ABELIAN VARIETIES WITH HECKE ALGEBRA ACTION

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Abstract. The action of a finite group G on an abelian variety $\mathcal{A}$ induces a decomposition of $\mathcal{A}$ into factors related to the rational irreducible representations of G, the so called isotypical decomposition of $\mathcal{A}$; when $\mathcal{A} = JZ$ is the Jacobian variety of a curve Z with G-action, for every subgroup H of G there is an induced canonical action of the corresponding Hecke algebra $\mathbb{Q}[H\backslash G/H]$ on the Jacobian of the quotient curve $Z_H = Z/H$, and a corresponding isotypical decomposition of JZ_H. These results have provided geometric and analytic information on the factors appearing in the isotypical decomposition of JZ and JZ_H.

In this paper we show that similar results hold for any abelian variety $\mathcal{A}$ with G-action: for any subgroup H of G there is a natural subvariety $\mathcal{A}_H$ of $\mathcal{A}$ fixed by H, such that $\mathbb{Q}[H\backslash G/H]$ acts on $\mathcal{A}_H$. We investigate the associated isotypical decomposition of $\mathcal{A}_H$, and find the decomposition of the analytic and the rational representations of the action of corresponding Hecke algebra on $\mathcal{A}_H$. We also show that the notion of Prym variety for covers of curves may be extended to abelian varieties, and describe its isotypical decomposition with respect to the action of a natural induced subalgebra of its endomorphism ring. We apply the results to the decomposition of the Jacobian and Prym varieties of the intermediate cover given by H, in the case of smooth projective curves with $G-$action. We work out several examples that give rise to families of principally polarized abeliana varieties, Jacobian and Prym varieties with large endomorphism rings.

1. Introduction

Let G be a finite group acting on an abelian variety $\mathcal{A}$; this action induces an algebra homomorphism $\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$.

Following [L-R] and [Ca-Ro], we define $\mathrm{Im}(\alpha) = \mathrm{Im}(\psi(m\alpha)) \subseteq \mathcal{A}$ for any $\alpha \in \mathbb{Q}[G]$, where m is any positive integer such that $m\alpha \in \mathbb{Z}[G]$. The decomposition of $1 \in \mathbb{Q}[G]$ as the sum of the central orthogonal idempotents e_i of $\mathbb{Q}[G]$ corresponding to the simple components of $\mathbb{Q}[G]$ induces a G-equivariant isogeny

$$\mathcal{A} \sim \mathcal{A}_1 \times \mathcal{A}_2 \times ... \times \mathcal{A}_r,$$

where $\mathcal{A}_i = \mathrm{Im}(e_i)$, the *isotypical decomposition* of $\mathcal{A}$, and G acts on $\mathcal{A}_i$ via the corresponding rational irreducible representation $\mathcal{W}_i$ of G.

Also, any decomposition of e_i as a sum of primitive orthogonal idempotents f_{i_k} induces an isogeny

$$\mathcal{A}_i \sim \mathcal{B}_{i_1} \times \mathcal{B}_{i_2} \times ... \times \mathcal{B}_{i_{n_i}},$$

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with $\mathcal{B}_{i_k} = \mathrm{Im}(f_{i_k})$.

Furthermore, the $\mathcal{B}_{i_1}, \mathcal{B}_{i_2}, ..., \mathcal{B}_{i_{n_i}}$ are all isogenous to each other. In this way the following G-equivariant isogeny decomposition for $\mathcal{A}$ is obtained

$$\mathcal{A} \sim \mathcal{B}_1^{n_1} \times \mathcal{B}_2^{n_2} \times ... \times \mathcal{B}_r^{n_r}.$$

The B_j were obtained as images of explicit idempotents in $\mathbb{Q}[G]$ in [Ca-Ro]. For the case $\mathcal{A} = JZ$, the Jacobian variety of a smooth projective curve Z with G-action, more information is known for the B_j. For instance, their dimension was obtained in [R], in terms of the fixed points of G in Z and their stabilizer subgroups.

It was also shown in [L-R] and [Ca-Ro] that if $H \leq N \leq G$ are subgroups of G with intermediate covering $F : Z_H \to Z_N$ where $Z_H = Z/H$ and $Z_N = Z/M$, then the Jacobians JZ_H and JZ_N, as well as the (generalized) Prym variety $P(Z_H/Z_N)$, defined as the orthogonal complement of $F^*(JZ_N)$ in JZ_H, admit similar isogeny decompositions. In fact, there exist non negative integers h_j and p_j such that

$$JZ_H \sim B_1^{h_1} \times \ldots \times B_r^{h_r}\ ,\ P(Z_H/Z_N) \sim B_1^{p_1} \times \ldots \times B_r^{p_r}\ .$$

It was further shown in [CLRR] and in [E] that the Hecke algebra $\mathbb{Q}[H\backslash G/H]$ acts naturally on JZ_H.

The decomposition of the induced actions of G on the analytic differentials on Z and on the first homology of Z into complex irreducible representations of G are known to be given respectively by the Chevalley-Weil formula and by the Lefschetz fixed point formula and the Eichler trace formula.

It is the aim of the present paper to generalize the above results to the case of abelian varieties $\mathcal{A}$ with a finite group G action: we associate to each subgroup $H \leq G$ a canonical subvariety $\mathcal{A}_H$ on which the Hecke algebra $\mathbb{Q}[H\backslash G/H]$ acts naturally, describe the isotypical decomposition of $\mathcal{A}_H$, compute the dimension of its factors, define the (generalized) Prym variety $P(\mathcal{A}_H/A_N)$ associated to subgroups $H \leq N \leq G$ and find its isotypical decomposition under the action of a natural subalgebra of $\mathbb{Q}[H\backslash G/H]$.

Our point of view for doing this is the following: Let G be a finite group acting on an abelian variety $\mathcal{A}$, and let H be a subgroup of G. For the central idempotent element of $\mathbb{Q}[H]$

$$p_H = \frac{1}{|H|}\sum_{h\in H} h,$$

consider the abelian subvariety of $\mathcal{A}$ given by

$$\mathcal{A}_H = \mathrm{Im}(p_H).$$

Let $\mathcal{H}_{H,\mathbb{Q}} = p_H\mathbb{Q}[G]p_H = \mathbb{Q}[H\backslash G/H]$ be the *Hecke algebra* over $\mathbb{Q}$ of H in G; namely the subalgebra of $\mathbb{Q}[G]$ consisting of the $\mathbb{Q}$-valued functions on G that are constant on each double coset of H in G. In Section 2 we recall the notation and describe the representation theory of $\mathcal{H}_{H,\mathbb{Q}}$.

The homomorphism $\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$ restricts to an algebra homomorphism

$$\psi_H : \mathcal{H}_{H,\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_H).$$

In Section 3 we show that the homomorphism ψ_H induces an $\mathcal{H}_{H,\mathbb{Q}}$-equivariant isogeny decomposition of $\mathcal{A}_H$ into factors related to the rational irreducible representations of $\mathcal{H}_{H,\mathbb{Q}}$, the *isotypical decomposition* of $\mathcal{A}_H$, and in Section 4 we obtain an isogeny decomposition of these $\mathcal{H}_{H,\mathbb{Q}}$-invariant factors, of the form

$$\mathcal{A}_H \sim \mathcal{B}_1^{a_1} \times \mathcal{B}_2^{a_2} \times \ldots \times \mathcal{B}_r^{a_r}.$$

In Section 5 we establish relations between the analytic and the rational representation of G on $\mathcal{A}$ and the corresponding representations of the Hecke algebra on $\mathcal{A}_H$.
In Section 6 we show that the notion of a generalized Prym variety for covers of curves may be extended to abelian varieties; we also describe its isotypical decomposition, and a natural subalgebra of its endomorphism ring.
In Section 7 we consider the case of curves with G-action and we apply these results to the Jacobian and Prym varieties of the intermediate cover given by H, as follows.

Let Z be a smooth projective curve (defined over the field of the complex numbers) on which G acts, and let Z_H denote the quotient of Z by H, for any subgroup $H \leq G$. Then there is a canonical action of the corresponding Hecke algebra $\mathcal{H}_{H,\mathbb{Q}}$ on JZ_H, the Jacobian variety of Z_H. We study this action, and obtain the corresponding isotypical decomposition of JZ_H, together with a description of the action on each factor. Formulae analogues to the Chevalley-Weil formula and for the decomposition of the rational representation are given.

If N denotes any subgroup of G containing H, there is a natural isogeny

$$JZ_H \sim JZ_N \times P(Z_H/Z_N),$$

where $P(Z_H/Z_N)$ is the (generalized) Prym variety of the cover $Z_H \to Z_N$. In Section 8 we describe the induced action on JZ_N and $P(Z_H/Z_N)$ by appropriate subalgebras of $\mathcal{H}_{H,\mathbb{Q}}$.

The paper concludes with several examples: the first is the description of a four dimensional family of principally polarized abelian varieties containing no Jacobians, admitting an action by the symmetric group of degree three; we find their isotypical decomposition, the decomposition of their analytic and rational actions, and a description of the isotypical factors in terms of a fixed subvariety and a generalized Prym variety, including their endomorphism rings.

The second example exhibits a series of Jacobian and classical Prym varieties with complex multiplication, of dimension 2^{m-3} for each $m \geq 3$.

The third example describes a series of Prym varieties of dimension $\dfrac{p(p-1)(q-1)}{2}$, whose endomorphism ring contains a copy of the square matrices of size $p(p-1)(q-1)$ over $\mathbb{Q}$, where p and q are odd prime numbers such that p divides $q-1$ but p^2 does not divide $q-1$.

We suppose throughout that all curves and abelian varieties are defined over the field of complex numbers. Moreover the curves will always be smooth and projective.

2. Preliminaries

2.1. **The group algebra.** Let G be a finite group. In order to fix the notation, we start by recalling some basic properties of representations of G and the Hecke algebra associated to any subgroup $H \leq G$ (see [C-R], [C-R1] and [Ca-Ro]). For any field F of characteristic zero we denote by $F[G]$ the group algebra of G over F; we identify the elements of $F[G]$ with the F-valued functions on G. In this paper the field F will be either the complex numbers $\mathbb{C}$ or the rational numbers $\mathbb{Q}$.

