

# A SIMPLY CONNECTED NUMERICAL CAMPEDELLI SURFACE WITH AN INVOLUTION

HEESANG PARK, DONGSOO SHIN, AND GIANCARLO URZÚA

**ABSTRACT.** We construct a simply connected minimal complex surface of general type with  $p_g = 0$  and  $K^2 = 2$  which has an involution such that the minimal resolution of the quotient by the involution is a simply connected minimal complex surface of general type with  $p_g = 0$  and  $K^2 = 1$ . In order to construct the example, we combine a double covering and  $\mathbb{Q}$ -Gorenstein deformation. Especially, we develop a method for proving unobstructedness for deformations of a singular surface by generalizing a result of Burns and Wahl which characterizes the space of first order deformations of a singular surface with only rational double points. We describe the stable model in the sense of Kollár and Shepherd-Barron of the singular surfaces used for constructing the example. We count the dimension of the invariant part of the deformation space of the example under the induced  $\mathbb{Z}/2\mathbb{Z}$ -action.

## 1. INTRODUCTION

One of the fundamental problems in the classification of complex surfaces is to find a new family of complex surfaces of general type with  $p_g = 0$ . In this paper we construct new simply connected *numerical Campedelli surfaces* with an involution, i.e. simply connected minimal complex surfaces of general type with  $p_g = 0$  and  $K^2 = 2$ , that have an automorphism of order 2.

There has been a growing interest for complex surfaces of general type with  $p_g = 0$  having an involution; cf. J. Keum-Y. Lee [12], Calabri-Ciliberto-Mendes Lopes [4], Calabri-Mendes Lopes-Pardini [5], Y. Lee-Y. Shin [17], Rito [23]. A classification of *numerical Godeaux surfaces* (i.e. minimal complex surfaces of general type with  $p_g = 0$  and  $K^2 = 1$ ) with an involution is given in Calabri-Ciliberto-Mendes Lopes [4]. It is known that the quotient surface of a numerical Godeaux surface by its involution is either rational or birational to an Enriques surface, and the bicanonical map of the numerical Godeaux surface factors through the quotient map.

However, the situation is more involved in the case of numerical Campedelli surfaces, because the bicanonical map may not factor through the quotient map; cf. Calabri-Mendes Lopes-Pardini [5]. In particular it can happen that the quotient is of general type. More precisely, let  $X$  be a numerical Campedelli surface with an involution  $\sigma$ . If  $\sigma$  has only fixed points and no fixed divisors, then the minimal resolution  $S$  of the quotient  $Y = X/\sigma$  is a numerical Godeaux surface and  $\sigma$  has only four fixed points; cf. Barlow [2]. Conversely, if  $S$  is of general type, then  $\sigma$  has only four fixed points and no fixed divisors; Calabri-Mendes Lopes-Pardini [5].

There are some examples of numerical Campedelli surfaces  $X$  with an involution  $\sigma$  having only four fixed points. Barlow [1] constructed examples with  $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} \oplus$

---

*Date:* December 31, 2011.

*2000 Mathematics Subject Classification.* 14J29, 14J10, 14J17, 53D05.

*Key words and phrases.* surface of general type, involution,  $\mathbb{Q}$ -Gorenstein smoothing.

$\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}$ . Barlow [2] also constructed examples with  $\pi_1(X) = \mathbb{Z}/5\mathbb{Z}$  whose minimal resolution of the quotient by the involution is the first example of a *simply connected* numerical Godeaux surface. Also all Catanese's surfaces [6] have such an involution and  $\pi_1 = \mathbb{Z}/5\mathbb{Z}$ . Recently Calabri, Mendes Lopes, and Pardini [5] constructed a numerical Campedelli surface with torsion  $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  and two involutions.

It is known that the orders of the algebraic fundamental groups of numerical Campedelli surfaces are at most 9 and the dihedral groups  $D_3$  and  $D_4$  cannot be realized. Recently, the existence question for numerical Campedelli surfaces with  $|\pi_1^{\text{alg}}| \leq 9$  was settled by the construction of examples with  $\pi_1^{\text{alg}} = \mathbb{Z}/4\mathbb{Z}$ ; Frapporti [10] and H. Park-J. Park-D. Shin [22]. Hence it would be an interesting problem to construct numerical Campedelli surfaces having an involution with  $\pi_1^{\text{alg}} = G$  for each given group  $G$  with  $|G| \leq 9$ . Especially we are concerned with the simply connected case because the fundamental groups of all the known examples with an involution have large order:  $|G| \geq 5$ . Furthermore the first example of *simply connected* numerical Campedelli surfaces is very recent (Y. Lee-J. Park [15]), but we have no information about the existence of an involution in their example. The main theorem of this paper is:

**Theorem** (Corollary 3.6). *There are simply connected minimal complex surfaces  $X$  of general type with  $p_g(X) = 0$  and  $K_X^2 = 2$  which have an involution  $\sigma$  such that the minimal resolution  $S$  of the quotient  $Y = X/\sigma$  is a simply connected minimal complex surface of general type with  $p_g(S) = 0$  and  $K_S^2 = 1$ .*

We also show that the minimal resolution  $S$  of the quotient  $Y = X/\sigma$  has a local deformation space of dimension 4 corresponding to deformations  $\mathcal{S}$  of  $S$  such that its general fiber  $\mathcal{S}_t$  is the minimal resolution of a quotient  $X_t/\sigma_t$  of a numerical Campedelli surface  $X_t$  by an involution  $\sigma_t$ ; Theorem 5.2. In addition, we show that the resolution  $S$  should be always simply connected if the double cover  $X$  is already simply connected; Proposition 3.7. Conversely Barlow [2] showed that if the resolution  $S$  is a simply connected numerical Godeaux surface then the possible order of the algebraic fundamental group of the double cover  $X$  is 1, 3, 5, 7, or 9. As far as we know, the example in Barlow [2] was the only one whose quotient is simply connected. It has  $\pi_1(X) = \mathbb{Z}/5\mathbb{Z}$  as mentioned earlier. Here we find an example with  $\pi_1(X) = 1$ . Hence it would be an intriguing problem in this context to construct an example with  $\pi_1(X) = \mathbb{Z}/3\mathbb{Z}$ .

In order to construct the examples, we combine a double covering and a  $\mathbb{Q}$ -Gorenstein smoothing method developed in Y. Lee-J. Park [15]. First we build singular surfaces by blowing up points and then contracting curves over a specific rational elliptic surface. These singular surfaces differ by contracting certain  $(-2)$ -curves. If we contract all of the  $(-2)$ -curves, we obtain a stable surface  $Y'$  in the sense of Kollár–Shepherd-Barron [14], and we prove that the space of  $\mathbb{Q}$ -Gorenstein deformations of  $Y'$  is smooth and 8 dimensional; Proposition 2.2. A  $(\mathbb{Q}$ -Gorenstein) smoothing of  $Y'$  in this space produces simply connected numerical Godeaux surfaces. In particular, the smoothing of  $Y'$  gives the existence of a two dimensional family of simply connected numerical Godeaux surfaces with six  $(-2)$ -curves; Corollary 2.3. We also prove that a four dimensional family in this space produces simply connected numerical Godeaux surfaces with a 2-divisible divisor consisting of four disjoint  $(-2)$ -curves; Theorem 2.4 and Theorem 5.2. These numerical Godeaux surfaces are used to construct the numerical Campedelli surfaces with an involution. The desired numerical Campedelli surfaces are obtained by taking double coverings of the numerical Godeaux surfaces branched along the four disjoint  $(-2)$ -curves; Theorem 3.4.

On the other hand we can also obtain the Campedelli family explicitly from a singular stable surface  $X'$ . It comes from blowing up points and contracting curves over a certain rational elliptic surface; Proposition 3.1. The  $\mathbb{Q}$ -Gorenstein space of deformations of  $X'$  is smooth and 6 dimensional; Proposition 3.3. In both Godeaux and Campedelli cases we compute  $H^2(\mathcal{T}) = 0$  to show no local-to-global obstruction to deform them; Theorem 2.1 and Theorem 3.2. This involves a new technique (Theorem 4.4) which generalizes a result of Burns-Wahl [3] describing the space of first order deformations of a singular complex surface with only rational double points.

