Spectral analysis of time changes of horocycle flows

R. Tiedra de Aldecoa[1*]

[1] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile*

E-mail: *rtiedra@mat.puc.cl*

Abstract

We prove (under the condition of A. G. Kushnirenko) that all time changes of the horocycle flow have purely absolutely continuous spectrum in the orthocomplement of the constant functions. This provides an answer to a question of A. Katok and J.-P. Thouvenot on the spectral nature of time changes of horocycle flows. Our proofs rely on positive commutator methods for self-adjoint operators.

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1 Introduction

The purpose of this note is to provide an answer to a question of A. Katok and J.-P. Thouvenot on the spectral nature of time changes of horocycle flows.

The set-up is the standard one. Consider the unit tangent bundle $M := T^1\Sigma$ of a finite volume Riemann surface Σ of genus ≥ 2. The 3-manifold M carries a probability measure μ which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ and the geodesic flow $\{F_{2,t}\}_{t\in\mathbb{R}}$. One associates to these flows vector fields X_j, Lie derivatives $\mathscr{L}_{X_j}$ and unitary groups $\{U_j(t)\}_{t\in\mathbb{R}}$ in $\mathsf{L}^2(M,\mu)$ in the usual way. It is a classical result that the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is strongly mixing (and hence ergodic) [15] and mixing of all orders [19], and that the unitary group $\{U_1(t)\}_{t\in\mathbb{R}}$ has countable Lebesgue spectrum [21]. Furthermore, A. G. Kushnirenko [18, Thm. 2] has proved (when Σ is compact) that all time changes of the horocycle flow are strongly mixing under a condition which holds if the time change is sufficiently small in the C^1 topology. Namely, if $f \in C^\infty(M)$ satisfies $f > 0$ and $f - \mathscr{L}_{X_2}(f) > 0$, then the flow of the vector field fX_1 is strongly mixing. This implies that the unitary group associated to fX_1 has purely continuous spectrum, except at 1, where it has a simple eigenvalue.

Nothing more is known about the spectral properties of the time change fX_1 (see the comments in [4, Sec. 1] and [18, Sec. 1]). However, as pointed out by A. Katok and J.-P. Thouvenot in [17, Sec. 6.3.1], it looks plausible that the unitary group associated to fX_1 has purely absolutely continuous or Lebesgue spectrum. In fact, A. Katok and J.-P. Thouvenot state as a conjecture the stability of the countable Lebesgue spectrum (see [17, Conject. 6.8]). In the present note, we give an answer to the first interrogation of these authors by proving that the unitary group associated to fX_1 has purely absolutely continuous spectrum outside $\{1\}$ under the condition of A. G. Kushnirenko.

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Our proof relies on a refined version [3, 23] of a commutator method introduced by É. Mourre [20]. It uses as a starting point the well-known commutation relation satisfied by the unitary groups of the horocycle flow and the geodesic flow:

$$U_2(s)U_1(t)U_2(-s)=U_1(\mathrm{e}^s\,t),\quad s,t\in\mathbb{R}. \tag{1.1}$$

To some extent, this approach has been suggested to us by the proof of A. G. Kushnirenko itself, since it already took advantage of commutator identities linking the vector fields X_1,X_2 and fX_1. We also aknowledge the influence of the article [12] on commutator methods for unitary operators, and we refer to [4, 9, 10, 11, 14, 16, 24] for related works on ergodic and spectral properties of time changes. In the future, we hope that commutators methods could be used to derive spectral properties of other classes of flows than the horocycle flows considered here.

Here is a brief description of the note. In Section 2, we recall some definitions and results on positive commutator methods for self-adjoint operators. In Section 3, we introduce a generalisation of the setting presented above: We consider on an abstract (possibly noncompact) n-manifold vector fields X_1,X_2 and flows $\{F_{1,t}\}_{t\in\mathbb{R}},\{F_{2,t}\}_{t\in\mathbb{R}}$ with unitary groups satisfying (1.1). Under an assumption generalising the one of A. G. Kushnirenko (see Assumption 3.2) we show that the self-adjoint operator associated to the time change fX_1 has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue (see Theorem 3.5). We use the theory of Section 2 to prove this result. In Section 4 we apply this abstract result to the horocycle flow on finite volume surfaces of constant negative curvature, taking into account the ergodicity of the horocycle flow. This leads to the desired result, namely, that the unitary group associated to a time change of the horocycle flow has purely absolutely continuous spectrum outside $\{1\}$, where it has a simple eigenvalue (see Theorem 4.2).