$F[G]$ is a semisimple algebra, whose simple components correspond bijectively with the elements of the set $\mathrm{Irr}_F(G)$, the irreducible F-representations of G, as we now recall.

The central idempotent $e_{\mathcal{V}}$ of $\mathbb{C}[G]$ that generates the simple subalgebra of $\mathbb{C}[G]$ corresponding to a complex irreducible representation $\mathcal{V}$ of G, and the central idempotent $e_{\mathcal{W}}$ of $\mathbb{Q}[G]$ that generates the simple subalgebra of $\mathbb{Q}[G]$ corresponding to a rational irreducible representation $\mathcal{W}$ of G, are respectively given by

$$e_{\mathcal{V}} = \frac{\dim_{\mathbb{C}}(\mathcal{V})}{|G|} \sum_{g \in G} \chi_{\mathcal{V}}(g^{-1}) g \, , \quad \text{and} \tag{2.1}$$

$$e_{\mathcal{W}} = \mathrm{tr}_{K_{\mathcal{V}}/\mathbb{Q}}\,(e_{\mathcal{V}}) = \frac{\dim_{\mathbb{C}}(\mathcal{V})}{|G|} \sum_{g \in G} \mathrm{tr}_{K_{\mathcal{V}}/\mathbb{Q}}(\chi_{\mathcal{V}}(g^{-1}))g \, ,$$

where $K_{\mathcal{V}} = \mathbb{Q}(\chi_{\mathcal{V}}(g) : g \in G)$ is the character field of $\mathcal{V}$, tr denotes the trace, and $\mathcal{V}$ is a complex irreducible representation of G Galois-associated to $\mathcal{W}$; that is, $\mathcal{W} \bigotimes_{\mathbb{Q}} \mathbb{C} \cong \sum_{\sigma \in \mathrm{Gal}(L_{\mathcal{V}}/\mathbb{Q})} \mathcal{V}^{\sigma}$, where $L_{\mathcal{V}}$ is the field of definition of $\mathcal{V}$.

Then the simple algebra $\mathbb{C}[G]e_{\mathcal{V}}$ affords the complex irreducible representation $\mathcal{V}$ with multiplicity $\dim_{\mathbb{C}}(\mathcal{V})$, and

$$\mathbb{C}[G] = \bigoplus_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{C}[G]\, e_{\mathcal{V}}$$

affords the regular representation of G.

In particular, the unit in $\mathbb{C}[G]$ decomposes as the sum of the central idempotents in $\mathbb{C}[G]$ as follows.

$$1_G = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} e_{\mathcal{V}}.$$

Similarly, the simple algebra $\mathbb{Q}[G]e_{\mathcal{W}}$ affords the rational irreducible representation $\mathcal{W}$ with multiplicity $n_{\mathcal{W}} = \frac{\dim_{\mathbb{C}}(\mathcal{V})}{s_{\mathcal{V}}}$, where $s_{\mathcal{V}} = [L_{\mathcal{V}} : K_{\mathcal{V}}]$ is the Schur index of $\mathcal{V}$, and $\mathcal{V}$ is a complex irreducible representation Galois-associated to $\mathcal{W}$, and

$$\mathbb{Q}[G] = \bigoplus_{\mathcal{W} \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathbb{Q}[G]\, e_{\mathcal{W}}$$

affords the regular representation of G.

In particular, the unit in $\mathbb{Q}[G]$ decomposes as the sum of the central idempotents in $\mathbb{Q}[G]$ as follows.

$$1_G = \sum_{\mathcal{W} \in \mathrm{Irr}_{\mathbb{Q}}(G)} e_{\mathcal{W}}.$$

2.2. **The Hecke algebra of a subgroup.** In this section we recall the notation and some known facts about the Hecke algebra for a subgroup H of a group G, following [C-R1] and [Ca-Ro].

Let H be a subgroup of a finite group G. Then the element

$$p_H = \frac{1}{|H|} \sum_{h \in H} h \tag{2.2}$$

is the central idempotent of $F[H]$ corresponding to the trivial representation of H. Moreover, the left ideal $F[G]p_H$ in $F[G]$ affords the F-representation of G induced by the trivial representation of H. In the sequel we denote this representation by Υ_H.

It is known that (see for instance [Ca-Ro, Lemma 4.3])

$$\Upsilon_H = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V}^H)\, \mathcal{V} = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} \langle \Upsilon_H, \mathcal{V} \rangle_G\, \mathcal{V} = \sum_{\mathcal{W} \in \mathrm{Irr}_{\mathbb{Q}}(G)} \frac{\dim_{\mathbb{C}}(\mathcal{V}^H)}{s_{\mathcal{V}}}\, \mathcal{W}, \tag{2.3}$$

where for each complex irreducible representation $\mathcal{V}$ of G, $\mathcal{V}^H$ denotes the subspace of $\mathcal{V}$ fixed by H, and for each rational irreducible representation $\mathcal{W}$ of G, $\mathcal{V}$ is a complex irreducible representation Galois-associated to $\mathcal{W}$, and $s_{\mathcal{V}}$ is the Schur index of $\mathcal{V}$. Here $\langle \cdot, \cdot \rangle_G$ denotes the usual inner product for two complex representations of G.

The F-algebra $\mathcal{H}_{H,F} = p_H F[G] p_H = F[H\backslash G/H]$, considered as a subalgebra of $F[G]$, consists of the F-valued functions on G which are constant on each double coset HgH of H in G. It is called the *Hecke algebra* over F of H in G, and its unit is p_H.

We now recall that there is a bijection from the set of all irreducible F-representations $\mathcal{U}$ of G such that $\dim_F(\mathcal{U}^H) \neq 0$ to the set of all irreducible F-representations $\widetilde{\mathcal{U}}$ of the semisimple F-algebra $\mathcal{H}_{H,F}$ (see for instance [C-R1, Chapter 11].

Given $\widetilde{\mathcal{U}}$ in $\mathrm{Irr}_F(\mathcal{H}_{H,F})$, the unique irreducible representation $\mathcal{U}$ of G such that $\widetilde{\mathcal{U}} = \mathcal{U}_{|\mathcal{H}_{H,F}}$ will be called *associated* to $\widetilde{\mathcal{U}}$. Note that $\dim_F(\widetilde{\mathcal{U}}) = \dim_F(\mathcal{U}^H)$. Furthermore, if $M_{\mathcal{U}}$ is a simple $F[G]$-module affording $\mathcal{U}$, then $p_H(M_{\mathcal{U}}) = M_{\widetilde{\mathcal{U}}}$ is a simple $\mathcal{H}_{H,F}$-module affording $\widetilde{\mathcal{U}}$.

We now recall some idempotents in $\mathcal{H}_{H,F}$ and their properties, that will prove useful later. For a complete proof see [Ca-Ro, Theorem 4.4].

Lemma 2.1. *For each $\mathcal{U}$ in $\mathrm{Irr}_F(G)$, set*

$$f_{H,\mathcal{U}} = p_H e_{\mathcal{U}} = e_{\mathcal{U}} p_H. \tag{2.4}$$

Then

- $f_{H,\mathcal{U}}^2 = f_{H,\mathcal{U}}$;

- $hf_{H,\mathcal{U}} = f_{H,\mathcal{U}} = f_{H,\mathcal{U}}h$ *for all* $h \in H$*; that is,* $f_{H,\mathcal{U}} \in \mathcal{H}_{H,F}$*;*
- $f_{H,\mathcal{U}} = 0$ *if and only if* $\dim_F(\mathcal{U}^H) = 0$.
- *If* $F = \mathbb{C}$*, then the left ideal* $\mathbb{C}[G]f_{H,\mathcal{U}}$ *affords the representation* $\mathcal{U}$ *with multiplicity* $\dim_{\mathbb{C}}(\mathcal{U}^H)$.

  If $F = \mathbb{Q}$*, then the left ideal* $\mathbb{Q}[G]f_{H,\mathcal{U}}$ *affords the representation* $\mathcal{U}$ *with multiplicity* $a_{\mathcal{U}} = \frac{\dim_{\mathbb{C}}(\mathcal{V}^H)}{s_{\mathcal{V}}}$*, where* $s_{\mathcal{V}}$ *is the Schur index of* $\mathcal{V}$*, and* $\mathcal{V}$ *is a complex irreducible representation Galois-associated to* $\mathcal{U}$*.*

An immediate consequence is the following result.

Corollary 2.2. *With notation as in the previous lemma, the central idempotents in* $\mathcal{H}_{H,F}$ *are given by* $\{f_{H,\mathcal{U}} : \widetilde{\mathcal{U}} \in \mathrm{Irr}_F(\mathcal{H}_{H,F})\}$*, the decomposition of the semisimple algebra* $\mathcal{H}_{H,F}$ *into simple components is given by*

$$\mathcal{H}_{H,F} = \bigoplus_{\widetilde{\mathcal{U}} \in \mathrm{Irr}_F(\mathcal{H}_{H,F})} \mathcal{H}_{H,F} f_{H,\mathcal{U}} = \bigoplus_{\widetilde{\mathcal{U}} \in \mathrm{Irr}_F(\mathcal{H}_{H,F})} f_{H,\mathcal{U}} \mathcal{H}_{H,F} f_{H,\mathcal{U}},$$

the decomposition of the unit p_H *of* $\mathcal{H}_{H,F}$ *into central idempotents is given by*

$$p_H = \sum_{\widetilde{\mathcal{U}} \in \mathrm{Irr}_F(\mathcal{H}_{H,F})} f_{H,\mathcal{U}},$$

and each simple component $\mathcal{H}_{H,F} f_{H,\mathcal{U}}$ *affords the irreducible representation* $\widetilde{\mathcal{U}}$ *with multiplicity*

$$a_{\mathcal{U}} = \begin{cases} \dim_{\mathbb{C}}(\mathcal{U}^H), & \text{if } \mathcal{U} \text{ is complex irreducible;} \\[2ex] \dfrac{\dim_{\mathbb{C}}(\mathcal{V}^H)}{s_{\mathcal{V}}}, & \text{if } \mathcal{U} \text{ is rational irreducible and } \mathcal{V} \text{ is a complex} \\ & \quad \text{irreducible representation Galois-associated to } \mathcal{U}. \end{cases}$$

2.3. **A construction of primitive rational idempotents invariant under a subgroup.** We are interested in further decomposing each central idempotent $f_{H,\mathcal{U}}$ in the simple algebra $\mathcal{H}_{H,\mathbb{Q}} f_{H,\mathcal{U}}$ constructed in the previous section, as a sum of H-invariant orthogonal *primitive* rational idempotents. We now provide an explicit construction for these invariant idempotents; they will be used later on to construct the basic blocks in the isotypical decomposition of the canonical abelian subvariety $\mathcal{A}_H$ associated to the subgroup H, for the abelian variety $\mathcal{A}$ with action of the group G.

