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Global invertibility of Sobolev maps

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Abstract

We define a class of Sobolev $W^{1,p}(\Omega,\mathbb{R}^n)$ functions, with $p > n-1$, such that its trace on $\partial\Omega$ is also Sobolev, and do not present cavitation in the interior or on the boundary. We show that if a function in this class has positive Jacobian and coincides on the boundary with an injective map, then the function is itself injective. We then prove the existence of minimizers within this class for the type of functionals that appear in nonlinear elasticity.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ and consider a map $\mathbf{u} : \bar{\Omega} \to \mathbb{R}^n$ such that $\mathbf{u}$ is locally invertible and coincides on $\partial\Omega$ with an invertible map $\mathbf{u}_0 : \bar{\Omega} \to \mathbb{R}^n$. A classic question in topology is to ascertain extra conditions on $\mathbf{u}$ to conclude that $\mathbf{u}$ is globally invertible.

The question of global invertibility was revitalized with the study of nonlinear elasticity. In that theory, $\mathbf{u} : \Omega \to \mathbb{R}^n$ represents the deformation of a body that occupies the set Ω in its reference configuration. Typically, $\mathbf{u}$ is assumed to be in some Sobolev space. A realistic deformation $\mathbf{u}$ is required to satisfy $\det D\mathbf{u} > 0$, because this condition guarantees the orientation-preserving character of the map. For sufficiently regular $\mathbf{u}$, that condition implies the local invertibility of $\mathbf{u}$, thanks to the inverse function theorem. The non-interpenetration of matter is a physical requirement on a deformation stating that two different material points cannot be mapped to the same point. This expresses the injectivity (or invertibility) of the deformation $\mathbf{u}$. It is also common in nonlinear elasticity to impose Dirichlet boundary conditions on $\mathbf{u}$, so that $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ for some deformation $\mathbf{u}_0 : \Omega \to \mathbb{R}^n$, which, again, has to be invertible. Hence, it is natural to ask whether $\mathbf{u}$ is also invertible.

The first result for global invertibility in Sobolev spaces was due to Ball [3]. He proved the invertibility of a Sobolev map $W^{1,p}$ with $p > n$ such that $\det D\mathbf{u} > 0$ and $\mathbf{u}$ coincides on $\partial\Omega$ with an invertible map. Developments and refinements of his result, as well as other approaches to invertibility in the context of nonlinear elasticity, are [8, 42, 43, 37, 35, 9, 23, 25, 26].

The starting point of this work is the paper by Barchiesi *et al.* [6], in which it is defined a class of Sobolev functions $W^{1,p}$ for $p > n-1$ for which *cavitation* (the formation of voids in the material) does not appear. For this class of functions, it is proved there a local invertibility theorem, thus extending the results by [17], done for the case $p \geq n$. In this paper we answer the question initially posed in this introduction: given a $\mathbf{u}$ which is in the class of [6] (hence locally invertible) and coincides with an invertible map on $\partial\Omega$, when is $\mathbf{u}$ globally invertible?

One of the extra conditions we require is further regularity on the trace $\mathbf{u}|_{\partial\Omega}$. To be precise, trace theory asserts that $\mathbf{u}|_{\partial\Omega}$ lies the space $W^{1-\frac{1}{p},p}$, while here we additionally impose that it is in $W^{1,p}$. In principle, the integrability exponent for the tangential derivatives of $\mathbf{u}|_{\partial\Omega}$ need not be the same p as for the integrability of $D\mathbf{u}$ in the interior. As will be seen in Theorem 9.1, our proof of global invertibility is valid as long as the integrability of the tangential derivatives on $\partial\Omega$ is above $n-1$. Since we also need $p \geq n-1$ in the interior, in particular to ensure that the cofactors are integrable, there is no substantial gain in presenting our results using an exponent for the boundary value different from that of the interior, so, for clarity of exposition, we shall work with maps $\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^n)$ such that $\mathbf{u}|_{\partial\Omega} \in W^{1,p}(\partial\Omega,\mathbb{R}^n)$.

The other fundamental extra condition for our analysis is longer to explain, and we start by recalling the class of [6]. An essential question in the existence theory in nonlinear elasticity is whether the distributional determinant $\operatorname{Det} D\mathbf{u}$ equals the pointwise determinant $\det D\mathbf{u}$. Equality $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ is expressed in a more concrete way as

$$\frac{1}{n} \operatorname{Div} [\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{u}(\mathbf{x})] = \det D\mathbf{u}(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.1}$$

where Div in the left-hand side denotes the distributional divergence. The following generalization of the previous formula is also of interest (see [33, 42, 34, 37]):

$$\operatorname{Div} [\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{u}(\mathbf{x}))] = \operatorname{div} \mathbf{g}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.2}$$

for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$. For smooth maps $\mathbf{u}$ both equalities hold, as a consequence of Piola's identity $\operatorname{Div} \operatorname{cof} D\mathbf{u} = \mathbf{0}$. When one writes down the definition of distributional divergence, formula (1.2) reads as

$$-\int_\Omega [\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{u}(\mathbf{x}))] \cdot D\phi(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_\Omega \det D\mathbf{u}(\mathbf{x})\, \phi(\mathbf{x}) \operatorname{div} \mathbf{g}(\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{1.3}$$

for all $\phi \in C_c^\infty(\Omega)$. The class of maps that were object of the study of [6] were those $\mathbf{u}$ in $W^{1,p}$ (with $p > n-1$) satisfying (1.3). Previous work [24, 25] showed that condition (1.3) means that $\mathbf{u}$ does not exhibit cavitation and does not create new surface. Hence, it was expected that those functions $\mathbf{u}$ are more regular than the typical Sobolev $W^{1,p}$ function, as confirmed by [6]. The passage from (1.2) to (1.3) is done, of course, by multiplication by a $\phi \in C_c^\infty(\Omega)$ and integration. If we instead multiply (1.2) by a $\phi \in C^\infty(\bar{\Omega})$ and integrate, what we obtain is

$$\begin{aligned} &\int_{\partial\Omega} \phi(\mathbf{x}) \left(\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{u}(\mathbf{x}))\right) \cdot \boldsymbol{\nu}(\mathbf{x})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) - \int_\Omega [\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{u}(\mathbf{x}))] \cdot D\phi(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &\quad = \int_\Omega \det D\mathbf{u}(\mathbf{x})\, \phi(\mathbf{x}) \operatorname{div} \mathbf{g}(\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \end{aligned} \tag{1.4}$$

where $\boldsymbol{\nu}$ is the unit exterior normal of $\partial\Omega$. Functions $\mathbf{u}$ in $W^{1,p}$ (with $p > n-1$) such that its trace $\mathbf{u}|_{\partial\Omega}$ is also in $W^{1,p}$ and satisfy (1.4) is the class of functions object of this study. It is for these functions that we prove the global invertibility theorem: if $\mathbf{u}, \mathbf{u}_0$ are functions of this class such that $\det D\mathbf{u} > 0$, $\det D\mathbf{u}_0 \geq 0$, $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$ and $\mathbf{u}_0$ is injective a.e., then $\mathbf{u}$ is injective a.e.

The functions satisfying (1.4) is a restricted class of those satisfying (1.3). Therefore, the maps studied in this work enjoy the regularity of those of [6], and, in some sense, they are also regular up to the boundary. Although we do not pursue the matter of boundary regularity on this paper, we show, by means of some examples, that condition (1.4) excludes *cavitation at the boundary* and *leakage at the boundary* (see [35]).

We also prove the existence of minimizers of a typical hyperelastic energy

$$\int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x} \tag{1.5}$$

in our class of functions, under Dirichlet boundary conditions. The main assumptions for W is that it is polyconvex in the last variable and it enjoys standard coercivity conditions.

The outline of this paper is as follows. In Section 2 we set the general notation of the paper. Section 3 recalls the definition of Sobolev function over a manifold, which in our case is the boundary of Ω. In Section 4 we define the space $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$ of Sobolev functions in Ω such that its trace is Sobolev, too. We then show that smooth functions are dense in that space. Section 5 starts with the definition of $\bar{\mathcal{A}}_p(\Omega)$, which is the set of maps $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ such that $\det D\mathbf{u} \in L^1(\Omega)$ and (1.4) holds for every $\phi \in C^1(\bar{\Omega})$ and $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$. Then we explain some examples taken from [35] to see that $\bar{\mathcal{A}}_p(\Omega)$ functions not only avoid cavitation in Ω, but also cavitation and leakage at the boundary. The main result of Section 6 is that $\bar{\mathcal{A}}_p(\Omega)$ functions can be extended to an open set $\tilde{\Omega} \supset \bar{\Omega}$ by a function in $\mathcal{A}_p(\tilde{\Omega})$ (the class studied in [6], for which (1.3) holds in $\tilde{\Omega}$), hence without cavities in $\tilde{\Omega}$. Section 7 begins by showing that regular functions up to the boundary, say $C^1(\bar{\Omega}, \mathbb{R}^n)$, are included in $\bar{\mathcal{A}}_p(\Omega)$. Then, it exploits the density and extension results of Sections 4 and 6 to prove that a large class of Sobolev maps with Sobolev trace are also in $\bar{\mathcal{A}}_p(\Omega)$. These inclusions parallel the results in classical elasticity [2, 42, 37] on classes of Sobolev functions for which (1.3) holds. In particular, based on [37], we show that $\bar{\mathcal{A}}_p(\Omega)$ contains all functions $\mathbf{u}$ in $W^{1,p}$ with trace in $W^{1,p}$ and such that $\operatorname{cof} D\mathbf{u}$ and its trace are in L^q, provided $p \geq n-1$ and $q \geq \frac{n}{n-1}$. In Section 8 we prove that $\bar{\mathcal{A}}_p(\Omega)$ functions $\mathbf{u}$ with $\det D\mathbf{u} \geq 0$ a.e. are bounded. Section 9 presents the main result of this paper: if $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ satisfies $\det D\mathbf{u} > 0$ a.e. and coincides on $\partial\Omega$ with an a.e. injective $\mathbf{u}_0 \in \bar{\mathcal{A}}_p(\Omega)$ with $\det D\mathbf{u}_0 \geq 0$ a.e., then $\mathbf{u}$ is injective a.e. We also show that the inverse $\mathbf{u}^{-1}$ is Sobolev $W^{1,1}$. In the final Section 10, we show the existence of minimizers of (1.5) in $\bar{\mathcal{A}}_p(\Omega)$. The proof of this fact is standard once we have shown that the class of functions satisfying (1.4) enjoys a compactness property under the relevant convergence.

2 Notation

In this section we set the general notation of the paper.

We will work in dimension $n \geq 2$. In all the article, Ω is always a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary. We will not state this explicitly in the statements. In some results, however, more regularity on the boundary of Ω is required, and in this case we will specify that. The letter $\boldsymbol{\nu}$ will always denote the unit exterior normal to Ω.

Vector-valued and matrix-valued quantities will be written in boldface. We will work in three different system of coordinates, which carry different notations. The reference configuration occupies Ω, and points in Ω are generically called $\mathbf{x}$. The deformed configuration occupies $\mathbf{u}(\Omega)$, and points in $\mathbf{u}(\Omega)$ are generically called $\mathbf{y}$. We will parametrize a relatively open set Γ of $\partial\Omega$ by the set $(0,r)^{n-1}$, and also a neighbourhood of Γ by $(0,r)^{n-1} \times (-\beta,\beta)$. Points in $(0,r)^{n-1}$ are called $\hat{\mathbf{z}}$, points in $(0,r)^{n-1} \times (-\beta,\beta)$ are called $\mathbf{z}$, and we will decompose $\mathbf{z}$ as $\mathbf{z} = (\hat{\mathbf{z}}, z_n)$. The canonical basis in $\mathbb{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

The closure of an $A \subset \mathbb{R}^n$ is denoted by $\bar{A}$, its boundary by ∂A, and its characteristic function by χ_A. Given two sets U, V of $\mathbb{R}^n$, we will write $U \subset\subset V$ if $\bar{U} \subset V$.

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, its determinant is denoted by $\det \mathbf{A}$. Its adjugate matrix is denoted by $\operatorname{adj} \mathbf{A}$, and the transpose of $\operatorname{adj} \mathbf{A}$ is the cofactor matrix $\operatorname{cof} \mathbf{A}$. Recall the formula

$$\mathbf{A} \operatorname{adj} \mathbf{A} = \operatorname{cof} \mathbf{A} \mathbf{A}^T = (\det \mathbf{A})\mathbf{I}, \tag{2.1}$$

where $\mathbf{I}$ denotes the identity matrix.

The inner (dot) product of vectors and of matrices will be denoted by $\cdot$. The norm of a vector or of a matrix is denoted by $|\cdot|$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ is the $n \times n$ matrix whose component (i,j) is $a_i\, b_j$.

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The abbreviation *a.e.* stands for *almost everywhere* or *almost every*; unless otherwise stated, it refers to the Lebesgue $\mathcal{L}^n$ measure. For $p \geq 1$ (the exponent p will always be finite), the Lebesgue L^p and Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the functions of class C^k, for k a positive integer or infinity, and their versions C^k_c of compact support. We will indicate the domain and target space, as in, for example, $L^p(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$; the corresponding norm is written $\|\cdot\|_{L^p(\Omega,\mathbb{R}^n)}$. Weak convergence in L^p or $W^{1,p}$ is indicated by $\rightharpoonup$. The identity function is denoted by $\mathbf{id}$, if the set is clear from the context, and by $\mathbf{id}_A$ to specify the set A of definition. The support of a function is indicated by supp.

The distributional derivative of a Sobolev function $\mathbf{u}$ is written $D\mathbf{u}$. The divergence in the reference configuration is written Div, and in the deformed configuration div. Both divergences are taken by rows when they refer to matrix-valued functions.

Given two sets A, B of $\mathbb{R}^n$, we write $A \subset B$ a.e. when $\mathcal{L}^n(A \setminus B) = 0$, and $A = B$ a.e. when $A \subset B$ a.e. and $B \subset A$ a.e.

A function $\mathbf{u} : \Omega \to \mathbb{R}^n$ is said to be *injective a.e.* if there exists $\Omega_1 \subset \Omega$ such that $\mathcal{L}^n(\Omega \setminus \Omega_1) = 0$ and $\mathbf{u}|_{\Omega_1}$ is injective.

We will use the following version of Federer's [15] area formula; see also [35, Prop. 2.6] for a modern formulation.

Proposition 2.1. *Let* $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$*. Then there exists a measurable set* $\Omega_0 \subset \Omega$ *with* $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$ *such that the following properties hold for any measurable* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and any measurable* $A \subset \Omega$*:*

a) Define $\mathcal{N}_{\mathbf{u},A} : \mathbb{R}^n \to \mathbb{N} \cup \{\infty\}$ *as follows:* $\mathcal{N}_{\mathbf{u},A}(\mathbf{y})$ *equals the number of* $\mathbf{x} \in \Omega_0 \cap A$ *such that* $\mathbf{u}(\mathbf{x}) = \mathbf{y}$*. Then,* $\mathcal{N}_{\mathbf{u},A}$ *is measurable and*

$$\int_A \varphi(\mathbf{u}(\mathbf{x}))\, |\det D\mathbf{u}(\mathbf{x})|\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y})\, \mathcal{N}_{\mathbf{u},A}(\mathbf{y})\, \mathrm{d}\mathbf{y},$$

whenever either integral exists.

b) Given $\psi : \Omega \to \mathbb{R}$ *measurable, the function* $\bar{\psi} : \mathbf{u}(\Omega_0) \to \mathbb{R}$ *defined by*

$$\bar{\psi}(\mathbf{y}) := \sum_{\mathbf{x} \in \Omega_0 \cap A :\, \mathbf{u}(\mathbf{x}) = \mathbf{y}} \psi(\mathbf{x})$$

is measurable and satisfies

$$\int_A \psi(\mathbf{x})\, \varphi(\mathbf{u}(\mathbf{x}))\, |\det D\mathbf{u}(\mathbf{x})|\, \mathrm{d}\mathbf{x} = \int_{\mathbf{u}(\Omega_0 \cap A)} \bar{\psi}(\mathbf{y})\, \varphi(\mathbf{y})\, \mathrm{d}\mathbf{y},$$

whenever the integral of the left-hand side exists.

We will mainly use $\mathcal{N}_{\mathbf{u},\Omega}$, which will be denoted by $\mathcal{N}_{\mathbf{u}}$.

Now we devote some paragraphs to explain the concepts we need from exterior algebra (see, e.g., [42, 35]).

The tangent space of $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$ is denoted by $T_{\mathbf{x}}\partial\Omega$, and the unit exterior normal to Ω at $\mathbf{x}$ by $\boldsymbol{\nu}(\mathbf{x})$. Since we are assuming that Ω is Lipschitz, both objects exist for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial\Omega$ (they exist everywhere if Ω is of class C^1).

Let V be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ and let $\mathbf{L} : V \to \mathbb{R}^n$ be linear. The space $\Lambda_{n-1}V$ consists of all alternating $(n-1)$-tensors on V. The transformation $\Lambda_{n-1}\mathbf{L} : \Lambda_{n-1}V \to \mathbb{R}^n$ is defined by

$$(\Lambda_{n-1}\mathbf{L})(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{n-1}) = \mathbf{L}\mathbf{a}_1 \wedge \cdots \wedge \mathbf{L}\mathbf{a}_{n-1}, \qquad \mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in V.$$

Here $\wedge$ denotes the exterior product between vectors in $\mathbb{R}^n$. Now, the one-dimensional subspace $\Lambda_{n-1}V$ is identified in a canonical way with the space generated by $\mathbf{v}$, one of the two unit normal vectors to V. Therefore, the linear transformation $\Lambda_{n-1}\mathbf{L}$ is determined by the value of $(\Lambda_{n-1}\mathbf{L})\,\mathbf{v}$, and, moreover, the formula

$$(\Lambda_{n-1}\mathbf{L})\,\mathbf{v} = (\operatorname{cof} \tilde{\mathbf{L}})\,\mathbf{v} \tag{2.2}$$

holds whenever $\tilde{\mathbf{L}} : \mathbb{R}^n \to \mathbb{R}^n$ is any linear map extending $\mathbf{L}$.

Now assume $\mathbf{u} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$ with $p \geq n-1$. Then, the tangential derivative $D\mathbf{u}(\mathbf{x}) : T_{\mathbf{x}}\partial\Omega \to \mathbb{R}^n$ exists for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial\Omega$, and $|D\mathbf{u}| \in L^p(\partial\Omega)$. Consequently, $\Lambda_{n-1}D\mathbf{u}(\mathbf{x})\,\boldsymbol{\nu}(\mathbf{x})$ exists for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial\Omega$, and $\Lambda_{n-1}D\mathbf{u}\,\boldsymbol{\nu} \in L^{\frac{p}{n-1}}(\partial\Omega, \mathbb{R}^n)$.

