ALMOST-ADDITIVE THERMODYNAMIC FORMALISM FOR COUNTABLE MARKOV SHIFTS

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ABSTRACT. This paper is devoted to extend the thermodynamic formalism theory to almost-additive sequences of continuos functions defined over topologically mixing, non compact, countable Markov shifts. Difficulties are two fold, on the one hand we have to deal with the lack of compactness of the phase space and on the other with the non-additivity of the sequence potentials. In this context, based on the work of Sarig and also on the work of Barreira, we introduce a definition of pressure. We prove that it satisfies the variational principle and hence it is good definition. Under certain combinatorial assumptions on the shift space (that of being BIP) we prove the existence and uniqueness of Gibbs measures. Applications are given, among others, to the study of maximal Lypaunov exponents of product of matrices and to obtain a formula for the Hausdorff dimension of certain geometrical constructions.

1. INTRODUCTION

This paper has two different starting points. On the one hand, we have the thermodynamic formalism developed for sub-additive and almost-additive sequences of continuous functions. The idea of this theory is to generalise classical results on thermodynamic formalism replacing the pressure of a continuous function with the pressure of a sequence of continuous functions. It was Falconer [F1] who introduced this set of ideas with the purpose of studying dimension theory of non-conformal systems. Recall that the relation between thermodynamic formalism and dimension theory has been extensively and successfully exploited ever since the pioneering work of Bowen [Bo2] (see the books [B4, F2, P, PrU] for recent developments of the theory). If an expanding dynamical system, $T : M \to M$, is conformal then it is possible to describe in great detail the Hausdorff dimension of dynamically defined subsets of the phase space. For instance, making use of the classic thermodynamic formalism, it is possible to study the size (e.g. Hausdorff dimension) of level sets determined by pointwise dimension of Gibbs measures or by Lyapunov exponents (see [B4, Chapter II and III]). The situation is far less developed if no conformal assumption is made on the system. There are several reasons for this, one of them being that, in the conformal setting, dynamically defined balls are *almost* balls and they form an optimal cover. However, in the non-conformal setting dynamically defined balls are, at best, ellipses. Therefore, it is likely that the dynamical cover

Date: November 11, 2011.
2000 *Mathematics Subject Classification.* 37D35, 37D25.
Key words and phrases. Thermodynamic formalism, equilibrium measures, Gibbs measures, almost-additive sequences.
GI was partially supported by Proyecto Fondecyt 1110040. YY was supported by Proyecto Fondecyt Postdoctorado 3090015 and Basal grant.

is not optimal. This is clearly related to the fact that the natural function used to estimate the dimension, namely the Jacobian, in the conformal setting satisfies

$$\|DT^{m+n}(x)\| = \|DT^m(T^n x)\| \|DT^n(x)\|,$$

where T^n denotes the $n-$th iterate of the map T and $\| \cdot \|$ is the operator norm. Whereas, if the map is not conformal we only have

$$\|DT^{m+n}(x)\| \leq \|DT^m(T^n x)\| \|DT^n(x)\|.$$

Therefore, the sequence defined by $\phi_n(x) = \log \|DT^n(x)\|$ is sub-additive (not additive) and a new thermodynamic formalism is required to deal with this situation. This was the main motivation of Falconer [F1]. This problem has attracted a great deal of attention over the last two decades and recently interesting developments have been obtained. We would like to single out the work of Barreira [B1, B2, B3] among many other substantial contributions to the theory. We will build up on his work. Our attention will be focused on a particular class of sub-additive sequences, namely, almost-additive sequences (see Definition 2.1). This condition is satisfied for example if $\mathcal{F} = \{\phi_n\}_n$ denotes the n-fold product of positive matrices and the associated dynamical system is a (compact) sub-shift of finite type defined over a finite alphabet (see [Fe1]). Under other cone type conditions this sequence is almost-additive also in non-compact settings (see Section 7 for precise statements). This remark immediately leads the way for applications in the dimension theory of non-conformal dynamical systems. It should be pointed out that even in the compact setting several results we present here are not valid under weaker additivity assumptions.

Our second starting point is the ergodic theory for countable Markov shifts. Uniformly hyperbolic dynamical systems have finite Markov partitions, see [Bo3, Chapter 3] and references therein for the case of discrete time dynamical systems. That type of coding allows for the proof of a great deal of fundamental results in ergodic theory. When systems are uniformly hyperbolic in most of the phase space but not in all of it, sometimes it is still possible to construct Markov partitions, although this time over countable alphabets. Probably, the best known example of such a situation being the Manneville-Pomeau map [PM, S2], which is an expanding interval map with a parabolic fixed point. Dynamical systems that can be coded using countable Markov partitions also occur in the study of one-dimensional real and complex multimodal maps. Indeed, a successful technique used to study the ergodic theory of these maps and overcome the lack of hyperbolicity due to the existence of critical points is the so called *inducing procedure* (see for example [BT, IT1, PS, PrR]). Given a multimodal map it is possible to associate an induced map (which is a generalisation of the first return time map) which possesses a countable Markov partition. The idea is to translate problems to this new system, solve them and then push the results back. The case of C^r diffeomorphisms defined over compact orientable smooth surface was recently studied by Sarig [S5]. He constructed countable Markov partitions for large invariant sets. While all the above examples are important, the most natural ones arise in number theory. Indeed, the Gauss map and the map associated to the Jacobi-Perron algorithm are both Markov over countable partitions (see [Ma1, Ma2, PW]). The thermodynamic formalism for countable Markov shifts has been developed by Mauldin and Urbański [MU1, MU2] and by Sarig [S1, S2, S3, S4] (see also [FFY, GS]). The main difficulty is that the phase space is no longer compact, therefore fixed point theorems used in the compact setting can not be applied here and new techniques have to be developed.

The aim of this paper is to bring these two theories together. We develop a thermodynamic formalism for almost-additive sequences of potentials defined over topologically mixing countable Markov shifts. In doing so, we construct a set of tools that can be used to tackle a wide range of problems for which, at present time, no machinery is available. Generalising the work of Barreira [B1, B2, B3], of Mauldin and Urbański [MU1, MU2] and that of Sarig [S1, S2, S3, S4], we define a notion of pressure for almost-additive sequences of functions defined over a (non-compact) countable Markov shift (see Section 2). We prove that this pressure satisfies the variational principle (see Theorem 3.1). This is an important result that relates objects of a topological nature (the pressure) with measure theoretic ones. The problem of the existence of Gibbs measures is also addressed, in Section 4. Under a combinatorial assumption on the shift (that of being BIP) we prove the existence of Gibbs measures (see Theorem 4.1). We stress that our definition and the variational principle hold for *any* topologically mixing countable Markov shift.

As an application of our results, we study the Maximal Lyapunov exponents of product of matrices (see Section 7). We consider a countable collection of $d \times d$ matrices, $\{A_1, A_2, \dots\}$, and a topologically mixing countable Markov shift (Σ, σ). If $w = (i_0, i_1, \dots) \in \Sigma$ define a sequence of functions by

$$\phi_n(w) = \log \|A_{i_{n-1}} \cdots A_{i_1} A_{i_0}\|.$$

The thermodynamic formalism for such class of sub-additive sequence has been extensively studied over the last years. Feng, in a series of articles [Fe1, Fe2, Fe3], has described in great detail the ergodic properties in the case in which (Σ, σ) is a topologically mixing sub-shift over a finite alphabet. We extend some of his results to this non-compact setting under a combinatorial assumption on the shift and an almost-additivity assumption on the sequence.

Another application we obtain is a formula relating the pressure for almost-additive sequences and the Hausdorff dimension of certain geometric constructions (see Section 8). We actually generalise results obtained by Barreira [B1].

Finally, a couple of examples are discussed. We obtain an explicit formula for the pressure of an almost-additive sequence of locally constant functions in the case of the full-shift (Section 5). We also discuss the thermodynamic formalism of an almost-additive sequence of continuous functions that naturally arises in the study of factor maps (see Section 6).

2. Definition of (almost-additive) Gurevich pressure

Let (Σ, σ) be a one-sided Markov shift over a countable alphabet S. This means that there exists a matrix $(t_{ij})_{S \times S}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma = \left\{x \in S^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}_0\right\}.$$

The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(x) = x'$, for $x = (x_n)_{n=0}^{\infty}, x' = (x'_n)_{n=0}^{\infty}, x'_n = x_{n+1}$ for all $n \in \mathbb{N}_0$. Sometimes we simply say that (Σ, σ) is a *countable Markov shift.* For an admissible word $i_0 \dots i_{n-1}$ of length n in Σ, we define a cylinder set $C_{i_0 \dots i_{n-1}}$ of length n by

$$C_{i_0 \cdots i_{n-1}} = \{x \in \Sigma : x_j = i_j \text{ for } 0 \le j \le n-1\}.$$

We equip Σ with the topology generated by the cylinders sets. We denote by $\mathcal{M}$ the set of σ-invariant Borel probability measures on Σ. We will always assume (Σ, σ) to be topologically mixing, that is, for every $a, b \in S$ there exists $N_{ab} \in \mathbb{N}$ such that for every $n > N_{ab}$ we have $C_a \cap \sigma^{-n} C_b \neq \emptyset$.

Definition 2.1. Let (Σ, σ) be a one-sided countable state Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is called *sub-additive* if for every $n, m \in \mathbb{N}, x \in \Sigma$, we have

$$0 < f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x). \tag{1}$$

Recall that if $\mathcal{F}$ is a sub-additive sequence, then by Kingman's sub-additive ergodic theorem [Ki], there exists a measurable function f with the following property: Let $\mu \in \mathcal{M}$. If $\log f_n : \Sigma \to \mathbb{R} \cup \{-\infty\}$ for all $n \in \mathbb{N}$ and $(\log f_1)^+ \in L^1(\mu)$, then for μ-almost every $x \in \Sigma$,

$$\lim_{n\to\infty} \frac{1}{n} \log f_n(x) = f(x) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \int \log f_n(x)\, d\mu = \int f(x)\, d\mu.$$

Definition 2.2. Let (Σ, σ) be a one-sided countable state Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is called *almost-additive* if there exists a constant $C \geq 0$ such that for every $n, m \in \mathbb{N}, x \in \Sigma$, we have

$$f_n(x) f_m(\sigma^n x) e^{-C} \leq f_{n+m}(x), \tag{2}$$

and

$$f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x) e^{C}. \tag{3}$$

In most parts of this paper, we will assume the sequence $\mathcal{F}$ to be almost-additive. It should be pointed out that even in the compact setting several results we present here are not valid under weaker assumptions.

One of the important ingredients in thermodynamic formalism is the regularity assumptions on the (sequence of) continuous functions. Several results depend upon this hypothesis. In the rest of paper, we will always assume the following regularity conditions.

Definition 2.3. Let (Σ, σ) be a one-sided countable Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be continuous. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is called a *Bowen* sequence if there exists $M \in \mathbb{R}^+$ such that

$$\sup\{A_n : n \in \mathbb{N}\} \leq M, \tag{4}$$

where

$$A_n = \sup\left\{ \frac{f_n(x)}{f_n(y)} : x, y \in \Sigma, x_i = y_i \text{ for } 0 \leq i \leq n-1 \right\}.$$

The definition above is related to a regularity assumption introduced by Bowen when studying classic thermodynamic formalism. Indeed, for $f \in C(\Sigma)$, let $V_n(f) := \sup\{|f(x) - f(x')| : x, x' \in \Sigma, x_i = x'_i, 0 \leq i \leq n-1\}$ and $\sigma_n f := \sum_{i=0}^{n-1} f \circ \sigma^i$. The *Bowen class* is defined by $\{f \in C(\Sigma) : \sup_{n\in\mathbb{N}} V_n(\sigma_n f) < \infty\}$ (see [Bo1, W4]). Most of the thermodynamic formalism for continuous functions is well developed when the function belongs to the Bowen class. We can think of a Bowen sequence as the natural generalisation of a function belonging to the Bowen class.

