ORBIFOLDS WITH SIGNATURE $(0; k, k^{n-1}, k^n, k^n)$

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Abstract. Two interesting problems that arise in the theory of closed Riemann surfaces are the following: (i) the computation of algebraic curves representing the surface, and (ii) to decide if the field of moduli is a field of definition.

In this paper we consider pairs (S, H), where S is a closed Riemann surface and H is a subgroup of $\mathrm{Aut}(S)$, the group of automorphisms of S, so that S/H is an orbifold with signature $(0; k, k^{n-1}, k^n, k^n)$ where k, $n \geq 2$ are integers.

In the case that S is the highest Abelian branched cover of S/H we provide explicit algebraic curves representing S. In the case that k is an odd prime, we also describe algebraic curves for some intermediate Abelian covers.

For $k = p \geq 3$ a prime and H a p-group, we prove that H is a p-Sylow subgroup of $\mathrm{Aut}(S)$, and if $p \geq 7$ we prove that H is normal in $\mathrm{Aut}(S)$. Also, when $n \neq 3$ we prove that the field of moduli in such cases is a field of definition. If, moreover, S is the highest Abelian branched cover of S/H, then we compute explicitly the field of moduli.

1. Introduction

A closed Riemann surface S of genus $g \geq 2$ may be described by many different objects; for instance, by algebraic curves (by the Riemann-Roch theorem [8]), by torsion free co-compact Fuchsian groups (by the Koebe-Poincaré uniformization theorem [16, 17, 20]), by Schottky groups (by the retrosection theorem [2, 17]), or by certain principally polarized Abelian varieties (by the Torelli theorem [23, 24]). In general, to provide different explicit representations for the same Riemann surface has been a difficult problem, in spite of huge efforts to solve it. It seems that Burnside [3] provided the first example of an algebraic curve and a Fuchsian group, both representing the same Riemann surface. In many cases, the group $\mathrm{Aut}(S)$ of automorphisms of S and its subgroups play a fundamental role to find algebraic curves representing S. For instance, if $S/\mathrm{Aut}(S)$ has signature of the form $(0; r, s, t)$, then in general it is not difficult to provide an explicit Fuchsian group and an explicit algebraic curve, both of them representing S.

A field of definition of S is a subfield $\mathbb{K}$ of $\mathbb{C}$ for which it is possible to find an irreducible non-singular projective algebraic curve representing S, defined by polynomials whose coefficients belongs to $\mathbb{K}$. If C is a algebraic curve describing S, then the field of moduli of S is defined as the fixed field of the group of field automorphisms σ of $\mathbb{C}$ such that C and C^σ are isomorphic, where C^σ is the algebraic curve defined as the zeroes of the polynomials obtained from the ones defining C after σ acts on their coefficients. The field of moduli is always contained in any field of definition, but it may happen that the field of moduli is not a field of definition.

In this article we study closed Riemann surfaces S admitting subgroups $H < \mathrm{Aut}(S)$ so that S/H has signature $(0; k, k^{n-1}, k^n, k^n)$, where n, $k \geq 2$ are integers. For $k = 2$ in [4, 9] these type of surfaces were considered to give examples of closed Riemann admitting topologically equivalent

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2000 *Mathematics Subject Classification.* 30F10, 30F40, 14H37.

The first and third author were supported by Fondecyt grants 1095165 and 1100767. The second author was supported by Fondecyt grant 1110001 and UTFSM grant 12.11.01.

but conformally non-equivalent cyclic groups of order 2^n.

In the general case, if S is the homology cover of S/H, then we compute the field of moduli and we give explicit algebraic curves for S. These explicit algebraic curves for homology covers allow us to find algebraic curves for those Riemann surfaces S admitting an Abelian group $G < \mathrm{Aut}(S)$ such that S/G has signature $(0; k, k^{n-1}, k^n, k^n)$. We describe such a situation for the case that k is a prime and $G \cong \mathbb{Z}_k \times \mathbb{Z}_{k^n}$. Also, for k an odd prime, we describe the group $\mathrm{Aut}(S)$ and we prove that the field of moduli of S is in fact a field of definition.

2. Preliminaries

2.1. **Orbifolds.** An *orbifold* is a tuple $\mathcal{O} = (S, \{(p_1, k_1), \dots, (p_n, k_n), \dots\})$ where (i) S is a Riemann surface, called the *Riemann surface structure* of $\mathcal{O}$, (ii) $p_1, p_2, \dots \in S$ is a collection of different isolated points, called the *cone points* of $\mathcal{O}$, and (iii) each $k_j \geq 2$ is an integer, called the *cone order* of p_j. An *orbifold* of signature $(\gamma; k_1, \dots, k_n)$ is given by an orbifold $\mathcal{O} = (S, \{(p_1, k_1), ..., (p_n, k_n)\})$ where S is a closed Riemann surface of genus γ. An orbifold without cone points is just a Riemann surface.

A *conformal homeomorphism* between two orbifolds is a conformal homeomorphism between the corresponding Riemann surface structures, sending cone points to cone points, and preserving the cone point orders. If both orbifolds are the same, then we speak about a *conformal automorphism* of the orbifold. We use the notation $\mathcal{O}_1 \cong \mathcal{O}_2$ to indicate that $\mathcal{O}_1$ and $\mathcal{O}_2$ are conformally equivalent orbifolds.

We denote by $\mathrm{Aut}_{\mathrm{orb}}(\mathcal{O})$ the group of conformal automorphisms of $\mathcal{O}$. If S is the conformal Riemann surface structure of $\mathcal{O}$, then we denote by $\mathrm{Aut}(S)$ its group of conformal automorphisms. There is a natural inclusion $\mathrm{Aut}_{\mathrm{orb}}(\mathcal{O}) < \mathrm{Aut}(S)$, but in general these two groups are different.

If $\mathcal{O}$ is an orbifold and $H < \mathrm{Aut}_{\mathrm{orb}}(\mathcal{O})$ acts discontinuously on the Riemann surface structure, then the quotient $\mathcal{O}/H$ may be seen again as an orbifold as follows. We denote by $\pi : \mathcal{O} \to \mathcal{O}/H$ the canonical quotient map. A cone point of $\mathcal{O}/H$ may be obtained in two different ways. In the first case, if $p \in \mathcal{O}$ is not a cone point and it has non-trivial H-stabilizer $H(p)$, then $\pi(p)$ is a cone point with order equal to the order of $H(p)$. In the second case, if $p \in \mathcal{O}$ is a cone point of order n and its H-stabilizer has order m, then $\pi(p)$ is a cone point with order equal to nm.

The orbifolds we consider in this paper are the *good orbifolds* in Thurston's terminology; they are obtained as quotient spaces R/F, where R is a Riemann surface and $F < \mathrm{Aut}(R)$ is a discontinuous group of conformal automorphisms of R. From now on we will identify R/F with $\mathcal{O}$ in order to simplify the notations; we will say that R/F is an orbifold.

2.2. **Homology covers.** Good orbifolds admit as (branched) universal cover either the Riemann sphere, the complex plane or the hyperbolic plane; this is a consequence of the classical uniformization theorem.

Consider a good orbifold $\mathcal{O} = (S, \{(p_1, k_1), ..., (p_n, k_n)\})$ of signature $(\gamma; k_1, ..., k_n)$.

The first (orbifold) fundamental group of $\mathcal{O}$ is

$$(1) \qquad \pi_1^{\mathrm{orb}}(\mathcal{O}) = \left\langle \alpha_1, \dots, \alpha_\gamma, \beta_1, \dots, \beta_\gamma, \delta_1, \dots, \delta_n : \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] \prod_{k=1}^{n} \delta_k = \delta_1^{k_1} = \cdots = \delta_n^{k_n} = 1 \right\rangle,$$

where $\pi_1(S) = \left\langle \alpha_1, \dots, \alpha_\gamma, \beta_1, \dots, \beta_\gamma : \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] = 1 \right\rangle$, with $[a, b] = aba^{-1}b^{-1}$, and the element δ_j represents a simple small loop around p_j in $S - \{p_1, \dots, p_n\}$, for each $j = 1, \dots, n$.

It is clear that to each normal subgroup N of finite index of $\pi_1^{\text{orb}}(\mathcal{O})$ there corresponds an orbifold $\widetilde{\mathcal{O}}$ and a finite group $H < \text{Aut}_{\text{orb}}(\widetilde{\mathcal{O}})$, so that $\mathcal{O} = \widetilde{\mathcal{O}}/H$. Observe that H is isomorphic to $\pi_1^{\text{orb}}(\mathcal{O})/N$.

When $N = \pi_1^{\text{orb}}(\mathcal{O})'$ (the derived subgroup of $\pi_1^{\text{orb}}(\mathcal{O})$), the corresponding cover orbifold $\widetilde{\mathcal{O}}$ is called the *homology orbifold cover* of $\mathcal{O}$. We will be interested only in the particular case when the homology orbifold cover is a closed Riemann surface, in which case we call it the *homology cover* of $\mathcal{O}$, and say that $\mathcal{O}$ is a *homology orbifold*.

Clearly, the homology orbifold cover of $\mathcal{O}$ is the homology cover if and only if $\pi_1^{\text{orb}}(\mathcal{O})'$ has finite index in $\pi_1^{\text{orb}}(\mathcal{O})$ and it acts freely on the universal cover space of $\mathcal{O}$. The finite index condition is equivalent to the condition that the underlying Riemann surface structure of $\mathcal{O}$ is the Riemann sphere; that is, to have $\gamma = 0$, and the free action condition is equivalent to the following (see [19])

$$\text{lcm}(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_n) = \text{lcm}(k_1, \ldots, k_n), \ \forall\, j = 1, \ldots, n. \tag{2}$$

Note that the homology cover (when it exists) is the highest abelian Galois cover of $\mathcal{O}$.

2.3. **Fuchsian groups.** The basic theory of Fuchsian groups may be found, for instance, in the classical book by Beardon [1]. A *co-compact Fuchsian group* acting on the upper half-plane $\mathbb{H}^2$ is a discrete group $\Gamma < \text{PSL}(2, \mathbb{R})$ such that $\mathbb{H}^2/\Gamma$ is an orbifold of some signature; that is, the underlying Riemann surface is a closed Riemann surface. It is known that a co-compact Fuchsian group Γ has a presentation of the form

$$\Gamma = \left\langle a_1, b_1, \ldots, a_\gamma, b_\gamma, \delta_1, \ldots, \delta_n : \prod_{j=1}^{\gamma}[a_j, b_j] \prod_{j=1}^{n} \delta_j = \delta_1^{k_1} = \ldots = \delta_n^{k_n} = 1 \right\rangle, \tag{3}$$

where γ and n are non-negative integers, the $k_j \geq 2$ are integers, and $2\gamma - 2 + n - \sum_{j=1}^{n} k_j^{-1} > 0$. The tuple $(\gamma; k_1, \ldots, k_n)$ is known as the *signature* of Γ (this is the signature of its quotient orbifold $\mathbb{H}^2/\Gamma$).

An orbifold $\mathcal{O}$ is *of hyperbolic type* if there is a co-compact Fuchsian group Γ so that $\mathcal{O} \cong \mathbb{H}^2/\Gamma$. By the Poincaré-Koebe uniformization theorem [16, 17, 20], every orbifold with signature $(\gamma; k_1, \ldots, k_n)$ is of hyperbolic type if and only if $2\gamma - 2 + n - \sum_{j=1}^{n} k_j^{-1} > 0$.

By the hyperbolic area of a Fuchsian group Γ (respectively, of a hyperbolic orbifold) of signature $(\gamma, n; k_1, \ldots, k_n)$ we refer to the hyperbolic area of a fundamental polygon domain for it; it is given by

$$A(\Gamma) = 2\pi\left(2\gamma - 2 + \sum_{j=1}^{n}(1 - \frac{1}{k_j})\right) \tag{4}$$

We say that a co-compact Fuchsian group Γ, with presentation (3), is a *homology Fuchsian group* if $\gamma = 0$ and it satisfies Maclachlan's conditions (2). In other words, homology Fuchsian groups are exactly those co-compact Fuchsian groups providing a Fuchsian uniformization of a hyperbolic homology orbifold of genus zero. If Γ is a homology Fuchsian group of signature $(0; k_1, \ldots, k_n)$, then the homology cover of the homology orbifold $\mathcal{O} = \mathbb{H}^2/\Gamma$ is $S = \mathbb{H}^2/\Gamma'$, where Γ' denotes the derived subgroup of Γ.

2.4. **Fields of moduli and fields of definition.** As a consequence of the Implicit Function Theorem, every irreducible non-singular projective algebraic curve defines a closed Riemann surface; conversely, by the Riemann-Roch Theorem, every closed Riemann surface may be described by an irreducible non-singular projective algebraic curve. It is this equivalence which allows the work at the analytical and at the algebraic settings in a parallel way.

