Commutator methods for the spectral analysis of uniquely ergodic dynamical systems

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Abstract

We present a method, based on commutator methods, for the spectral analysis of uniquely ergodic dynamical systems. When applicable, it leads to the absolute continuity of the spectrum of the corresponding unitary operators. As an illustration, we consider time changes of horocycle flows, skew products over translations and Furstenberg transformations. For time changes of horocycle flows, we obtain absolute continuity under assumptions weaker than the ones to be found in the literature, and for skew products over translations and Furstenberg transformations, we obtain countable Lebesgue spectrum under assumptions not previously covered in the literature.

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1 Introduction

Commutator methods in the sense of É. Mourre [45] and their various extensions (see for instance [4, 9, 15, 24, 37, 47, 50]) are a very efficient tool for the spectral analysis of self-adjoint operators. They have been fruitfully applied to numerous models in mathematical physics and pure mathematics. Even so, it is only recently that an analogue of the theory has been developed for unitary operators (see [5] for the first article on the topic, and [17] for an optimal formulation of the theory). Accordingly, the absence of general works on commutator methods for the spectral analysis of unitary operators from ergodic theory is not a surprise. Our purpose here is to start filling this gap by presenting an abstract class of uniquely ergodic dynamical systems to which commutator methods apply. The class in question is simple enough to be described in terms of commutators and general enough to contain interesting examples of uniquely ergodic dynamical systems. We hope that the examples treated in this paper, together with the simplicity of our approach, will motivate other works on commutator methods for the spectral analysis of (uniquely ergodic) dynamical systems.

The content of the paper is the following. In Section 2, we recall the needed definitions and results on commutator methods, both for self-adjoint and unitary operators. Then, we exhibit a general class of unitary operators which are shown to have purely absolutely continuous spectrum thanks to commutator methods. Also, we explain why typical examples of such unitary operators are Koopman operators induced by uniquely ergodic transformations. After that, we dedicate the rest of the paper to applications of the theory of Section 2. We consider time changes of horocycle flows in Section 3, skew products over translations in Section 4 and Furstenberg transformations in Section 5.

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In Theorem 3.7 of Section 3, we show that time changes of horocycle flows on compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions. Our result holds for time changes of class C^3, which is the weakest regularity assumption under which this absolute continuity has been established (see the discussion after Theorem 3.7 for a comparison with recent results of G. Forni and C. Ulcigrai [18] and of the author [51]). In Theorem 4.4 of Section 4, we prove that skew products over translations on compact metric abelian Banach Lie groups have countable Lebesgue spectrum in the orthocomplement of functions depending only on the first variable. Our result holds for cocycle functions being differentiable along the flow generated by the translations and with corresponding derivative satisfying a Dini-type condition (see Assumption 4.1 for details). In the case of skew products on tori, this complements previous results of A. Iwanik, M. Lemańzyk and D. Rudolph [33, 35] in one dimension and a previous result of A. Iwanik [34] in higher dimensions (see Corollary 4.5 and the discuss on that follows). Finally, in Theorem 5.3 of Section 5, we show that Furstenberg transformations have countable Lebesgue spectrum in the orthocomplement of functions depending only on the first variable for perturbations with partial derivatives satisfying a Dini-type condition. This complements Corollary 3 of [35], where the same result is shown for perturbations with partial derivatives being of bounded variation.

2 Commutator methods for uniquely ergodic dynamical systems

We present in this section a method, based on commutator methods, for the spectral analysis of uniquely ergodic dynamical systems. We start by recalling some facts on commutator methods borrowed from [4], [17] and [50] (see also the original paper [45] of É. Mourre).

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ linear in the second argument, denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\|\cdot\|$ for the norm on $\mathcal{H}$ and the norm on $\mathscr{B}(\mathcal{H})$. Let also A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and take $S\in\mathscr{B}(\mathcal{H})$. For any $k\in\mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S\in C^k(A)$, if the map

$$\mathbb{R}\ni t\mapsto \mathrm{e}^{-itA}\,S\,\mathrm{e}^{itA}\in\mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k=1$, one has $S\in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A)\ni\varphi\mapsto\big\langle\varphi,iSA\varphi\big\rangle-\big\langle A\varphi,iS\varphi\big\rangle\in\mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[iS,A]$ the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the function (2.1) at $t=0$.