We also remark that our definition of Bowen sequence restricted to a compact set is equivalent to the definition of sequences of bounded variation given by Barreira in [B2] (see also [M] for similar conditions). It is plausible that results presented in Sections 2 and 3 can be extended to a larger class of sequences of continuous functions satisfying a tempered condition of the type $\lim_{n\to\infty} A_n/n = 0$ (see [B2] for more about this condition).

The aim of this section is to provide a good definition of pressure for almost-additive sequences of continuous functions defined over a countable Markov shift. There exist several definitions of pressure for sub-additive (and hence for almost-additive) sequences defined over compact spaces. Indeed, Falconer [F1] gave a definition that behaves well for sub-shifts of finite type defined over finite alphabets. Cao, Feng and Huang [CFH] gave one based on $(n, \epsilon)-$sets and Barreira [B1, B2] studied a definition using the theory of dimension-like characteristics developed by Pesin [P]. Mummert also gave a definition in the same spirit [M]. We stress that our situation completely differs from the above since our phase space is no longer compact. It is important that our definition does not depend upon the metric (as in the case of Cao et al [CFH]), so that it satisfies a variational principle (for a discussion of this issue see Section 3). Also note that we cannot continuously extend $\mathcal{F}$ to any compactification of the space Σ (which is what is needed if we want to extend the definitions of Barreira [B1, B2] or Mummert [M]) because the sequence $\mathcal{F}$ is not assumed to be bounded. Finally, note that if we extend the definition given by Falconer [F1] to the countable Markov shift setting, then only a narrow class of shifts would satisfy the variational principle (see [Gu1, Gu2]).

For countable Markov shifts, the thermodynamic formalism has been developed by Mauldin and Urbański [MU1, MU2] for a certain class of Markov shifts with combinatorics close to that of the full-shift and in full generality by Sarig [S1, S2, S3, S4]. The definition we propose is both a generalisation of the Gurevich pressure for Markov shifts over a countable alphabet introduced by Sarig in [S1] and the pressure for almost-additive sequences on compact spaces introduced independently by Barreira in [B2] and by Mummert [M].

Definition 2.4. Let (Σ, σ) be a topologically mixing countable state Markov shift. For $a \in S$ and an almost-additive Bowen sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ, we define the *partition function* by

$$Z_n(\mathcal{F}, a) = \sum_{\sigma^n x = x} f_n(x) \chi_{C_a}(x), \tag{5}$$

where $\chi_{C_a}(x)$ is the characteristic function of the cylinder C_a.

Definition 2.5. Let (Σ, σ) be a topologically mixing countable state Markov shift. The *Gurevich pressure* of an almost-additive sequence $\mathcal{F}$ on Σ is defined by

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a). \tag{6}$$

The limit in equation (6) exists and does not depend on the choice of $a \in S$. Indeed, see Theorem 2.1 for details.

Remark 2.1. If $f : \Sigma \to \mathbb{R}$ is a continuous function of summable variations, define the additive sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ where $f_n(x) = e^{f(x)+f(\sigma x)+\cdots+f(\sigma^{n-1}x)}$ for every $n \in \mathbb{N}, x \in \Sigma$. Then the (almost-additive) Gurevich pressure of $\mathcal{F}$ coincides with the Gurevich pressure of f defined by Sarig in [S1]. Also, if the Markov shift (Σ, σ) is defined over a finite alphabet, then the (almost-additive) Gurevich pressure coincides with the almost-additive pressure introduced by Barreira in [B2](compare also with Mummert's definition in [M]).

The rest of this section is devoted to prove that the Gurevich pressure is actually well defined (see Theorem 2.1). Throughout the rest of this section, we assume that S is a countable alphabet and (Σ, σ) is a topologically mixing countable Markov shift. For a Bowen sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ, we let M and $A_n, n \in \mathbb{N}$ be defined as in Definition 2.3. Our main goal in this section is to prove the following theorem.

Theorem 2.1. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ. Then for $a \in S$,*

(1) *The limit*
$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a)$$
exists and it is not minus infinity.

(2) *If $\|L_{f_1} 1\|_{\infty} < \infty$, then*
$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a) \neq \infty.$$

(3) *The limit*
$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a)$$
is independent of the symbol $a \in S$.

To prove Theorem 2.1 we need the following Lemmas.

Lemma 2.1. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a Bowen sequence on Σ that satisfies equation* (2)*. Then there exists a constant $k \in \mathbb{R}$ such that for every $a \in S, n, m \in \mathbb{N}$, we have*
$$Z_n(\mathcal{F}, a) Z_m(\mathcal{F}, a) \leq Z_{n+m}(\mathcal{F}, a) e^k.$$

Proof. For $a \in S$, let $x \in \Sigma \cap C_a$ be a periodic point of period n, that is $\sigma^n x = x$. Then we can write $x = (x_0, x_1, \ldots, x_{n-1}, x_0, x_1, \ldots)$, where $x_0 = a$. Let A denote the admissible word $x_0 \ldots x_{n-1}$. Consider now $\tilde{x} \in \Sigma \cap C_a$, a periodic point of period m. Again, we can write $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{m-1}, \tilde{x}_0, \tilde{x}_1 \ldots)$, where $\tilde{x}_0 = a$. Let B denote the admissible word $\tilde{x}_0 \ldots \tilde{x}_{m-1}$. Let us consider now the point $x' \in \Sigma \cap C_a$ obtained by concatenating the admissible words A and B, that is
$$x' = ABABABABABAB \cdots := (AB)^{\infty}.$$
Clearly $\sigma^{n+m} x' = x'$. Since $\mathcal{F}$ satisfies equation (2), we obtain
$$e^{-C} f_n(x') f_m(\sigma^n x') \leq f_{n+m}(x'),$$
which implies that
$$e^{-C} f_n(x') f_m(\sigma^n x') f_n(x) f_m(\tilde{x}) \leq f_{n+m}(x') f_n(x) f_m(\tilde{x}).$$
Hence
$$f_n(x) f_m(\tilde{x}) \leq e^C f_{n+m}(x') \frac{f_n(x) f_m(\tilde{x})}{f_n(x') f_m(\sigma^n x')}.$$
Since for every $0 \leq i \leq m-1$ we have $(\sigma^n x')_i = \tilde{x}_i$, we obtain
$$f_n(x) f_m(\tilde{x}) \leq e^C f_{n+m}(x') A_n A_m \leq e^C M^2 f_{n+m}(x').$$
The result now follows setting $e^k = e^C M^2$. □

Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on Σ and let $m \in \mathbb{N}$. For $w : \Sigma \to \mathbb{R}$ a continuous function, we define

$$(L_{\mathcal{F}}^m w)(x) := \sum_{\sigma^m z = x} f_m(z) w(z) \quad \text{for } x \in \Sigma.$$

Let $Y \subset \Sigma$ be a topologically mixing finite state Markov shift. Then for a sequence $\mathcal{F}|_Y := \{\log f_n|_Y\}_{n=1}^{\infty}$, we have

$$\left(L_{\mathcal{F}|_Y}^m w\right)(y) = \sum_{\sigma^m z = y} f_m(z) w(z) \chi_Y(z) \quad \text{for } y \in Y.$$

Lemma 2.2. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a Bowen sequence on Σ. Let $Y \subset \Sigma$ be a topologically mixing finite state Markov shift such that $Y \cap C_a \neq \emptyset, a \in S$. Define*

$$Z_n(Y, \mathcal{F}, a) = \sum_{\sigma^n x = x} f_n(x) \chi_{C_a \cap Y}(x).$$

Then for each $a \in S, x \in C_a \cap Y, n \in \mathbb{N}$,

$$\frac{1}{M} L_{\mathcal{F}|_Y}^n (\chi_{C_a})(x) \leq Z_n(Y, \mathcal{F}, a) \leq M L_{\mathcal{F}|_Y}^n (\chi_{C_a})(x).$$

Proof. For $a \in S$, let $x \in C_a \cap Y$ and write $x = a\tilde{x}$. Then

$$L_{\mathcal{F}|_Y}^n (\chi_{C_a})(a\tilde{x}) = \sum_{\sigma^n z = a\tilde{x}} f_n|_Y(z) \chi_{C_a \cap Y}(z) = \sum_{z \in Y, z = a z_1 z_2 \dots z_{n-1} a \tilde{x}} f_n|_Y(z).$$

Let $\overline{x} \in C_a \cap Y$ be such that $\sigma^n \overline{x} = \overline{x}$. Then we can write the point $\overline{x} = (a, \overline{x}_1, \dots, \overline{x}_{n-1}, a, \overline{x}_1, \dots, \overline{x}_{n-1}, \dots)$. Now set $a z_1 z_2 \dots z_{n-1} = a \overline{x}_1 \dots \overline{x}_{n-1}$. In this way, we have $a \overline{x}_1 \dots \overline{x}_{n-1} a \tilde{x} \in C_a \cap Y$. Since $\mathcal{F}$ is in the Bowen class,

$$\frac{f_n(\overline{x})}{f_n|_Y(a \overline{x}_1 \dots \overline{x}_{n-1} a \tilde{x})} \leq M.$$

Therefore,

$$Z_n(Y, \mathcal{F}, a) \leq M L_{\mathcal{F}|_Y}^n (\chi_{C_a})(x).$$

In order to prove the other inequality, for $a y_1 \dots y_{n-1} a \tilde{x} \in Y$, we define the point $x' = (a, y_1, \dots, y_{n-1}, a, y_1 \dots, y_{n-1}, a, \dots)$ of period n. In this way, $x' \in C_a \cap Y$ and $\sigma^n x' = x$. Therefore,

$$\frac{f_n|_Y(a y_1 \dots y_{n-1} a \tilde{x})}{f_n(x')} \leq M.$$

Hence

$$\frac{1}{M} L_{\mathcal{F}|_Y}^n (\chi_{C_a})(x) \leq Z_n(Y, \mathcal{F}, a).$$

□

Lemma 2.3. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a Bowen sequence on Σ that satisfies equation* (2) *and let $Y \subset \Sigma$ be a topologically mixing finite state Markov shift. Then there exists $\beta > 0$ such that for each $a \in S, x \in C_a \cap Y, n \in \mathbb{N}$,*

$$\left(L_{\mathcal{F}|_Y}^n \chi_{C_a}\right)(x) \geq \beta^n e^{-(n-1)C}.$$

Moreover,

$$Z_n(Y, \mathcal{F}, a) \geq \frac{1}{M} \beta^n e^{-(n-1)C},$$

and so

$$Z_n(\mathcal{F}, a) \geq \frac{1}{M} \beta^n e^{-(n-1)C},$$

where C is defined as in equation (2).