Let C be an irreducible non-singular projective algebraic curve, say defined by homogeneous polynomials $P_1,\ldots, P_r$, each one with coefficients in a subfield $K < \mathbb{C}$. Let g denote the genus of the closed Riemann surface corresponding to C. If $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the group of field automorphisms of $\mathbb{C}$, then we may consider the new polynomials $P_1^{\sigma},\ldots, P_r^{\sigma}$, where the coefficients of P_j^{σ} are the corresponding images under σ of the coefficients of the original polynomial P_j. The algebraic curve C^{σ}, defined by these new polynomials, is still an irreducible non-singular projective algebraic curve, and it defines a new closed Riemann surface of genus g. It is not difficult to see that if $\widetilde{C}$ is another irreducible non-singular projective algebraic curve that is birationally equivalent to C, then C^{σ} and $\widetilde{C}^{\sigma}$ are also birationally equivalent. Therefore, a natural action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ is defined on the moduli space of genus g. The stabilizer of the moduli class of C under such action is the subgroup

$$K_C = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : C \cong C^{\sigma}\} < \mathrm{Aut}(\mathbb{C}/\mathbb{Q}).$$

The fixed field of K_C, denoted by $\mathcal{M}(C)$, is called the *field of moduli* of C.

A subfield $\mathbb{K}$ of $\mathbb{C}$ is called a *field of definition* of C if there is an irreducible non-singular projective algebraic curve $\widetilde{C}$ defined over $\mathbb{K}$ which is birationally equivalent to C. At this point it is important to note that it is not clear that given a field of definition $L < \mathbb{C}$ of C there is a smaller subfield $F < L$ which is again a field of definition of C.

The field of moduli $\mathcal{M}(C)$ is contained in any field of definition of C, and it coincides with the intersection of all fields of definitions of C [18]. Moreover, there is a field of definition of C which is an extension of finite degree of the field of moduli [6, 11].

If $g = 0$, then $C \cong \mathbb{P}^1$, so in this case $\mathcal{M}(C) = \mathbb{Q}$ is a field of definition. If $g = 1$, then C is equivalent to an (affine) elliptic curve $E_\eta = \{y^2 = x(x-1)(x-\eta)\}$, where $\eta \in \mathbb{C} - \{0, 1\}$. If $j(\eta) = (1-\eta+\eta^2)^3/\eta^2(\eta-1)^2$ is its j-invariant and $a(\eta) = 27j(\eta)/(j(\eta)-1)$, then E_η is also described by $D_\eta = \{y^2 = 4x^3 - a(\eta)x - a(\eta)\}$. It follows that $\mathbb{Q}(j(\eta))$ is a field of definition for E_η. Moreover, if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and $E_\eta^{\sigma} = E_{\sigma(\eta)}$ is conformally equivalent to E_η, then they must have the same j-invariant; that is, $\sigma(j(\eta)) = j(\eta)$. It follows that $\mathcal{M}(C) = \mathcal{M}(E_\eta) = \mathbb{Q}(j(\eta))$ is also a field of definition.

In genus $g \geq 2$, the situation is more difficult. There are examples for which the field of moduli is not a field of definition [7, 15, 21]; all of the examples there are hyperelliptic curves. It is stated in [7] that there are examples of non-hyperelliptic Riemann surfaces with the same properties, but no explicit one was given. An explicit example of a non-hyperelliptic Riemann surface of genus $g = 17$ which cannot be defined over $\mathbb{R}$ and whose field of moduli is inside $\mathbb{R}$ is given in [13] (this example is related to the hyperelliptic example in [7]).

A. Weil [24] provided the following sufficient and necessary conditions for the moduli field to be a field of definition.

Theorem 1 ([24])**.** *Let C be an irreducible non-singular projective algebraic curve defined over a finite Galois extension L of its field of moduli $\mathcal{M}(C)$. If for every $\sigma \in \mathrm{Aut}(L/\mathcal{M}(C))$ there is a biholomorphism $f_\sigma : C \to C^{\sigma}$ defined over L such that the compatibility condition $f_{\tau\sigma} = f_\sigma^{\tau} \circ f_\tau$ holds for all $\sigma, \tau \in \mathrm{Aut}(L/\mathcal{M}(C))$, then there exists an irreducible non-singular projective algebraic curve E defined over $\mathcal{M}(C)$ and there exists a biregular map $R : C \to E$, defined over L, such that $R^{\sigma} \circ f_\sigma = R$.*

As a consequence of Theorem 1, it follows that if C has no non-trivial automorphism, then it may be defined over its field of moduli. Unfortunately, if C has non-trivial automorphisms, then it is a very difficult task to verify if Weil's conditions hold. But if $C/\operatorname{Aut}(C)$ has signature of the form $(0; a, b, c)$ (quasiplatonic surfaces, or platonic if some cone order is equal to 2), then C may be defined over its field of moduli [5, 25].

Consider a (branched) holomorphic covering between closed Riemann surfaces, say $f : X \to Y$. Assume X and Y are given by fixed algebraic curves and that Y is defined over $\mathcal{M}(X)$. For each $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathcal{M}(X))$ we may consider the (branched) holomorphic covering $f^\sigma : X^\sigma \to Y^\sigma = Y$. We say that they are equivalent, denoted by $\{f^\sigma : X^\sigma \to Y\} \cong \{f : X \to Y\}$, if there is a holomorphic isomorphism $\phi_\sigma : X \to X^\sigma$ so that $f^\sigma \circ \phi_\sigma = f$. The *field of moduli* of $f : X \to Y$, denoted by $\mathcal{M}(f : X \to Y)$, is the fixed field of the subgroup

$$K(f : X \to Y) = \{\sigma \in \operatorname{Aut}(\mathbb{C}/\mathcal{M}(X)) : \{f^\sigma : X^\sigma \to Y\} \cong \{f : X \to Y\}\}.$$

It is clear from the definition that $\mathcal{M}(X) < \mathcal{M}(f : X \to Y)$, but in general they may be different fields. For the particular case that $Y = X/\operatorname{Aut}(X)$ and X has genus at least two, the following is well known (a direct consequence of Theorem 1).

Theorem 2 (Dèbes-Emsalem [6])**.** *If X is an irreducible non-singular projective algebraic curve of genus $g \geq 2$, then there exists an irreducible non-singular projective algebraic curve B, defined over $\mathcal{M}(X)$, and there exists a Galois cover $f : X \to B$, with* $\operatorname{Aut}(X)$ *as Deck group, so that $\mathcal{M}(f : X \to B) = \mathcal{M}(X)$. Moreover, if B_f denotes the branch locus of f and if $B - B_f$ contains at least one $\mathcal{M}(X)$-rational point, then $\mathcal{M}(X)$ is also a field of definition of X. Such a curve B is called a canonical model of $X/\operatorname{Aut}(X)$.*

3. Main Results

Let S be a closed Riemann surface and let $H_1, H_2 < \operatorname{Aut}(S)$. We say that H_1 and H_2 are (weakly) *topologically equivalent* (respectively, *conformally equivalent*) if there is an orientation preserving self-homeomorphism (respectively, conformal automorphism) $h : S \to S$ so that $H_2 = fH_1f^{-1}$. If $H < \operatorname{Aut}(S)$, then we denote by $\operatorname{Aut}_H(S)$ the normalizer of H in $\operatorname{Aut}(S)$.

3.1. p-groups of automorphisms.

We are interested in regular p^{n+1}-covers of orbifolds of type $(0; p, p^{n-1}, p^n, p^n)$, where $n \geq 2$ and p is an odd prime.

The interest in these type of examples is that in [4, 9] it has been constructed examples of closed Riemann surfaces S admitting topologically equivalent but conformally non-equivalent cyclic groups of order 2^{n+1}, where $n \geq 2$, so the quotient of S by the 2-group generated these two cyclic subgroups is an orbifold with signature $(0; 2, 2^n, 2^{n+1}, 2^{n+1})$.

Let S be a closed Riemann surface and let $H < \operatorname{Aut}(S)$ be a p-group such that S/H has signature of the form $(0; p, p^{n-1}, p^n, p^n)$, with $n \geq 2$. There is a regular branched cover $P : S \to \widehat{\mathbb{C}}$, with H as Deck group.

If $n \geq 3$, then (up to left composition by a suitable Möbius transformation) we may assume that the branch values of P are ∞ of order p, 0 of order p^{n-1}, and 1 and some $\lambda \in \mathbb{C} - \{0, 1\}$ are the ones of order p^n. In this case, the choice of λ is not unique, but the only other possible choice is $1/\lambda$.

If $n = 2$, then again (up to left composition by a suitable Möbius transformation) we may assume that the branch values of P of order p are ∞ and 0, the ones of order p^n are 1 and some $\lambda \in \mathbb{C} - \{0, 1\}$. Again the choice of λ is not unique, but the only other possible choice is $1/\lambda$.

Theorem 3. *Let $p \geq 3$ be a prime and let $n \geq 2$ be an integer. Consider a closed Riemann surface S with a subgroup $H < \mathrm{Aut}(S)$ such that H is a p-group with S/H of signature $(0; p, p^{n-1}, p^n, p^n)$. Let $\lambda \in \mathbb{C} - \{0, 1\}$ be as defined above. Then the following properties hold.*

(1) *H is a p-Sylow subgroups of $\mathrm{Aut}(S)$. In particular, if $H_1, H_2 < \mathrm{Aut}(S)$ are p-groups with S/H_j of signature $(0; p, p^{n-1}, p^n, p^n)$, with $n \geq 2$, then H_1 and H_2 are conformally equivalent.*

(2) *If $n \geq 3$, then*
   (a) $\mathrm{Aut}_H(S) = H$, *for* $\lambda \neq -1$.
   (b) $[\mathrm{Aut}_H(S) : H] \in \{1, 2\}$, *for* $\lambda = -1$.

(3) *If $n = 2$, then*
   (a) $[\mathrm{Aut}_H(S) : H] \in \{1, 2\}$, *for* $\lambda \neq -1$.
   (b) $[\mathrm{Aut}_H(S) : H] \in \{1, 2, 4\}$, *for* $\lambda = -1$.

(4) *If $p \geq p_0$, where*
   (a) *$p_0 = 7$ for $n = 2$, and*
   (b) *$p_0 = 5$ for $n \geq 3$,*

   then $\mathrm{Aut}_H(S) = \mathrm{Aut}(S)$.

***Remark* 4.** In the case $\lambda = -1$ and $n \geq 3$, part (2) of Theorem 3 asserts that either $\mathrm{Aut}_H(S) = H$ or $[\mathrm{Aut}_H(S) : H] = 2$. In the last case, $S/\mathrm{Aut}_H(S)$ has signature $(0; 2p, 2p^{n-1}, p^n)$, which is a maximal signature [22], so $\mathrm{Aut}_H(S) = \mathrm{Aut}(S)$.

3.2. **Normality condition.**

Let S be a closed Riemann surface and $H < \mathrm{Aut}(S)$. Let $\mathcal{M}(S, H)$ denote the locus in the moduli space $\mathcal{M}(S)$ of S consisting of those classes of Riemann surfaces $\widehat{S}$ admitting a group $\widehat{H}$ of conformal automorphisms, which is topologically equivalent to H. In general, one should expect that $\mathcal{M}(S, H)$ is a singular variety. The following shows that this is not the case if H is a p-group and S/H has signature $(0; p, p^{n-1}, p^n, p^n)$.

Corollary 5. *Let $p \geq 3$ be a prime and let $n \geq 2$ be an integer. Consider a closed Riemann surface S and let $H < \mathrm{Aut}(S)$ be a p-group such that S/H has signature $(0; p, p^{n-1}, p^n, p^n)$. Then $\mathcal{M}(S, H)$ is a normal subvariety of $\mathcal{M}(S)$.*

Proof. The normality condition for $\mathcal{M}(S, H)$ is equivalent to the following property: Given any two pairs (S_1, H_1) and (S_2, H_2), where S_j is a closed Riemann surface (of the same genus as S) and H_j is a p-group of conformal automorphisms of S_j so that S_j/H_j has signature $(0; p, p^{n-1}, p^n, p^n)$, and there is an orientation preserving homeomorphism $f : S_1 \to S_2$ with $fH_1f^{-1} = H_2$, then f may be replaced by a biholomorphism with the same properties. This property is exactly what part (1) of Theorem 3 states.