**Notations.** A cyclic quotient singularity (germ at  $(0,0)$  of)  $\mathbb{C}^2/G$ , where  $G = \langle (x,y) \mapsto (\zeta x, \zeta^q y) \rangle$  with  $\zeta$  a  $m$ -th primitive root of 1,  $1 < q < m$ , and  $(q,m) = 1$ , is denoted by  $\frac{1}{m}(1,q)$ .  $A_m$  means  $\frac{1}{m}(1,m-1)$ . A  $(-1)$ -curve (or  $(-2)$ -curve) in a smooth surface is an embedded  $\mathbb{CP}^1$  with self-intersection  $-1$  (respectively,  $-2$ ). Throughout this paper we use the same letter to denote a curve and its proper transform under a birational map. A singularity of class  $T$  is a quotient singularity which admits a  $\mathbb{Q}$ -Gorenstein one parameter smoothing. They are either rational double points or  $\frac{1}{dn^2}(1, dna-1)$  with  $1 < a < n$  and  $(n,a) = 1$ ; see Kollár-Shepherd-Barron [14, §3]. For a normal variety  $X$  its tangent sheaf  $\mathcal{T}_X$  is  $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ . The dimension of  $H^i$  is  $h^i$ .

**Acknowledgements.** The authors would like to thank Professor Yongnam Lee for helpful discussion during the work, careful reading of the draft version, and many valuable comments. The authors also wish to thank Professor Jenia Tevelev for indicating a mistake in an earlier version of this paper. Heesang park was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korean Government (2011-0012111). Dongsoo Shin was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korean Government (2010-0002678). Giancarlo Urzúa was supported by a FONDECYT Inicio grant funded by the Chilean Government (11110047).

## 2. NUMERICAL GODEAUX SURFACES WITH A 2-DIVISIBLE DIVISOR

In this section we construct a family of simply connected numerical Godeaux surfaces having a 2-divisible divisor consisting of four disjoint  $(-2)$ -curves by smoothing a singular surface  $\tilde{Y}$ ; Theorem 2.4. This is the key to construct numerical Campedelli surfaces with an involution. In addition, we describe the explicit stable model of the singular surface  $\tilde{Y}$ . In fact, we construct a rational normal projective surface  $Y'$  with four singularities  $A_3, A_3, \frac{1}{8^2}(1, 8 \cdot 5 - 1), \frac{1}{7^2}(1, 7 \cdot 4 - 1)$  and  $K_{Y'}$  ample. Hence  $Y'$  is a stable surface (cf. Kollár-Shepherd-Barron [14], Hacking [11]). We will prove that the versal  $\mathbb{Q}$ -Gorenstein deformation space  $\text{Def}^{\text{QG}}(Y')$  (cf. Hacking [11, §3]) is smooth and 8 dimensional, and that the  $\mathbb{Q}$ -Gorenstein smoothings of  $Y'$  are simply connected numerical Godeaux surfaces. In particular, this shows that there are simply connected numerical Godeaux surfaces whose canonical model has precisely two  $A_3$  singularities; Corollary 2.3. Furthermore a four dimensional family in  $\text{Def}^{\text{QG}}(Y')$  produces the above simply connected numerical Godeaux surfaces with a 2-divisible divisor consisting of four disjoint  $(-2)$ -curves; Theorem 2.4 and Theorem 5.2.

**2.1. A rational elliptic surface  $E(1)$ .** We start with a rational elliptic surface  $E(1)$  with an  $I_8$ -singular fiber, an  $I_2$ -singular fiber, and two nodal singular fibers. In fact we will use the same rational elliptic surface  $E(1)$  in the papers H. Park-J. Park-D. Shin [20, 21].

However, we need to sketch the construction of  $E(1)$  to show the relevant curves that will be used to build the singular surfaces  $\tilde{Y}$  and  $Y'$ .

Let  $L_1, L_2, L_3$ , and  $\ell$  be lines in  $\mathbb{CP}^2$  and let  $c$  be a smooth conic in  $\mathbb{CP}^2$  given by the following equations. They intersect as in Figure 1.

$$\begin{aligned} L_1 : 2y - 3z = 0, \quad L_2 : y + \sqrt{3}x = 0, \quad L_3 : y - \sqrt{3}x = 0 \\ \ell : x = 0, \quad c : x^2 + (y - 2z)^2 - z^2 = 0. \end{aligned}$$

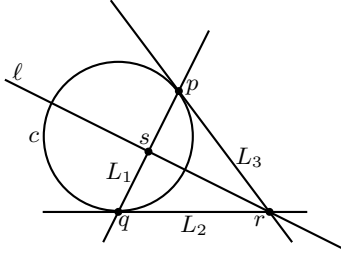


FIGURE 1. A pencil of cubic curves

We consider the pencil of cubics

$$\{\lambda(y - \sqrt{3}x)(y + \sqrt{3}x)(2y - 3z) + \mu x(x^2 + (y - 2z)^2 - z^2) \mid [\lambda : \mu] \in \mathbb{CP}^1\}$$

generated by the two cubic curves  $L_1 + L_2 + L_3$  and  $\ell + c$ . This pencil has four base points  $p, q, r, s$ , and four singular members corresponding to  $[\lambda : \mu] = [1 : 0], [0 : 1], [2 : 3\sqrt{3}], [2 : -3\sqrt{3}]$ . The latter two singular members are nodal curves, denoted by  $F_1$  and  $F_2$  respectively. They have nodes at  $[-\sqrt{3} : 0 : 1]$  and  $[\sqrt{3} : 0 : 1]$ , respectively.

In order to obtain a rational elliptic surface  $E(1)$  from the pencil, we resolve all base points (including infinitely near base-points) of the pencil by blowing-up 9 times as follows. We first blow up at the points  $p, q, r, s$ . Let  $e_1, e_2, e_3, e_4$  be the exceptional divisors over  $p, q, r, s$ , respectively. We blow up again at the three points  $e_1 \cap L_3, e_2 \cap L_2, e_3 \cap \ell$ . Let  $e_5, e_6, e_7$  be the exceptional divisors over the intersection points, respectively. We finally blow up at each intersection points  $e_5 \cap c$  and  $e_6 \cap c$ . Let  $e_8$  and  $e_9$  be the exceptional divisors over the blown-up points. We then get a rational elliptic surface  $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$  over  $\mathbb{CP}^1$ ; see Figure 2.

The four exceptional curves  $e_4, e_7, e_8, e_9$  are sections of the elliptic fibration  $E(1)$ , which correspond to the four base points  $s, r, p, q$ , respectively. The elliptic fibration  $E(1)$  has one  $I_8$ -singular fiber  $\sum_{i=1}^8 B_i$  containing all  $L_i$  ( $i = 1, 2, 3$ ):  $B_2 = L_2, B_3 = L_3$ , and  $B_5 = L_1$ ; cf. Figure 2. We will use frequently the sum  $B = B_1 + B_2 + B_3 + B_4$ , which will be shown to be 2-divisible. The surface  $E(1)$  has also one  $I_2$ -singular fiber consisting of  $\ell$  and  $c$ , and it has two more nodal singular fibers  $F_1$  and  $F_2$ .

There is a special bisection on  $E(1)$ . Let  $M$  be the line in  $\mathbb{CP}^2$  passing through the point  $r = [0 : 0 : 1]$  and the two nodes  $[-\sqrt{3} : 0 : 1]$  and  $[\sqrt{3} : 0 : 1]$  of  $F_1$  and  $F_2$ . Since  $M$  meets every member in the pencil at three points but it passes through only one base point, the proper transform of  $M$  is a bisection of the elliptic fibration  $E(1) \rightarrow \mathbb{CP}^1$ ; cf. Figure 2. Note that  $M^2 = 0$  in  $E(1)$ .



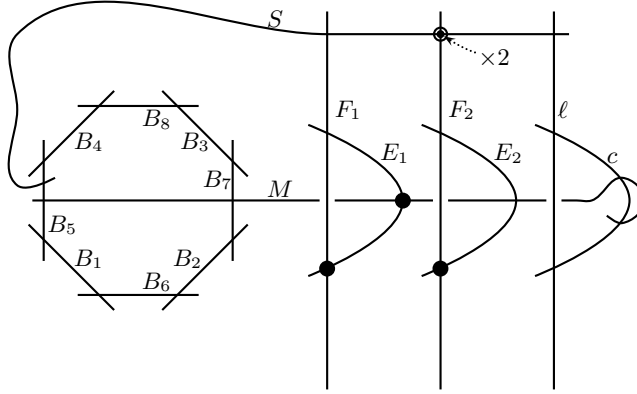


FIGURE 3. A blown-up rational elliptic surface  $W = E(1) \# 2\overline{\mathbb{CP}}^2$

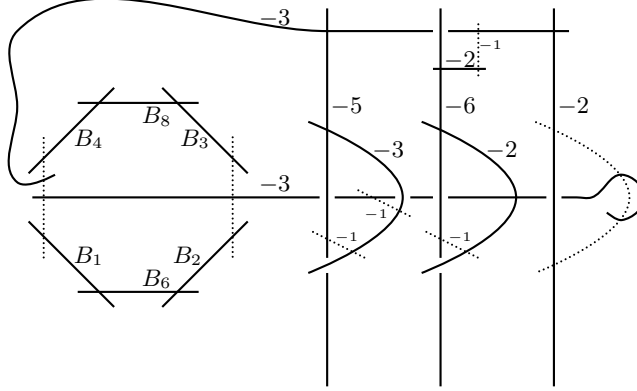


FIGURE 4. A rational surface  $Z = E(1) \# 7\mathbb{CP}^2$

contract  $C_{8,5}$ ,  $C_{7,4}$ ,  $B_4 + B_8 + B_3$  and  $B_1 + B_6 + B_2$  in  $Z$  to obtain  $Y'$ . It has the singularities  $\frac{1}{8^2}(1, 8 \cdot 5 - 1)$ ,  $\frac{1}{7^2}(1, 7 \cdot 4 - 1)$ , and two  $A_3 = \frac{1}{4}(1, 3)$ 's. Let  $\alpha': Z \rightarrow Y'$  be the contraction.