Proof. For $x \in Y \cap C_a$, we have that

$$\left(L^n_{\mathcal{F}|_Y} \chi_{C_a}\right)(x) = \sum_{\sigma^n z = x} f_n|_Y(z) \chi_{C_a \cap Y}(z) = \sum_{z \in Y, z = a z_1 \dots z_{n-1} x} f_n|_Y(z).$$

If $x = a\tilde{x}$, we have

$$\left(L^n_{\mathcal{F}|_Y} \chi_{C_a}\right)(a\tilde{x}) = \sum_{z \in Y, z = a z_1 z_2 \dots z_{n-1} a\tilde{x}} f_n|_Y(z) \geq \min_{z \in Y} f_n|_Y(z).$$

By virtue of equation (2) we see that for every $x \in \Sigma$,

$$\begin{aligned} f_n(x) &\geq e^{-C} f_{n-1}(\sigma x) f_1(x) \geq e^{-2C} f_{n-2}(\sigma^2 x) f_1(\sigma x) f_1(x) \\ &\geq \dots \\ &\geq e^{-(n-1)C} f_1(\sigma^{n-1} x) f_1(\sigma^{n-2} x) \dots f_1(\sigma x) f_1(x). \end{aligned}$$

Since the space Y is compact and invariant, we set $\beta = \min_{z \in Y} f_1|_Y(z)$. Then, for every $y \in Y$ we have

$$f_n(y) \geq e^{-(n-1)C} \beta^n.$$

From the above equation, we can conclude that

$$\left(L^n_{\mathcal{F}|_Y} \chi_{C_a}\right)(x) \geq \beta^n e^{-(n-1)C}.$$

The above result together with Lemma 2.2 implies that

$$Z_n(Y, \mathcal{F}, a) \geq \frac{1}{M} \beta^n e^{-(n-1)C},$$

and so

$$Z_n(\mathcal{F}, a) \geq \frac{1}{M} \beta^n e^{-(n-1)C}.$$

□

Given $f : \Sigma \to \mathbb{R}$ a continuous function, the *transfer operator* L_f applied to function $g : \Sigma \to \mathbb{R}$ is formally defined by

$$(L_f g)(x) := \sum_{\sigma z = x} f(z) g(z) \quad \text{for every } x \in \Sigma. \tag{7}$$

In the following lemma, we will make use of the supremum norm of a continuous function. For a function $g : \Sigma \to \mathbb{R}$, define $\|g\|_\infty = \sup\{|g(x)| : x \in \Sigma\}$.

Lemma 2.4. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ be a Bowen sequence on Σ that satisfies equation* (3)*. Then there exists $\tilde{M} > 0$ such that for any $a \in S, n \in \mathbb{N}$*

$$Z_n(\mathcal{F}, a) \leq \tilde{M} e^{C(n-1)} \|L_{f_1} 1\|^n_\infty,$$

where C is defined as in equation (3)*.*

Proof. Let $a \in S$. In order to prove this lemma, it suffices to show that there exists $\tilde{M} > 0$ such that for every $x \in \Sigma$,

$$Z_n(\mathcal{F}, a) \leq \tilde{M} \left(L^n_{\mathcal{F}} 1\right)(x) \leq \tilde{M} \left(L^n_{f_1} 1\right)(x) e^{C(n-1)} \leq \tilde{M} e^{C(n-1)} \|L_{f_1} 1\|^n_\infty.$$

We can show the first inequality using a similar argument as the one used in the proof of Lemma 2.2 (replace Y by Σ).

Now we show the second inequality. Note that if $x = a\tilde{x} \in C_a$, we have that

$$(L^n_{\mathcal{F}}1)(x) = \sum_{\sigma^n z = a\tilde{x}} f_n(z) = \sum_{z\in\Sigma, z=z_0z_1\dots z_{n-1}a\tilde{x}} f_n(z).$$

From equation (3), we have

$$f_n(z) \leq e^C f_1(z) f_{n-1}(\sigma z) \leq e^{2C} f_1(z) f_1(\sigma z) f_{n-2}(\sigma^2 z) \tag{8}$$

$$\leq e^{(n-1)C} f_1(z) f_1(\sigma z) \dots f_1(\sigma^{n-1} z). \tag{9}$$

On the other hand, the iterations of the transfer operator (see equation (7))

$$\left(L^n_{f_1}1\right)(x) = \sum_{i_n\dots i_1 x\in\Sigma} f_1(i_1x) f_1(i_2i_1x)\dots f_1(i_n\dots i_1 x). \tag{10}$$

Therefore,

$$\begin{aligned}(L^n_{\mathcal{F}}1)(a\tilde{x}) &= \sum_{\sigma^n z = a\tilde{x}} f_n(z) \leq \sum_{\sigma^n z = a\tilde{x}} f_1(z)\dots f_1(\sigma^{n-1}z)e^{(n-1)C} \text{ (by (8))}\\ &= \sum_{z\in\Sigma, z=z_0z_1\dots z_{n-1}a\tilde{x}} f_1(z_{n-1}a\tilde{x}) f_1(z_{n-2}z_{n-1}a\tilde{x})\dots f_1(z_0\dots z_{n-1}a\tilde{x})e^{(n-1)C}\\ &= e^{(n-1)C} L^n_{f_1}1(a\tilde{x}) \text{ (by (10))},\end{aligned}$$

which implies the second inequality. Finally, a direct computation shows that

$$\left(L^n_{f_1}1\right)(x) \leq \|L_{f_1}1\|^n_\infty \text{ for every } x \in C_a.$$

□

Proof of Theorem 2.1 *(1) (2)* The fact that the limit exists follows from Lemma 2.1. It follows from Lemmas 2.1 and 2.3 that the limit is not minus infinity. Theorem 2.1 (2) is a consequence of Lemma 2.4.

Proof of Theorem 2.1 *(3)* It is independent of the symbol $a \in S$ because (Σ, σ) is topologically mixing. Indeed, we will prove that given any two symbols $a, b \in S$ there exist constants $C' > 0$ and $k(a, b) \in \mathbb{N}$ such that

$$Z_n(\mathcal{F}, a) \leq C' Z_{n+2k(a,b)}(\mathcal{F}, b),$$

from which the result follows.

Let $x = (a, x_1, x_2, \dots, x_{n-1}, a, \dots) \in C_a$ be a periodic point of period n, that is $\sigma^n x = x$. Since Σ is topologically mixing, there exists $N_{ba} \in \mathbb{N}$ such that for any $m \geq N_{ba}$ there exists an admissible word of length $(m-1)$ given by $by_1\dots y_{m-2}$ such that $by_1\dots y_{m-2}a$ is an admissible word of length m. Similarly, there exists $N_{ab} \in \mathbb{N}$ such that for any $m \geq N_{ab}$ there exists an admissible word of length $(m-1)$ given by $az_1\dots z_{m-2}$ such that $az_1\dots z_{m-2}b$ is an admissible word of length m. Set $k = \max\{N_{ba}, N_{ab}\}$ and $m = k+1$. Now consider the periodic point $\tilde{x} \in \Sigma$ satisfying $\sigma^{n+2k}\tilde{x} = \tilde{x}$, where

$$\tilde{x} = (b, y_1, \dots, y_{m-2}, a, x_1, x_2, \dots, x_{n-1}, a, z_1, \dots, z_{m-2}, b, \dots).$$

Note that

$$f_n(x) = \frac{f_n(x)}{f_{n+2k}(\tilde{x})} f_{n+2k}(\tilde{x}). \tag{11}$$

Since $\mathcal{F}$ satisfies equation (2), we have that

$$\frac{f_n(x)}{f_n(\sigma^k\tilde{x})} \geq \frac{f_n(x)f_k(\tilde{x})}{f_{n+k}(\tilde{x})e^C} \geq \frac{f_n(x)f_k(\tilde{x})f_k(\sigma^{n+k}\tilde{x})}{e^{2C}f_{n+2k}(\tilde{x})}$$
$$\geq \left(\frac{f_n(x)}{e^{2C}f_{n+2k}(\tilde{x})}\right)\frac{1}{M^2}\sup\left\{f_k(u) : u \in C_{by_1\dots y_{k-1}}\right\}\sup\left\{f_k(u) : u \in C_{az_1\dots z_{k-1}}\right\}.$$

Recall that $f_k > 0$. Therefore, there exist positive constants C_1, C_2 such that

$$C_1 \leq \sup\left\{f_k(u) : u \in C_{by_1\dots y_{k-1}}\right\} \text{ and } C_2 \leq \sup\left\{f_k(u) : u \in C_{az_1\dots z_{k-1}}\right\}.$$

Hence

$$\frac{f_n(x)}{f_{n+2k}(\tilde{x})} \leq \frac{e^{2C}M^2}{C_1C_2}\frac{f_n(x)}{f_n(\sigma^k\tilde{x})}.$$

Therefore, using inequality (11)

$$f_n(x) \leq \frac{e^{2C}M^3}{C_1C_2}f_{n+2k}(\tilde{x}).$$

Thus there exists a constant $C' > 0$ such that

$$\sum_{\sigma^n x = x} f_n(x)\chi_{C_a}(x) \leq C' \sum_{\sigma^{n+2k}x=x} f_{n+2k}(x)\chi_{C_b}(x).$$

.

Remark 2.2. Let us stress that Lemma 2.4 holds under the assumption that the Bowen sequence $\mathcal{F}$ satisfies equation (3). On the other hand, Lemmas 2.1 and 2.3 hold when the Bowen sequence $\mathcal{F}$ satisfies equation (2).

3. The variational principle

One of the major results in the classical thermodynamic formalism is that the topological pressure satisfies the variational principle [W1]. It states that if we consider a dynamical system defined on a compact metric space and a continuous function ϕ the following equality holds,

$$P(\phi) = \sup\left\{h(\mu) + \int \phi \, d\mu : \mu \in \mathcal{M}\right\}. \tag{12}$$

In this context, the topological pressure is defined by means of $(n, \epsilon)-$separated sets (see [W2, Chapter 9]). This notion depends upon the metric. Two uniformly equivalent metrics yield the same value of the pressure (see [W2, p.171] for precise statement and proofs). Since in the compact setting two equivalent metrics are uniformly equivalent, the value of the pressure does not depend upon the metric. Recently, following the same approach, Cao, Feng and Huang [CFH] defined the pressure and proved the variational principle for sub-additive sequences of continuous functions defined on a compact metric space. Under some strong assumptions Falconer [F1] also obtained a variational principle. This assumptions were notably relaxed in the work of Barreira [B2] and Mummert [M].

In the non-compact setting, the definition of topological pressure obtained using $(n, \epsilon)-$separated sets has several problems. Most notably, equivalent metrics can yield different values for the topological pressure. Let us stress that the right hand side of the equality (12) only depends on the Borel structure of the space and not on the metric. Therefore, a notion of pressure satisfying the variational principle should not depend upon the metric of the space.

The definition we proposed in the previous section does not depend on the metric. We stress again that it is both a generalisation of the notion introduced by Sarig [S1] and of a formula for the pressure of almost-additive sequences obtained by Barreira [B2] and Mummert [M] (both of which satisfy a version of equality (12)). The main result of this section is that the Gurevich pressure satisfies the variational principle. We also prove that it is well approximated by its restriction to compact invariant sets.

Recall that, when (Σ, σ) is a topologically mixing finite state Markov shift and $\mathcal{F}$ is an almost-additive sequence on Σ satisfying the tempered condition $\lim_{n\to\infty}(\log A_n)/n = 0$ (where A_n is defined in Definition 2.3), Barreira [B2, Theorem 2] proved that the pressure he defined satisfies the following formula:

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x) \right).$$

Thus,

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x) \right) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x)\chi_{C_a}(x) \right).$$

We prove now that the almost-additive pressure defined over a countable Markov shift can be approximated by the pressure on finite Markov shifts.