□

3.3. **Homology rigidity.**

Corollary 6. *Every Riemann orbifold of signature $(0; p, p^{n-1}, p^n, p^n)$, where $p \geq 3$ is a prime and $n \geq 2$ is an integer, is uniquely determined, up to conformal equivalence, by its homology cover Riemann surface. cover Riemann surface.*

Proof. A consequence of part (1) in Theorem 3. □

Remark 7 (Torelli's theorem). Let $\mathcal{O}$ be a Riemann orbifold of signature $(0; p, p^{n-1}, p^n, p^n)$, where $p \geq 3$ is a prime and $n \geq 2$ is an integer. As any two homology covers of $\mathcal{O}$ are conformally equivalent Riemann surfaces, we may define the Jacobian of $\mathcal{O}$, denoted by $J(\mathcal{O})$, as the Jacobian of any of these covers. It follows that $J(\mathcal{O})$ is uniquely determined, up to equivalence of principally polarized Abelian varieties, by $\mathcal{O}$. As a consequence of Torelli's theorem, $J(\mathcal{O})$ determines the conformal class of the homology cover of $\mathcal{O}$ and, by Corollary 6, it also determines the conformal class of $\mathcal{O}$. In this way, a kind of Torelli's theorem is obtained for this class of Riemann orbifolds. We may wonder how to describe the Jacobian of $\mathcal{O}$ in terms of multivalued holomorphic differential forms so that it looks more similar to the construction for the case of Riemann surfaces. In order to do this, we use as homology the orbifold homology group $H_1^{\mathrm{orb}}(\mathcal{O}) = \pi_1^{\mathrm{orb}}(\mathcal{O})/\pi_1^{\mathrm{orb}}(\mathcal{O})'$, and as holomorphic forms those multivalued holomorphic forms whose liftings to the homology cover define the holomorphic one forms of it.

3.4. **Algebraic curves in the Abelian case.** Curves for the hyperelliptic homology covers and for the homology covers of homology orbifolds with triangular signature have been described in [12]. Algebraic curves for the homology covers of orbifolds with signature of the form $(0; k, \ldots, k)$ have been obtained in [10]. We next provide the algebraic curves for the homology covers of orbifolds with signature $(0; k, k^{n-1}, k^n, k^n)$, where $k, n \geq 2$ are integers. As a consequence of the results in [12], the homology covers of such orbifolds cannot be hyperelliptic. Note that if R is the homology cover of such an orbifold $\mathcal{O}$, then $\mathcal{O} = R/H$, where $H \cong \mathbb{Z}_k \times \mathbb{Z}_{k^{n-1}} \times \mathbb{Z}_{k^n}$.

Theorem 8. *Let $k, n \geq 2$ be integers and let $\mathcal{O}$ be a Riemann orbifold with signature $(0; k, k^{n-1}, k^n, k^n)$. Denote by R an homology cover of $\mathcal{O}$, let $H < \mathrm{Aut}(R)$ be so that $R/H = \mathcal{O}$, and let $P : R \to \mathcal{O}$ be the Galois cover with H as Deck group. We may assume (up to a Möbius transformation) that the cone points of $\mathcal{O}$ (that is, the branch values of P) are given by the points 0, 1, ∞ and $\lambda \in \mathbb{C} - \{0, 1\}$. We may also assume that ∞ is the cone point of order k, that 0 is the cone point of order k^{n-1} and that 1 and λ are the cone points of order k^n.*

Then R is represented by the (singular) projective algebraic curve

$$C_\lambda : \left\{ \begin{array}{rcl} z_0^k z_3^{k^n-k} + z_1^{k^{n-1}} z_3^{k^n-k^{n-1}} + z_2^{k^n} & = & 0 \\ \lambda z_0^k z_3^{k^{n-1}-k} + z_1^{k^{n-1}} + z_3^{k^{n-1}} & = & 0 \end{array} \right\} \subset \mathbb{P}^3 \ ;$$

H is generated by the projective linear transformations

$$\begin{array}{rcl} a_0([z_0 : z_1 : z_2 : z_3]) & = & [\rho_1 z_0 : z_1 : z_2 : z_3] \\ b_0([z_0 : z_1 : z_2 : z_3]) & = & [z_0 : \rho_{n-1} z_1 : z_2 : z_3] \\ c_0([z_0 : z_1 : z_2 : z_3]) & = & [z_0 : z_1 : \rho_n z_2 : z_3], \end{array}$$

where $\rho_s = e^{2\pi i/k^s}$, for each positive integer s, and the branched covering map P is represented in this model by

$$P([z_0 : z_1 : z_2 : z_3]) = -\left(\frac{z_1^{k^{n-1}}}{z_0^k z_3^{k^{n-1}-k}} \right).$$

The only singular point of the above curve is $[1 : 0 : 0 : 0]$.

Theorem 8 may be used to find algebraic curves for closed Riemann surfaces S admitting an Abelian group $G < \mathrm{Aut}(S)$ whose quotient orbifold S/G has signature of the form $(0; k, k^{n-1}, k^n, k^n)$. In fact, let $Q : S \to S/G = \mathcal{O}$ be a regular Abelian branched cover with G as Deck group. Let R be the homology cover of $\mathcal{O}$, let $P : R \to \mathcal{O}$ be the regular Abelian branched cover, with Deck group $H < \mathrm{Aut}(R)$. Then there exists a subgroup $K < H$, acting freely on R and so that $G \cong H/K$, and there exists a regular unbranched cover $F : R \to S$, with K as Deck group, satisfying that $P = Q \circ F$. As we have explicit curves for R and an explicit presentation for H, the classical invariant theory permits to obtain explicit algebraic curves for S and an explicit presentation of G. We show an application in the next section.

3.5. **Families with Galois group of order** p^{n+1}.

As mentioned before, we are interested in regular p^{n+1}-covers of orbifolds of type $(0; p, p^{n-1}, p^n, p^n)$, where $n \geq 2$ and p is an odd prime. In Section 9 we will see that the algebraic structure of the corresponding groups of order p^{n+1} is restricted to only two algebraic types: a direct or a semi-direct product of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_p$. The geometric types (classified by either geometric signature or generating vector for the corresponding action) are more varied: four different types are found in each algebraic case.

We study the corresponding families of Riemann surfaces, giving their algebraic curves in the abelian case.

The next result makes the above more explicit for the case when $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, where p is a prime. As we will see in its proof, this is a heavy computational procedure, but not a hard one.

Theorem 9. *Let S be a closed Riemann surface S admitting a group $G < \mathrm{Aut}(S)$ such that $G = \langle A, B : A^p = B^{p^n} = [A, B] = 1\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ and $\mathcal{O} = S/G$ is a Riemann orbifold with signature $(0; p, p^{n-1}, p^n, p^n)$, where $n \geq 2$ and p is a prime. Let R be an homology cover of $\mathcal{O}$, let $H < \mathrm{Aut}(R)$ be so that $R/H = \mathcal{O}$. Let $K < H$ be the normal subgroup so that $S = R/K$ and $G = H/K$.*

(1) *If $K \cong \mathbb{Z}_{p^{n-1}}$, then there exist $\beta \in \{1, 2, \ldots, p^{n-1} - 1\}$, $\alpha \in \{0, 1, \ldots, p - 1\}$ and $q \in \{1, \ldots, [(p^n - 1)/p]\}$, with $(\beta, p) = 1 = (p, q)$, such that a (singular) projective algebraic curve representation of S is given by either of the following two families.*
   (a) *If $\alpha = 0$, then there exists λ in $\mathbb{C}$, with $\lambda \neq 0, 1$, such that*

$$S : \left\{ \begin{array}{rcl} (\lambda - 1)w_0^p - w_1^p + w_3^p & = & 0 \\ (-1)^{q+1}(w_0^p + w_1^p)^q w_1^{p^{n-1}-\beta} + w_2^{p^{n-1}} w_3^{qp-\beta} & = & 0 \end{array} \right\} \subset \mathbb{P}^3$$

and the action of G is generated by the following projective linear transformations

$$\begin{array}{rcl} A([w_0 : w_1 : w_2 : w_3]) & = & [\rho_1 w_0 : w_1 : w_2 : w_3] \\ B([w_0 : w_1 : w_2 : w_3]) & = & [w_0 : \rho_1 w_1 : \rho_n^{p^{n-1}-\beta} w_2 : w_3] \end{array}$$

where $\rho_k = e^{2\pi i/p^k}$. The regular branched covering map $Q : S \to S/G$ in this model is represented by

$$Q([w_0 : w_1 : w_2 : w_3]) = \frac{w_0^p + w_1^p}{w_0^p}.$$

The singular points of the above curve are given by the $(p + 1)$ points $[0 : 0 : 1 : 0]$ and $[1 : 0 : 0 : (1 - \lambda)^{1/p}]$.

(b) *If $\alpha > 0$, then there exists λ in $\mathbb{C}$, with $\lambda \neq 0, 1$, such that*

$$S : \left\{ \begin{array}{rcl} v_1^{p^{n-1}} + \dfrac{(-1)^{q+1}}{(\lambda-1)^q}(\lambda v_1^p - v_3^p)^q v_1^{p^{n-1}-\beta} v_3^{\beta - pq} & = & 0 \\ v_2^p v_3^{p(p^r-\beta)+\alpha p - p} + \dfrac{(-1)^{\alpha+1}}{(\lambda-1)^{\alpha+p^r-\beta}}(v_0^p - v_3^p)^{p^r-\beta}(\lambda v_0^p - v_3^p)^{\alpha} & = & 0. \end{array} \right\} \subset \mathbb{P}^3$$

and the group G is generated by the transformations

$$\begin{array}{rcl} A([v_0 : v_1 : v_2 : v_3]) & = & [v_0 : v_1 : \rho_1^{p^r-\beta} v_2 : v_3] \\ B([v_0 : v_1 : v_2 : v_3]) & = & [\rho_n^{p^{n-1}} v_0 : \rho_n^{p^{n-1}-\beta} v_1 : v_2 : v_3] \end{array}$$

The regular branched covering map $Q : S \to S/G$ in this model is represented by

$$Q([v_0 : v_1 : v_2 : v_3]) = \frac{\lambda v_0^p - v_3^p}{v_0^p + v_3^p}.$$

(2) *If $K \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$, then there exist λ in $\mathbb{C}$, with $\lambda \neq 0, 1$, integers $\gamma, \nu \in \{1, \ldots, p-1\}$ such that a (singular) projective algebraic curve representation of S is provided by the following plane projective curve*

$$\left\{ \frac{(-1)^{p^{n-1}(p-\gamma)}}{\lambda^{p^{n-1}(p-\gamma)+1}} \left(u_0^p + u_2^p\right)^{p^{n-1}(p-\gamma)} u_1^{p^2 \nu} \left((\lambda-1)u_0^p - u_2^p\right) + u_1^{p^n} u_2^{p^n(p-\gamma-1)+p+p^2\nu} = 0 \right\} \subset \mathbb{P}^2.$$

and the group G is generated by the transformations

$$\begin{array}{rcl} A([u_0 : u_1 : u_2]) & = & [\rho_1 u_0 : u_1 : u_2] \\ B([u_0 : u_1 : u_2]) & = & [u_0 : \rho_n u_1 : u_2] \end{array}$$

The regular branched covering map $Q : S \to S/G$ in this model is represented by

$$Q([u_0 : u_1 : u_2]) = \frac{\lambda u_0^p}{u_0^p + u_1^p}.$$

3.6. **Field of moduli.** If S is a closed Riemann surface, then it follows from the Riemann-Roch's theorem that S may be described by an irreducible non-singular projective algebraic curve C. It is clear from the definition that we may define the field of moduli of S as the field of moduli of C and a field of definition of S as a field of definition of C.

Theorem 10. *Let $p \geq 3$ be a prime, $n \geq 3$ be an integer, S be a closed Riemann surface, and $H < \mathrm{Aut}(S)$ be a p-group with S/H of signature $(0; p, p^{n-1}, p^n, p^n)$. Then S may be defined over its field of moduli.*

Remark 11. Under the hypotheses of Theorem 10, if $\mathrm{Aut}_{\mathrm{orb}}(S/H)$ is non-trivial, then S/H admits an extra conformal involution J such that $(S/H)/\langle J \rangle$ is the orbifold whose underlying Riemann surface is $\widehat{\mathbb{C}}$, with exactly three cone points (of orders $2p$, $2p^{n-1}$ and p^n). It follows that S is a Belyi curve and hence it may be defined over a finite extension of $\mathbb{Q}$.

Our next result computes the field of moduli for the homology covers of orbifolds with signature $(0; p, p^{n-1}, p^n, p^n)$, where $p \geq 3$ is a prime and $n \geq 2$.