We recall from [Ca-Ro, Corollary 3.6] that for each rational irreducible representation $\mathcal{W}$ of G, explicit orthogonal primitive idempotents f_j, $1 \leq j \leq n$, may be found such that

$$e_{\mathcal{W}} = f_1 + \ldots + f_n \, .$$

For each subgroup $H \leq G$, multiplying this last equality on the right by p_H we obtain

$$f_{H,\mathcal{W}} = f_1 \, p_H + \ldots + f_n \, p_H .$$

However, the $f_j\, p_H$ need not be idempotents (some mat be zero), they may generate the same ideal, and this is not the sought decomposition. To find this decomposition, set $J_j = \mathbb{Q}[G] f_j p_H$, and renumber the f_j so that the first $a_{\mathcal{W}} = \dfrac{\dim_{\mathbb{C}}(\mathcal{V}^H)}{s_{\mathcal{V}}}$ of them satisfy $f_j\, p_H \neq 0$, where $\mathcal{V}$ is a complex irreducible representation Galois-associated to $\mathcal{W}$, and also so that $J_j \neq J_k$ for $1 \leq j \neq k \leq a_{\mathcal{W}}$. Lemma 2.1 implies that one can find precisely $a_{\mathcal{W}}$ of the ideals J_j satisfying these conditions.

Then each J_j is a minimal left ideal in the simple algebra $\mathbb{Q}[G]\, f_{H,\mathcal{W}}$, for $1 \leq j \leq a_{\mathcal{W}}$, and

$$\mathbb{Q}[G]\, f_{H,\mathcal{W}} = \bigoplus_{j=1}^{a_{\mathcal{W}}} J_j\,. \tag{2.5}$$

Hence there exist unique primitive idempotents $\{v_j\}_{j=1}^{a_{\mathcal{W}}}$ in $\mathbb{Q}[G]\, f_{H,\mathcal{W}}$ such that each v_j generates J_j and

$$f_{H,\mathcal{W}} = v_1 + \ldots + v_{a_{\mathcal{W}}}\,. \tag{2.6}$$

Note that each v_j is invariant under right multiplication by any h in H, since J_j is; v_j is also invariant under right multiplication, because $h\, v_j$ is in J_j, the left hand side of (2.6) is invariant under multiplication by any h, and the decomposition (2.6) is unique.

Therefore $v_j \in \mathcal{H}_{H,\mathbb{Q}}$, and it is also a primitive idempotent there, as follows from [C-R, Corollary 11.23]. We have thus shown how to explicitly construct the required H-invariant primitive idempotents v_j in $\mathcal{H}_{H,\mathbb{Q}}$ satisfying (2.6).

3. The isotypical decomposition of $\mathcal{A}_H$

Let G be a finite group acting on an abelian variety $\mathcal{A}$; this action induces an algebra homomorphism

$$\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}).$$

Since this homomorphism is canonical, we will denote the elements of $\mathbb{Q}[G]$ and their images by the same letter. As mentioned in the introduction, for any $\alpha \in \mathbb{Q}[G]$ we define $\mathrm{Im}(\alpha) = \mathrm{Im}(\psi(m\alpha)) \subseteq \mathcal{A}$ where m is some positive integer such that $m\alpha \in \mathbb{Z}[G]$. It is clear that $\mathrm{Im}(\alpha)$ is an abelian subvariety of $\mathcal{A}$, which does not depend on m.

Let H be a subgroup of G. Consider the subvariety of $\mathcal{A}$ given by

$$\mathcal{A}_H = \mathrm{Im}(p_H).$$

We call $\mathcal{A}_H$ the *canonical subvariety of* $\mathcal{A}$ *fixed* by H.

It is clear that H acts trivially on $\mathcal{A}_H$, and that in the case H is a normal subgroup of G the factor group G/H acts on $\mathcal{A}_H$. In the general case $H \leq G$, consider the Hecke algebra $\mathcal{H}_{H,\mathbb{Q}} = p_H\mathbb{Q}[G]p_H$; then the homomorphism ψ restricts to an algebra homomorphism

$$\psi_H : \mathcal{H}_{H,\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_H). \tag{3.1}$$

In order to study the action of $\mathcal{H}_{H,\mathbb{Q}}$ on $\mathcal{A}_H$, it is useful to make a few comments about the structure of the Hecke algebra of H in G.

Let $\mathrm{Irr}_{\mathbb{Q}}(G) = \{\mathcal{W}_1, \mathcal{W}_2, ..., \mathcal{W}_r\}$ be the set of all rational irreducible representations of G. Consider Υ_H the representation of G induced by the trivial representation of H. Recall from (2.3) that Υ_H decomposes as follows

$$\Upsilon_H \cong a_1\mathcal{W}_1 \oplus a_2\mathcal{W}_2 \oplus ... \oplus a_r\mathcal{W}_r \, ,$$

with $a_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^H)}{s_{\mathcal{V}_i}}$, where $\mathcal{V}_i$ is a complex irreducible representation Galois-associated to $\mathcal{W}_i$.

Let (renumbering if necessary) $\{W_1, \mathcal{W}_2, ..., \mathcal{W}_t\}$ denote the set of all rational irreducible representations $\mathcal{W}_i$ such that $a_i \neq 0$. We recall from Section 2.2 that there is a bijection from this set to the set $\{\widetilde{\mathcal{W}}_1, \widetilde{\mathcal{W}}_2, ...\widetilde{\mathcal{W}}_t\}$ of all rational irreducible representations of $\mathcal{H}_{H,\mathbb{Q}}$.

According to Corollary 2.2, the unit $p_H \in \mathcal{H}_{H,\mathbb{Q}}$ decomposes as follows

$$p_H = f_{H,\mathcal{W}_1} + f_{H,\mathcal{W}_2} + ... + f_{H,\mathcal{W}_t}, \tag{3.2}$$

with $f_{H,\mathcal{W}_i}$ the unit in the simple subalgebra of $\mathcal{H}_{H,\mathbb{Q}}$ corresponding to $\widetilde{\mathcal{W}}_i$.

Proposition 3.1. *Set* $\mathcal{A}_{H,\widetilde{\mathcal{W}}_i} = \mathrm{Im}(f_{H,\mathcal{W}_i}) \subseteq \mathcal{A}_H$ *for* $i = 1, ..., t$. *Then*

(1) *The subvarieties* $\mathcal{A}_{H,\widetilde{\mathcal{W}}_i}$ *are* $\mathcal{H}_{H,\mathbb{Q}}$*-invariant, and the action of* $\mathcal{H}_{H,\mathbb{Q}}$ *on* $\mathcal{A}_{H,\widetilde{\mathcal{W}}_i}$ *is given by the representation* $\widetilde{\mathcal{W}}_i$*; in fact, the homomorphism* ψ_H *of* (3.1) *restricts to*

$$\psi_{H,\mathcal{W}_i} : f_{H,\mathcal{W}_i}\,(\mathbb{Q}[G])\, f_{H,\mathcal{W}_i} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_{H,\widetilde{\mathcal{W}}_i}).$$

(2) *There is an* $\mathcal{H}_{H,\mathbb{Q}}$*-equivariant isogeny*

$$\mathcal{A}_{H,\widetilde{\mathcal{W}}_1} \times \mathcal{A}_{H,\widetilde{\mathcal{W}}_2} \times ... \times \mathcal{A}_{H,\widetilde{\mathcal{W}}_t} \to \mathcal{A}_H \, . \tag{3.3}$$

As in [L-R] for the case of a group action, we call (3.3) the *isotypical decomposition* of $\mathcal{A}_H$. It is unique up to permutation of the factors.

Proof. (1) By Corollary 2.2, $f_{H,\mathcal{W}_i}$ is a central idempotent in $\mathcal{H}_{H,\mathbb{Q}}$.

Furthermore,

$$p_H g p_H \left(\mathcal{A}_{H,\widetilde{\mathcal{W}}_i}\right) = p_H g p_H \left(f_{H,\mathcal{W}_i}\left(\mathcal{A}_H\right)\right) = f_{H,\mathcal{W}_i}\left(p_H g p_H\left(\mathcal{A}_H\right)\right) \subseteq \mathcal{A}_{H,\widetilde{\mathcal{W}}_i}$$

for all $g \in G$. Hence, $\mathcal{A}_{H,\widetilde{\mathcal{W}}_i}$ is $\mathcal{H}_{H,\mathbb{Q}}$-invariant. The second assertion follows from the fact that the idempotent element $f_{H,\mathcal{W}_i}$ affords the representation $\widetilde{\mathcal{W}}_i$ with multiplicity a_i.