Remark 1.1. *After the completion of this note, the author was informed about the work [13] by G. Forni and C. Ulcigrai on time changes of horocycle flows (not available when this note was put online). The work [13] establishes (for compact surfaces and for time changes in a Sobolev space of order* $>11/2$*) the absolute continuity of time changes of horocycle flows without assuming the condition of A. G. Kushnirenko and also shows that the maximal spectral type is equivalent to Lebesgue. Here, we establish with other methods (for surfaces of finite volume and for time changes of class* C^2*) the absolute continuity of time changes of horocycle flows under the condition of A. G. Kushnirenko.*

2 Positive commutator methods

We recall in this section some facts on positive commutator methods borrowed from [3] and [23] (see also the original paper [20] of É. Mourre). Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}}$, and denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. Let also A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and $S\in\mathscr{B}(\mathcal{H})$. For any $k\in\mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S\in C^k(A)$, if the map

$$\mathbb{R}\ni t\mapsto \mathrm{e}^{-itA}\,S\,\mathrm{e}^{itA}\in\mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k=1$, one has $S\in C^1(A)$ if the quadratic form

$$\mathcal{D}(A)\ni\varphi\mapsto\langle\varphi,iSA\varphi\rangle_{\mathcal{H}}-\langle A\varphi,iS\varphi\rangle_{\mathcal{H}}\in\mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[iS,A]$ the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the function (2.1) at $t=0$.

If H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$. If H is of class $C^1(A)$, then the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H-z)^{-1} A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, (H-z)^{-1}\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

extends continuously to a bounded form defined by the operator $\big[(H-z)^{-1}, A\big] \in \mathscr{B}(\mathcal{H})$. Furthermore, the set $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, H\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous in the topology of $\mathcal{D}(H)$ [3, Thm. 6.2.10(b)]. This form extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a continuous operator $[H, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. In addition, the following relation holds in $\mathscr{B}(\mathcal{H})$ (see [3, Eq. (6.2.24)]):

$$\big[(H-z)^{-1}, A\big] = -(H-z)^{-1}[H, A](H-z)^{-1}. \tag{2.2}$$

Let $E^H(\,\cdot\,)$ denote the spectral measure of the self-adjoint operator H, and assume that H is of class $C^1(A)$. Then, the operator $E^H(J)\big[iH, A\big]E^H(J)$ is bounded and self-adjoint for each bounded Borel set $J \subset \mathbb{R}$. If there exists a number $a > 0$ such that

$$E^H(J)\big[iH, A\big]E^H(J) \geq aE^H(J),$$

then one says that H satisfies a strict Mourre estimate on J. The main consequence of such an estimate is to imply a limiting absorption principle for H on J if H is also of class $C^2(A)$. This in turns implies that H has no singular spectrum in J. We recall here a version of this result valid even if H has no spectral gap (see [3, Sec. 7.1.2] and [23, Thm. 0.1] for the most general version of this result):

Theorem 2.1. *Let H and A be self-ajoint operators in a Hilbert space $\mathcal{H}$. Suppose that H is of class $C^2(A)$ and satisfies a strict Mourre estimate on a bounded Borel set $J \subset \mathbb{R}$. Then, H has no singular spectrum in J.*

3 Spectral analysis of time changes for abstract flows

Let M be a C^∞ manifold of dimension $n \geq 1$ with volume form Ω, and let $\{F_{j,t}\}_{t\in\mathbb{R}}$, $j = 1, 2$, be (nontrivial) C^∞ complete flows on M preserving the measure μ_Ω induced by Ω. Then, it is known that the operators

$$U_j(t)\varphi := \varphi \circ F_{j,t}, \quad \varphi \in C_c^\infty(M),$$

define strongly continuous unitary groups $\{U_j(t)\}_{t\in\mathbb{R}}$ in the Hilbert space $\mathcal{H} := \mathsf{L}^2(M, \mu_\Omega)$ (here $C_c^\infty(M)$ stands for the space of C^∞ functions with compact support in M). Since $C_c^\infty(M)$ is dense in $\mathcal{H}$ and left invariant by $\{U_j(t)\}_{t\in\mathbb{R}}$, it follows from Nelson's theorem [2, Prop. 5.3] that the generator of the group $\{U_j(t)\}_{t\in\mathbb{R}}$