A condition slightly stronger than the inclusion $S\in C^1(A)$ is provided by the following definition: S belongs to $C^{1+0}(A)$, with notation $S\in C^{1+0}(A)$, if $S\in C^1(A)$ and

$$\int_0^1\frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}[A,S]\,\mathrm{e}^{itA}-[A,S]\big\|<\infty.$$

If we regard $C^1(A)$, $C^{1+0}(A)$ and $C^2(A)$ as subspaces of $\mathscr{B}(\mathcal{H})$, then we have the inclusions

$$C^2(A)\subset C^{1+0}(A)\subset C^1(A)\subset\mathscr{B}(\mathcal{H}).$$

Now, if H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H-z)^{-1}\in C^k(A)$ for some $z\in\mathbb{C}\setminus\sigma(H)$. So, H is of class $C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A)\ni\varphi\mapsto\big\langle\varphi,(H-z)^{-1}A\varphi\big\rangle-\big\langle A\varphi,(H-z)^{-1}\varphi\big\rangle\in\mathbb{C}$$

extends continuously to a bounded form defined by the operator $[(H-z)^{-1},A]\in\mathscr{B}(\mathcal{H})$. In such a case, the set $\mathcal{D}(H)\cap\mathcal{D}(A)$ is a core for H and the quadratic form

$$\mathcal{D}(H)\cap\mathcal{D}(A)\ni\varphi\mapsto\big\langle H\varphi,A\varphi\big\rangle-\big\langle A\varphi,H\varphi\big\rangle\in\mathbb{C}$$

is continuous in the topology of $\mathcal{D}(H)$ [4, Thm. 6.2.10(b)]. This form then extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a continuous operator $[H, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. In addition, the following relation holds in $\mathscr{B}(\mathcal{H})$:

$$\big[(H-z)^{-1}, A\big] = -(H-z)^{-1}[H, A](H-z)^{-1}. \tag{2.2}$$

Let $E^H(\,\cdot\,)$ denote the spectral measure of the self-adjoint operator H, and assume that H is of class $C^1(A)$. Then, for each bounded Borel set $J \subset \mathbb{R}$ the operator $E^H(J)[iH, A]E^H(J)$ is bounded and self-adjoint. If there exist a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that

$$E^H(J)[iH, A]E^H(J) \geq aE^H(J) + K, \tag{2.3}$$

then one says that H satisfies a Mourre estimate on J and that A is a conjugate operator for H on J. Also, one says that H satisfies a strict Mourre estimate on J if (2.3) holds with $K = 0$. The main consequence of a strict Mourre estimate is to imply a limiting absorption principle for H on J if H is also of class $C^{1+0}(A)$. This in turns implies that H has no singular spectrum in J. If H only satisfies a Mourre estimate on J, then the same holds up to the possible presence of a finite number of eigenvalues in J, each one of finite multiplicity. We recall here a version of these results (see [4, Sec. 7.1.2] and [50, Thm. 0.1] for more details):

Theorem 2.1. *Let H and A be self-ajoint operators in a Hilbert space $\mathcal{H}$, with H of class $C^{1+0}(A)$. Suppose there exist a bounded Borel set $J \subset \mathbb{R}$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that*

$$E^H(J)[iH, A]E^H(J) \geq aE^H(J) + K. \tag{2.4}$$

Then, H has at most finitely many eigenvalues in J, each one of finite multiplicity, and H has no singularly continuous spectrum in J. Furthermore, if (2.4) *holds with $K = 0$, then H has no singular spectrum in J.*

Similar notations and results exist in the case of a unitary operator $U \in C^1(A)$ with spectral measure $E^U(\,\cdot\,)$ and spectrum $\sigma(U) \subset \mathbb{S}^1 \equiv \{z \in \mathbb{C} \mid |z| = 1\}$. Namely, one says that U satisfies a Mourre estimate on a Borel set $\Theta \subset \mathbb{S}^1$ if there exists a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that

$$E^U(\Theta)U^*[A, U]E^U(\Theta) \geq aE^U(\Theta) + K. \tag{2.5}$$

Also, one says that U satisfies a strict Mourre estimate on Θ if (2.5) holds with $K = 0$. Furthermore, one has the following result on the spectral nature of U (see [17, Thm. 2.7 & Rem. 2.8] for a more general version of this result):

Theorem 2.2. *Let U and A be respectively a unitary and a self-ajoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^{1+0}(A)$. Suppose there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and a compact operator $K \in \mathscr{B}(\mathcal{H})$ such that*

$$E^U(\Theta)U^*[A, U]E^U(\Theta) \geq aE^U(\Theta) + K. \tag{2.6}$$

Then, U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singularly continuous spectrum in Θ. Furthermore, if (2.6) *holds with $K = 0$, then U has no singular spectrum in Θ.*