We will use the following version of the change of variables formula for surface integrals (see, e.g., [35, Prop. 2.7] for a more general statement).

Proposition 2.2. *Let* $S \subset \mathbb{R}^n$ *be an orientable Lipschitz manifold of dimension* $n-1$ *oriented by the unit vector field* $\boldsymbol{\nu}$*, and let* $\mathbf{H} : S \to \mathbb{R}^n$ *be a bi-Lipschitz homeomorphism onto* $\mathbf{H}(S)$*. Then, for every bounded and* $\mathcal{H}^{n-1}$*-measurable* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$*, and any* $\mathcal{H}^{n-1}$*-measurable subset* $A \subset S$*,*

$$\int_A \mathbf{g}(\mathbf{H}(\mathbf{x})) \cdot (\Lambda_{n-1}D\mathbf{H}(\mathbf{x})\,\boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbf{H}(A)} \mathbf{g}(\mathbf{y}) \cdot \frac{\Lambda_{n-1}D\mathbf{H}(\mathbf{H}^{-1}(\mathbf{y}))\,\boldsymbol{\nu}(\mathbf{H}^{-1}(\mathbf{y}))}{|\Lambda_{n-1}D\mathbf{H}(\mathbf{H}^{-1}(\mathbf{y}))\,\boldsymbol{\nu}(\mathbf{H}^{-1}(\mathbf{y}))|}\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}).$$

3 Sobolev functions on the boundary

There are several ways to define Sobolev functions over a manifold, all of which are equivalent if the manifold is smooth enough. Here we have chosen the approach of [29, Sect. 6], which does not use partitions of unity, since it lends itself to simpler proofs in our context.

In this work, the only manifold we will deal with is $\partial\Omega$ or a relatively open subset of it, so we restrict our exposition to this case.

We start by recalling the definition of functions of class $C^{k,\alpha}(\bar{\Omega})$. Given an integer $k \geq 0$ and $\alpha \in [0,1]$, the set $C^{k,\alpha}(\Omega)$ is composed by those functions u defined in Ω that are k times differentiable in Ω and u and all its derivatives up to order k are locally Hölder continuous of exponent α if $\alpha > 0$, and continuous if $\alpha = 0$. The set $C^{k,\alpha}(\bar{\Omega})$ is composed by those functions u defined in $\bar{\Omega}$ that admit a $C^{k,\alpha}$ extension to an open set $\tilde{\Omega} \supset \bar{\Omega}$.

A proper rigid transformation is an affine map in $\mathbb{R}^n$ of the form $\mathbf{x} \mapsto \mathbf{R}\,\mathbf{x} + \mathbf{a}$ with $\mathbf{R}$ an orthonormal matrix with determinant 1 and $\mathbf{a} \in \mathbb{R}^n$. The collection of all proper rigid transformation in $\mathbb{R}^n$ is denoted by $SE(n)$.

The next concept defines an open set of class $C^{k,\alpha}$. In this work, the minimum regularity required for Ω will be Lipschitz, and that is why we assume $k + \alpha \geq 1$.

Definition 3.1. Let $k \geq 0$ be an integer and $\alpha \in [0,1]$ be such that $k + \alpha \geq 1$. We say that the bounded open set Ω is of class $C^{k,\alpha}$ when there exist

$$r > 0, \quad \beta > 0, \quad m \in \mathbb{N}, \quad a_1, \ldots, a_m \in C^{k,\alpha}([0,r]^{n-1}), \quad \mathbf{A}_1, \ldots, \mathbf{A}_m \in SE(n)$$

such that, when we define

$$\begin{aligned}
\Gamma_i &:= \mathbf{A}_i^{-1}\left(\left\{(\hat{\mathbf{z}}, z_n) \in (0,r)^{n-1} \times \mathbb{R} : z_n = a_i(\hat{\mathbf{z}})\right\}\right), \\
U_i^+ &:= \mathbf{A}_i^{-1}\left(\left\{(\hat{\mathbf{z}}, z_n) \in (0,r)^{n-1} \times \mathbb{R} : a_i(\hat{\mathbf{z}}) < z_n < a_i(\hat{\mathbf{z}}) + \beta\right\}\right), \\
U_i^- &:= \mathbf{A}_i^{-1}\left(\left\{(\hat{\mathbf{z}}, z_n) \in (0,r)^{n-1} \times \mathbb{R} : a_i(\hat{\mathbf{z}}) - \beta < z_n < a_i(\hat{\mathbf{z}})\right\}\right),
\end{aligned}$$

we have that

$$\partial\Omega = \bigcup_{i=1}^m \Gamma_i, \qquad \bigcup_{i=1}^m U_i^+ \subset \Omega \quad \text{and} \quad \bigcup_{i=1}^m U_i^- \subset \mathbb{R}^n \setminus \bar{\Omega}.$$

For each $i \in \{1, \ldots, m\}$, the set Γ_i is relatively open in $\partial\Omega$, and the sets U_i^+ and U_i^- are open. Denote

$$U_i := U_i^+ \cup \Gamma_i \cup U_i^-, \qquad i \in \{1, \ldots, m\},$$

which is an open set. The collection $\{U_i\}_{i=1}^m$ is an open cover of $\partial\Omega$. Consider, additionally, any open set $U_0 \subset\subset \Omega$ such that $\bar{\Omega} \subset \bigcup_{i=0}^m U_i$.

For each $i \in \{1, \ldots, m\}$, define $\mathbf{Q}_i : [0,r]^{n-1} \times [-\beta, \beta] \to \mathbb{R}^n$ by $\mathbf{Q}_i(\hat{\mathbf{z}}, z_n) := (\hat{\mathbf{z}}, a_i(\hat{\mathbf{z}}) + z_n)$, and $\mathbf{G}_i := \mathbf{A}_i^{-1} \circ \mathbf{Q}_i$. Then, $\mathbf{G}_i$ is injective and

$$\begin{aligned}
&\mathbf{G}_i\left((0,r)^{n-1} \times (-\beta, \beta)\right) = U_i, && \mathbf{G}_i\left((0,r)^{n-1} \times (0, \beta)\right) = U_i^+, \\
&\mathbf{G}_i\left((0,r)^{n-1} \times [0, \beta)\right) = \Gamma_i \cup U_i^+, && \mathbf{G}_i\left((0,r)^{n-1} \times \{0\}\right) = \Gamma_i.
\end{aligned}$$

It is known that in this case, $\partial\Omega$ is an orientable $C^{k,\alpha}$ manifold, and that for a.e. $\hat{\mathbf{z}} \in (0,r)^{n-1}$, the unit exterior normal of Ω at $\mathbf{G}_i(\hat{\mathbf{z}}, 0)$ is

$$\boldsymbol{\nu}(\mathbf{G}_i(\hat{\mathbf{z}}, 0)) = \frac{\operatorname{cof} D\mathbf{G}_i(\hat{\mathbf{z}}, 0)(-\mathbf{e}_n)}{|\operatorname{cof} D\mathbf{G}_i(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n|} = \frac{\Lambda_{n-1} D(\mathbf{G}_i|_{(0,r)^{n-1} \times \{0\}})(\hat{\mathbf{z}}, 0)(-\mathbf{e}_n)}{\left|\Lambda_{n-1} D(\mathbf{G}_i|_{(0,r)^{n-1} \times \{0\}})(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n\right|}. \tag{3.1}$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $n-1$ coordinates, and let $\eta : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the map $\eta(\hat{\mathbf{z}}) = (\hat{\mathbf{z}}, 0)$. For any function u defined on Γ_i, we define the map $\mathcal{L}_i(u) : \pi(\mathbf{G}_i^{-1}(\Gamma_i)) \to \mathbb{R}$ by $\mathcal{L}_i(u) = u \circ \mathbf{G}_i \circ \eta$.

The following criterion is useful: a function $u : \Omega \to \mathbb{R}$ is in $W^{1,p}(\Omega)$ if and only if $u \in W^{1,p}(U_0)$ and $u \circ \mathbf{G}_i \in W^{1,p}((0,r)^{n-1} \times (0,\beta))$ for all $i \in \{1,\dots,m\}$. Moreover, for a function u defined in U_i^+, the expression

$$u \mapsto \|u \circ \mathbf{G}_i\|_{W^{1,p}((0,r)^{n-1}\times(0,\beta))} \tag{3.2}$$

defines a norm in $W^{1,p}(U_i^+)$ equivalent to the usual one.

From now on, the minimum regularity for Ω will be $C^{0,1}$, i.e., Lipschitz. We will only state the regularity of Ω when more than $C^{0,1}$ is needed.

Definition 3.2. Let $p \geq 1$. We denote by $W^{1,p}(\partial\Omega)$ the set of functions $u : \partial\Omega \to \mathbb{R}$ such that $\mathcal{L}_i(u) \in W^{1,p}((0,r)^{n-1})$ for all $i \in \{1,\dots,m\}$, equipped with the norm

$$\|u\|_{W^{1,p}(\partial\Omega)} := \sum_{i=1}^{m} \|\mathcal{L}_i(u)\|_{W^{1,p}((0,r)^{n-1})}\,.$$

Similarly, for each $i \in \{1,\dots,m\}$, the space $W^{1,p}(\Gamma_i)$ is the set of functions $u : \Gamma_i \to \mathbb{R}$ such that $\mathcal{L}_i(u) \in W^{1,p}((0,r)^{n-1})$, equipped with the norm $\|u\|_{W^{1,p}(\Gamma_i)} := \|\mathcal{L}_i(u)\|_{W^{1,p}((0,r)^{n-1})}$.

It can be shown that $W^{1,p}(\partial\Omega)$ is a Banach space and that its definition and its structure as a Banach space do not depend on the particular description of $\partial\Omega$ given by Definition 3.1.

By Definition 3.2, the map

$$\begin{aligned} W^{1,p}(\partial\Omega) &\to W^{1,p}((0,r)^{n-1})^m \\ u &\mapsto (\mathcal{L}_1(u),\dots,\mathcal{L}_m(u)) \end{aligned}$$

is an isometry, with the aid of which one can easily show the following result.

Lemma 3.3. *Let $p \geq 1$. For each $j \in \mathbb{N}$, let $u_j, u \in W^{1,p}(\partial\Omega)$. We have that $u_j \rightharpoonup u$ in $W^{1,p}(\partial\Omega)$ as $j \to \infty$ if and only if $\mathcal{L}_i(u_j) \rightharpoonup \mathcal{L}_i(u)$ in $W^{1,p}((0,r)^{n-1})$ for all $i = 1,\dots,m$.*

The following result is well known: it states the density of smooth functions in $W^{1,p}(\partial\Omega)$. Its proof relies on the density of smooth functions in $W^{1,p}((0,r)^{n-1})^m$ together with a standard use of partitions of unity.

Proposition 3.4. *Let $p \geq 1$. Assume Ω is of class $C^{k,\alpha}$ for some integer $k \geq 0$ and some $\alpha \in [0,1]$. Then $C^{k,\alpha}(\partial\Omega)$ is dense in $W^{1,p}(\partial\Omega)$.*

We finish this section with the following convention. If $u \in W^{1,p}(\partial\Omega)$ for some $p > n-1$, then, by Morrey's embedding, u admits a representative that is continuous. Without further mention, we will always assume that u itself is the continuous representative.

4 Sobolev functions with Sobolev trace

Let $p \geq 1$ and $u \in W^{1,p}(\Omega)$. We denote by $u|_{\partial\Omega}$ the trace of u on $\partial\Omega$, which is known to belong to $L^p(\partial\Omega)$. With a slight abuse of notation, we say that u belongs to $W^{1,p}(\partial\Omega)$ when its trace belongs to $W^{1,p}(\partial\Omega)$, and we will write $u \in W^{1,p}(\partial\Omega)$. With this notation, the intersection space $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$ is the set of $u \in W^{1,p}(\Omega)$ such that $u|_{\partial\Omega} \in W^{1,p}(\partial\Omega)$. It is equipped with the usual norm of an intersection:

$$\|u\|_{W^{1,p}(\Omega)\cap W^{1,p}(\partial\Omega)} := \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\partial\Omega)}\,,$$

and a standard reasoning shows that $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$ is a Banach space. Similarly, if Γ is a relatively open subset of $\partial\Omega$ we will use the spaces $W^{1,p}(\Gamma)$ and $W^{1,p}(\Omega) \cap W^{1,p}(\Gamma)$. The corresponding notation for $\mathbb{R}^n$-valued functions is $W^{1,p}(\Omega,\mathbb{R}^n) \cap W^{1,p}(\partial\Omega,\mathbb{R}^n)$.

The following elementary result shows that the extension to $(0,r)^{n-1}\times(0,\beta)$ by projection onto $(0,r)^{n-1}\times\{0\}$ of a Sobolev function defined on $(0,r)^{n-1}\times\{0\}$ is itself Sobolev.

Lemma 4.1. *Let $p \geq 1$. Let $r, \beta > 0$ and call $D := (0,r)^{n-1} \times (0,\beta)$ and $\Gamma := (0,r)^{n-1} \times \{0\}$. Then the map $E : W^{1,p}(\Gamma) \to W^{1,p}(D)$ defined by $Eu := u \circ \eta \circ \pi$ is linear and bounded. In fact,*

$$\frac{\partial(Eu)}{\partial z_i} = \frac{\partial u}{\partial z_i} \circ \eta \circ \pi, \text{ for } i = 1, \dots, n-1, \text{ and } \frac{\partial(Eu)}{\partial z_n} = 0. \tag{4.1}$$

Moreover, $(Eu)|_\Gamma = u$.

Proof. Call $\tilde{u} := Eu$. Clearly, $\tilde{u} \in L^p(D)$, since

$$\int_D |\tilde{u}(\mathbf{z})|^p \,\mathrm{d}\mathbf{z} = \beta \int_{(0,r)^{n-1}} |u(\hat{\mathbf{z}}, 0)|^p \,\mathrm{d}\hat{\mathbf{z}}. \tag{4.2}$$

Now let $\varphi \in C^1_c(D)$. Then, for each $i \in \{1, \dots, n-1\}$,

$$\begin{aligned}\int_D \tilde{u}(\mathbf{z}) \frac{\partial \varphi}{\partial z_i}(\mathbf{z}) \,\mathrm{d}\mathbf{z} &= \int_0^\beta \int_{(0,r)^{n-1}} u(\hat{\mathbf{z}}, 0) \frac{\partial \varphi}{\partial z_i}(\hat{\mathbf{z}}, z_n) \,\mathrm{d}\hat{\mathbf{z}} \,\mathrm{d}z_n \\ &= -\int_0^\beta \int_{(0,r)^{n-1}} \frac{\partial u}{\partial z_i}(\hat{\mathbf{z}}, 0)\, \varphi(\hat{\mathbf{z}}, z_n) \,\mathrm{d}\hat{\mathbf{z}} \,\mathrm{d}z_n = -\int_D \frac{\partial u}{\partial z_i}(\eta(\pi(\mathbf{z})))\, \varphi(\mathbf{z}) \,\mathrm{d}\mathbf{z},\end{aligned}$$

whereas

$$\int_D \tilde{u}(\mathbf{z}) \frac{\partial \varphi}{\partial z_n}(\mathbf{z}) \,\mathrm{d}\mathbf{z} = \int_{(0,r)^{n-1}} u(\hat{\mathbf{z}}, 0) \int_0^\beta \frac{\partial \varphi}{\partial z_n}(\hat{\mathbf{z}}, z_n) \,\mathrm{d}z_n \,\mathrm{d}\hat{\mathbf{z}} = 0.$$

Therefore, (4.1) holds. As in (4.2), we have that

$$\int_D \left| \frac{\partial \tilde{u}}{\partial z_i}(\mathbf{z}) \right|^p \,\mathrm{d}\mathbf{z} = \beta \int_{(0,r)^{n-1}} \left| \frac{\partial u}{\partial z_i}(\hat{\mathbf{z}}, 0) \right|^p \,\mathrm{d}\hat{\mathbf{z}}, \text{ for } i = 1, \dots, n-1. \tag{4.3}$$

Equalities (4.1), (4.2) and (4.3) show that $\tilde{u} \in W^{1,p}(D)$ and that the map E is linear and bounded. As $\tilde{u}|_\Gamma = u$ for every $u \in W^{1,p}(\Gamma) \cap C(\bar{\Gamma})$, by continuity of the trace operator this equality holds for every $u \in W^{1,p}(\Gamma)$. □

It is easy to see that the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$ is $W^{1,p}_0(\Omega)$. A more interesting result is the density of smooth functions in $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$. The key tool for its proof is a result of Fonseca and Malý [19] that allows one to modify the boundary values of a function without increasing significantly its norm.

Proposition 4.2. *Let $p \geq 1$. Let $k \geq 0$ be an integer and $\alpha \in [0,1]$ be such that $k + \alpha \geq 1$. Let Ω be a bounded open set with a $C^{k,\alpha}$ boundary. Then $C^{k,\alpha}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$.*

Proof. Let $r, \beta > 0$. Call $D := (0,r)^{n-1} \times (0,\beta)$ and $\Gamma := (0,r)^{n-1} \times \{0\}$. The first part of the proof consists in showing that, given $u \in W^{1,p}(D) \cap W^{1,p}(\Gamma)$, there exists a $C^{k,\alpha}(\bar{D})$ function arbitrarily close to u in $W^{1,p}(D) \cap W^{1,p}(\Gamma)$.

As $u \in W^{1,p}(\Gamma)$, the function $u|_\Gamma \circ \eta$ is in $W^{1,p}((0,r)^{n-1})$. Therefore, there exists a sequence $\{\tilde{w}_k\}_{k \in \mathbb{N}} \subset C^\infty([0,r]^{n-1})$ tending to $u|_\Gamma \circ \eta$ in $W^{1,p}((0,r)^{n-1})$. For each $k \in \mathbb{N}$, define $w_k : \bar{D} \to \mathbb{R}$ as $w_k := \tilde{w}_k \circ \pi$. By Lemma 4.1, $w_k \in W^{1,p}(D)$, and it is immediate to see that $w_k \to u|_\Gamma \circ \eta \circ \pi$ in $W^{1,p}(D)$ as $k \to \infty$.