$$H_j\varphi := \text{s-}\lim_{t\to 0} it^{-1}\{U_j(t) - 1\}\varphi, \quad \varphi \in \mathcal{D}(H_j) := \Big\{\varphi \in \mathcal{H} \mid \lim_{t\to 0} |t|^{-1}\big\|\{U_j(t) - 1\}\varphi\big\|_{\mathcal{H}} < \infty\Big\}$$

is essentially self-adjoint on $C_c^\infty(M)$. In fact, a direct calculation shows that

$$H_j\varphi := -i\mathscr{L}_{X_j}\varphi, \quad \varphi \in C_c^\infty(M),$$

where X_j is the (divergence-free) vector field associated to $\{F_{j,t}\}_{t\in\mathbb{R}}$ and $\mathscr{L}_{X_j}$ the corresponding Lie derivative. Now, suppose that there exists a C^1 isomorphism $e : (\mathbb{R}, +) \to \big((0, \infty), \cdot\big)$ such that

$$U_2(s)U_1(t)U_2(-s) = U_1\big(e(s)t\big) \quad \text{for all } s, t \in \mathbb{R}. \tag{3.1}$$

Then, for each $t \neq 0$, $U_1(t)$ has homogeneous Lebesgue spectrum (that is, the spectrum $\sigma(H_1)$ of H_1 covers $\mathbb{R}$, and $\sigma(H_1) \setminus \{0\}$ is purely Lebesgue with uniform multiplicity, see [17, Prop. 1.23]). Furthermore, if $\mu_\Omega(M) < \infty$, then any constant function on M is an eigenvector of $U_1(t)$ with eigenvalue 1 (in some cases, as when the system $(M, \mu_\Omega, F_{1,t})$ is ergodic, 1 is even a simple eigenvalue of $U_1(t)$). By applying the strong derivative $i\,\mathrm{d}/\mathrm{d}t$ at $t = 0$ in (3.1), one gets that $U_2(s)H_1U_2(-s)\varphi = e(s)H_1\varphi$ for each $\varphi \in C_\mathrm{c}^\infty(M)$. Since $C_\mathrm{c}^\infty(M)$ is a core for H_1, one infers that H_1 is H_2-homogeneous in the sense of [7]; namely,

$$U_2(s)H_1U_2(-s) = e(s)H_1 \quad \text{on} \quad \mathcal{D}(H_1). \tag{3.2}$$

It follows that H_1 is of class $C^\infty(H_2)$ with

$$\big[iH_1, H_2\big] = e'(0)H_1. \tag{3.3}$$

Now, consider a C^1 vector field with the same orientation and colinear to the vector field X_1, that is, a vector field fX_1 where $f \in C^1(M)$ satisfies $f \geq \delta_f$ for some $\delta_f > 0$ and $f \in \mathsf{L}^\infty(M)$. The vector field fX_1 has the same integral curves as X_1, but with reparametrised time coordinate. Indeed, it is known (see [8, Chap. 2.2], [16, Sec. 1] and [22, Sec. 5.1]) that the formula

$$t = \int_0^{h(p,t)} \frac{\mathrm{d}s}{f\big(F_{1,s}(p)\big)}, \quad (p,t) \in M \times \mathbb{R},$$

defines for each $p \in M$ a strictly increasing function $\mathbb{R} \ni t \mapsto h(p,t) \in \mathbb{R}$ satisfying $h(p,0) = 0$ and $\lim_{t\to\pm\infty} h(p,t) = \pm\infty$. Furthermore, the implicit function theorem implies that the map $t \mapsto h(p,t)$ is C^1 with $\frac{\mathrm{d}}{\mathrm{d}t}h(p,t) = f\big(F_{1,h(p,t)}(p)\big)$. Therefore, the function $\mathbb{R} \ni t \mapsto \widetilde{F}_{1,t}(p) \in M$ given by $\widetilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$ satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{F}_1(p,t) = (fX_1)_{\widetilde{F}_1(p,t)}, \quad \widetilde{F}_1(p,0) = p,$$

meaning that $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is the flow of fX_1. Since the divergence $\mathrm{div}_{\Omega/f}(fX_1)$ of fX_1 with respect to the volume form Ω/f is zero (see [1, Prop. 2.5.23]), the operators

$$\widetilde{U}_1(t)\,\varphi := \varphi \circ \widetilde{F}_{1,t}, \quad \varphi \in C_\mathrm{c}(M),$$

define a strongly continuous unitary group $\big\{\widetilde{U}_1(t)\big\}_{t\in\mathbb{R}}$ in the Hilbert space $\widetilde{\mathcal{H}} := \mathsf{L}^2(M, \mu_\Omega/f)$. Its generator $\widetilde{H} := -i\,\mathscr{L}_{fX_1}$ is essentially self-adjoint on $C_\mathrm{c}^1(M) \subset \widetilde{\mathcal{H}}$ due to Nelson's theorem.

