that the elongation and coalescence of the cavities experimentally and numerically observed for large loads can only take place after all the cavities have attained a size comparable to the space they have available in the reference configuration. Based on the previous work of Henao & Serfaty, who apply the Ginzburg-Landau theory for superconductivity to the cavitation problem, this paper shows how the study of the interaction of the cavities is connected to the following more basic question: for what cavitation sites $a_1, \dots, a_n$ and areas $v_1, \dots, v_n$ does there exist and incompressible and invertible deformation producing cavities of those areas originating from those points. In order to use the incompressible flow of Dacorogna & Moser to answer that question, it is necessary to study first how do the elliptic regularity estimates for the Neumann problem in domains with circular holes depend on the geometry of the domain.

1 Introduction

1.1 Regularity estimates

The problem of study is

$$\begin{cases} div\, v = 0 & \text{in } E, \\ v(x) = g(x)\nu(x) & \text{on } \partial E\ , \end{cases} \tag{1}$$

where

$$E = B(z_0, r_0) \setminus \bigcup_{k=1}^{n} \overline{B(z_k, r_k)} \subset \mathbb{R}^2, \tag{2}$$

$\nu(x)$ is its unit outward normal, and $g \in C^{1,\alpha}\left(\bigcup_{k=0}^{n} \partial B(z_k, r_k)\right)$ for some $\alpha \in (0,1)$. The datum must be compatible with the equation:

$$\int_{\partial B(z_0,r_0)} g = \sum_{k=1}^{n} \int_{\partial B(z_k,r_k)} g. \tag{3}$$

We study the dependence on the geometry of E of the regularity estimates for (1), motivated by a free boundary problem arising in the analysis of cavitation (as described in the next subsection). We find that the estimates do not blow up provided that the radii of the holes, their distance to the outer boundary and the distance between them do not become too small compared to the domain size. To obtain quantitative estimates, we assume throughout that

$$\begin{gathered} \forall i \geq 1\ r_i \geq d, \\ \forall i \geq 1\ B(z_i, r_i + d) \subset B(z_0, r_0), \text{ and} \\ \min_{\substack{i,j \geq 1 \\ i \neq j}} \operatorname{dist}(\overline{B(z_i, r_i)}, \overline{B(z_j, r_j)}) \geq 2d, \end{gathered} \tag{4}$$

for some generic length d. We also set

$$C_P(E) := \sup \left\{ \|\phi\|_{L^2(E)} : \phi \in H^1(E) \text{ s.t. } \|D\phi\|_{L^2(E)} = 1 \text{ and } \int_E \phi = 0 \right\}. \tag{5}$$

Theorem 1. *Let $n \in \mathbb{N}$ and $0 < \delta < 1$. There exists a universal constant $C_3(\delta)$ such that whenever $z_0, \ldots, z_n \in \mathbb{R}^2$ and $d, r_0, \ldots, r_n > 0$ satisfy $\frac{d}{r_0} \geq \delta$ and* (4)*, we have that for every g verifying* (3) *it is possible to construct a solution to* (1) *for which*

$$\|v\|_\infty \leq C_3 \Bigg(\left(\left(\frac{r_0}{d}\right)^{1+\alpha} + B \left(\frac{r_0}{d^2}\right)^3 + B^2 \left(\frac{r_0}{d^3}\right)^3 \right) \|g\|_\infty + \left(\frac{r_0^{2\alpha+1}}{d^{\alpha+1}} + B \frac{r_0^{2+\alpha}}{d^5} \right) [g]_{0,\alpha} \Bigg),$$

$$\|Dv\|_\infty \leq C_3 \left(C_1 \|g\|_\infty + C_2 [g]_{0,\alpha} + \frac{r_0^\alpha}{d^\alpha} \|g'\|_\infty + \frac{r_0^{2\alpha}}{d^\alpha} [g']_{0,\alpha} \right),$$

where

$$B = B(E) := |E|^{\frac{1}{2}} C_P(E)\Big(d^{-\frac{1}{2}} C_P(E) + d^{\frac{1}{2}}\Big) n^{\frac{1}{2}} r_0^{\frac{1}{2}}, \tag{6}$$

$C_1 = r_0^{1+\alpha} d^{-\alpha-2} + B r_0^3 d^{-7} + B^2 r_0^3 d^{-10}$, *and* $C_2 = r_0^{2\alpha+1} d^{-2-\alpha} + B r_0^{2+\alpha} d^{-6}$.

The theorem is proved in Section 2.5, using the strategy of Dacorogna & Moser [DM90]: in it a solution to the linearised incompressibility boundary value problem (1) is found by adding two harmonic maps arising from coupled Neumann problems (one giving the right normal velocity g on the boundary and the other cancelling out the tangential velocities appearing in the first step). The result that the regularity estimates do not blow up as long as the connected components of ∂E are far apart comes from Theorem 1 and the fact that (4) ensures that the Poincaré constant $C_P(E)$ remains bounded (see Theorem 3).

1.2 Cavitation and spherical symmetry

The motivation comes from the modelling of cavitation (the sudden formation and expansion of cavities) in materials that strongly resist changes in volume. The first experimental studies in elastomers are due to Gent & Lindley [GL59], who also theoretically estimated the hydrostatic load for rupture by solving the non-linearised equilibrium equations for an infinitely thick elastic shell under the assumption of radial symmetry. The first analysis of the evolution of a cavity (beyond its nucleation) was due to Ball [Bal82]; he showed that the one-parameter family of deformations

$$u(x) = \sqrt[n]{|x|^n + L^n}\,\frac{x}{|x|}, \quad L \geq 0, \quad n = 2, 3 \tag{7}$$

constitutes a stable branch of weak solutions to the incompressible elasticity equations, which bifurcates from the homogeneous deformation at the dead-load predicted by Gent & Lindley. The radial symmetry assumption, which persisted in this pioneering work, was finally removed by Müller & Spector [MS95] and Sivaloganathan & Spector [SS00]; they proved the existence of minimizers of the elastic energy allowing for all sorts of cavitation configurations. Lopez-Pamies, Idiart & Nakamura [LPIN11] and Negrón-Marrero & Sivaloganathan [NMS12] discussed the onset of cavitation under non-symmetric loadings. Mora-Corral [MC14] studied the quasistatic evolution of cavitation. We refer to [FGLP, KFLP18, PLPRC, KRCLP], the Introduction in [HS13], and the references therein for a more complete guide through the extensive literature on this fracture mechanism.

The analyses [SS10a, SS10b, HS13] and the numerical study [LL11b] suggest that the cavities inside an elastic body prefer to adopt a spherical shape when pressurised by large and multiaxial external tensions, regardless of their shape and size at the onset of fracture (or in the rest state, if they existed already). In particular, given any open $\mathcal{B} \subset \mathbb{R}^2$; any small $\varepsilon > 0$; any finite collection $a_1, \dots, a_n \in \mathcal{B}$ of cavitation points; and any incompressible and invertible[1]

[1] In the topological sense of Müller & Spector's condition INV, see Definition 2 and [MS95].

deformation map $u : \mathcal{B} \setminus \bigcup_1^n \overline{B}_\varepsilon(a_i) \to \mathbb{R}^2$; using the arguments in [HS13] it can be seen[2] that

$$\int_{\mathcal{B}\setminus\bigcup_1^n \overline{B}_\varepsilon(a_i)} \frac{|Du|^2-1}{2} dx \geq \sum_1^n v_i \log\frac{R}{\varepsilon} + \sum_{i=1}^n v_i D_i^2 \log\frac{\min\{d_i, \sqrt{v_i D_i^2}\}}{\varepsilon} - C \tag{8}$$

where C is a universal constant and R, d_i, v_i, and D_i respectively denote $R := \mathrm{dist}(\{a_1, \ldots, a_n\}, \partial\mathcal{B})$; the distance to the nearest cavitation point (or to $\partial\mathcal{B}$ should the outer boundary be closer to a_i); the area of a cavity coming from $B_\varepsilon(a_i)$; and the Fraenkel asymmetry [FMP08] of the same cavity (which measures how far is it from being a circle). The first term on the right-hand side is the exact cost of a radially-symmetric cavitation; the prefactor of $|\log\varepsilon|$ in the second term, on the other hand, is strictly positive if and only if the cavities are not circular. This shows that it is very expensive to produce non-circular cavities (as stated above[3]).

In spite of the previous energetic consideration, if the external load is too large then an important geometric obstruction frustrates the desire of producing only spherical cavities. Although this is already explained in [HS13], let us briefly describe the situation. Consider again a body that is only two dimensional; that is furthermore a disk; that is subject to the displacement condition $u(x) = \lambda x \ \forall\, x \in \partial B_{R_0}$, for some $\lambda > 1$ (R_0 being the domain radius); and that can open only two cavities. A necessary condition for circular cavities of areas v_1 and v_2 to be opened is that they be disjoint and enclosed by the deformed outer boundary. This is possible only when the sum $2\sqrt{\frac{v_1}{\pi}} + 2\sqrt{\frac{v_2}{\pi}}$ of their diameters is less than the outer diameter $2\lambda R_0$. On the other hand, if the body is incompressible (if none of its parts can change its volume), the area occupied by the material after and before the deformation must coincide:

$$\pi(\lambda R_0)^2 - (v_1 + v_2) = \pi R_0^2 - O(\varepsilon^2) \tag{9}$$

(the term of order ε^2 accounts for the eventual preexisting microvoids). Hence, the necessary condition reads

$$2\sqrt{v_1 v_2} \leq \pi R_0^2 - O(\varepsilon^2). \tag{10}$$

It follows that, for instance, if $\lambda > \sqrt{2}$ then the body cannot open two equally big circular cavities.

The conflict between the geometric obstruction due to incompressibility and the energetic cost of distorted cavities raises the question of:

What is the maximum load compatible with the opening of only spherical cavities?

In order to address this question, first we need to take the following into account. It does not lead far to think of the load as just a scalar: it is more appropriate to consider the whole combination of the displacement condition at

[2] A sketch of the proof can be found in Section 3.2.

[3] The result is in 2D but suggests that the same occurs in 3D elasticity.

the outer boundary; the cavitation sites in the reference configuration; and the size that each cavity is expected to attain; as the load. Consider, for example, the following trivial observation: Equations (9) and (10) impose no limit on λ if v_1 and v_2 are taken to be, respectively, as $(\lambda^2 - 1) \cdot \pi R_0^2 + O(\varepsilon^2)$ and zero. The obstruction arises when all the n cavities grow from a size of order ε to a size of order 1, which corresponds naturally to the situation in quasistatic and dynamic loadings (once a cavity forms and grows it is not expected to shrink back; healing is possible, however, upon compression and/or unloading [FGLP, KFLP18, PLPRC, KRCLP]).

There is a simple geometric condition that is necessary for a $2D$ body to open only circular cavities of areas $v_1, \dots, v_n$ at the points $a_1, \dots, a_n \in \mathcal{B}$: *there must exist an evolution of the domain itself (i.e. of the centers and radii of the cavities and of the outer boundary) such that the total enclosed volume is preserved.* This paper's answer to the question of the previous paragraph is that *this simple geometric condition is also sufficient for the existence of an incompressible and invertible deformation opening round cavities of the desired sizes at the desired sites.* This is made explicit in Theorem 4 of Section 3.3.

The reason to study how does the Hölder estimates for the Neumann problem depend on the geometry of these domains with circular holes is that such study is what is required in order to prove the existence of the above-mentioned deformation using the Dacorogna-Moser flow [DM90]. The details are in the proof of the theorem.

1.3 Void coalescence

There is an extensive literature about the coalescence of voids in elastomers and in ductile materials. On the experimental side, see, e.g., [Gen91, PLLPRC17, PCSE06]. On the numerical and modelling side, and restricting our attention, for concreteness, to the case of elastomers, see both [XH11, LL11b, LL11a, LL12, LRCLP15], which focus on the building-up of tension before coalescence (only Sobolev maps are considered in the energy minimization), and the SBV models [HMCX16, KFLP18] (based on [BFM08] and the analyses [HMC11, HMC12, HMC15, HMCX15]), where the interaction can be followed all the way up to the nucleation and propagation of cracks.

What is observed during the quasistatic loading of a confined elastomer is that cavities eventually lose their spherical shape as the load increases, and begin to interact with other cavities until they merge into micro-cracks. It follows that if for a certain load it is possible to prove that the cavities formed inside the body are close to spherical, then that load constitutes a lower bound for the load at which the voids begin to coalesce. For 2D neo-Hookean materials, such radial symmetry result can in fact be obtained, as shown by Henao & Serfaty [HS13], using the methods and ideas developed for Ginzburg-Landau superconductivity. The existence question addressed in this paper, namely, that of determining for what loads there exists at least one deformation having finite energy and opening only round cavities (regardless of whether it is energy minimizing), happens to play an important role in that more complete radial symmetry statement. For those loads it can be shown that the cavities opened by the actual energy minimizers are also close to being circular. Consequently, by finding out a condition on the load sufficient to ensure that deformations with round cavities still exist (which is what we do in Theorem 4, as explained in the previous section), we have, at the same time, obtained a lower bound for the coalescence load in the 2D neo-Hookean model. This is what lies behind Theorem 5 in Section 3.4.

We end this Introduction by mentioning Corollary 3.2, which illustrates what loads satisfy the geometric condition for the opening of only round cavities. It motivates Theorem 6, a modified version of Theorem 5 where a slightly more general variational problem is considered. This theorem brings some evidence to the conjecture, implicitly present already in [BM84, HS13], that *any given cavity will retain its spherical shape as long as its radius, after the deformation, remains smaller or comparable to the distance, in the undeformed configuration, to the nearest cavitation point*[4], and that *no coalescence ought to take place until all the cavities have attained that critical size.*

2 Hölder regularity for the incompressibility equation in a moving domain

2.1 Notation

Excision of holes off the elastic body

We begin by studying the Hölder regularity of the classical 2D singular integrals in a generic annulus:

$$\Omega := \{x \in \mathbb{R}^2 : R < |x| < R + d\}. \tag{11}$$

For calculations that have to be made away from $\partial\Omega$, we work in

$$\Omega' := \{x \in \mathbb{R}^2 : R + \frac{1}{3}d < |x| < R + \frac{2}{3}d\}. \tag{12}$$

The study is carried out in Sections 2.2 and 2.3. The results will be applied in Section 3.3 to the cavitation problem, where a certain evolution of an elastic body will be considered. The domain occupied by the body changes during the evolution; we obtain thus a one-parameter family of domains $B(0, tR_0) \setminus \bigcup_1^n \overline{B}(z_i(t), L_i(t))$. For reasons that will become clear there, the treatment near the cavities differs from that in the rest of the domain; because of this, we cut some holes $\overline{B}(z_i(t), r_i(t))$ off the domain, with $r_i(t) > L_i(t)$ (they contain both the cavity $\overline{B}(z_i(t), L_i(t))$ and the material that surrounds it). Throughout the paper we will be consistent in the distinction between the use made of the word *hole* and the use of the word *cavity.* The analysis of this section will apply to the smaller sets $E(t) := B(0, tR_0) \setminus \bigcup_1^n \overline{B}(z_i(t), r_i(t))$, whereby the generic radius R

[4] Or the outer boundary, if it is closer.

in (11) corresponds to the radius $r_i(t)$ of one of the holes of $E(t)$. The role of the generic length d is that of giving a uniform lower bound for the width of an annular neighbourhood of the excised hole that is still contained in the domain.

In Proposition 2.18 negative powers of the radii of the holes are obtained. It is for this reason that in the final result (see (4) and Theorem 1) not only the distances between the holes but also their radii are assumed to be greater than the generic length d; the velocity field v is then be controlled, in the appropriate norms, by inverse powers of d. In some intermediate results, knowing that the radius is greater than d simplifies the estimates (e.g. in Lemma 2.6 we obtain $\|Du\|_\infty \leq CR\|f\|_\infty$ instead of $\|Du\|_\infty \leq C(R+d)\|f\|_\infty$). This is why the hypothesis $R \geq Cd$ is added througout the whole section.

Function spaces and Green's function

We fix a value of $\alpha \in (0,1)$ and work with the norms $\|f\|_\infty := \sup|f(x)|$ and

$$[f]_{0,\alpha} := \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}, \qquad \|f\|_{0,\alpha} := \|f\|_\infty + [f]_{0,\alpha},$$

$$[f]_{1,\alpha} := \sup_{x\neq y}\frac{|Df(x)-Df(y)|}{|x-y|^\alpha}, \qquad \|f\|_{1,\alpha} := \|f\|_\infty + \|Df\|_\infty + [f]_{1,\alpha}.$$

The function g in (1) will belong to

$$C^{0,\alpha}_{per} := \{g \in C^{0,\alpha}_{loc}(\mathbb{R}) : g \text{ is } 2\pi\text{-periodic}\}.$$

The inversion of $x \in \mathbb{R}^2$ with respect to $B(0,R)$ is $x^* = \frac{R^2}{|x|^2}x$. Set

$$\Phi(x) := \frac{-1}{2\pi}\log(|x|), \quad \phi^x(y) := \frac{1}{2\pi}ln(|y-x^*|) - \frac{|y|^2}{4\pi R^2}, \quad G_N(x,y) := \Phi(x)-\phi^x(y).$$

The expression $u_{,\beta}$ stands for $\partial_\beta u = \frac{\partial u}{\partial x_\beta}$.