For each $j, k \in \mathbb{N}$ with $\beta j > 2$, we apply [19, Lemma 2.4] and find $z_{jk} \in W^{1,p}(D)$ and open sets $V_{jk} \subset (0,r)^{n-1} \times (1/j, \beta)$ and $W_{jk} \subset (0,r)^{n-1} \times (0, 2/j)$ such that

$$D = V_{jk} \cup W_{jk}, \qquad z_{jk} = u \text{ in } D \setminus W_{jk}, \qquad z_{jk} = w_k \text{ in } D \setminus V_{jk}$$

and

$$\|z_{jk}\|_{W^{1,p}(V_{jk} \cap W_{jk})} \leq C \left(\|u\|_{W^{1,p}((0,r)^{n-1} \times (1/j, 2/j))} + \|w_k\|_{W^{1,p}((0,r)^{n-1} \times (1/j, 2/j))} \right), \tag{4.4}$$

for a constant $C > 0$ not depending on j or k. Obviously, $z_{jk}|_\Gamma = w_k|_\Gamma$, hence

$$\sup_{j\in\mathbb{N}} \lim_{k\to\infty} \|z_{jk} - u\|_{W^{1,p}(\Gamma)} = 0. \tag{4.5}$$

On the other hand, thanks to (4.4),

$$\begin{aligned}\|z_{jk} - u\|_{W^{1,p}(D)} &= \|z_{jk} - u\|_{W^{1,p}(V_{jk}\cap W_{jk})} + \|w_k - u\|_{W^{1,p}(W_{jk}\setminus V_{jk})} \\ &\le \|z_{jk}\|_{W^{1,p}(V_{jk}\cap W_{jk})} + \|w_k\|_{W^{1,p}(W_{jk}\setminus V_{jk})} + \|u\|_{W^{1,p}(W_{jk})} \\ &\le (C+1)\left(\|u\|_{W^{1,p}((0,r)^{n-1}\times(0,2/j))} + \|w_k\|_{W^{1,p}((0,r)^{n-1}\times(0,2/j))}\right),\end{aligned}$$

so

$$\limsup_{k\to\infty} \|z_{jk} - u\|_{W^{1,p}(D)} \le (C+1)\left(\|u\|_{W^{1,p}((0,r)^{n-1}\times(0,2/j))} + \|u|_\Gamma \circ \eta \circ \pi\|_{W^{1,p}((0,r)^{n-1}\times(0,2/j))}\right)$$

and, consequently,

$$\lim_{j\to\infty} \limsup_{k\to\infty} \|z_{jk} - u\|_{W^{1,p}(D)} = 0. \tag{4.6}$$

Equalities (4.5) and (4.6) show that given $\varepsilon > 0$ there exist $j, k \in \mathbb{N}$ for which

$$\|z_{jk} - u\|_{W^{1,p}(D)\cap W^{1,p}(\Gamma)} \le \varepsilon. \tag{4.7}$$

Observe now that $z_{jk} - w_k \in W^{1,p}(D)$ with $(z_{jk} - w_k)|_\Gamma = 0$, so there exists $f \in C^\infty(\bar{D})$ with $\operatorname{supp} f \cap \Gamma = \varnothing$ such that $\|f - z_{jk} + w_k\|_{W^{1,p}(D)} \le \varepsilon$. The proof of this fact (see, e.g., [14, Th. 5.5.2], [1, Th. 5.37] or [30, Th. 15.29]) is usually done for a half-space, but the same proof works for a rectangle, as is in our case with D. Then, thanks to (4.7), the function $f + w_k \in C^\infty(\bar{D})$ satisfies

$$\|f + w_k - u\|_{W^{1,p}(\Gamma)} = \|z_{jk} - u\|_{W^{1,p}(\Gamma)} \le \varepsilon$$

and

$$\|f + w_k - u\|_{W^{1,p}(D)} \le \|f - z_{jk} + w_k\|_{W^{1,p}(D)} + \|z_{jk} - u\|_{W^{1,p}(D)} \le 2\varepsilon.$$

This proves the result claimed at the beginning of the proof.

Now assume, as in the statement, that Ω is a bounded open set with a $C^{k,\alpha}$ boundary, and let $u \in W^{1,p}(\Omega) \cap W^{1,p}(\partial\Omega)$. We use the notation of Section 3. Then $u \circ \mathbf{G}_i \in W^{1,p}(D) \cap W^{1,p}(\Gamma)$ for each $i \in \{1, \dots, m\}$. Let $\varepsilon > 0$. By the result of the first part of the proof, for each $i \in \{1, \dots, m\}$ there exists $\phi_i \in C^{k,\alpha}(\bar{D})$ such that

$$\|u \circ \mathbf{G}_i - \phi_i\|_{W^{1,p}(D)\cap W^{1,p}(\Gamma)} \le \varepsilon. \tag{4.8}$$

Moreover, there exists $\phi_0 \in C_c^\infty(\Omega)$ such that $\|u - \phi_0\|_{W^{1,p}(U_0)} \le \varepsilon$. Let $\{\varphi_i\}_{i=0}^m \subset C^\infty(\mathbb{R}^n, [0,1])$ be a partition of unity as follows: $\operatorname{supp} \varphi_i \subset U_i$ for each $i \in \{0, \dots, m\}$ and $\sum_{i=0}^m \varphi_i = 1$ in $\bar{\Omega}$. We define $\phi : \bar{\Omega} \to \mathbb{R}$ as

$$\phi(\mathbf{x}) := \varphi_0(\mathbf{x})\, \phi_0(\mathbf{x}) + \sum_{\substack{i=1\\ \mathbf{x}\in \bar{U}_i^+}}^m \varphi_i(\mathbf{x}) \left(\phi_i(\mathbf{G}_i^{-1}(\mathbf{x}))\right).$$

Then $\phi \in C^{k,\alpha}(\bar{\Omega})$ and, having in mind that $\sum_{i=0}^m \varphi_i = 1$,

$$\begin{aligned}\|u - \phi\|_{W^{1,p}(\Omega)} &\le \|\varphi_0\,(u - \phi_0)\|_{W^{1,p}(U_0)} + \sum_{i=1}^m \left\|\varphi_i\left(u - \phi_i \circ \mathbf{G}_i^{-1}\right)\right\|_{W^{1,p}(U_i^+)} \\ &\le C\left(\|u - \phi_0\|_{W^{1,p}(U_0)} + \sum_{i=1}^m \left\|u - \phi_i \circ \mathbf{G}_i^{-1}\right\|_{W^{1,p}(U_i^+)}\right),\end{aligned}$$

where C only depends on the Lipschitz norms of $\varphi_0, \dots, \varphi_m$. By the equivalence of norms given by (3.2), we have that, for some constant $M > 0$ and all $i \in \{1, \dots, m\}$,

$$\left\| u - \phi_i \circ \mathbf{G}_i^{-1} \right\|_{W^{1,p}(U_i^+)} \le M \left\| u \circ \mathbf{G}_i - \phi_i \right\|_{W^{1,p}(D)} \le M \varepsilon,$$

thanks to (4.8). Similarly,

$$\|u - \phi\|_{W^{1,p}(\partial\Omega)} \le \sum_{i=1}^m \left\| \varphi_i \left(u - \phi_i \circ \mathbf{G}_i^{-1} \right) \right\|_{W^{1,p}(\Gamma_i)} \le C \sum_{i=1}^m \left\| u - \phi_i \circ \mathbf{G}_i^{-1} \right\|_{W^{1,p}(\Gamma_i)},$$

since multiplication by a Lipschitz function is also a continuous operation in $W^{1,p}(\Gamma_i)$. Using now Definition 3.2, we note that

$$\left\| u - \phi_i \circ \mathbf{G}_i^{-1} \right\|_{W^{1,p}(\Gamma_i)} = \left\| \mathcal{L}_i(u) - \phi_i \circ \eta \right\|_{W^{1,p}((0,r)^{n-1})} = \left\| u \circ \mathbf{G}_i - \phi_i \right\|_{W^{1,p}(\Gamma)} \le \varepsilon,$$

which concludes the proof. □

5 A class of functions not admitting cavitation

We define the class of functions that are object of our work.

Definition 5.1. Let $p \ge n - 1$.

a) Given $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $\det D\mathbf{u} \in L^1(\Omega)$ and $\mathbf{f} \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, we define

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) := \int_\Omega \left[\operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det D\mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right] \mathrm{d}\mathbf{x}. \tag{5.1}$$

We define $\mathcal{A}_p(\Omega)$ as the set of $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that $\det D\mathbf{u} \in L^1(\Omega)$ and

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) = 0 \qquad \text{for all } \mathbf{f} \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n). \tag{5.2}$$

b) Given $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ and $\mathbf{f} \in C_c^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$, we define

$$\mathcal{F}(\mathbf{u}, \mathbf{f}) := \int_{\partial\Omega} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot \left(\Lambda_{n-1} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}) \right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}).$$

We define $\bar{\mathcal{A}}_p(\Omega)$ as the set of $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ such that $\det D\mathbf{u} \in L^1(\Omega)$ and $\mathcal{E}(\mathbf{u}, \mathbf{f}) = \mathcal{F}(\mathbf{u}, \mathbf{f})$ for all $\mathbf{f} \in C_c^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$.

In equation (5.1), $D\mathbf{f}(\mathbf{x}, \mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot, \mathbf{y})$ evaluated at $\mathbf{x} \in \Omega$, while $\operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x}, \cdot)$ evaluated at $\mathbf{y} \in \mathbb{R}^n$. When we want to underline the dependence on the domain, we will sometimes write $\mathcal{E}_\Omega(\mathbf{u}, \mathbf{f})$ and $\mathcal{F}_{\partial\Omega}(\mathbf{u}, \mathbf{f})$.

The functional $\mathcal{E}$ was introduced in [23] to measure the creation of new surface of a deformation. Condition (5.2) was shown in [25, Th. 4.6] to be equivalent to the requirement that $\mathbf{u}$ does not exhibit cavitation, or, in general [24, Th. 3], that $\mathbf{u}$ does not create new surface.

The class $\mathcal{A}_p(\Omega)$ was introduced by Müller [33] without naming it. It was also used in Giaquinta *et al.* [20, Def. 3.2.1.3 and Prop. 3.2.4.1]. Both references, as well as [42, 34, 37, 25], show that it is a suitable class for doing calculus of variations in nonlinear elasticity. The definition of the class $\bar{\mathcal{A}}_p(\Omega)$ is new, although it has an antecessor in [27].

We observe that condition b) of Definition 5.1 is a global version of condition a). While in [6] it was studied the local invertibility in the class $\mathcal{A}_p(\Omega)$, in this paper we study the global invertibility in the class $\bar{\mathcal{A}}_p(\Omega)$. Obviously, $\bar{\mathcal{A}}_p(\Omega) \subset \mathcal{A}_p(\Omega)$, whereas by using similar arguments as in [23, Th. 2] (see [38, Lemma 3.3.1] for a full proof), one has that $\mathbf{u}|_U \in \bar{\mathcal{A}}_p(U)$ for any $\mathbf{u} \in \mathcal{A}_p(\Omega)$ and "almost all" open sets $U \subset\subset \Omega$

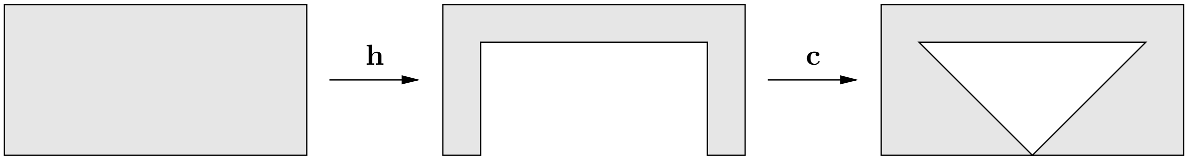

Figure 1: Deformation of Example 5.3: cavitation at the boundary with regular trace

with a C^2 boundary. In Section 7 we will see that smooth functions up to the boundary are in $\bar{\mathcal{A}}_p(\Omega)$, as well as a large class of Sobolev maps with Sobolev trace.

Due to the density in $C_c^1(\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ of sums of functions of separate variables (see, e.g., [31, Cor. 1.6.5]), one can see that condition (5.2) is equivalent to

$$\int_\Omega [\operatorname{cof} D\mathbf{u}(\mathbf{x})\cdot(\mathbf{g}(\mathbf{u}(\mathbf{x}))\otimes D\phi(\mathbf{x})) + \det D\mathbf{u}(\mathbf{x})\,\phi(\mathbf{x})\operatorname{div}\mathbf{g}(\mathbf{u}(\mathbf{x}))]\,\mathrm{d}\mathbf{x} = 0, \tag{5.3}$$

for all $\phi\in C_c^1(\Omega)$ and $\mathbf{g}\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$ (this equivalence was also proved in [25, Th. 4.6]). Similarly, condition $\mathcal{E}(\mathbf{u},\mathbf{f})=\mathcal{F}(\mathbf{u},\mathbf{f})$ for all $\mathbf{f}\in C_c^1(\bar\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ is equivalent to

$$\mathcal{E}(\mathbf{u},\phi\,\mathbf{g}) = \mathcal{F}(\mathbf{u},\phi\,\mathbf{g}) \qquad \text{for all } \phi\in C^1(\bar\Omega) \text{ and } \mathbf{g}\in C_c^1(\mathbb{R}^n,\mathbb{R}^n), \tag{5.4}$$

where $\phi\,\mathbf{g}\in C_c^1(\bar\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ denotes the function $(\phi\,\mathbf{g})(\mathbf{x},\mathbf{y})=\phi(\mathbf{x})\,\mathbf{g}(\mathbf{y})$.

In the following examples we show how functions in $\bar{\mathcal{A}}_p(\Omega)$ do not exhibit cavitation at the boundary.

Example 5.2 (Cavitation at the boundary with discontinuous trace). Consider $r:[0,1]\to\mathbb{R}$ of class C^1 with $\inf r'>0$ and $r(0)>0$. Let $\Omega := \{\mathbf{x}\in\mathbb{R}^n : |\mathbf{x}|<1,\ x_n>0\}$ and define $\mathbf{u}:\Omega\to\mathbb{R}^n$ as

$$\mathbf{u}(\mathbf{x}) := r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}.$$

It is easy to check (see [4, Lemma 4.1], if necessary) that $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^n)$ if and only if $p<n$, and $\mathbf{u}\in W^{1,p}(\partial\Omega,\mathbb{R}^n)$ if and only if $p<n-1$. Therefore, when p is required to be $p\geq n-1$, we have that $\mathbf{u}\notin\bar{\mathcal{A}}_p(\Omega)$.

The essence of the previous example is that $\mathbf{u}|_{\partial\Omega}$ is discontinuous. Now we show an example, taken from Müller and Spector [35, Sect. 11], in which a cavity is formed at the boundary, yet the trace of the function is regular.

Example 5.3 (Cavitation at the boundary with regular trace). Let $\Omega=(-1,1)\times(0,1)$. Define $\mathbf{h}:\Omega\to\mathbb{R}^2$ as

$$\mathbf{h}(\mathbf{x}) := \frac{|\mathbf{x}|_\infty+3}{4|\mathbf{x}|_\infty}\mathbf{x},$$

where $|\cdot|_\infty$ stands for the max-norm of a vector, and $\mathbf{c}:\mathbb{R}^2\to\mathbb{R}^2$ as

$$\mathbf{c}(x_1,x_2) := \begin{cases} (\operatorname{sgn} x_1(1-4(1-|x_1|)(1-x_2)),x_2) & \text{if } x_2<\frac{3}{4}, \\ (x_1,x_2) & \text{if } x_2\geq\frac{3}{4}. \end{cases}$$

Consider $\mathbf{u}:=\mathbf{c}\circ\mathbf{h}$; see Figure 1 for its representation. What happens here is that, after the cavity has been formed at the boundary (map $\mathbf{h}$), the body is stretched in a vicinity of the cavity so as to close it (map $\mathbf{c}$). Then $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^2)$ for all $p<2$. Moreover, $\mathbf{u}$ is injective a.e. and $\det D\mathbf{u}>0$ a.e. As $\mathbf{u}$ is locally Lipschitz in Ω, we have that $\mathbf{u}\in\mathcal{A}_p(\Omega)$. We also have that $\mathbf{u}|_{\partial\Omega}=\mathbf{id}|_{\partial\Omega}$, so in particular $\mathbf{u}\in W^{1,p}(\partial\Omega,\mathbb{R}^2)$ for all $p\geq1$. We shall show that $\mathbf{u}\notin\bar{\mathcal{A}}_p(\Omega)$. Note that $\mathbf{id}\in\bar{\mathcal{A}}_p(\Omega)$ (by Lemma 7.1 below, or just by a straightforward calculation), hence for any $\mathbf{f}\in C_c^1(\bar\Omega\times\mathbb{R}^2,\mathbb{R}^2)$ we have that

$$\mathcal{E}(\mathbf{id},\mathbf{f}) = \mathcal{F}(\mathbf{id},\mathbf{f}) = \mathcal{F}(\mathbf{u},\mathbf{f}).$$

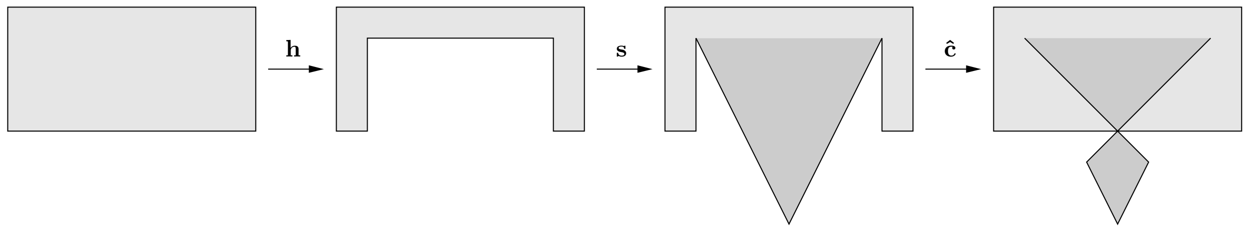

Figure 2: Deformation of Example 5.4: leakage at the boundary

Consequently, in order to show that $\mathbf{u} \notin \bar{\mathcal{A}}_p(\Omega)$ it suffices to check that $\mathcal{E}(\mathbf{id}, \mathbf{f}) \neq \mathcal{E}(\mathbf{u}, \mathbf{f})$. For this, we choose an $\mathbf{f} \in C_c^1(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R}^2)$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{y}$ for $(\mathbf{x}, \mathbf{y}) \in \bar{\Omega} \times \bar{\Omega}$, and note that $\mathbf{u}(\bar{\Omega}) \subset \bar{\Omega}$. For this choice we have

$$\mathcal{E}(\mathbf{id}, \mathbf{f}) = \int_\Omega \mathrm{d}\mathbf{x} = \mathcal{L}^2(\Omega) = 2 \quad \text{and} \quad \mathcal{E}(\mathbf{u}, \mathbf{f}) = \int_\Omega \det D\mathbf{u} \, \mathrm{d}\mathbf{x} = \mathcal{L}^2(\mathbf{u}(\Omega)) = \frac{23}{16}.$$

In the last example, taken again from [35, Sect. 11], we show a deformation presenting *leakage at the boundary*, and see how functions in $\bar{\mathcal{A}}_p(\Omega)$ cannot exhibit this phenomenon.