Theorem 12. *Let $p \geq 3$ be a prime and $n \geq 2$ be an integer. For each $\lambda \in \mathbb{C} - \{0, 1\}$, let C_λ as in Theorem 8 with $k = p$. Then the following properties hold.*

(1) $C_\lambda \cong C_\mu$ *for* $\lambda, \mu \in \mathbb{C} - \{0, 1\}$ *if and only if* $\mu \in \{\lambda, 1/\lambda\}$*;*
(2) $\mathcal{M}(C_\lambda) = \mathbb{Q}(\lambda + \lambda^{-1})$*: and*
(3) $\mathcal{M}(C_\lambda)$ *is a field of definition for* C_λ*.*

Theorem 12 will be proved using arguments similar to those given by Dèbes-Emsalem in the proof of Theorem 2. In our case, we do not consider the quotient by the full group of automorphisms, but just the quotient by the Abelian group H in Theorem 8.

4. Proof of Theorem 3

4.1. **Proof of part (1).** As previously noted, there is a regular branched cover $P : S \to \widehat{\mathbb{C}}$, with H as Deck group, so that its branch values are ∞ of order p, 0 of order p^{n-1}, 1 of order p^n and λ of order p^n. Let us denote by $\mathcal{O}_\lambda$ the orbifold whose underlying Riemann surface is $\widehat{\mathbb{C}}$ and whose cone points are ∞ of order p, 0 of order p^{n-1}, 1 of order p^n and λ of order p^n; that is, $\mathcal{O}_\lambda = S/H$.

If H is not a p-Sylow subgroup, then there is some $H \lhd K < \mathrm{Aut}(S)$, where K is a p-group and $[K : H] = p$. It follows that there is an automorphism of order $p \geq 3$ of the orbifold $\mathcal{O}_\lambda$. As there are no three cone points with the same order, this is impossible.

4.2. **Proof of parts (2) and (3).** If $n \geq 3$, then it is easy to see that

$$\mathrm{Aut}_{\mathrm{orb}}(\mathcal{O}_\lambda) = \begin{cases} \{I\}, & \lambda \in \mathbb{C} - \{0, \pm 1\}, \\ \langle \tau(z) = -z \rangle, & \lambda = -1. \end{cases}$$

Since $\mathrm{Aut}_H(S)/H < \mathrm{Aut}_{\mathrm{orb}}(\mathcal{O}_\lambda)$, it follows that

$$\mathrm{Aut}_H(S) = \begin{cases} H, & \lambda \in \mathbb{C} - \{0, \pm 1\}, \\ K, & \lambda = -1. \end{cases}$$

where $[K : H] \in \{1, 2\}$.

If $n = 2$, then

$$\mathrm{Aut}_{\mathrm{orb}}(\mathcal{O}_\lambda) = \begin{cases} \langle \alpha(z) = \lambda/z \rangle, & \lambda \in \mathbb{C} - \{0, \pm 1\}, \\ \langle \tau(z) = -z, \beta(z) = -1/z \rangle, & \lambda = -1. \end{cases}$$

Again as $\mathrm{Aut}_H(S)/H < \mathrm{Aut}_{\mathrm{orb}}(\mathcal{O}_\lambda)$, it follows that

$$\mathrm{Aut}_H(S) = \begin{cases} \widehat{H}, & \lambda \in \mathbb{C} - \{0, \pm 1\}, \\ \widehat{K}, & \lambda = -1. \end{cases}$$

where $[\widehat{H} : H] \in \{1, 2\}$ and $[\widehat{K} : H] \in \{1, 2, 4\}$.

4.3. **Proof of part (4).** As a consequence of the results in [14], there exists a prime p_0 such that the group H is a normal subgroup in $\mathrm{Aut}(S)$ for $p \geq p_0$; that is, $\mathrm{Aut}(S) = \mathrm{Aut}_H(S)$. Next, we proceed to prove that p_0 may be chosen as desired.

Let $p \geq 3$ be any odd prime. We already know that H is a p-Sylow subgroup of $\mathrm{Aut}(S)$ and that S/H has signature $(0; p, p^{n-1}, p^n, p^n)$. If $S/\mathrm{Aut}(S)$ has signature of the form $(0; a, b, c, d)$, then it follows from Singerman's list of maximal Fuchsian groups [22] that $(0; a, b, c, d) = (0; p, p^{n-1}, p^n, p^n)$ and, in particular, that $H = \mathrm{Aut}(S)$.

Thus we need only take care of the case when $S/\mathrm{Aut}(S)$ has signature of the form $(0; r, s, t)$. In this case, at least one of the values r, s, t should be a multiple of p^n. We may assume $t = kp^n$, where k is a positive integer. We may also assume that $2 \le r \le s$ and, moreover, that if $r = 2$, then $s \ge 3$. Let $D = [\mathrm{Aut}(S) : H]$. If $D = 2$, then clearly $\mathrm{Aut}_H(S) = \mathrm{Aut}(S)$.

From now on assume that $D \ge 3$. Riemann-Hurwitz (hyperbolic area comparison) asserts that

$$D\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{kp^n}\right) = 2 - \frac{1}{p} - \frac{1}{p^{n-1}} - \frac{2}{p^n}, \tag{5}$$

where both sides are necessarily positive.

Lemma 13. *If*

(1) *either $p \ge 7$, or*
(2) *$p \in \{3, 5\}$ and $n \ge 3$,*

then $D \le 11$.

Proof. Assume $D \ge 12$. As $(r, s) \ne (2, 2)$, it follows from (5) that

$$D\left(\frac{1}{6} - \frac{1}{kp^n}\right) \le 2 - \frac{1}{p} - \frac{1}{p^{n-1}} - \frac{2}{p^n}.$$

Since $\left(\frac{1}{6} - \frac{1}{kp^n}\right)$ is positive, the last inequality implies that

$$k \le \frac{12}{2 + p + p^{n-1}}.$$

Therefore, if $p \ge 7$ then

$$k \le \frac{12}{2 + p + p^{n-1}} \le \frac{12}{2 + 2p} \le \frac{3}{4} < 1,$$

and if $p \in \{3, 5\}$ and $n \ge 3$ then

$$k \le \frac{12}{2 + p + p^{n-1}} \le \frac{12}{2 + 3 + 3^2} \le \frac{6}{7} < 1,$$

obtaining a contradiction in all cases. □

The following Proposition gives the desired result.

Proposition 14. (1) *If $n \ge 2$, then $p_0 \le 7$.*
(2) *If $n \ge 3$, then $p_0 \le 5$.*

Proof. Let us denote by N_p be the number of p-Sylow subgroups of $\mathrm{Aut}(S)$. We need to prove that $N_p = 1$, if either (i) $p \ge 7$ is prime and $n \ge 2$ or if (ii) $p \ge 5$ is a prime and $n \ge 3$.

As $N_p \equiv 1 \mod p$, we may write $N_p = 1 + pL_p$, where L_p is a non-negative integer.

If we assume that $N_p > 1$, then $N_p \ge 1 + p$. As N_p divides $|\mathrm{Aut}(S)| = D|H|$, it follows that N_p must divide D.

If $p \ge 11$, then $N_p \ge 12$; as $D \le 11$ by Lemma 13, we obtain a contradiction.

For the remaining cases, we will make use of the following equality, obtained from (5),

$$\left(D\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)p^n+p^{n-1}+p+2=\frac{D}{k}\in\{1,\ldots,D\}. \tag{6}$$

Note that both sides in this equality are positive integers.

If $p=7$, since $D\leq 11$ by Lemma 13, we must have that $L_7=1$ and $N_7=D=8$. If either $r,s\geq 3$ or $r=2$ and $s\geq 4$, then

$$\left(8\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)\geq 0$$

and the left side of (6) is bigger than 8, a contradiction to the fact that the right side should be less or equal to D.

We are left with the case $r=2$ and $s=3$. But in this case the left side of (6) equals

$$\left(8\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)7^n+7^{n-1}+9<0,$$

again a contradiction.

Now we consider $p=5$ and $n\geq 3$. In this case either (i) $L_5=1$ and $N_5=D=6$ or (ii) $L_5=2$ and $N_5=D=11$.

For $D=6$, if either (a) $r,s\geq 3$ or (b) $r=2$ and $s\geq 6$, then

$$\left(6\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)\geq 0$$

and the left side of (6) is bigger than D, a contradiction. The remaining cases are $r=2$ and $3\leq s\leq 5$. But in these cases we have

$$\left(6\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)5^n+5^{n-1}+7<0,$$

again a contradiction.

For $D=11$, if either (a) $r,s\geq 3$ or (b) $r=2$ and $s\geq 4$, then

$$\left(11\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)\geq 0$$

and the left side of (6) is bigger than D, a contradiction. The remaining cases are $r=2$ and $s=3,4$. But in these cases we have

$$\left(11\left(1-\frac{1}{r}-\frac{1}{s}\right)-2\right)5^n+5^{n-1}+7<0,$$

again a contradiction.

□

5. Proof of Theorem 8

Let R be the homology cover of an orbifold $\mathcal{O}$ with signature $(0;k,k^{n-1},k^n,k^n)$, where $k,n\geq 2$. The closed Riemann surface R admits a group $H<\mathrm{Aut}(R)$, where $H\cong\mathbb{Z}_k\times\mathbb{Z}_{k^{n-1}}\times\mathbb{Z}_{k^n}$ and such that $R/H=\mathcal{O}$.

First consider the Riemann orbifold $\mathcal{O}^*$ obtained from $\mathcal{O}$, but assuming all cone points of order k^n. The homology cover of this new orbifold is a closed Riemann surface S admitting a group

$H^* < \mathrm{Aut}(S)$, $H^* \cong \mathbb{Z}_{k^n} \times \mathbb{Z}_{k^n} \times \mathbb{Z}_{k^n}$, and such that $O^* = S/H^*$. It is known (see [10]) that an algebraic curve representation of S is given by

$$\widehat{C} : \left\{ \begin{array}{rcl} x_0^{k^n} + x_1^{k^n} + x_2^{k^n} & = & 0 \\ \lambda x_0^{k^n} + x_1^{k^n} + x_3^{k^n} & = & 0 \end{array} \right\} \subset \mathbb{P}^3,$$

that H^* is generated by the projective transformations

$$a([x_0 : x_1 : x_2 : x_3]) = [\rho_n x_0 : x_1 : x_2 : x_3]$$
$$b([x_0 : x_1 : x_2 : x_3]) = [x_0 : \rho_n x_1 : x_2 : x_3]$$
$$c([x_0 : x_1 : x_2 : x_3]) = [x_0 : x_1 : \rho_n x_2 : x_3]$$

and that the holomorphic map

$$\pi : \widehat{C} \to \widehat{\mathbb{C}} : [x_0 : x_1 : x_2 : x_3] \mapsto -\left(\frac{x_1}{x_0}\right)^{k^n}$$

has degree k^{3n} and is a branched regular cover with H^* as Deck group. In this case, $\pi(\mathrm{Fix}(a)) = \infty$, $\pi(\mathrm{Fix}(b)) = 0$, $\pi(\mathrm{Fix}(c)) = 1$ and $\pi(\mathrm{Fix}(abc)) = \lambda$.

Now consider the subgroup of H^* given by $K = \langle a^k, b^{k^{n-1}} \rangle \cong \mathbb{Z}_{k^{n-1}} \times \mathbb{Z}_k$, and set $O_0 = S/K$. The group $H_0 = H^*/K$ is a group of conformal automorphism of O_0, $H_0 \cong H$, and $O_0/H_0 = O^*$.

Clearly, if R_0 denotes the underlying Riemann surface structure of the Riemann orbifold O_0, then R_0/H_0 is the Riemann orbifold O. In this way, since any two homology covers of O are conformally equivalent, we may assume $R = R_0$.