Proposition 3.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ. Then*

$$P(\mathcal{F}) = \sup\{P(\mathcal{F}|_Y) : Y \subset \Sigma \textit{ a topologically mixing finite state Markov shift}\}.$$

Proof. Let $Y \subset \Sigma$ be a topologically mixing finite state Markov shift. Then clearly

$$\begin{aligned} P(\mathcal{F}|_Y) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x)\chi_{(Y\cap C_a)}(x) \right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} f_n(x)\chi_{C_a}(x) \right) = P(\mathcal{F}). \end{aligned}$$

Therefore

$$P(\mathcal{F}) \geq \sup\{P(\mathcal{F}|_Y) : Y \subset \Sigma \text{ a topologically mixing finite state Markov shift}\}.$$

In order to prove the reverse inequality, we assume that $P(\mathcal{F}) < \infty$. The other case can be proved in a similar way. The following proof is close to that of Theorem 2 [S1] and we identify a countable alphabet S with $\mathbb{N}$. Since $\mathcal{F}$ is an almost-additive Bowen sequence, let us assume that $\mathcal{F}$ satisfies equations (2), (3) and (4). We have

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\mathcal{F}, a) = \sup_n \frac{1}{n} \log Z_n(\mathcal{F}, a).$$

Hence given $\epsilon > 0$, there exists $m > C/\epsilon$ (where C is defined as in Definition 2.1) $m \in \mathbb{N}$ such that

$$P(\mathcal{F}) < \frac{1}{m} \log Z_m(\mathcal{F}, a) + \epsilon.$$

On the other hand, note that every periodic orbit belongs to a set of the form $\Sigma \cap \Sigma_M$, where Σ_M is the full-shift on M symbols. Therefore,

$$Z_m(\mathcal{F}, a) = \lim_{M\to\infty} Z_m(\Sigma_M \cap \Sigma, \mathcal{F}, a).$$

Clearly the sequence in the limit is increasing in M. Hence, given $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$\frac{1}{m} \log Z_m(\mathcal{F}, a) < \frac{1}{m} \log Z_m(\Sigma_M \cap \Sigma, \mathcal{F}, a) + \epsilon.$$

Adding a finite number of states to $\{1, 2, \ldots, M\}$, it is possible to construct a topologically mixing (finite state) Markov shift $Y \subset \Sigma$ such that

$$\frac{1}{m} \log Z_m(\mathcal{F}, a) < \frac{1}{m} \log Z_m(Y, \mathcal{F}, a) + \epsilon.$$

Indeed, this follows since for every symbol i, j belonging to the set $\{1, 2, \cdots, M\}$, there exists $\{b_1^{ij}, b_2^{ij}, \ldots, b_{n(i,j)}^{ij}\} \subset \mathbb{N}$ such that the word $ib_1^{ij} b_2^{ij} \ldots b_{n(i,j)}^{ij} j$ is an admissible word in Σ. Adding all the symbols $b_m^{ij} \in \mathbb{N}$ obtained this way and taking the closure on Σ, we obtain the required topologically mixing finite state Markov shift Y.

Now set $a_n = \log Z_n(Y, F, a)$. Then, by equation (2), $a_n + a_m \leq a_{n+m} + C$. Letting $n = km + r$, for $r = 0, 1, \ldots, k-1$, we obtain

$$\frac{ka_m + a_r}{km + r} \leq \frac{a_{km+r} + kC}{km + r} \leq \frac{a_n}{n} + \epsilon.$$

Letting $n \to \infty$, we obtain

$$\frac{1}{m} \log Z_m(Y, \mathcal{F}, a) \leq P(F|_Y) + \epsilon.$$

Therefore, we have

$$P(\mathcal{F}) \leq P(Y, \mathcal{F}) + 3\epsilon.$$

Hence

$$P(\mathcal{F}) \leq \sup\{P(\mathcal{F}|_Y) : Y \subset \Sigma \text{ a topologically mixing finite state Markov shift }\}.$$

□

The following result is a consequence of Proposition 3.1.

Corollary 3.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ. Then*

$$P(\mathcal{F}) = \sup\{P(\mathcal{F}|_K) : K \subset \Sigma \textit{ compact and } \sigma^{-1}(K) = K\}.$$

Corollary 3.2. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F}$ be an almost-additive Bowen sequence on Σ with $||L_{f_1} 1||_\infty < \infty$. Then the pressure function $t \mapsto P(t\mathcal{F})$ is convex.*

Proof. The pressure function defined on a compact invariant set is convex. Therefore, the result follows because the supremum of convex functions is a convex function. □

Our definition of pressure satisfies the variational principle.

Theorem 3.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F}$ be an almost-additive Bowen sequence on Σ, with $\sup f_1 < \infty$. Then*

$$\begin{aligned} P(\mathcal{F}) &= \sup\left\{h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu : \mu \in \mathcal{M} \textit{ and } \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu \neq -\infty\right\} \\ &= \sup\left\{h(\mu) + \int \lim_{n\to\infty} \frac{1}{n} \log f_n \, d\mu : \mu \in \mathcal{M} \textit{ and } \int \lim_{n\to\infty} \frac{1}{n} \log f_n \, d\mu \neq -\infty\right\}. \end{aligned}$$

Proof. If the pressure is infinite, $P(\mathcal{F})=\infty$, the variational principle holds. Indeed, by virtue of Proposition 3.1, there exists a sequence of topologically mixing finite state Markov shifts $\{Y_n\}_{n=1}^{\infty}$, with $Y_n \subset \Sigma$, for every $n \in \mathbb{N}$, such that

$$\infty = P(\mathcal{F}) = \lim_{n\to\infty} \sup\left\{ h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu : \mu \in \mathcal{M}_{Y_n}\right\}$$
$$\leq \sup\left\{ h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu : \mu \in \mathcal{M} \text{ and } \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu \neq -\infty\right\}.$$

In the rest of the proof we will assume $P(\mathcal{F})<\infty$. Since $\mathcal{F}$ is almost-additive, we assume that it satisfies equations (2) and (3). We first show the second equality in Theorem 3.1 by proving for any $\mu \in \mathcal{M}$,

$$\lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu = \int \lim_{n\to\infty}\frac{1}{n}\log f_n \ d\mu.$$

To see this, set $g_n(x) = f_n(x)e^C$. Then $\{\log g_n\}_{n=1}^{\infty}$ satisfies the subadditivity condition (equation (1) in Definition 2.1). Since $\sup f_1 < \infty$, we have $\log g_n : \Sigma \to \mathbb{R}\cup\{-\infty\}$ for all $n \in \mathbb{N}$ and $(\log g_1)^+ \in L^1(\mu)$. Therefore, Kingman's subadditive ergodic theorem implies the result.

Now we will show the first equality. For a compact subset $Y \subset \Sigma$, denote by $\mathcal{M}_Y$ the set of σ-invariant Borel probability measures on Y. Note that Barreira [B2] proved the variational principle for the case when Σ is a finite state Markov shift. Therefore, by virtue of Proposition 3.1, there exists a sequence of topologically mixing finite state Markov shifts $\{Y_n\}_{n=1}^{\infty}$, with $Y_n \subset \Sigma$, for every $n \in \mathbb{N}$, such that

$$P(\mathcal{F}) = \lim_{n\to\infty} P(\mathcal{F}|_{Y_n}) = \lim_{n\to\infty} \sup\left\{ h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu : \mu \in \mathcal{M}_{Y_n}\right\}$$
$$\leq \sup\left\{ h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu : \mu \in \mathcal{M} \text{ and } \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu \neq -\infty\right\}.$$

In order to prove the other inequality, we adapt the proof of [S1, Theorem 3]. We need a version of [S1, Lemma 4] for sequences of functions. In the following proof, we identify a countable alphabet S with $\mathbb{N}$. For $m \in \mathbb{N}$, set $C_{\geq m} = \{x \in \Sigma : x_0 \geq m\}$ and let $\alpha_m = \{C_1, \ldots, C_{m-1}, C_{\geq m}\}$ (see Section 2 for the notation of cylinder sets). Let $\mu \in \mathcal{M}$. Then

$$\lim_{m\to\infty}\left(h_\mu(\sigma,\alpha_m) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu\right) = h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n \ d\mu.$$

Fix $m \in \mathbb{N}$ and set $\beta = \alpha_m$. Let $\beta_0^n = \bigvee_{i=0}^n \sigma^{-i}(\beta)$. For each $a_i \in \beta$, let $E_{a_0 \dots a_n} := \cap_{k=0}^n \sigma^{-k}(a_k)$. For $E \in \beta_0^n$ define $f_n[E] = \sup\{f_n(x) : x \in E\}$. We have that

$$
\begin{aligned}
&\frac{1}{n}\left(H_\mu(\beta_0^n) + \int \log f_n d\mu\right) \\
&\leq \frac{1}{n}\sum_{a,b\in\beta} \mu(a\cap\sigma^{-n}b)\left(\sum_{\substack{E\subseteq a\cap\sigma^{-n}b,\\ E\in\beta_0^n}} \mu(E|a\cap\sigma^{-n}b)\log\frac{f_n[E]}{\mu(E)}\right) \\
&\leq \frac{1}{n}\left(\sum_{a,b\in\beta}\mu(a\cap\sigma^{-n}b)\log(\sum_{\substack{E\in a\cap\sigma^{-n}b\\ E\in\beta_0^n}} f_n[E])\right) + \frac{1}{n}H_\mu\left(\beta\vee\sigma^{-n}\beta\right),
\end{aligned}
$$

where the last inequality follows from [W1, Lemma 9.9]. For $a, b \in \beta$, set

$$
P_n(a,b) = \frac{1}{n}\log\left(\sum_{E\subseteq a\cap\sigma^{-n}b, E\in\beta_0^n} f_n[E]\right).
$$

Thus we obtain

$$
\frac{1}{n}H_\mu(\beta_0^n) + \frac{1}{n}\int \log f_n d\mu \leq \left(\sum_{a,b\in\beta}\mu(a\cap\sigma^{-n}b)P_n(a,b)\right) + \frac{2H_\mu(\beta)}{n}. \tag{13}
$$

In what follows, we will obtain an upper bound for $\limsup_{n\to\infty} P_n(a,b)$. This bound will be different depending on whether both a and b belong to the set $\{C_1, \dots, C_{m-1}\}$ or not. Let $M > 0$ be such that $\sup\{f_n(y)/f_n(x) : x_i = y_i, 0 \leq i \leq n-1\} \leq M$ for all $n \in \mathbb{N}$.

Lemma 3.1. *Under the assumptions of Theorem* 3.1*, we have*

1. *If $a, b \neq C_{\geq m}$, then* $\limsup_{n\to\infty} P_n(a,b) \leq P(\mathcal{F})$.
2. *If $a = C_{\geq m}$ or $b = C_{\geq m}$, then there exists $C' \in \mathbb{R}$ such that*
$$\limsup_{n\to\infty} P_n(a,b) \leq C'.$$

Proof. Our arguments are similar to those in [S1] and make use of the ideas in Theorem 2.1 (3). We first show (1). We claim that there exist constants $A, k > 0$ such that

$$
\sum_{E\subseteq a\cap\sigma^{-n}b, E\in\beta_0^n} f_n[E] \leq A \sum_{\sigma^{n+2k}x=x, x\in a} f_{n+2k}(x). \tag{14}
$$

Let $E_{ad_1\dots d_{n-1}b} \subset a \cap \sigma^{-n}b$. For convenience, let $a = C_{N_1}$ and $b = C_{N_2}$, where $1 \leq N_1, N_2 \leq m-1$. Take a point $x = (x_0, \dots, x_n, \dots) \in E_{ad_1\dots d_{n-1}b}$ such that $f_n[E_{ad_1\dots d_{n-1}b}] \leq 2f_n(x)$. Then $x \in C_{N_1\bar{d}_1\bar{d}_2\cdots\bar{d}_{n-1}N_2} \subset E_{ad_1\dots d_{n-1}b}$, for some $\bar{d}_i \in \mathbb{N}, C_{\bar{d}_i} \subseteq C_{d_i}, 1 \leq i \leq n-1$. Using the same arguments used to prove Theorem 2.1 (3), we construct $\tilde{x} \in C_{N_1}$ such that $\sigma^{n+2k}\tilde{x} = \tilde{x}$, in the following way. Since $N_1\bar{d}_1 \dots \bar{d}_{n-1}N_2$ is an admissible word of length $(n+1)$ in Σ, using the same notation as in the proof of Theorem 2.1 (3) , set $k = \max\{N_{N_1N_1}, N_{N_2N_1}\}$. Define $y_1 \dots y_{k-1}$ and $z_1 \dots z_{k-1}$ so that $N_1y_1 \dots y_{k-1}N_1$ and $N_2z_1 \dots z_{k-1}N_1$ are allowable words in Σ. Set $A = N_1 y_1 \dots y_{k-1} N_1 \bar{d}_1 \dots \bar{d}_{n-1} N_2 z_1 \dots z_{k-1}$ and define $\tilde{x} = A^\infty$. Clearly,

$$f_n[E_{ad_1\dots d_{n-1}b}] \le \frac{2f_n(x)}{f_{n+2k}(\tilde{x})} f_{n+2k}(\tilde{x}).$$