Example 5.4 (Leakage at the boundary). Let Ω and $\mathbf{h}$ be as in Example 5.3. Define $\mathbf{s} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\mathbf{s}(x_1, x_2) := \begin{cases} (x_1, 1 - (1 - x_2)(7 - 8|x_1|)) & \text{if } |x_1| < \frac{3}{4}, \\ (x_1, x_2) & \text{if } |x_1| \geq \frac{3}{4}, \end{cases}$$

and $\hat{\mathbf{c}} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\hat{\mathbf{c}}(x_1, x_2) := \begin{cases} (\operatorname{sgn} x_1 (1 - 4(1 - |x_1|))(1 - x_2), x_2) & \text{if } 0 \leq x_2 < \frac{3}{4}, \ \frac{3}{4} < |x_1|, \\ \left(\frac{8x_1x_2}{4x_2+3}, x_2\right) & \text{if } |x_1| < \frac{4x_2+3}{8}, \ 0 \leq x_2 < \frac{3}{4}, \\ \left(\frac{-8x_1x_2}{4x_2+3}, x_2\right) & \text{if } |x_1| < \frac{4x_2+3}{8}, \ -\frac{1}{4} < x_2 \leq 0, \\ (x_1, x_2) & \text{elsewhere.} \end{cases}$$

Consider $\mathbf{u} := \hat{\mathbf{c}} \circ \mathbf{s} \circ \mathbf{h}$; see Figure 2 for its representation. We can see the process of leakage at the boundary: a cavity is first formed at the boundary (map $\mathbf{h}$), then material close to the surface of the cavity is stretched down (map $\mathbf{s}$) and then the material closes the cavity but leaves part of the material outside the boundary (map $\hat{\mathbf{c}}$). One can check that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p < 2$, $\mathbf{u}$ is injective a.e., $\det D\mathbf{u} > 0$ a.e., $\mathbf{u}$ is locally Lipschitz in Ω and $\mathbf{u}|_{\partial\Omega} = \mathbf{id}|_{\partial\Omega}$. Reasoning as in Example 5.3, in order to show that $\mathbf{u} \notin \bar{\mathcal{A}}_p(\Omega)$ it suffices to check that $\mathcal{L}^2(\Omega) \neq \mathcal{L}^2(\mathbf{u}(\Omega))$. In this case we have $\mathcal{L}^2(\Omega) = 2$ and $\mathcal{L}^2(\mathbf{u}(\Omega)) = \frac{35}{16}$.

Traditionally, the way that a model prohibits cavitation at the boundary and leakage at the boundary has been to assume an extension $\tilde{\mathbf{u}}$ of $\mathbf{u}$ to an open set $\tilde{\Omega} \supset \bar{\Omega}$ so that $\tilde{\mathbf{u}}$ does not present cavitation in $\tilde{\Omega}$. Moreover, $\tilde{\mathbf{u}}$ was assumed to be a diffeomorphism in $\tilde{\Omega} \setminus \bar{\Omega}$; see [42, 40, 41, 22, 27, 28]. With the examples above, we can see that the class $\bar{\mathcal{A}}_p(\Omega)$ also avoids cavitation and leakage at the boundary, in principle, without the need of any extension. Nevertheless, we will in fact prove in the next section that $\bar{\mathcal{A}}_p(\Omega)$ functions can be extended to an $\tilde{\Omega}$ by a function without cavities on $\tilde{\Omega}$.

6 Extension

In this section we prove that functions in $\bar{\mathcal{A}}_p(\Omega)$ can be extended to an open set $\tilde{\Omega} \supset \bar{\Omega}$ by a function in $\mathcal{A}_p(\tilde{\Omega})$. We start with a continuation of Lemma 4.1; recall the maps π and η of Section 3.

Lemma 6.1. *Let $p \geq 1$. Let $r, \beta > 0$ and call $D := (0,r)^{n-1} \times (0,\beta)$ and $\Gamma := (0,r)^{n-1} \times \{0\}$. Then the map $E : W^{1,p}(\Gamma, \mathbb{R}^n) \to W^{1,p}(D, \mathbb{R}^n)$ defined by $E\mathbf{u} := \mathbf{u} \circ \eta \circ \pi$ is linear and bounded. Moreover, $\det D(E\mathbf{u}) = 0$ and $(E\mathbf{u})|_\Gamma = \mathbf{u}$. If, in addition, $\Lambda_{n-1} D\mathbf{u}\, \mathbf{e}_n \in L^q(\Gamma, \mathbb{R}^n)$ for some $q \geq 1$ then $\operatorname{cof} D(E\mathbf{u}) \in L^q(D, \mathbb{R}^{n\times n})$ and $\|\operatorname{cof} D(E\mathbf{u})\|_{L^q(D,\mathbb{R}^{n\times n})} = \beta^{1/q} \|\Lambda_{n-1} D\mathbf{u}\, \mathbf{e}_n\|_{L^q(\Gamma,\mathbb{R}^n)}$.*

Proof. Call $\tilde{\mathbf{u}} := E\mathbf{u}$. By Lemma 4.1, E is linear and bounded,

$$\frac{\partial \tilde{\mathbf{u}}}{\partial z_i} = \frac{\partial \mathbf{u}}{\partial z_i} \circ \eta \circ \pi, \text{ for } i \in \{1, \dots, n-1\}, \text{ and } \frac{\partial \tilde{\mathbf{u}}}{\partial z_n} = \mathbf{0}, \tag{6.1}$$

which imply that $\det D\tilde{\mathbf{u}} = 0$. It was also shown in Lemma 4.1 that $(E\mathbf{u})|_\Gamma = \mathbf{u}$.

Now assume $\Lambda_{n-1} D\mathbf{u}\, \mathbf{e}_n \in L^q(\Gamma, \mathbb{R}^n)$ for some $q \geq 1$. Equalities (6.1) can be written as $D\tilde{\mathbf{u}}(\mathbf{z}) = D\mathbf{u}(\hat{\mathbf{z}}, 0) \circ \eta \circ \pi$ for a.e. $\mathbf{z} \in D$. Moreover, for a.e. $\hat{\mathbf{z}} \in (0,r)^{n-1}$, the map $D\mathbf{u}(\hat{\mathbf{z}}, 0) \circ \eta \circ \pi : \mathbb{R}^n \to \mathbb{R}^n$ is a linear extension of the map $D\mathbf{u}(\hat{\mathbf{z}}, 0) : \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^n$. Therefore, by (2.2),

$$\Lambda_{n-1} D\mathbf{u}(\hat{\mathbf{z}}, 0)\, \mathbf{e}_n = \operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{z})\, \mathbf{e}_n. \tag{6.2}$$

Consequently,

$$\int_D |\operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{z})\, \mathbf{e}_n|^q \, \mathrm{d}\mathbf{z} = \beta \int_{(0,r)^{n-1}} |\Lambda_{n-1} D\mathbf{u}(\hat{\mathbf{z}}, 0)\, \mathbf{e}_n|^q \, \mathrm{d}\hat{\mathbf{z}} = \beta \|\Lambda_{n-1} D\mathbf{u}\, \mathbf{e}_n\|^q_{L^q(\Gamma,\mathbb{R}^n)}.$$

On the other hand,

$$\operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{z})\, \mathbf{e}_i = \mathbf{0}, \qquad i \in \{1, \dots, n-1\}, \tag{6.3}$$

for a.e. $\mathbf{z} \in (0,r)^{n-1} \times (0,\beta)$, due to (6.1). Therefore, $\|\operatorname{cof} D\tilde{\mathbf{u}}\|^q_{L^q(D,\mathbb{R}^{n\times n})} = \beta \|\Lambda_{n-1} D\mathbf{u}\, \mathbf{e}_n\|^q_{L^q(\Gamma,\mathbb{R}^n)}$. □

The extension that we seek requires the following property for our domains (see [35, Sect. 9]).

Definition 6.2. We say that the open set Ω is extendable if Ω is bounded, has a Lipschitz boundary and there exist a set N with $\partial\Omega \subset N \subset \mathbb{R}^n \setminus \Omega$, a $\delta > 0$ and a bi-Lipschitz homeomorphism $\mathbf{w} : \partial\Omega \times (-\delta, 0] \to N$ onto N such that $\mathbf{w}(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$.

It is known that C^2 open sets are extendable (see, e.g., [13, Th. 16.25.2]); in fact, so are $C^{1,1}$ open sets (see, e.g., [12, Th. 4.3.2]) and one can show that the assumption of piecewise $C^{1,1}$ suffices.

It is important to notice that, in the context of Definition 6.2, the set $\Omega \cup N$ is open. To see that, we recall Section 3 and, for each $i \in \{1, \dots, m\}$, define $\boldsymbol{\phi}_i : U_i \to \mathbb{R}^n$ as

$$\boldsymbol{\phi}_i(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in U_i^+ \\ \mathbf{w}\left(\mathbf{G}_i(\hat{\mathbf{z}}, 0), \frac{\delta}{\beta} z_n\right) & \text{if } \mathbf{x} = \mathbf{G}_i(\hat{\mathbf{z}}, z_n) \text{ for some } (\hat{\mathbf{z}}, z_n) \in (0,r)^{n-1} \times (-\beta, 0]. \end{cases}$$

It is immediate to check that $\boldsymbol{\phi}_i$ is a homeomorphism. Consequently, by the invariance of domain theorem, $\boldsymbol{\phi}_i(U_i)$ is open. Now, $\boldsymbol{\phi}_i(U_i) = U_i^+ \cup \mathbf{w}(\Gamma_i \times (-\delta, 0])$, and, hence, the set

$$U_0 \cup \bigcup_{i=1}^m \left(U_i^+ \cup \mathbf{w}(\Gamma_i \times (-\delta, 0])\right) = \Omega \cup \mathbf{w}(\partial\Omega \times (-\delta, 0]) = \Omega \cup N$$

is open.

The fundamental extension property of this section is part *b)* of the following proposition.

Proposition 6.3. *Let $p \geq 1$. Let Ω be an extendable open set. Then there exist an open set $\tilde{\Omega} \supset \bar{\Omega}$ and a linear bounded operator $E : W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) \to W^{1,p}(\tilde{\Omega}, \mathbb{R}^n)$ such that $E\mathbf{u} = \mathbf{u}$ a.e. in Ω, $\det D(E\mathbf{u}) = 0$ a.e. in $\tilde{\Omega} \setminus \Omega$ and the following hold:*

a) If $\Lambda_{n-1} D\mathbf{u}\,\boldsymbol{\nu} \in L^q(\partial\Omega, \mathbb{R}^n)$ *for some* $q \geq 1$ *then* $\operatorname{cof} D(E\mathbf{u}) \in L^q(\tilde{\Omega} \setminus \Omega, \mathbb{R}^{n\times n})$ *and*

$$\|\operatorname{cof} D(E\mathbf{u})\|_{L^q(\tilde{\Omega}\setminus\Omega,\mathbb{R}^{n\times n})} \leq C \,\|\Lambda_{n-1} D\mathbf{u}\,\boldsymbol{\nu}\|_{L^q(\partial\Omega,\mathbb{R}^n)}$$

for some constant $C > 0$ *independent of* $\mathbf{u}$.

b) Let $p \geq n-1$. *Then,* $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ *if and only if* $E\mathbf{u} \in \mathcal{A}_p(\tilde{\Omega})$.

Proof. Let N and $\mathbf{w}$ be as in Definition 6.2. Define the open set $\tilde{\Omega} := \Omega \cup N$. Let $\tilde{\pi} : \partial\Omega \times (-\delta, 0] \to \partial\Omega$ denote the projection onto $\partial\Omega$. Define $\tilde{\mathbf{u}} : \tilde{\Omega} \to \mathbb{R}^n$ as

$$\tilde{\mathbf{u}} := \begin{cases} \mathbf{u} & \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} \circ \tilde{\pi} \circ \mathbf{w}^{-1} & \text{in } N. \end{cases}$$

Recall the notation of Section 3. Fix $i \in \{1, \dots, m\}$ and notice (thanks to the invariance of domain theorem and the use of the map $\mathbf{G}_i \circ \eta$) that $\mathbf{w}(\Gamma_i \times (-\delta, 0))$ is open. We show first that $\tilde{\mathbf{u}} \in W^{1,p}(\mathbf{w}(\Gamma_i \times (-\delta, 0)), \mathbb{R}^n)$. The process is better understood with the following commutative diagram:

$$\begin{tikzcd}
\mathbf{w}(\Gamma_i \times (-\delta,0)) \arrow[r, "\mathbf{w}^{-1}"] \arrow[dr, "\boldsymbol{\psi}_i"'] & \Gamma_i \times (-\delta,0) \arrow[r, "\tilde{\pi}"] & \Gamma_i \arrow[r, "\mathbf{u}|_{\partial\Omega}"] & \mathbb{R}^n \\
& (0,r)^{n-1} \times (-\delta,0) \arrow[u, "(\mathbf{G}_i\circ\eta)\times\mathbf{id}"'] \arrow[r, "\pi"] & (0,r)^{n-1}, \arrow[u, "\mathbf{G}_i\circ\eta"'] \arrow[ur, "\mathcal{L}_i(\mathbf{u})"'] &
\end{tikzcd} \tag{6.4}$$

where $\boldsymbol{\psi}_i := \big((\mathbf{G}_i \circ \eta)^{-1} \times \mathbf{id}_{(-\delta,0)}\big) \circ \mathbf{w}^{-1}$. A key observation is that $\boldsymbol{\psi}_i$ is a bi-Lipschitz homeomorphism between open sets of $\mathbb{R}^n$. By Definition 3.2, $\mathcal{L}_i(\mathbf{u}) \in W^{1,p}((0,r)^{n-1}, \mathbb{R}^n)$, so $\mathcal{L}_i(\mathbf{u}) \circ \pi \in W^{1,p}((0,r)^{n-1} \times \{0\}, \mathbb{R}^n)$. By Lemma 6.1, $\mathcal{L}_i(\mathbf{u}) \circ \pi \in W^{1,p}((0,r)^{n-1} \times (-\delta, 0), \mathbb{R}^n)$, since $\pi \circ \eta \circ \pi = \pi$. As $\boldsymbol{\psi}_i$ is bi-Lipschitz, we obtain, having in mind the commutativity of diagram (6.4), that $\tilde{\mathbf{u}} \in W^{1,p}(\mathbf{w}(\Gamma_i \times (-\delta, 0)), \mathbb{R}^n)$. Since this is true for all $i \in \{1, \dots, m\}$, we conclude that $\tilde{\mathbf{u}}$ is Sobolev $W^{1,p}$ in

$$\bigcup_{i=1}^m \mathbf{w}(\Gamma_i \times (-\delta, 0)) = \mathbf{w}(\partial\Omega \times (-\delta, 0)) = N \setminus \partial\Omega.$$

It is immediate to check that the map $\mathbf{u} \mapsto \tilde{\mathbf{u}}$ is linear, and the same reasoning above, as well as Definition 3.2, show that, for some constants $c_1, c_2 > 0$,

$$\begin{aligned}
\|\tilde{\mathbf{u}}\|_{W^{1,p}(N\setminus\partial\Omega,\mathbb{R}^n)} &\leq \sum_{i=1}^m \big\|\mathbf{u}|_{\partial\Omega} \circ \tilde{\pi} \circ \mathbf{w}^{-1}\big\|_{W^{1,p}(\mathbf{w}(\Gamma_i\times(-\delta,0)),\mathbb{R}^n)} \leq c_1 \sum_{i=1}^m \|\mathcal{L}_i(\mathbf{u}) \circ \pi\|_{W^{1,p}((0,r)^{n-1}\times(-\delta,0),\mathbb{R}^n)} \\
&\leq c_2 \sum_{i=1}^m \|\mathcal{L}_i(\mathbf{u}) \circ \pi\|_{W^{1,p}((0,r)^{n-1}\times\{0\},\mathbb{R}^n)} = c_2 \sum_{i=1}^m \|\mathcal{L}_i(\mathbf{u})\|_{W^{1,p}((0,r)^{n-1},\mathbb{R}^n)} = c_2 \|\mathbf{u}\|_{W^{1,p}(\partial\Omega,\mathbb{R}^n)},
\end{aligned}$$

so the map $\mathbf{u} \mapsto \tilde{\mathbf{u}}$ is bounded from $W^{1,p}(\partial\Omega, \mathbb{R}^n)$ to $W^{1,p}(N \setminus \partial\Omega, \mathbb{R}^n)$. Now, the trace of $\tilde{\mathbf{u}}|_{N\setminus\partial\Omega}$ on $\partial\Omega$ is $\mathbf{u}|_{\partial\Omega}$: this is obvious if $\mathbf{u}|_{\partial\Omega}$ is continuous and follows from the continuity of the trace operator for any $\mathbf{u} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$. Therefore, $\tilde{\mathbf{u}} \in W^{1,p}(\tilde{\Omega}, \mathbb{R}^n)$ and the map $E : W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) \to W^{1,p}(\tilde{\Omega}, \mathbb{R}^n)$ defined as $E\mathbf{u} := \tilde{\mathbf{u}}$ is linear and bounded.

By Lemma 6.1, $\det D(\mathcal{L}_i(\mathbf{u}) \circ \pi) = 0$ a.e. in $(0,r)^{n-1} \times (-\delta, 0)$. By the chain rule and the fact that $\boldsymbol{\psi}_i$ is bi-Lipschitz, we infer that $\det D\tilde{\mathbf{u}} = 0$ a.e. in $\mathbf{w}(\Gamma_i \times (-\delta, 0))$. Since this is true for all $i \in \{1, \dots, m\}$, we conclude that $\det D\tilde{\mathbf{u}} = 0$ a.e. in $\tilde{\Omega} \setminus \Omega$.