(2) Since

$$p_H = f_{H,\mathcal{W}_1} + f_{H,\mathcal{W}_2} + ... + f_{H,\mathcal{W}_t}$$

and $f_{H,\mathcal{W}_i}$ is a central idempotent in $\mathcal{H}_{H,\mathbb{Q}}$, the addition map induces the required $\mathcal{H}_{H,\mathbb{Q}}$-equivariant isogeny. □

4. A Decomposition of the components $\mathcal{A}_{H,\widetilde{\mathcal{W}_i}}$ of A_H

Let G be a finite group acting on an abelian variety $\mathcal{A}$. Let $\{W_1, \mathcal{W}_2, ..., \mathcal{W}_r\}$ be the set of all rational irreducible representations of G. According to [L-R, Prop. 1.1], the isotypical decomposition of $\mathcal{A}$ is given by

$$\mathcal{A} \sim \mathcal{A}_{\mathcal{W}_1} \times \mathcal{A}_{\mathcal{W}_2} \times ... \times \mathcal{A}_{\mathcal{W}_r},$$

where $\mathcal{A}_{\mathcal{W}_i} = \mathrm{Im}(e_{\mathcal{W}_i})$ and $e_{\mathcal{W}_i}$ is the unit of the corresponding simple components of $\mathbb{Q}[G]$. Also, by [L-R, Th. 2.2] and [Ca-Ro, Section 5] we have

$$\mathcal{A} \sim \mathcal{B}_{\mathcal{W}_1}^{n_1} \times \mathcal{B}_{\mathcal{W}_2}^{n_2} \times .... \times \mathcal{B}_{\mathcal{W}_r}^{n_r}, \tag{4.1}$$

with $n_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i)}{s_{\mathcal{V}_i}}$, where $\mathcal{V}_i$ is a complex irreducible representation of G Galois-associated to $\mathcal{W}_i$. The last isogeny is obtained by considering the decomposition of each $e_{\mathcal{W}_i}$ as a sum of primitive orthogonal idempotents f_i in the corresponding simple component $\mathbb{Q}[G]e_{\mathcal{W}_i}$ of $\mathbb{Q}[G]$, and letting $\mathcal{B}_{\mathcal{W}_i} = \mathrm{Im}(f_i)$.

Given H a subgroup of G, we are interested to obtain the corresponding decomposition of the isotypical factors $\mathcal{A}_{H,\widetilde{\mathcal{W}_i}}$ for the canonical subvariety $\mathcal{A}_H$ fixed by H given in Proposition 3.1:

$$\mathcal{A}_H \sim \mathcal{A}_{H,\widetilde{\mathcal{W}_1}} \times \mathcal{A}_{H,\widetilde{\mathcal{W}_2}} \times ... \times \mathcal{A}_{H,\widetilde{\mathcal{W}_t}}.$$

Let $\mathcal{A}_{H,\widetilde{\mathcal{W}}}$ denote any one of them. Then

$$\mathcal{A}_{H,\widetilde{\mathcal{W}}} = \mathrm{Im}(f_{H,\mathcal{W}}) = \mathrm{Im}(p_H e_{\mathcal{W}}) \subseteq \mathrm{Im}(e_{\mathcal{W}}) = \mathcal{A}_{\mathcal{W}},$$

since $\mathcal{A}_{\mathcal{W}} = \mathrm{Im}(e_{\mathcal{W}})$ is G-invariant.

According to Section 2.3 we can write $f_{H,\mathcal{W}}$ as a sum of primitive orthogonal rational idempotents in $\mathcal{H}_{H,\mathbb{Q}}$, all left and right invariant under multiplication by each element of H, as follows.

$$f_{H,\mathcal{W}} = p_H e_{\mathcal{W}} = e_{\mathcal{W}} p_H = f_{H,1,\mathcal{W}} + \ldots + f_{H,a,\mathcal{W}}. \tag{4.2}$$

Consider the abelian subvarieties of $A_{H,\widetilde{\mathcal{W}}}$ defined by

$$\mathcal{B}_{H,k,\widetilde{\mathcal{W}}} = \mathrm{Im}(f_{H,k,\mathcal{W}}) \subseteq A_{H,\widetilde{\mathcal{W}}} \subseteq \mathcal{A}_{\mathcal{W}} \tag{4.3}$$

for $1 \leq k \leq a$.

Proposition 4.1. *Let $\mathcal{A}_{H,\widetilde{\mathcal{W}}}$ be an isotypical factor of $\mathcal{A}_H$ in the decomposition given in Proposition 3.1 and $\mathcal{B}_{H,k,\widetilde{\mathcal{W}}}$ as in (4.3). Then*

(1) *There is an isogeny*

$$\mathcal{B}_{H,1,\widetilde{\mathcal{W}}} \times \mathcal{B}_{H,2,\widetilde{\mathcal{W}}} \times \ldots \times \mathcal{B}_{H,a,\widetilde{\mathcal{W}}} \to \mathcal{A}_{H,\widetilde{\mathcal{W}}}.$$

(2) *The subvarieties $\mathcal{B}_{H,1,\widetilde{\mathcal{W}}}, \mathcal{B}_{H,2,\widetilde{\mathcal{W}}}, \ldots, \mathcal{B}_{H,a,\widetilde{\mathcal{W}}}$ are all isogenous to each other, as well as to the corresponding factor $B_{\mathcal{W}}$ in* (4.1).

Proof. (1) According to (4.2) we have

$$f_{H,\mathcal{W}} = f_{H,1,\mathcal{W}} + \ldots + f_{H,a,\mathcal{W}},$$

with all $f_{H,k,\mathcal{W}}$ left and right invariant under multiplication by H. Then the addition map gives an isogeny

$$\mathcal{B}_{H,\,1,\widetilde{\mathcal{W}}} \times \mathcal{B}_{H,\,2,\widetilde{\mathcal{W}}} \times \ldots \times \mathcal{B}_{H,\,a,\widetilde{\mathcal{W}}} \to \mathcal{A}_{H,\widetilde{\mathcal{W}}}$$

(2) This assertion follows from the fact that all $f_{H,k,\mathcal{W}}$ are primitive idempotents in the simple component of $\mathbb{Q}[G]$ corresponding to $\mathcal{W}$, and hence the minimal left ideals they generate are all isomorphic to each other.

□

Combining Propositions 3.1 and 4.1 we obtain the main result of this section

Theorem 4.2. *Let G be a finite group acting on an abelian variety $\mathcal{A}$ and H be a subgroup of G. Let $\{\widetilde{\mathcal{W}_1}, \widetilde{\mathcal{W}_2}, \ldots, \widetilde{W_t}\}$ be the set of the all irreducible rational representations of the Hecke algebra $\mathcal{H}_{H,\mathbb{Q}}$.*

Then there are subvarieties $\mathcal{B}_{H,\widetilde{\mathcal{W}_1}}, \mathcal{B}_{H,\widetilde{\mathcal{W}_2}}, \ldots, \mathcal{B}_{H,\widetilde{\mathcal{W}_t}}$ of the canonical subvariety $\mathcal{A}_H$ fixed by H, and an $\mathcal{H}_{H,\mathbb{Q}}$-equivariant isogeny

$$\mathcal{A}_H \sim \mathcal{B}^{a_1}_{H,\widetilde{\mathcal{W}_1}} \times \mathcal{B}^{a_2}_{H,\widetilde{\mathcal{W}_2}} \times \ldots \times \mathcal{B}^{a_t}_{H,\widetilde{\mathcal{W}_t}}, \tag{4.4}$$

with $a_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^H)}{s_{\mathcal{V}_i}}$, where $s_{\mathcal{V}_i}$ is the Schur index of $\mathcal{V}_i$ and $\mathcal{V}_i$ is a complex irreducible representation Galois-associated to $\mathcal{W}_i$. Furthermore, the subvarieties $\mathcal{B}^{a_k}_{H,\widetilde{\mathcal{W}_k}}$ are $\mathcal{H}_{H,\mathbb{Q}}$-invariant, and the action of $\mathcal{H}_{H,\mathbb{Q}}$ on $\mathcal{B}^{a_k}_{H,\widetilde{\mathcal{W}_k}}$ is given by the representation $\widetilde{\mathcal{W}_k}$, for all $1 \leq k \leq t$.

In particular, when $H = \{1\}$ we obtain Theorem 2.2 of [L-R]; see also [Ca-Ro] Section 5.

5. Analytic and rational representations

Let G be a finite group acting on an abelian variety $\mathcal{A}$. This action induces a complex linear representation ρ_a of the group G on $H^1(\mathcal{A}, \mathbb{C})$, the *analytic representation.* Also, the induced action of G on $H^1(\mathcal{A}, \mathbb{Q})$ gives a rational linear representation ρ_r of G, the *rational representation.*

In this section we study the corresponding complex representation $\widetilde{\rho}_{\mathrm{a}}$ of the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}} = \mathbb{C}[H\backslash G/H]$ on $H^1(\mathcal{A}_H, \mathbb{C})$, and rational representation $\widetilde{\rho}_{\mathrm{r}}$ of the rational Hecke algebra $\mathcal{H}_{H,\mathbb{Q}} = \mathbb{Q}[H\backslash G/H]$ on $H^1(\mathcal{A}_H, \mathbb{Q})$, where H is any subgroup of G and $\mathcal{A}_H$ is the canonical subvariety of $\mathcal{A}$ fixed by H.

Remark 5.1. Let ρ denote the representation of G on $H^1(\mathcal{A}, F)$ (analytic or rational) and let M denote an $F[G]$-module affording the representation ρ. By [GR, p. 202, Corollaire], there are an isomorphism of $F[G]$-modules

$$H^1(\mathcal{A}, F) \simeq M$$

and isomorphisms of $\mathcal{H}_{H,F}$-modules

$$H^1(\mathcal{A}_H, F) \simeq H^1(\mathcal{A}, F)^H \simeq p_H(M). \tag{5.1}$$

Proposition 5.2. *Let G be a finite group acting on an abelian variety $\mathcal{A}$ and H a subgroup of G. Let ρ_a be the analytic representation and ρ_r be the rational representation of G, on $H^1(\mathcal{A}, \mathbb{C})$ and $H^1(\mathcal{A}, \mathbb{Q})$ respectively.*

(1) *If the decomposition of ρ_a as a sum of complex irreducible representations of G is given by*
$$\rho_a = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} n_{\mathcal{V}} \, \mathcal{V},$$
then the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ acts on $H^1(\mathcal{A}_H, \mathbb{C})$ with analytic representation
$$\widetilde{\rho}_a = \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} n_{\mathcal{V}} \, \widetilde{\mathcal{V}}.$$

(2) *If the decomposition of $\rho_r \otimes_{\mathbb{Q}} \mathbb{C}$ as a sum of complex irreducible representations of G is given by*
$$\rho_r \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} m_{\mathcal{V}} \, \mathcal{V},$$
then the Hecke algebra $\mathcal{H}_{H,\mathbb{Q}}$ acts on $H^1(\mathcal{A}_H, \mathbb{Q})$ with rational representation $\widetilde{\rho}_r$, and $\widetilde{\rho}_r \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as a sum of complex irreducible representations of $\mathcal{H}_{H,\mathbb{C}}$ as follows
$$\widetilde{\rho}_r \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} m_{\mathcal{V}} \, \widetilde{\mathcal{V}}.$$

Proof. Let $M = H^1(\mathcal{A}, F)$ be the $F[G]$-module affording the representation ρ of G, where $\rho = \rho_{\mathrm{a}}$ if $F = \mathbb{C}$ and $\rho = \rho_{\mathrm{r}}$ if $F = \mathbb{Q}$. Then
$$M = \bigoplus_{\mathcal{U} \in \mathrm{Irr}_F(G)} n_{\mathcal{U}} M_{\mathcal{U}}$$
where $M_{\mathcal{U}}$ is a simple $F[G]$-module affording the representation $\mathcal{U} \in \mathrm{Irr}_F(G)$.