Note that $x_i = (\sigma^k \tilde{x})_i$ for $0 \le i \le n-1$. Therefore, we obtain

$$\frac{f_n(x)}{f_n(\sigma^k \tilde{x})} \le M.$$

Approximating $f_n(x)/f_{n+2k}(\sigma^k \tilde{x})$ by an argument similar to that in the proof of Theorem 2.1 (3), we obtain (14). Therefore, we have

$$\begin{aligned} \limsup_{n\to\infty} P_n(a,b) &\le \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^{n+2k} x = x, x \in C_{N_1}} f_{n+2k}(x) \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log Z(\mathcal{F}, N_1) = P(\mathcal{F}). \end{aligned}$$

Next we show Lemma 3.1 (2). We consider the case when $a = b = C_{\ge m}$. Other cases can be shown similarly. Let $n \ge 3$ be fixed. Write $\{E \in \beta_0^n : E \subset a \cap \sigma^{-n} b\} = \cup_{i,j,k} A_{i,j,k}$, where

$$A_{i,j,k} = \left\{ E_{a^i d_1 \dots d_j b^k} \in \beta_0^n : a = b = C_{\ge m}, d_1, d_j \ne C_{\ge m} \right\},$$

$i + j + k = n + 1$. We first consider the case when $j \ge 2$. For $j = 0, 1$, we make a similar argument. Define for each i, j, k,

$$S_{i,j,k} = \sum_{E \in A_{i,j,k}} f_n[E].$$

We first find an upper bound for $S_{i,j,k}$. Fix i, j and k. Let $E_{a^i d_1 \dots d_j b^k} \in A_{i,j,k}$. For a point $x \in E_{a^i d_1 \dots d_j b^k}$, we have by equation (3)

$$f_n(x) \le f_1(x) f_{n-1}(\sigma x) e^C \le f_1(x) f_1(\sigma x) f_{n-2}(\sigma^2 x) e^{2C} \le \dots . \le ||f_1||_\infty^{n-j} e^{C(n-j)} f_j(\sigma^i x).$$

Now let $d_1 = C_{N_1}, d_j = C_{N_2}, 1 \le N_1, N_2 \le m-1$ and $E_{a^i d_1 \dots d_j b^k} \in A_{i,j,k}$. Take a point $x \in E_{a^i d_1 \dots d_j b^k}$ such that $f_n[E_{a^i d_1 \dots d_j b^k}] \le 2 f_n(x)$. Call it $\bar{x}_{a^i d_1 \dots d_j b^k}$. Then $\bar{x}_{a^i d_1 \dots d_j b^k} \in C_{x_1 \dots x_i N_1 y_2 \dots y_{j-1} N_2 z_1 \dots z_k}$, where $x_l \ge m$ for $1 \le l \le i$, $z_l \ge m$ for $1 \le l \le k$, $y_l \ge 1$ for $2 \le l \le j-1$. Therefore,

$$\begin{aligned} f_n[E_{a^i d_1 \dots d_j b^k}] &\le 2 f_n(\bar{x}_{a^i d_1 \dots d_j b^k}) \le 2 ||f_1||_\infty^{n-j} e^{C(n-j)} f_j(\sigma^i \bar{x}_{a^i d_1 \dots d_j b^k}) \\ &\le 2 ||f_1||_\infty^{n-j} e^{C(n-j)} \sup\{ f_j(x) : x \in C_{N_1 y_2 \dots y_{j-1} N_2} \} \\ &\le 2 ||f_1||_\infty^{n-j} e^{C(n-j)} M f_j(x), \end{aligned}$$

for any $x \in C_{N_1 y_2 \dots y_{j-1} N_2}$. Consider a point $N_2 z = (N_2, z_0, \dots, z_n, \dots) \in \Sigma$ and let it be fixed. Denote by $B_j(\Sigma)$ the set of admissible words of length j in Σ. Then for $d_1 = C_{N_1}, d_j = C_{N_2}$,

$$\sum_{E_{a^i d_1 \dots d_j b^k} \in A_{i,j,k}} f_n[E_{a^i d_1 \dots d_j b^k}] \le 2 ||f_1||_\infty^{n-j} e^{C(n-j)} M \sum_{\substack{x = N_1 \dots N_2 z \in \Sigma \\ N_1 \dots N_2 \in B_j(\Sigma)}} f_j(x).$$

By an argument similar to that used to prove Lemma 2.4, we obtain

$$\sum_{\substack{x = N_1 \dots N_2 z \in \Sigma \\ N_1 \dots N_2 \in B_j(\Sigma)}} f_j(x) \le e^{C(j-1)} ||f_1||_\infty L_{f_1}^{j-1} \chi_{C_{N_1}}(N_2 z) \le e^{C(j-1)} ||f_1||_\infty ||L_{f_1} 1||_\infty^{j-1}.$$

Since $N_1, N_2 \in \{1, \dots, m-1\}$, we have

$$S_{i,j,k} \quad = \sum_{E \in A_{i,j,k}} f_n[E] \le 2||f_1||_\infty^{n-j} e^{C(n-j)} (m-1)^2 M e^{C(j-1)} ||f_1||_\infty ||L_{f_1} 1||_\infty^{j-1}.$$

The above inequality also holds for $j = 1$. For each fixed n, consider $j \ge 1$ such that the right hand side of the above inequality takes the maximal value at j. Call it j_n. For $j = 0$ (and so $i+k = n+1$), it is easy to see that $S_{i,0,k} \le 2||f_1||_\infty^n e^{(n-1)C}$. Let

$$B_n = \max\{2||f_1||_\infty^n e^{(n-1)C}, 2||f_1||_\infty^{n-j_n} e^{C(n-j_n)} (m-1)^2 M e^{C(j_n-1)} ||f_1||_\infty ||L_{f_1} 1||_\infty^{j_n-1}\}.$$

Since $1 \le i, k \le n+1, 0 \le j \le n-1$

$$P_n(a,b) = \frac{1}{n} \log \left(\sum_{i,j,k} S_{i,j,k} \right) \le \frac{1}{n} \log (n+1)^3 B_n.$$

Therefore,

$$\begin{aligned} &\limsup_{n\to\infty} P_n(a,b) \\ &\le \max \left\{ \log ||f_1||_\infty + \log ||L_{f_1} 1||_\infty + 2C, 2C, \log ||f_1||_\infty + 2C, \log ||L_{f_1} 1||_\infty + 2C \right\}. \end{aligned}$$

□

For completeness, we will include the final part of the proof of [S4, Theorem 4.4] (see also [S1, Theorem 3]). By Lemma 3.1 and (13),

$$\begin{aligned} h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu &\le \limsup_{n\to\infty} \left(\sum_{a,b\in\beta} \mu(a \cap \sigma^{-n} b) P_n(a,b) \right) \\ &\le \limsup_{n\to\infty} \left(P(\mathcal{F}) \sum_{a,b \ne C_{\ge m}} \mu(a \cap \sigma^{-n} b) + C' \sum_{a = C_{\ge m} \text{ or } b = C_{\ge m}} \mu(a \cap \sigma^{-n} b) \right) \\ &\le P(\mathcal{F}) + C'(\mu(C_{\ge m}) + \mu(\sigma^{-n}(C_{\ge m}))) \le P(\mathcal{F}) + 2C'\mu(C_{\ge m}). \end{aligned}$$

Letting $m \to \infty$, we obtain the result. □

The set of σ-invariant Borel probability measures, $\mathcal{M}$, is a very large and complicated convex, non-compact set. Indeed, it strictly contains a countable family of Poulsen simplexes, that is, infinite dimensional compact and convex sets with the property that the extreme points are dense in the set. It is therefore a major problem in the ergodic theory of countable Markov shifts to choose relevant invariant measures. The variational principle provides a criteria for making that choice.

4. Gibbs measures

In this section, we prove the existence of Gibbs measures for an almost-additive sequence of continuous functions under certain assumptions. In order to do so, we require an additional assumption on the combinatorial structure of the Markov shift. This is a necessary assumption in the classical thermodynamical formalism for countable Markov shifts (see [MU2, S3]). Let us start with some basic definitions.

Definition 4.1. Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive sequence on Σ. A measure $\mu \in \mathcal{M}$ is said to be an *equilibrium measure* for $\mathcal{F}$ if

$$P(\mathcal{F}) = h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \ d\mu.$$

Definition 4.2. Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive sequence on Σ. A measure $\mu \in \mathcal{M}$ is said to be *Gibbs* for $\mathcal{F}$ if there exist constants $C_0 > 0$ and $P \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ and every $x \in C_{i_0 \dots i_{n-1}}$ we have

$$\frac{1}{C_0} \leq \frac{\mu(C_{i_0 \dots i_{n-1}})}{\exp(-nP) f_n(x)} \leq C_0.$$

There is a special class of Markov shifts having a combinatorial structure similar to that of the full-shift, that will be important for us.

Definition 4.3. A countable Markov shift (Σ, σ) is said to satisfy the *big images and preimages property (BIP property)* if there exists $\{b_1, b_2, \dots, b_n\}$ in the alphabet S such that

$$\forall a \in S \ \exists i, j \text{ such that } t_{b_i a} t_{a b_j} = 1.$$

It was shown by Mauldin and Urbański [MU2] and also by Sarig [S3] that there is a combinatorial obstruction to the existence of Gibbs measures corresponding to a continuous function of summable variations. Indeed, if (Σ, σ) is a topologically mixing countable Markov shift that does not satisfy the BIP property, then no continuous function can have Gibbs measures. It should also be noticed that in the compact setting of Markov shifts over a finite alphabet, Gibbs measures are always equilibrium measures. This is no longer true in the non-compact setting of countable Markov shifts. Indeed, a Gibbs measure μ for a continuous function ϕ could satisfy $h(\mu) = \infty$ and $\int \phi \ d\mu = -\infty$. In such a situation, the measure μ is not an equilibrium measure for ϕ (see [S3] for comments and examples). Of course, this type of phenomena can also occur in our context.

Note that if $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is an almost-additive Bowen sequence defined on Σ satisfying $\sum_{a \in S} \sup f_1|_{C_a} < \infty$, then implies $||L_{f_1} 1||_{\infty} < \infty$. In particular the pressure is finite, $P(\mathcal{F}) < \infty$, and the variational principle (see Theorem 3.1) holds. The main result of this section is the following.

Theorem 4.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift with the BIP property. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence defined on Σ satisfying $\sum_{a \in S} \sup f_1|_{C_a} < \infty$. Then there is a Gibbs measure μ for $\mathcal{F}$ and it is mixing. Moreover, If $h(\mu) < \infty$, then it is the unique equilibrium measure for $\mathcal{F}$.*

Proof. The proof is inspired on [MU2, Lemma 2.8] and [B2, Lemmas 1, 2 and Theorem 5]. Those results need to be modified and adapted to the almost-additive setting and to the case of a non-compact phase space. We identify a countable alphabet S with the set $\mathbb{N}$.