Now we show *a)* and, accordingly, assume that $\Lambda_{n-1} D\mathbf{u}\,\boldsymbol{\nu} \in L^q(\partial\Omega, \mathbb{R}^n)$. Fix $i \in \{1, \dots, m\}$ and recall that $\mathcal{L}_i(\mathbf{u}) \circ \pi \in W^{1,p}((0,r)^{n-1} \times \{0\}, \mathbb{R}^n)$. Now we prove $\Lambda_{n-1} D(\mathcal{L}_i(\mathbf{u}) \circ \pi)\, \mathbf{e}_n \in L^q((0,r)^{n-1} \times \{0\}, \mathbb{R}^n)$.

We have that $\mathbf{G}_i : (0,r)^{n-1} \times \{0\} \to \Gamma_i$ is a bi-Lipschitz homeomorphism. Therefore, for a.e. $\hat{\mathbf{z}} \in (0,r)^{n-1}$, the linear map $D\mathbf{G}_i(\hat{\mathbf{z}}, 0) : \mathbb{R}^{n-1} \times \{0\} \to T_{\mathbf{G}_i(\hat{\mathbf{z}},0)}\Gamma_i$ is an isomorphism, where $T_{\mathbf{G}_i(\hat{\mathbf{z}},0)}\Gamma_i$ denotes the tangent space of Γ_i at $\mathbf{G}_i(\hat{\mathbf{z}}, 0)$. Consequently, the linear map $\Lambda_{n-1} D\mathbf{G}_i(\hat{\mathbf{z}}, 0) : \Lambda_{n-1}(\mathbb{R}^{n-1} \times \{0\}) \to \Lambda_{n-1}(T_{\mathbf{G}_i(\hat{\mathbf{z}},0)}\Gamma_i)$

is also an isomorphism. With the identifications $\Lambda_{n-1}(\mathbb{R}^{n-1} \times \{0\})$ with the space spanned by $\mathbf{e}_n$ and $\Lambda_{n-1}(T_{\mathbf{G}_i(\hat{\mathbf{z}},0)}\Gamma_i)$ with the space spanned by $\boldsymbol{\nu}(\mathbf{G}_i(\hat{\mathbf{z}},0))$ (a unit normal vector of Γ_i at $\mathbf{G}_i(\hat{\mathbf{z}},0)$), we have that the linear map $\Lambda_{n-1}D\mathbf{G}_i(\hat{\mathbf{z}},0)$ acts by multiplication by $\pm|\Lambda_{n-1}D\mathbf{G}_i(\hat{\mathbf{z}},0)\,\mathbf{e}_n|$. On the other hand, as $\mathbf{G}_i$ is bi-Lipschitz, we have that

$$\operatorname*{ess\,sup}_{\hat{\mathbf{z}}\in(0,r)^{n-1}} |\Lambda_{n-1}D\mathbf{G}_i(\hat{\mathbf{z}},0)\,\mathbf{e}_n| < \infty.$$

Moreover, the same reasoning also applies to $\mathbf{G}_i^{-1}$. In conclusion, an $\mathbb{R}^n$-valued Sobolev map $\mathbf{v}$ defined in $(0,r)^{n-1}\times\{0\}$ satisfies $\Lambda_{n-1}D\mathbf{v}\,\mathbf{e}_n \in L^q((0,r)^{n-1}\times\{0\},\mathbb{R}^n)$ if and only if $\Lambda_{n-1}D(\mathbf{v}\circ\mathbf{G}_i^{-1})\,\boldsymbol{\nu} \in L^q(\Gamma_i,\mathbb{R}^n)$; in addition, their L^q norms are comparable. Since we are assuming $\Lambda_{n-1}D(\mathbf{u}|_{\partial\Omega})\,\boldsymbol{\nu} \in L^q(\partial\Omega,\mathbb{R}^n)$, we have that $\Lambda_{n-1}D(\mathbf{u}|_{\Gamma_i})\,\boldsymbol{\nu} \in L^q(\Gamma_i,\mathbb{R}^n)$ and, consequently, $\Lambda_{n-1}D(\mathcal{L}_i(\mathbf{u})\circ\pi)\,\mathbf{e}_n \in L^q((0,r)^{n-1}\times\{0\},\mathbb{R}^n)$. Moreover, the L^q norms of $\Lambda_{n-1}D(\mathbf{u}|_{\Gamma_i})\,\boldsymbol{\nu}$ and $\Lambda_{n-1}D(\mathcal{L}_i(\mathbf{u})\circ\pi)\,\mathbf{e}_n$ are comparable.

Once we know $\Lambda_{n-1}D(\mathcal{L}_i(\mathbf{u})\circ\pi)\,\mathbf{e}_n \in L^q((0,r)^{n-1}\times\{0\},\mathbb{R}^n)$, we can apply Lemma 6.1 and obtain that $\operatorname{cof} D(\mathcal{L}_i(\mathbf{u})\circ\pi) \in L^q((0,r)^{n-1}\times(-\delta,0),\mathbb{R}^{n\times n})$. Moreover, the L^q norm of $\operatorname{cof} D(\mathcal{L}_i(\mathbf{u})\circ\pi)$ is comparable to the L^q norm of $\Lambda_{n-1}D(\mathcal{L}_i(\mathbf{u})\circ\pi)\,\mathbf{e}_n$. By the chain rule and the fact that $\boldsymbol{\psi}_i$ is bi-Lipschitz, we infer that $\operatorname{cof} D\tilde{\mathbf{u}} \in L^q(\mathbf{w}(\Gamma_i\times(-\delta,0)),\mathbb{R}^{n\times n})$. Moreover, the L^q norm of $\operatorname{cof} D\tilde{\mathbf{u}}$ is comparable to the L^q norm of $\operatorname{cof} D(\mathcal{L}_i(\mathbf{u})\circ\pi)$. As this is true for all $i\in\{1,\dots,m\}$, we conclude that $\operatorname{cof} D\tilde{\mathbf{u}} \in L^q(\tilde{\Omega}\setminus\Omega,\mathbb{R}^{n\times n})$ and the estimate of *a)* follows.

Now we show *b)* and, first, assume $\mathbf{u}\in\bar{\mathcal{A}}_p(\Omega)$. Let $\tilde{\phi}\in C_c^1(\tilde{\Omega})$ and $\mathbf{g}\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$. Thanks to (5.3), in order to show that $\tilde{\mathbf{u}}\in\mathcal{A}_p(\tilde{\Omega})$ it remains to see that $\mathcal{E}(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g})=0$. For each $i\in\{1,\dots,m\}$ let Γ_i' be a relatively open set of $\partial\Omega$ such that $\Gamma_i'\subset\Gamma_i$, the family $\{\Gamma_i'\}_{i=1}^m$ is disjoint and

$$\mathcal{H}^{n-1}\left(\partial\Omega\setminus\bigcup_{i=1}^m\Gamma_i'\right)=0.$$

Call $I_i' := (\mathbf{G}_i\circ\eta)^{-1}(\Gamma_i')$. As $\mathbf{u}\in\bar{\mathcal{A}}_p(\Omega)$, we have

$$\mathcal{E}(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g}) = \mathcal{E}_\Omega(\mathbf{u},\tilde{\phi}\,\mathbf{g}) + \mathcal{E}_N(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g}) = \mathcal{F}_{\partial\Omega}(\mathbf{u},\tilde{\phi}\,\mathbf{g}) + \mathcal{E}_N(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g}) = \sum_{i=1}^m \Big(\mathcal{F}_{\Gamma_i'}(\mathbf{u},\tilde{\phi}\,\mathbf{g}) + \mathcal{E}_{\mathbf{w}(\Gamma_i'\times(-\delta,0))}(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g})\Big).$$

Now fix $i\in\{1,\dots,m\}$ and define $\mathbf{F}_i : (0,r)^{n-1}\times(-\delta,0)\to\mathbf{w}(\Gamma_i\times(-\delta,0))$ as $\mathbf{F}_i := \boldsymbol{\psi}_i^{-1}$, i.e., $\mathbf{F}_i = \mathbf{w}\circ\big((\mathbf{G}_i\circ\eta)\times\mathbf{id}_{(-\delta,0)}\big)$. Then $\mathbf{F}_i$ is a bi-Lipschitz homeomorphism between open sets of $\mathbb{R}^n$. We claim that $\det D\mathbf{F}_i>0$ a.e.; in order to show that, it is enough to check that $\mathbf{F}_i$ preserves the orientation, that is, the normals. More explicitly, it suffices to show that for a.e. $\hat{\mathbf{z}}\in(0,r)^{n-1}$, the unit exterior normal of Ω at $\mathbf{F}_i(\hat{\mathbf{z}},0)$ is a positive multiple of $\Lambda_{n-1}D(\mathbf{F}_i|_{(0,r)^{n-1}\times\{0\}})(\hat{\mathbf{z}},0)(-\mathbf{e}_n)$. Thanks to Definition 6.2, we have that

$$\mathbf{F}_i|_{(0,r)^{n-1}\times\{0\}} = \mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}}. \tag{6.5}$$

Therefore, $\Lambda_{n-1}D(\mathbf{F}_i|_{(0,r)^{n-1}\times\{0\}})(\hat{\mathbf{z}},0) = \Lambda_{n-1}D(\mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}})(\hat{\mathbf{z}},0)$ for a.e. $\hat{\mathbf{z}}\in(0,r)^{n-1}$. According to (3.1), the exterior normal at $\mathbf{F}_i(\hat{\mathbf{z}},0)$ is a positive multiple of $\Lambda_{n-1}D\mathbf{F}_i(\hat{\mathbf{z}},0)(-\mathbf{e}_n)$. We thus conclude that $\det D\mathbf{F}_i>0$ a.e.

Changing variables and using that $\det D\tilde{\mathbf{u}}=0$ in $\tilde{\Omega}\setminus\Omega$, we have

$$\begin{aligned}\mathcal{E}_{\mathbf{w}(\Gamma_i'\times(-\delta,0))}(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g}) = \mathcal{E}_{\mathbf{F}_i(I_i'\times(-\delta,0))}(\tilde{\mathbf{u}},\tilde{\phi}\,\mathbf{g}) &= \int_{\mathbf{F}_i(I_i'\times(-\delta,0))} \mathbf{g}(\tilde{\mathbf{u}}(\mathbf{x}))\cdot\Big(\operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{x})\,D\tilde{\phi}(\mathbf{x})\Big)\,\mathrm{d}\mathbf{x}\\ &= \int_{I_i'\times(-\delta,0)} \mathbf{g}(\tilde{\mathbf{u}}(\mathbf{F}_i(\mathbf{z})))\cdot\Big(\operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{F}_i(\mathbf{z}))\,D\tilde{\phi}(\mathbf{F}_i(\mathbf{z}))\Big)\det D\mathbf{F}_i(\mathbf{z})\,\mathrm{d}\mathbf{z}.\end{aligned} \tag{6.6}$$

Now we observe that for a.e. $\mathbf{z}\in I_i'\times(-\delta,0)$,

$$\operatorname{cof} D\tilde{\mathbf{u}}(\mathbf{F}_i(\mathbf{z}))\,D\tilde{\phi}(\mathbf{F}_i(\mathbf{z}))\det D\mathbf{F}_i(\mathbf{z}) = \operatorname{cof} D\left(\tilde{\mathbf{u}}\circ\mathbf{F}_i\right)(\mathbf{z})\,D\left(\tilde{\phi}\circ\mathbf{F}_i\right)(\mathbf{z}),$$

because of formula (2.1), as well as the chain rule. Now define $\mathbf{f}_i := (\mathbf{u}|_{\partial\Omega} \circ \mathbf{F}_i)|_{I_i' \times \{0\}}$. Then $(\tilde{\mathbf{u}} \circ \mathbf{F}_i)(\hat{\mathbf{z}}, z_n) = \mathbf{f}_i(\hat{\mathbf{z}}, 0)$ for any $(\hat{\mathbf{z}}, z_n) \in I_i' \times (-\delta, 0)$. Therefore, as in (6.2)–(6.3), for any $(\hat{\mathbf{z}}, z_n) \in I_i' \times (-\delta, 0)$,

$$\operatorname{cof} D\,(\tilde{\mathbf{u}} \circ \mathbf{F}_i)\,(\hat{\mathbf{z}}, z_n)\,\mathbf{e}_i = \mathbf{0} \text{ for } i \in \{1, \dots, n-1\}, \text{ and } \operatorname{cof} D\,(\tilde{\mathbf{u}} \circ \mathbf{F}_i)\,(\hat{\mathbf{z}}, z_n)\,\mathbf{e}_n = \Lambda_{n-1} D\mathbf{f}_i(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n.$$

Consequently,

$$\operatorname{cof} D\,(\tilde{\mathbf{u}} \circ \mathbf{F}_i)\,(\hat{\mathbf{z}}, z_n)\, D\left(\tilde{\phi} \circ \mathbf{F}_i\right)(\hat{\mathbf{z}}, z_n) = \frac{\partial(\tilde{\phi} \circ \mathbf{F}_i)}{\partial z_n}(\hat{\mathbf{z}}, z_n)\,\Lambda_{n-1} D\mathbf{f}_i(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n$$

and, hence, thanks to (6.6) and the fundamental theorem of Calculus,

$$\begin{aligned}\mathcal{E}_{\mathbf{w}(\Gamma_i' \times (-\delta, 0))}(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) &= \int_{I_i'} \mathbf{g}(\mathbf{f}_i(\hat{\mathbf{z}}, 0)) \cdot (\Lambda_{n-1} D\mathbf{f}_i(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n) \int_{-\delta}^{0} \frac{\partial(\tilde{\phi} \circ \mathbf{F}_i)}{\partial z_n}(\hat{\mathbf{z}}, z_n)\,\mathrm{d}z_n\,\mathrm{d}\hat{\mathbf{z}} \\ &= \int_{I_i'} (\tilde{\phi} \circ \mathbf{F}_i)(\hat{\mathbf{z}}, 0)\,\mathbf{g}(\mathbf{f}_i(\hat{\mathbf{z}}, 0)) \cdot (\Lambda_{n-1} D\mathbf{f}_i(\hat{\mathbf{z}}, 0)\,\mathbf{e}_n)\,\mathrm{d}\hat{\mathbf{z}}.\end{aligned} \tag{6.7}$$

Now we compute $\mathcal{F}_{\Gamma_i'}(\mathbf{u}, \tilde{\phi}\,\mathbf{g})$ through the change of variables given by $\mathbf{G}_i^{-1} : \Gamma_i' \to I_i' \times \{0\}$; applying Proposition 2.2 we obtain

$$\begin{aligned}\mathcal{F}_{\Gamma_i'}(\mathbf{u}, \tilde{\phi}\,\mathbf{g}) &= \int_{\Gamma_i'} \tilde{\phi}(\mathbf{x})\,\mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot \Lambda_{n-1} D\mathbf{u}(\mathbf{x})\,\boldsymbol{\nu}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \\ &= \int_{I_i' \times \{0\}} \tilde{\phi}(\mathbf{G}_i(\mathbf{z}))\,\mathbf{g}(\mathbf{u}(\mathbf{G}_i(\mathbf{z}))) \cdot \frac{\Lambda_{n-1} D\mathbf{u}(\mathbf{G}_i(\mathbf{z}))\,\boldsymbol{\nu}(\mathbf{G}_i(\mathbf{z}))}{|\Lambda_{n-1} D(\mathbf{G}_i^{-1}|_{\Gamma_i'})(\mathbf{G}_i(\mathbf{z}))\,\boldsymbol{\nu}(\mathbf{G}_i(\mathbf{z}))|}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).\end{aligned} \tag{6.8}$$

Now, using (3.1) and (6.5), we have that, for $\mathcal{H}^{n-1}$-a.e. $\mathbf{z} \in I_i' \times \{0\}$,

$$\begin{aligned}&\frac{\Lambda_{n-1} D\mathbf{u}(\mathbf{G}_i(\mathbf{z}))\,\boldsymbol{\nu}(\mathbf{G}_i(\mathbf{z}))}{|\Lambda_{n-1} D(\mathbf{G}_i^{-1}|_{\Gamma_i'})(\mathbf{G}_i(\mathbf{z}))\,\boldsymbol{\nu}(\mathbf{G}_i(\mathbf{z}))|} = \frac{\Lambda_{n-1} D\mathbf{u}(\mathbf{G}_i(\mathbf{z}))\,\Lambda_{n-1} D(\mathbf{G}_i|_{I_i' \times \{0\}})(\mathbf{z})(-\mathbf{e}_n)}{|\Lambda_{n-1} D(\mathbf{G}_i^{-1}|_{\Gamma_i'})(\mathbf{G}_i(\mathbf{z}))\,\Lambda_{n-1} D(\mathbf{G}_i|_{I_i' \times \{0\}})(\mathbf{z})\,\mathbf{e}_n|} \\ &= \frac{\Lambda_{n-1} D(\mathbf{u} \circ \mathbf{G}_i|_{I_i' \times \{0\}})(\mathbf{z})(-\mathbf{e}_n)}{|\mathbf{e}_n|} = \Lambda_{n-1} D\mathbf{f}_i(\mathbf{z})(-\mathbf{e}_n).\end{aligned} \tag{6.9}$$

Therefore, using again (6.5),

$$\begin{aligned}\mathcal{F}_{\Gamma_i'}(\mathbf{u}, \tilde{\phi}\,\mathbf{g}) &= \int_{I_i' \times \{0\}} \tilde{\phi}(\mathbf{G}_i(\mathbf{z}))\,\mathbf{g}(\mathbf{u}(\mathbf{G}_i(\mathbf{z}))) \cdot \Lambda_{n-1} D\mathbf{f}_i(\mathbf{z})(-\mathbf{e}_n)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \\ &= \int_{I_i' \times \{0\}} \tilde{\phi}(\mathbf{F}_i(\mathbf{z}))\,\mathbf{g}(\mathbf{f}_i(\mathbf{z})) \cdot \Lambda_{n-1} D\mathbf{f}_i(\mathbf{z})(-\mathbf{e}_n)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).\end{aligned} \tag{6.10}$$

Comparing (6.7) and (6.10), we conclude that $\mathcal{F}_{\Gamma_i'}(\mathbf{u}, \tilde{\phi}\,\mathbf{g}) + \mathcal{E}_{\mathbf{w}(\Gamma_i' \times (-\delta, 0))}(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) = 0$ and, hence $\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) = 0$.