According to Section 2.2 and Remark 5.1, $H^1(\mathcal{A}_H, F) \simeq p_H(M)$ is an $\mathcal{H}_{H,F}$-module affording the representation $\widetilde{\rho}$ of $\mathcal{H}_{H,F}$. In this way, we obtain the decomposition
$$p_H(M) = \bigoplus_{\mathcal{U} \in \mathrm{Irr}_F(G)} n_{\mathcal{U}} p_H(M_{\mathcal{U}}) = \bigoplus_{\widetilde{\mathcal{U}} \in \mathrm{Irr}_F(\mathcal{H}_{H,F})} n_{\mathcal{U}} M_{\widetilde{\mathcal{U}}}$$
where $M_{\widetilde{\mathcal{U}}} := p_H(M_{\mathcal{U}})$ is a simple $\mathcal{H}_{H,F}$-module affording the representation $\widetilde{\mathcal{U}}$ associated to $\mathcal{U}$, and the result is proved. $\square$

Remark 5.3. Let G be a group acting on an abelian variety $\mathcal{A}$. Suppose that
$$\rho_{\mathrm{a}} = \sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} n_{\mathcal{V}} \, \mathcal{V} \, .$$

Since the regular representation of G, induced from the trivial representation of the trivial subgroup $\{1_G\}$, decomposes as

$$\Upsilon_{\{1_G\}} = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V})\, \mathcal{V},$$

it follows that the dimension of $\mathcal{A}$ is given by

$$\dim_{\mathbb{C}}(\mathcal{A}) = \rho_{\mathrm{a}}(1_G) = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} n_{\mathcal{V}} \dim_{\mathbb{C}}(\mathcal{V}) = \left\langle \rho_{\mathrm{a}}, \Upsilon_{\{1_G\}} \right\rangle_G . \tag{5.2}$$

A similar argument is used to compute the dimension of the canonical subvariety $\mathcal{A}_H$ fixed by a subgroup H of G, as follows.

Corollary 5.4. *Let G be a group acting on an abelian variety $\mathcal{A}$ with analytic representation ρ_a, and let H a subgroup of G.*

Then the dimension of $\mathcal{A}_H$ is given by

$$\dim_{\mathbb{C}}(\mathcal{A}_H) = \langle \rho_a, \Upsilon_H \rangle_G \ , \tag{5.3}$$

where Υ_H denotes the representation of G induced by the trivial representation of H.

Proof. Recall from (2.3) that

$$\Upsilon_H = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V}^H)\, \mathcal{V} = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\widetilde{\mathcal{V}})\, \mathcal{V}\, .$$

Then

$$\dim_{\mathbb{C}}(\mathcal{A}_H) = \widetilde{\rho}_{\mathrm{a}}(p_H) = \sum_{\widetilde{\mathcal{V}}\in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} n_{\mathcal{V}} \dim_{\mathbb{C}}(\widetilde{\mathcal{V}}) = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} n_{\mathcal{V}} \dim_{\mathbb{C}}(\mathcal{V}^H) = \langle \rho_{\mathrm{a}}, \Upsilon_H \rangle_G \ .$$

□

Note that this dimension depends on the geometry of the action of G on $\mathcal{A}$, and not only on the abstract group G nor its subgroup H, as seen from the fact that it is computed from the analytic representation ρ_{a} of G.

In a similar vein, we can compute the dimension of the B_j appearing in the isotypical decompositions (4.1) and (4.4), as follows.

Corollary 5.5. *Let G be a group acting on an abelian variety $\mathcal{A}$, and let H denote a subgroup of G. Assume that the decomposition of $\rho_r \otimes_{\mathbb{Q}} \mathbb{C}$ as a sum of complex irreducible representations of G is given by*

$$\rho_r \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} m_{\mathcal{V}}\, \mathcal{V}.$$

For any rational irreducible representation $\widetilde{\mathcal{W}}$ of $\mathcal{H}_{H,\mathbb{Q}}$ consider its corresponding isotypical factor $B_{H,\widetilde{\mathcal{W}}}$ in the decomposition (4.4).

Then

$$2 \dim_{\mathbb{C}}(\mathcal{B}_{H,\widetilde{\mathcal{W}}}) = m_{\mathcal{V}}\, s_{\mathcal{V}}\, |K_{\mathcal{V}} : \mathbb{Q}| \, ,$$

where $s_{\mathcal{V}}$ is the Schur index of $\mathcal{V}$, $\mathcal{V}$ is a complex irreducible representation of G Galois-associated to $\mathcal{W}$, and $\mathcal{W}$ is the rational irreducible representation of G associated to $\widetilde{\mathcal{W}}$.

Proof. With the given notation, it follows from Proposition 4.1 that $\dim_{\mathbb{C}}(\mathcal{B}_{H,\widetilde{\mathcal{W}}}) = \dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{W}})$ with $\mathcal{B}_{\mathcal{W}}$ as in (4.1).

It follows from

$$\mathcal{W} \otimes \mathbb{C} \cong s_{\mathcal{V}} \bigoplus_{\sigma \in \mathrm{Gal}(K_{\mathcal{V}}/\mathbb{Q})} V^{\sigma}$$

that

$$\rho_{\mathrm{r}} \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\mathcal{U} \in \mathrm{Irr}_{\mathbb{C}}(G)} m_{\mathcal{U}} \mathcal{U} = \sum_{\mathcal{W} \in \mathrm{Irr}_{\mathbb{Q}}(G)} \frac{m_{\mathcal{V}}}{s_{\mathcal{V}}} \mathcal{W}\,.$$

Since $\rho_{\mathrm{r}}(G)$ acts on $B_{\mathcal{W}}^{\frac{\dim_{\mathbb{C}} \mathcal{V}}{s_{\mathcal{V}}}}$ via the representation $\mathcal{W}$, we obtain the equality

$$2\,\frac{\dim_{\mathbb{C}} \mathcal{V}}{s_{\mathcal{V}}} \dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{W}}) = \frac{m_{\mathcal{V}}}{s_{\mathcal{V}}} \dim_{\mathbb{C}}(\mathcal{W} \otimes \mathbb{C}) = \frac{m_{\mathcal{V}}}{s_{\mathcal{V}}}\, s_{\mathcal{V}}\, |K_{\mathcal{V}} : \mathbb{Q}| \dim_{\mathbb{C}}(\mathcal{V})\,,$$

and the result follows. □

6. Complementary Prym variety

The simplest case of a group action on a curve was studied in [W] and [M], namely the group with two elements $G = \langle j : j^2 \rangle$ acting on a curve Z. Then the Jacobian variety JZ of Z has an involution j acting on it and, furthermore, according to the isotypical decomposition of JZ, we have

$$JZ \sim \mathcal{B}_1 \times \mathcal{B}_2,$$

where $\mathcal{B}_1 = \mathrm{Im}(1 + j)$ and $\mathcal{B}_2 = \mathrm{Im}(1 - j)$. In this case G acts trivially on $\mathcal{B}_1 \sim JZ_G$. This decomposition was already observed and used by Schottky and Jung in [S-J]. Later, $\mathcal{B}_2$ was called by Mumford the Prym variety $P(Z/Z_G)$ of the given cover $Z \to Z_G$, where Z_G denotes the quotient of Z by G.

In this section we extend the definition of Prym variety to abelian varieties with group action.

Remark 6.1. Let G be a finite group acting on an abelian variety $\mathcal{A}$. Given two subgroups H and N of G with $H \leq N \leq G$, consider the canonical subvarieties $\mathcal{A}_H$ and $\mathcal{A}_N$ of $\mathcal{A}$ fixed by H and N respectively. Since

$$p_N = p_H p_N = p_N p_H$$

we have that

$$\mathcal{A}_N \subseteq \mathcal{A}_H \quad \text{and} \quad p_N \in \mathcal{H}_{N,\mathbb{Q}} \subseteq \mathcal{H}_{H,\mathbb{Q}}\ \ , \text{ with } \quad \psi_N : \mathcal{H}_{N,\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_N)$$

The next result shows the existence of a natural complement of $\mathcal{A}_N$ inside of $\mathcal{A}_H$, and describes a natural subalgebra of $\mathcal{H}_{H,\mathbb{Q}}$ acting on it.

Theorem 6.2. *Let G be a finite group acting on an abelian variety $\mathcal{A}$ and consider two subgroups H and N of G with $H \leq N \leq G$.*

For the idempotent $q = p_H - p_N \in \mathcal{H}_{H,\mathbb{Q}} \subseteq \mathbb{Q}[G]$, set

$$P\left(\mathcal{A}_H/\mathcal{A}_N\right) = \mathrm{Im}(q) \subseteq \mathcal{A}_H.$$

Then

(1) *There is an isogeny*

$$\mathcal{A}_N \times P\left(\mathcal{A}_H/\mathcal{A}_N\right) \to \mathcal{A}_H.$$

(2) *The homomorphism $\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$ restricts to an algebra homomorphism*

$$\psi_P : q\,\mathcal{H}_{H,\mathbb{Q}}\,q \to \mathrm{End}_{\mathbb{Q}}(P\left(\mathcal{A}_H/\mathcal{A}_N\right)).$$

(3) *Consider the decomposition of $\mathcal{A}_H$ provided by Theorem 4.2*

$$\mathcal{A}_H \sim \mathcal{B}_{H,\mathcal{W}_1}^{a_1} \times \mathcal{B}_{H,\mathcal{W}_2}^{a_2} \times \ldots \times \mathcal{B}_{H,\mathcal{W}_t}^{a_t}.$$

Then the following decomposition of $P\left(\mathcal{A}_H/\mathcal{A}_N\right)$ holds:

$$P\left(\mathcal{A}_H/\mathcal{A}_N\right) \sim \mathcal{B}_{H,\mathcal{W}_1}^{c_1} \times \mathcal{B}_{H,\mathcal{W}_2}^{c_2} \times \ldots \times \mathcal{B}_{H,\mathcal{W}_t}^{c_t},$$

with $0 \leq c_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^H)}{s_{\mathcal{V}_i}} - \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^N)}{s_{\mathcal{V}_i}}$, where $\mathcal{V}_i$ is a complex irreducible representation Galois-associated to $\mathcal{W}_i$. Furthermore $\psi_P(q\mathcal{H}_{H,\mathbb{Q}}q)$ acts on $\mathcal{B}_{H,\mathcal{W}_j}^{c_j}$ by $\widetilde{\mathcal{W}_j}$.