In Section 4 we will prove that the obstruction spaces to local-to-global deformations of the singular surfaces  $\tilde{Y}$ ,  $Y$  and  $Y'$  vanish. That is:

**Theorem 2.1.**  $H^2(\tilde{Y}, \mathcal{T}_{\tilde{Y}}) = 0$ ,  $H^2(Y, \mathcal{T}_Y) = 0$ , and  $H^2(Y', \mathcal{T}_{Y'}) = 0$ .

The singular surface  $Y'$  is the stable model of the singular surfaces  $\tilde{Y}$  and  $Y$ :

**Proposition 2.2.** *The surface  $Y'$  has  $K_{Y'}^2 = 1$ ,  $p_g(Y') = 0$ , and  $K_{Y'}$  is ample. The space  $\text{Def}^{\text{QG}}(Y')$  is smooth and 8 dimensional. A  $\mathbb{Q}$ -Gorenstein smoothing of  $Y'$  is a simply connected canonical surface of general type with  $p_g = 0$  and  $K^2 = 1$ .*

*Proof.* For a surface  $Y'$  with only singularities of type  $T$  we have

$$K_{Y'}^2 = K_Z^2 + \sum_{p \in \text{Sing}(Y')} s_p - \sum_{p \in \text{Sing}(Y')} \mu_p,$$

where  $s_p$  is the number of exceptional curves over  $p$  and  $\mu_p$  is the Milnor number of  $p$ . In our case,  $K_{Y'}^2 = -7 + 4 + 4 = 1$ . We have  $p_g(Y') = q(Y') = 0$  because of the rationality of the singularities.

We now compute  $\alpha'^*(K_{Y'})$  in a  $\mathbb{Q}$ -numerically effective way. Let  $F$  be the general fiber of the elliptic fibration in  $Z$ . Let  $E_3, E_4, E_5$  be the exceptional curves over  $M \cap E_1, F_1 \cap E_1, F_2 \cap E_2$  respectively. Let  $E_6, E_7$  be the exceptional curves over  $F_2 \cap S$  with  $E_7^2 = -1$ . Then,  $K_Z \sim -F + \sum_{i=1}^7 E_i + E_3 + E_4 + E_5 + E_7$ . We also have  $F \sim F_1 + 2E_1 + 2E_3 + 3E_4 \sim F_2 + 2E_2 + 3E_5 + E_6 + E_7$ . Writing  $F \equiv \frac{1}{2}F + \frac{1}{2}F$  in  $K_Z$  and adding the discrepancies from  $p_1$  and  $p_2$ , we obtain

$$\begin{aligned} \alpha'^*(K_{Y'}) &\equiv \frac{3}{8}F_1 + \frac{5}{14}F_2 + \frac{5}{8}E_1 + \frac{5}{7}E_2 + E_3 \\ &\quad + \frac{1}{2}E_4 + \frac{1}{2}E_5 + \frac{13}{14}E_6 + \frac{3}{2}E_7 + \frac{4}{7}M + \frac{3}{8}\ell + \frac{6}{8}S. \end{aligned}$$

We now intersect  $\alpha'^*(K_{Y'})$  with all the curves in its support, which are not contracted by  $\alpha'$ , to check that  $K_{Y'}$  is nef. Moreover, if  $\Gamma \cdot \alpha'^*(K_{Y'}) = 0$  for a curve  $\Gamma$  not contracted by  $\alpha'$ , then  $\Gamma$  is a component of a fiber in the elliptic fibration which does not intersect any curve in the support. This is because  $F_1, E_1, E_3$ , and  $E_4$  belong to the support, and they are the components of a fiber. One easily checks that  $\Gamma$  does not exist, proving that  $K_{Y'}$  is ample. Therefore any  $\mathbb{Q}$ -Gorenstein smoothing of  $Y'$  over a (small) disk will produce canonical surfaces; cf. Kollár-Mori [13, p. 34].

To compute the fundamental group of a  $\mathbb{Q}$ -Gorenstein smoothing we use the recipe in Y. Lee-J. Park [15]. We follow the argument as in Y. Lee-J. Park [15, p. 493]. Consider the normal circles around  $M$  and  $E_1$ . We can compare them through the transversal sphere  $E_3$ . Since the orders of the circles are 49 and 64, which are coprime, we obtain that both end up being trivial.

The smoothness of  $\text{Def}^{\text{QG}}(Y')$  follows from Theorem 2.1 and Hacking [11, §3]. To compute the dimension, we observe that if  $\mathcal{Y}' \rightarrow \Delta$  is a  $\mathbb{Q}$ -Gorenstein smoothing of  $\mathcal{Y}'_0 = Y'$  and  $\mathcal{T}_{\mathcal{Y}'|\Delta}$  is the dual of  $\Omega^1_{\mathcal{Y}'|\Delta}$ , then  $\mathcal{T}_{\mathcal{Y}'|\Delta}$  restricts to  $\mathcal{Y}'_t$  as  $\mathcal{T}_{\mathcal{Y}'_t}$  (tangent bundle of  $\mathcal{Y}'_t$ ) when  $t \neq 0$ , and  $\mathcal{T}_{\mathcal{Y}'|\Delta}|_{\mathcal{Y}'_0} \subset \mathcal{T}_{\mathcal{Y}'_0}$  with cokernel supported at the singular points of  $\mathcal{Y}'_0$ ; cf. Wahl [25]. Then the flatness of  $\mathcal{T}_{\mathcal{Y}'|\Delta}$  and semicontinuity in cohomology plus the fact that  $H^2(Y', \mathcal{T}_{Y'}) = 0$  gives  $H^2(\mathcal{Y}'_t, \mathcal{T}_{\mathcal{Y}'_t}) = 0$  for any  $t$ . But then, since  $\mathcal{Y}'_t$  is of general type, Hirzebruch-Riemann-Roch Theorem says

$$H^1(\mathcal{Y}'_t, \mathcal{T}_{\mathcal{Y}'_t}) = 10\chi(\mathcal{Y}'_t, \mathcal{O}_{\mathcal{Y}'_t}) - 2K_{\mathcal{Y}'_t}^2 = 10 - 2 = 8.$$

This proves the claim.  $\square$

**Corollary 2.3.** *There is a two dimensional family of simply connected canonical numerical Godeaux surfaces with two  $A_3$  singularities.*

*Proof.* We consider the sequence

$$0 \rightarrow H^1(Y', \mathcal{T}_{Y'}) \rightarrow T_{\text{QG}, Y'}^1 \rightarrow H^0(Y', \mathcal{T}_{\text{QG}, Y'}^1) \rightarrow 0$$

at the end of Hacking [11, §3]. We just proved that  $T_{\text{QG}, Y'}^1$  is 8 dimensional, and we know that  $H^0(Y', \mathcal{T}_{\text{QG}, Y'}^1)$  is 8 dimensional, since each  $A_3$  gives 3 dimensions and each  $p_i$  gives 1 dimension. Therefore  $H^1(Y', \mathcal{T}_{Y'}) = 0$ . To produce the claimed family we need to smooth up at the same time  $p_1$  and  $p_2$ .  $\square$

A simply connected numerical Godeaux surfaces with a 2-divisible divisor consisting of four disjoint  $(-2)$ -curves is obtained from a  $\mathbb{Q}$ -Gorenstein smoothing of the singular surface  $\tilde{Y}$ :