Since (Σ, σ) is topologically mixing and has the BIP property, there exist $k \in \mathbb{N}$ and a finite collection W of admissible words of length k such that for any $a, b \in S$, there exists $w \in W$ such that awb is admissible (see [S3, p.1752] and [MU2]). Let A be the transition matrix for Σ. By rearranging the set $\mathbb{N}$, there is an increasing sequence $\{l_n\}_{n=1}^{\infty}$ such that the matrix $A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$ is primitive. Let Y_{l_n} be the topologically mixing finite state Markov shift with the transition matrix $A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$. Then there exists $p \in \mathbb{N}$ such that for all $n \geq p$, Y_{l_n} contains all admissible words in W. We denote by $B_n(Y_l)$ the set of admissible words of length n in Y_l. Since $\mathcal{F}$ is almost-additive, there exists $C > 0$ such that for each $n, m \in \mathbb{N}, x \in \Sigma$, equations (2) and (3) hold. In the proof of the following claim, we continue to use W and k defined as above.

Claim 4.1. *For $Y_{l_n} \subset \Sigma$, $n \geq p$, there is a unique equilibrium measure for $\mathcal{F}|_{Y_{l_n}}$ and it is Gibbs for $\mathcal{F}|_{Y_{l_n}}$. Moreover, the constant C_0 (see Definition 4.2) can be chosen in such a way that C_0 is independent of Y_{l_n}.*

Proof of the Claim. Since $\mathcal{F}$ is a Bowen sequence, we assume that it satisfies equation (4). Clearly, $\mathcal{F}|_{Y_{l_n}}$ is an almost-additive Bowen sequence on $(Y_{l_n}, \sigma|_{Y_{l_n}})$. Therefore, the first part of the claim is immediate by [B2, Theorem 5] or [M, Theorem 6]. Slightly modifying the proof of [B2, Lemmas 1, 2 and Theorem 5], we will obtain the second part of the claim. We only show the first step of the proof to see how we can get a uniform constant C_0. By the assumptions, any admissible word in W is an admissible word in Y_{l_n} for all $n \geq p$. Fix Y_{l_n}, $n \geq p$, and call it Y. Define $\alpha_n^Y = \sum_{i_0\cdots i_{n-1}\in B_n(Y)} \sup\{f_n|_Y(y) : y \in C_{i_0\dots i_{n-1}}\}$. For $l \in \mathbb{N}$, let ν_l be the Borel probability measure on Y defined by

$$\nu_l(C_{i_0\dots i_{l-1}}) = \frac{\sup\{f_l|_Y(y) : y \in C_{i_0\dots i_{l-1}}\}}{\alpha_l^Y}.$$

Let $n \in \mathbb{N}$ and $l \geq n + k$. For any admissible words $i_0 \cdots i_{n-1}$ and $j_0 \dots j_{l-k-1}$ in Y, there exists $m_0 \dots m_{k-1} \in W$ such that $i_0 \dots i_{n-1} m_0 \dots m_{k-1} j_0 \dots j_{l-k-1}$ is an admissible word in Y. For $w \in W$, let $N_w = \sup\{f_k(z) : z \in C_w\}$ and $\bar{N} = \min\{N_w : w \in W\}$. For any $y = (y_0, \dots, y_n, \dots) \in Y \subset \Sigma$ with $y_0 \dots y_{k-1} = w \in W$, we have

$$\frac{N_w}{f_k|_Y(y)} \leq M.$$

By (2) and (4), for each $y \in C_{i_0\dots i_{n-1}m_0\dots m_{k-1}j_0\dots j_{l-k-1}}$, we have

$$\begin{aligned} f_{l+n}|_Y(y) &\geq f_n|_Y(y) f_k|_Y(\sigma^n y) f_{l-k}|_Y(\sigma^{n+k} y) e^{-2C} \\ &\geq \frac{\bar{N} e^{-2C}}{M^3} \sup\{f_n|_Y(y) : y \in C_{i_0\dots i_{n-1}}\} \sup\{f_{l-k}|_Y(y) : y \in C_{j_0\dots j_{l-k-1}}\}. \end{aligned}$$

For each fixed $i_0 \dots i_{n-1} \in B_n(Y)$, we have

$$\sum_{t_0\dots t_{l-1}} \sup\{f_{n+l}|_Y(y) : y \in C_{i_0\dots i_{n-1}t_0\dots t_{l-1}}\} \tag{15}$$

$$\geq \sum_{j_0\dots j_{l-k-1}} \frac{\bar{N} e^{-2C}}{M^3} \sup\{f_n|_Y(y) : y \in C_{i_0\dots i_{n-1}}\} \sup\{f_{l-k}|_Y(y) : y \in C_{j_0\dots j_{l-k-1}}\} \tag{16}$$

$$= \frac{\bar{N} e^{-2C}}{M^3} \sup\{f_n|_Y(y) : y \in C_{i_0\dots i_{n-1}}\} \alpha_{l-k}^Y. \tag{17}$$

Since $\alpha_l^Y \leq e^C \alpha_n^Y \alpha_{l-n}^Y$, we obtain

$$\alpha_{n+l}^Y \geq \frac{\bar{N} e^{-3C} \alpha_n^Y \alpha_l^Y}{M^3 \alpha_k^Y}.$$

We note that by (3)

$$\alpha_k^Y = \sum_{i_0 \dots i_{k-1}} \sup\{f_k|_Y(z) : z \in C_{i_0 \dots i_{k-1}}\} \leq e^{(k-1)C} (\sum_{i \in \mathbb{N}} f_1|_{C_i})^k < \infty.$$

Therefore, there exists $C_1 > 0$ such that

$$\alpha_{n+l}^Y \geq C_1 \alpha_n^Y \alpha_l^Y, \tag{18}$$

and clearly C_1 does not depend on l_n. By [B2], we know that

$$P(\mathcal{F}|_Y) = \lim_{n \to \infty} \frac{1}{n} \log \alpha_n^Y.$$

Since

$$\lim_{n \to \infty} \frac{1}{n} \log \alpha_n^Y = \inf_{n \in \mathbb{N}} \frac{1}{n} \log e^C \alpha_n^Y,$$

we have $e^C \alpha_n^Y \geq e^{P(\mathcal{F}|_Y)n}$. Similarly, by (18),

$$P(\mathcal{F}|_Y) = \sup_{n \in \mathbb{N}} \frac{1}{n} \log C_1 \alpha_n^Y.$$

Therefore,

$$C_1 \alpha_n^Y \leq e^{P(\mathcal{F}|_Y)n} \leq e^C \alpha_n^Y. \tag{19}$$

Using (15), (18) and (19), similar arguments to those in [B2] show that there exist constants $C_2, C_3 > 0$, both independent of l_n, such that for all $n \in \mathbb{N}$

$$C_2 \leq \frac{\nu_l(C_{i_0 \dots i_{n-1}})}{e^{-nP(\Phi(\mathcal{F}|_Y))} f_n|_Y(y)} \leq C_3, i_0 \dots i_{n-1} \in B_n(Y). \tag{20}$$

Now we take a convergent subsequence $\{\nu_{l_k}\}_{k=1}^\infty$ of $\{\nu_l\}_{l=1}^\infty$ and let ν be the limit point. Then ν satisfies equation (20) (by replacing ν_l by ν). Modifying arguments in [B2] by using the property of bounded variation and the BIP property as seen in the above arguments, we conclude that we can choose a constant C_0 (in Definition 4.2) for the σ_Y-invariant ergodic Gibbs measure μ_Y for $F|_Y$ such that C_0 is independent of l_n. □

By the claim above, for each fixed l_n,$n \geq p$, $\mathcal{F}|_{Y_{l_n}}$ has a unique equilibrium state μ_{l_n} which is Gibbs, i.e., for each $q \in \mathbb{N}$, there exist $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$\tilde{C}_1 \leq \frac{\mu_{l_n}(C_{i_0 \cdots i_{q-1}})}{e^{-qP(\mathcal{F}|_{Y_{l_n}})} f_q|_{Y_{l_n}}(y)} \leq \tilde{C}_2, \text{ for } y \in C_{i_0 \dots i_{q-1}}, i_0 \dots i_{q-1} \in B_q(Y_{l_n}). \tag{21}$$

We first show that the sequence $\{\mu_{l_n}\}_{n=p}^\infty$ of σ-invariant Borel probability measures on Σ is tight. Let $\pi_k : \Sigma \to \mathbb{N}$ be the projection onto the k-th coordinate. Then for each $a \in \mathbb{N}$,

$$\begin{aligned}\mu_{l_n}(\pi_k^{-1}(a)) &= \sum_{i_0 \dots i_{k-2} a \in B_k(Y_{l_n})} \mu_{l_n}(C_{i_0 \cdots i_{k-2} a}) \\ &\leq \tilde{C}_2 \sum_{i_0 \dots i_{k-2} a \in B_k(Y_{l_n}), y \in C_{i_0 \cdots i_{k-2} a}} e^{-kP(\mathcal{F}|_{Y_{l_n}})} f_k|_{Y_{l_n}}(y)(by(21)),\end{aligned}$$

where the summation is taken over a set containing a point y from each cylinder set $C_{i_0\dots i_{k-2}a}$. By (3),

$$f_k|_{Y_{l_n}}(y) \le f_{k-1}|_{Y_{l_n}}(y) f_1|_{Y_{l_n}}\left(\sigma^{k-1}y\right) e^C \le \left(\sup f_1|_{C_a}\right) f_{k-1}|_{Y_{l_n}}(y) e^C.$$

Thus

$$\begin{aligned}
&\sum_{i_0\dots i_{k-2}a \in B_k(Y_{l_n}), y\in C_{i_0\dots i_{k-2}a}} f_k|_{Y_{l_n}}(y) \\
&\le \left(\sup f_1|_{C_a}\right) e^C \sum_{i_0\dots i_{k-2}a \in B_k(Y_{l_n}), y\in C_{i_0\dots i_{k-2}a}} f_{k-1}|_{Y_{l_n}}(y) \\
&\le \left(\sup f_1|_{C_a}\right) e^{C(k-1)} \sum_{i_0\dots i_{k-2}a \in B_k(Y_{l_n})} \left(f_1|_{Y_{l_n}}(y) f_1|_{Y_{l_n}}(\sigma y)\dots f_1|_{Y_{l_n}}(\sigma^{k-2}y)\right) \\
&\le \left(\sup f_1|_{C_a}\right) e^{C(k-1)} \sum_{i_0\dots i_{k-2}a \in B_k(Y_{l_n})} \sup f_1|_{C_{i_0}} \dots \sup f_1|_{C_{i_{k-2}}} \\
&\le \left(\sup f_1|_{C_a}\right) e^{C(k-1)} \left(\sum_{i\in\mathbb{N}} \sup f_1|_{C_i}\right)^{k-1}.
\end{aligned}$$

Hence

$$\mu_{l_n}(\pi_k^{-1}(a)) \quad \le \tilde{C}_2 e^{-kP(\mathcal{F}|_{Y_{l_n}})} \left(\sup f_1|_{C_a}\right) e^{C(k-1)} (\textstyle\sum_{i\in\mathbb{N}} \sup f_1|_{C_i})^{k-1}.$$

Now set $N = P(\mathcal{F}|_{Y_{l_1}})$ if $P(\mathcal{F}) < 0$, and $N = \min\{-P(\mathcal{F}), -|P(F|_{Y_{l_1}})|\}$ otherwise. Then we obtain

$$\mu_{l_n}(\pi_k^{-1}[a+1,\infty)) \le \tilde{C}_2 e^{-kN + C(k-1)} \left(\sum_{i\in\mathbb{N}} \sup f_1|_{C_i}\right)^{k-1} \sum_{i>a} \sup f_1|_{C_i}.$$

Since $\sum_{i\in\mathbb{N}} \sup f_1|_{C_i} < \infty$, for each given $\epsilon > 0, k \in \mathbb{N}$, we can find $n_k \in \mathbb{N}$ with the property of

$$\tilde{C}_2 e^{-kN + C(k-1)} \left(\sum_{i\in\mathbb{N}} \sup f_1|_{C_i}\right)^{k-1} \sum_{i>n_k} \sup f_1|_{C_i} \le \frac{\epsilon}{2^k}.$$

Therefore, for any $l_n, k \in \mathbb{N}$,

$$\mu_{l_n}\left(\pi_k^{-1}[n_k+1,\infty)\right) \le \frac{\epsilon}{2^k},$$

and so

$$\mu_{l_n}\left(\Sigma \cap \prod_{k\ge 1}[1,n_k]\right) \ge 1 - \sum_{k\ge 1} \mu_{l_n}\left(\pi_k^{-1}([n_k+1,\infty))\right) \ge 1-\epsilon.$$

Since $\Sigma\cap\prod_{k\ge1}[1,n_k]$ is a compact subset of Σ, by Prohorov's theorem, the sequence $\{\mu_{l_n}\}_{n=p}^{\infty}$ is tight. Therefore, there exists a convergent subsequence $\{\mu_{l_{n_k}}\}_{k=1}^{\infty}$ of $\{\mu_{l_n}\}_{n=p}^{\infty}$. We denote by μ a limit point of this subsequence. Since it is a limit point of a sequence of invariant measures on Σ, μ is also σ-invariant on Σ. By the property (21), letting $l_n \to \infty$, we obtain for $q \in \mathbb{N}$ and each $y \in C_{i_0\dots i_{q-1}}, i_0\dots i_{q-1} \in B_q(\Sigma)$,

$$\tilde{C}_1 \le \frac{\mu\left(C_{i_0\cdots i_{q-1}}\right)}{e^{-qP(\mathcal{F})} f_q(y)} \le \tilde{C}_2. \tag{22}$$

Therefore, μ is a Gibbs measure for $\mathcal{F}$.