In the next lemma, we introduce two auxiliary operators which will be useful for the spectral analysis of $\widetilde{H}$.

Lemma 3.1. *Let $f \in C^1(M)$ be such that $f \geq \delta_f$ for some $\delta_f > 0$ and $f \in \mathsf{L}^\infty(M)$. Then,*

(a) The operator

$$\mathscr{U} : \mathcal{H} \to \widetilde{\mathcal{H}}, \quad \varphi \mapsto f^{1/2}\varphi,$$

is unitary with adjoint $\mathscr{U}^* : \widetilde{\mathcal{H}} \to \mathcal{H}$ given by $\mathscr{U}^*\psi = f^{-1/2}\psi$.

(b) The operator

$$H\varphi := f^{1/2}H_1f^{1/2}\varphi, \quad \varphi \in C_\mathrm{c}^1(M),$$

is essentially self-adjoint in $\mathcal{H}$, and the closure of H (which we denote by the same symbol) is unitarily equivalent to $\widetilde{H}$.

(c) For each $z \in \mathbb{C} \setminus \mathbb{R}$, the operator $H_1 + zf^{-1}$ is invertible with bounded inverse, and satisfies

$$(H+z)^{-1} = f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}f^{-1/2}. \tag{3.4}$$

Proof. Point (a) follows from a direct calculation taking into account the boundedness of f from below and from above. For (b), observe that

$$H\varphi = f^{-1/2} f H_1 f^{1/2} \varphi = \mathscr{U}^* \widetilde{H} \mathscr{U} \varphi$$

for each $\varphi \in \mathscr{U}^* C^1_c(M)$. So, H is essentially self-adjoint on $\mathscr{U}^* C^1_c(M) \equiv C^1_c(M)$, and the closure of H is unitarily equivalent to $\widetilde{H}$. To prove (c), take $z \equiv \lambda + i\mu \in \mathbb{C} \setminus \mathbb{R}$, $\varphi \in \mathcal{D}\big(H_1 + zf^{-1}\big) \equiv \mathcal{D}(H_1)$ and $\{\varphi_n\} \subset C^\infty_c(M)$ such that $\lim_n \|\varphi - \varphi_n\|_{\mathcal{D}(H_1)} = 0$. Then, it follows from (b) that

$$\big\|\big(H_1 + zf^{-1}\big)\varphi\big\|^2_{\mathcal{H}} = \lim_n \big\|f^{-1/2}(H+z)f^{-1/2}\varphi_n\big\|^2_{\mathcal{H}} \geq \inf_{p\in M} f^{-2}(p)\,\mu^2 \big\|\varphi\big\|^2_{\mathcal{H}},$$

and thus $H_1 + zf^{-1}$ is invertible with bounded inverse (see [2, Lemma 3.1]). Now, to show (3.4), take $\psi = (H+z)\zeta$ with $\zeta \in C^1_c(M)$, observe that

$$(H+z)^{-1}\psi - f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1} f^{-1/2}\psi = 0, \tag{3.5}$$

and then use the density of $(H+z)C^1_c(M)$ in $\mathcal{H}$ to extend the identity (3.5) to all of $\mathcal{H}$. □

The operators H and $\widetilde{H}$ are unitarily equivalent due to Lemma 3.1(b). Therefore, one can either work with H in $\mathcal{H}$ or with $\widetilde{H}$ in $\widetilde{\mathcal{H}}$ to determine the spectral properties associated with the time change fX_1. For convenience, we present in the sequel our results for the operator H. We start by collecting all the necessary assumptions on the function f.