In order to find an algebraic curve representation for R_0 we proceed as follows. First, we consider the affine curve representation of S defined by $x = x_0/x_3$, $y = x_1/x_3$ and $z = x_2/x_3$; that is,

$$\widehat{C_0} = \left\{ \begin{array}{l} x^{k^n} + y^{k^n} + z^{k^n} = 0 \\ \lambda x^{k^n} + y^{k^n} + 1 = 0 \end{array} \right\} \subset \mathbb{C}^3$$

and the action of H^* is generated by the linear transformations

$$a(x, y, z) = (\rho_n x, y, z)$$
$$b(x, y, z) = (x, \rho_n y, z)$$
$$c(x, y, z) = (x, y, \rho_n z)$$

The subalgebra of $\langle a^k, b^{k^{n-1}} \rangle$ invariant polynomials, $\mathbb{C}[x, y, z]^{\langle a^k, b^{k^{n-1}} \rangle}$, is generated by the monomials $x^{k^{n-1}}$, y^k and z. It follows that the holomorphic map

$$F : \mathbb{C}^3 \to \mathbb{C}^3$$
$$(x, y, z) \mapsto (x^{k^{n-1}}, y^k, z) = (u, v, w)$$

is a regular branched covering with $\langle a^k, b^{k^{n-1}} \rangle$ as Deck group, and therefore $F(\widehat{C_0})$ provides an affine algebraic curve representation of R, given by

$$F(\widehat{C_0}) = \left\{ \begin{array}{rcl} u^k + v^{k^{n-1}} + w^{k^n} & = & 0 \\ \lambda u^k + v^{k^{n-1}} + 1 & = & 0 \end{array} \right\} \subset \mathbb{C}^3 .$$

where the action of $H = H^*/K$ is generated by

$$a_0(u, v, w) = (\rho_1 u, v, w),$$
$$b_0(u, v, w) = (u, \rho_{n-1} v, w),$$
$$c_0(u, v, w) = (u, v, \rho_n w).$$

If we consider the projective space $\mathbb{P}^3$ with coordinates $[z_0 : z_1 : z_2 : z_3]$, and we set

$$u = \frac{z_0}{z_3},\ v = \frac{z_1}{z_3},\ w = \frac{z_2}{z_3},$$

then we obtain that R is represented by the projective algebraic curve

$$C = \left\{ \begin{array}{rcl} z_0^k z_3^{k^n-k} + z_1^{k^{n-1}} z_3^{k^n-k^{n-1}} + z_2^{k^n} & = & 0 \\ \lambda z_0^k z_3^{k^{n-1}-k} + z_1^{k^{n-1}} + z_3^{k^{n-1}} & = & 0 \end{array} \right\} \subset \mathbb{P}^3.$$

As the branched covering map $P : R \to R/H$ must satisfy that $\pi = P \circ F$ and

$$F([x_0 : x_1 : x_2 : x_3]) = [x_0^{k^{n-1}} : x_1^k x_3^{k^{n-1}-k} : x_2 x_3^{k^{n-1}-1} : x_3^{k^{n-1}}]\,,$$

then

$$P([z_0 : z_1 : z_2 : z_3]) = -\left(\frac{z_1^{k^{n-1}}}{z_0^k z_3^{k^{n-1}-k}} \right).$$

6. Proof of Theorem 9

Consider a closed Riemann surface S admitting a group $G < \mathrm{Aut}(S)$ such that $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ and $O = S/G$ is a Riemann orbifold with signature $(0; p, p^{n-1}, p^n, p^n)$, where $n \geq 2$ and p is an odd prime. Denote by $P : S \to O$ the natural holomorphic branched cover with G as Deck group.

In this section we will find algebraic curves representing S and the action of G on them.

Let R be the homology cover of O, and let $Q : R \to O = R/H$ be the branched regular covering with H as Deck group, where $H = \mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^n}$.

Since G is abelian, there is a subgroup $K < H$ such that $S = R/K$ (and hence K acts freely on R), $G = H/K$, and there is a regular holomorphic covering $T : R \to S$ with K as Deck group and $Q = P \circ T$.

Consider the affine algebraic curve C_0 representing R, obtained from Theorem 8 by making $z_3 = 1$,

$$C_0 = \left\{ \begin{array}{l} z_0^p + z_1^{p^{n-1}} + z_2^{p^n} = 0 \\ \lambda z_0^p + z_1^{p^{n-1}} + 1 = 0 \end{array} \right\} \subset \mathbb{C}^3,$$

in which case the group H is generated by

$$\begin{array}{rcl} a_0(z_0, z_1, z_2) & = & (\rho_1 z_0, z_1, z_2) \\ b_0(z_0, z_1, z_2) & = & (z_0, \rho_{n-1} z_1, z_2) \\ c_0(z_0, z_1, z_2) & = & (z_0, z_1, \rho_n z_2) \end{array}$$

6.1. **Algebraic structure of K.** We next describe the algebraic structure of K. At this point we should note that, using the model of R given in Theorem 8, the transformations in H acting with fixed points on S are exactly the ones that belong to $\langle a_0 \rangle \cup \langle b_0 \rangle \cup \langle c_0 \rangle \cup \langle a_0 b_0 c_0 \rangle$.

Proposition 15. *Consider the algebraic model of (R, H) provided by Theorem 8. Let $K < H$ be such that K acts freely on R and $H/K \cong \mathbb{Z}_p \times \mathbb{Z}_{p^n}$. Then, either*

(1) $\mathbb{Z}_{p^{n-1}} \cong K = \langle a_0^{\alpha} b_0 c_0^{pq} \rangle$, *where* $(p, q) = 1$ *and* $0 \leq \alpha \leq p-1$; *or*
(2) $\mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p \cong K = \langle b_0^{-p} c_0^{p^2 v} \rangle \times \langle a_0 c_0^{p^{n-1}\gamma} \rangle$, *where* $(p, v) = 1$ *and* $1 \leq \gamma \leq p-1$.

Proof. Consider a surjective homomorphism

$$\Phi : H \to J = \mathbb{Z}_p \times \mathbb{Z}_{p^n}$$

with $K = \ker(\Phi)$ acting freely on R. Note that the order of K is p^{n-1}.

Then

a) $K \cap \langle a_0 \rangle = \{I\}$, which implies that $\Phi(a_0)$ has order p;
b) $K \cap \langle b_0 \rangle = \{I\}$, which implies that $\Phi(b_0)$ has order p^{n-1};
c) $K \cap \langle c_0 \rangle = \{I\}$, which implies that $\Phi(c_0)$ has order p^n; and
d) $K \cap \langle a_0 b_0 c_0 \rangle = \{I\}$, which implies that $\Phi(a_0)\Phi(b_0)\Phi(c_0)$ has order p^n.

Hence the subgroups of J given by $\langle \Phi(b_0) \rangle$ and $\langle \Phi(c_0) \rangle$ have respective indices p^2 and p, and there are two cases to be considered, as follows.

Case i). Assume $\langle \Phi(b_0) \rangle \subset \langle \Phi(c_0) \rangle$. Then there exists $1 \leq u \leq p-1$ such that $\Phi(b_0) = \Phi(c_0^{pu})$, in which case $h = b_0 c_0^{-pu}$ is an element of K of order p^{n-1}, and therefore $K = \langle h \rangle$ is cyclic of the form given in case (1).

Case ii). Assume $\langle \Phi(b_0) \rangle \not\subset \langle \Phi(c_0) \rangle$.

Then we have the following commutative diagram of subgroup inclusions and corresponding indices

$$\begin{tikzcd}
& J & \\
\langle \Phi(c_0) \rangle \arrow[ur, "p"] & & \langle \Phi(b_0) \rangle \arrow[ul, "p^2"'] \\
& \langle \Phi(c_0) \rangle \cap \langle \Phi(b_0) \rangle \arrow[ul, "p^2"] \arrow[ur, "p"'] &
\end{tikzcd}$$

and it follows that

$$\langle \Phi(c_0) \rangle \cap \langle \Phi(b_0) \rangle = \langle \Phi(c_0^{p^2}) \rangle = \langle \Phi(b_0^p) \rangle \,.$$

Hence there exists v such that $h_0 = c_0^{p^2 v} b_0^{-p}$ is in K, and h_0 has order p^{n-2}. Also note that $(v, p) = 1$, since otherwise an adequate power of h_0 would be a nontrivial power of b_0 in K. It follows that there are two possibilities for K, either $K \cong \mathbb{Z}_{p^{n-1}}$ or $K = \langle h_0 \rangle \times \langle t \rangle \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$.

Subcase K is not cyclic. As previously noted, in this case $K = \langle h_0 \rangle \times \langle t \rangle \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$, where $h_0 = c_0^{p^2 v} b_0^{-p}$ and $(p, v) = 1$. As $t \in H$ has order p, it has the form $t = a_0^{\alpha} b_0^{\beta p^{n-2}} c_0^{\gamma p^{n-1}}$, where $\alpha, \beta, \gamma \in \{0, 1, ..., p-1\}$.

Let us assume $\alpha = 0$. If $\gamma = 0$, then $t \in \langle b_0 \rangle$. As K acts freely on R, necessarily $t = 1$ and we get a contradiction. If $(\gamma, p) = 1$, then we may assume $t = b_0^{\beta p^{n-2}} c_0^{p^{n-1}}$ (by considering an appropriate power of the original t), hence $\tilde{h} = t h_0^{-p^{n-3}} = b_0^{(\beta + v) p^{n-2}} \in K \cap \langle b_0 \rangle$. Again, as K acts freely, $\tilde{h}$ must be trivial, and t would belong to $\langle h_0 \rangle$, again a contradiction. Then we have proved that $\alpha > 0$.

Since t has order p, we may replace t by a suitable power of it in order to assume that $t = a_0 b_0^{\beta p^{n-2}} c_0^{\gamma p^{n-1}}$.

We now claim that we may assume $\beta = 0$. Indeed, if $\beta > 0$, then $t h_0^{\beta p^{n-3}} = a_0 c_0^{p^{n-1}(\gamma + v)}$ is an element of order p in K that does not belong to $\langle h_0 \rangle$.

Therefore we may write $t = a_0 c^{p^{n-1}\gamma}$, and observe that $1 \leq \gamma \leq p-1$ because $K \cap \langle a_0 \rangle = \{I\}$. This is case (2).

Subcase K is cyclic. In this case, $K = \langle h \rangle \cong \mathbb{Z}_{p^{n-1}}$. Let us write

$$h = a_0^{\alpha} b_0^{\beta} c_0^{\gamma}$$

where $\alpha \in \{0, 1, \dots, p-1\}$, $\beta \in \{0, 1, \dots, p^{n-1}-1\}$, $\gamma \in \{0, 1, \dots, p^n - 1\}$.

The condition $c_0^{\gamma p^{n-1}} = h^{p^{n-1}} = 1$ ensures that $\gamma \equiv 0 \mod p$. It follows that either $\gamma = 0$ or $\gamma = p^s q$, where $s \in \{1, \dots, n-1\}$ and $(p, q) = 1$.

Next, we need to ensure that, for $\delta \in \{1, 2, \dots, p^{n-1} - 1\}$, no power h^{δ} acts with fixed points in C; that is, $h^{\delta} \notin \langle a_0 \rangle \cup \langle b_0 \rangle \cup \langle c_0 \rangle \cup \langle a_0 b_0 c_0 \rangle$.

But if $\gamma = 0$ then $h^p = b_0^{p\beta}$ is a nontrivial element of the group generated by b_0, a contradiction. Similarly, if $s > 1$ then $h^{p^{n-s}} = b_0^{\beta p^{n-s}}$ is a nontrivial element of the group generated by b_0, a contradiction.

Therefore $h = a_0^{\alpha} b_0^{\beta} c_0^{pq}$, with $(p, q) = 1$, and it follows that h^{δ} is not in $\langle b_0 \rangle$.

But if $\beta \equiv 0 \mod p$, then $h^{p^{n-2}} = c^{qp^{n-2}}$ is a nontrivial element of the group generated by c_0, a contradiction. Hence $(p, \beta) = 1$, and h^{δ} is not in $\langle c_0 \rangle$.

We note that $h^{\delta} \in \langle a_0 \rangle$ implies that $\beta\delta \equiv 0 \mod p^{n-1}$, and since $(\beta, p) = 1$, to have $\delta \equiv 0 \mod p^{n-1}$, which is not possible by our choice for δ.

The condition $h^{\delta} \in \langle a_0 b_0 c_0 \rangle$ implies that $\beta\delta \equiv pq\delta \mod p^{n-1}$, from which $(\beta - pq)\delta \equiv 0 \mod p^{n-1}$, and then $\delta \equiv 0 \mod p^{n-1}$, which is not possible by our choice for δ.

By taking an appropriate power of h, we may assume that

$$K = \langle a_0^{\alpha} b_0 c_0^{pq} \rangle,$$

where $(p, q) = 1$.