2.2 Estimates in the interior of the domain

The following regularity estimates for harmonic functions can be found in [Eva10, Thm. 2.2.7]

Lemma 2.1. *Let v be weakly harmonic in $B(x,d)$, then:*
$\|v\|_{L^\infty(B(x,\frac{d}{2}))} \leq Cd^{-2}\|v\|_{L^1(B(x,d))}$.
$\left\|D^\beta v\right\|_{L^\infty(B(x,\frac{d}{2}))} \leq Cd^{-2-|\beta|}\|v\|_{L^1(B(x,d))}$.

Proposition 2.2. *: Let v be harmonic in the distributional sense in Ω and $R \geq Cd$, then we have the folllowing estimates :*
$\|v\|_{L^\infty(\Omega')} \leq Cd^{-2}\|v\|_{L^1(\Omega)}$.
$[v]_{0,\alpha(\Omega')} \leq Cd^{-3}R^{1-\alpha}\|v\|_{L^1(\Omega)}$.
$\left\|D^\beta v\right\|_{L^\infty(\Omega')} \leq Cd^{-2-|\beta|}\|v\|_{L^1(\Omega)}$.
$[v]_{1,\alpha(\Omega')} \leq Cd^{-4}R^{1-\alpha}\|v\|_{L^1(\Omega)}$.

Proof. The first and third estimates follow from the previous lemma. To prove the second estimate note that using polar coordinates we get (for $r \in (R + \frac{1}{3}d, R + \frac{2}{3}d)$ and $\theta_1, \theta_2 \in [-\pi, \pi]$, such that $|\theta_1 - \theta_2| \leq \pi$):

$$|v(re^{i\theta_1}) - v(re^{i\theta_2})| \leq \int_{\theta_1}^{\theta_2} \left| \frac{d}{d\theta} \left(v(re^{i\theta}) \right) \right| d\theta \leq \int_{\theta_1}^{\theta_2} \left| \frac{\partial v}{\partial x_1} \right| r |\sin(\theta)| + \left| \frac{\partial v}{\partial x_2} \right| r |\cos(\theta)| d\theta$$

$$\leq Cd^{-3} \|v\|_{L^1(\Omega)} r|\theta_1 - \theta_2| \leq Cd^{-3} \|v\|_{L^1(\Omega)} |re^{i\theta_1} - re^{i\theta_2}|^\alpha R^{1-\alpha},$$

since $r|\theta_1 - \theta_2| \leq \frac{\pi}{2}|re^{i\theta_1} - re^{i\theta_2}|$ (recall that $\frac{2}{\pi^2} \leq \frac{1-\cos(\theta)}{\theta^2} \leq \frac{1}{2}$, for $\theta \in [-\pi, \pi]$) and $|re^{i\theta_1} - re^{i\theta_2}| \leq 2r \leq CR$.
Moreover, for $\theta \in [-\pi, \pi]$ and $r_1, r_2 \in [R + \frac{1}{3}d, R + \frac{2}{3}d]$, we have:

$$|v(r_1 e^{i\theta}) - v(r_2 e^{i\theta})| \leq \int_{r_1}^{r_2} \left| \frac{d}{dr} \left(v(re^{i\theta}) \right) \right| dr \leq \int_{r_1}^{r_2} \left| \frac{\partial v}{\partial x_1} \right| |\cos(\theta)| + \left| \frac{\partial v}{\partial x_2} \right| |\sin(\theta)| dr$$

$$\leq Cd^{-3} \|v\|_{L^1(\Omega)} |r_1 - r_2| \leq Cd^{-3} \|v\|_{L^1(\Omega)} |r_1 e^{i\theta} - r_2 e^{i\theta}|^\alpha R^{1-\alpha}$$

Now, for $r_1, r_2 \in [R + \frac{1}{3}d, R + \frac{2}{3}d]$, $r_1 \leq r_2$ and $\theta_1, \theta_2 \in [-\pi, \pi]$, such that $|\theta_1 - \theta_2| \leq \pi$, we have:

$$|v(r_1 e^{i\theta_1}) - v(r_2 e^{i\theta_2})| \leq |v(r_1 e^{i\theta_1}) - v(r_1 e^{i\theta_2})| + |v(r_1 e^{i\theta_2}) - v(r_2 e^{i\theta_2})|$$

$$\leq Cd^{-3} R^{1-\alpha} \|v\|_{L^1(\Omega)} (|r_1 e^{i\theta_1} - r_1 e^{i\theta_2}|^\alpha + |r_1 e^{i\theta_2} - r_2 e^{i\theta_2}|^\alpha)$$

$$\leq Cd^{-3} R^{1-\alpha} \|v\|_{L^1(\Omega)} (|r_1 e^{i\theta_1} - r_2 e^{i\theta_2}|^\alpha + |r_1 e^{i\theta_1} - r_2 e^{i\theta_2}|^\alpha),$$

since $|r_1 e^{i\theta_1} - r_2 e^{i\theta_2}|^2 = (r_1 - r_2)^2 + 2r_1 r_2 (1 - \cos(\theta_1 - \theta_2)) \geq 2r_1^2 (1 - \cos(\theta_1 - \theta_2)) = |r_1 e^{i\theta_1} - r_1 e^{i\theta_2}|^2$ and $|r_1 e^{i\theta_1} - r_2 e^{i\theta_2}| \geq |r_1 - r_2|$. The proof of the fourth estimate is analogous. □

Lemma 2.3. *Let $R \geq Cd$, v be harmonic in Ω and ζ a cut-off function with support within $|x| < R + \frac{2}{3}d$ and equal to 1 for $|x| \leq R + \frac{1}{3}d$, then:*
$[\Delta(v\zeta)]_{0,\alpha(\mathbb{R}^2)} \leq CR^{1-\alpha} d^{-5} \|v\|_{L^1(\Omega)}$.
$\|\Delta(v\zeta)\|_{\infty(\mathbb{R}^2)} \leq Cd^{-4} \|v\|_{L^1(\Omega)}$.

Proof. It is clear that we can choose ζ to be such that: $|D^k \zeta| \leq C_k d^{-k}$ (and then $[\zeta]_{k,\alpha(\Omega')} \leq C_{k+1} d^{-k-1} R^{1-\alpha}$ since $\zeta \in C_c^\infty(B(0, R + d))$). Then, using Proposition 2.4 and the estimates for ζ we get:

$$|\Delta(v\zeta)| \leq 2|\nabla v \cdot \nabla \zeta| + |v \Delta \zeta| \leq Cd^{-4} \|v\|_{L^1(\Omega)} .$$

On the other hand:

$$[\Delta(v\zeta)]_{0,\alpha(\Omega')} \leq 2[\nabla v \cdot \nabla \zeta]_{0,\alpha(\Omega')} + [v \Delta \zeta]_{0,\alpha(\Omega')}.$$

Now note that:

$$[v_{,\beta} \cdot \zeta_{,\beta}]_{0,\alpha(\Omega')} \leq [v_{,\beta}]_{0,\alpha(\Omega')} \|\zeta_{,\beta}\|_{\infty(\Omega')} + [\zeta_{,\beta}]_{0,\alpha(\Omega')} \|v_{,\beta}\|_{\infty(\Omega')}$$

$$\leq Cd^{-4}R^{1-\alpha}\left\|v\right\|_{L^1(\Omega)}\cdot d^{-1}+Cd^{-2}R^{1-\alpha}\cdot d^{-3}\left\|v\right\|_{L^1(\Omega)}.$$

Furthermore:

$$[v\Delta\zeta]_{0,\alpha(\Omega')}\leq [v]_{0,\alpha(\Omega')}\left\|\Delta\zeta\right\|_{\infty(\Omega')}+[\Delta\zeta]_{0,\alpha(\Omega')}\left\|v\right\|_{\infty(\Omega')}$$

$$\leq Cd^{-3}R^{1-\alpha}\left\|v\right\|_{L^1(\Omega)}\cdot d^{-2}+Cd^{-3}R^{1-\alpha}\cdot d^{-2}\left\|v\right\|_{L^1(\Omega)}.$$

Hence:

$$[\Delta(v\zeta)]_{0,\alpha(\Omega')}\leq Cd^{-5}R^{1-\alpha}\left\|v\right\|_{L^1(\Omega)}.$$

Now if $x\in\Omega'$ and $y\in\mathbb{R}^2\setminus\overline{\Omega'}$, there exists $t\in(0,1)$ such that $z=tx+(1-t)y\in\partial\Omega'$, then we have

$$|\Delta(v\zeta)(x)-\Delta(v\zeta)(y)|\leq|\Delta(v\zeta)(x)-\Delta(v\zeta)(z)|+|\Delta(v\zeta)(z)-\Delta(v\zeta)(y)|$$

$$=|\Delta(v\zeta)(x)-\Delta(v\zeta)(z)|\leq CR^{1-\alpha}d^{-5}\left\|v\right\|_{L^1(\Omega)}|x-z|^\alpha$$

$$=CR^{1-\alpha}d^{-5}\left\|v\right\|_{L^1(\Omega)}(1-t)^\alpha|x-y|^\alpha\leq CR^{1-\alpha}d^{-5}\left\|v\right\|_{L^1(\Omega)}|x-y|^\alpha$$

(Clearly if $x,y\in\mathbb{R}^2\setminus\overline{\Omega'}$, $|\Delta(v(x)\zeta(x))-\Delta(v(y)\zeta(y))|=0$). Finally, we get:

$$[\Delta(\zeta v)]_{0,\alpha(\mathbb{R}^2)}\leq CR^{1-\alpha}d^{-5}\left\|v\right\|_{L^1(\Omega)}.$$

□

2.3 Estimates near circular boundaries

Proposition 2.4. *Let v be harmonic in Ω and ζ be a cut-off function with support within $|x|<R+\frac{2}{3}d$ and equal to 1 for $|x|\leq R+\frac{1}{3}d$. Then, if $u=\zeta v$:*

$$u(x)=C-\int_{\partial B_R}\frac{\partial u}{\partial\nu}\left(\Phi(y-x)-\phi^x(y)\right)dS(y)-\int_\Omega\Delta u\left(\Phi(y-x)-\phi^x(y)\right)dy.$$

Proof. Let us proceed as in [Eva10]:

$$\int_{\Omega\setminus B_\varepsilon(x)}\Delta u(y)\Phi(y-x)-u(y)\Delta_y\Phi(y-x)dy=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\Phi(y-x)-\frac{\partial\Phi}{\partial\nu}(y-x)u(y)dS(y)$$

$$+\int_{\partial B_\varepsilon(x)}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)-\frac{\partial u}{\partial\nu}\Phi(y-x)dS(y),$$

letting $\varepsilon\to0$ (and using the fact that u vanishes outside $B_{R+\frac{2}{3}d}$), we get:

$$\int_\Omega\Delta u(y)\Phi(y-x)dy=\int_{\partial B_R}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)-\frac{\partial u}{\partial\nu}\Phi(y-x)dS(y)-u(x).$$

Hence:

$$u(x)=\int_{\partial B_R}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)-\frac{\partial u}{\partial\nu}\Phi(y-x)dS(y)-\int_\Omega\Delta u(y)\Phi(y-x)dy,$$

with the normal pointing outside B_R. Now (as can be seen in [DiB09]), if a function $\phi^x(y)$ satisfies:

$$\begin{cases} -\Delta_y \phi^x(y) = k & \text{if } y \in \Omega, \\ \dfrac{\partial \phi^x}{\partial \nu} = \dfrac{\partial \Phi}{\partial \nu}(y-x) & \text{if } y \in \partial B_R \ , \end{cases} \tag{13}$$

with k being a constant, then:

$$\int_\Omega \Delta_y \phi^x(y) u(y) - \Delta u \phi^x(y) dy = \int_{\partial \Omega} u(y) \frac{\partial}{\partial \nu} \phi^x(y) - \phi^x(y) \frac{\partial u}{\partial \nu} dS(y)$$

$$= \int_{\partial B_R} \phi^x(y) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial}{\partial \nu} \Phi(y-x) dS(y) = k \int_\Omega u dy - \int_\Omega \Delta u \phi^x(y) dy,$$

where we have used (13). Finally, replacing in the expression for $u(x)$, we obtain:

$$u(x) = C - \int_{\partial B_R} \frac{\partial u}{\partial \nu} \left(\Phi(y-x) - \phi^x(y)\right) dS(y) - \int_\Omega \Delta u \left(\Phi(y-x) - \phi^x(y)\right) dy.$$

It is easy to see that $\phi^x(y) = \frac{1}{2\pi} \log(|y - x^*|) - \frac{|y|^2}{4\pi R^2}$ satisfies (13) using the identity $|x_1||x_2 - x_1^*| = |x_2||x_1 - x_2^*|$. $\square$

Proposition 2.5. *Let $f \in C_c^{0,\alpha}(\Omega')$, $R \geq Cd$ and $u = \int_{\mathbb{R}^2} f(y)\Phi(x-y)dy$, then:*
$\|Du\|_{\infty(\mathbb{R}^2)} \leq CR \|f\|_\infty$.
$[Du]_{0,\alpha(B(0,R+d)\setminus\overline{B(0,R)})} \leq CR^{1-\alpha} \|f\|_\infty$
$\left\|\partial^2_{\beta\gamma} u\right\|_{\infty(B(0,R+d)\setminus\overline{B(0,R)})} \leq CR^\alpha [f]_{0,\alpha(\mathbb{R}^2)} + \frac{\delta_{\beta\gamma}}{2} \|f\|_\infty$.
$[D^2 u]_{0,\alpha(B(0,R+d)\setminus\overline{B(0,R)})} \leq C[f]_{0,\alpha(\mathbb{R}^2)}$.

Proof. Let us estimate the first derivative:

$$|u_{,\beta}| \leq \|f\|_\infty \int_{\Omega'} \frac{dy}{|x-y|} \leq C \|f\|_\infty \int_0^{2R+\frac{5}{3}d} dr \leq CR \|f\|_\infty ,$$

Now let us estimate the Holdër seminorm of the derivatives: let

$$v_\rho = \int_{\mathbb{R}^2 \setminus B(x,\rho)} f(y) \Phi_{,\beta}(x-y) dy,$$

with $\rho \in (0, 2(R+d))$, then:

$$|u_{,\beta} - v_\rho| \leq C \|f\|_\infty \int_{B(x,\rho)} |x-y|^{-1} dy \leq C \|f\|_\infty \int_{B(x,\rho)} |x-y|^{-1} dy$$

$$\leq C \|f\|_\infty \rho \leq C \|f\|_\infty \rho^\alpha R^{1-\alpha}.$$

On the other hand:

$$\frac{\partial v_\rho}{\partial \gamma} = \int_{\mathbb{R}^2 \setminus B(x,\rho)} f(y) \Phi_{,\beta\gamma}(x-y) dy - \int_{\partial B(x,\rho)} f(y) \Phi_{,\beta}(x-y) \nu_\gamma dS(y),$$

therefore:

$$\left|\frac{\partial v_\rho}{\partial \gamma}\right| \le C\left\|f\right\|_\infty \left(\int_{\mathbb{R}^2\setminus B(x,\rho)} |x-y|^{-2}dy + \int_{\partial B(x,\rho)} |x-y|^{-1}dS(y)\right)$$

$$\le C\left\|f\right\|_\infty \left(1+\int_{B(x,2(R+d))\setminus B(x,\rho)} |x-y|^{-2}dy\right)$$

$$\le C\left\|f\right\|_\infty \left(1+\left|\log\left(\frac{R}{\rho}\right)\right|\right) \le C\left\|f\right\|_\infty \left(1+\left(\frac{R}{\rho}\right)^{1-\alpha}\right).$$

(Note that $\frac{R}{\rho}\in(\frac{1}{2},\infty)$). Finally, if $|x-y|=\rho$:

$$|u_{,\beta}(x)-u_{,\beta}(y)| \le |u_{,\beta}(x)-v_\rho(x)| + |v_\rho(x)-v_\rho(y)| + |v_\rho(y)-u_{,\beta}(y)|$$

$$\le C\left\|f\right\|_\infty \rho^\alpha R^{1-\alpha} + C|x-y|\left\|f\right\|_\infty \left(1+\left(\frac{R}{\rho}\right)^{1-\alpha}\right)$$

$$\le C\left\|f\right\|_\infty \rho^\alpha R^{1-\alpha},$$

where we have used that $\rho \le CR$.