We call $P\left(\mathcal{A}_H/\mathcal{A}_N\right)$ the *complementary Prym* variety of $\mathcal{A}_N$ inside of $\mathcal{A}_H$ (see also [B-L], p. 125). This result generalizes Proposition 3.7 of [L-R] and Proposition 5.2 and Corollary 5.4 of [Ca-Ro], where a similar result was proved in the case $\mathcal{A} = JZ$.

Proof. (1) Since the unit p_H of $\mathcal{H}_{H,\mathbb{Q}}$ decomposes as a sum of idempotent elements in $\mathcal{H}_{H,\mathbb{Q}}$ as follows

$$p_H = p_N + (p_H - p_N) = p_N + q,$$

the addition map induces the required isogeny.

(2) The assertion follows from the fact that q is an idempotent in $\mathcal{H}_{H,\mathbb{Q}} \subseteq \mathbb{Q}[G]$.

(3) The last assertion is an immediate consequence of the following facts

$$\Upsilon_H = a_1\mathcal{W}_1 \oplus a_2\mathcal{W}_2 \oplus \ldots \oplus a_t\mathcal{W}_t, \quad \text{and}$$
$$\Upsilon_N = b_1\mathcal{W}_1 \oplus b_2\mathcal{W}_2 \oplus \ldots \oplus b_t\mathcal{W}_t,$$

with $a_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^H)}{s_{\mathcal{V}_i}} \geq b_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^N)}{s_{\mathcal{V}_i}}$. □

We mention the following interesting particular cases of complementary Prym varieties.

Remark 6.3. If $H = \{1\}$ and $\{1\} \neq N$ is any subgroup of G, then

$$\mathcal{A} \sim \mathcal{A}_N \times P\left(\mathcal{A}/\mathcal{A}_N\right);$$

in this case, we see that $\mathcal{A}_N$, the canonical subvariety of $\mathcal{A}$ fixed by N, has a complement given by a projector in $\mathbb{Q}[G]$, namely $P(\mathcal{A}/\mathcal{A}_N) = \mathrm{Im}(1_G - p_N)$. Furthermore, if $N = G$, then

$$\mathcal{A} \sim \mathcal{A}_G \times P(\mathcal{A}/\mathcal{A}_G),$$

and G acts trivially on $\mathcal{A}_G$ and non trivially on $P(\mathcal{A}/\mathcal{A}_G)$, as in the case of classical Prym varieties.

7. Action on Curves and Jacobians

In this section we fix a smooth projective curve Z defined over the field of the complex numbers such that the group G acts on Z. Then G acts on $H^1(JZ, \mathbb{C})$, where JZ is the Jacobian variety of Z, this is ρ_{a}, the analytic representation of G. The decomposition of ρ_{a} into complex irreducible representations of G is provided by the well known Chevalley-Weil formula, that computes the multiplicity of any given complex irreducible representation of G in ρ_{a}.

There is also a natural action of G on $H^1(JZ, \mathbb{Q})$ gives ρ_{r}, the rational representation of G. The Lefschetz fixed point formula and the Eichler trace formula have been used by many authors to compute the character of ρ_{r}, as well as the decomposition of this representation as a sum of rational representations related to the geometry of the action, see for instance [E, M, B].

Using the fact that the subvariety $p_H(JZ) = (JZ)_H$ of JZ is isogenous to the Jacobian variety JZ_H, in this section we study the action of the corresponding Hecke algebra on JZ_H by applying the results of Section 5, to the complex representation $\widetilde{\rho}_{\mathrm{a}}$ of the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ on $H^1(JZ_H, \mathbb{C})$ and the rational representation $\widetilde{\rho}_{\mathrm{r}}$ of the rational Hecke algebra $\mathcal{H}_{H,\mathbb{Q}}$ on $H^1(JZ_H, \mathbb{Q})$, where H is any subgroup of G and Z_H denotes the quotient of Z by H.

7.1. **On Jacobians of intermediate covers given by subgroups.** Consider a subgroup H of G. If we denote the quotients of Z by H and G by Z_H and Z_G respectively, we have the following diagram of covers of curves

(7.1)
$$\begin{tikzcd} Z \arrow[dd, "\pi"'] \arrow[dr, "\pi_H"] & \\ & Z_H \arrow[dl, "\varphi_H"] \\ Z_G & \end{tikzcd}$$

The group action of G on Z induces an algebra homomorphism

$$\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JZ).$$

As mentioned before, this homomorphism restricts to an algebra homomorphism

$$\psi_H : \mathcal{H}_{H,\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(p_H(JZ)). \tag{7.2}$$

Now the pull-back map $\pi_H^* : JZ_H \to p_H(JZ)$ and the restriction of the norm map $\mathrm{Nm}(\pi_H) : p_H(JZ) \to JZ_H$ are isogenies satisfying $\mathrm{Nm}(\pi_H) \circ \pi_H^* = |H| 1_{JZ_H}$. This implies that the composition

$$\varepsilon : \mathcal{H}_{H,\mathbb{Q}} \to \mathrm{End}_{\mathbb{Q}}(JZ_H), \quad \varphi \mapsto \frac{1}{|H|} \mathrm{Nm}(\pi_H) \circ \varphi \circ \pi_H^* \tag{7.3}$$

is a homomorphism of $\mathbb{Q}$-algebras (see [CLRR] and [E]). Moreover, Remark 5.1 implies that

$$\varepsilon(\mathcal{H}_{H,\mathbb{Q}}) \cong \psi_H(\mathcal{H}_{H,\mathbb{Q}}).$$

To end this subsection, we recall the following result on intermediate covers given by a subgroup, see [R].

Remark 7.1. If $f : Z \to Z_G$ is a Galois cover of curves, with Galois group G and monodromy $(g_1, \dots, g_s)$, then for any subgroup H of G, the genus of the quotient Z_H is given by

$$g(Z_H) = [G : H](g(Z_G) - 1) + 1 + \frac{1}{2} \sum_{j=1}^{s} \left([G : H] - |H \backslash G / \langle g_j \rangle| \right) \ . \tag{7.4}$$

7.2. **The analytic representation.** Let g_j denote an element of G of order c_j that represents the local monodromy of f at the branch point Q_j. For a given $\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)$ and $\zeta_{c_j} = \exp(2\pi\imath/c_j)$ a primitive c_j-th root of unity, let N_{jk} denote the number of eigenvalues of $\mathcal{V}(g_j)$ that are equal to $\zeta_{c_j}^k$. We write $\langle s \rangle = s - \lfloor s \rfloor$ for the fractional part of a rational number s. Then the multiplicity $n_{\mathcal{V}}$ of the given complex irreducible representation $\mathcal{V}$ in the analytic representation ρ_{a} of G is provided by the Chevalley-Weil formula, see [C-W].

Theorem 7.2. *(Chevalley-Weil) Let $f : Z \to Z_G$ be a Galois covering of curves, with Galois group G, and monodromy $(g_1, \dots, g_s)$. Let the symbols c_j and N_{jk} be defined as above.*

Then the multiplicity $n_{\mathcal{V}}$ of a given complex irreducible representation $\mathcal{V}$ of G in the analytic representation ρ_a of G on $H^0(Z, \Omega_Z^1)$ is given by

$$n_{\mathcal{V}} = \dim_{\mathbb{C}}(\mathcal{V})(g(Z_G) - 1) + \sum_{j=1}^{s} \sum_{k=0}^{c_j - 1} N_{jk} \left\langle \frac{-k}{c_j} \right\rangle + \eta \,, \tag{7.5}$$

where η is equal to 1 if $\mathcal{V}$ is the trivial representation, and equal to 0 otherwise.

Applying Proposition 5.2, we can describe the corresponding result for the complex representation $\widetilde{\rho}_{\mathrm{a}}$ of the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ on $H^1(JZ_H, \mathbb{C})$.

Corollary 7.3. *Assume the hypotheses of Theorem 7.2, and let $H \leq G$. Then the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ acts on $H^1(JZ_H, \mathbb{C})$ with analytic representation*

$$\widetilde{\rho}_a = \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} n_{\mathcal{V}} \, \widetilde{\mathcal{V}} \,,$$

where $n_{\mathcal{V}}$ is given by (7.5) for $\mathcal{V}$ the complex irreducible representation associated to $\widetilde{\mathcal{V}}$.

7.3. **The rational representation.** In this section we study the rational representation of G given by the action of G on $H^1(JZ,\mathbb{Q})$ and the rational representation of $\mathcal{H}_{H,\mathbb{Q}}$ given by the action of $\mathcal{H}_{H,\mathbb{Q}}$ on $H^1(JZ_H,\mathbb{Q})$.

The Lefschetz fixed point formula and the Eichler trace formula are usually used to prove the following result (see for instance [B], [E] and [M]).

Theorem 7.4. *Let $f: Z \to Z_G$ be a Galois covering of curves, with Galois group G, and monodromy $(g_1,\dots,g_s)$. Then*

$$\rho_r = 2\mathcal{V}_0 + (2g(Z_G)-2+s)\Upsilon_{\{1_G\}} - \sum_{j=1}^{s}\Upsilon_{\langle g_j\rangle}, \tag{7.6}$$

where $\mathcal{V}_0$ denotes the trivial representation of G.