Now note that in this case $1 \leq \alpha \leq p-1$, since $\alpha = 0$ implies that $\Phi(b_0) = \Phi(c_0)^{-pq}$ is an element of $\langle \Phi(c_0) \rangle$, which is a contradiction, as we are in case ii). This is case (1). □

6.2. **The cyclic case.** As a consequence of Proposition 15, we may assume

$$K = \langle a_0^{\alpha} b_0 c_0^{pq} \rangle,$$

where $(p, q) = 1$ and $\alpha \in \{0, 1, \dots, p-1\}$. Note that

$$a_0^{\alpha} b_0 c_0^{pq}(z_0, z_1, z_2) = (\rho_1^{\alpha} z_0, \rho_{n-1} z_1, \rho_{n-1}^{q} z_2).$$

6.2.1. *The case $\alpha = 0$.* We next search for polynomials in $\mathbb{C}[z_0, z_1, z_2]^K$. We first note that $z_0 \in \mathbb{C}[z_0, z_1, z_2]^K$. Next, we search for polynomials of the form $z_1^u z_2^v \in \mathbb{C}[z_0, z_1, z_2]^K$, where $u, v \in \{0, 1, \dots, p^{n-1}\}$. The invariance property obligates to have that the values u and v must satisfy the relation

$$u + vq \equiv 0 \mod p^{n-1}.$$

As $(p, q) = 1$, we have that some of those polynomials are given by

$$z_1^{p^{n-1}},\ z_2^{p^{n-1}},\ z_1^q z_2^{p^{n-1}-1}.$$

Let us consider the holomorphic map

$$F : \mathbb{C}^3 \to \mathbb{C}^4$$

$$F(z_0, z_1, z_2) = (z_0, z_1^{p^{n-1}}, z_2^{p^{n-1}}, z_1^q z_2^{p^{n-1}-1}) = (x_1, x_2, x_3, x_4).$$

Let us note that $x_4/x_3 = z_1^q/z_2$. As $(p^{n-1}, q) = 1$, it follows the existence of integers a, b so that $aq + bp^{n-1} = 1$, that is, $z_1 = (z_1^q)^a (z_1^{p^{n-1}})b = (x_4/x_3)^a x_2^b$. It follows that z_1 is uniquely determined by the tuple (x_1, x_2, x_3, x_4) and a choice for z_2. In particular, as z_0 is uniquely determined by x_1, one sees that the map F has degree p^{n-1} and it is K-invariant. In this way, an affine algebraic curve defining $F(C_0)$ is given by

$$F(C_0) = \left\{ \begin{array}{rcl} x_1^p + x_2 + x_3^p & = & 0 \\ \lambda x_1^p + x_2 + 1 & = & 0 \\ x_4^{p^{n-1}} - x_2^q x_3^{p^{n-1}-1} & = & 0 \end{array} \right\} \subset \mathbb{C}^4$$

and a projective one is provided by taking $x_1 = y_0/y_4, x_2 = y_1/y_4, x_3 = y_2/y_4, x_4 = y_3/y_4$, where $[y_0 : y_1 : y_2 : y_3, y_4] \in \mathbb{P}^4$, as follows

$$\left\{ \begin{array}{rcl} y_0^p + y_1 y_4^{p-1} + y_2^p & = & 0 \\ \lambda y_0^p + y_1 y_4^{p-1} + y_4^p & = & 0 \\ y_3^{p^{n-1}} - y_1^q y_2^{p^{n-1}-1} y_4^{1-q} & = & 0 \end{array} \right\} \subset \mathbb{P}^4$$

The map F is, in projective coordinates, given as

$$F([z_0 : z_1 : z_2 : z_3]) = [z_0 z_3^{p^{n-1}-1} : z_1^{p^{n-1}} : z_2^{p^{n-1}} : z_1^q z_2^{p^{n-1}-1} z_3^{1-q} : z_3^{p^{n-1}}] = [y_0 : y_1 : y_2 : y_3 : y_4].$$

As, by the first equality above,

$$y_1 = -\left(\frac{y_0^p + y_2^p}{y_4^{p-1}} \right),$$

the above also provides the (bi-rational) algebraic curve

$$\left\{ \begin{array}{rcl} (\lambda - 1) y_0^p - y_2^p + y_4^p & = & 0 \\ (-1)^{q+1} (y_0^p + y_2^p)^q y_2^{p^{n-1}-1} + y_3^{p^{n-1}} y_4^{qp-1} & = & 0 \end{array} \right\} \subset \mathbb{P}^3.$$

By making the change of coordinates $w_0 = y_0, w_1 = y_2, w_2 = y_3, w_3 = y_4$, the above is written as follows

$$\left\{ \begin{array}{rcl} (\lambda - 1) w_0^p - w_1^p + w_3^p & = & 0 \\ (-1)^{q+1} (w_0^p + w_1^p)^q w_1^{p^{n-1}-1} + w_2^{p^{n-1}} w_3^{qp-1} & = & 0 \end{array} \right\} \subset \mathbb{P}^3$$

and the map F is given as

$$F([z_0 : z_1 : z_2 : z_3]) = [z_0 z_3^{p^{n-1}-1} : z_2^{p^{n-1}} : z_1^q z_2^{p^{n-1}-1} z_3^{1-q} : z_3^{p^{n-1}}] = [w_0 : w_1 : w_2 : w_3].$$

In this case, the group $G = H/K$ is generated by the transformations

$$\begin{array}{rcl} A_1([w_0 : w_1 : w_2 : w_3]) & = & [\rho_1 w_0 : w_1 : w_2 : w_3] \\ B_1([w_0 : w_1 : w_2 : w_3]) & = & [w_0 : w_1 : \rho_{n-1}^q w_2 : w_3] \\ C_1([w_0 : w_1 : w_2 : w_3]) & = & [w_0 : \rho_1 w_1 : \rho_n^{p^{n-1}-1} w_2 : w_3] \end{array}$$

Notice that the elements $A = A_1$ and $B = C_1$ also generates G as desired. As the branched covering map $Q : S \to S/G$ must satisfy that $P = Q \circ F$, where $P : R \to R/H$ is (as in Theorem 8) given by

$$P([z_0 : z_1 : z_2 : z_3]) = -\left(\frac{z_1^{p^{n-1}}}{z_0^p z_3^{p^{n-1}-p}} \right),$$

and since

$$-\left(\frac{z_1^{p^{n-1}}}{z_0^p z_3^{p^{n-1}-p}}\right) = -\left(\frac{y_1 y_4^{p-1}}{y_0^p}\right) = \frac{y_0^p + y_2^p}{y_0^p} = \frac{w_0^p + w_1^p}{w_0^p},$$

we obtain

$$Q([w_0 : w_1 : w_2 : w_3]) = \frac{w_0^p + w_1^p}{w_0^p}.$$

6.2.2. *The case* $\alpha \in \{1, 2 \ldots, p-1\}$. Next, we search for polynomials of the form $z_0^t z_1^u z_2^v \in \mathbb{C}[z_0, z_1, z_2]^K$, where $t \in \{0, 1, \ldots, p-1\}$ and $u, v \in \{0, 1, \ldots, p^{n-1}\}$. The invariance property obligates to have that the values u and v must satisfy the relation

$$t\alpha p^{n-2} + u + vq \equiv 0 \mod p^{n-1}.$$

As $(p, q) = (\alpha, p) = 1$, we have that some of those polynomials are given by

$$z_0^p,\ z_1^{p^{n-1}},\ z_2^{p^{n-1}},\ z_1^q z_2^{p^{n-1}-1},\ z_0^{p-1} z_1^{\alpha p^{n-2}}.$$

Let us consider the holomorphic map

$$F : \mathbb{C}^3 \to \mathbb{C}^5$$

$$F(z_0, z_1, z_2) = (z_0^p, z_1^{p^{n-1}}, z_2^{p^{n-1}}, z_1^q z_2^{p^{n-1}-1}, z_0^{p-1} z_1^{\alpha p^{n-2}}) = (x_1, x_2, x_3, x_4, x_5).$$

Let us note that $x_4/x_3 = z_1^q/z_2$. As $(p^{n-1}, q) = 1$, it follows the existence of integers a, b so that $aq + bp^{n-1} = 1$, from where $z_1 = (z_1^q)^a (z_1^{p^{n-1}})^b = (x_4/x_3)^a x_2^b z_2$. It follows that z_1 is uniquely determined by the tuple $(x_1, x_2, x_3, x_4, x_5)$ and a choice for z_2.

As z_0^p is uniquely determined by x_1, and $z_0^{p-1} z_1^{\alpha p^{n-2}}$ is uniquely determined by x_2, x_3, x_4, x_5 and a choice of z_2, we have that z_0 is also uniquely determined by the previous data.

All the above permits to see that the map F has degree p^{n-1} and it is K-invariant. In this way, an affine algebraic curve defining $F(C_0)$ is given by

$$F(C_0) = \left\{ \begin{array}{rcl} x_1 + x_2 + x_3^p & = & 0 \\ \lambda x_1 + x_2 + 1 & = & 0 \\ x_4^{p^{n-1}} - x_2^q x_3^{p^{n-1}-1} & = & 0 \\ x_5^p - x_1^{p-1} x_2^{\alpha} & = & 0. \end{array} \right\} \subset \mathbb{C}^5$$

We may write $x_2 = -(x_1 + x_3^p)$. In this way, writing $u_1 = x_1$, $u_2 = x_3$, $u_3 = x_4$ and $u_4 = x_5$, the above curve is

$$\left\{ \begin{array}{rcl} (\lambda - 1)u_1 - u_2^p + 1 & = & 0 \\ u_3^{p^{n-1}} + (-1)^{q+1}(u_1 + u_2^p)^q u_2^{p^{n-1}-1} & = & 0 \\ u_4^p + (-1)^{\alpha+1} u_1^{p-1} (u_1 + u_2^p)^{\alpha} & = & 0. \end{array} \right\} \subset \mathbb{C}^4$$

Now, we may write

$$u_1 = \frac{1}{\lambda - 1}(u_2^p - 1),$$

and setting $y_1 = u_2$, $y_2 = u_3$ and $y_3 = u_4$, the above curve is

$$\left\{ \begin{array}{rcl} y_2^{p^{n-1}} + \dfrac{(-1)^{q+1}}{(\lambda-1)^q}(\lambda y_1^p - 1)^q y_1^{p^{n-1}-1} & = & 0 \\ y_3^p + \dfrac{(-1)^{\alpha+1}}{(\lambda-1)^{\alpha+p-1}}(y_1^p - 1)^{p-1}(\lambda y_1^p - 1)^{\alpha} & = & 0. \end{array} \right\} \subset \mathbb{C}^3$$

and F is of the form

$$F(z_0, z_1, z_2) = (z_2^{p^{n-1}}, z_1^q z_2^{p^{n-1}-1}, z_0^{p-1} z_1^{\alpha p^{n-2}}) = (y_1, y_2, y_3).$$

Writing $y_1 = v_0/v_3$, $y_2 = v_1/v_3$ and $y_3 = v_2/v_3$, we obtain the projective model

$$\left\{ \begin{array}{rcl} v_1^{p^{n-1}} v_3^{pq-1} + \dfrac{(-1)^{q+1}}{(\lambda-1)^q}(\lambda v_0^p - v_3^p)^q v_0^{p^{n-1}-1} & = & 0 \\ v_2^p v_3^{p^2+p(\alpha-2)} + \dfrac{(-1)^{\alpha+1}}{(\lambda-1)^{\alpha+p-1}}(v_0^p - v_3^p)^{p-1}(\lambda v_0^p - v_3^p)^{\alpha} & = & 0. \end{array} \right\} \subset \mathbb{P}^3$$

and for $n \geq 3$ we have that $\max\{p^{n-1}, p^{n-1}+q-1, \alpha p^{n-2}+p-1\} = p^{n-1}+q-1$ and therefore $F : \mathbb{P}^3 \to \mathbb{P}^3$ is given as follows.

$$F([z_0 : z_1 : z_2 : z_3]) = [z_2^{p^{n-1}} z_3^{q-1} : z_1^q z_2^{p^{n-1}-1} : z_0^{p-1} z_1^{\alpha p^{n-2}} z_3^{p^{n-1}+q-p-\alpha p^{n-2}} : z_3^{p^{n-1}+q-1}].$$

In the case $n = 2$ a similar formula may be given for F; the maximum value above is $p+q-1$ if $q \geq \alpha$ and $p+\alpha-1$ otherwise.

Continuing with $n \geq 3$, the group $G = H/K$ is generated by the transformations

$$\begin{array}{rcl} A_2([v_0 : v_1 : v_2 : v_3]) & = & [v_0 : v_1 : \rho_1^{p-1} v_2 : v_3] \\ B_2([v_0 : v_1 : v_2 : v_3]) & = & [v_0 : \rho_{n-1}^q v_1 : \rho_1^{\alpha} v_2 : v_3] \\ C_2([v_0 : v_1 : v_2 : v_3]) & = & [\rho_1 v_0 : \rho_n^{p^{n-1}-1} v_1 : v_2 : v_3] \end{array}$$

Notice that the elements $A = A_2$ and $B = C_2$ also generates G as desired. As the branched covering map $Q : S \to S/G$ must satisfy that $P = Q \circ F$, where $P : R \to R/H$ is (as in Theorem 8) given by

$$\begin{array}{rl} P([z_0 : z_1 : z_2 : z_3]) = -\left(\dfrac{z_1^{p^{n-1}}}{z_0^p z_3^{p^{n-1}-p}} \right) = -\left(\dfrac{x_2}{x_1} \right) = \dfrac{u_1 + u_2^p}{u_1} = & \\ = 1 + \dfrac{(\lambda-1)u_2^p}{(u_2^p - 1)} = 1 + \dfrac{(\lambda-1)y_1^p}{(y_1^p - 1)} = & \\ = 1 + \dfrac{(\lambda-1)v_0^p}{v_0^p - v_3^p}, & \end{array}$$

we obtain

$$Q([v_0 : v_1 : v_2 : v_3]) = \frac{\lambda v_0^p - v_3^p}{v_0^p + v_3^p}.$$

6.3. **The non-cyclic case.** In this case,

$$K = \langle b_0^{-p} c_0^{p^2 v}, a_0 c_0^{\gamma p^{n-1}} \rangle,$$

where $(p, v) = 1$ and $\gamma \in \{1, 2, \dots, p-1\}$.