To prove the third estimate, first note that the second derivatives of u are given by:

$$u_{,\beta\gamma} = \lim_{\rho\to 0^+}\int_{\mathbb{R}^2\setminus B(x,\rho)} \Phi_{,\beta\gamma}(x-y)f(y)dy - \frac{\delta_{\beta\gamma}}{2}f.$$

Since $f\in C_c^{0,\alpha}$ (and using the fact that $\int_{\partial B(0,1)}\Phi_{,\beta\gamma}(z)dS(z)=0$, and $\int_A \Phi_{,\beta\gamma}(z)dz = 0$ if A is any annulus centered at the origin), the absolute value of the singular integral is bounded by:

$$\left|\lim_{\rho\to 0^+}\int_{B(x,2R+\frac{5}{3}d)\setminus B(x,\rho)} (f(y)-f(x))\Phi_{,\beta\gamma}(x-y)dy\right|$$

$$\le \lim_{\rho\to 0^+}\int_{\partial B(0,1)} |\Phi_{,\beta\gamma}(\omega)|dS(\omega)\int_\rho^{2R+\frac{5}{3}d} r^{\alpha-1}dr[f]_{0,\alpha} \le CR^\alpha[f]_{0,\alpha};$$

that proves the second result (obviously we have $\left\|\frac{\delta_{ij}}{2}f\right\|_\infty \le \frac{\delta_{ij}}{2}\left\|f\right\|_\infty$). To prove the last estimate, we proceed as in [Mor66, Thm. 2.6.4]: first note that if $\Phi_{,ij}(x)=\Delta(x)$, $\omega(x)=u_{,ij}(x)+\frac{\delta_{ij}}{n}f(x)$, $n=2$, and

$$\omega_\rho(x) = \int_{\mathbb{R}^n\setminus B(x,\rho)} \Delta(x-\xi)f(\xi)d\xi,$$

then:

$$|\omega_\sigma(x)-\omega_\rho(x)| \le \int_{B(x,\rho)\setminus B(x,\sigma)} |\Delta(x-\xi)|[f]_{0,\alpha}|x-\xi|^\alpha d\xi \le CM_0[f]_{0,\alpha}\rho^\alpha,$$

being $M_0 = \sup_{|x|=1} |\Delta(x)|$. If we let $\sigma \to 0$, we obtain:

$$|\omega(x) - \omega_\rho(x)| \leq CM_0[f]_{0,\alpha}\rho^\alpha.$$

Let $M = 3R + 3d$ and $M_1 = \sup_{|x|=1} |\nabla\Delta(x)|$. The derivatives of ω_ρ are given by:

$$\omega_{\rho,\beta}(x) = \int_{\mathbb{R}^n \setminus B(x,\rho)} \Delta_{,\beta}(x-\xi)f(\xi)d\xi - \int_{\partial B(x,\rho)} \Delta(x-\xi)f(\xi)d\xi_\beta^{'}$$

$$= \int_{B(x,M)\setminus B(x,\rho)} \Delta_{,\beta}(x-\xi)(f(\xi)-f(x))d\xi + \int_{\partial B(x,M)} \Delta(x-\xi)(f(\xi)-f(x))d\xi_\beta^{'}$$

$$+ \int_{\partial B(x,\rho)} \Delta(x-\xi)(f(x)-f(\xi))d\xi_\beta^{'}$$

Note that:

$$\int_{\partial B(x,M)} \Delta(x-\xi)f(\xi)d\xi_\beta^{'} = 0.$$

Let $x, z \in B(0, R+d)$ and $\rho = |x-z|$,then:

$$|\nabla\omega_\rho| \leq C(M_0 + M_1)[f]_{0,\alpha}(\rho^{\alpha-1} + M^{\alpha-1}) \leq C(M_0 + M_1)[f]_{0,\alpha}\rho^{\alpha-1}.$$

Thus (applying the mean value theorem):

$$|\omega(x)-\omega(z)| \leq |\omega(x)-\omega_\rho(x)|+|\omega_\rho(x)-\omega_\rho(z)|+|\omega_\rho(z)-\omega(z)| \leq C(M_0+M_1)[f]_{0,\alpha}\rho^\alpha;$$

that yields: $[\omega]_{0,\alpha} \leq C(M_0 + M_1)[f]_{0,\alpha}$. □

Lemma 2.6. *Let* $u = \int_{\mathbb{R}^2} f(y)\log|x^* - y|dy$ *with* $f \in C_c^{0,\alpha}(B_{R+\frac{2}{3}d} \setminus \overline{B_{R+\frac{d}{3}}})$, $R \geq Cd$. *Then:*
$\|Du\|_{L^\infty(B_{R+d}\setminus\overline{B_R})} \leq CR\|f\|_\infty$.
$[Du]_{0,\alpha(B_{R+d}\setminus\overline{B_R})} \leq CR^{2-\alpha}d^{-1}\|f\|_\infty$.
$\|D^2u\|_{L^\infty(B_{R+d}\setminus\overline{B_R})} \leq CRd^{-1}\|f\|_\infty$.
$[D^2u]_{0,\alpha(B_{R+d}\setminus\overline{B_R})} \leq CR^{2-\alpha}d^{-2}\|f\|_\infty$.

Proof. Using the identity $|x_1||x_1^* - x_2| = |x_2||x_1 - x_2^*|$, let us first note that:

$$\log|y - x^*| = \log|y^* - x| + \log|y| - \log|x|, \tag{14}$$

this implies that:

$$u = C + \int_{\mathbb{R}^2} \log|x - y^*|f(y)dy - \log|x|\int_{\mathbb{R}^2} f(y)dy,$$

then:

$$|u_{,\beta}| \leq C\int_{\Omega'} \frac{|f(y)|dy}{|x-y^*|} + \frac{C}{|x|}\|f\|_\infty Rd \leq C\int_{\Omega'} \frac{|f(y)|dy}{|x| - |y^*|} + \frac{C}{|x|}\|f\|_\infty Rd$$

$$\le CRd\frac{\|f\|_\infty}{R-\frac{R^2}{R+\frac{d}{3}}}+Cd\|f\|_\infty \le CR\|f\|_\infty .$$

The other estimates are proved analogously (for the Hölder continuity we can use the same argument as in Proposition 2.2). □

Proposition 2.7. *Let* $f \in C_c^{0,\alpha}(B_{R+\frac{2}{3}d}\backslash\overline{B_{R+\frac{d}{3}}})$, $R \ge Cd$ *and* $u = \int_{\mathbb{R}^2} f(y)G_N(x,y)dy$, *then (in* $B_{R+d}\setminus\overline{B_R}$*) :*
$\|Du\|_\infty \le CR\|f\|_\infty$.
$[Du]_{0,\alpha} \le CR^{2-\alpha}d^{-1}\|f\|_\infty$.
$\left\|D^2u\right\|_\infty \le C(Rd^{-1}\|f\|_\infty + R^\alpha[f]_{0,\alpha})$.
$[D^2u]_{0,\alpha} \le C(R^{2-\alpha}d^{-2}\|f\|_\infty + [f]_{0,\alpha})$.

Proof. It follows from Proposition 2.5 and Lemma 2.6. □

Lemma 2.8. *Let* $g \in C^{0,\alpha}_{per}$, $\phi \in [0,2\pi]$, $1 < r_2 < r_1$. *Then:*

$$|\omega(r_1e^{i\phi}) - \omega(r_2e^{i\phi})| \le Cr_1[g]_{0,\alpha}|r_1-r_2|^\alpha,$$

where

$$\omega := \int_{-\pi}^{\pi} g(\tau+\phi)\frac{r\sin(\tau)d\tau}{r^2+1-2r\cos(\tau)} \tag{15}$$

Proof. Note that:

$$|\omega(r_1e^{i\phi}) - \omega(r_2e^{i\phi})| = \left|\int_{r_2}^{r_1}\frac{\partial\omega}{\partial r}dr\right| \le \int_{r_2}^{r_1}\left|\frac{\partial\omega}{\partial r}\right|dr.$$

On the other hand:

$$\frac{\partial\omega}{\partial r}(re^{i\phi}) = \int_{-\pi}^{\pi} g(\tau+\phi)\frac{(1-r^2)\sin(\tau)d\tau}{((1-r)^2+2r(1-\cos(\tau)))^2}$$

$$= \int_{-\pi}^{\pi}(g(\tau+\phi)-g(\phi))\frac{(1-r^2)\sin(\tau)d\tau}{((1-r)^2+2r(1-\cos(\tau)))^2},$$

where we have used that $\sin(\tau)$ is odd. Moreover:

$$\left|\int_{|\tau|\le r-1}(g(\tau+\phi)-g(\phi))\frac{(1-r^2)\sin(\tau)d\tau}{((r-1)^2+2r(1-\cos(\tau)))^2}\right|$$

$$\le \int_{|\tau|\le r-1}\frac{2r_1(r-1)[g]_{0,\alpha}|\tau|^{1+\alpha}}{((r-1)^2+2r(1-\cos(\tau)))^2} \le \int_{|\tau|\le r-1}\frac{Cr_1[g]_{0,\alpha}(r-1)^{2+\alpha}}{(r-1)^4}d\tau$$

$$= Cr_1[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

Recall that $\frac{2}{\pi^2}|\tau|^2 \le 1-\cos(\tau) \le \frac{1}{2}|\tau|^2$ for $\tau \in (-\pi,\pi)$. To estimate the rest of the integral, it suffices to note that:

$$\left|\int_{r-1\le|\tau|\le\pi}(g(\tau+\phi)-g(\phi))\frac{(1-r^2)\sin(\tau)d\tau}{((r-1)^2+2r(1-\cos(\tau)))^2}\right|$$

$$\le \int_{r-1\le|\tau|\le\pi} 2r_1(r-1)[g]_{0,\alpha}\frac{|\tau|^{1+\alpha}}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau$$

$$\le \int_{r-1\le|\tau|\le\pi} Cr_1(r-1)[g]_{0,\alpha}\frac{|\tau|^{1+\alpha}}{4|\tau|^4}d\tau \le (r-1)Cr_1[g]_{0,\alpha}\int_{r-1\le|\tau|\le\pi}|\tau|^{\alpha-3}d\tau$$

$$\le Cr_1(r-1)(r-1)^{\alpha-2} = Cr_1[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

Finally:

$$|\omega(r_1e^{i\phi})-\omega(r_2e^{i\phi})| \le \int_{r_2}^{r_1}\left|\frac{\partial\omega}{\partial r}\right|dr \le Cr_1[g]_{0,\alpha}\int_{r_2}^{r_1}(r-1)^{\alpha-1}dr \le Cr_1[g]_{0,\alpha}|r_1-r_2|^{\alpha}.$$

(Recall that $|x|^\alpha$ is locally Hölder continuous in $[0,\infty)$.) □

Lemma 2.9. *Let $g \in C^{0,\alpha}_{per}$, $r > 1$, ω as in (15), and $x_1, x_2 \in \mathbb{R}^2$ such that $|x_1| = |x_2| = r$. Then:*

$$|\omega(x_1)-\omega(x_2)| \le Cr^2[g]_{0,\alpha}(r-1)^{\alpha-1}|x_1-x_2|.$$

Proof. Let $1 < r \le 2$ and $|\phi_1-\phi_2| \le \pi$, if we define $K_r(\tau) = \frac{\sin(\tau)}{1+r^2-2r\cos(\tau)}$ then:

$$\omega(re^{i\phi}) = r\int_{-\pi}^{\pi} g(\tau+\phi)K_r(\tau)d\tau = -r\int_{-\pi}^{\pi} g(\tau)K_r(\phi-\tau)d\tau.$$

The derivative of K_r is given by:

$$\frac{\cos(\tau)(1+r^2)-2r}{(1+r^2-2r\cos(\tau))^2} = \left(1-\frac{(1+r)^2(1-\cos(\tau))}{(r-1)^2+2r(1-\cos(\tau))}\right)(1+r^2-2r\cos(\tau))^{-1}.$$

Since:

$$\left|\frac{\cos(\tau)(1+r^2)-2r}{(r-1)^2+2r(1-\cos(\tau))}\right| \le 1+\frac{(1+r)^2(1-\cos(\tau))}{2r(1-\cos(\tau))} \le Cr,$$

we have:

$$\left|\frac{\partial K_r}{\partial\tau}(\tau)\right| \le \frac{Cr}{(r-1)^2+2r(1-\cos(\tau))} \le C'r|\tau|^{-2}, \text{if } |\tau| \le \pi.$$

Let $\rho = |\phi_1-\phi_2| \le \pi$, then:

$$\left|\frac{\partial\omega}{\partial\phi}\right| \le r\left|\int_{-\pi}^{\pi}(g(\tau)-g(\phi))K_r'(\phi-\tau)d\tau\right|$$

$$\le Cr^2[g]_{0,\alpha}\int_{|\tau-\phi|\le r-1}\frac{|\tau-\phi|^\alpha}{(r-1)^2}d\tau + Cr^2[g]_{0,\alpha}\int_{r-1\le|\tau-\phi|\le\pi}|\phi-\tau|^{\alpha-2}d\tau$$

$$\le Cr^2(r-1)^{\alpha-1}[g]_{0,\alpha}.$$

Now using the fundamental theorem of calculus:

$$|\omega(re^{i\phi_1}) - \omega(re^{i\phi_2})| \le \int_{\phi_1}^{\phi_2} Cr^2(r-1)^{\alpha-1}[g]_{0,\alpha} d\phi$$

$$= Cr^2(r-1)^{\alpha-1}[g]_{0,\alpha}|\phi_1 - \phi_2| \le Cr^2(r-1)^{\alpha-1}[g]_{0,\alpha}|re^{i\phi_1} - re^{i\phi_2}|.$$

$\square$

Proposition 2.10. *Let $g \in C^{0,\alpha}_{per}$, ω as in* (15), *and $x_1, x_2 \in \mathbb{R}^2$ such that $1 < |x_2| \le |x_1| \le 2$. Then:*

$$|\omega(x_1) - \omega(x_2)| \le C[g]_{0,\alpha}|x_1 - x_2|^\alpha.$$

(i.e. $[\omega]_{0,\alpha} \le C[g]_{0,\alpha}$).

Proof. Set $x_1 = r_1 e^{i\phi_1}$, $x_2 = r_2 e^{i\phi_2}$, $|\phi_1 - \phi_2| \le \pi$, $\rho := |x_1 - x_2|$.

Case $r_1 - 1 \ge \rho$: by Lemmas 2.8 and 2.9 :

$$|\omega(x_1) - \omega(x_2)| \le |\omega(r_1e^{i\phi_1}) - \omega(r_1e^{i\phi_2})| + |\omega(r_1e^{i\phi_2}) - \omega(r_2e^{i\phi_2})|$$

$$\le Cr_1[g]_{0,\alpha}(r_1-1)^{\alpha-1}|r_1e^{i\phi_1} - r_1e^{i\phi_2}| + Cr_1[g]_{0,\alpha}||x_1| - |x_2||^\alpha$$

$$\le 2C[g]_{0,\alpha}\rho^{\alpha-1}(|r_1e^{i\phi_1} - r_2e^{i\phi_2}| + |r_2e^{i\phi_2} - r_1e^{i\phi_2}|) + 2C[g]_{0,\alpha}|x_1 - x_2|^\alpha$$

$$\le C[g]_{0,\alpha}(\rho^{\alpha-1}(\rho+\rho) + \rho^\alpha).$$

Case $r_1 - 1 < \rho$: set $r := 1 + \rho$. Note that since $r_2 < r_1 < 2$, then $r = 1 + |x_1 - x_2| < 1 + r_1 + r_2 \le 5$

$$|\omega(x_1)-\omega(x_2)| \le |\omega(r_1e^{i\phi_1})-\omega(re^{i\phi_1})|+|\omega(re^{i\phi_1})-\omega(re^{i\phi_2})|+|\omega(re^{i\phi_2})-\omega(r_2e^{i\phi_2})|$$

$$\le 2 \cdot 5C[g]_{0,\alpha}|r - r_1|^\alpha + 5C[g]_{0,\alpha}(r-1)^{\alpha-1}|re^{i\phi_1} - re^{i\phi_2}|,$$

since $r_2 > 1$, then $r - r_2 = \rho - (r_2 - 1) < \rho$. On the other hand: $|re^{i\phi_1} - re^{i\phi_2}| \le |r - r_1| + |x_1 - x_2| + |r_2 - r| < 3\rho$ and $(r-1)^{\alpha-1} = \rho^{\alpha-1}$ by definition of r. This completes the proof.

$\square$

Proposition 2.11. *Let $g \in C^{0,\alpha}_{per}$, ω as in* (15), *and $x_1, x_2 \in \mathbb{R}^2$ such that $1 < |x_2| \le |x_1| \le 2$. Then:*

$$\|\omega\|_\infty \le C[g]_{0,\alpha}.$$

Proof. It is easy to see that:

$$|\omega| \le C[g]_{0,\alpha} \int_{-\pi}^{\pi} \frac{|\tau|^{1+\alpha}}{|\tau|^2} d\tau \le C[g]_{0,\alpha}.$$

$\square$

Lemma 2.12. *Let $x = re^{i\phi}$ and $y = e^{i\tau}$. Let u be given by:*

$$u(re^{i\phi}) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{g(\tau)d\tau}{|x-y|^2}, \tag{16}$$

then: $\|u\|_\infty \leq C\,\|g\|_\infty$.