Now we prove the converse, so we assume $\tilde{\mathbf{u}} \in \mathcal{A}_p(\tilde{\Omega})$. Let $\phi \in C^1(\bar{\Omega})$ and $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$. Due to (5.4), we want to show that $\mathcal{E}_\Omega(\mathbf{u}, \phi\,\mathbf{g}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}, \phi\,\mathbf{g})$. We consider any extension $\tilde{\phi} \in C_c^1(\tilde{\Omega})$ of ϕ. As $\tilde{\mathbf{u}} \in \mathcal{A}_p(\tilde{\Omega})$, we have

$$0 = \mathcal{E}_{\tilde{\Omega}}(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) = \mathcal{E}_\Omega(\mathbf{u}, \phi\,\mathbf{g}) + \mathcal{E}_N(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}).$$

The calculation of $\mathcal{E}_N(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g})$ done in the first part of the proof of *b)* remains valid, and is summarized as

$$\mathcal{E}_N(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) = \sum_{i=1}^{m} \mathcal{E}_{\mathbf{w}(\Gamma_i' \times (-\delta, 0))}(\tilde{\mathbf{u}}, \tilde{\phi}\,\mathbf{g}) = -\sum_{i=1}^{m} \mathcal{F}_{\Gamma_i'}(\mathbf{u}, \tilde{\phi}\,\mathbf{g}) = -\mathcal{F}_{\partial\Omega}(\mathbf{u}, \tilde{\phi}\,\mathbf{g}).$$

Therefore, $\mathcal{E}_\Omega(\mathbf{u}, \phi\,\mathbf{g}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}, \phi\,\mathbf{g})$ and $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$. $\square$

7 Regular functions are included

The classical approach to the existence of minimizers in nonlinear elasticity for polyconvex integrands (see [2]) is based on the satisfaction of Piola's identity $\operatorname{Div}\operatorname{cof} D\mathbf{u} = \mathbf{0}$, its consequence $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ (see (1.1)) and its generalization (1.2). Thus, much of the work in nonlinear elasticity developed in the last decades has focused on ascertaining sufficient conditions for the validity of (1.1) or (1.2). The fact that functions $\mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$ for $p \geq n$ satisfy (1.1) is due to [39]. This equality was rediscovered by [2], who also proved it in the case $p \geq n-1$ and $\operatorname{cof} D\mathbf{u} \in L^q(\Omega, \mathbb{R}^{n\times n})$ with $q \geq \frac{p}{p-1}$. The case $p \geq n-1$ and $q \geq \frac{n}{n-1}$ was covered in [37].

In our context, functions in $\bar{\mathcal{A}}_p(\Omega)$ require the satisfaction of (1.2) not only in the sense of distributions, but when it is multiplied by a $\phi \in C^\infty(\bar\Omega)$ and integrated by parts (see (1.4)). In this section we prove the analogue of the results mentioned above [39, 2, 37] to our case. We start with C^1 functions.

Lemma 7.1. *Let $p \geq n-1$. Then $C^1(\bar\Omega, \mathbb{R}^n) \subset \bar{\mathcal{A}}_p(\Omega)$.*

Proof. Let $\mathbf{f} \in C^1_c(\bar\Omega \times \mathbb{R}^n, \mathbb{R}^n)$. Assume first that $\mathbf{u} \in C^2(\bar\Omega, \mathbb{R}^n)$. Then, as a consequence of Piola's identity, for all $\mathbf{x} \in \bar\Omega$,

$$\operatorname{div}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) + D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot \operatorname{cof} D\mathbf{u}(\mathbf{x}) = \operatorname{Div}\left[(\operatorname{adj} D\mathbf{u}(\mathbf{x}))\, \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))\right],$$

hence by the divergence theorem we obtain $\mathcal{E}(\mathbf{u}, \mathbf{f}) = \mathcal{F}(\mathbf{u}, \mathbf{f})$.

Now assume $\mathbf{u} \in C^1(\bar\Omega, \mathbb{R}^n)$, and consider an extension $\tilde{\mathbf{u}} \in C^1(\tilde\Omega, \mathbb{R}^n)$ of $\mathbf{u}$ to an open set $\tilde\Omega$ containing $\bar\Omega$. By mollification, there exist an open set $\Omega' \subset \tilde\Omega$ containing $\bar\Omega$ and a sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ in $C^\infty(\Omega', \mathbb{R}^n)$ such that $\mathbf{u}_j \to \mathbf{u}$ and $D\mathbf{u}_j \to D\mathbf{u}$, as $j \to \infty$, uniformly in compact subsets of Ω'. By the result of the first part of the proof, $\mathcal{E}_\Omega(\mathbf{u}_j, \mathbf{f}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}_j, \mathbf{f})$ for all $j \in \mathbb{N}$. On the other hand, the convergences above imply $\lim_{j\to\infty} \mathcal{E}_\Omega(\mathbf{u}_j, \mathbf{f}) = \mathcal{E}_\Omega(\mathbf{u}, \mathbf{f})$ and $\lim_{j\to\infty} \mathcal{F}_{\partial\Omega}(\mathbf{u}_j, \mathbf{f}) = \mathcal{F}_{\partial\Omega}(\mathbf{u}, \mathbf{f})$. □

With the aid of Proposition 4.2, we can easily prove that $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ is contained in $\bar{\mathcal{A}}_p(\Omega)$ when $p \geq n$.

Proposition 7.2. *Let Ω be a bounded open set of $\mathbb{R}^n$ of class C^1 and let $p \geq n$. Then $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) \subset \bar{\mathcal{A}}_p(\Omega)$.*

Proof. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ and $\mathbf{f} \in C^1_c(\bar\Omega \times \mathbb{R}^n, \mathbb{R}^n)$. By Proposition 4.2, there exists a sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}} \subset C^1(\bar\Omega, \mathbb{R}^n)$ such that $\mathbf{u}_j \to \mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$, a.e. in Ω and $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$. Hence

$$\begin{aligned}
&\operatorname{cof} D\mathbf{u}_j \to \operatorname{cof} D\mathbf{u} \text{ in } L^{\frac{p}{n-1}}(\Omega, \mathbb{R}^{n\times n}), && \det D\mathbf{u}_j \to \det D\mathbf{u} \text{ in } L^{\frac{p}{n}}(\Omega),\\
&\operatorname{cof} D\mathbf{u}_j\, \boldsymbol{\nu} \to \operatorname{cof} D\mathbf{u}\, \boldsymbol{\nu} \text{ in } L^{\frac{p}{n-1}}(\partial\Omega, \mathbb{R}^n), && \mathbf{f}(\mathbf{x}, \mathbf{u}_j(\mathbf{x})) \to \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \text{ a.e. } \mathbf{x} \in \Omega,\\
&D\mathbf{f}(\mathbf{x}, \mathbf{u}_j(\mathbf{x})) \to D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \text{ a.e. } \mathbf{x} \in \Omega, && \mathbf{f}(\mathbf{x}, \mathbf{u}_j(\mathbf{x})) \to \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))\ \mathcal{H}^{n-1}\text{-a.e. } \mathbf{x} \in \partial\Omega
\end{aligned}$$

as $j \to \infty$. By Lemma 7.1, $\mathcal{E}(\mathbf{u}_j, \mathbf{f}) = \mathcal{F}(\mathbf{u}_j, \mathbf{f})$ for all $j \in \mathbb{N}$. Taking limits as $j \to \infty$, by a standard convergence result (see, e.g., [18, Prop. 2.61]), we obtain $\mathcal{E}(\mathbf{u}, \mathbf{f}) = \mathcal{F}(\mathbf{u}, \mathbf{f})$. □

The passage from $p \geq n$ to $p \geq n-1$ is as follows; we adapt the proof of [33, Lemma 2].

Proposition 7.3. *Let Ω be a bounded open set of class C^1. Let $p \geq n-1$ and $q \geq \frac{p}{p-1}$. Then $\{\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) : \operatorname{cof} D\mathbf{u} \in L^q(\Omega, \mathbb{R}^{n\times n})\} \subset \bar{\mathcal{A}}_p(\Omega)$.*

Proof. Formula (2.1) and Hölder's inequality imply that $\det D\mathbf{u} \in L^1(\Omega)$.

First we show that for all $\mathbf{v}, \mathbf{b} \in C^1(\bar\Omega, \mathbb{R}^n)$ one has

$$\int_\Omega \operatorname{cof} D\mathbf{v} \cdot D\mathbf{b} = \int_{\partial\Omega} \mathbf{b} \cdot (\operatorname{cof} D\mathbf{v}\, \boldsymbol{\nu}). \tag{7.1}$$

Formula (7.1) immediately holds under the assumption $\mathbf{v} \in C^2(\bar{\Omega}, \mathbb{R}^n)$, since, as a consequence of Piola's identity $\operatorname{Div} \operatorname{cof} D\mathbf{v} = \mathbf{0}$ we have

$$D\mathbf{b}(\mathbf{x}) \cdot \operatorname{cof} D\mathbf{v}(\mathbf{x}) = \operatorname{Div}\left[(\operatorname{adj} D\mathbf{v}(\mathbf{x}))\, \mathbf{b}(\mathbf{x})\right], \qquad \mathbf{x} \in \bar{\Omega},$$

and (7.1) follows by an integration by parts. If $\mathbf{v} \in C^1(\bar{\Omega}, \mathbb{R}^n)$, by approximation with C^2 functions, as in the proof of Lemma 7.1, we obtain that (7.1) is valid for $\mathbf{v} \in C^1(\bar{\Omega}, \mathbb{R}^n)$.

Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ satisfy $\operatorname{cof} D\mathbf{u} \in L^q(\Omega, \mathbb{R}^{n\times n})$. By Proposition 4.2, there exists a sequence $\{\mathbf{v}_j\}_{j\in\mathbb{N}} \subset C^1(\bar{\Omega}, \mathbb{R}^n)$ such that $\mathbf{v}_j \to \mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$. Therefore, $\operatorname{cof} D\mathbf{v}_j \to \operatorname{cof} D\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^{n\times n})$ and $\operatorname{cof} D\mathbf{v}_j\, \boldsymbol{\nu} \to \Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu}$ in $L^1(\partial\Omega, \mathbb{R}^n)$. Passing to the limit in (7.1) (with $\mathbf{v}$ replaced with $\mathbf{v}_j$) we obtain that

$$\int_\Omega \operatorname{cof} D\mathbf{u} \cdot D\mathbf{b} = \int_{\partial\Omega} \mathbf{b} \cdot (\Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu}). \tag{7.2}$$

Now consider $\mathbf{a} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$. By Proposition 4.2, there exists a sequence $\{\mathbf{b}_j\}_{j\in\mathbb{N}} \subset C^1(\bar{\Omega}, \mathbb{R}^n)$ such that $\mathbf{b}_j \to \mathbf{a}$ in $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$. Therefore, $D\mathbf{b}_j \to D\mathbf{a}$ in $L^p(\Omega, \mathbb{R}^{n\times n})$ and $\mathbf{b}_j \to \mathbf{a}$ in $L^p(\partial\Omega, \mathbb{R}^n)$. Passing to the limit in (7.2) (with $\mathbf{b}$ replaced with $\mathbf{b}_j$) we obtain that

$$\int_\Omega \operatorname{cof} D\mathbf{u} \cdot D\mathbf{a} = \int_{\partial\Omega} \mathbf{a} \cdot (\Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu}). \tag{7.3}$$

Now let $\phi \in C^1(\bar{\Omega})$ and $\mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, and define $\mathbf{a} := \phi\, (\mathbf{g} \circ \mathbf{u})$. By the chain rule for Sobolev functions, $\mathbf{g} \circ \mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ so $\mathbf{a} \in W^{1,p}(\Omega, \mathbb{R}^n)$; the same argument shows that $\mathbf{a} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$. Therefore, $\mathbf{a} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ and, moreover,

$$D\mathbf{a} = (\mathbf{g} \circ \mathbf{u}) \otimes D\phi + \phi\, (D\mathbf{g} \circ \mathbf{u})\, D\mathbf{u}.$$

Plugging this expression in (7.3) yields

$$\int_\Omega \operatorname{cof} D\mathbf{u} \cdot ((\mathbf{g} \circ \mathbf{u}) \otimes D\phi + \phi\, (D\mathbf{g} \circ \mathbf{u})\, D\mathbf{u}) = \int_{\partial\Omega} \phi\, (\mathbf{g} \circ \mathbf{u}) \cdot (\Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu}).$$

Now, using (2.1) we obtain that

$$\operatorname{cof} D\mathbf{u} \cdot (\phi\, (D\mathbf{g} \circ \mathbf{u})\, D\mathbf{u}) = \phi \det D\mathbf{u}\, (\operatorname{div} \mathbf{g} \circ \mathbf{u}),$$

which shows that $\mathcal{E}(\mathbf{u}, \phi\, \mathbf{g}) = \mathcal{F}(\mathbf{u}, \phi\, \mathbf{g})$. Thus (see (5.4)), $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$. ☐

In the case $p < n$, the reduction from the exponent $q \geq \frac{p}{p-1}$ to $q \geq \frac{n}{n-1}$ is more delicate and cannot be based solely on an approximation argument. Instead of adapting the proof of [37], we use the extension property of Proposition 6.3 to provide a quick derivation of the following result from that of [37].

Proposition 7.4. *Let Ω be an extendable open set. Let $p \geq n-1$ and $q \geq \frac{n}{n-1}$. Then $\{\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) : \operatorname{cof} D\mathbf{u} \in L^q(\Omega, \mathbb{R}^{n\times n}),\ \Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu} \in L^q(\partial\Omega, \mathbb{R}^n)\} \subset \bar{\mathcal{A}}_p(\Omega)$.*

Proof. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ be such that $\operatorname{cof} D\mathbf{u} \in L^q(\Omega, \mathbb{R}^{n\times n})$ and $\Lambda_{n-1} D\mathbf{u}\, \boldsymbol{\nu} \in L^q(\partial\Omega, \mathbb{R}^n)$. Taking determinants in (2.1) show that $\det D\mathbf{u} \in L^1(\Omega)$; see [37, Eq. (1.4)], if necessary. Let $\tilde{\mathbf{u}}$ be the extension of $\mathbf{u}$ to an open set $\tilde{\Omega} \supset \bar{\Omega}$ given by Proposition 6.3. We then have that $\tilde{\mathbf{u}} \in W^{1,p}(\tilde{\Omega}, \mathbb{R}^n)$ and $\operatorname{cof} D\tilde{\mathbf{u}} \in L^q(\tilde{\Omega}, \mathbb{R}^{n\times n})$. By the result of [37, Th. 3.2], $\tilde{\mathbf{u}} \in \mathcal{A}_p(\tilde{\Omega})$. Again by Proposition 6.3, $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$. ☐

8 Boundedness

In this section we prove that functions $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ with $\det D\mathbf{u} \geq 0$ a.e. are bounded. This is the global version of [6, Th. 4.1], where it was proved that functions $\mathbf{u} \in \mathcal{A}_p(\Omega)$ with $\det D\mathbf{u} \geq 0$ a.e. are locally bounded.

We start with the definition of topological image (see [42]). Given $\mathbf{u} \in C(\partial\Omega, \mathbb{R}^n)$, we define $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ as the set of $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial\Omega)$ such that $\deg(\mathbf{u}, \Omega, \mathbf{y}) \neq 0$. Here deg is the Brouwer degree (see, e.g., [11, 16]). It is known that $\deg(\mathbf{u}, \Omega, \cdot)$ is zero in the unbounded component of $\mathbb{R}^n \setminus \mathbf{u}(\partial\Omega)$ (see, e.g., [11, Sect. 5.1]); consequently, $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ is bounded. It is also open because of the continuity of the degree. Now, given $\mathbf{u} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$ with $p > n-1$, we have that $\mathbf{u}$ admits a continuous representative, so that $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ is also defined.

The following result is due to [36, Prop. 2.1] (see also [35, Prop. 2.1]) and computes the distributional derivative of the degree.

Proposition 8.1. *Let $p > n-1$ and $\mathbf{u} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$. Then, for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \deg(\mathbf{u}, \Omega, \mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\partial\Omega} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\Lambda_{n-1} D\mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}).$$

We make the following observation about Proposition 2.1.

Remark 8.2. Let $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$ and let Ω_0 be the set of Proposition 2.1. This set Ω_0 is not uniquely defined, although it can be given a precise definition (see [35, Eq. (2.10)]). We fix any Ω_0 with that property, and note that in this paper the specific choice of Ω_0 is not relevant because of the following properties:

a) Any $\Omega_0' \subset \Omega_0$ with $\mathcal{L}^n(\Omega \setminus \Omega_0') = 0$ satisfies the same properties of Proposition 2.1.

b) If Ω_1 is another subset of Ω with the properties listed in Proposition 2.1, then for any measurable $A \subset \Omega$, we have that $\mathbf{u}(A \cap \Omega_0) = \mathbf{u}(A \cap \Omega_1)$ a.e., and the two definitions of $\mathcal{N}_{\mathbf{u},A}$ corresponding to Ω_0 and Ω_1 coincide a.e.

We now define the geometric image (see [35]).

Definition 8.3. Given $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$, let Ω_0 be the set of Proposition 2.1. We define the geometric image of Ω under $\mathbf{u}$, denoted by $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega)$, as $\mathbf{u}(\Omega_0)$.

With these definitions, we are able to present the main result of this section; its proof follows that of [6, Th. 4.1]. Recall the function $\mathcal{N}_{\mathbf{u}}$ from Proposition 2.1.

Proposition 8.4. *Let $p > n-1$. If $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ with $\det D\mathbf{u} \geq 0$ a.e. then $\deg(\mathbf{u}, \Omega, \cdot) = \mathcal{N}_{\mathbf{u}}$ a.e., $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega) = \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ a.e. and $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^n)$.*

Proof. Fix $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ and consider $\mathbf{g}$ also as a function in $C_c^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$ with no dependence on the first set of variables. Using Proposition 8.1, we find that

$$\int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \deg(\mathbf{u}, \Omega, \mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\partial\Omega} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \mathcal{F}(\mathbf{u}, \mathbf{g}).$$

On the other hand, by Proposition 2.1, and using that $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$,

$$\int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathcal{N}_{\mathbf{u}}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\Omega} \operatorname{div} \mathbf{g}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \mathcal{E}(\mathbf{u}, \mathbf{g}) = \mathcal{F}(\mathbf{u}, \mathbf{g}).$$

We thus obtain that

$$\int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \deg(\mathbf{u}, \Omega, \mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathcal{N}_{\mathbf{u}}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

This equality being true for all $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ means that the distributional derivatives of $\deg(\mathbf{u}, \Omega, \cdot)$ and $\mathcal{N}_{\mathbf{u}}$ coincide, so there exists $c \in \mathbb{Z}$ such that $\mathcal{N}_{\mathbf{u}} - \deg(\mathbf{u}, \Omega, \cdot) = c$ a.e.