**Theorem 2.4.** (a) *There is a  $\mathbb{Q}$ -Gorenstein smoothing  $\widetilde{\mathcal{Y}} \rightarrow \Delta$  over a disk  $\Delta$  with central fiber  $\widetilde{\mathcal{Y}}_0 = \widetilde{Y}$  and an effective divisor  $\mathcal{B} \subset \widetilde{\mathcal{Y}}$  such that the restriction to a fiber  $\widetilde{Y}_t$  over  $t \in \Delta$*

$$B_t := \mathcal{B} \cap \widetilde{Y}_t = B_{1,t} + B_{2,t} + B_{3,t} + B_{4,t}$$

*is 2-divisible in  $\widetilde{Y}_t$  consisting of four disjoint  $(-2)$ -curves and  $B_0 = B$ .*

(b) *There is a  $\mathbb{Q}$ -Gorenstein deformation  $\mathcal{Y} \rightarrow \Delta$  of  $Y$  with central fiber  $\mathcal{Y}_0 = Y$  such that a fiber  $Y_t$  over  $t \neq 0$  has four ordinary double points as its only singularities and the minimal resolution of  $Y_t$  is the corresponding fiber  $\widetilde{Y}_t$  of  $\widetilde{\mathcal{Y}} \rightarrow \Delta$ .*

*Proof.* We apply a similar method in Y. Lee-J. Park [16]. Since any local deformations of the singularities of  $Y$  can be globalized by Theorem 2.1, there are  $\mathbb{Q}$ -Gorenstein deformations of  $Y$  over a disk  $\Delta$  which keep all four ordinary double points and smooth up  $p_1$  and  $p_2$ . Let  $\mathcal{Y} \rightarrow \Delta$  be such deformation, with  $Y$  as its central fiber, and  $\mathcal{Y}_t$  ( $t \neq 0$ ) a normal projective surface with four  $A_1$ s as its only singularities. We resolve simultaneously these four singularities in each fiber  $\mathcal{Y}_t$ . We then get a family  $\widetilde{\mathcal{Y}} \rightarrow \Delta$  that is a  $\mathbb{Q}$ -Gorenstein smoothing of the central fiber  $\widetilde{Y}$ , which shows that a  $\mathbb{Q}$ -Gorenstein smoothing of  $Y$  can be lifted to a  $\mathbb{Q}$ -Gorenstein smoothing of the pair  $(Y, B)$ , i.e. the 2-divisible divisor  $B$  on  $\widetilde{Y}$  is extended to an effective divisor  $\mathcal{B} \subset \widetilde{\mathcal{Y}}$ .

We finally show that the effective divisor  $B_t$  is 2-divisible in  $\widetilde{Y}_t$  for  $t \neq 0$ . According to Manetti [18, Lemma 2], the natural restriction map  $r_t : \text{Pic}(\widetilde{\mathcal{Y}}) \rightarrow \text{Pic}(\widetilde{Y}_t)$  is injective for every  $t \in \Delta$  and bijective for  $0 \in \Delta$ . Here we are using that  $p_g(\widetilde{Y}) = q(\widetilde{Y}) = 0$ . Since the divisor  $B$  is nonsingular,  $\mathcal{B}_t$  is also nonsingular. Since  $B \sim 2L$  in (2.1), it follows that  $B_t \sim 2L_t$ , where  $L$  is extended to a line bundle  $\mathcal{L} \subset \widetilde{\mathcal{Y}}$  and  $L_t$  is the corresponding restriction.  $\square$

### 3. NUMERICAL CAMPEDELLI SURFACES WITH AN INVOLUTION

The main purpose of this section is to construct simply connected numerical Campedelli surfaces with an involution. Along the way, we will introduce a rational normal projective surface  $X'$  with 6 singularities (two  $A_1$ , two  $\frac{1}{8^2}(1, 8 \cdot 5 - 1)$ , and two  $\frac{1}{7^2}(1, 7 \cdot 4 - 1)$ ) and  $K_{X'}$  ample. A certain four dimensional  $\mathbb{Q}$ -Gorenstein deformation of  $X'$  will produce numerical Campedelli surfaces with an involution.

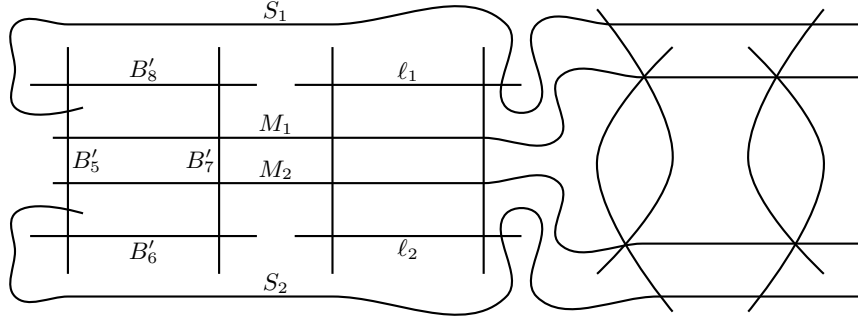
Recall that the rational surface  $Z$  has a 2-divisible divisor  $B = B_1 + B_2 + B_3 + B_4$ ; cf. (2.1). Let  $V$  be the double cover of  $Z$  branched along the divisor  $B$ , where the double cover is given by the data  $B \sim 2L$ ,  $L = h - e_3 - e_5 - e_6$ . We denote the double covering by  $\psi : V \rightarrow Z$ . The surface  $V$  has two  $C_{8,5}$ 's and two  $C_{7,4}$ 's.

On the other hand the surface  $V$  can be obtained from a certain rational elliptic surface by blowing-ups, as we now explain. The morphism  $\psi : V \rightarrow Z$  blows down to a double cover  $\psi' : V' \rightarrow E(1)$  branched along  $B$ . The ramification divisor  $\psi'^{-1}(B)$  consists of four disjoint  $(-1)$ -curves  $R_1, \dots, R_4$ . We blow down them from  $V'$  to obtain a surface  $E(1)'$ ; cf. Figure 5. In Figure 5 the pull-back of the  $B_i$  are the  $B'_i$ , of the curve  $\ell$  is  $\ell_1 + \ell_2$ , of the section  $S$  is  $S_1 + S_2$ , and of the double section  $M$  is  $M_1 + M_2$ . Each  $I_2$  in  $E(1)'$  is the pull-back of each  $I_1$  in  $E(1)$ .

Note that the surface  $E(1)'$  has an elliptic fibration structure with two  $I_4$ -singular fibers and two  $I_2$ -singular fibers. In fact, the surface  $E(1)'$  can be obtained from the pencil of cubics in  $\mathbb{CP}^2$

$$\{\lambda x(y-z)(x-2z) + \mu y(x-z)(y-2z) \mid [\lambda : \mu] \in \mathbb{CP}^1\}$$



FIGURE 5. Relevant curves in the elliptic rational surface  $E(1)'$ 

where the  $I_4$ -singular fibers come from  $x(y-z)(x-2z)=0$  and  $y(x-z)(y-2z)=0$ , and the  $I_2$ -singular fibers come from  $(x+y-2z)(xy-yz-xz)=0$  and  $(x-y)(xy-xz+2z^2-yz)=0$ . The two double sections  $M_1$  and  $M_2$  are defined by the lines  $x = (1 + \sqrt{-1})z$  and  $x = (1 - \sqrt{-1})z$ . In summary:

**Proposition 3.1.** *The surface  $V'$  is the blow-up at four nodes of one  $I_4$ -singular fiber of the rational elliptic fibration  $E(1)' \rightarrow \mathbb{CP}^1$ . Hence the surface  $V$  can be obtained from  $V'$  by blowing-up in the obvious way.*

Let  $\tilde{X}$  be the double cover of the singular surface  $\tilde{Y}$  branched along the divisor  $B$ . Note that the surface  $\tilde{X}$  is a normal projective surface with four singularities of class  $T$  whose resolution graphs consist of two  $C_{8,5}$ 's and two  $C_{7,4}$ 's. The ramification divisor in  $\tilde{X}$  consists of the four disjoint  $(-1)$ -curves  $R_1, \dots, R_4$ . Let  $\tilde{\phi} : \tilde{X} \rightarrow \tilde{Y}$  be the double covering. On the other hand the surface  $\tilde{X}$  can be obtained from the rational surface  $V$  by contracting the two  $C_{8,5}$ 's and two  $C_{7,4}$ 's. Let  $\tilde{\beta} : V \rightarrow \tilde{X}$  be the contraction morphism.