Proof. This is immediate from the well-known formula (see [Gam01]):

$$\frac{r^2-1}{2\pi}\int_{-\pi}^{\pi}\frac{d\tau}{1+r^2-2r\cos(\tau)} = sgn(r-1). \tag{17}$$

$\square$

Lemma 2.13. *Let $g \in C^{0,\alpha}_{per}$, $r > 1$, $|\phi_1 - \phi_2| \leq \pi$ and u as in* (16)*. Then:*

$$|u(re^{i\phi_1}) - u(re^{i\phi_2})| \leq C[g]_{0,\alpha}|re^{i\phi_1} - re^{i\phi_2}|.$$

Proof. First note that (thanks to (17)):

$$u(re^{i\phi}) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} g(\tau)\frac{d\tau}{|x-y|^2} = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{g(\tau+\phi)-g(\phi)}{1+r^2-2r\cos(\tau)}d\tau - g(\phi),$$

then:

$$|u(re^{i\phi_1}) - u(re^{i\phi_2})| \leq [g]_{0,\alpha}|\phi_1-\phi_2|^\alpha + \frac{r^2-1}{2\pi}\int_{-\pi}^{\pi}\frac{|g(\tau+\phi_1)-g(\tau+\phi_2)|}{1+r^2-2r\cos(\tau)}d\tau$$

$$\leq [g]_{0,\alpha}|\phi_1-\phi_2|^\alpha + [g]_{0,\alpha}|\phi_1-\phi_2|^\alpha\frac{r^2-1}{2\pi}\frac{2\pi}{r^2-1} \leq C'[g]_{0,\alpha}|re^{i\phi_1}-re^{i\phi_2}|^\alpha.$$

$\square$

Lemma 2.14. *Let $g \in C^{0,\alpha}_{per}$, u as in* (16)*, $1 < r_2 < r_1 \leq 2$. Then:*

$$|u(r_1e^{i\phi}) - u(r_2e^{i\phi})| \leq C[g]_{0,\alpha}|r_1-r_2|^\alpha.$$

Proof. Note that:

$$\frac{d}{dr}\left(\frac{1-r}{1+r^2-2r\cos(\tau)}\right) = \frac{(r-1)^2-2(1-\cos(\tau))}{((r-1)^2+2r(1-\cos(\tau)))^2},$$

also:

$$\frac{d}{dr}\left(\frac{(1+r)(1-r)}{(1-r)^2+2r(1-\cos(\tau))}\right) = (1+r)\frac{d}{dr}\left(\frac{1-r}{1+r^2-2r\cos(\tau)}\right)$$
$$+\frac{1-r}{1+r^2-2r\cos(\tau)}.$$

We want to prove $\left|\frac{\partial u}{\partial r}\right| \leq C(r-1)^{\alpha-1}$, for $r \in (1,2)$. For that, it suffices to estimate the following integrals:

$$\left|(r-1)\int_{-\pi}^{\pi}(g(\tau+\phi)-g(\phi))\frac{d\tau}{(r-1)^2+2r(1-\cos(\tau))}\right| \leq C\pi^{\alpha}[g]_{0,\alpha}(r-1)\frac{2\pi}{r^2-1}$$
$$\leq C[g]_{0,\alpha} \leq C[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

Now let us estimate the second integral for $|\tau| \leq r-1$:

$$2\left|\int_{|\tau|\leq r-1}(g(\tau+\phi)-g(\phi))\frac{1-\cos(\tau)}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau\right|$$
$$\leq C[g]_{0,\alpha}\int_{|\tau|\leq r-1}\frac{|\tau|^{\alpha+2}}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau$$
$$\leq C[g]_{0,\alpha}\int_{|\tau|\leq r-1}\frac{|\tau|^{\alpha+2}}{(r-1)^4}d\tau \leq C'[g]_{0,\alpha}\frac{(r-1)^{\alpha+3}}{(r-1)^4} = C'[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

Then for $r-1 \leq |\tau| \leq \pi$:

$$2\left|\int_{r-1\leq|\tau|\leq\pi}(g(\tau+\phi)-g(\phi))\frac{1-\cos(\tau)}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau\right|$$
$$\leq [g]_{0,\alpha}C\int_{r-1\leq|\tau|\leq\pi}\frac{|\tau|^{\alpha+2}}{(2|\tau|^2)^2}d\tau \leq C'((r-1)^{\alpha-1}-\pi^{\alpha-1}) \leq C'[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

Finally, let us estimate the last integral for $|\tau| \leq r-1$:

$$(r-1)^2\left|\int_{|\tau|\leq r-1}\frac{g(\tau+\phi)-g(\phi)}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau\right|$$
$$\leq [g]_{0,\alpha}C(r-1)^2\int_{|\tau|\leq r-1}\frac{|\tau|^{\alpha}}{(r-1)^4}d\tau \leq C'[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

At last for $r-1 \leq |\tau| \leq \pi$:

$$(r-1)^2\left|\int_{r-1\leq|\tau|\leq\pi}\frac{g(\tau+\phi)-g(\phi)}{((r-1)^2+2r(1-\cos(\tau)))^2}d\tau\right|$$
$$\leq C[g]_{0,\alpha}(r-1)^2\int_{r-1\leq|\tau|\leq\pi}\frac{|\tau|^{\alpha}}{|\tau|^4}d\tau \leq C'[g]_{0,\alpha}(r-1)^2((r-1)^{\alpha-3}-\pi^{\alpha-3})$$
$$\leq C'[g]_{0,\alpha}(r-1)^{\alpha-1}.$$

In conclusion, we have:

$$|u(r_1e^{i\phi})-u(r_2e^{i\phi})| = \left|\int_{r_2}^{r_1}\frac{\partial u}{\partial r}dr\right| \leq \int_{r_2}^{r_1}\left|\frac{\partial u}{\partial r}\right|dr \leq C[g]_{0,\alpha}\int_{r_2}^{r_1}(r-1)^{\alpha-1}dr$$
$$\leq C'[g]_{0,\alpha}|r_1-r_2|^{\alpha},$$

and the result follows from the above. □

Proposition 2.15. *Let $g \in C^{0,\alpha}_{per}$, u as in* (16) $1 < r_1 \leq r_2 \leq 2$, *and* $|\phi_1 - \phi_2| \leq \pi$. *Then:*

$$|u(r_1 e^{i\phi_1}) - u(r_2 e^{i\phi_2})| \leq C[g]_{0,\alpha}|r_1 e^{i\phi_1} - r_2 e^{i\phi_2}|^\alpha.$$

(i.e. $[u]_{0,\alpha(B(0,2)\setminus B(0,1))} \leq C[g]_{0,\alpha(\partial B(0,1))}$*).*

Proof. Note that from the previous propositions we get:

$$\begin{aligned}|u(r_1 e^{i\phi_1}) - u(r_2 e^{i\phi_2})| &\leq |u(r_1 e^{i\phi_1}) - u(r_1 e^{i\phi_2})| + |u(r_1 e^{i\phi_2}) - u(r_2 e^{i\phi_2})| \\ &\leq C[g]_{0,\alpha(\partial B(0,1))}|r_1 e^{i\phi_1} - r_1 e^{i\phi_2}|^\alpha + C[g]_{0,\alpha(\partial B(0,1))}|r_1 e^{i\phi_2} - r_2 e^{i\phi_2}|^\alpha \\ &\leq C[g]_{0,\alpha(\partial B(0,1))}|r_1 e^{i\phi_1} - r_2 e^{i\phi_2}|^\alpha + C[g]_{0,\alpha(\partial B(0,1))}\,|r_2 - r_1|^\alpha \\ &\leq C[g]_{0,\alpha(\partial B(0,1))}|r_1 e^{i\phi_1} - r_2 e^{i\phi_2}|^\alpha,\end{aligned}$$

because if θ is the angle between $r_1 e^{i\phi_1}$ and $r_2 e^{i\phi_2}$, we have:

$$\begin{aligned}|r_1 e^{i\phi_1} - r_2 e^{i\phi_2}|^2 - |r_1 e^{i\phi_1} - r_1 e^{i\phi_2}|^2 &= r_2^2 - r_1^2 - 2r_1 r_2\cos(\theta) + 2r_1^2\cos(\theta) \\ &= (r_2 - r_1)(r_1 + r_2 - 2r_1\cos(\theta)) \geq (r_2 - r_1)^2 \geq 0.\end{aligned}$$

□

Proposition 2.16. *Let $g \in C^{1,\alpha}_{per}$, u as in* (16)*, then:*

$$\left\|\frac{\partial u}{\partial x_\beta}\right\|_{\infty(B(0,2)\setminus B(0,1))} \leq C\,\|g'\|_{0,\alpha(\partial B(0,1))}.$$

Moreover:

$$\left[\frac{\partial u}{\partial x_\beta}\right]_{0,\alpha(B(0,2)\setminus B(0,1))} \leq C\,\|g'\|_{0,\alpha(\partial B(0,1))}.$$

Proof. Set $x = re^{i\phi} \in B(0,2)\setminus\overline{B(0,1)}$, $y = e^{i\tau}$. Let $P(x;\tau) = \frac{1-|x|^2}{|x-y|^2}$, then:

$$D_x(P(x;\tau)) = D_x\left(\frac{1-|x|^2}{|x-y|^2}\right) = -2\left(\frac{x(|x-y|^2 + 1 - |x|^2) - y(1-|x|^2)}{|x-y|^4}\right).$$

Now, for $x \in B(0,2)\setminus\overline{B(0,1)}$, we have (due to the dominated convergence theorem):

$$D_x(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_x\left(P(x;\tau)\right) g(\tau)d\tau.$$

In addition, the derivatives of P are given by (note that we use $\tau = (\tau-\phi)+\phi$ and $|x-y|^2 = 1 + r^2 - 2r\cos(\tau-\phi)$):

$$\frac{\partial P}{\partial x_1} = -2\frac{\cos(\phi)(2r - (r^2+1)\cos(\tau-\phi)) + \sin(\phi)(1-r^2)\sin(\tau-\phi)}{(1 + r^2 - 2r\cos(\tau-\phi))^2}$$

$$\frac{\partial P}{\partial x_2} = -2\frac{\sin(\phi)(2r - (r^2+1)\cos(\tau-\phi)) - \cos(\phi)(1-r^2)\sin(\tau-\phi)}{(1 + r^2 - 2r\cos(\tau-\phi))^2}.$$

Furthermore:

$$\int_{-\pi}^{\pi} \frac{2r-(r^2+1)\cos(\tau-\phi)}{(1+r^2-2r\cos(\tau-\phi))^2} g(\tau)d\tau = -\int_{-\pi}^{\pi} \frac{d}{d\tau}\left(\frac{\sin(\tau-\phi)}{1+r^2-2r\cos(\tau-\phi)}\right) g(\tau)d\tau$$

$$= \int_{-\pi}^{\pi} \frac{\sin(\tau-\phi)}{1+r^2-2r\cos(\tau-\phi)} g'(\tau)d\tau = \int_{-\pi}^{\pi} \frac{\sin(\tau)}{1+r^2-2r\cos(\tau)} g'(\tau+\phi)d\tau.$$

Moreover:

$$\int_{-\pi}^{\pi} \frac{(1-r^2)\sin(\tau-\phi)}{(1+r^2-2r\cos(\tau-\phi))^2} g(\tau)d\tau$$

$$= -\frac{1-r^2}{2r}\int_{-\pi}^{\pi} \frac{d}{d\tau}\left(\frac{1}{1+r^2-2r\cos(\tau-\phi)}\right) g(\tau)d\tau$$

$$= \frac{1}{2r}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\tau-\phi)} g'(\tau)d\tau.$$

From the above, it is easy to conclude the result (using the estimates from the previous propositions and that $[\frac{\sin(\phi)}{r}]_{0,\alpha(B(0,2)\setminus B(0,1))} \le C$, $[\frac{\cos(\phi)}{r}]_{0,\alpha(B(0,2)\setminus B(0,1))} \le C$). □

Proposition 2.17. *Let $g \in C^{1,\alpha}(\partial B_1)$ and $u(x) = \int_{\partial B_1} g(y)\log|y-x|dS(y)$, then (for $1 < |x| < 2$) :*
$\|Du\|_\infty \le C(\|g\|_\infty + [g]_{0,\alpha})$.
$[Du]_{0,\alpha} \le C(\|g\|_\infty + [g]_{0,\alpha})$.
$\|D^2u\|_\infty \le C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha})$.
$[D^2u]_{0,\alpha} \le C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha})$.

Proof. : The gradient of u is given by:

$$Du(x) = \int_{-\pi}^{\pi} g(\tau)\frac{x-y}{|x-y|^2}d\tau,$$

with $y = (\cos(\tau), \sin(\tau))$ and $x = |x|e^{i\phi}$. Now, if $e_r(\tau) = (\cos(\tau), \sin(\tau))$ and $e_\tau(\tau) = (-\sin(\tau), \cos(\tau))$, we have:

$$Du(x) = \int_{-\pi}^{\pi} g(\tau)e_r(\tau)\frac{|x|\cos(\tau-\phi)-1}{|x-y|^2}d\tau - \int_{-\pi}^{\pi} g(\tau)e_\tau(\tau)\frac{|x|\sin(\tau-\phi)}{|x-y|^2}d\tau.$$

Note that $g_1 := g(\tau)e_r(\tau)$ and $g_2 := g(\tau)e_\tau(\tau)$ are $C^{1,\alpha}$ as functions of τ.
If we call v_1 and v_2 to the first and second integral respectively, we get:

$$v_1(x) = \frac{-1}{2}\int_{-\pi}^{\pi} g_1(\tau)\left(1 + \frac{1-|x|^2}{|x-y|^2}\right)d\tau.$$

On the other hand we have:

$$v_2 = \frac{1}{2}\int_{-\pi}^{\pi} g_2(\tau)\frac{d}{d\tau}\left(\log(|x-y|^2)\right)d\tau = \frac{-1}{2}\int_{-\pi}^{\pi} \frac{d}{d\tau}g_2(\tau)\log\left(|x-y|^2\right)d\tau$$

$$+\frac{1}{2}g_2(\tau)\log\left(|x-y|^2\right)\Big|_{\tau=-\pi}^{\tau=\pi} = -\int_{-\pi}^{\pi}\frac{d}{d\tau}g_2(\tau)\log\left(|x-y|\right)d\tau.$$

If we repeat the argument (to each component) we get:

$$D(v_2^{(j)}) = \frac{1}{2}\int_{-\pi}^{\pi} g_2'^{(j)}(\tau)e_r(\tau)\left(1+\frac{1-|x|^2}{|x-y|^2}\right)d\tau + \int_{-\pi}^{\pi} g_2'^{(j)}(\tau)e_\tau(\tau)\frac{|x|\sin(\tau-\phi)}{|x-y|^2}d\tau.$$

It is easy to see (using the estimates from the previous propositions) that:

$$|Du| \leq C(\|g\|_\infty + [g]_{0,\alpha}).$$

Moreover:

$$|D^2u| \leq C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha}).$$

Furthermore:

$$[D^2u]_{0,\alpha} \leq C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha}).$$

(It may be useful to know the following estimates, where β represents either r or τ :
$[g_k']_{0,\alpha} \leq C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha})$.
$[g_k'^{(j)}e_\beta]_{0,\alpha} \leq C(\|g\|_\infty + [g]_{0,\alpha} + \|g'\|_\infty + [g']_{0,\alpha})$.
$[g_k]_{0,\alpha} \leq C(\|g\|_\infty + [g]_{0,\alpha})$.
$[e_\beta]_{0,\alpha} \leq C$). ☐

Proposition 2.18. *Let $g \in C^{1,\alpha}(\partial B_R)$ and $u = \int_{\partial B_R} g\log|y-x|dS$, then (for $R < |x| < R+d$, with $d \leq R$) :*
$\|Du\|_\infty \leq C(\|g\|_\infty + R^\alpha[g]_{0,\alpha})$.
$[Du]_{0,\alpha} \leq C(R^{-\alpha}\|g\|_\infty + [g]_{0,\alpha})$.
$\left\|D^2u\right\|_\infty \leq C(R^{-1}\|g\|_\infty + R^{\alpha-1}[g]_{0,\alpha} + \|g'\|_\infty + R^\alpha[g']_{0,\alpha})$.
$[D^2u]_{0,\alpha} \leq C(R^{-1-\alpha}\|g\|_\infty + R^{-1}[g]_{0,\alpha} + R^{-\alpha}\|g'\|_\infty + [g']_{0,\alpha})$.

Proof. It follows by a rescaling argument. ☐

Proposition 2.19. *Let $u = \int_{\partial B_R} gG_N(x,y)dS(y)$, then:*
$\|Du\|_{\infty(B(0,R+d)\setminus\overline{B(0,R)})} \leq C(\|g\|_\infty + R^\alpha[g]_{0,\alpha})$.
$[Du]_{0,\alpha(B(0,R+d)\setminus\overline{B(0,R)})} \leq C(R^{-\alpha}\|g\|_\infty + [g]_{0,\alpha})$.
$\left\|D^2u\right\|_{\infty(B(0,R+d)\setminus\overline{B(0,R)})} \leq C(R^{-1}\|g\|_\infty + R^{\alpha-1}[g]_{0,\alpha} + \|g'\|_\infty + R^\alpha[g']_{0,\alpha})$.
$[D^2u]_{0,\alpha(B(0,R+d)\setminus\overline{B(0,R)})} \leq C(R^{-1-\alpha}\|g\|_\infty + R^{-1}[g]_{0,\alpha} + R^{-\alpha}\|g'\|_\infty + [g']_{0,\alpha})$.