Furthermore, the multiplicity $m_{\mathcal{V}}$ of a given complex irreducible representation $\mathcal{V}$ of G in $\rho_r \otimes_{\mathbb{Q}} \mathbb{C}$ is given by

$$m_{\mathcal{V}} = (2g(Z_G)-2+s)\dim_{\mathbb{C}}(\mathcal{V}) - \sum_{j=1}^{s}\dim_{\mathbb{C}}(\mathcal{V}^{\langle g_j\rangle}) + 2\eta\,, \tag{7.7}$$

where η is equal to 1 if $\mathcal{V}$ is the trivial representation, and equal to 0 otherwise.

Remark 7.5. The theorem says that the decomposition of $\rho_{\mathrm{r}} \otimes_{\mathbb{Q}} \mathbb{C}$ as sum of complex irreducible representations of G is given by

$$\rho_{\mathrm{r}} \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} m_{\mathcal{V}}\,\mathcal{V}\,,$$

with $m_{\mathcal{V}}$ given by (7.7). In particular, we obtain

$$\begin{aligned} 2g(Z) = \rho_{\mathrm{r}}(1_G) &= \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} m_{\mathcal{V}}\ \dim_{\mathbb{C}}(\mathcal{V}) \\ &= 2 + \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} \left[(2g(Z_G)-2+s)\dim_{\mathbb{C}}(\mathcal{V}) - \sum_{j=1}^{s}\dim_{\mathbb{C}}(\mathcal{V}^{\langle g_j\rangle})\right]\ \dim_{\mathbb{C}}(\mathcal{V}) \\ &= 2 + 2(g(Z_G)-1)|G| + \sum_{j=1}^{s}(|G| - [G:\langle g_j\rangle])\,, \end{aligned} \tag{7.8}$$

and recover the Riemann-Hurwitz formula, where the last equality holds because

$$\sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} (\dim_{\mathbb{C}}(\mathcal{V}))^2 = |G|\ ,\ \text{and}\qquad \sum_{\mathcal{V}\in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V}^{\langle g_j\rangle})\ \dim_{\mathbb{C}}(\mathcal{V}) = [G:\langle g_j\rangle]\,.$$

Applying Proposition 5.2, we obtain the multiplicity of any complex irreducible representation of the complex Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ in the decomposition of the representation obtained by the action of $\mathcal{H}_{H,\mathbb{Q}}$ on $H^1(JZ_H,\mathbb{Q})$.

Corollary 7.6. *Assume the hypotheses of Theorem 7.4, and let $H \leq G$. Then the Hecke algebra $\mathcal{H}_{H,\mathbb{Q}}$ acts on $H^1(JZ_H, \mathbb{Q})$ with a rational representation $\widetilde{\rho}_r$ and $\widetilde{\rho}_r \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as sum of complex irreducible representations of $\mathcal{H}_{H,\mathbb{C}}$ as follows*

$$\widetilde{\rho}_r \otimes_{\mathbb{Q}} \mathbb{C} = \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} m_{\mathcal{V}}\, \widetilde{\mathcal{V}}\,,$$

where $m_{\mathcal{V}}$ is given by (7.7) for $\mathcal{V}$ the complex irreducible representation of G associated to $\widetilde{\mathcal{V}}$.

Remark 7.7. Analogously to (7.8), we obtain (compare with (7.4))

$$\begin{aligned} 2g(Z_H) = \widetilde{\rho}_{\mathrm{r}}(p_H) &= \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} m_{\mathcal{V}} \dim_{\mathbb{C}}(\widetilde{\mathcal{V}}) \\ &= 2 + \sum_{\widetilde{\mathcal{V}} \in \mathrm{Irr}_{\mathbb{C}}(\mathcal{H}_{H,\mathbb{C}})} \left[(2g(Z_G) - 2 + s) \dim_{\mathbb{C}}(\mathcal{V}) - \sum_{j=1}^{s} \dim_{\mathbb{C}}(\mathcal{V}^{\langle g_j \rangle}) \right] \dim_{\mathbb{C}}(\mathcal{V}^H) \\ &= 2 + 2[G:H](g(Z_G) - 1) + \sum_{j=1}^{s} \left([G:H] - [H\backslash G/\langle g_j \rangle] \right) , \end{aligned}$$

where the last equality holds because

$$\sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V}) \dim_{\mathbb{C}}(\mathcal{V}^H) = [G:H],$$

and

$$\sum_{\mathcal{V} \in \mathrm{Irr}_{\mathbb{C}}(G)} \dim_{\mathbb{C}}(\mathcal{V}^{\langle g_j \rangle}) \dim_{\mathbb{C}}(\mathcal{V}^H) = [H\backslash G/\langle g_j \rangle].$$

8. Isotypical decomposition of intermediate Jacobians and Pryms

In this Section we extend some results on the isotypical decomposition of Jacobian and Prym varieties of intermediate covers given by subgroups, in the case of curves with automorphisms, obtained in [Ca-Ro] and [R].

The (generalized) Prym variety $P(X/Y)$ of any cover of curves $f: X \to Y$ is the orthogonal complement of $f^*(JY)$ in JX with respect to the canonical polarization of JX. In this way we have an isogeny

$$JX \sim JY \times P(X/Y)$$

Assume Z is the Galois cover of f, with Galois group G and let H be a subgroup of G such that $X = Z_H$. Then

$$JZ_H \sim JZ_G \times P(Z_H/Z_G)$$

In general, if we consider two subgroups H and N of G with $H \leq N \leq G$, then we have the following diagram of curves and covers, together with the corresponding diagrams of Jacobians and homomorphisms.

$$
\begin{tikzcd}
Z \arrow[dr] \arrow[ddd] & & \\
& Z_H = X \arrow[dr, "F"] & \\
& & Z_N \arrow[dll] \\
Z_G = Y & &
\end{tikzcd}
\qquad
\begin{tikzcd}
JZ & & \\
& JZ_H \arrow[ul] \arrow[dr, bend left, "\mathrm{Nm}\, F"] & \\
& & JZ_N \arrow[ul, bend left, "F^*"] \\
JZ_G \arrow[uuu] \arrow[urr] & &
\end{tikzcd}
$$

We have already seen that then the Hecke algebras $\mathcal{H}_{H,\mathbb{Q}}$ and $\mathcal{H}_{N,\mathbb{Q}}$ act on JZ_H and JZ_N respectively; observe that $\mathcal{H}_{N,\mathbb{Q}}$, as subalgebra of $\mathcal{H}_{H,\mathbb{Q}}$, acts on $F^*(JZ_N) \subset JZ_H$. The (generalized) Prym variety $P(Z_H/Z_N)$ of the cover $F : Z_H \to Z_N$ is the orthogonal complement of $F^*(JZ_N)$ in JZ_H with respect to the canonical polarization of JZ_H. In this way we have an isogeny

$$JZ_H \sim JZ_N \times P(Z_H/Z_N).$$

Remark 8.1. Let $f : Z \to Z_G$ be a Galois cover of curves, with Galois group G, and monodromy $(g_1, \dots, g_s)$. Consider two subgroups H and N of G with $H \leq N \leq G$.

Then the following are known results, (for instance see [L-R], [Ca-Ro], [R]).

(1) The isotypical decomposition of JZ is given by

$$JZ \sim JZ_G \times \mathcal{B}_{\mathcal{W}_2}^{\frac{\dim_{\mathbb{C}}(\mathcal{V}_2)}{s_{\mathcal{V}_2}}} \times \dots \times \mathcal{B}_{\mathcal{W}_r}^{\frac{\dim_{\mathbb{C}}(\mathcal{V}_r)}{s_{\mathcal{V}_r}}},$$

since $\mathcal{B}_{\mathcal{W}_1} \sim JZ_G$ for $\mathcal{W}_1$ the trivial representation of G.

(2) The isotypical decomposition of JZ_H is given by

$$JZ_H \sim JZ_G \times \mathcal{B}_{\mathcal{W}_2}^{\frac{\dim_{\mathbb{C}}(\mathcal{V}_2^H)}{s_{\mathcal{V}_2}}} \times \dots \times \mathcal{B}_{\mathcal{W}_r}^{\frac{\dim_{\mathbb{C}}(\mathcal{V}_r^H)}{s_{\mathcal{V}_r}}}.$$

(3) The isotypical decomposition of $P(Z_H/Z_N)$ is given as follows.

$$P(Z_H/Z_N) \sim \mathcal{B}_{\mathcal{W}_2}^{c_2} \times \dots \times \mathcal{B}_{\mathcal{W}_r}^{c_r}$$

(4) For all $i > 1$ we have

$$\dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{W}_i}) = s_{\mathcal{V}_i}|K_{\mathcal{V}_i} : \mathbb{Q}| \left(\dim_{\mathbb{C}}(\mathcal{V}_i)(g(Z_G) - 1) + \frac{1}{2} \left(\sum_{j=1}^{s} \dim_{\mathbb{C}}(\mathcal{V}_i) - \dim_{\mathbb{C}}(\mathcal{V}_i^{\langle g_j \rangle}) \right) \right)$$

where $\mathcal{V}_i$ is a complex irreducible representation Galois-associated to $\mathcal{W}_i$, $s_{\mathcal{V}_i}$ is the Schur index and $K_{\mathcal{V}_i}$ is the character field of $\mathcal{V}_i$, and $c_i = \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^H)}{s_{\mathcal{V}_i}} - \frac{\dim_{\mathbb{C}}(\mathcal{V}_i^N)}{s_{\mathcal{V}_i}}$. These facts were generalized in the previous sections to the case of the abelian varieties with group action.

Remark 8.2. Let $f : Z \to Z_G$ be a Galois cover of curves, with Galois group G, and monodromy $(g_1, \dots, g_s)$. Let $\mathcal{W}$ be a rational irreducible representation of G and $\mathcal{V}$ a complex irreducible representation of G Galois-associated to $\mathcal{W}$. Theorem 7.4 and Remark 8.1 item (4) allow us to determine $m_{\mathcal{V}}$, the multiplicity of $\mathcal{V}$ in $\rho_{\mathrm{r}} \otimes_{\mathbb{Q}} \mathbb{C}$, and $\dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{W}})$, the dimension of the subvariety $B_{\mathcal{W}}$ associated to $\mathcal{W}$ in the isotypical decomposition of JZ. Applying Corollary 5.5, we can verify that

$$2\dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{W}}) = s_{\mathcal{V}}\, m_{\mathcal{V}}\, |K_{\mathcal{V}} : \mathbb{Q}|.$$

Our next result provides a relation between the complementary Prym variety defined in Section 5 and the generalized Prym variety of a cover of curves.