Let  $X$  be the surface obtained by blowing down the four  $(-1)$ -curves  $R_1, \dots, R_4$  from  $\tilde{X}$ . We denote the blowing-down morphism by  $\beta : \tilde{X} \rightarrow X$ . Then there is a double covering  $\phi : X \rightarrow Y$  branched along the four ordinary double points  $q_1, \dots, q_4$ . Finally, let  $X'$  be the contraction of the  $(-2)$ -curves  $B'_8$  and  $B'_6$  in  $X$ . Let  $\beta' : V \rightarrow X'$  be the contraction. To sum up, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 E(1)' & \longleftarrow & V' & \longleftarrow & V & \xrightarrow{\tilde{\beta}} & \tilde{X} & \xrightarrow{\beta} & X & \longrightarrow & X' \\
 & & \downarrow \psi' & & \downarrow \psi & & \downarrow \tilde{\phi} & & \downarrow \phi & & \\
 & & E(1) & \longleftarrow & Z & \xrightarrow{\tilde{\alpha}} & \tilde{Y} & \xrightarrow{\alpha} & Y & \longrightarrow & Y'
 \end{array}$$

We will show in Section 4 the obstruction spaces to local-to-global deformations of the singular surfaces  $\tilde{X}$ ,  $X$  and  $X'$  vanish:

**Theorem 3.2.**  $H^2(\tilde{X}, \mathcal{T}_{\tilde{X}}) = 0$ ,  $H^2(X, \mathcal{T}_X) = 0$ , and  $H^2(X', \mathcal{T}_{X'}) = 0$ .

The singular surface  $X'$  is the stable model of  $\tilde{X}$  and  $X$ :

**Proposition 3.3.** *The surface  $X'$  has  $K_{X'}^2 = 2$ ,  $p_g(X') = 0$ , and  $K_{X'}$  ample. The space  $\text{Def}^{QG}(X')$  is smooth and 6 dimensional. A  $\mathbb{Q}$ -Gorenstein smoothing of  $X'$  is a simply connected canonical surface of general type with  $p_g = 0$  and  $K^2 = 2$ .*

*Proof.* The proof goes as the one for  $Y'$  in Proposition 2.2, using the explicit model we have for  $V$  by blowing-up  $E(1)'$  in Proposition 3.1. One can check that an intersection computation as in Proposition 2.2 verifies ampleness for  $K_{X'}$ .  $\square$

The proof of the next main result follows easily from Theorem 2.4.

**Theorem 3.4.** *There exist  $\mathbb{Q}$ -Gorenstein smoothings  $\tilde{\mathcal{X}} \rightarrow \Delta$  of  $\tilde{X}$  and  $\mathcal{X} \rightarrow \Delta$  of  $X$  that are compatible with the  $\mathbb{Q}$ -Gorenstein deformations of  $\tilde{\mathcal{Y}} \rightarrow \Delta$  of  $\tilde{Y}$  and  $\mathcal{Y} \rightarrow \Delta$  of  $Y$  in Theorem 2.4, respectively; that is, the double coverings  $\tilde{\phi} : \tilde{X} \rightarrow \tilde{Y}$  and  $\phi : X \rightarrow Y$  extend to the double coverings  $\tilde{\phi}_t : \tilde{X}_t \rightarrow \tilde{Y}_t$  and  $\phi_t : X_t \rightarrow Y_t$  between the fibers of the  $\mathbb{Q}$ -Gorenstein deformations.*

*Remark 3.5.* By Theorem 3.2, the obstruction  $H^2(X, \mathcal{T}_X)$  to local-to-global deformations of the singular surface  $X$  vanishes. The point of the above theorem is that there is a  $\mathbb{Q}$ -Gorenstein smoothing of the cover  $X$  that is compatible with the  $\mathbb{Q}$ -Gorenstein deformation of the base  $Y$ .

**Corollary 3.6.** *A general fiber  $X_t$  of the  $\mathbb{Q}$ -Gorenstein smoothing  $\mathcal{X} \rightarrow \Delta$  of  $X$  is a simply connected numerical Campedelli surface with an involution  $\sigma_t$  such that the minimal resolution of the quotient  $Y_t = X_t / \sigma_t$  is a simply connected numerical Godeaux surface.*

**3.1. The fundamental group of the quotient by an involution.** Let  $X$  be a minimal complex surface of general type with  $p_g = 0$  and  $K^2 = 2$ . Suppose that the group  $\mathbb{Z}/2\mathbb{Z}$  acts on  $X$  with just 4 fixed points. Let  $Y = X / (\mathbb{Z}/2\mathbb{Z})$  be the quotient and let  $S \rightarrow Y$  be the minimal resolution of  $Y$ . Barlow [2, Proposition 1.3] proved that if  $\pi_1^{\text{alg}}(S) = 1$  then  $|\pi_1^{\text{alg}}(X)| = 1, 3, 5, 7, 9$ . Conversely:

**Proposition 3.7.** *If  $X$  is simply connected, then so is  $S$ .*

*Proof.* Let  $f : X \rightarrow Y$  be the quotient map. Then  $f$  is a double covering branched along the four ordinary double points of  $Y$ . Let  $T$  be the blow-up of  $X$  at the four ramification points. Denote the four exceptional divisors by  $E_1, \dots, E_4$ . Then there is a double covering  $g : T \rightarrow S$  branched along the four  $(-2)$ -curves  $R_1, \dots, R_4$  which is an extension of the covering  $f : X \rightarrow Y$ . Let  $X_0 = T \setminus \bigcup_{i=1}^4 E_i$  and  $Y_0 = S \setminus \bigcup_{i=1}^4 R_i$ . Then  $g|_{X_0} : X_0 \rightarrow Y_0$  is an étale double covering. Since  $\pi_1(X_0) = \pi_1(X) = 1$ , we have  $\pi_1(Y_0) = \mathbb{Z}/2\mathbb{Z}$ . Therefore  $\pi_1(S)$  is trivial or  $\mathbb{Z}/2\mathbb{Z}$  by van Kampen theorem.

Suppose that  $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$ . Then there is an étale double covering  $h : U \rightarrow S$  such that  $U$  is simply connected and connected. Then the fiber product of  $T$  and  $U$  over  $S$  is an étale double cover of  $T$ . Let  $\phi : T \times_Y U \rightarrow U$  be the double covering, which is branched along the eight  $(-2)$ -curves  $h^*(\sum_{i=1}^4 R_i)$ . That is, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & T & \xleftarrow[2:1]{\text{étale}} & T \times_Y U \\ f \downarrow & & g \downarrow & & \phi \downarrow \\ Y & \longleftarrow & S & \xleftarrow{h} & U \end{array}$$

Set  $U_0 = U \setminus h^*(\sum_{i=1}^4 R_i)$ . Since  $h : U_0 \rightarrow Y_0$  is étale and  $\pi_1(Y_0) = \mathbb{Z}/2\mathbb{Z}$ , it follows that  $\pi_1(U_0) = 1$ . Furthermore, since  $T$  is simply connected, the fiber product  $T \times_Y U$  is diffeomorphic to the disjoint union  $T \sqcup T$ . Then  $\phi : X_0 \sqcup X_0 \rightarrow U_0$  is an étale double covering and  $\pi_1(U_0) = 1$  and  $U_0$  is connected. Therefore  $X_0 \sqcup X_0$  is diffeomorphic to  $U_0 \sqcup U_0$ . Note that  $\partial X_0$  is diffeomorphic to the disjoint union of 3-spheres  $S^3$ . However  $\partial U_0$  is diffeomorphic to the disjoint union of the lens space  $L(2, 1)$ , which is a contradiction.  $\square$

## 4. THE OBSTRUCTION SPACES TO LOCAL-TO-GLOBAL DEFORMATIONS

In this section we prove Theorem 2.1 which says that the obstruction spaces to local-to-global deformations of the singular surfaces  $\tilde{Y}$ ,  $Y$ , and  $Y'$  vanish. That is, we will prove that  $H^2(\tilde{Y}, \mathcal{T}_{\tilde{Y}}) = H^2(Y, \mathcal{T}_Y) = H^2(Y', \mathcal{T}_{Y'}) = 0$ . At the end, we also prove the analogues, Theorem 3.2, for  $\tilde{X}$ ,  $X$ , and  $X'$ .