In order to show that μ is ergodic, we use similar arguments to those used to prove [B2, Lemma 2]. Let $i_0 \dots i_{n-1}$ and $j_0 \dots j_{l-1}$ be fixed admissible words in Σ. Let k and $\bar{N}$ be defined as in the proof of claim 4.1. We also define α_n^{Σ} as we defined α_n^{Y} by replacing Y by Σ. Then for $m-n \geq k$,

$$\begin{aligned}
&\mu(C_{i_0\dots i_{n-1}} \cap f^{-m}(C_{j_0\dots j_{l-1}})) \\
&= \sum_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}\in B_{m+l}(\Sigma)} \mu(C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}) \\
&\geq \tilde{C}_1 e^{-(m+l)P(\mathcal{F})} \sum_{y\in C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}} f_m(y) f_l(\sigma^m y) f_{m-n}(\sigma^n y) e^{-2C} \\
&\geq \frac{e^{-2C}\tilde{C}_1 e^{-(m+l)P(\mathcal{F})}}{M^2} \sup\{f_n(y) : y \in C_{i_0\dots i_{n-1}}\} \sup\{f_l(y) : y\in C_{j_0\dots j_{l-1}}\} \\
&\sum_{y\in C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}} f_{m-n}(\sigma^n y),
\end{aligned}$$

where in the second and third inequalities each summation is taken over a set containing a point y from each cylinder set $C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}$ in Σ. Then

$$\begin{aligned}
&\sum_{y\in C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}} f_{m-n}(\sigma^n y) \\
&\geq e^{-2C} \sum_{y\in C_{i_0\dots i_{n-1}k_0\dots k_{m-n-1}j_0\dots j_{l-1}}} f_k(\sigma^n y) f_{m-n-2k}(\sigma^{n+k} y) f_k(\sigma^{m-k} y) \\
&\geq \frac{e^{-2C}\bar{N}^2}{M^3}\alpha^{\Sigma}_{m-n-2k}.
\end{aligned}$$

We claim that, for a symbol $a \in \mathbb{N}$, there exists a constant $\bar{C} > 0$ depending on a such that

$$\bar{C}\alpha_n^{\Sigma} \leq \sum_{z\in C_a, \sigma^{n+2k+1}z=z} f_{n+2k+1}(z). \tag{23}$$

Clearly,

$$f_{n+2k+1}(z) \geq f_{k+1}(z) f_n(\sigma^{k+1}z) f_k(\sigma^{n+k+1}z) e^{-2C}.$$

Therefore,

$$\begin{aligned}
\sum_{z\in C_a, \sigma^{n+2k+1}z=z} f_{n+2k+1}(z) &\geq \frac{\bar{N}^2 e^{-2C}}{M} \sup\{f_1(z) : z \in C_a\} \sum_{z\in C_{z_0\dots z_{n-1}}} f_n(z) \\
&\geq \frac{\bar{N}^2 e^{-2C}}{M^2} \sup\{f_1(z) : z\in C_a\}\alpha_n^{\Sigma},
\end{aligned}$$

where the summation in the right hand side of the first inequality is taken over a set containing a point z from each cylinder set $C_{z_0\dots z_{n-1}}$. This proves (23). By the definition of the pressure, we obtain

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log \alpha_n^{\Sigma}.$$

Thus we have $e^C\alpha_n^{\Sigma} \geq e^{P(\mathcal{F})n}$ by using the same proof used to prove Claim 4.1. Therefore, using (22), it is easy to see that there exists a constant $\tilde{C}_3 > 0$ such that

$$\mu(C_{i_0\dots i_{n-1}} \cap f^{-m}(C_{j_0\dots j_{l-1}})) \geq \tilde{C}_3 \mu(C_{i_0\dots i_{n-1}})\mu(C_{j_0\dots j_{l-1}}).$$

Therefore, μ ergodic and thus it is the unique Gibbs measure for $\mathcal{F}$. If $h(\mu) < \infty$, using the proof of [MU2, Theorem 3.5](replace $S_n f$ and $P(f)$ by f_n and $P(\mathcal{F})$ respectively), it is the unique equilibrium measure for $\mathcal{F}$. The fact that the measure μ is mixing is fairly standard and follows as in the final part of the proof of [B2, Theorem 5]. $\square$

5. Example 1: The full-shift

Let us consider the full-shift on a countable alphabet, that is (Σ_F, σ), where

$$\Sigma_F := \{(x_i)_{i=0}^{\infty} : x_i \in \mathbb{N}\}.$$

The good combinatorial properties of this shift allow us to make some explicit computations. Indeed, let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that $\lambda_j \in (0,1)$ and $\sum_{j=1}^{\infty} \lambda_j < \infty$. Let $\{\log c_n\}_{n=1}^{\infty}$ be an almost-additive sequence of real numbers, that is, there exists a constant $C > 0$ such that

$$e^{-C} c_n c_m \leq c_{n+m} \leq e^{C} c_n c_m.$$

For $n \in \mathbb{N}$, define $f_n : \Sigma_F \to \mathbb{R}$ by

$$f_n(x) = c_n \lambda_{i_0} \lambda_{i_1} \cdots \lambda_{i_{n-1}}, \text{ for } x \in C_{i_0 \dots i_{n-1}}.$$

Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$. Then $\mathcal{F}$ is an almost-additive Bowen sequence on Σ_F. By definition we have

$$\begin{aligned} P(\mathcal{F}) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_0, i_1, \dots, i_{n-1} \in \mathbb{N}} c_n \lambda_{i_0} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \right) \\ &= \lim_{n\to\infty} \frac{\log c_n}{n} + \log \left(\sum_{i=1}^{\infty} \lambda_i \right). \end{aligned}$$

Clearly, Proposition 3.1 and the variational principle (Theorem 3.1) hold for $\mathcal{F}$. Since $\sum_{i\in\mathbb{N}} \sup f_1|_{C_i} = c_1 \sum_{i\in\mathbb{N}} \lambda_i < \infty$, by Theorem 4.1, there exist a Gibbs measure μ for $\mathcal{F}$ which is mixing. If $h(\mu) < \infty$, then it is the unique equilibrium measure for $\mathcal{F}$.

Similarly, we obtain an explicit formula for the pressure function for $t\mathcal{F} = \{t \log f_n\}_{n=1}^{\infty}$, with $t \in \mathbb{R}$, namely

$$P(t\mathcal{F}) = \lim_{n\to\infty} \frac{t \log c_n}{n} + \log \left(\sum_{i=1}^{\infty} {\lambda_i}^t \right).$$

In particular, there exists $t' > 0$ such that

$$P(t\mathcal{F}) = \begin{cases} \infty & \text{if } t < t'; \\ \text{finite}, & \text{if } t > t'. \end{cases}$$

Moreover, for $t > t'$ the pressure function $t \to P(t\mathcal{F})$ is real analytic, convex and decreasing.

6. Example 2: A factor map

Let $\pi : \Sigma_1 \to \Sigma_2$ be a one-block factor map between sub-shifts of finite type defined over finite alphabets. If the measure μ is an equilibrium measure for $\phi : \Sigma_1 \to \mathbb{R}$ it is natural to enquire whether the measure $\pi(\mu)$ is an equilibrium measure for a potential defined on Σ_2 related to ϕ. Walters established such a relation under the assumption that a so called *compensation* function to exists (see [W3] for precise definitions and details). Using sub-additive thermodynamic formalism, Yayama [Y] was able to establish the same relation between equilibrium measures, without the need of a compensation function to exists (related results have been obtained by Feng in [Fe4]). The present example exhibits one of the pathologies that one might encounter when trying to extend this type of results to the countable Markov shift setting. The sequence of potentials considered is the ones used in Yayama's proof and in this particular case the pressure is infinite. It also shows a natural way of obtaining almost-additive sequences of potentials.

Let A be the matrix defined by $A = (a_{ij})_{\mathbb{N}_0\times\mathbb{N}_0}$ where $a_{i0} = a_{0j} = 1$, for all $i, j \in \mathbb{N}_0$, and $a_{ij} = 0$ otherwise. Let B be the matrix defined by $B = (b_{ij})_{\mathbb{N}\times\mathbb{N}}$, $b_{i1} = b_{1j} = 1$, for $i, j \in \mathbb{N}$, and $b_{ij} = 0$ otherwise. Let X, Y be the topologically mixing countable Markov shifts with the BIP property determined by the transition matrix A, B respectively. Let $\pi : X \to Y$ be a one-block factor map defined by $\pi(k) = k/2 + 1$ if k is even, and $\pi(k) = (k-1)/2 + 1$ if k is odd. For an admissible word $y_1 \dots y_n$ of length n on Y, denote by $|\pi^{-1}[y_1 \dots y_n]|$ the number of admissible words of length n in X that are mapped to $y_1 \dots y_n$ by π. Define $\phi_n : Y \to \mathbb{R}$ by $\phi_n(y) = |\pi^{-1}[y_1 \dots y_n]|$ and let $\mathcal{F} = \{-\log \phi_n\}_{n=1}^{\infty}$. Let A_1 be the transition matrix of the symbols of $\pi^{-1}\{1\} = \{0, 1\}$ and a be the largest eigenvalue of the matrix A_1. It is easy to see that in general a point y in Y is given by $y = 1^{n_1} i_1 1^{n_2} i_2 \dots$ or $y = i_1 1^{n_2} i_2 \dots$, where $i_1, i_2, \dots \geq 2, n_1, n_2, \dots \geq 1$ and that there exist $C_1, C_2 > 0$ such that for any $l \in \mathbb{N}$,

$$C_1 2^l a^{n_1+\dots+n_l} \leq |\pi^{-1}[1^{n_1} i_1 1^{n_2} \dots i_{l-1} 1^{n_l} i_l]| \leq C_2 2^l a^{n_1+\dots+n_l}.$$

We can approximate the upper bounds and lower bounds of $|\pi^{-1}[1^{n_1} i_1 1^{n_2} \dots i_{l-1} 1^{n_l}]|$, $|\pi^{-1}[i_1 1^{n_2} \dots i_{l-1} 1^{n_l} i_l]|$, and $|\pi^{-1}[i_1 1^{n_2} \dots i_{l-1} 1^{n_l}]|$ similarly. Therefore, $\mathcal{F}$ is an almost-additive Bowen sequence on Y and Proposition 3.1 holds. However, since the entropy of (Y, σ) is infinite and the functions $(1/n) \log \phi_n$ are uniformly bounded (below and above) we have that $P(\mathcal{F}) = \infty$.