Assumption 3.2 (Time change). *The function $f \in C^2(M)$ is such that*

(i) $f \geq \delta_f$ for some $\delta_f > 0$,

(ii) the functions $f, \mathscr{L}_{X_1}(f), \mathscr{L}_{X_2}(f)$, $\mathscr{L}_{X_1}\big(\mathscr{L}_{X_2}(f)\big)$ and $\mathscr{L}_{X_2}\big(\mathscr{L}_{X_2}(f)\big)$ belong to $\mathsf{L}^\infty(M)$,

(iii) the function $g := \dfrac{e'(0)f - \mathscr{L}_{X_2}(f)}{2f}$ satisfies $g \geq \delta_g$ for some $\delta_g > 0$.

If M is compact, then (ii) is automatically verified and (i) and (iii) are satisfied if f and $e'(0)f - \mathscr{L}_{X_2}(f)$ are strictly positive functions. Therefore, Assumption 3.2 reduces to the assumptions of A. G. Kushnirenko [18, Thm. 2].

In the next lemma, we prove regularity properties of H and H^2 with respect to H_2 which will be useful when deriving the strict Mourre estimate.

Lemma 3.3. *Let f satisfy Assumption 3.2, take $\alpha \in \{\pm 1/2, \pm 1\}$ and let $z \in \mathbb{C} \setminus \mathbb{R}$. Then,*

(a) the multiplication operators g^α and f^α satisfy $g^\alpha, f^\alpha \in C^1(H_2)$ and $g^\alpha \in C^1(H)$ with

$$\big[ig^\alpha, H_2\big] = -\alpha\, g^{\alpha-1}\mathscr{L}_{X_2}(g), \quad \big[if^\alpha, H_2\big] = -\alpha f^{\alpha-1}\mathscr{L}_{X_2}(f) \quad \text{and} \quad \big[ig^\alpha, H\big] = -\alpha f g^{\alpha-1}\mathscr{L}_{X_1}(g),$$

(b) $(H+z)^{-1} \in C^1(H_2)$ with $\big[i(H+z)^{-1}, H_2\big] = -(H+z)^{-1}(Hg + gH)(H+z)^{-1}$,

(c) $\big(H^2+1\big)^{-1} \in C^1(H_2)$ with $\big[i\big(H^2+1\big)^{-1}, H_2\big] = -\big(H^2+1\big)^{-1}\big(H^2g + 2HgH + gH^2\big)\big(H^2+1\big)^{-1}$,

(d) $\big(H^2+1\big)^{-1} \in C^2(H_2)$.

Proof. (a) Simple computations using the linearity of $\mathscr{L}_{X_2}$ and the bound $f \geq \delta_f$ imply that

$$\mathscr{L}_{X_2}(f^{1/2}) = \tfrac{1}{2} f^{-1/2}\mathscr{L}_{X_2}(f).$$

Thus, one has for each $\varphi \in C_c^\infty(M)$

$$\langle H_2\varphi, f^{1/2}\varphi\rangle_{\mathcal{H}} - \langle \varphi, f^{1/2}H_2\varphi\rangle_{\mathcal{H}} = \langle \varphi, [H_2, f^{1/2}]\varphi\rangle_{\mathcal{H}} = \langle \varphi, -\tfrac{i}{2}\, f^{-1/2}\mathscr{L}_{X_2}(f)\varphi\rangle_{\mathcal{H}}.$$

Since $f^{-1/2}\mathscr{L}_{X_2}(f) \in \mathsf{L}^\infty(M)$, it follows by the density of $C_c^\infty(M)$ in $\mathcal{D}(H_2)$, that $f^{1/2} \in C^1(H_2)$ with $[H_2, f^{1/2}] = -\frac{i}{2}\, f^{-1/2}\mathscr{L}_{X_2}(f)$. The other identities can be shown similarly.

(b) Let $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}$. Then, one infers from Equations (3.2) and (3.4) that

$$\begin{aligned}
&\mathrm{e}^{-itH_2}(H+z)^{-1}\,\mathrm{e}^{itH_2}\,\varphi\\
&= \mathrm{e}^{-itH_2}\, f^{-1/2}\,\mathrm{e}^{itH_2}\,\big(e(t)H_1 + z\,\mathrm{e}^{-itH_2}\, f^{-1}\,\mathrm{e}^{itH_2}\big)^{-1}\,\mathrm{e}^{-itH_2}\, f^{-1/2}\,\mathrm{e}^{itH_2}\,\varphi.
\end{aligned}$$