Proof. Thanks to (14) we have:

$$G_N(x,y) = -\frac{1}{\pi}\log|y-x| + \frac{1}{2\pi}\log\frac{|x|}{R} - \frac{|y|^2}{4\pi R^2}.$$

The estimates for u then follow from Proposition 2.18 and estimates for $\log|x|$ (recall that for the Hölder continuity, we can proceed as in Proposition 2.2). ☐

Lemma 2.20. *Let* $\phi \in H^1(B_{\rho_2} \backslash \overline{B_{\rho_1}})$ *for some* $0 < \rho_1 < \rho_2$. *Then (for* $i = 1, 2$*):*

$$\int_{\partial B_{\rho_i}} \phi^2(x) dS(x) \leq \frac{8}{\rho_2 - \rho_1} \int_{B_{\rho_2} \backslash \overline{B_{\rho_1}}} \phi^2(x) dx + 4(\rho_2 - \rho_1) \int_{B_{\rho_2} \backslash \overline{B_{\rho_1}}} |D\phi|^2(x) dx$$

Proof. i) First we estimate $\int_{\partial B_{\rho_1}} \phi^2 dS$. Given $\varepsilon > 0$, let $\eta \in C^\infty(\overline{B_{\rho_2}} \setminus B_{\rho_1})$ be such that $\eta = 0$ on ∂B_{ρ_2}, $\eta = 1$ on ∂B_{ρ_1} and $|D\eta| \leq \frac{1+\varepsilon}{\rho_2 - \rho_1}$.

$$\int_{\partial B_{\rho_1}} \phi^2(x) dS(x) = \rho_1 \int_{S^1} \left(\int_{\rho_1}^{\rho_2} \frac{d}{ds}((\eta\phi)(sz)) ds \right)^2 dS(z)$$

$$\leq 2\rho_1(\rho_2 - \rho_1) \int_{S^1} \int_{\rho_1}^{\rho_2} (|\phi D\eta|^2 + |\eta D\phi|^2) ds dS(z)$$

$$\leq 2 \left(\frac{(1+\varepsilon)^2}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} \int_{S^1} \phi^2(sz) s dS(z) ds + (\rho_2 - \rho_1) \int_{\rho_1}^{\rho_2} \int_{S^1} |D\phi|^2(sz) s dS(z) ds \right).$$

ii) To estimate $\int_{\partial B_{\rho_2}} \phi^2 dS$, we consider first the case in which $\rho_1 \geq \frac{\rho_2}{2}$: given $\varepsilon > 0$, let $\eta \in C^\infty(\overline{B_{\rho_2}} \setminus B_{\rho_1})$ be such that $\eta = 1$ on ∂B_{ρ_2}, $\eta = 0$ on ∂B_{ρ_1} and $|D\eta| \leq \frac{1+\varepsilon}{\rho_2 - \rho_1}$.

$$\int_{\partial B_{\rho_2}} \phi^2(x) dS(x) = \rho_2 \int_{S^1} \left(\int_{\rho_1}^{\rho_2} \frac{d}{ds}((\eta\phi)(sz)) ds \right)^2 dS(z)$$

$$\leq 2\rho_2(\rho_2 - \rho_1) \int_{S^1} \int_{\rho_1}^{\rho_2} (|\phi D\eta|^2 + |\eta D\phi|^2) ds dS(z)$$

$$\leq 4 \left(\frac{(1+\varepsilon)^2}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} \int_{S^1} \phi^2(x) s dS(x) ds + (\rho_2 - \rho_1) \int_{\rho_1}^{\rho_2} \int_{S^1} |D\phi|^2(x) s dS(x) ds \right).$$

Case in which $\rho_1 < \frac{\rho_2}{2}$: by the previously considered case, since $H^1(B_{\rho_2} \setminus \overline{B_{\rho_1}}) \subset H^1(B_{\rho_2} \setminus \overline{B_{\frac{\rho_2}{2}}})$ we have that

$$\int_{\partial \rho_2} \phi^2 dx \leq \frac{4}{\rho_2 - \frac{\rho_2}{2}} \int_{B_{\rho_2} \backslash B_{\frac{\rho_2}{2}}} \phi^2 dx + 4 \left(\rho_2 - \frac{\rho_2}{2} \right) \int_{B_{\rho_2} \backslash B_{\frac{\rho_2}{2}}} |D\phi|^2 dx$$

$$\leq \frac{8}{\rho_2 - \rho_1} \int_{B_{\rho_2} \backslash \overline{B_{\rho_1}}} \phi^2(x) dx + 4(\rho_2 - \rho_1) \int_{B_{\rho_2} \backslash \overline{B_{\rho_1}}} |D\phi|^2(x) dx.$$

□

2.4 The Neumann problem

The L^1 norm

Proposition 2.21. *Let* E, d, *and* B *be as in* (2), (4), *and* (6). *Suppose*

$$\begin{cases} \Delta u = 0 \text{ in } E, \\ \dfrac{\partial u}{\partial \nu} = g \text{ on } \partial E \end{cases}$$

and $\int_E u(y)dy = 0$. *Then:* $\|u\|_{L^1(E)} \leq C \cdot B\|g\|_\infty$.

Proof. First note that:

$$\int_E |u|dy \leq |E|^{\frac{1}{2}}\|u\|_{L^2(E)} \leq C_P|E|^{\frac{1}{2}}\|Du\|_{L^2(E)},$$

$C_P(E)$ being Poincaré's constant (5). Integrating by parts we get:

$$\int_E u\Delta u dy = \int_{\partial E} ugdS(y) - \int_E |Du|^2 dy = 0.$$

Moreover:

$$\int_E |Du|^2 dy \leq \|g\|_{L^2(\partial E)}\|u\|_{L^2(\partial E)}.$$

Using Cauchy's inequality, we get:

$$\|Du\|_{L^2(E)} \leq \frac{1}{2^{\frac{1}{2}}}\left(A\|g\|_{L^2(\partial E)} + \frac{\|u\|_{L^2(\partial E)}}{A}\right).$$

Furthermore, using 2.20 and the Poincaré's constant, we obtain:

$$\int_{\partial E} u^2 dS = \sum_{k=0}^{n}\int_{\partial B(z_k,r_k)} u^2 dS \leq C\left(\int_{B(z_0,r_0)\setminus B(z_0,r_0-d)} d^{-1}u^2 + d|Du|^2 dy\right)$$

$$+C\left(\sum_{k=1}^{n}\int_{B(z_k,r_k+d)\setminus B(z_k,r_k)} d^{-1}u^2 + d|Du|^2 dy\right)$$

$$\leq C\left(d^{-1}\int_E u^2 dy + \int_E d|Du|^2 dy\right) \leq C(d^{-1}C_P^2 + d)\int_E |Du|^2 dy$$

Choosing $A = 2^{\frac{1}{2}}C(d^{\frac{-1}{2}}C_P + d^{\frac{1}{2}})$ we deduce that:

$$\|Du\|_{L^2(E)} \leq 2^{\frac{1}{2}}A\|g\|_{L^2(\partial E)} \leq C(d^{\frac{-1}{2}}C_P + d^{\frac{1}{2}})n^{\frac{1}{2}}r_0^{\frac{1}{2}}\|g\|_\infty.$$

Finally, we obtain:

$$\|u\|_{L^1(E)} \leq C \cdot |E|^{\frac{1}{2}}C_P(d^{\frac{-1}{2}}C_P + d^{\frac{1}{2}})n^{\frac{1}{2}}r_0^{\frac{1}{2}}\|g\|_\infty.$$

□

Regularity near the holes

Proposition 2.22. *Let B and u be as in* (6) *and Proposition 2.21, then, if* $A = \cup_{k=1}^n B(z_k, r_k + \frac{d}{3}) \setminus \overline{B(z_k,r_k)}$, *we have:*
$\|Du\|_{L^\infty(A)} \leq C\left(1 + Bd^{-4}r_0\right)\|g\|_\infty + Cr_0^\alpha[g]_{0,\alpha}$.
$[Du]_{0,\alpha(B(z_k,r_k+\frac{d}{3})\setminus\overline{B(z_k,r_k)})} \leq C\left(Bd^{-5}r_0^{2-\alpha} + d^{-\alpha}\right)\|g\|_\infty + C[g]_{0,\alpha}$.
$\|D^2u\|_{L^\infty(A)} \leq C\left(Bd^{-5}r_0 + d^{-1}\right)\|g\|_\infty + Cd^{\alpha-1}[g]_{0,\alpha} + C\|g'\|_\infty + Cr_0^\alpha[g']_{0,\alpha}$.
$[D^2u]_{0,\alpha(B(z_k,r_k+\frac{d}{3})\setminus\overline{B(z_k,r_k)})} \leq C\left(Bd^{-6}r_0^{2-\alpha} + d^{-1-\alpha}\right)\|g\|_\infty + Cd^{-1}[g]_{0,\alpha} + Cd^{-\alpha}\|g'\|_\infty + C[g']_{0,\alpha}$.

Proof. It follows from Proposition 2.4, Proposition 2.19, Proposition 2.7, Lemma 2.3 and Proposition 2.21 (recall that $r_i \geq d$). $\square$

Interior regularity

Proposition 2.23. *Let E, d, and B be as in* (2), (4), *and* (6). *Let u be harmonic in E and $E' = B(z_0, r_0 - \frac{d}{3}) \setminus \bigcup_{k=1}^{n} B(z_k, r_k + \frac{d}{3})$, then:*
$\|u\|_{L^\infty(E')} \leq Cd^{-2} \|u\|_{L^1(E)} \leq CBd^{-2} \|g\|_\infty$.
$[u]_{0,\alpha(E')} \leq Cd^{-3} r_0^{1-\alpha} \|u\|_{L^1(E)} \leq CBd^{-3} r_0^{1-\alpha} \|g\|_\infty$.
$\left\|D^\beta u\right\|_{L^\infty(E')} \leq Cd^{-2-|\beta|} \|u\|_{L^1(E)} \leq CBd^{-2-|\beta|} \|g\|_\infty$.
$[u]_{1,\alpha(E')} \leq Cd^{-4} r_0^{1-\alpha} \|u\|_{L^1(E)} \leq CBd^{-4} r_0^{1-\alpha} \|g\|_\infty$.
$[D^2 u]_{0,\alpha(E')} \leq Cd^{-5} r_0^{1-\alpha} \|u\|_{L^1(E)} \leq CBd^{-5} r_0^{1-\alpha} \|g\|_\infty$.

Proof. It follows from local regularity for harmonic functions and Proposition 2.2 (using triangle inequality at most $2n+1$ times): join x and z with a straight line, then the segment intersects at most the n holes. In that case, join the points using segments of the above straight line and segments of circles of the form $\partial B(z_k, r_k + \frac{d}{3})$ (for straight lines use local estimates for harmonic functions and for circles use Proposition 2.2). $\square$

Regularity near the exterior boundary

In the next proposition and lemma, R should be thought of as $r_0 - d$, hence $\{x : R < |x| < R + d\}$ is the the part of the d-neighbourhood of the exterior boundary that lies inside E.

Proposition 2.24. *Let v be harmonic in Ω and ζ be a cut-off function equal to* 0 *for $|x| \leq R + \frac{d}{3}$ and equal to* 1 *for $R + \frac{2}{3}d \leq |x|$, then, if $u = \zeta v$:*

$$u(x) = C + \int_{\partial B_R} \frac{\partial u}{\partial \nu} \left(\Phi(y - x) - \phi^x(y)\right) dS(y) - \int_\Omega \Delta u \left(\Phi(y - x) - \phi^x(y)\right) dy.$$

Proof. This can be shown using the same techniques as in the proof of Proposition 2.4. $\square$

The proofs of the following two results, are similar to the proof of Lemma 2.3 and Proposition 2.19, respectively:

Lemma 2.25. *Let $R \geq Cd$, v be harmonic in Ω and ζ be a cut-off function equal to* 0 *for $|x| \leq R + \frac{d}{3}$ and equal to* 1 *for $R + \frac{2}{3}d \leq |x|$, then:*
$[\Delta(v\zeta)]_{0,\alpha(\mathbb{R}^2)} \leq CR^{1-\alpha} d^{-5} \|v\|_{L^1(\Omega)}$.
$\|\Delta(v\zeta)\|_{\infty(\mathbb{R}^2)} \leq Cd^{-4} \|v\|_{L^1(\Omega)}$.

Proposition 2.26. *Let $u = \int_{\partial B_{r_0}} g G_N(x, y) dS(y)$, then:*
$\|Du\|_{\infty(B(0,r_0) \setminus \overline{B(0, r_0 - \frac{d}{3})})} \leq C(\|g\|_\infty + r_0^\alpha [g]_{0,\alpha})$.
$[Du]_{0,\alpha(B(0,r_0) \setminus \overline{B(0, r_0 - \frac{d}{3})})} \leq C(r_0^{-\alpha} \|g\|_\infty + [g]_{0,\alpha})$.

$\left\|D^2u\right\|_{\infty(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(r_0^{-1}\left\|g\right\|_\infty + r_0^{\alpha-1}[g]_{0,\alpha} + \left\|g'\right\|_\infty + r_0^{\alpha}[g']_{0,\alpha})$.
$[D^2u]_{0,\alpha(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(r_0^{-1-\alpha}\left\|g\right\|_\infty + r_0^{-1}[g]_{0,\alpha} + r_0^{-\alpha}\left\|g'\right\|_\infty + [g']_{0,\alpha})$.

Proposition 2.27. *Let B and u be as in Proposition 2.21, then, we have:*
$\left\|Du\right\|_{\infty(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(1+Bd^{-4}r_0)\left\|g\right\|_\infty + Cr_0^{\alpha}[g]_{0,\alpha}$.
$[Du]_{0,\alpha(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(r_0^{-\alpha} + Bd^{-5}r_0^{2-\alpha})\left\|g\right\|_\infty + C[g]_{0,\alpha}$.
$\left\|D^2u\right\|_{\infty(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(r_0^{-1} + Bd^{-5}r_0)\left\|g\right\|_\infty + Cr_0^{\alpha-1}[g]_{0,\alpha} + C\left\|g'\right\|_\infty + Cr_0^{\alpha}[g']_{0,\alpha}$.
$[D^2u]_{0,\alpha(B(0,r_0)\setminus\overline{B(0,r_0-\frac{d}{3})})} \le C(r_0^{-1-\alpha} + Bd^{-6}r_0^{2-\alpha})\left\|g\right\|_\infty + Cr_0^{-1}[g]_{0,\alpha} + Cr_0^{-\alpha}\left\|g'\right\|_\infty + C[g']_{0,\alpha}$.

Proof. First note that the hypothesis: $B(z_i, r_i + d) \subset B(z_0, r_0)$ and $r_i \ge d$ for all $i \in \{1, \dots, n\}$, implies that $r_0 \ge 2d$. Hence, the hypothesis $R \ge Cd$ for some $C > 0$ is satisfied when $R = r_0 - d$. The estimates then follow from Proposition 2.24, Proposition 2.26, Proposition 2.7, Lemma 2.25 and Proposition 2.21. □

Global regularity

Theorem 2. *Let B and u be as in Proposition 2.21, then, we have:*
$\left\|Du\right\|_{\infty(E)} \le C(1+Bd^{-4}r_0)\left\|g\right\|_\infty + Cr_0^{\alpha}[g]_{0,\alpha}$.
$[Du]_{0,\alpha(E)} \le C(d^{-\alpha} + Bd^{-5}r_0^{2-\alpha})\left\|g\right\|_\infty + C[g]_{0,\alpha}$.
$\left\|D^2u\right\|_{\infty(E)} \le C(d^{-1} + Bd^{-5}r_0)\left\|g\right\|_\infty + Cd^{\alpha-1}[g]_{0,\alpha} + C\left\|g'\right\|_\infty + Cr_0^{\alpha}[g']_{0,\alpha}$.
$[D^2u]_{0,\alpha(E)} \le C(d^{-1-\alpha} + Bd^{-6}r_0^{2-\alpha})\left\|g\right\|_\infty + Cd^{-1}[g]_{0,\alpha} + Cd^{-\alpha}\left\|g'\right\|_\infty + C[g']_{0,\alpha}$.

Proof. It follows from Proposition 2.22, Proposition 2.23 and Proposition 2.27. □

Poincaré's constant

Recall the definition of $C_P(E)$ in (5).

Theorem 3. *Let $n \in \mathbb{N}$ and $0 < \delta < 1$. There exists a universal constant $C(\delta)$ such that*

$$C_P(E) \le C(\delta) r_0$$

for $E = B(z_0, r_0) \setminus \bigcup_{i=1}^{n} B(z_i, r_i)$, whenever $z_0, ...z_n \in \mathbb{R}^2$ and $d, r_0, ..., r_n > 0$ satisfy $\frac{d}{r_0} \ge \delta$ and (4).

Remark. Note that $n \le \delta^{-2}$, because (4) implies $\bigcup_{i=1}^{n} B(z_i, r_i + r_0\delta) \subset B(z_0, r_0)$, which yields $n(r_0\delta)^2 \le \sum_{i=1}^{n}(r_i + r_0\delta)^2 \le r_0^2$.