Theorem 8.3. *Let $f : Z \to Z_G$ be a Galois cover of curves, with Galois group G and $\psi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JZ)$ the induced algebra homomorphism. Consider two subgroups H and N of G with $H \leq N \leq G$, and set $q = p_H - p_N$. Then*

$$P(Z_H/Z_N) \sim P\left((JZ)_H/(JZ)_N\right) = \mathrm{Im}(q);$$

that is, the generalized Prym variety $P(Z_H/Z_N)$ is isogenous to the complementary Prym of $(JZ)_N$ inside of $(JZ)_H$, the abelian subvarieties of JZ fixed by N and H respectively.

In particular, according to Theorem 6.2, we have an algebra homomorphism

$$\psi_P : q\,\mathbb{Q}[G]\,q \to \mathrm{End}_{\mathbb{Q}}(P\left(Z_H/Z_N\right)).$$

Proof. This is an immediate consequence of Theorem 6.2 and item (3) of Remark 8.1. $\square$

9. Examples

Example 9.1. Let $G = \langle a, b \ / \ a^3 = b^2 = abab = 1\rangle$ be the symmetric group of degree three, and consider the integral representation ϕ of G given by

$$\phi(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad ; \quad \phi(b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the symplectic representation θ of G, given by

$$\theta : G \to \mathrm{Sp}(8, \mathbb{Z}) \ , \quad \theta(g) = \begin{pmatrix} \phi(g) & 0 \\ 0 & {}^t\phi(g)^{-1} \end{pmatrix}$$

The existence of Riemann matrices fixed under the action of $\theta(G)$ was shown in [CGR]. In this way we obtain a four dimensional family $\mathcal{F}$ of principally polarized abelian varieties of dimension four admitting the given $G-$action. Let $\mathcal{A} \in \mathcal{F}$. We have that

$$\rho_a = \phi = 2\mathcal{V}_0 + \mathcal{V}_2 \quad \text{and} \quad \rho_r = 2\phi = 4\mathcal{V}_0 + 2\mathcal{V}_2,$$

where $\mathcal{V}_0$ is the trivial representation and $\mathcal{V}_2$ is the irreducible representation of degree two of G.

Applying Corollary 5.5, we have that

$$\dim_{\mathbb{C}}(\mathcal{B}_0) = 2 \quad ; \quad \dim_{\mathbb{C}}(\mathcal{B}_2) = 1, \quad \text{and} \quad \mathcal{A} \sim \mathcal{B}_0 \times \mathcal{B}_2^2.$$

Since $\dim_{\mathbb{C}}(\mathcal{B}_0) = 2$, it follows from the Riemann-Hurwitz equation that $\mathcal{A}$ is not a Jacobian variety.

Consider the subgroup $H = \langle b \rangle$ of G. Then $\Upsilon_H = \mathcal{V}_0 + \mathcal{V}_2$ and $\mathcal{A}_H \sim \mathcal{A}_G \times P(\mathcal{A}_H/\mathcal{A}_G)$ where $\mathcal{A}_G = \mathcal{B}_0$ and $P(\mathcal{A}_H/\mathcal{A}_G) \sim \mathcal{B}_2$.

Example 9.2. Consider the group

$$G_m = \langle a, b \mid a^{2^m} = b^2 = 1, bab = a^d \rangle$$

where $m \geq 3$ and $d = 2^{m-1} - 1$ (note that $d^2 \equiv 1 \bmod 2^m$). As a semidirect product of the subgroup $\langle a \rangle$ by the subgroup $\langle b \rangle$, G_m is a group of order 2^{m+1}.

Consider the complex irreducible representations of G defined for all odd numbers k such that $1 \leq k \leq 2^m - 1$ by

$$\mathcal{V}_k(a) = \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^{kd} \end{pmatrix}, \qquad \mathcal{V}_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ξ is a primitive 2^m-th-root of unity.

Then

$$\langle \Upsilon_H, \mathcal{V}_k \rangle_G = 1$$

for the subgroup $H = \langle b \rangle \leq G_m$ and all k as above.

Consider $Z \to \mathbb{P}^1$ a Galois covering with Galois group G_m ramified over 3 points of $\mathbb{P}^1$, with monodromy $g_1 = a$, $g_2 = b$ and $g_3 = (ab)^{-1}$. These curves were studied in [CLR].

Applying the Chevalley-Weil formula, the multiplicity of $\mathcal{V}_k$ in the analytic representation ρ_a on $H^1(JZ, \mathbb{C})$ is $n_{\mathcal{V}_k} = 1$ for all odd numbers k such that $1 \leq k \leq 2^{m-2} - 1$. Hence

$$\rho_a = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{m-2}-1} \mathcal{V}_k \ , \quad \text{and} \quad g(Z) = \rho_a(1_G) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{m-2}-1} \dim_{\mathbb{C}}(\mathcal{V}_k) = 2\phi(2^{m-2}) = 2^{m-2}.$$

Therefore the analytic representation $\widetilde{\rho}_{\rm a}$ of the Hecke algebra $\mathcal{H}_{H,\mathbb{C}}$ on $H^1(JZ_H, \mathbb{C})$ decomposes as follows

$$\widetilde{\rho}_{\rm a} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{m-2}-1} \widetilde{\mathcal{V}}_k \ , \quad \text{and} \quad g(Z_H) = \widetilde{\rho}_{\rm a}(p_H) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{m-2}-1} \dim_{\mathbb{C}}(\widetilde{\mathcal{V}}_k) = \phi(2^{m-2}) = 2^{m-3},$$

where $\widetilde{\mathcal{V}}_k$ is the associated representation of $\mathcal{V}_k$, and $\dim_{\mathbb{C}}(\widetilde{\mathcal{V}}_k) = 1$.

Since

$$\langle \Upsilon_{\langle a \rangle}, \mathcal{V}_k \rangle_G = 0 \quad \text{and} \quad \langle \Upsilon_{\langle ab \rangle}, \mathcal{V}_k \rangle_G = 0$$

for all odd numbers k such that $1 \leq k \leq 2^m - 1$, the multiplicity m_k of $\mathcal{V}_k$ in $\rho_{\rm r} \otimes_{\mathbb{Q}} \mathbb{C}$ (and of $\widetilde{\mathcal{V}}_k$ in $\widetilde{\rho}_{\rm r} \otimes_{\mathbb{Q}} \mathbb{C}$) is given by

$$m_{\mathcal{V}_k} = (-2+3) \dim_{\mathbb{C}}(V_k) - \sum_{j=1}^{3} \dim_{\mathbb{C}}(\mathcal{V}_k^{\langle g_j \rangle}) = 1\,.$$

Let $\mathcal{W}$ be the rational irreducible representation of G obtained as

$$\mathcal{W} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^m-1} \mathcal{V}_k.$$

Then the action of $\mathcal{H}_{H,\mathbb{Q}}$ on $JZ_H \sim \mathcal{B}_{\mathcal{W}}$ is given by $\widetilde{\mathcal{W}}$, and the action of G on $JZ \sim \mathcal{B}^2_{\mathcal{W}}$ is given by $\mathcal{W}$.

Example 9.3. Let p and q be odd prime numbers such that $p \,/\, q-1$ but $p^2 \nmid q-1$.

Consider the group G of order qp^2 given by

$$G = \langle a,\ b \ / \ a^q = b^{p^2} = 1,\ b^{-1}ab = a^k \rangle$$

where $1 < k < q$ and $k^p \equiv 1 \mod q$. The group G has five rational irreducible representations. Among them we consider $\mathcal{W}_4$ and $\mathcal{W}_5$ such that

$\dim_{\mathbb{Q}}(\mathcal{W}_4) = q-1$	$\dim_{\mathbb{C}} \mathcal{V}_4 = p$	$\lvert K_{\mathcal{V}_4} : \mathbb{Q}\rvert = \dfrac{(q-1)}{p}$	$s_{\mathcal{V}_4} = 1$
$\dim_{\mathbb{Q}}(\mathcal{W}_5) = p(p-1)(q-1)$	$\dim_{\mathbb{C}} \mathcal{V}_5 = p$	$\lvert K_{\mathcal{V}_5} : \mathbb{Q}\rvert = \dfrac{(q-1)(p-1)}{p}$	$s_{\mathcal{V}_5} = p$

,

where $\mathcal{V}_i$ is a complex irreducible representation Galois-associated to $\mathcal{W}_i$.

Consider $Z \to \mathbb{P}^1$ a Galois covering with Galois group G ramified over 3 points of $\mathbb{P}^1$ with monodromy $g_1 = a$, $g_2 = b$ and $g_3 = (ab)^{-1}$. Let $H = \{1_G\}$ and $N = \langle\, b^p \,\rangle$. Then, according to (7.4) we have

$g(Z) = 1 + \dfrac{1}{2}qp^2 - \dfrac{1}{2}p^2 - q$	$g(Z_N) = \dfrac{(p-2)(q-1)}{2}$

,

by Remark 8.1 we have

$JZ \sim \mathcal{B}_4^p \times \mathcal{B}_5$	$JZ_N \sim \mathcal{B}_4^p$	$JZ \sim JZ_N \times P(Z/Z_N)$	$P(Z/Z_N) \sim \mathcal{B}_5$

,

and by Corollary 5.5 we have

$$\dim_{\mathbb{C}}(P(Z/Z_N)) = \frac{p(p-1)(q-1)}{2}.$$

Finally, applying Theorem 6.2 we obtain that

$$\mathbb{M}_u(\mathbb{Q}) \cong \psi_P\left((1_G - p_N)\mathbb{Q}[G](1_G - p_N)\right) \subseteq \mathrm{End}_{\mathbb{Q}}(P\,(Z/Z_N)), \ \text{ with } u = p(p-1)(q-1).$$

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