Now we show that $\mathrm{im_G}(\mathbf{u},\Omega) \subset \mathrm{im_T}(\mathbf{u},\Omega)$ a.e. Notice, by the classic result of [32] (see also [42, Th. 1] or [35, Prop. 2.7]), that $\mathcal{L}^n(\mathbf{u}(\partial\Omega)) = 0$. For all $\mathbf{y} \in \mathrm{im_G}(\mathbf{u},\Omega) \setminus (\mathrm{im_T}(\mathbf{u},\Omega) \cup \mathbf{u}(\partial\Omega))$ we have $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) \geq 1$ and $\deg(\mathbf{u},\Omega,\mathbf{y}) = 0$, so if

$$\mathcal{L}^n(\mathrm{im_G}(\mathbf{u},\Omega) \setminus \mathrm{im_T}(\mathbf{u},\Omega)) > 0 \tag{8.1}$$

we would obtain $c \geq 1$. Thus, for a.e. $\mathbf{y} \in \mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u},\Omega)$ we would have $\deg(\mathbf{u},\Omega,\mathbf{y}) = 0$, so $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) = c \geq 1$ and, hence, $\mathbf{y} \in \mathrm{im_G}(\mathbf{u},\Omega)$. Therefore, $\mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u},\Omega) \subset \mathrm{im_G}(\mathbf{u},\Omega)$ a.e., so, using that $\mathrm{im_T}(\mathbf{u},\Omega)$ is bounded, as well as Proposition 2.1,

$$\infty = \mathcal{L}^n\left(\mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u},\Omega)\right) \leq \mathcal{L}^n\left(\mathrm{im_G}(\mathbf{u},\Omega)\right) \leq \int_{\mathrm{im_G}(\mathbf{u},\Omega)} \mathcal{N}_{\mathbf{u}}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_\Omega \det D\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

which contradicts the fact that $\det D\mathbf{u} \in L^1(\Omega)$; this contradiction comes from assumption (8.1). Thus, $\mathrm{im_G}(\mathbf{u},\Omega) \subset \mathrm{im_T}(\mathbf{u},\Omega)$ a.e.; consequently, $\mathbf{u} \in L^\infty(\Omega,\mathbb{R}^n)$.

Now, for a.e. $\mathbf{y} \in \mathbb{R}^n \setminus \mathrm{im_T}(\mathbf{u},\Omega)$ we have $\deg(\mathbf{u},\Omega,\mathbf{y}) = 0$, $\mathbf{y} \notin \mathrm{im_G}(\mathbf{u},\Omega)$ and, hence, $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) = 0$. Thus $c = 0$ and, hence, $\deg(\mathbf{u},\Omega,\cdot) = \mathcal{N}_{\mathbf{u}}$ a.e., which shows $\mathrm{im_G}(\mathbf{u},\Omega) = \mathrm{im_T}(\mathbf{u},\Omega)$ a.e. □

9 Global invertibility

The previous section provides us with all the ingredients to prove the global invertibility result of this paper.

Theorem 9.1. *Let $p > n-1$. Let $\mathbf{u},\mathbf{u}_0 \in \bar{\mathcal{A}}_p(\Omega)$ satisfy $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$, $\det D\mathbf{u} > 0$ a.e., $\det D\mathbf{u}_0 \geq 0$ a.e. and $\mathbf{u}_0$ is injective a.e. Then $\mathbf{u}$ is injective a.e. and $\mathrm{im_G}(\mathbf{u},\Omega) = \mathrm{im_G}(\mathbf{u}_0,\Omega)$ a.e.*

Proof. According to Proposition 8.4, $\deg(\mathbf{u},\Omega,\cdot) = \mathcal{N}_{\mathbf{u}}$ a.e., $\deg(\mathbf{u}_0,\Omega,\cdot) = \mathcal{N}_{\mathbf{u}_0}$ a.e. Now, as the degree only depends on the boundary values, we have that $\deg(\mathbf{u},\Omega,\cdot) = \deg(\mathbf{u}_0,\Omega,\cdot)$. Therefore, $\mathcal{N}_{\mathbf{u}} = \mathcal{N}_{\mathbf{u}_0}$ a.e. As $\mathbf{u}_0$ is injective a.e., there exists $\Omega_1 \subset \Omega_0$ with $\mathcal{L}^n(\Omega \setminus \Omega_1) = 0$ such that $\mathbf{u}_0|_{\Omega_1}$ is injective. Therefore, for all $\mathbf{y} \in \mathbf{u}_0(\Omega_1)$ there is exactly one $\mathbf{x} \in \Omega_1$ such that $\mathbf{u}_0(\mathbf{x}) = \mathbf{y}$, while for all $\mathbf{y} \notin \mathbf{u}_0(\Omega_1)$ there is no $\mathbf{x} \in \Omega_1$ such that $\mathbf{u}_0(\mathbf{x}) = \mathbf{y}$. Thanks to Remark 8.2, this shows that $\mathcal{N}_{\mathbf{u}_0} = \chi_{\mathrm{im_G}(\mathbf{u}_0,\Omega)}$ a.e. Altogether, $\mathcal{N}_{\mathbf{u}} = \chi_{\mathrm{im_G}(\mathbf{u}_0,\Omega)}$ a.e. Consequently, there exists a set $N \subset \mathbb{R}^n$ of zero measure such that for all $\mathbf{y} \in \mathrm{im_G}(\mathbf{u}_0,\Omega) \setminus N$ we have $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) = 1$ and $\mathbf{y} \in \mathrm{im_G}(\mathbf{u},\Omega)$, while for all $\mathbf{y} \notin \mathrm{im_G}(\mathbf{u}_0,\Omega) \cup N$ we have $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) = 0$ and $\mathbf{y} \notin \mathrm{im_G}(\mathbf{u},\Omega)$. This shows that $\mathrm{im_G}(\mathbf{u},\Omega) \setminus N = \mathrm{im_G}(\mathbf{u}_0,\Omega) \setminus N$ and that $\mathbf{u}$ is injective in

$$\{\mathbf{x} \in \Omega_0 : \mathbf{u}(\mathbf{x}) \in \mathrm{im_G}(\mathbf{u}_0,\Omega) \setminus N\} = \{\mathbf{x} \in \Omega_0 : \mathbf{u}(\mathbf{x}) \notin N\}.$$

Thus, $\mathrm{im_G}(\mathbf{u},\Omega) = \mathrm{im_G}(\mathbf{u}_0,\Omega)$ a.e. Moreover, by Proposition 2.1 and the fact $\det D\mathbf{u} > 0$ a.e., we have that the set $\{\mathbf{x} \in \Omega_0 : \mathbf{u}(\mathbf{x}) \in N\}$ has measure zero. This concludes that $\mathbf{u}$ is injective a.e. □

Once we know that $\mathbf{u}$ is injective, we can define a.e. its inverse. Pointwise definitions of the inverse can be found in [42, 24, 26, 6], but for our purposes, the following a.e. definition suffices.

Definition 9.2. Let $p > n-1$. Let $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ be injective a.e. Let Ω_0 be the set of Proposition 2.1. Let $\Omega_1 \subset \Omega_0$ satisfy $\mathcal{L}^n(\Omega \setminus \Omega_1) = 0$ and $\mathbf{u}|_{\Omega_1}$ is injective. The inverse $\mathbf{u}^{-1} : \mathrm{im_T}(\mathbf{u},\Omega) \to \mathbb{R}^n$ is defined a.e. as $\mathbf{u}^{-1}(\mathbf{y}) = \mathbf{x}$, for each $\mathbf{y} \in \mathbf{u}(\Omega_1)$, and where $\mathbf{x} \in \Omega_1$ satisfies $\mathbf{u}(\mathbf{x}) = \mathbf{y}$.

As happened with Definition 8.3, the set Ω_1 is not uniquely defined, but if Ω_1' is another such set with the same properties, then the definitions of the inverse corresponding to Ω_1 and Ω_1' coincide a.e., since $\mathbf{u}(\Omega_1) = \mathbf{u}(\Omega_1') = \mathrm{im_T}(\mathbf{u},\Omega)$ a.e.

The Sobolev regularity of the inverse has been proved in several places under slightly different assumptions; see [3, 42, 26]. We present the corresponding result in our context.

Theorem 9.3. *Let $p > n-1$. Let $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ be injective a.e. with $\det D\mathbf{u} > 0$ a.e. Then $\mathbf{u}^{-1} \in W^{1,1}(\mathrm{im_T}(\mathbf{u},\Omega),\mathbb{R}^n)$ and $D\mathbf{u}^{-1}(\mathbf{y}) = D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1}$ for a.e. $\mathbf{y} \in \mathrm{im_T}(\mathbf{u},\Omega)$.*

Proof. The measurability of $\mathbf{u}^{-1}$ is a consequence of Proposition 2.1. By Definition 9.2, $\mathbf{u}^{-1}(\mathbf{u}(\Omega_1)) = \Omega_1$, so $\mathbf{u}^{-1} \in L^\infty(\operatorname{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$.

In order to calculate the distributional derivative of $\mathbf{u}^{-1}$, we let $\mathbf{G} \in C^1_c(\operatorname{im_T}(\mathbf{u}, \Omega), \mathbb{R}^{n\times n})$. Call the rows of $\mathbf{G}$ by $\mathbf{g}_1, \ldots, \mathbf{g}_n$. As $\operatorname{im_T}(\mathbf{u}, \Omega) = \mathbf{u}(\Omega_1)$ a.e., we can apply Proposition 2.1, as well as the definition of $\mathcal{E}$, to obtain

$$\int_{\operatorname{im_T}(\mathbf{u},\Omega)} \mathbf{u}^{-1}(\mathbf{y}) \cdot \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\mathbf{u}(\Omega_1)} \mathbf{u}^{-1}(\mathbf{y}) \cdot \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_\Omega \mathbf{x} \cdot \operatorname{div} \mathbf{G}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$
$$= \sum_{i=1}^n \int_\Omega x_i \operatorname{div} \mathbf{g}_i(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \sum_{i=1}^n \left[\mathcal{E}(\mathbf{u}, x_i \, \mathbf{g}_i) - \int_\Omega \operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot (\mathbf{g}_i(\mathbf{u}(\mathbf{x})) \otimes \mathbf{e}_i) \, \mathrm{d}\mathbf{x} \right].$$

Of course, for each $i \in \{1, \ldots, n\}$ we have denoted by x_i the map $\mathbf{x} \mapsto x_i$. As $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$, we have that $\mathcal{E}(\mathbf{u}, x_i \, \mathbf{g}_i) = \mathcal{F}(\mathbf{u}, x_i \, \mathbf{g}_i)$. Now, $\operatorname{supp} \mathbf{G} \subset \operatorname{im_T}(\mathbf{u}, \Omega)$ and $\operatorname{im_T}(\mathbf{u}, \Omega) \cap \mathbf{u}(\partial\Omega) = \varnothing$, so $\operatorname{supp} \mathbf{G} \cap \mathbf{u}(\partial\Omega) = \varnothing$. Consequently, $\mathbf{G}(\mathbf{u}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in \partial\Omega$ and, hence, $\mathcal{F}(\mathbf{u}, x_i \, \mathbf{g}_i) = 0$ for each $i \in \{1, \ldots, n\}$. On the other hand, changing variables again and using elementary matrix properties, we find that

$$\sum_{i=1}^n \int_\Omega \operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot (\mathbf{g}_i(\mathbf{u}(\mathbf{x})) \otimes \mathbf{e}_i) \, \mathrm{d}\mathbf{x} = \int_\Omega \operatorname{adj} D\mathbf{u}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}$$
$$= \int_{\operatorname{im_G}(\mathbf{u},\Omega)} \frac{\operatorname{adj} D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))}{\det D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))} \cdot \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \int_{\operatorname{im_T}(\mathbf{u},\Omega)} D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1} \cdot \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Altogether,

$$\int_{\operatorname{im_T}(\mathbf{u},\Omega)} \mathbf{u}^{-1}(\mathbf{y}) \cdot \operatorname{div} \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = -\int_{\operatorname{im_T}(\mathbf{u},\Omega)} D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1} \cdot \mathbf{G}(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

which shows that

$$D\mathbf{u}^{-1}(\mathbf{y}) = D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))^{-1} = \frac{\operatorname{adj} D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))}{\det D\mathbf{u}(\mathbf{u}^{-1}(\mathbf{y}))}, \qquad \text{a.e. } \mathbf{y} \in \operatorname{im_T}(\mathbf{u}, \Omega).$$

Using Proposition 2.1 again, we get

$$\int_{\operatorname{im_T}(\mathbf{u},\Omega)} \left| D\mathbf{u}^{-1}(\mathbf{y}) \right| \mathrm{d}\mathbf{y} = \int_\Omega \left| \operatorname{adj} D\mathbf{u}(\mathbf{x}) \right| \mathrm{d}\mathbf{x} < \infty,$$

so $\mathbf{u}^{-1} \in W^{1,1}(\operatorname{im_T}(\mathbf{u}, \Omega), \mathbb{R}^n)$, which concludes the proof. □

In this paper we have not used Müller and Spector's [35] condition INV of invertibility. We just mention that, as a consequence of [6, Lemma 5.1], functions in $\mathcal{A}_p(\Omega)$ that are injective a.e. automatically satisfy INV. In particular, the function $\mathbf{u}$ of Theorem 9.1 satisfies INV.

In order to apply Theorem 9.1, it is useful to know a sufficient condition for which $\operatorname{im_T}(\mathbf{u}_0, \Omega) = \mathbf{u}_0(\Omega)$. The following result is well known, but the usual proof (see, e.g., [16, Th. 3.35]) assumes that $\mathbf{u}$ is a homeomorphism from $\bar{\Omega}$, so we provide a proof in the case (as in [3, Th. 1]) in which $\mathbf{u}$ is only a homeomorphism from Ω: this allows for *self-contact at the boundary*; see [8].

Proposition 9.4. *Let* $\mathbf{u} \in C(\bar{\Omega}, \mathbb{R}^n)$ *be such that* $\mathbf{u}|_\Omega$ *is injective. Then* $|\deg(\mathbf{u}, \Omega, \cdot)| = \chi_{\mathbf{u}(\Omega)}$ *and* $\operatorname{im_T}(\mathbf{u}, \Omega) = \mathbf{u}(\Omega)$.

Proof. Recall from the invariance of domain theorem that $\mathbf{u}|_\Omega$ is an open map. We first show that $\mathbf{u}(\partial\Omega) \cap \mathbf{u}(\Omega) = \varnothing$. Indeed, assume, for a contradiction, that there exists $\mathbf{y} \in \mathbf{u}(\partial\Omega) \cap \mathbf{u}(\Omega)$. As $\mathbf{y} \in \mathbf{u}(\Omega)$, there exists an open $U \subset\subset \Omega$ such that $\mathbf{y} \in \mathbf{u}(U)$. Since $\mathbf{y} \in \mathbf{u}(\partial\Omega)$, there also exists $\mathbf{x} \in \partial\Omega$ such that $\mathbf{y} = \mathbf{u}(\mathbf{x})$. Let $\{\mathbf{x}_j\}_{j\in\mathbb{N}}$ be a sequence in Ω converging to $\mathbf{x}$. Then $\mathbf{u}(\mathbf{x}_j) \to \mathbf{y}$ as $j \to \infty$. On the one hand, as $\mathbf{u}(U)$ is open and $\mathbf{y} \in \mathbf{u}(U)$, for j large we have that $\mathbf{u}(\mathbf{x}_j) \in \mathbf{u}(U)$. On the other hand, since $\mathbf{x}_j \to \mathbf{x}$ with $\mathbf{x} \in \partial\Omega$

and $\bar{U} \cap \partial\Omega = \varnothing$, we have that $\mathbf{x}_j \notin U$ for large j. The facts $\mathbf{u}(\mathbf{x}_j) \in \mathbf{u}(U)$ and $\mathbf{x}_j \in \Omega \setminus U$ contradict the injectivity of $\mathbf{u}|_\Omega$. We thus conclude that $\mathbf{u}(\partial\Omega) \cap \mathbf{u}(\Omega) = \varnothing$.

As a consequence of the existence property of the degree (see, e.g., [11, Th. 3.1(d4)]), if $\chi_{\mathbf{u}(\Omega)}(\mathbf{y}) = 0$ for some $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial\Omega)$ then $\deg(\mathbf{u}, \Omega, \mathbf{y}) = 0$; therefore, $\operatorname{im_T}(\mathbf{u}, \Omega) \subset \mathbf{u}(\Omega)$. For the other inclusion, assume $\mathbf{y} \in \mathbf{u}(\Omega)$. As shown above, $\mathbf{y} \notin \mathbf{u}(\partial\Omega)$. There exists an open set $U \subset\subset \Omega$ such that $\mathbf{y} \in \mathbf{u}(U)$. By the injectivity of $\mathbf{u}|_\Omega$, we have that $\mathbf{y} \notin \mathbf{u}(\Omega \setminus U)$, so $\mathbf{y} \notin \mathbf{u}(\bar{\Omega} \setminus U)$. By the excision property of the degree (see, e.g., [11, Th. 3.1(d7)]), $\deg(\mathbf{u}, U, \mathbf{y}) = \deg(\mathbf{u}, \Omega, \mathbf{y})$. Now, $\mathbf{u}|_{\bar{U}}$ is a homeomorphism onto its image, so, as a consequence of the product formula for the degree (see, e.g., [16, Th. 3.35]), $|\deg(\mathbf{u}, U, \mathbf{y})| = 1$. Therefore, $|\deg(\mathbf{u}, \Omega, \mathbf{y})| = |\deg(\mathbf{u}, U, \mathbf{y})| = 1$ and, hence, $\mathbf{y} \in \operatorname{im_T}(\mathbf{u}, \Omega)$. □

We finish this section with a variant of Theorem 9.5. The difference relies that in the next result, the boundary datum $\mathbf{u}_0$ is not assumed to be in $\bar{\mathcal{A}}_p(\Omega)$ or satisfy $\det D\mathbf{u}_0 \geq 0$ a.e., but instead $\mathbf{u}_0 \in C(\bar{\Omega}, \mathbb{R}^n)$ and $\mathbf{u}_0|_\Omega$ is injective.