Proof. 1. Let us see first that it is enough to consider the case when $r_0 = 1$ and $z_0 = 0$. Suppose there exists such a constant $C(\delta)$ for domains with outer radius equal to 1. Now consider a general E (with r_0 not necessarily equal to 1). Let

$\phi \in H^1(E)$ be such that $\|D\phi\|_{L^2(E)} = 1$ and $\int_E \phi = 0$. Set $\psi(\omega) := \phi(z_0 + r_0\omega)$, $\omega \in \hat{E}$ where $\hat{E} = \frac{E - z_0}{r_0}$. We have that

$$\int_{\hat{E}} |D\psi|^2 = r_0^2 \int_{\hat{E}} |D\phi(z_0 + r_0\omega)|^2 d\omega = \int_E |D\phi|^2 dy = 1$$

Clearly, we also have that $\int_{\hat{E}} \psi = 0$. Then, by assumption, $\|\psi\|_{L^2} \leq C(\delta)$ (note that if E satisfies the conditions in the statement then also does $\hat{E}$).
Hence

$$\int_E \phi^2(y)dy = \int_{\hat{E}} \phi^2(z_0 + r_0\omega) r_0^2 d\omega = r_0^2 \|\psi\|^2_{L^2(\hat{E})} \leq C(\delta)^2 r_0^2;$$

since ϕ is arbitrary, this yields $C_P(E) \leq C(\delta) r_0$.
2. Looking for a contradiction, suppose there exist $0 < \delta < 1$, a sequence of domains $(E_j)_{j\in\mathbb{N}}$ with unit outer radius and a sequence $(\phi_j)_{j\in\mathbb{N}}$ such that for all j:
a)$\phi_j \in H^1(E_j)$.
b)$\|D\phi_j\|_{L^2(E_j)} < \frac{1}{j}$, $\|\phi_j\|_{L^2(E_j)} = 1$.
c)$\int_{E_j} \phi_j = 0$.
d)Each E_j satisfies the conditions in the statement of the theorem.
Let $\tilde{\phi_j}$ denote the extensions of ϕ_j to $B(0,1)$. We have that $\|\tilde{\phi_j}\|_{L^2(E_j)} \leq 2\|\phi_j\|_{L^2(E_j)} = 2$ and $\int_{B(0,1)} |D\tilde{\phi_j}|^2 \leq C\left(\delta^{-2} + \frac{1}{j}\right)$.
Taking a subsequence, we obtain that $\tilde{\phi_j} \overset{H^1}{\rightharpoonup} \phi$ for some $\phi \in H^1(B(0,1))$. Also, a subsequence can be taken such that the centers $z_i^{(j)}$ and the radii $r_i^{(j)}$ of the holes of E_j converge. Set E be the limit domain. Clearly $|E \Delta E_j| \to 0$.
For every $E' = B(0,1) \setminus \overline{\bigcup_{i=1}^{i=n} B(z_i, r_i')}$ such that $E' \subset\subset E$ and such that the disks $B(z_i, r_i' + \frac{\delta}{2})$ are disjoint and contained in $B(0, 1 - \frac{\delta}{2})$, we have that $D\tilde{\phi_j} = D\phi_j \to 0$ in $L^2(E')$ (because $\|D\phi_j\|_{L^2(E')} \leq \|D\phi_j\|_{L^2(E_j)} < \frac{1}{j}$ since $E' \subset E_j$ for sufficiently large j).
By uniqueness of weak limits, $D\phi \equiv 0$ in every such E'. Indeeed, for every $\eta \in C_c^\infty(E')$ we have that

$$\left|\int_{E'} \eta \partial_\alpha \phi\right| = \left|\int_{B(0,1)} \eta \partial_\alpha \phi\right| = \left|\lim_{j\to\infty} \int_{B(0,1)} \eta \partial_\alpha \tilde{\phi_j}\right| = \lim_{j\to\infty} \left|\int_{B(0,1)} \eta \partial_\alpha \tilde{\phi_j}\right|$$

$$= \lim_{j\to\infty} \left|\int_{E'} \eta \partial_\alpha \tilde{\phi_j}\right| \leq \limsup_{j\to\infty} \|\eta\|_{L^2(E')} \|D\tilde{\phi_j}\|_{L^2(E')} = 0.$$

By the fundamental theorem of the calculus of variations, $\partial_\alpha \phi = 0$ in E'.
It follows that $\phi|_{E'}$ is constant for every such E'. If E', E'' are two such domains and $E' \subset E''$, clearly the constant value of $\phi|_{E'}$ must coincide with the constant value of $\phi|_{E''}$, hence ϕ is constant in E.
Since $H^1(B(0,1)) \subset\subset L^q(B(0,1))$ we can assume that for some $q > 2$ $\tilde{\phi_j} \to \phi$

strongly in L^q. Thus

$$1 = \lim_{j\to\infty} \int_{E_j} \phi_j^2 = \lim_{j\to\infty} \int_{B(0,1)} \tilde{\phi_j}^2 \chi_{E_j} = \int_{B(0,1)} \phi^2 \chi_E$$

($\tilde{\phi_j}^2 \to \phi^2$ in $L^{\frac{q}{2}}$ and $\chi_{E_j} \to \chi_E$ in $L^{\left(\frac{q}{2}\right)'}$).
Analogously,

$$0 = \lim_{j\to\infty} \int_{E_j} \phi_j = \lim_{j\to\infty} \int_{B(0,1)} \tilde{\phi}_j \chi_{E_j} = \int_{B(0,1)} \phi \chi_E$$

Hence $\phi = 0$ in E (because ϕ was constant), but this contradicts that $\int_{B(0,1)} \phi^2 \chi_E = 1$. This completes the proof.

□

2.5 The incompressibility equation

Proof of Theorem 1. We follow the strategy of Dacorogna-Moser [DM90] which consists in solving first

$$\begin{cases} \Delta\phi = 0 & \text{in } E, \\ \dfrac{\partial\phi}{\partial\nu} = g(x) & \text{on } \partial E\ , \end{cases} \tag{18}$$

with $\int_E \phi = 0$ and then choosing $v = D\phi + D^\perp\psi$ where $D^\perp\psi := (\partial_{z_2}\psi, -\partial_{z_1}\psi)$ is a divergence-free covector field that cancels out the tangential parts of $D\phi$ on $\partial B_i, \forall i$. Concretely $\psi(z) = \varphi(z) - \zeta\left(\frac{2dist(z,\partial E)}{d}\right)\varphi(q(z))$ where φ is the solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in } E, \\ \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\phi}{\partial\tau} & \text{on } \partial E\ , \end{cases} \tag{19}$$

$$q(z) = \begin{cases} r_k \frac{z-z_k}{|z-z_k|} + z_k & \text{if } |z-z_k| < r_k + \frac{d}{2} \\ r_0 \frac{z}{|z|} & \text{if } |z| > r_0 - \frac{d}{2} \end{cases} \tag{20}$$

and ζ is a cutoff function such that $0 \le \zeta \le 1$, $\zeta(0) = 1$ and $\zeta(1) = 0$.

Using Theorem 2 we get the following estimates:

$$\|D\varphi\|_\infty \le C\left((1 + Bd^{-4}r_0)\left\|\frac{\partial\phi}{\partial\tau}\right\|_\infty + r_0^\alpha \left\|\frac{\partial\phi}{\partial\tau}\right\|_{0,\alpha}\right)$$

$$\|D^2\varphi\|_\infty \le C\left((d^{-1} + Bd^{-5}r_0)\left\|\frac{\partial\phi}{\partial\tau}\right\|_\infty + d^{\alpha-1}\left[\frac{\partial\phi}{\partial\tau}\right]_{0,\alpha} + \left\|\frac{\partial^2\phi}{\partial\tau^2}\right\|_\infty + r_0^\alpha\left[\frac{\partial^2\phi}{\partial\tau^2}\right]_{0,\alpha}\right).$$

Now, it is easy to see that:

$$\left\|\frac{\partial\phi}{\partial\tau}\right\|_{\infty} \leq \|D\phi\|_{\infty}$$

$$\left[\frac{\partial\phi}{\partial\tau}\right]_{0,\alpha} \leq C\left(d^{-\alpha}\|D\phi\|_{\infty} + [D\phi]_{0,\alpha}\right)$$

$$\left\|\frac{\partial^2\phi}{\partial\tau^2}\right\|_{\infty} \leq C\left(d^{-1}\|D\phi\|_{\infty} + \|D^2\phi\|_{\infty}\right)$$

$$\left[\frac{\partial^2\phi}{\partial\tau^2}\right]_{0,\alpha} \leq C\left(d^{-1-\alpha}\|D\phi\|_{\infty} + d^{-1}\left[D\phi\right]_{0,\alpha} + d^{-\alpha}\|D^2\phi\|_{\infty} + \left[D^2\phi\right]_{0,\alpha}\right).$$

Moreover:

$$\left\|\frac{\partial\phi}{\partial\tau}\right\|_{\infty} \leq C\left((1+Bd^{-4}r_0)\|g\|_{\infty} + r_0^{\alpha}[g]_{0,\alpha}\right)$$

$$\left[\frac{\partial\phi}{\partial\tau}\right]_{0,\alpha} \leq C\left((d^{-\alpha} + Bd^{-5}r_0^{2-\alpha})\|g\|_{\infty} + \frac{r_0^{\alpha}}{d^{\alpha}}[g]_{0,\alpha}\right)$$

$$\left\|\frac{\partial^2\phi}{\partial\tau^2}\right\|_{\infty} \leq C\left((d^{-1} + Bd^{-5}r_0)\|g\|_{\infty} + r_0^{\alpha}d^{-1}[g]_{0,\alpha} + \|g'\|_{\infty} + r_0^{\alpha}[g']_{0,\alpha}\right)$$

$$\left[\frac{\partial^2\phi}{\partial\tau^2}\right]_{0,\alpha} \leq C\left((d^{-1-\alpha} + Bd^{-6}r_0^{2-\alpha})\|g\|_{\infty} + d^{-1}\frac{r_0^{\alpha}}{d^{\alpha}}[g]_{0,\alpha} + d^{-\alpha}\|g'\|_{\infty} + \frac{r_0^{\alpha}}{d^{\alpha}}[g']_{0,\alpha}\right).$$

From the above we deduce that:

$$\|D\varphi\|_{\infty} \leq C\left((r_0^{\alpha}d^{-\alpha} + Bd^{-5}r_0^2 + B^2d^{-8}r_0^2)\|g\|_{\infty} + \left(\frac{r_0^{2\alpha}}{d^{\alpha}} + Bd^{-4}r_0^{1+\alpha}\right)[g]_{0,\alpha}\right)$$

$$\|D^2\varphi\|_{\infty} \leq C\left(A_1\|g\|_{\infty} + A_2[g]_{0,\alpha} + \frac{r_0^{\alpha}}{d^{\alpha}}\|g'\|_{\infty} + \frac{r_0^{2\alpha}}{d^{\alpha}}[g']_{0,\alpha}\right),$$

where $A_1 = \left(\frac{r_0}{d}\right)^{\alpha} d^{-1} + Bd^{-6}r_0^2 + B^2d^{-9}r_0^2$ and $A_2 = \frac{r_0^{2\alpha}}{d^{1+\alpha}} + Bd^{-5}r_0^{1+\alpha}$

On the other hand, it is easy to see that:

$$\|D\psi\|_{\infty} \leq C\left(\frac{1}{d}\|\varphi\|_{\infty} + \|D\varphi\|_{\infty}\right)$$

$$\|D^2\psi\|_{\infty} \leq C\left(\frac{1}{d^2}\|\varphi\|_{\infty} + \frac{1}{d}\|D\varphi\|_{\infty} + \|D^2\varphi\|_{\infty}\right).$$

Note that using the fundamental theorem of calculus one can obtain (using that there exists a point where φ vanishes): $\|\varphi\|_{\infty} \leq Cr_0\|D\varphi\|_{\infty}$. Finally the result follows by adding the estimates for φ. $\square$

3 Estimates for cavitation

3.1 Preliminaries

Topological image and condition INV

We give a succint definition of the topological image (see [HS13] for more details).

Definition 1. *Let $u \in W^{1,p}(\partial B(x,r), \mathbb{R}^2)$ for some $x \in \mathbb{R}^2$, $r > 0$, and $p > 1$. Then*

$$\operatorname{im}_{\mathrm{T}}(u, B(x,r)) := \{y \in \mathbb{R}^2 : \deg(u, \partial B(x,r), y) \neq 0\}.$$

Given $u \in W^{1,p}(E, \mathbb{R}^2)$ and $x \in E$, there is a set $R_x \subset (0, \infty)$, which coincides a.e. with $\{r > 0 : B(x,r) \subset E\}$, such that $u|_{\partial B(x,r)} \in W^{1,p}$ and both $\deg(u, \partial B(x,r), \cdot)$ and $\operatorname{im}_{\mathrm{T}}(u, B(x,r))$ are well defined for all $r \in R_x$.

Definition 2. *We say that u satisfies condition INV if for every $x \in E$ and every $r \in R_x$*

(i) $u(z) \in \operatorname{im}_{\mathrm{T}}(u, B(x,r))$ for a.e. $z \in B(x,r) \cap E$ and

(ii) $u(z) \in \mathbb{R}^2 \setminus \operatorname{im}_{\mathrm{T}}(u, B(x,r))$ for a.e. $z \in E \setminus B(x,r)$.

If u satisfies condition INV then $\{\operatorname{im}_{\mathrm{T}}(u, B(x,r)) : r \in R_x\}$ is increasing in r for every x.

Definition 3. *Given $a \in E$ we define*

$$\operatorname{im}_{\mathrm{T}}(u, a) := \bigcap_{r \in R_a} \operatorname{im}_{\mathrm{T}}(u, B(a,r)).$$

Analogously, if $u \in W^{i,p}$ is defined and satisfies condition INV in a domain of the form $E = \mathcal{B} \setminus \bigcup_1^n B(z_i, r_i)$, then we define

$$\operatorname{im}_{\mathrm{T}}(u, B(z_i, r_i)) = \bigcap_{\substack{r \in R_{z_i} \\ r > r_i}} \operatorname{im}_{\mathrm{T}}(u, B(z,r)).$$

Distributional Jacobian

Definition 4. *Given $u \in W^{1,2}(E, \mathbb{R}^2) \cap L^\infty_{loc}(E, \mathbb{R}^2)$ its distributional Jacobian is defined as the distribution*

$$\langle \operatorname{Det} Du, \phi \rangle := -\frac{1}{2} \int_E u(x) \cdot (\operatorname{cof} Du(x)) D\phi(x) \, \mathrm{d}x, \quad \phi \in C_c^\infty(E).$$

3.2 The cost of distortion

We show how to adapt the proof of [HS13, Prop. 1.1] in order to obtain the refined estimate (8) (which, as mentioned in the Introduction, shows clearly that round cavities are energetically preferred).

Proof of (8). Equations (3.3)-(3.4) in [HS13] show that

$$\int_{\mathcal{B}\setminus\bigcup \overline{B}_\varepsilon(a_i)} \frac{|Du|^2-1}{2} dx \geq \sum_1^n v_i \log \frac{R/2}{n\varepsilon} + C\int_{t_0}^{s_0} \left(\sum_{B\in\mathcal{B}(t)} |E_B| D(E_B)^2 \right) \frac{dt}{t},$$

where t_0, s_0, $\mathcal{B}(t)$ and E_B are as in the proof of [HS13, Prop. 1.1] (E_B is an abbreviated notation for $\mathrm{im_T}(u, B)$; it is the union of the the cavities opened from B and of region occupied, in the deformed configuration, by the material points in B). In the ball construction giving rise to the collection $\mathcal{B}(t)$, the radius $r(B)$ of every ball $B \in \mathcal{B}(t)$ is such that $r(B) \geq t/n$. Let r_i be the radius of the largest among all the disks in the ball construction that are obtained as simple dilations of $B_\varepsilon(a_i)$ (that is, before any merging process takes place). If $B(a_i, \frac{d_i}{2})$ is disjoint with all balls in $\mathcal{B}(s_0)$, then there is no loss of generality in assuming that $r_i = \frac{d_i}{2}$. If that were not the case, then $B(a_i, r_i)$ merges with other ball of $\mathcal{B}(t)$ precisely at 'time' t. Since $r \leq t$ holds true both for $r = r_i$ and for the radius of the other ball with which it merges, and since the other ball necesssarily contains other cavitation points, it follows that $d_i < 3t$. Since $r_i = r(B)$ for the ball $B = B(a_i, r_i) \in \mathcal{B}(t)$, by the observation at the beginning of this paragraph we know that $r_i \geq t/n$. Therefore, $r_i \geq \frac{d_i}{3n}$. Taking this into account it can be seen that the above estimate can be replaced by

$$\int_{\mathcal{B}\setminus\bigcup \overline{B}_\varepsilon(a_i)} \frac{|Du|^2-1}{2} dx \geq \sum_1^n v_i \log \frac{R/2}{n\varepsilon} + C\sum_1^n \int_\varepsilon^{r_i} |E(a_i,r)| D(E(a_i,r))^2 \frac{dr}{r}.$$

By decreasing r_i, if necessary, it can be assumed that $r_i < \sqrt{\frac{v_i D_i^2}{24\pi}}$. By virtue of [HS13, Lemma 3.6.(ii)], this suffices to conclude that $|E(a_i,r)| D(E(a_i,r))^2 \geq \frac{1}{2} v_i D_i^2$ for all $r \in (\varepsilon, r_i)$. This completes the proof. □

3.3 Deformations opening only round cavities

A simple necessary geometric condition

Definition 5. *Let $n \in \mathbb{N}$, $R_0 > 0$, and $\mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$. We say that $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ is a configuration attainable through an evolution of circular cavities (or, more briefly, an* attainable configuration*) if $a_i \in \mathcal{B}$ and $v_i > 0$ for all $i \in \{1, \dots, n\}$, and there exist evolutions*

- $z_i \in C^1([1, \lambda], \mathbb{R}^2)$ *of the cavity centres, and*
- $L_i : [1, \lambda] \to [0, \infty)$ *of the cavity radii,*

where λ is given by

$$\sum_{i=1}^{n} v_i = (\lambda^2 - 1)\pi R_0^2,$$

such that

$$\sum_{i=1}^{n} \pi L_i^2(t) = (t^2 - 1)\pi R_0^2 \qquad \forall\, t \in [1, \lambda] \tag{21}$$

and for each $i \in \{1, \dots, n\}$

i) L_i^2 *belongs to* $C^1([1, \lambda], [0, \infty))$*;*

ii) $z_i(1) = a_i$ *and* $L_i(1) = 0$*;*

iii) $\pi L_i^2(\lambda) = v_i$*; and*

iv) for all $t \in [1, \lambda]$ *the disks* $\overline{B(z_i(t), L_i(t))}$ *are disjoint and contained in* $B(0, tR_0)$*.*

Although other time parametrizations are of course possible for the evolution of the centres and the radii in the above definition, we have chosen the stretch factor at the outer boundary $\partial\mathcal{B}$ as our parameter.