Remark 2.3. *If $U = \mathrm{e}^{-iH}$ for some bounded self-adjoint operator $H \in C^{1+0}(A)$, then one can use indifferently the self-adjoint or the unitary formulation of commutator methods. Indeed, in such a case one has for each $\varphi \in \mathcal{H}$ that*

$$\big(U^* \mathrm{e}^{itA} U \mathrm{e}^{-itA} - 1\big)\varphi = \mathrm{e}^{iH}\left[\mathrm{e}^{itA}, \mathrm{e}^{-iH}\right]\mathrm{e}^{-itA}\varphi = i\int_0^1 \mathrm{d}s\, \mathrm{e}^{isH}\int_0^t \mathrm{d}u\, \mathrm{e}^{iuA}[iH, A]\,\mathrm{e}^{i(t-u)A}\,\mathrm{e}^{-isH}\varphi$$

which implies that

$$\left\|\frac{U^* \mathrm{e}^{itA} U \mathrm{e}^{-itA} - 1}{t} - i\int_0^1 \mathrm{d}s\, \mathrm{e}^{isH}[iH, A]\,\mathrm{e}^{-isH}\right\| \leq \sup_{u \in [0,t]} \big\|\mathrm{e}^{iuA}[iH, A]\,\mathrm{e}^{i(t-u)A} - [iH, A]\big\|.$$

So, one infers that $U \in C^1(A)$ with $U^*[A,U] = \int_0^1 \mathrm{d}s\, \mathrm{e}^{isH}[iH,A]\,\mathrm{e}^{-isH}$, which in turns implies the inclusion $U \in C^{1+0}(A)$. Moreover, one has for any Borel set $J \subset \mathbb{R}$ the equality

$$E^H(J) = E^U(\Theta) \quad \textit{with} \quad \Theta := \big\{\, \mathrm{e}^{-i\lambda} \in \mathbb{S}^1 \mid \lambda \in J \big\}.$$

Therefore, one obtains the following equivalences of Mourre estimates:

$$\begin{aligned} &E^H(J)[iH,A]E^H(J) \ge aE^H(J) + K \quad \textit{with } J \subset \mathbb{R} \textit{ a bounded Borel set} \\ \iff\ &E^U(\Theta)U^*[A,U]E^U(\Theta) \ge aE^U(\Theta) + \int_0^1 \mathrm{d}s\, \mathrm{e}^{isH}\, K\, \mathrm{e}^{-isH} \quad \textit{with } \Theta := \big\{\, \mathrm{e}^{-i\lambda} \in \mathbb{S}^1 \mid \lambda \in J \big\}. \end{aligned}$$

Now, suppose for a moment that there exists a self-adjoint operator A with domain $\mathcal{D}(A)$ such that $U \in C^1(A)$ and $[A,U] = U$. Then, one has $U \in C^k(A)$ for each $k \in \mathbb{N}$ and $U^*[A,U] = 1$. In particular, $U \in C^{1+0}(A)$ and U satisfies a strict Mourre estimate on all of $\mathbb{S}^1$. Thus, Theorem 2.2 applies and one deduces that U has purely absolutely continuous spectrum. In fact, one can check that the conditions $U \in C^1(A)$ and $[A,U] = U$ imply that $\mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} = \mathrm{e}^{-it}\, U$ for each $t \in \mathbb{R}$. So, the operator U is unitarily equivalent to $\mathrm{e}^{-it}\, U$ for each $t \in \mathbb{R}$, and thus has purely Lebesgue spectrum covering the whole circle $\mathbb{S}^1$. No need of commutator methods whatsoever.

But, now assume that the situation is more general in the sense that we are only able to find a self-adjoint operator A such that $U \in C^1(A)$ and $[A,U] = UF + GU$ for some self-adjoint operators $F, G \in \mathscr{B}(\mathcal{H})$. In this case, no simple trick permits to obtain Lebesgue spectrum (since it would be obviously wrong). Moreover, we only get the relation

$$U^*[A,U] = F + U^*GU$$

which do not lead to any explicit Mourre estimate, unless we impose some positivity condition on the operator $F + U^*GU$. Fortunately, in certain situations, it is sufficient to modify appropriately the operator A in order to get the desired positivity. Let us explain how to proceed. Since $U \in C^1(A)$, we know from standard results (see [4, Prop. 5.1.5-5.1.6]) that $U^k \in C^1(A)$ and $U^k\, \mathcal{D}(A) = \mathcal{D}(A)$ for each $k \in \mathbb{Z}$. Therefore, the operator $\frac{1}{n}\sum_{k=0}^{n-1} U^{-k}\big[A, U^k\big]$ is bounded for each $n \in \mathbb{N}^*$, and the operator

$$A_n\, \varphi := \frac{1}{n}\sum_{k=0}^{n-1} U^{-k} A U^k\, \varphi = \frac{1}{n}\sum_{k=0}^{n-1} U^{-k}\big[A, U^k\big]\varphi + A\varphi, \quad \varphi \in \mathcal{D}(A_n) := \mathcal{D}(A),$$