So, one gets from point (a), Equation (3.4) and Lemma 3.1(b) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}(H+z)^{-1}\,\mathrm{e}^{itH_2}\,\varphi\Big|_{t=0}\\
&= [if^{-1/2}, H_2]\big(H_1 + zf^{-1}\big)^{-1}f^{-1/2}\varphi + f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}[if^{-1/2}, H_2]\varphi\\
&\qquad - f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}\big\{e'(0)H_1 + z[if^{-1}, H_2]\big\}\big(H_1 + zf^{-1}\big)^{-1}f^{-1/2}\varphi\\
&= \frac{1}{2}\, f^{-1}\mathscr{L}_{X_2}(f)(H+z)^{-1}\varphi + \frac{1}{2}\,(H+z)^{-1}f^{-1}\mathscr{L}_{X_2}(f)\varphi\\
&\qquad - (H+z)^{-1}\big\{e'(0)H + zf^{-1}\mathscr{L}_{X_2}(f)\big\}(H+z)^{-1}\varphi\\
&= \frac{1}{2}\,(H+z)^{-1}Hf^{-1}\mathscr{L}_{X_2}(f)(H+z)^{-1}\varphi + \frac{1}{2}\,(H+z)^{-1}f^{-1}\mathscr{L}_{X_2}(f)H(H+z)^{-1}\varphi\\
&\qquad - (H+z)^{-1}e'(0)H(H+z)^{-1}\varphi\\
&= -(H+z)^{-1}(Hg+gH)(H+z)^{-1}\varphi,
\end{aligned}$$

which implies the claim.

(c) Let $\varphi \in \mathcal{H}$. Then, it follows from point (b) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}\,\big(H^2+1\big)^{-1}\,\mathrm{e}^{itH_2}\,\varphi\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}(H+i)^{-1}\,\mathrm{e}^{itH_2}\,\mathrm{e}^{-itH_2}(H-i)^{-1}\,\mathrm{e}^{itH_2}\,\varphi\Big|_{t=0}\\
&= -(H+i)^{-1}(Hg+gH)(H+i)^{-1}(H-i)^{-1}\varphi - (H+i)^{-1}(H-i)^{-1}(Hg+gH)(H-i)^{-1}\varphi\\
&= -\big(H^2+1\big)^{-1}\big(H^2g + 2HgH + gH^2\big)\big(H^2+1\big)^{-1}\varphi,
\end{aligned}$$

which implies the claim.

(d) Direct computations using point (c) show that

$$\begin{aligned}
&[i(H^2+1)^{-1}, H_2]\\
&= -(H^2+1)^{-1}\big\{(H^2+1)g + 2(H+i)g(H-i) + 2i(H+i)g - 2ig(H-i) + g(H^2+1)\big\}(H^2+1)^{-1}\\
&= -2\,\mathrm{Re}\big\{g(H^2+1)^{-1} + 2i(H-i)^{-1}g(H^2+1)^{-1} + (H-i)^{-1}g(H+i)^{-1}\big\}.
\end{aligned}$$

Morevover, we know from points (a)-(c) that the operators $(H^2+1)^{-1}$, $(H+i)^{-1}$, g and $(H-i)^{-1}$ belong to $C^1(H_2)$. So, one infers from standard results on the space $C^1(H_2)$ (see [3, Prop. 5.1.5]) that $[i(H^2+1)^{-1}, H_2]$ also belongs to $C^1(H_2)$. □

In order to apply the theory of Section 2, one has to prove at some point a positive commutator estimate. Usually, one proves it for the operator H under study. But in our case, the commutator $[iH, H_2] = Hg + gH$ appearing in Lemma 3.3(b) (which is the simplest nontrivial commutator in our set-up) does not exhibit any explicit positivity. By contrast, the commutator $\big[iH^2, H_2\big] = H^2 g + 2HgH + gH^2$ of Lemma 3.3(c) is made of the positive operators g, H^2 and HgH, and thus $\big[iH^2, H_2\big]$ is more likely to be positive as a whole. The formalisation of this intuition is the content of the next lemma.