At first the vanishing of the obstruction spaces of a singular surface can be proved by the vanishing of the second cohomologies of a certain logarithmic tangent sheave on the minimal resolution of the singular surface:

**Proposition 4.1** (Y. Lee-J. Park [15, Theorem 2]). *If  $\pi : T \rightarrow S$  be the minimal resolution of a normal projective surface  $S$  with only quotient singularities, and  $D$  is the reduced exceptional divisor of the resolution  $\pi$ , then  $h^2(S, \mathcal{T}_S) = h^2(T, \mathcal{T}_T(-\log D))$ .*

**Proposition 4.2** (Flenner-Zaidenberg [9, Lemma 1.5]). *Let  $T$  be a nonsingular surface and let  $D$  be a simple normal crossing divisor in  $T$ . Let  $f : T' \rightarrow T$  be the blow-up of  $T$  at a point  $p$  of  $D$ . Let  $D' = f^*(D)_{\text{red}}$ . Then  $h^2(T', \mathcal{T}_{T'}(-\log D')) = h^2(T, \mathcal{T}_T(-\log D))$ .*

We can add or remove disjoint  $(-1)$ -curves.

**Proposition 4.3.** *Let  $T$  be a nonsingular surface and let  $D$  be a simple normal crossing divisor in  $T$ . Let  $E$  be a  $(-1)$ -curve in  $T$  such that  $D+E$  is again simple normal crossing. Then  $h^2(T, \mathcal{T}_T(-\log(D+E))) = h^2(T, \mathcal{T}_T(-\log D))$ .*

We can also add or remove disjoint exceptional divisors of rational double points. The following theorem may give a new general way to prove unobstructedness for deformations of surfaces.

**Theorem 4.4.** *Let  $S$  be a normal projective surface with only rational double points  $q_1, \dots, q_n$  as singularities. Let  $\pi : \tilde{S} \rightarrow S$  be the minimal resolution of  $S$  with exceptional reduced divisor  $M = \sum_{i=1}^n \pi^{-1}(q_i)$ . Let  $C$  be a simple normal crossing divisor such that  $C \cap M = \emptyset$ . Then  $h^2(\tilde{S}, \mathcal{T}_{\tilde{S}}(-\log(C+M))) = h^2(\tilde{S}, \mathcal{T}_{\tilde{S}}(-\log C))$ .*

*Proof.* Let  $M = \sum_i M_i$  and  $C = \sum_i C_i$  be the prime decompositions of  $M$  and  $C$ . We have three short exact sequences:

$$\begin{aligned} 0 &\rightarrow \mathcal{T}_{\tilde{S}}(-\log M) \rightarrow \mathcal{T}_{\tilde{S}} \rightarrow \oplus \mathcal{N}_{M_i/\tilde{S}} \rightarrow 0, \\ 0 &\rightarrow \mathcal{T}_{\tilde{S}}(-\log(C+M)) \rightarrow \mathcal{T}_{\tilde{S}}(-\log M) \rightarrow \oplus \mathcal{N}_{C_i/\tilde{S}} \rightarrow 0, \\ 0 &\rightarrow \mathcal{T}_{\tilde{S}}(-\log(C+M)) \rightarrow \mathcal{T}_{\tilde{S}}(-\log C) \rightarrow \oplus \mathcal{N}_{M_i/\tilde{S}} \rightarrow 0. \end{aligned}$$

We then have the following commutative diagram of cohomologies:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(\mathcal{T}_{\tilde{S}}(-\log(C+M))) & \longrightarrow & H^1(\mathcal{T}_{\tilde{S}}(-\log C)) & \xrightarrow{\phi} & \oplus H^1(\mathcal{N}_{M_i/\tilde{S}}) \\ & & \downarrow & & \downarrow \xi & & \parallel \\ 0 & \longrightarrow & H^1(\mathcal{T}_{\tilde{S}}(-\log M)) & \longrightarrow & H^1(\mathcal{T}_{\tilde{S}}) & \xrightarrow{\psi} & \oplus H^1(\mathcal{N}_{C_i/\tilde{S}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \zeta & & \\ & & \oplus H^1(\mathcal{N}_{C_i/\tilde{S}}) & \xlongequal{\quad} & \oplus H^1(\mathcal{N}_{C_i/\tilde{S}}) & & \end{array}$$

Here all horizontal and vertical sequences are exact. Especially the second row is a short exact sequence, which we explain now briefly: It is shown in Burns-Wahl [3, pp. 70–72] (see also Wahl [24, §6]) that the composition

$$H_M^1(\mathcal{T}_{\tilde{S}}) \rightarrow H^1(\mathcal{T}_{\tilde{S}}) \rightarrow \oplus H^1(\mathcal{N}_{M_i/\tilde{S}})$$

is an isomorphism because the  $q_i$ 's are *rational double points*; hence, one has a direct sum decomposition

$$H^1(\mathcal{T}_{\tilde{S}}) = H^1(\mathcal{T}_{\tilde{S}}(-\log M)) \oplus H_M^1(\mathcal{T}_{\tilde{S}}) \quad (4.1)$$

and an isomorphism  $H^2(\mathcal{T}_{\tilde{S}}) \cong H^2(\mathcal{T}_{\tilde{S}}(-\log M))$ . Therefore the second row is exact.

In order to prove the assertion, it is enough to show that

$$\phi : H^1(\mathcal{T}_{\tilde{S}}(-\log C)) \rightarrow \oplus H^1(\mathcal{N}_{M_i/\tilde{S}})$$

is surjective. Let  $\alpha \in \oplus H^1(\mathcal{N}_{M_i/\tilde{S}})$ . Since  $\psi$  is surjective, we have  $\psi(\beta) = \alpha$  for some  $\beta \in H^1(\mathcal{T}_{\tilde{S}})$ . By (4.1) we have

$$\beta = \gamma + \alpha'$$

for some  $\gamma \in H^1(\mathcal{T}_{\tilde{S}}(-\log M))$  and  $\alpha' \in H_M^1(\mathcal{T}_{\tilde{S}})$  such that  $\alpha'$  is mapped to  $\alpha$  under the composition  $H_M^1(\mathcal{T}_{\tilde{S}}) \rightarrow H^1(\mathcal{T}_{\tilde{S}}) \rightarrow \oplus H^1(\mathcal{N}_{M_i/\tilde{S}})$ . Since  $\alpha'$  is supported on  $M$  and  $C \cap M = \emptyset$ , its image  $\zeta(\alpha')$  under  $\zeta : H^1(\mathcal{T}_{\tilde{S}}) \rightarrow \oplus H^1(\mathcal{N}_{C_i/\tilde{S}})$  vanishes. Therefore  $\zeta(\beta - \gamma) = \zeta(\alpha') = 0$ , and so

$$\beta - \gamma = \xi(\delta)$$

for some  $\delta \in H^1(\mathcal{T}_{\tilde{S}}(-\log C))$ ; hence,  $\phi(\delta) = \psi(\xi(\delta)) = \alpha$ , which shows that  $\phi$  is surjective.  $\square$

**Proposition 4.5** (Esnault-Viehweg [8, 2.3]). *Let  $E = \sum_{i=1}^n C_i$  be a simple normal crossing divisor on a smooth surface  $T$ . Then one has the following exact sequences:*

$$(a) \quad 0 \rightarrow \Omega_T^1 \rightarrow \Omega_T^1(\log E) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{C_i} \rightarrow 0.$$

$$(b) \quad 0 \rightarrow \Omega_T^1(\log E) \rightarrow \Omega_T^1(\log(E - C_1))(C_1) \rightarrow \Omega_{C_1}^1(E|_{C_1}) \rightarrow 0.$$

*Proof of Theorem 2.1.* We first claim that

$$H^2(W, \mathcal{T}_W(-\log(F_1 + F_2))) = 0.$$

By duality, we have to show that

$$h^2(W, \mathcal{T}_W(-\log(F_1 + F_2))) = h^0(W, \Omega_W^1(\log(F_1 + F_2))(K_W)) = 0.$$

Since  $K_W \sim -F_2 + E_1 - E_2$ , it follows by Proposition 4.5 that

$$\begin{aligned} H^0(W, \Omega_W^1(\log(F_1 + F_2))(K_W)) &= H^0(W, \Omega_W^1(\log F_1)(K_W + F_2)) \\ &= H^0(W, \Omega_W^1(\log F_1)(E_1 - E_2)) \\ &\subset H^0(W, \Omega_W^1(\log F_1)(E_1)) \\ &= H^0(W, \Omega_W^1(\log(F_1 + E_1))). \end{aligned}$$

Hence it suffices to show that  $H^0(W, \Omega_W^1(\log(F_1 + E_1))) = 0$ . On the other hand, we obtain a long exact sequence from Proposition 4.5:

$$H^0(W, \Omega_W) \rightarrow H^0(W, \Omega_W^1(\log(F_1 + E_1))) \rightarrow H^0(F_1, \mathcal{O}_{F_1}) \oplus H^0(E_1, \mathcal{O}_{E_1}) \xrightarrow{\delta} H^1(W, \Omega_W)$$

Since  $H^0(W, \Omega_W) = 0$ , it is enough to show that the connecting homomorphism  $\delta$  is injective. Note the map  $\delta$  is the first Chern class map. But  $F_1$  and  $E_1$  are linearly independent in the Picard group of  $W$ ; hence, the map  $\delta$  is injective. Therefore the claim follows.