Theorem 9.5. *Let $p > n-1$. Let $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ and $\mathbf{u}_0 \in C(\bar{\Omega}, \mathbb{R}^n)$ satisfy $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$, $\det D\mathbf{u} > 0$ a.e., and $\mathbf{u}_0|_\Omega$ is injective. Then $\mathbf{u}$ is injective a.e. and $\operatorname{im_G}(\mathbf{u}, \Omega) = \mathbf{u}_0(\Omega)$ a.e.*

Proof. According to Proposition 8.4, $\deg(\mathbf{u}, \Omega, \cdot) = \mathcal{N}_{\mathbf{u}}$ a.e. and $\operatorname{im_G}(\mathbf{u}, \Omega) = \operatorname{im_T}(\mathbf{u}, \Omega)$ a.e. As the degree only depends on the boundary values, $\deg(\mathbf{u}, \Omega, \cdot) = \deg(\mathbf{u}_0, \Omega, \cdot)$ and $\operatorname{im_T}(\mathbf{u}, \Omega) = \operatorname{im_T}(\mathbf{u}_0, \Omega)$. By Proposition 9.4, $|\deg(\mathbf{u}_0, \Omega, \cdot)| = \chi_{\mathbf{u}_0(\Omega)}$ and $\operatorname{im_T}(\mathbf{u}_0, \Omega) = \mathbf{u}_0(\Omega)$. This string of equalities show that $\mathcal{N}_{\mathbf{u}} = \chi_{\mathbf{u}_0(\Omega)}$ a.e. and $\operatorname{im_G}(\mathbf{u}, \Omega) = \mathbf{u}_0(\Omega)$ a.e.

From the equality $\mathcal{N}_{\mathbf{u}} = \chi_{\operatorname{im_G}(\mathbf{u}, \Omega)}$ a.e. we infer that there exists a set $N \subset \mathbb{R}^n$ of zero measure such that for all $\mathbf{y} \in \operatorname{im_G}(\mathbf{u}, \Omega) \setminus N$ we have $\mathcal{N}_{\mathbf{u}}(\mathbf{y}) = 1$, which shows that $\mathbf{u}$ is injective in $\{\mathbf{x} \in \Omega_0 : \mathbf{u}(\mathbf{x}) \notin N\}$, where Ω_0 is the set of Definition 8.3. By Proposition 2.1 and the fact $\det D\mathbf{u} > 0$ a.e., we have that the set $\{\mathbf{x} \in \Omega_0 : \mathbf{u}(\mathbf{x}) \in N\}$ has measure zero. This concludes that $\mathbf{u}$ is injective a.e. □

10 Existence of minimizers

In this final section, we show how the class $\bar{\mathcal{A}}_p(\Omega)$ is suitable for proving existence of minimizers in nonlinear elasticity.

We first show the weak continuity of $\Lambda_{n-1} D\mathbf{u}\,\boldsymbol{\nu}$ in $W^{1,p}(\partial\Omega, \mathbb{R}^n)$. This result has been proved in [7, Prop. 15], but we provide a self-contained proof in our context.

Proposition 10.1. *Let $p > n-1$. For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$, and assume that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}(\partial\Omega, \mathbb{R}^n)$ as $j \to \infty$. Then $\Lambda_{n-1} D\mathbf{u}_j\,\boldsymbol{\nu} \rightharpoonup \Lambda_{n-1} D\mathbf{u}\,\boldsymbol{\nu}$ in $L^{\frac{p}{n-1}}(\partial\Omega, \mathbb{R}^n)$ as $j \to \infty$.*

Proof. By Lemma 3.3, $\mathcal{L}_i(\mathbf{u}_j) \rightharpoonup \mathcal{L}_i(\mathbf{u})$ in $W^{1,p}((0,r)^{n-1})$ for each $i \in \{1, \ldots, m\}$. Fix any such i. For each $k \in \{1, \ldots, n\}$, denote by M_k the number $(-1)^{k+n}$ times the $n-1$ minor of an $n \times (n-1)$ matrix obtained by deleting the row k. A standard property on the weak continuity of the minors (see, e.g., [10, Th. 8.20]) shows that $M_k(D(\mathcal{L}_i(\mathbf{u}_j))) \rightharpoonup M_k(D(\mathcal{L}_i(\mathbf{u})))$ in $L^{\frac{p}{n-1}}((0,r)^{n-1})$. Now fix any $\beta > 0$ and consider the map $\mathcal{L}_i(\mathbf{u}_j) \circ \pi : (0,r)^{n-1} \times (0, \beta) \to \mathbb{R}^n$, which satisfies $D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi)(\mathbf{z}) = D(\mathcal{L}_i(\mathbf{u}_j))(\hat{\mathbf{z}}) \circ \pi$ and

$$\operatorname{cof} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi)(\mathbf{z}) = \begin{pmatrix} 0 & \cdots & 0 & M_1(D(\mathcal{L}_i(\mathbf{u}_j))(\hat{\mathbf{z}})) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & M_n(D(\mathcal{L}_i(\mathbf{u}_j))(\hat{\mathbf{z}})) \end{pmatrix}.$$

Therefore, $\operatorname{cof} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi) \rightharpoonup \operatorname{cof} D(\mathcal{L}_i(\mathbf{u}) \circ \pi)$ in $L^{\frac{p}{n-1}}((0,r)^{n-1} \times (0,\beta))$ as $j \to \infty$. Moreover, by (2.2), we have that

$$\Lambda_{n-1} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi|_{(0,r)^{n-1} \times \{0\}})(\hat{\mathbf{z}}, 0)\, \mathbf{e}_n = \operatorname{cof} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi)(\hat{\mathbf{z}}, 0)\, \mathbf{e}_n,$$

so

$$\Lambda_{n-1} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi|_{(0,r)^{n-1} \times \{0\}})\, \mathbf{e}_n \rightharpoonup \Lambda_{n-1} D(\mathcal{L}_i(\mathbf{u}) \circ \pi|_{(0,r)^{n-1} \times \{0\}})\, \mathbf{e}_n \quad \text{in } L^{\frac{p}{n-1}}((0,r)^{n-1} \times \{0\}) \qquad (10.1)$$

as $j \to \infty$.

In order to show $\Lambda_{n-1} D\mathbf{u}_j \boldsymbol{\nu} \rightharpoonup \Lambda_{n-1} D\mathbf{u} \boldsymbol{\nu}$ in $L^{\frac{p}{n-1}}(\partial\Omega, \mathbb{R}^n)$ as $j \to \infty$ it suffices to check that the convergence holds in $L^{\frac{p}{n-1}}(\Gamma_i, \mathbb{R}^n)$ for each $i \in \{1, \dots, m\}$, so fix any such i and consider any $\boldsymbol{\psi} \in L^q(\Gamma_i, \mathbb{R}^n)$, where q is the conjugate exponent of $\frac{p}{n-1}$. Performing the change of variables given by $\mathbf{G}_i^{-1} : \Gamma_i \to (0,r)^{n-1} \times \{0\}$, in a similar way as in (6.8)–(6.9), we have that

$$\begin{aligned}
&\int_{\Gamma_i} \boldsymbol{\psi}(\mathbf{x}) \cdot (\Lambda_{n-1} D\mathbf{u}_j(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \\
&= \int_{(0,r)^{n-1}\times\{0\}} (\boldsymbol{\psi} \circ \mathbf{G}_i)(\mathbf{z}) \cdot \left(\Lambda_{n-1} D \left(\mathbf{u}_j \circ \mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}} \right)(\mathbf{z})(-\mathbf{e}_n) \right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).
\end{aligned} \tag{10.2}$$

Now, $\boldsymbol{\psi} \circ \mathbf{G}_i \in L^q((0,r)^{n-1} \times \{0\})$, since $\mathbf{G}_i$ is bi-Lipschitz. In addition, $\mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}} = \mathbf{G}_i \circ \eta \circ \pi|_{(0,r)^{n-1}\times\{0\}}$, so $\mathbf{u}_j \circ \mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}} = \mathcal{L}_i(\mathbf{u}_j) \circ \pi|_{(0,r)^{n-1}\times\{0\}}$, and, hence, for $\mathcal{H}^{n-1}$-a.e. $\mathbf{z} \in (0,r)^{n-1} \times \{0\}$,

$$\Lambda_{n-1} D \left(\mathbf{u}_j \circ \mathbf{G}_i|_{(0,r)^{n-1}\times\{0\}} \right)(\mathbf{z})(-\mathbf{e}_n) = \Lambda_{n-1} D(\mathcal{L}_i(\mathbf{u}_j) \circ \pi|_{(0,r)^{n-1}\times\{0\}})(\mathbf{z})\,(-\mathbf{e}_n). \tag{10.3}$$

Now we consider (10.2) and (10.3), and realize that the same formulas hold when $\mathbf{u}$ replaces $\mathbf{u}_j$. With (10.1) we conclude that

$$\lim_{j\to\infty} \int_{\Gamma_i} \boldsymbol{\psi}(\mathbf{x}) \cdot (\Lambda_{n-1} D\mathbf{u}_j(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\Gamma_i} \boldsymbol{\psi}(\mathbf{x}) \cdot (\Lambda_{n-1} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}).$$

This shows that $\Lambda_{n-1} D\mathbf{u}_j \boldsymbol{\nu} \rightharpoonup \Lambda_{n-1} D\mathbf{u} \boldsymbol{\nu}$ in $L^{\frac{p}{n-1}}(\Gamma_i, \mathbb{R}^n)$ as $j \to \infty$ and concludes the proof. □

We present now a compactness result for the class $\bar{\mathcal{A}}_p(\Omega)$, which is a consequence of the main result of [23] (see also [6, Prop. 6.1]).

Proposition 10.2. *Let $p > n-1$. Let $\{\mathbf{u}_j\}_{j\in\mathbb{N}} \subset \bar{\mathcal{A}}_p(\Omega)$ be such that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n)$ and $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ is equiintegrable. Then there exists $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ such that, for a subsequence,*

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \cap W^{1,p}(\partial\Omega, \mathbb{R}^n) \quad \text{and} \quad \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \text{ in } L^1(\Omega)$$

as $j \to \infty$.

Proof. For a subsequence (not relabelled), we have that there exist $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$, $\mathbf{v} \in W^{1,p}(\partial\Omega, \mathbb{R}^n)$ and $\theta \in L^1(\Omega)$ such that

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^n), \quad \mathbf{u}_j \rightharpoonup \mathbf{v} \text{ in } W^{1,p}(\partial\Omega, \mathbb{R}^n) \quad \text{and} \quad \det D\mathbf{u}_j \rightharpoonup \theta \text{ in } L^1(\Omega).$$

as $j \to \infty$. Taking a further subsequence, we have that $\mathbf{u}_j \to \mathbf{u}$ a.e., while by the weak continuity of the cofactors (see, e.g., [10, Th. 8.20]) we have that $\operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u}$ in $L^1(\Omega, \mathbb{R}^{n\times n})$. By the continuity of the traces, $\mathbf{u}_j|_{\partial\Omega} \rightharpoonup \mathbf{u}|_{\partial\Omega}$ in $L^p(\partial\Omega, \mathbb{R}^n)$, hence $\mathbf{v} = \mathbf{u}|_{\partial\Omega}$. Thanks to [23, Th. 3], we have $\theta = \det D\mathbf{u}$ a.e., since $\bar{\mathcal{A}}_p(\Omega)$ functions satisfy (5.2). Now let $\mathbf{f} \in C_c^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$. The above convergences imply by a standard result (see, e.g., [18, Prop. 2.61]),

$$\lim_{j\to\infty} \mathcal{E}(\mathbf{u}_j, \mathbf{f}) = \mathcal{E}(\mathbf{u}, \mathbf{f}).$$

On the other hand, by Proposition 10.1, $\Lambda_{n-1} D\mathbf{u}_j \boldsymbol{\nu} \rightharpoonup \Lambda_{n-1} D\mathbf{u} \boldsymbol{\nu}$ in $L^{\frac{p}{n-1}}(\partial\Omega, \mathbb{R}^n)$. Moreover, we can assume, by taking a subsequence, that $\mathbf{u}_j \to \mathbf{u}$ $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$. Again, a standard convergence result shows that

$$\lim_{j\to\infty} \mathcal{F}(\mathbf{u}_j, \mathbf{f}) = \mathcal{F}(\mathbf{u}, \mathbf{f}).$$

Since $\mathcal{E}(\mathbf{u}_j, \mathbf{f}) = \mathcal{F}(\mathbf{u}_j, \mathbf{f})$, we conclude that $\mathcal{E}(\mathbf{u}, \mathbf{f}) = \mathcal{F}(\mathbf{u}, \mathbf{f})$ and, consequently, $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$. □

Proposition 10.2 will be used in Theorem 10.3 with more restrictive assumptions; namely, when $\mathbf{u}_j|_{\partial\Omega}$ is the same for all $j \in \mathbb{N}$. We have, nevertheless, proved Proposition 10.2 in its generality because in this way it shows the compactness of $\bar{\mathcal{A}}_p(\Omega)$ in the relevant topology.

We have all the ingredients to prove the existence of minimizers in $\bar{\mathcal{A}}_p(\Omega)$ for functionals of the form

$$I(\mathbf{u}) = \int_\Omega W(\mathbf{x}, \mathbf{u}(\mathbf{x}), D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x} \tag{10.4}$$

under the key assumption of polyconvexity of W in the last variable. This kind of functionals appear naturally in the theory of nonlinear elasticity, and much of the work in this area in the last decades has been focused on proving the existence of minimizers (see, e.g., [2, 33, 20, 21, 34, 37]). In this context, $\mathbf{u}$ represents the deformation of the body, which occupies Ω in its reference configuration. The term W accounts for the elastic energy (which typically depends only on $\mathbf{x}$ and $D\mathbf{u}$) and external forces (which typically depend only on $\mathbf{x}$ and $\mathbf{u}$).

We recall the concept of polyconvexity (see, e.g., [10]). We denote by $\mathbb{R}^{n\times n}_+$ the set of $n \times n$ matrices with positive determinant. Let τ be the number of minors of an $n \times n$ matrix; we call $\mathbb{R}^\tau_+ := \mathbb{R}^{\tau-1} \times (0, \infty)$ and denote by $M(F) \in \mathbb{R}^\tau$ the collection of all the minors of an $F \in \mathbb{R}^{n\times n}$ in a given order such that its last component is $\det F$. A function $W_0 : \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ is *polyconvex* if there exists a convex function $\Phi : \mathbb{R}^\tau_+ \to \mathbb{R}$ such that $W_0(F) = \Phi(M(F))$ for all $F \in \mathbb{R}^{n\times n}_+$.

The existence result is as follows. Thanks to Proposition 10.2, its proof is by now standard.

Theorem 10.3. *Let $p > n - 1$. Let $\mathbf{u}_0 \in \bar{\mathcal{A}}_p(\Omega)$ be injective a.e. and satisfy $\det D\mathbf{u}_0 \geq 0$ a.e. Let $W : \Omega \times \mathrm{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega) \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable, where $\mathcal{B}^n$ and $\mathcal{B}^{n\times n}$ and denote the Borel sigma-algebras in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively.

b) $W(\mathbf{x}, \cdot, \cdot)$ is lower semicontinuous for a.e. $\mathbf{x} \in \Omega$.

c) For a.e. $\mathbf{x} \in \Omega$ and every $\mathbf{y} \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega)$, the function $W(\mathbf{x}, \mathbf{y}, \cdot)$ is polyconvex.

d) There exist a constant $c > 0$, an $a \in L^1(\Omega)$ and a Borel function $h : (0, \infty) \to [0, \infty)$ such that

$$\lim_{t\searrow 0} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty$$

and

$$W(\mathbf{x}, \mathbf{y}, \mathbf{F}) \geq a(\mathbf{x}) + c\,|\mathbf{F}|^p + h(\det \mathbf{F}) \qquad \text{for a.e. } \mathbf{x} \in \Omega, \text{ all } \mathbf{y} \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega) \text{ and all } \mathbf{F} \in \mathbb{R}^{n\times n}_+.$$

Let $\mathcal{A}$ be the set of $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ such that $\det D\mathbf{u} > 0$ a.e. and $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$. Suppose $\mathcal{A} \neq \varnothing$. Define I as in (10.4) and assume that I is not identically infinity in $\mathcal{A}$. Then there exists a minimizer of I in $\mathcal{A}$, and any element of $\mathcal{A}$ is injective a.e.

Proof. The fact that any element of $\mathcal{A}$ is injective a.e. is a consequence of Theorem 9.1.

Let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of I in $\mathcal{A}$. By Proposition 8.4 and Theorem 9.1, $\mathbf{u}_j(\mathbf{x}) \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega)$ for a.e. $\mathbf{x} \in \Omega$ and all $j \in \mathbb{N}$. Assumption *d)* and De la Vallée-Poussin's criterion imply that $\{D\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$ and $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ is equiintegrable. By the L^∞ bound on $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ given by $\mathrm{im}_{\mathrm{T}}(\mathbf{u}_0, \Omega)$, we have that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$. By Proposition 10.2, there exists $\mathbf{u} \in \bar{\mathcal{A}}_p(\Omega)$ such that for a subsequence (not relabelled),

$$\mathbf{u}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^n) \quad \text{and} \quad \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \text{ in } L^1(\Omega) \quad \text{as } j \to \infty.$$

Clearly, $\det D\mathbf{u} \geq 0$ a.e. If $\det D\mathbf{u}$ were zero in a set A of positive measure, then we would have (for a subsequence) $\det D\mathbf{u}_j \to 0$ a.e. in A as $j \to \infty$; by *d)*, we would obtain $h(\det D\mathbf{u}_j) \to \infty$ a.e. in A, so, by Fatou's lemma and *d)* again, we would get $I(\mathbf{u}_j) \to \infty$, which is a contradiction. Therefore, $\det D\mathbf{u} > 0$ a.e. Moreover, the boundary condition is also preserved under the limit, so $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$ and, hence, $\mathbf{u} \in \mathcal{A}$.

A standard lower semicontinuity result for polyconvex functionals (see, e.g., [5, Th. 5.4] or [18, Th. 7.5]) shows that $I(\mathbf{u}) \leq \liminf_{j\to\infty} I(\mathbf{u}_j)$. Therefore, $\mathbf{u}$ is a minimizer of I in $\mathcal{A}$ and the proof is concluded. ☐

According to Theorem 9.5 and Proposition 9.4, in Theorem 10.3 the assumption that $\mathbf{u}_0 \in \bar{\mathcal{A}}_p(\Omega)$ is injective a.e. and satisfies $\det D\mathbf{u}_0 \geq 0$ a.e. can be replaced by $\mathbf{u}_0 \in C(\bar{\Omega}, \mathbb{R}^n)$ with $\mathbf{u}_0|_\Omega$ injective.

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