Lemma 3.4 (Strict Mourre estimate for H^2)**.** *Let f satisfy Assumption 3.2 and let J be a bounded Borel set in $(0,\infty)$. Then,*

$$E^{H^2}(J)\big[iH^2, H_2\big]E^{H^2}(J) \ge aE^{H^2}(J) \quad \textit{with} \quad a := 2\delta_g \cdot \inf(J) > 0.$$

Proof. We know from Equation (2.2) and Lemma 3.3(c) that

$$E^{H^2}(J)\big[iH^2, H_2\big]E^{H^2}(J) = E^{H^2}(J)\big(H^2 g + 2HgH + gH^2\big)E^{H^2}(J).$$

We also know from Assumption 3.2(iii) that

$$E^{H^2}(J) 2HgH E^{H^2}(J) \ge aE^{H^2}(J) \quad \text{with} \quad a = 2\delta_g \cdot \inf(J) > 0.$$

Therefore, it is sufficient to show that $E^{H^2}(J)\big(H^2 g + gH^2\big)E^{H^2}(J) \ge 0$.

So, for any $\varepsilon > 0$ let $H^2_\varepsilon := H^2\big(\varepsilon^2 H^2 + 1\big)^{-1}$ and $H^\pm_\varepsilon := H(\varepsilon H \pm i)^{-1}$. Then, the inclusion $g^{1/2} \in C^1(H)$ of Lemma 3.3(a) implies that

$$\text{s-}\lim_{\varepsilon\searrow 0}\big[H^\pm_\varepsilon, g^{1/2}\big] = \pm\,\text{s-}\lim_{\varepsilon\searrow 0}(\varepsilon H \pm i)^{-1}\big[iH, g^{1/2}\big](\varepsilon H \pm i)^{-1} = \pm i\big[g^{1/2}, H\big].$$

Therefore, for each $\varphi \in \mathcal{H}$ it follows that

$$\begin{aligned}
&\big\langle \varphi, E^{H^2}(J)\big(H^2 g + gH^2\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H^2_\varepsilon g^{1/2}g^{1/2} + g^{1/2}g^{1/2}H^2_\varepsilon\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(\big[H^2_\varepsilon, g^{1/2}\big]g^{1/2} + 2\,g^{1/2}H^2_\varepsilon g^{1/2} + g^{1/2}\big[g^{1/2}, H^2_\varepsilon\big]\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&\ge \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(\big[H^2_\varepsilon, g^{1/2}\big]g^{1/2} + g^{1/2}\big[g^{1/2}, H^2_\varepsilon\big]\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H^+_\varepsilon\big[H^-_\varepsilon, g^{1/2}\big]g^{1/2} + \big[H^+_\varepsilon, g^{1/2}\big]H^-_\varepsilon g^{1/2}\\
&\qquad\qquad + g^{1/2}\big[g^{1/2}, H^+_\varepsilon\big]H^-_\varepsilon + g^{1/2}H^+_\varepsilon\big[g^{1/2}, H^-_\varepsilon\big]\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H\big[H, g^{1/2}\big]g^{1/2} + \big[H^+_\varepsilon, g^{1/2}\big]g^{1/2}H^-_\varepsilon + \big[H^+_\varepsilon, g^{1/2}\big]\big[H^-_\varepsilon, g^{1/2}\big]\\
&\qquad\qquad + g^{1/2}\big[g^{1/2}, H\big]H + H^+_\varepsilon g^{1/2}\big[g^{1/2}, H^-_\varepsilon\big] + \big[g^{1/2}, H^+_\varepsilon\big]\big[g^{1/2}, H^-_\varepsilon\big]\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \big\langle \varphi, E^{H^2}(J)\big(H\big[H, g^{1/2}\big]g^{1/2} + \big[H, g^{1/2}\big]g^{1/2}H + 2\big[H, g^{1/2}\big]^2 + g^{1/2}\big[g^{1/2}, H\big]H\\
&\qquad\qquad + Hg^{1/2}\big[g^{1/2}, H\big]\big)E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&= \big\langle \varphi, E^{H^2}(J)2\big[H, g^{1/2}\big]^2 E^{H^2}(J)\varphi\big\rangle_{\mathcal{H}}\\
&\ge 0,
\end{aligned}$$

which implies the claim. $\square$

Using the previous results for H^2, one can finally determine spectral properties of H :

Theorem 3.5 (Spectral properties of H)**.** *Let f satisfy Assumption 3.2. Then, H has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue.*

Proof. We know from Lemmas 3.3(d) and 3.4 that $\big(H^2+1\big)^{-1} \in C^2(H_2)$ and that H^2 satisfies a strict Mourre estimate on each bounded Borel subset of $(0,\infty)$. It follows by Theorem 2.1 that H^2 has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue. Accordingly, the Hilbert space $\mathcal{H}$ admits the orthogonal decomposition

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\rm ac}(H^2),$$

with $\mathcal{H}_{\rm ac}(H^2)$ the subspace of absolute continuity of H^2.