Let  $B' = B_4 + B_8 + B_3 + B_1 + B_6 + B_2$ . By Theorem 4.4 we have

$$h^2(W, \mathcal{T}_W(-\log(F_1 + F_2))) = h^2(W, \mathcal{T}_W(-\log(F_1 + F_2 + B' + M + \ell))).$$

We use Propositions 4.3 and 4.2 to obtain

$$h^2(W, \mathcal{T}_W(-\log(F_1 + F_2 + B' + M + \ell))) = h^2(Z, \mathcal{T}_Z(-\log(F_1 + F_2 + B' + M + \ell + S + E'))),$$

where  $E' = E_1 + E_2 + E_6$ . In this way, it follows by the above claim that

$$h^2(Z, \mathcal{T}_Z(-\log(F_1 + F_2 + B' + M + \ell + S + E'))) = h^2(W, \mathcal{T}_W(-\log(F_1 + F_2))) = 0.$$

Then, by Proposition 4.1, we have  $H^2(Y, \mathcal{T}_{Y'}) = 0$ . Notice we can modify  $B'$  to obtain vanishing for  $H^2(\tilde{Y}, \mathcal{T}_{\tilde{Y}})$  and  $H^2(Y, \mathcal{T}_Y)$  as well.  $\square$

We now prove that  $H^2(\tilde{X}, \mathcal{T}_{\tilde{X}}) = H^2(X, \mathcal{T}_X) = H^2(X', \mathcal{T}_{X'}) = 0$ .

*Proof of Theorem 3.2.* We will use our explicit model of  $X'$  in Proposition 3.1. The proof goes along the same lines as the proof of the above Theorem 2.1. We may only need to mention that we start with the elliptic fibration  $E(1)'$ , and  $2 I_2$  fibers (instead of  $2 I_1$ 's).  $\square$

## 5. THE INVARIANT PART OF THE DEFORMATION SPACE

The involution of a general fiber  $X_t$  induced by the double covering  $\phi_t : X_t \rightarrow Y_t$  extends to a  $\mathbb{Z}/2\mathbb{Z}$ -action on the deformation space of  $X_t$ . We will count the dimension of the subspace of  $H^1(X_t, \mathcal{T}_{X_t})$  which is fixed by the  $\mathbb{Z}/2\mathbb{Z}$ -action; Theorem 5.2.

Let  $\alpha_t : \tilde{Y}_t \rightarrow Y_t$  be the minimal resolution and let  $\beta_t : \tilde{X}_t \rightarrow X_t$  be the blowing-up at the four ramification points. We then have the following commutative diagram where the vertical morphisms are double covers:

$$\begin{array}{ccc} \tilde{X}_t & \xrightarrow{\beta_t} & X_t \\ \tilde{\phi}_t \downarrow & & \downarrow \phi_t \\ \tilde{Y}_t & \xrightarrow{\alpha_t} & Y_t \end{array}$$

Recall that the branch divisor  $B_t$  of the double covering  $\tilde{\phi}_t$  consists of four disjoint  $(-2)$ -curves  $B_{1,t}, \dots, B_{4,t}$  and the corresponding ramification divisor  $R_t$  consists of four disjoint  $(-1)$ -curves  $R_{1,t}, \dots, R_{4,t}$ . As before the involution of  $\tilde{X}_t$  induced by the double covering  $\tilde{\phi}_t : \tilde{X}_t \rightarrow \tilde{Y}_t$  extends to a  $\mathbb{Z}/2\mathbb{Z}$ -action on the deformation space of  $\tilde{X}_t$ .

**Lemma 5.1.**  $\dim H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t})^{\mathbb{Z}/2\mathbb{Z}} = 4$ .

*Proof.* By Pardini [19, Lemma 4.2], the invariant part of  $(\tilde{\phi}_t)_* \mathcal{T}_{\tilde{X}_t}$  under the  $\mathbb{Z}/2\mathbb{Z}$ -action is  $\mathcal{T}_{\tilde{Y}_t}(-\log B_t)$ . Therefore we have

$$H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t})^{\mathbb{Z}/2\mathbb{Z}} \cong H^1(\tilde{Y}_t, \mathcal{T}_{\tilde{Y}_t}(-\log B_t)) \cong H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^1(\log B_t)(K_{\tilde{Y}_t})).$$

We know that  $\tilde{\phi}_t^*(\Omega_{\tilde{X}_t}^1(K_{\tilde{X}_t})) = \Omega_{\tilde{Y}_t}^1(\log B_t)(K_{\tilde{Y}_t}) \oplus \Omega_{\tilde{Y}_t}^1(K_{\tilde{Y}_t} + L_t)$ , where  $2L_t \sim B_t$  defines the double cover  $\tilde{\phi}_t$ . Therefore  $H^2(\mathcal{T}_{\tilde{X}_t}) = H^0(\Omega_{\tilde{X}_t}^1(K_{\tilde{X}_t})) = H^0(\tilde{\phi}_t^*(\Omega_{\tilde{X}_t}^1(K_{\tilde{X}_t}))) = 0$  by

Theorem 3.3. Then we have  $H^0(\Omega_{\tilde{Y}_t}^1(\log B_t)(K_{\tilde{Y}_t})) = 0$ . Hence, by Proposition 4.5, there is a short exact sequence

$$0 \rightarrow H^0(B_t, \mathcal{O}_{B_t}(K_{\tilde{Y}_t})) \rightarrow H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^1(K_{\tilde{Y}_t})) \rightarrow H^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^1(\log B_t)(K_{\tilde{Y}_t})) \rightarrow 0.$$

Since  $B_t$  consists of four disjoint  $(-2)$ -curves, we have  $h^0(B_t, \mathcal{O}_{B_t}(K_{\tilde{Y}_t})) = 4$ . By Proposition 2.2, we know that  $h^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^1(K_{\tilde{Y}_t})) = 8$ . Therefore  $h^1(\tilde{Y}_t, \Omega_{\tilde{Y}_t}^1(\log B_t)(K_{\tilde{Y}_t})) = 4$ .  $\square$

**Theorem 5.2.** *The subspace of the deformation space of  $X_t$  invariant under the  $\mathbb{Z}/2\mathbb{Z}$ -action is four dimensional.*

*Proof.* We apply a similar strategy in Werner [26, §4]. We have the exact sequence

$$0 \rightarrow \mathcal{T}_{\tilde{X}_t}(-\log R) \rightarrow \mathcal{T}_{\tilde{X}_t} \rightarrow \oplus \mathcal{N}_{R_{i,t}, \tilde{X}} \rightarrow 0.$$

Since each  $R_{i,t}$  is a  $(-1)$ -curve, we have  $H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t}) = H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t}(-\log R))$ . On the other hand, it follows by Catanese [7, Lemma 9.22] that

$$H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t}(-\log R)) = H^1(X_t, \mathcal{T}_{X_t} \otimes \mathcal{I}) \quad (5.1)$$

where  $\mathcal{I}$  is the ideal sheaf of the four points in  $X_t$  obtained by contacting the exceptional divisors  $R_{1,t}, \dots, R_{4,t}$ .

Let  $P$  be the set of these four points. From the ideal sequence, we have

$$0 \rightarrow H^0(P, \mathcal{T}_{X_t} \otimes \mathcal{O}_P) \rightarrow H^1(X_t, \mathcal{T}_{X_t} \otimes \mathcal{I}) \rightarrow H^1(X_t, \mathcal{T}_{X_t}) \rightarrow 0.$$

Therefore the invariant parts of each space satisfies:

$$0 \rightarrow H^0(P, \mathcal{T}_{X_t} \otimes \mathcal{O}_P)^{\mathbb{Z}/2\mathbb{Z}} \rightarrow H^1(X_t, \mathcal{T}_{X_t} \otimes \mathcal{I})^{\mathbb{Z}/2\mathbb{Z}} \rightarrow H^1(X_t, \mathcal{T}_{X_t})^{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0.$$

According to Werner [26, p. 1523], we have  $H^0(P, \mathcal{T}_{X_t} \otimes \mathcal{O}_P)^{\mathbb{Z}/2\mathbb{Z}} = 0$ . Therefore it follows by (5.1) and Lemma 5.1 that

$$\dim H^1(X_t, \mathcal{T}_{X_t})^{\mathbb{Z}/2\mathbb{Z}} = \dim H^1(X_t, \mathcal{T}_{X_t} \otimes \mathcal{I})^{\mathbb{Z}/2\mathbb{Z}} = \dim H^1(\tilde{X}_t, \mathcal{T}_{\tilde{X}_t}(-\log R))^{\mathbb{Z}/2\mathbb{Z}} = 4. \quad \square$$

## 6. ANOTHER EXAMPLE

We briefly describe another rational surface  $Z$  which makes it possible to construct simply connected numerical Campedelli surfaces with an involution as before. The associated Godeaux surfaces come from a rational surface  $Y'$  with  $K_{Y'}$  ample having three  $A_1$ -singularities, one  $A_3$ -singularity, and only one singularity of class  $T$ .