We have that

$$b_0^{-p} c_0^{p^2 v}(z_0, z_1, z_2) = (z_0, \rho_{n-2}^{-1} z_1, \rho_{n-2}^{v} z_2)$$

$$a_0 c_0^{\gamma p^{n-1}}(z_0, z_1, z_2) = (\rho_1 z_0, z_1, \rho_1^{\gamma} z_2)$$

Clearly, $z_0^A z_1^B z_2^C \in \mathbb{C}[z_0, z_1, z_2]^K$ if and only if

$$\left\{ \begin{array}{rcl} A + C\gamma & \equiv & 0 \mod p \\ Cv - B & \equiv & 0 \mod p^{n-2}. \end{array} \right.$$

In this way,

$$z_0^p, z_1^{p^{n-2}}, z_0^{p-\gamma} z_1^v z_2 \in \mathbb{C}[z_0, z_1, z_2]^K.$$

Let us consider the map

$$F : \mathbb{C}^3 \to \mathbb{C}^3$$

$$F(z_0, z_1, z_2) = (z_0^p, z_1^{p^{n-2}}, z_0^{p-\gamma} z_1^v z_2) = (x_1, x_2, x_3).$$

If we fix (x_1, x_2, x_3), then we have p choices for z_0 ($z_0^p = x_1$) and p^{n-2} choices for z_1 ($z_1^{p^{n-2}} = x_2$). Once we have made such choices, the value of z_2 is uniquely determined from $z_0^{p-\gamma} z_1^v z_2 = x_3$. It follows that F has degree p^{n-1} and is K-invariant us desired.

The algebraic curve $F(C_0)$ is provided by

$$F(C_0) = \left\{ \begin{array}{rcl} x_1^{p^{n-1}(p-\gamma)} x_2^{p^2 v}(x_1 + x_2^p) + x_3^{p^n} & = & 0 \\ \lambda x_1 + x_2^p + 1 & = & 0 \end{array} \right\} \subset \mathbb{C}^3.$$

As

$$x_1 = -\frac{(1 + x_2^p)}{\lambda}$$

this curve is also represented by, taking $y_1 = x_2$ and $y_2 = x_3$,

$$\left\{ \frac{(-1)^{p^{n-1}(p-\gamma)}}{\lambda^{p^{n-1}(p-\gamma)}} \left(1 + y_1^p\right)^{p^{n-1}(p-\gamma)} y_1^{p^2 v} \left(y_1^p - \frac{(1 + y_1^p)}{\lambda}\right) + y_2^{p^n} = 0 \right\} \subset \mathbb{C}^2.$$

A projectivization of this plane curve is given by, using the projective coordinates $[u_0 : u_1 : u_2] \in \mathbb{P}^2$ and taking $y_1 = u_0/u_2$ and $y_2 = u_1/u_2$, the following one

$$\left\{ \frac{(-1)^{p^{n-1}(p-\gamma)}}{\lambda^{p^{n-1}(p-\gamma)+1}} \left(u_0^p + u_2^p\right)^{p^{n-1}(p-\gamma)} u_1^{p^2 v} \left((\lambda - 1)u_0^p - u_2^p\right) + u_1^{p^n} u_2^{p^n(p-\gamma-1)+p+p^2 v} = 0 \right\} \subset \mathbb{P}^2.$$

In this case, the transformations a_0, b_0 and c_0 define the transformations

$$\begin{array}{rcl} A_3([u_0 : u_1 : u_2]) & = & [u_0 : \rho_1^{p-\gamma} u_1 : u_2] \\ B_3([u_0 : u_1 : u_2]) & = & [\rho_1 u_0 : \rho_{n-1}^v u_1 : u_2] \\ C_3([u_0 : u_1 : u_2]) & = & [u_0 : \rho_n u_1 : u_2] \end{array}$$

Notice that the elements $A = C_3^{-vp} B_3$ and $B = C_3$ also generates G as desired. As

$$P(z_0, z_1, z_2) = -\left(\frac{z_1^{p^{n-1}}}{z_0^p} \right) = \frac{\lambda y_1^p}{1 + y_1^p},$$

we obtain that

$$Q([u_0 : u_1 : u_2]) = \frac{\lambda u_0^p}{u_0^p + u_1^p}.$$

7. Proof of Theorem 10

Let C be a non-singular projective algebraic curve admitting a p-group H of conformal automorphisms of C with C/H of signature $(0; p, p^{n-1}, p^n, p^n)$ and let $P : C \to C/H = \widehat{\mathbb{C}}$ be a holomorphic branched covering with H as Deck group. We may assume the branch values of P are given by ∞ or order p, 0 of order p^{n-1} and 1 and $\lambda \in \mathbb{C} - \{0, 1\}$ the ones of order p^n. We notice that

$$\mathrm{Aut}_{\mathrm{orb}}(S/H) = \begin{cases} \{I\}, & \lambda \neq -1 \\ \langle J(z) = -z \rangle, & \lambda = -1. \end{cases}$$

Let $K_C = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : C^\sigma \cong C\}$. For each $\sigma \in K_C$ there is a biholomorphism $f_\sigma : C \to C^\sigma$. As H^σ is unique up to conjugation in $\mathrm{Aut}(C^\sigma)$, by Theorem 3, we may assume that $f_\sigma H f_\sigma^{-1} = H^\sigma$. It follows that there is a Möbius transformation M_σ so that $P^\sigma \circ f_\sigma = M_\sigma \circ P$. The transformation M_σ is uniquely determined by f_σ. As M_σ must preserve the cone points and their orders, it follows that $M_\sigma(\infty) = \infty$, $M_\sigma(0) = 0$ and that $\{1, \lambda_\sigma\} = \{M_\sigma(1), M_\sigma(\lambda)\}$, where $\lambda_\sigma \in \mathbb{C} - \{0, 1\}$ is branch value of order p^n of $P^\sigma : C^\sigma \to \widehat{\mathbb{C}}$ (in fact, $\lambda_\sigma = \sigma(\lambda)$). It follows that either (i) $M_\sigma = I$, in which case $\lambda_\sigma = \lambda$ or (ii) $M_\sigma(z) = z/\lambda$, in which case $\lambda_\sigma = 1/\lambda$.

7.1. Let us assume, from now on, that $\lambda \neq -1$.

Lemma 16. *Let $\lambda \neq -1$ and $\sigma \in K_C$. If there is another biholomorphism $\widehat{f_\sigma} : C \to C^\sigma$ such that $\widehat{f_\sigma} H \widehat{f_\sigma}^{-1} = H^\sigma$, then $\widehat{f_\sigma} = h \circ f_\sigma$, for some $h \in H$.*

Proof. If there is another biholomorphism $\widehat{f_\sigma} : C \to C^\sigma$ such that $\widehat{f_\sigma} H \widehat{f_\sigma}^{-1} = H^\sigma$, then $f_\sigma^{-1} \circ \widehat{f_\sigma} \in \mathrm{Aut}(C)$ normalizes H. In this way, $f_\sigma^{-1} \circ \widehat{f_\sigma}$ induces an element of $\mathrm{Aut}_{\mathrm{orb}}(S/H)$. As this last group is trivial, we obtain that $f_\sigma^{-1} \circ \widehat{f_\sigma} \in H$. □

As a consequence of Lemma 16, M_σ is uniquely determined by σ and, in particular, the collection $\{M_\sigma : \sigma \in K_C\}$ satisfies Weil's conditions in Theorem 1. Hence, there is an isomorphism $R : \widehat{\mathbb{C}} \to B$, where B is defined over $\mathcal{M}(C)$, with the property that $R = R^\sigma \circ M_\sigma$ for every $\sigma \in K_C$.

Let us consider the Galois cover $Q : C \to B$, where $Q = R \circ P$. We note that, for $\sigma \in K_C$, it holds that (as $P^\sigma = P$)

$$Q^\sigma \circ f_\sigma = R^\sigma \circ P^\sigma \circ f_\sigma = R \circ M_\sigma^{-1} \circ M_\sigma \circ P \circ f_\sigma^{-1} \circ f_\sigma = R \circ P = Q.$$

Now we follow Dèbes-Emsalem's arguments [6]. Assume we are able to find a point $b \in B$ which is $\mathcal{M}(C)$-rational and so that b is not a branch value of the Galois covering Q. Fix a point $c \in C$ so that $Q(c) = b$. It follows that the H-stabilizer of c is trivial. We have the points $\sigma(c), f_\sigma(c) \in C^\sigma$. As

$$Q^\sigma(\sigma(c)) = \sigma(Q(c)) = \sigma(b) = b,$$

and

$$Q^\sigma(f_\sigma(c)) = Q(c) = b,$$

it follows that there is some $h_\sigma \in H$ so that $h_\sigma(f_\sigma(c)) = \sigma(c)$. Moreover, as a consequence of Lemma 16 and the fact that c has trivial stabilizer in H, such $h_\sigma \in H$ is unique. In this way, we may assume that $f_\sigma(c) = \sigma(c)$ and, by the above, such an isomorphism is uniquely determined by σ. Again, by the uniqueness, this new family $\{f_\sigma : \sigma \in K_\lambda\}$ satisfies Weil's conditions and, by Theorem 1, C is definable over its field of moduli.

In this way, in order to finish our proof, we only need to find a $\mathcal{M}(C)$-rational point on B outside the branch set. This is equivalent to find a point $r \in \widehat{\mathbb{C}} - \{\infty, 0, 1, \lambda\}$ with the property that $R(r) = \sigma(R(r))$, for every $\sigma \in K_C$. As $\sigma(R(r)) = R^{\sigma}(\sigma(r)) = R(M_{\sigma}^{-1}(\sigma(r)))$, we need to find a point $r \in \mathbb{C} - \{0, 1, \lambda\}$ such that

$$M_{\sigma}(r) = \sigma(r).$$

In this way, we need to find a point $r \in \mathbb{C} - \{0, 1, \lambda\}$ so that

(1) if $\sigma(\lambda) = \lambda$, then $\sigma(r) = r$; and
(2) if $\sigma(\lambda) = 1/\lambda$, then $\sigma(r) = r/\lambda$.

Condition (1) asserts that we need to find $r \in \mathbb{Q}(\lambda)$. Clearly, any point of the form $r = \alpha(1 + \lambda)$, where $\alpha \in \mathbb{Q}$ satisfies (1) and (2).

7.2. Let us now consider the case $\lambda = -1$. We have, see Remark 4, that either (i) $\mathrm{Aut}_H(C) = H$ or (ii) $\mathrm{Aut}(C) = \mathrm{Aut}_H(C)$ and $[\mathrm{Aut}(C) : H] = 2$.

In case (i) we may proceed as in the case $\lambda \neq -1$ as Lemma 16 still valid in this situation (the normalizer of H in $\mathrm{Aut}(C)$ is H).

In case (ii) we have that $C/\mathrm{Aut}(C) = (C/H)/\langle J \rangle$, that is, C is quasiplatonic, so it is defined over its field of moduli.

8. Proof of Theorem 12

Since

$$C_{\lambda} = \left\{ \begin{array}{rcl} z_0^p z_3^{p^n - p} + z_1^{p^{n-1}} z_3^{p^n - p^{n-1}} + z_2^{p^n} & = & 0 \\ \lambda z_0^p z_3^{p^{n-1} - p} + z_1^{p^{n-1}} + z_3^{p^{n-1}} & = & 0 \end{array} \right\} \subset \mathbb{P}^3$$

and

$$P([z_0 : z_1 : z_2 : z_3]) = -\left(\frac{z_1^{k^{n-1}}}{z_0^k z_3^{k^{n-1} - k}} \right),$$

then, for each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, one has that $C_{\lambda}^{\sigma} = C_{\sigma(\lambda)}$ and $P^{\sigma} = P$.