Examples

The following examples give a sense about which configurations $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ are attainable through an evolution of circular cavities. We begin with a general result; the more concrete examples are obtained as its corollaries.

Lemma 3.1. *Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$, $v_1, \dots, v_n > 0$. Let $\lambda > 1$ be such that $(\lambda^2 - 1)\pi R_0^2 = \sum v_i$. Set*

$$\sigma = \min \left\{ \min_i \frac{\left(1 - \frac{|a_i|}{R_0}\right)^2}{\frac{v_i}{\sum v_k}}, \min_{i \neq j} \frac{|a_i - a_j|^2}{R_0^2 \left(\sqrt{\frac{v_i}{\sum v_k}} + \sqrt{\frac{v_j}{\sum v_k}} \right)^2} \right\}. \tag{22}$$

Then both in the case $\sigma \geq 1$ and in the case when $\sigma < 1$ and $\lambda^2 < \frac{1}{1-\sigma}$ the configuration $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ is attainable through an evolution of circular cavities.

Proof. For every $t \in [1, \lambda]$ and every $i \in \{1, \dots, n\}$ set

$$z_i(t) := ta_i, \quad L_i(t) := \sqrt{(t^2 - 1)\frac{v_i}{\sum v_k}} \cdot R_0. \tag{23}$$

We only need to check that the $\overline{B(z_i(t), L_i(t))}$ are disjoint and contained in $B(0, tR_0)$ for all t (the remaining conditions in Definition 5 are immediately verified). Both in the case $\sigma \geq 1$ and in the case $\sigma < 1$ and $\lambda^2 < \frac{1}{1-\sigma}$ we have that

$$1 - \lambda^{-2} < \sigma.$$

As a consequence, we obtain that

$$1 - t^{-2} < \sigma \quad \forall\, t \in [1, \lambda].$$

Hence,

$$1 - t^{-2} < \frac{\left(1 - \frac{|a_i|}{R_0}\right)^2}{\frac{v_i}{\sum v_k}} \quad \forall\, i$$

and

$$1 - t^{-2} < \frac{|a_i - a_j|^2}{R_0^2 \left(\sqrt{\frac{v_i}{\sum v_k}} + \sqrt{\frac{v_j}{\sum v_k}}\right)^2} \quad \forall\, i \neq j.$$

It is easy to see that the first inequality is equivalent to

$$L_i(t)^2 < t^2 (R_0 - |a_i|)^2$$

which in turn says that $L_i(t) + |z_i(t)| < tR_0$ (i.e., each $\overline{B(z_i(t), L_i(t))} \subset B(0, tR_0)$). Analogously, the second inequality is equivalent to

$$(\sqrt{L_i(t)} + \sqrt{L_j(t)})^2 < t^2 |a_i - a_j|^2$$

which in turn says that $L_i(t) + L_j(t) < |z_i(t) - z_j(t)|$ (i.e., the disks are disjoint). This completes the proof. $\square$

In the case when $v_1 = v_2 = \cdots = v_n$,

$$\sigma = \frac{n\pi \min\left\{\min_i (R_0 - |a_i|)^2, \min_{i \neq j} \left(\frac{|a_i - a_j|}{2}\right)^2\right\}}{\pi R_0^2}. \tag{24}$$

This is the packing density of the largest disjoint collection of the form $\{B(a_i, \rho) : i \in \{1, \ldots, n\}\}$ contained in $\mathcal{B}$ (same ρ for all i). There is an extensive literature on the famous circle packing problem; for example, it is known [Mel94] that when $n = 11$ the maximum packing density is

$$\frac{11}{\left(1 + \frac{1}{\sin \frac{\pi}{9}}\right)^2} \approx 0.7145,$$

which yields the upper bound

$$\lambda < \sqrt{\frac{(1 + \sin \frac{\pi}{9})^2}{1 + 2 \sin \frac{\pi}{9} - 10 \sin^2 \frac{\pi}{9}}} \approx 1.8714$$

for which our above construction is able to produce attainable configurations with 11 cavities of equal size.

Corollary 3.1. *Given any $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$, and $1 \le \lambda < \frac{1}{\sqrt{1-\sigma}}$, where σ is the maximum packing density (24), it is possible to attain the configuration of cavities of equal size compatible with the boundary condition $u(x) = \lambda x$ for $x \in \partial B(0, R_0)$ (namely, $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ with $v_1 = \dots = v_n = \frac{\pi R_0^2}{n}(\lambda^2 - 1)$).*

Corollary 3.2. *Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$. If $d_1, \dots, d_n > 0$ are such that the disks $\overline{B}(a_i, d_i)$ are disjoint and contained in $B(0, R_0)$, then the configuration $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$, with*

$$v_i = \pi d_i^2 \cdot \frac{1}{1 - \frac{\sum \pi d_k^2}{\pi R_0^2}} \quad \forall i \in \{1, \dots, n\},$$

is attainable.

Proof. We begin by noting that if $v_1, \dots, v_n$ are proportional to the areas of disks of radii $d_1, \dots, d_n$ then there is a simple sufficient condition for the hypothesis $\lambda^2 < \frac{1}{1-\sigma}$ in Lemma 3.1 to be satisfied. Indeed, suppose

$$\exists\, s > 0\ \forall\, i \in \{1, \dots, n\}\ v_i = \frac{s}{\sum_{k=1}^n \pi d_k^2} \pi d_i^2. \tag{25}$$

Then $\sigma > 1 - \lambda^{-2}$ if and only if

$$\forall i: \ \frac{\left(1 - \frac{|a_i|}{R_0}\right)^2}{\frac{\pi d_i^2}{\sum \pi d_k^2}} > 1 - \lambda^{-2} \quad \text{and} \quad \forall i \neq j: \ \frac{|a_i - a_j|^2}{R_0^2 \frac{(d_i + d_j)^2}{\sum d_k^2}} > 1 - \lambda^{-2}.$$

This is equivalent to

$$(1 - \lambda^{-2}) \frac{\pi R_0^2}{\sum_k \pi d_k^2} < \min \left\{ \min_i \ \left(\frac{R_0 - |a_i|}{d_i} \right)^2, \min_{i \neq j} \frac{|a_i - a_j|^2}{(d_i + d_j)^2} \right\}.$$

The minimum on the right-hand side is greater than one because the disks $\overline{B}(a_i, d_i)$ are disjoint and contained in $B(0, R_0)$. Hence, thanks to Lemma 3.1, for $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ to be attainable it suffices that $(1 - \lambda^{-2}) \le \frac{\sum_k \pi d_k^2}{\pi R_0^2}$, i.e.,

$$\lambda^2 \le \frac{1}{1 - \frac{\sum \pi d_k^2}{\pi R_0^2}}. \tag{26}$$

Recall that $\sum v_k = (\lambda^2 - 1)\pi R_0^2$, due to incompressibility. Since $\sum v_k = s$, the expression for v_i in (25) may be rewritten as

$$v_i = (\lambda^2 - 1)\pi R_0^2 \frac{\pi d_i^2}{\sum \pi d_k^2}. \tag{27}$$

The conclusion then follows by choosing the maximum value of λ in (26). $\square$

In the case of only one cavity, all loads are attainable, even if the cavitation point is close to the boundary.

Proposition 3.2. *All configurations with $n = 1$ are attainable.*

Proof. Let $a \in B(0, R_0)$ and $\lambda > 1$. We are to show that evolutions $t \in [1, \lambda] \mapsto z(t)$ and $t <\in [1, \lambda] \mapsto L(t)$ of the cavity's center and radius exist such that z and L^2 are C^1, $z(1) = a$, $L(1) = 0$, $\forall t : \pi L^2(t) = (t^2 - 1)\pi R_0^2$, and $\forall t : \overline{B(z(t), L(t))} \subset B(0, tR_0)$. It suffices to take $L(t) := \sqrt{t^2 - 1}R_0$ and $z(t) := (\lambda - \sqrt{\lambda^2 - 1})a$, which are well defined actually for all $t \in [1, \infty)$. $\square$

Existence theorem

Theorem 4. *Let $n \in \mathbb{N}$, $R_0 > 0$, and $\mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$. Suppose that the configuration $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ is attainable. Let $\lambda > 1$ be given by $\sum v_i = (\lambda^2 - 1)\pi R_0^2$. Then there exists $u \in \bigcap_{1 \leq p < 2} W^{1,p}(\mathcal{B}, \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathcal{B} \setminus \{a_1, \ldots, a_n\}, \mathbb{R}^2)$ satisfying*

- *the invertibility condition (INV) of Definition 2;*
- *$u(x) = \lambda x$ for $x \in \partial\mathcal{B}$;*
- *$\det Du(x) = 1$ for a.e. $x \in \mathcal{B}$;*
- *the cavities $\mathrm{im_T}(u, a_i)$ (as defined in Definition 1) are disks of areas v_i, for all $i \in \{1, \ldots, n\}$;*
- *there exists a constant $C = C\big(n, R_0, (a_i)_{i=1}^n, (v_i)_{i=1}^n\big)$ such that for all $\varepsilon < 0$*

$$\int_{\mathcal{B} \setminus \bigcup_1^n \overline{B}_\varepsilon(a_i)} \frac{|Du|^2}{2} dx \leq C + \left(\sum_{i=1}^n v_i \right) |\log \varepsilon|. \tag{28}$$

Proof. Let $z_i : [1, \lambda] \to \mathbb{R}^2$ and $L_i : [1, \lambda] \to [0, \infty)$, $i \in \{1, \ldots, n\}$, be as in Definition 5. By continuity, there exist $R_1, \ldots, R_n > 0$ such that for

$$r_i(t) := \sqrt{L_i(t)^2 + R_i^2}, \quad t \in [1, \lambda], \quad i \in \{1, \ldots, n\} \tag{29}$$

the balls $\overline{B(z_i(t), r_i(t))}$ are disjoint and contained in $B(0, r_0(t))$, with

$$r_0(t) := tR_0,$$

for every $t \in [1, \lambda]$.

Near each cavitation point (to be precise, in $\{x : \ \epsilon \leq |x - a_i| \leq R_i\}$), we work with the unique radially symmetric deformations creating cavities of the desired sizes. This is Proposition 3.3. Then, in order to 'glue' these symmetric independent cavitations, we build an incompressible deformation far from the cavities, using the flow of Dacorogna & Moser [DM90] and the fine estimates of the previous section. This is Proposition 3.4. The conclusion follows by combining both propositions. $\square$

Proposition 3.3. *Let $u : \mathcal{B} \to \mathbb{R}^2$ be such that for every i and $0 < r < R_i$*

$$u(a_i + re^{i\theta}) = z_i(\lambda) + \sqrt{L_i(\lambda)^2 + r^2}e^{i\theta}.$$

Then $u|_{\bigcup B(a_i,R_i)}$ is one-to-one a.e., satisfies $\det Du \equiv 1$ a.e., and is such that $|\operatorname{im}_{\mathrm{T}}(u, B_\varepsilon(a_i))| = v_i + \pi\varepsilon^2$ for all i and

$$\int_{\bigcup\{x:\varepsilon<|x-a_i|<R_i\}} \frac{|Du|^2}{2}\,\mathrm{d}x \le \sum_i \pi R_i^2 + \sum_i v_i \log R_i + \left(\sum_{i=1}^n v_i\right) |\log\varepsilon|$$

for every small $\varepsilon > 0$.

Proof. Given $i \in \{1, \dots, n\}$, $r \in (0, R_i)$ and $\theta \in [0, 2\pi]$

$$Du(a_i + re^{i\theta}) = \frac{r}{\sqrt{L_i(\lambda)^2 + r^2}} e^{i\theta} \otimes e^{i\theta} + \sqrt{1 + \frac{L_i(\lambda)^2}{r^2}} ie^{i\theta} \otimes ie^{i\theta}. \qquad (30)$$

Hence $\det Du \equiv 1$ and

$$\int_{\bigcup\{x:\varepsilon<|x-a_i|<R_i\}} \frac{|Du|^2}{2}\,\mathrm{d}x \le \sum_i \int_\varepsilon^{R_i} \left(1 + \left(1 + \frac{L_i(\lambda)^2}{r^2}\right)\right) \cdot \pi r\,\mathrm{d}r. \qquad (31)$$

$\square$

Proposition 3.4. *Let $n \in \mathbb{N}$ and $\mathcal{B} = B(0, R_0) \subset \mathbb{R}^2$. Suppose that the configuration $\Big((a_i)_{i=1}^n, (v_i)_{i=1}^n\Big)$ is attainable. There exists $u_{ext} \in H^1(\mathcal{B} \setminus \bigcup_1^n B(a_i, R_i), \mathbb{R}^2)$, where the R_i are as in (29), satisfying $u_{ext}(x) = \lambda x$ on $\partial\mathcal{B}$; $\det Du_{ext} \equiv 1$ in $\mathcal{B} \setminus \bigcup_1^n B(a_i, R_i)$; condition (INV); and*

$$u_{ext}(a_i + R_i e^{i\theta}) = z_i(\lambda) + \sqrt{L_i(\lambda)^2 + R_i^2} e^{i\theta}, \quad \forall\, i \in \{i, \dots, n\}\ \forall\, \theta \in [0, 2\pi].$$

Proof. We proceed as follows:

- We fix the notation to describe the growth of the (boundaries of the) circular holes (corresponding to the disks $B(a_i, R_i)$ of Proposition 3.3 which are not analyzed in Theorem 3.4 and are, thus, removed from $\mathcal{B}$).
- At each instant we build a velocity field for the material points by superposing two auxiliary fields, one that increases the radii $r_i(t)$ of the excised holes and another that deals with the evolution of their centers $z_i(t)$.
- The trajectory of each material point is obtained as the solution of the ODE that establishes its relation to the previously constructed instantaneous velocity fields.
- We explain why the resulting deformation is injective and incompressible.

Evolution of the domains. For every $t \in [1, \lambda]$ set

$$E(t) := B(0, tR_0) \setminus \bigcup_{i=1}^{n} B(z_i(t), r_i(t))$$

where $r_i(t)$ is defined in (29). By continuity, there exists $d > 0$ (independent of t) such that (4) is satisfied, for every $t \in [1, \lambda]$, with z_i replaced with $z_i(t)$ and r_i replaced with $r_i(t)$. Regarding $r_0(t) = tR_0$, note that $r_0(t) \leq \lambda R_0$ for all $t \in [1, \lambda]$. Hence, setting $\delta := \frac{d}{2\lambda R_0}$ (which depends on n, R_0, $(a_i)_{i=1}^n$ and $(v_i)_{i=1}^n$ but not on t) we have that $\frac{d}{r_0(t)} \geq \delta \ \forall t \in [1, \lambda]$. In particular, by Theorem 3 there exists C such that $C_P\Big(E(t)\Big) \leq C \cdot r_0(t)$ for all t. This implies that $B\Big(E(t)\Big) \leq C$ for some C independent of t, where $B\Big(E(t)\Big)$ is that of (6).

A velocity field that accounts for the increase in the radii $r_i(t)$. Consider a fixed $t \in [1, \lambda]$. Define $g : \partial E(t) \to \mathbb{R}$ by

$$g(y) = \frac{\mathrm{d}r_i(t)}{\mathrm{d}t} \quad \forall\, y \in \partial B(z_i(t), r_i(t)), \quad i \in \{0, 1, \dots, n\}.$$

Clearly (21) and (29) imply (3). We have thus all the hypotheses of Theorem 1, which yields the existence of $v_t \in C^{2,\alpha}(\overline{E(t)}, \mathbb{R}^2)$ such that

$$\operatorname{div} v_t \equiv 0 \text{ in } E(t) \tag{32}$$

$$v_t\Big(z_i(t) + r_i(t)e^{i\theta}\Big) = \frac{\mathrm{d}r_i(t)}{\mathrm{d}t} e^{i\theta} \quad \forall\, i, \theta \tag{33}$$

$$\|Dv_t\|_\infty \leq C\|g\|_\infty, \tag{34}$$

where $C = C\big(n, R_0, (a_i)_{i=1}^n, (v_i)_{i=1}^n\big)$. Recall that $L_i^2 \in C^1([1, \lambda], [0, \infty))$ (by Definition 5), so

$$\|g\|_\infty = \max_i \left| \frac{\frac{\mathrm{d}}{\mathrm{d}t}(L_i^2(t))}{r_i(t)} \right| \leq \frac{C}{\min_i R_i}$$

is bounded above indepedently of t.