Now, the function $\lambda \mapsto \lambda^2$ has the Luzin N property on $\mathbb{R}$; namely, if J is a Borel subset of $\mathbb{R}$ with Lebesgue measure zero, then J^2 also has Lebesgue measure zero. It follows that $\mathcal{H}_{\rm ac}(H^2) \subset \mathcal{H}_{\rm ac}(H)$, with $\mathcal{H}_{\rm ac}(H)$ the subspace of absolute continuity of H (see Proposition 29, Section 3.5.4 of [5]). Since $\ker(A^2) = \ker(A)$ for all self-adjoint operators A, we thus infer that

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\rm ac}(H^2) \subset \ker(H) \oplus \mathcal{H}_{\rm ac}(H).$$

So, one necessarily has $\mathcal{H} = \ker(H) \oplus \mathcal{H}_{\rm ac}(H)$, meaning that H has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue. $\square$

4 Spectral analysis of time changes of horocycle flows

In this section, we apply the results of Section 3 to time changes of horocycle flows on finite volume surfaces of constant negative curvature.

Let Σ be a finite volume Riemann surface of genus ≥ 2 and let $M := T^1\Sigma$ be the unit tangent bundle of Σ. The 3-manifold M carries a probability measure μ_Ω (induced by a canonical volume form Ω) which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ and the geodesic flow $\{F_{2,t}\}_{t\in\mathbb{R}}$. Both flows correspond to right translations on M when M is identified with a homogeneous space $\Gamma \setminus \mathsf{PSL}(2;\mathbb{R})$, for some lattice Γ in $\mathsf{PSL}(2;\mathbb{R})$ (see [6, Sec. II.3 & Sec. IV.1] for details). We denote by $\{U_1(t)\}_{t\in\mathbb{R}}$ and $\{U_2(t)\}_{t\in\mathbb{R}}$ the corresponding unitary groups in $\mathcal{H} := \mathsf{L}^2(M,\mu_\Omega)$, and we write X_j (resp. H_j) for the vector field (resp. self-adjoint generator) associated to $\{U_j(t)\}_{t\in\mathbb{R}}$, $j=1,2$ (see Section 3). It is a classical result that the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is strongly mixing (and hence ergodic) [15] and mixing of all orders [19], and that $U_1(t)$ has countable Lebesgue spectrum for each $t\neq 0$ (see [17, Prop. 2.2] and [21]). Moreover, the identity (3.1) holds with $e:\mathbb{R}\to(0,\infty)$ the exponential, *i.e.*

$$U_2(s)\,U_1(t)\,U_2(-s) = U_1(\mathrm{e}^s\,t) \quad \text{for all } s,t\in\mathbb{R}$$

(here we consider the negative horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}} \equiv \{F^-_{1,t}\}_{t\in\mathbb{R}}$, but everything we say can be adapted to the positive horocycle flow by inverting a sign, see [6, Rem. IV.1.2]).

Now, consider a time change fX_1 of X_1 with $f\in C^2(M)$ satisfying Assumption 3.2 (with $e'(0)=1$), let H be the self-adjoint operator as in Lemma 3.1(b), and let $\widetilde{H}$ be the self-adjoint operator associated to fX_1. If M is compact, then Assumption 3.2 reduces to the following:

Assumption 4.1. *The functions $f\in C^2(M)$ and $f-\mathscr{L}_{X_2}(f)\in C^1(M)$ are strictly positive.*

Under Assumption 3.2, the flows $\{F_{1,t}\}_{t\in\mathbb{R}}$, $\{F_{2,t}\}_{t\in\mathbb{R}}$ and the function f satisfy all the hypotheses of Section 3. Therefore, Theorem 3.5 implies that the operator H has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue. Since H and $\widetilde{H}$ are unitarily equivalent, this also holds for the operator $\widetilde{H}$. Now, the flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is ergodic. So, we know from the theory of time changes on probability spaces [22, Sec. 5.1] that the time-changed flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is also ergodic. Therefore, the operator $\widetilde{H}$ has a simple eigenvalue at 0. Putting these information together, one obtains the following result:

Theorem 4.2. *Let f satisfy Assumption 3.2 with $e'(0) = 1$ (or simply Assumption 4.1 if M is compact). Then, the self-adjoint operator $\widetilde{H}$ associated to the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.*

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