Let $K_{\lambda} = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : C_{\lambda} \cong C_{\sigma(\lambda)}\}$, so $\mathcal{M}(C_{\lambda}) = \mathrm{Fix}(K_{\lambda})$.

If $\sigma \in K_{\lambda}$, then there is an isomorphism $f_{\sigma} : C_{\lambda} \to C_{\sigma(\lambda)}$. As a consequence of Theorem 3, we may assume $f_{\sigma} H f_{\sigma}^{-1} = H$. So, there is a Möbius transformation M_{σ} such that $M_{\sigma} \circ P = P^{\sigma} \circ f_{\sigma}$. As M_{σ} must preserve the cone points and their orders, one has that

$$M_{\sigma}(\infty) = \infty, M_{\sigma}(0) = 0, M_{\sigma}\{1, \lambda\} = \{1, \sigma(\lambda)\}.$$

It follows, from the the two first equalities in the above, that $M_{\sigma}(z) = Lz$, for a suitable $L \in \mathbb{C} - \{0\}$. The equality $M_{\sigma}\{1, \lambda\} = \{1, \sigma(\lambda)\}$ asserts that either (1) $L = 1$ and $\sigma(\lambda) = \lambda$ or (2) $L = \sigma(\lambda)$ and $\sigma(\lambda) = 1/\lambda$. As a consequence, we have proved (1) and (2).

Part (3) is consequence of Theorem 10.

9. Galois groups of order p^{n+1}

In this section, we consider those groups G of order $|G| = p^{n+1}$ acting on compact Riemann surfaces with signature $(0; p, p^{n-1}, p^n, p^n)$, for any odd prime p.

The algebraic structure for these groups is determined by the following result.

Proposition 17. *Let p be an odd prime number and let $G < \mathrm{Aut}(S)$ be a group of order $|G| = p^{n+1}$ acting on a compact Riemann surface S with S/G of signature $(0; p, p^{n-1}, p^n, p^n)$.*

Then G is isomorphic to either

(1)
$$\mathbb{Z}_{p^n} \times \mathbb{Z}_p \quad , or$$

(2)
$$\langle x, y : x^{p^n} = y^p = 1, y^{-1}xy = x^{p^{n-1}+1} \rangle .$$

***Remark* 18.** Note that in the first case we have provided, in Theorem 9, algebraic curves for S. In the second case explicit algebraic curves are more complicated, but we will study this problem elsewhere.

Proof. First notice that G has a presentation of the form

$$G = \left\langle x_1, x_2, x_3, x_4 : x_1^{p^n} = x_2^{p^n} = x_3^{p^{n-1}} = x_4^p = x_1x_2x_3x_4 = 1, R \right\rangle$$

where R denotes other relations.

Therefore G cannot be cyclic, since otherwise it could not be generated by elements of the given orders.

Moreover, G has a cyclic subgroup of order p^n, which is normal because it has index p, and therefore G is isomorphic to

$$G \cong \mathbb{Z}_{p^n} \rtimes_\sigma \mathbb{Z}_p = \langle x \rangle \rtimes_\sigma \langle y \rangle$$

where $\sigma(x) = x^u$ with $u^p = 1 \bmod p^n$. The only solutions for u are $u = 1$ and the powers of $u = p^{n-1} + 1$, and the result follows. ◻

***Remark* 19.** We will denote the groups appearing in Proposition 17 as follows.

$$G_u = \left\langle x, y : x^{p^n} = y^p = 1, y^{-1}xy = x^u \right\rangle \tag{7}$$

with $u = 1$ or $u = 1 + p^{n-1}$, and we will study the families of algebraic curves admitting G_u actions with signature $(0; p^n, p^n, p^{n-1}, p)$.

Lemma 20. *Consider the groups G_u given by* (7) *and*

$$\Gamma = \left\langle a_0,\ b_0, c_0\ d_0 \quad : \quad a_0^p = b_0^{p^{n-1}} = c_0^{p^n} = d_0^{p^n} = a_0b_0c_0d_0 = 1 \right\rangle . \tag{8}$$

Assume $\Phi : \Gamma \twoheadrightarrow G_u$ is an epimorphism such that $K = \ker\Phi$ is torsion-free.

Then either

I) $K = \left\langle\left\langle b_0c_0^{-pq}a_0^{-\alpha}, a_0^{-1}c_0a_0c_0^{-u^s} \right\rangle\right\rangle$, *with* $0 \le \alpha \le p-1,\ \ 0 < s < p$ *and* $(q, p) = 1$, *or*

II) $K = \left\langle\left\langle a_0c_0^{-p^{n-1}v}, b_0^pc_0^{-p^2q}, b_0^{-1}c_0b_0c_0^{-u^s} \right\rangle\right\rangle$, *with* $1 \le v \le p-1,\ \ 0 < s < p$ *and* $(q, p) = 1$,

where $\langle\langle\cdot\rangle\rangle$ denotes the normal closure in Γ.

Proof. Since K is torsion-free, we obtain that

a) $K \cap \langle a_0 \rangle = \{1\}$, and it follows that $y_1 = \Phi(a_0)$ has order p;
b) $K \cap \langle b_0 \rangle = \{1\}$, and it follows that $y_2 = \Phi(b_0)$ has order p^{n-1};
c) $K \cap \langle c_0 \rangle = \{1\}$, and it follows that $y_3 = \Phi(c_0)$ has order p^n;

d) $K \cap \langle a_0 b_0 c_0 \rangle = \{1\}$, and it follows that $y_4 = \Phi(d_0)$ has order p^n.

Since Φ is an epimorphism, $\{y_1, y_2, y_3, y_4\}$ generate G_u. But clearly $y_4 = (y_1 y_2 y_3)^{-1}$, and therefore $\{y_1, y_2, y_3\}$ generate G_u.

We now examine the following two cases separately.

Case I) Suppose $\langle y_1, y_3 \rangle = G_u$.

We have that $G_u = \langle y_3 \rangle \rtimes_{u^s} \langle y_1 \rangle$ for some $0 < s < p$. Also $y_2 = y_1^{\alpha} y_3^{pq}$ with $(q, p) = 1$. Hence $y_2 y_3^{-pq} y_1^{-\alpha} = \Phi(b_0 c_0^{-pq} a_0^{-\alpha}) = 1$ and it follows that $b_0 c_0^{-pq} a_0^{-\alpha} \in K$.

Furthermore $\Phi(a_0^{-1} c_0 a_0 c_0^{-u^s}) = y_1^{-1} y_3 y_1 y_3^{-u^s} = 1$ and it follows that $a_0^{-1} c_0 a_0 c_0^{-u^s} \in K$.

Then, checking the order of $\Gamma / \langle\langle b_0 c_0^{-pq} a_0^{-\alpha}, a_0 c_0^{-1} a_0^{-1} c_0^{-u^s} \rangle\rangle$, we obtain the required

$$K = \left\langle\left\langle b_0 c_0^{-pq} a_0^{-\alpha}, a_0 c_0^{-1} a_0^{-1} c_0^{-u^s} \right\rangle\right\rangle.$$

Case II) Suppose $\langle y_1, y_3 \rangle < G_u$.

Then $y_1 = y_3^{p^{n-1} v}$ with $(v, p) = 1$, since $\langle y_3 \rangle$ is a maximal subgroup of G_u. Hence $a_0 c_0^{-p^{n-1} v} \in K$.

It this case $\langle y_2, y_3 \rangle = G_u = \langle y_3 \rangle \rtimes_{u^s} \langle y_2 \rangle$ for some $0 < s < p$. Hence $y_2^{-1} y_3 y_2 y_3^{-u^s} = 1$ from where $b_0^{-1} c_0 b_0 c_0^{-u^s} \in K$.

Finally, $y_2^p = y_3^{p^2 q}$ with $(q, p) = 1$, from where $b_0^p c_0^{-p^2 q} \in K$.

Again, checking the order of $\Gamma / \langle\langle a_0 c_0^{-p^{n-1} v}, b_0^p c_0^{-p^2 q}, b_0^{-1} c_0 b_0 c_0^{-u^s} \rangle\rangle$ we obtain $K = \langle\langle a_0 c_0^{-p^{n-1} v}, b_0^p c_0^{-p^2 q}, b_0^{-1} c_0 b_0 c_0^{-u^s} \rangle\rangle$. □

Considering the above notation for the elements $y_1 = \Phi(a_0)$, $y_2 = \Phi(b_0)$, $y_3 = \Phi(c_0)$ and $y_4 = \Phi(d_0)$ in G_u, we have the following result, which states that examples for both cases as Proposition 17 exist, by the Riemann existence theorem.

Corollary 21. *If the group G_u, with $u = 1$ or $u = 1 + p^{n-1}$, acts on a compact Riemann surface with signature $(0; p, p^{n-1}, p^n, p^n)$, then a generating vector for the action may be chosen to be exactly of one of the following forms.*

a) $(y_1, y_1^{\alpha} y_3^{pq}, y_3, y_3^{-1-pq} y_1^{-1-\alpha})$, *with* $(q, p) = 1$ *and* $1 \leq \alpha \leq p - 2$.
b) $(y_1, y_3^{pq}, y_3, y_3^{-1-pq} y_1^{-1})$, *with* $(q, p) = 1$.
c) $(y_1, y_1^{-1} y_3^{pq}, y_3, y_3^{-1-pq})$, *with* $(q, p) = 1$
d) $(y_3^{p^{n-1} v}, y_2, y_3, y_3^{-1-p^{n-1} v} y_2^{-1})$

In the first three cases the order of y_1 is p, the order of y_3 is p^n and $y_1^{-1} y_3 y_1 = y_3^{u^s}$ with $0 < s < p$. In the last case y_2 has order p^{n-1}, y_3 has order p^n, $y_2^p = y_3^{q p^2}$ and $y_2^{-1} y_3 y_2 = y_3^{u^s}$ with $0 < s < p$.

The following table gives the genera of some intermediate curves, where g_L denotes the genus of the quotient of S by the subgroup $L \leq \mathrm{Aut}(S)$.

generating vector	$u = 1 + p^{n-1}$	$u = 1$
$(y_1, y_1^{\alpha} y_3^{pq}, y_3, y_3^{-1-pq} y_1^{-1-\alpha})$	$g_{\langle y_3\rangle} = \frac{p-1}{2}$ $g_{\langle y_3^{-1-pq} y_1^{-1-\alpha}\rangle} = \frac{p-1}{2}$ $g_{\langle y_1\rangle} = \frac{2p^n - p^{n-2}(2p-1)-p}{2}$ $g_{\langle y_3^p\rangle} = p^2 - 2p + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = p - 1$	$g_{\langle y_3\rangle} = \frac{p-1}{2}$ $g_{\langle y_3^{-1-pq} y_1^{-1-\alpha}\rangle} = \frac{p-1}{2}$ $g_{\langle y_1\rangle} = \frac{p^n - p}{2}$ $g_{\langle y_3^p\rangle} = p^2 - 2p + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = p - 1$
$(y_1, y_3^{pq}, y_3, y_3^{-1-pq} y_1^{-1})$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-pq} y_1^{-1}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{2p^n - p^{n-2}(2p-1)-p}{2}$ $g_{\langle y_3^p\rangle} = \frac{p^2-3p}{2} + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = \frac{p-1}{2}$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-pq} y_1^{-1}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{p^n - p}{2}$ $g_{\langle y_3^p\rangle} = \frac{p^2-3p}{2} + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = \frac{p-1}{2}$
$(y_1, y_1^{-1} y_3^{pq}, y_3, y_3^{-1-pq})$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-pq}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{2p^n - p^{n-2}(2p-1)-p}{2}$ $g_{\langle y_3^p\rangle} = p^2 - 2p + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = p - 1$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-pq}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{p^n - p}{2}$ $g_{\langle y_3^p\rangle} = p^2 - 2p + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = p - 1$
$(y_3^{p^{n-1} v}, y_2, y_3, y_3^{-1-p^{n-1} v} y_2^{-1})$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-p^{n-1} v} y_2^{-1}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{2p^n - p^{n-1} - p}{2}$ $g_{\langle y_3^p\rangle} = \frac{p^2-3p}{2} + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = \frac{p-1}{2}$	$g_{\langle y_3\rangle} = 0$ $g_{\langle y_3^{-1-p^{n-1} v} y_2^{-1}\rangle} = 0$ $g_{\langle y_1\rangle} = \frac{2p^n - p^{n-1} - p}{2}$ $g_{\langle y_3^p\rangle} = \frac{p^2-3p}{2} + 1$ $g_{\langle y_3^p, y_1\rangle} = 0$ $g_M = \frac{p-1}{2}$

where M is any cyclic maximal subgroup acting freely.

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