A velocity field for the translation of the excised holes. Let $\eta \in C_c^\infty([0, 1))$ be such that $\eta(0) = 1$ and $\eta'(0) = 0$. Define

$$w(y) := \begin{cases} \eta\left(\frac{r - r_i(t)}{d}\right) \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} \cdot (rie^{i\theta}), & \text{if } y = z_i(t) + re^{i\theta},\ r_i(t) \leq r < r_i(t) + d; \\ 0 & \text{in other case} \end{cases}$$

and

$$\tilde{v}_t(y) := D^\perp w(y), \qquad y \in \overline{E(t)}.$$

Then

$$\operatorname{div} \tilde{v}_t \equiv 0 \text{ in } E(t) \tag{35}$$

$$\tilde{v}_t(y) = \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} \text{ on } \partial B(z_i(t), r_i(t)) \tag{36}$$

and

$$\|D\tilde{v}_t\|_\infty = \max_i \left\| \left(d^{-2}\eta'' e^{i\theta} \otimes e^{i\theta} + (dr)^{-1}\eta' ie^{i\theta} \otimes ie^{i\theta}\right) \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} \cdot (rie^{i\theta}) \right.$$
$$\left. + d^{-1}\eta' \left(\frac{\mathrm{d}z_i(t)}{\mathrm{d}t}\right)^\perp \otimes e^{i\theta} \right\|_\infty$$
$$\leq C(d^{-2} \cdot (\lambda R_0) + d^{-1}) \left| \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} \right|,$$

which again is bounded uniformly in t since $z_i \in C^1([1,\lambda], \mathbb{R}^2)$.

Definition of u_{ext} and energy bounds. For every $x \in \mathcal{B} \setminus \bigcup_1^n B(a_i, R_i)$ and every $t \in [1, \lambda]$ let $f(x,t)$ be the solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial f}{\partial t}(x,t) = v_t(f(x,t)) + \tilde{v}_t(f(x,t)) \\ &f(x,1) = x. \end{aligned} \tag{37}$$

It can be seen (as in Dacorogna & Moser [DM90]) that the above autonomous ODE indeed has a well defined solution with enough regularity in time and space (in spite of the fact that the velocity fields are defined in changing domains). Moreover,

$$f(a_i + R_i e^{i\theta}, t) = z_i(t) + r_i(t) e^{i\theta} \quad \forall\, i, \theta$$

and

$$f(R_0 e^{i\theta}, t) = tR_0 e^{i\theta}$$

thanks to the boundary conditions for v_t and $\tilde{v}_t$. Define u_{ext} by

$$u_{ext}(x) := f(x, \lambda), \quad x \in \mathcal{B} \setminus \bigcup_1^n B(a_i, R_i).$$

For every $i \in \{i, \dots, n\}$ and $\theta \in [0, 2\pi]$

$$u_{ext}(a_i + R_i e^{i\theta}) = z_i(\lambda) + \sqrt{L_i(\lambda)^2 + R_i^2}\, e^{i\theta}$$

since $r_i(\lambda) = \sqrt{L_i(\lambda)^2 + R_i^2}$. Also $u_{ext}(x) = \lambda x$ on $\partial\mathcal{B}$.

The resulting deformation u_{ext} is incompressible because

$$\begin{aligned} \frac{\partial}{\partial t} \det D_x f(x,t) &= \operatorname{cof} D_x f(x,t) \cdot D_x \frac{\partial f}{\partial t}(x,t) \\ &= \operatorname{cof} D_x f(x,t) \cdot D_x((v_t + \tilde{v}_t) \circ f)(x,t) \\ &= \operatorname{cof} D_x f(x,t) \cdot (D_y(v_t + \tilde{v}_t)(f(x,t)) D_x f(x,t)) \\ &= (\operatorname{cof} D_x f(x,t) (D_x f(x,t))^T) \cdot D_y(v_t + \tilde{v}_t)(f(x,t)) \\ &= (\det D_x f(x,t)) I \cdot D_y(v_t + \tilde{v}_t)(f(x,t)) \end{aligned}$$

and the right-hand side is zero since $\operatorname{div}(v_t + \tilde{v}_t) \equiv 0$.

To see that $u_{ext} \in H^1$ it is enough to observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int |D_x f(x,t)|^2\,\mathrm{d}x &= \int D_x f(x,t)\cdot D_x\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x \\ &= \int D_x f(x,t)\cdot ((D_y(v_t+\tilde{v}_t))(f(x,t))D_x f(x,t)),\end{aligned}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |D_x f(x,t)|^2\,\mathrm{d}x \leq \underbrace{(\sup_t \|Dv_t + D\tilde{v}_t\|_{L^\infty(E(t))})}_{:=C}\int |D_x f(x,t)|^2\,\mathrm{d}x.$$

This implies that $e^{-Ct}\int |D_x f(x,t)|^2$ decreases with t. Consequently,

$$\int |Du_{ext}|^2 \leq e^{C(\lambda-1)}\int |I|^2\,\mathrm{d}x < \infty.$$

Finally, Ball's global invertibility theorem [Bal81] shows that u_{ext} is one-to-one a.e. which combined with the previous energy estimate and [BHMC17, Lemma 5.1] yields that u_{ext} satisfies condition INV.

□

3.4 Lower bound for the coalescence load

3.4.1 Main result

In this section we will recall briefly [HS13] where the question of when do the cavities begin to lose their round shape and eventually coalesce had already been studied. Most of their analysis was valid in the case of an arbitrarily large number of cavities; however, the main results could only be stated when $n = 2$. For the general case one ingredient was missing: the existence theorem we obtained in the previous subsection. As will be explained shortly, the conclusion that can now be obtained is that if a load $\left((a_i)_{i=1}^n, (v_i)_{i=1}^n\right)$ is attainable through an evolution of circular cavities (Definition 5) then that load is not large enough to trigger the coalescence of the cavities.

Theorem 5. *Let* $n \in \mathbb{N}$ *and* $\mathcal{B} := B(0, R_0) \subset \mathbb{R}^2$. *Suppose that the configuration* $\left((a_i)_{i=1}^n, (v_i)_{i=1}^n\right)$ *is attainable. Let* $\lambda > 1$ *be given by* $\sum_1^n v_i = (\lambda^2 - 1)\pi R_0^2$. *Let* $\varepsilon_j \to 0$ *be a sequence that we will denote in what follows simply by* ε. *Set* $\mathcal{B}_\varepsilon := \mathcal{B} \setminus \bigcup_{i=1}^n \overline{B}_\varepsilon(a_i)$. *Assume that for every* ε *the map* u_ε *minimizes* $\int_{\mathcal{B}_\varepsilon} |Du|^2\,\mathrm{d}x$ *among all* $u \in H^1(\mathcal{B}_\varepsilon; \mathbb{R}^2)$ *satisfying*

- *the invertibility condition (INV) of Definition 2;*
- $u(x) = \lambda x$ *for* $x \in \partial\mathcal{B}$;
- $\det Du(x) = 1$ *for a.e.* $x \in \mathcal{B}_\varepsilon$;

- *and* $|\operatorname{im_T}(u, B_\varepsilon(a_i))| = v_i + \pi\varepsilon^2$ *for all* $i \in \{1, \dots, n\}$.

Then there exists a subsequence (not relabelled) and $u \in \bigcap_{1\le p<2} W^{1,p}(\mathcal{B}, \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathcal{B} \setminus \{a_1, \dots, a_n\}, \mathbb{R}^2)$ *such that*

- $u_\varepsilon \rightharpoonup u$ *in* $H^1_{\mathrm{loc}}(\mathcal{B} \setminus \{a_1, \dots, a_n\}, \mathbb{R}^2)$;
- $\operatorname{Det} Du_\varepsilon \overset{*}{\rightharpoonup} \operatorname{Det} Du$ *in* $\mathcal{B} \setminus \{a_1, \dots, a_n\}$*; locally in the sense of measures (where* $\operatorname{Det} Du$ *is the distributional Jacobian of Definition 4);*
- $\operatorname{Det} Du = \sum_{i=1}^n v_i \delta_{a_i} + \mathcal{L}^2$ *in* $\mathcal{B}$ *(where* $\mathcal{L}^2$ *is the Lebesgue measure);*
- *The cavities* $\operatorname{im_T}(u, a_i)$ *(as defined in Definition 1) are disks of area* v_i*, for all* $i \in \{1, \dots, n\}$*;*
- $|\operatorname{im_T}(u_\varepsilon, B_\varepsilon(a_i)) \triangle \operatorname{im_T}(u, a_i)| \to 0$ *as* $\varepsilon \to 0$ *for* $i \in \{1, \dots, n\}$.

3.4.2 Blow-up rate of the Dirichlet energy

The conclusions of the theorem are the same as those in [HS13, Thm. 1.9]; this should come as no surprise since the former is obtained by applying the latter. What differs is that the conclusions are obtained under a different set of hypotheses.

The main assumption in [HS13, Thm. 1.9] is that a constant C (independent of ε) exists such that

$$\int_{\mathcal{B}_\varepsilon} \frac{|Du_\varepsilon|^2}{2}\,\mathrm{d}x \le C + \left(\sum_{i=1}^n v_i\right) |\log\varepsilon|. \tag{38}$$

Recall that the cost of opening round cavities of areas $v_1, \dots, v_n$ is $(\sum v_i)|\log\varepsilon|$. In a sense, this is to be expected since the singularity in the gradient of a map creating a cavity from a single point $a \in \mathcal{B}$ is at least of the order of

$$|Du(x)| \sim \frac{L}{r}, \quad r = |x - a|$$

where L is such that πL^2 equals the area of the created cavity. In light of (8), condition (38) yields that all the distortions are zero, hence all cavities are round, as stated in the theorem. From (8) we see that leaving the space of deformations that open only round cavities comes with an energetic cost of order $|\log\varepsilon|$ (in addition to the $(\sum v_i)|\log\varepsilon|$ common to all maps in the admissible space). Therefore, the elongation and coalescence of voids corresponds to *a higher energy regime*; condition (8), in contrast, characterizes the lowest energy regime where the Dirichlet enegy blows up at no more that the stated rate of $(\sum v_i)|\log\varepsilon|$, which corresponds to loads not large enough so as to initiate the merging of cavities.

The question remained open as to for what loads the energy upper bound (38) is fulfiled. Henao & Serfaty [HS13] solved this for the case of two cavities, using explicit constructions of incompressible maps opening cavities of all possible sizes from a pair of arbitrary cavitation points. The novelty on this work is that we now solve the nonlinear equation of incompressibility for an arbitrarily large number of cavities. As seen in the previous sections, instead of the explicit constructions we use the flow of Dacorogna & Moser [DM90], combined with a careful study of the dependence on the geometry of the regularity estimates for the Neumann problem. As follows from Theorem 4, our conclusion is that a sufficient condition for a load configuration to lie in the lowest energy regime (38) is that it be attainable through an evolution of circular cavities.

Proof of Theorem 5. By Theorem 4 there exists u, defined in all of $\mathcal{B}\setminus\{a_1, \dots, a_n\}$, which is a radially symmetric cavitation in a neighbourhood of each a_i. For each $\varepsilon > 0$ let $\tilde{u}_\varepsilon$ denote the restriction of u to $\mathcal{B}_\varepsilon$. Since the sequence $(\tilde{u}_\varepsilon)_\varepsilon$ clearly fulfils (38), and since $\int |Du_\varepsilon| \leq \int |D\tilde{u}_\varepsilon|$ (because, by hypothesis, the u_ε are energy minimizers), the sequence $(u_\varepsilon)_\varepsilon$ also satisfies (38). The result then follows by applying the arguments [HS13, Thm. 1.9]. $\square$

3.4.3 Critical boundary displacement for coalescence

As mentioned in p. 5, the problem of interest is to understand how cavities will continue to evolve once they have attained a certain size of order 1 (that is, a size much larger than the one at the rest state -or at the onset of fracture). More concretely, we shall consider variational problems with a constraint of the form

$$v_i \geq \upsilon_i \quad \forall i \in \{1, \dots, n\} \tag{39}$$

for the areas of the cavities, where the minimum areas $\upsilon_i > 0$ are specified a priori. Note that this is important also in light of Proposition 3.2: if it is possible for all except one of the υ_i to be equal to zero, then nothing regarding void coalescence can be deduced from our analysis. Theorem 5 treats the problem of opening cavities of prespecified areas $v_1, \dots, v_n$. The constraint (39) allows us to consider the more general variational problem of minimizing the Dirichlet energy in the space $\mathcal{A}_\varepsilon$ of maps $u \in H^1(\mathcal{B}_\varepsilon; \mathbb{R}^2)$ satisfying

- the invertibility condition (INV) of Definition 2;
- $u(x) = \lambda x$ for $x \in \partial\mathcal{B}$;
- $\det Du(x) = 1$ for a.e. $x \in \mathcal{B}_\varepsilon$;
- $|\operatorname{im_T}(u, B_\varepsilon(a_i))| \geq \upsilon_i + \pi\varepsilon^2$ for all $i \in \{1, \dots, n\}$.

(40)

For these variational problems we finally obtain a result in terms only of the displacement of the outer boundary.

Theorem 6. *Let $n \in \mathbb{N}$, $\mathcal{B} = B(0, R_0) \subset \mathbb{R}^2$, and $a_1, \dots, a_n \in \mathcal{B}$ be given. Let $\overline{B}(a_1, d_1)$, ..., $\overline{B}(a_n, d_n)$ be a disjoint collection of closed balls contained in $\mathcal{B}$. Let $\sigma^* := \frac{\sum_k \pi d_k^2}{\pi R_0^2}$ denote its associated packing density. Let $\upsilon_1, \dots, \upsilon_n > 0$ be* *given and suppose that*

$$\upsilon_i < \pi d_i^2 \cdot \frac{1}{1-\sigma^*} \text{ for each } i \in \{1, \dots, n\}. \tag{41}$$

Then there exists $\lambda_0 \in (1, \frac{1}{\sqrt{1-\sigma^*}})$ *such that given any*

$$\lambda_0 \leq \lambda < \frac{1}{\sqrt{1-\sigma^*}}; \tag{42}$$

any sequence $\varepsilon_j \to 0$ *(which we denote in what follows simply by* ε*); and any sequence* $(u_\varepsilon)_\varepsilon$ *of minimizers of* $\int_{\mathcal{B}_\varepsilon} |Du|^2 dx$ *in the spaces* $\mathcal{A}_\varepsilon$ *of* (40)*; the maps* u_ε *tend to produce only round cavities in the limit as* $\varepsilon \to 0$ *(i.e., all of the conclusions of Theorem 5 are obtained).*

Remarks. 1. Based on the discussions of this section, we interpret the value of λ found in (42), namely, $\left(1 - \frac{\sum_k \pi d_k^2}{\pi R_0^2}\right)^{-\frac{1}{2}}$, as a lower bound for the coalescence load for this problem.

2. Thinking of a quasistatic loading, the theorem says that even if n cavities have already formed and grown inside the body, it is still possible to continue loading it without entering the stage of void coalescence provided that their current radii $\sqrt{\frac{\upsilon_i}{\pi}}$ are less than $\frac{d_i}{\sqrt{1-\sigma^*}}$. As mentioned at the end of the Introduction, this suggests that if even one of the cavities has not yet attained that characteristic size then no coalescence should be expected (because that cavity still has room to grow as a round cavity, sustaining itself the global effect of the increment in the external load).

3. Observe that $\frac{1}{1-\sigma^*} \to \infty$ as $\sigma^* \to 1^-$. This has an effect both on the coalescence load (which is larger than $\sqrt{\frac{1}{1-\sigma^*}}$) and on the critical final radius $\frac{d_i}{\sqrt{1-\sigma^*}}$ for a circular cavity. This suggests that the energetically most favourable situation is when the space available in the reference configuration $B(0, R_0)$ is optimally distributed among all the balls $B(a_i, d_i)$. This occurs either when the body opens only one cavity, or at the other end when the body opens a larger and larger number of smaller cavities. The second possibility is more realistic, due to the dynamic and irreversible nature of the fracture processes and due to local vs. global minimization considerations. What prevents an arbitrarily large number of cavities from being created are the energies required for fracture (see [MC14]) and the tension associated to the presence of an ever increasing inner surface (which is especially large since what matters is its state in the deformed configuration, as pointed out by Müller & Spector [MS95]).

Proof. We proceed as in the proof of Theorem 5, except that now $v_1, \dots, v_n$ are to be found such that $v_i \geq \upsilon_i$ for all i and the configuration $\left((a_i)_{i=1}^n, (v_i)_{i=1}^n\right)$

is attainable. Choose the $v_1, \dots, v_n$ given by (27). Thanks to (26), $\lambda < \frac{1}{\sqrt{1-\sigma^*}}$ is enough to ensure that the configuration is attainable. From (27) we also see that $v_i \geq \upsilon_i$ if and only if $\lambda^2 \geq 1 + \frac{\upsilon_i}{\pi d_i^2}\sigma^*$. This holds for each i if and only if

$$\lambda \geq \lambda_0 := \sqrt{1 + \left(\max_i \frac{\upsilon_i}{\pi d_i^2}\right)\sigma^*}.$$

Note, in turn, that $\lambda_0 < \frac{1}{\sqrt{1-\sigma^*}}$ (which is necessary for (42) to be meaningful) if and only if (41) is satisfied. The conclusion now follows by applying the arguments in [HS13, Thm. 1.9]; the hypothesis on the blow-up rate of the energy (as $\varepsilon \to 0$) is satisfied thanks to Theorem 4. $\square$

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