**6.1. A rational surface  $Z = E(1) \# 8\overline{\mathbb{CP}^2}$ .** The elliptic fibration  $E(1)$  is the one in Section 2. In the construction of  $Z$ , we will use the sections  $e_4, e_7, e_8$  among the four sections of  $E(1)$ . We denote the sections  $e_4, e_7, e_8$  by  $S_1, S_2, S_3$ , respectively. We first blow up at the two nodes of the nodal singular fibers  $F_1$  and  $F_2$  so that we obtain a blown-up rational elliptic surface  $W = E(1) \# 2\overline{\mathbb{CP}^2}$ ; Figure 6. Let  $E_1$  and  $E_2$  be the exceptional curves over the nodes of  $F_1$  and  $F_2$ , respectively. We further blow up at each two marked points  $\bullet$  and blow up four times at the marked point  $\odot$  in Figure 6. We then get a rational surface  $Z = E(1) \# 8\overline{\mathbb{CP}^2}$ ; Figure 7. There exists one linear chain of  $\mathbb{CP}^1$ 's in  $Z$  whose dual graph is

$$\begin{array}{ccccccccccc} -6 & -1 & -8 & -2 & -2 & -3 & -2 & -2 & -2 \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ F_1 & S_3 & F_2 & S_1 & \ell & S_2 & E_5 & E_6 & E_7 \end{array}$$

Notice that the  $(-1)$ -curve  $S_3$  is contracted in the way down, which fixes the configuration so that we obtain one singular point of class  $T$  whose resolution graph is given by

$$C_{24,5} := \begin{array}{c} \circ^5 - \circ^7 - \circ^2 - \circ^2 - \circ^3 - \circ^2 - \circ^2 - \circ^2. \end{array}$$

The divisor  $B_1 + B_2 + B_3 + B_4$  is the 2-divisible one as before. The  $C_{24,5}$  and the  $(-2)$ -curves  $B_3, B_4, B_1 + B_6 + B_2$ , and  $M$  are contracted to obtain a singular surface  $Y'$ . One can use again Theorem 4.4 to show that the space  $\text{Def}^{\text{QG}}(Y')$  is smooth (of dimension 8).

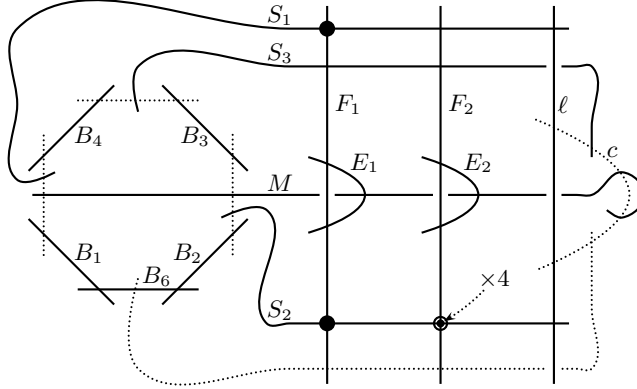


FIGURE 6. A blown-up rational elliptic surface  $W = E(1) \# 2\overline{\mathbb{CP}^2}$

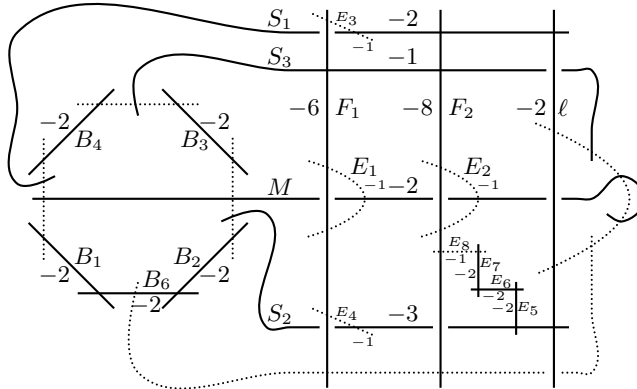


FIGURE 7. A rational surface  $Z = E(1) \# 8\overline{\mathbb{CP}^2}$

## REFERENCES

- [1] R. Barlow, *Some new surfaces with  $p_g = 0$* , Duke Math. J. **51** (1984), 889–904.
- [2] ———, *A simply connected surface of general type with  $p_g = 0$* , Invent. Math. **79** (1985), 293–301.
- [3] D. Burns, J. Wahl, *Local contributions to global deformations of surfaces*, Invent. Math. **26** (1974), 67–88.
- [4] A. Calabri, C. Ciliberto, M. Mendes Lopes, *Numerical Godeaux surfaces with an involution*, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1605–1632.
- [5] A. Calabri, M. Mendes Lopes, R. Pardini, *Involutions on numerical Campedelli surfaces*, Tohoku Math. J. (2) **60** (2008), no. 1, 1–22.

- [6] F. Catanese, *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Invent. Math. **63** (1981), no. 3, 433–465.
- [7] ———, *Moduli of algebraic surfaces*, Lecture Notes in Math. **1337**, Springer, 1988.
- [8] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar **20**, Birkhäuser Verlag.
- [9] H. Flenner, M. Zaidenberg,  *$\mathbb{Q}$ -acyclic surfaces and their deformations*, Contemp. Math. **162** (1994), 143–208.
- [10] D. Frapporti, *Mixed surfaces, new surfaces of general type with  $p_g = 0$  and their fundamental group*, arXiv:1105.1259.
- [11] P. Hacking, *Compact moduli spaces of surfaces of general type*, arXiv:1107.2717.
- [12] J. Keum, Y. Lee, *Fixed locus of an involution acting on a Godeaux surface*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 2, 205–216.
- [13] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, **134**, Cambridge University Press, Cambridge, 1998.
- [14] J. Kollár, N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), 299–338.
- [15] Y. Lee, J. Park, *A simply connected surface of general type with  $p_g = 0$  and  $K^2 = 2$* , Invent. Math. **170** (2007), 483–505.
- [16] ———, *A construction of Horikawa surface via  $\mathbb{Q}$ -Gorenstein smoothings*, Math. Z. **267** (2011), no. 1-2, 15–25.
- [17] Y. Lee, Y. Shin, *Involutions on a surface of general type with  $p_g = q = 0$ ,  $K^2 = 7$* , arXiv:1003.3595.
- [18] M. Manetti, *Normal degenerations of the complex projective plane*, J. Reine Angew. Math. **419** (1991), 89–118.
- [19] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.
- [20] H. Park, J. Park, D. Shin, *A simply connected surface of general type with  $p_g = 0$  and  $K^2 = 3$* , Geom. Topol. **13** (2009), no. 2, 743–767.
- [21] ———, *A simply connected surface of general type with  $p_g = 0$  and  $K^2 = 4$* , Geom. Topol. **13** (2009), no. 3, 1483–1494.
- [22] ———, *A complex surface of general type with  $p_g = 0$ ,  $K^2 = 2$  and  $H_1 = \mathbb{Z}/4\mathbb{Z}$* , arXiv:1012.5871, to appear in Trans. Amer. Math. Soc.
- [23] C. Rito, *Involutions on surfaces with  $p_g = q = 0$  and  $K^2 = 3$* , to appear in Geom. Dedicata.
- [24] J. Wahl, *Vanishing theorems for resolutions of surface singularities*, Invent. Math. **31** (1975), 17–41.
- [25] ———, *Smoothings of normal surface singularities*, Topology **20** (1981), no. 3, 219–246.
- [26] C. Werner, *A four-dimensional deformation of a numerical Godeaux surface*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1515–1525.

SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA  
*E-mail address*: hspark@kias.re.kr

DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, DAEJEON 305-764, KOREA &  
 SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA  
*E-mail address*: ds shin@cnu.ac.kr

FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE.  
*E-mail address*: urzua@mat.puc.cl