7. Maximal Lyapunov exponents of product of matrices

Let A, B be two square matrices of size $d \times d$. Let U be the $d-$dimensional column vector having each coordinate equal to 1. Consider the following norm

$$\|A\| := U^t A U. \tag{24}$$

Let $\{A_1, A_2, \dots\}$ be a countable collection of $d \times d$ matrices and let (Σ, σ) be a topologically mixing countable Markov shift. If $w = (i_0, i_1, \dots) \in \Sigma$, define the sequence of functions by

$$\phi_n(w) = \|A_{i_{n-1}} \cdots A_{i_1} A_{i_0}\|.$$

Since

$$\|AB\| \leq \|A\|\|B\|,$$

the sequence $\mathcal{F} = \{\log \phi_n\}_{n=1}^{\infty}$ is sub-additive on Σ. The study of this type of functions began with the work of Bellman [Be] and flourished with the seminal work of Furstenberg and Kesten [FK] who in 1960 considered the case of finitely many square matrices $\{A_1, A_2, \dots, A_m\}$ and the full-shift on m symbols. They proved that if $\mu \in \mathcal{M}$ is ergodic then μ-almost everywhere the following equality holds:

$$\lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu = \lim_{n\to\infty} \frac{1}{n} \log \phi_n(w).$$

Kingman [Ki], eight years later, proved his famous sub-additive ergodic theorem from which the above result follows. The number

$$\lambda(w) := \lim_{n\to\infty} \frac{1}{n} \log \phi_n(w),$$

is called *Maximal Lyapunov exponent of* w, whenever the limit exists. It is a fundamental dynamical quantity whose study arises in a wide range of different context, e.g. Schrödinger operators [AJ], smooth cocycles [AV], Hausdorff dimension of measures [Fe3]. Actually, its effective computation is also of interest [Po]. Recently, in a series of papers Feng [Fe1, Fe2, Fe3] studied dimension theory and thermodynamic formalism for maps $M : \Sigma \to L(\mathbb{R}^d, \mathbb{R}^d)$, where $L(\mathbb{R}^d, \mathbb{R}^d)$ denotes the space of $d \times d$ matrices. The techniques developed in the previous section allow us to generalise some of the results obtained by Feng to this non-compact setting.

In this generality, the sequence $\{\log \phi_n\}_{n=1}^{\infty}$ is only sub-additive (not necessarily almost-additive) and the shift (Σ, σ) need not to satisfy the BIP property. Under certain additional assumptions, we are able to prove the existence of Gibbs measures.

Proposition 7.1. *Let (Σ, σ) be a countable Markov shift satisfying the BIP condition. Let $\{A_1, A_2, \dots\}$ be a countable collection of $d \times d$ matrices having strictly positive entries. For $n \in \mathbb{N}$, define $\phi_n : \Sigma \to \mathbb{R}$ by*

$$\phi_n(w) = \|A_{i_{n-1}} \cdots A_{i_1} A_{i_0}\|$$

for $w = (i_0, i_1, \dots) \in \Sigma$. If $\mathcal{F} = \{\log \phi_n\}_{n=1}^{\infty}$ is an almost-additive sequence on Σ with $\sum_{i=1}^{\infty} \|A_i\| < \infty$, then

$$P(\mathcal{F}) = \sup \left\{ h(\mu) + \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu : \mu \in \mathcal{M} \text{ and } \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n d\mu \neq -\infty \right\},$$

and there exists a Gibbs measure μ for $\mathcal{F}$ which is mixing.

Proof. Note that the continuous functions ϕ_n are locally constant over cylinders of length n. Therefore, the Bowen condition is satisfied. Moreover,

$$\sum_{a \in S} \sup \phi_1|_{C_a} = \sum_{i=1}^{\infty} \|A_i\| < \infty.$$

Since the system satisfies the BIP condition, the result follows from Theorem 4.1. □

Remark 7.1. If the measure μ in Proposition 7.1 is such that $h(\mu) < \infty$, then μ is the unique equilibrium measure for $\mathcal{F}$.

The assumption $\sum_{i=1}^{\infty} \|A_i\| < \infty$ implies that $\lim_{n\to\infty} \|A_n\| = 0$. Therefore, almost-additivity cannot be obtained (in general) as in [Fe1, Lemma 2.1], even if all the entries are positive. Nevertheless, the same proof obtained by Feng in [Fe1, Lemma 2.1] gives us

Lemma 7.1. *Let $\{A_1, A_2, \dots\}$ be a countable collection of $d \times d$ matrices having strictly positive entries. Suppose there exists a constant $C > 0$ with the property that for every $k \in \mathbb{N}$ the following holds:*

$$\frac{\min_{i,j}(A_k)_{i,j}}{\max_{i,j}(A_k)_{i,j}} \geq C.$$

Then the sequence $\mathcal{F} = \{\log \phi_n\}_{n=1}^{\infty}$ is almost-additive on Σ, where $\phi_n : \Sigma \to \mathbb{R}$ is defined as in Proposition 7.1 *.*

The above condition is related to the cone condition studied by Barreira and Gelfert in [BG]. Alternative conditions ensuring almost-additivity of $\mathcal{F}$ have been obtained by Feng [Fe3, Proposition 2.8] and, in a slightly different setting, by Falconer and Sloan [FS, Corollary 2.3].

Remark 7.2. Under the assumptions of Proposition 7.1, there exists a positive number $t' > 0$ such that the pressure function $t \to P(t\mathcal{F})$ has the following form:

$$P(t\mathcal{F}) = \begin{cases} \infty & \text{if } t < t'; \\ \text{finite, convex and decreasing} & \text{if } t > t'. \end{cases}$$

8. A Bowen formula

In this section, we apply the results obtained in order to prove a formula that relates the pressure with the Hausdorff dimension of a geometric construction. This formula generalises previous results by Barreira [B1] to the countable setting.

Let us consider the following geometric construction in the interval. For every $n \in \mathbb{N}$, let $\Delta_n \subset [0,1]$ be a closed interval of length r_n. Assume that the intervals do not overlap, that is, if $m \neq n$ then $\Delta_m \cap \Delta_n = \emptyset$. For each $k \in \mathbb{N}$, choose again a family of non-overlapping closed intervals $\{\Delta_{kn}\}_{n \in \mathbb{N}}$ with $\Delta_{kn} \subset \Delta_k$. Denote the length of Δ_{kn} by r_{kn}. Iterating this procedure, for each interval $\Delta_{i_0 \dots i_{n-1}}$, we obtain a countable family of non-overlapping closed intervals $\{\Delta_{i_0 \dots i_{n-1} m}\}_{m \in \mathbb{N}}$ with $\Delta_{i_0 \dots i_{n-1} m} \subset \Delta_{i_0 \dots i_{n-1}}$. Denote by $r_{i_0 \dots i_{n-1} m}$ the length of $\{\Delta_{i_0 \dots i_{n-1} m}\}$. We define the limit set

$$\mathcal{K} = \bigcap_{n=1}^{\infty} \bigcup_{(i_0 \cdots i_{n-1})} \Delta_{i_0 \dots i_{n-1}}.$$

This geometric construction can be coded by a full-shift on a countable alphabet (Σ_F, σ). That is, there exists a homeomorphism $\psi : \Sigma_F \to \mathcal{K}$.

Denote by $\phi_n : \Sigma \to \mathbb{R}$ the function defined by $\phi_n(x) = \log r_{i_0 \dots i_{n-1}}$ if $x \in C_{i_0 \dots i_{n-1}}$ and by $\mathcal{F} = \{\phi_n\}_{n=1}^{\infty}$.

Theorem 8.1. *Let $\mathcal{K}$ be a geometric construction as above. Assume that there exists $C > 0$ such that for every $n, m \in \mathbb{N}$*

$$r_{i_0 \dots i_{n-1}} r_{i_n \cdots i_{m-1}} e^{-C} \leq r_{i_0 \cdots i_{n+m-1}} \leq r_{i_0 \dots i_{n-1}} r_{i_n \cdots i_{m-1}} e^{C}.$$

Then

$$\dim_H(\mathcal{K}) = \inf\{t \in \mathbb{R} : P(t\mathcal{F}) \leq 0\}.$$

Proof. Let us start with the lower bound. We consider a subset of $K_n \subset \mathcal{K}$ defined by the projection of ψ restricted to the compact sub-shift $\Sigma_n \subset \Sigma$, where Σ_n is the full-shift on $\{1, 2, \dots, n\}$. Denote by

$$R_n := \min\{r_1, r_2, \dots, r_n\}.$$

By the almost-additivity assumption on the radii we obtain that if $i_j \in \{1, 2, \dots, n\}$ then

$$\frac{r_{i_0 i_2 \dots i_n}}{r_{i_0 i_2 \dots i_{n-1}}} \geq R_n e^{-C} := \delta_n$$

Then, a result proved by Barreira in [B1] (see also [B4, p.35]) implies that

$$\dim_H K_n = t_{K_n},$$

where $t_{K_n} \in \mathbb{R}$ is the unique root of the equation $P(t\mathcal{F}|_{K_n}) = 0$. By the approximation property of the pressure (see Theorem 3.1), we obtain that

$$\lim_{n \to \infty} \dim_H K_n = \inf\{t \in \mathbb{R} : P(t\mathcal{F}) \leq 0\}.$$

Since $K_n \subset \mathcal{K}$, this proves the lower bound.

In order to prove the upper bound, we make use of the natural cover. Let $s \in \mathbb{R}$ be such that $\inf\{t \in \mathbb{R} : P(t\mathcal{F}) \leq 0\} < s$. Since the pressure of $s\mathcal{F}$ is negative there exists $L < 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \sum_{(i_0 \dots i_{n-1}) \in \mathbb{N}^n} (r_{i_0 \dots i_{n-1}})^s = P(s\mathcal{F}) < L < 0.$$

Hence, for sufficiently large values of $n \in \mathbb{N}$ we have that

$$\sum_{(i_0 \dots i_{n-1}) \in \mathbb{N}^n} (r_{i_0 \dots i_{n-1}})^s < e^{nL}.$$

Since $L < 0$ we have that $\lim_{n \to \infty} e^{nL} = 0$. Note that for each $n \in \mathbb{N}$ the family $\{\Delta_{i_0 \dots i_{n-1}} : i_0 \dots i_{n-1} \in \mathbb{N}^n\}$ is a cover of $\mathcal{K}$. Moreover, as n tends to infinity the diameters of the sets $\Delta_{i_0 \dots i_{n-1}}$ converge to zero. This implies that the $s-$Hausdorff measure of $\mathcal{K}$ is zero. Therefore, we obtain the desired upper bound. $\square$

It should be pointed out that, even in the additive case, it can happen that the equation $P(t\mathcal{K}) = 0$ does not have a root (see the work of Mauldin and Urbański [MU1] and that of Iommi [Io] for explicit examples).

Remark 8.1. If the equation $P(t\mathcal{F}) = 0$ has a root and the Gibbs measure corresponding to $(\dim_H \mathcal{K})\mathcal{F}$ is an equilibrium measure, then we obtain an almost-additive version of Ledrappier-Young formula (see [LY]) . Indeed,

$$P((\dim_H \mathcal{K})\mathcal{F}) = h(\mu) + \dim_H \mathcal{K} \left(\lim_{n \to \infty} \frac{1}{n} \int \phi_n \, d\mu \right) = 0.$$

Therefore,

$$\dim_H \mathcal{K} = -\frac{h(\mu)}{\lim_{n \to \infty} \frac{1}{n} \int \phi_n \, d\mu}.$$

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