is self-adjoint. Furthermore, a simple calculation shows that $U \in C^1(A_n)$ with

$$U^*[A_n, U] = U^*\,\frac{1}{n}\sum_{k=0}^{n-1} U^{-k}\,[A,U]\,U^k = \frac{1}{n}\sum_{k=0}^{n-1} U^{-k} F U^k + U^*\left(\frac{1}{n}\sum_{k=0}^{n-1} U^{-k}\, G U^k\right) U \equiv F_n + U^*\, G_n\, U, \tag{2.7}$$

which in turns implies that $U \in C^{1+0}(A_n)$ if the operators F_n and G_n satisfy

$$\int_0^1 \frac{\mathrm{d}t}{t}\, \big\|\, \mathrm{e}^{-itA_n}\, F_n\, \mathrm{e}^{itA_n} - F_n \big\| < \infty \qquad \text{and} \qquad \int_0^1 \frac{\mathrm{d}t}{t}\, \big\|\, \mathrm{e}^{-itA_n}\, G_n\, \mathrm{e}^{itA_n} - G_n \big\| < \infty. \tag{2.8}$$

Now, even if $F + U^*GU$ is not a strictly positive operator, the averaged operator $F_n + U^*\, G_n\, U$ may converge in norm as $n \to \infty$ to a strictly positive operator. In such a case, the r.h.s. of (2.7) would be strictly positive for n big enough. Accordingly, one would obtain a strict Mourre estimate on all of $\mathbb{S}^1$, and thus conclude by Theorem 2.2 that U has purely absolutely continuous spectrum if F_n and G_n satisfy (2.8) (if the operator A_n is bounded, one can skip the verification of (2.8) thanks to a theorem of C. R. Putnam, see [48, Thm. 2.3.2]).

The convergence in norm of the averaged operator $F_n + U^*\, G_n\, U$ is similar to the uniform convergence of Birkhoff sums for uniquely ergodic transformations (it is also similar to the norm convergence of Birkhoff sums for uniquely ergodic automorphisms of C^*-algebras, as defined in [1, Sec. 1]). Therefore, it is quite natural to particularise the previous construction to the case where F and G are multiplication operators and U is a unitary operator generated by a uniquely ergodic transformation. So, let $T : X \to X$ be a uniquely ergodic homeomorphism on a compact metric space X with normalised Haar measure μ, and let U_T be the unitary Koopman operator in $\mathcal{H} := \mathsf{L}^2(X,\mu)$ given by

$$U_T : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto \varphi \circ T.$$

Furthermore, assume that there exists a self-adjoint operator A such that $U_T \in C^1(A)$ and $[A, U_T] = U_T f + g U_T$ for some functions $f, g \in C(X;\mathbb{R})$ (here we identify the functions f and g with the corresponding multiplication operators). Then, (2.7) reduces to

$$(U_T)^*[A_n, U_T] = \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^{-k} + (U_T)^* \left(\frac{1}{n}\sum_{k=0}^{n-1} g \circ T^{-k}\right) U_T \equiv f_n + (U_T)^* g_n U_T. \tag{2.9}$$

Since T is uniquely ergodic, the Birkhoff sums f_n and g_n converge uniformly to $\int_X \mathrm{d}\mu\, f$ and $\int_X \mathrm{d}\mu\, g$, respectively. So, if $\int_X \mathrm{d}\mu\,(f+g) > 0$, then the r.h.s. of (2.9) is strictly positive for n big enough. Accordingly, one obtains a strict Mourre estimate on all of $\mathbb{S}^1$, and one concludes by Theorem 2.2 that U_T has purely absolutely continuous spectrum if f_n and g_n satisfy

$$\int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itA_n} f_n \, \mathrm{e}^{itA_n} - f_n \big\| < \infty \qquad \text{and} \qquad \int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itA_n} g_n \, \mathrm{e}^{itA_n} - g_n \big\| < \infty.$$

Obviously, this last construction can be adapted to various other situations as when one allows a compact perturbation, or when the r.h.s. of the identity $[A, U_T] = U_T f + g U_T$ involves another combination of operators, or when one has a continuous flow of homeomorphisms $\{T_t\}_{t\in\mathbb{R}}$ generating a strongly continuous group of unitary operators $\{U_t\}_{t\in\mathbb{R}}$. In the latter case, it might be more convenient to replace the discrete averages $A_n = \frac{1}{n}\sum_{k=0}^{n-1}(U_T)^{-k} A (U_T)^k$ by the continuous averages $A_L := \frac{1}{L}\int_0^L \mathrm{d}t\, U_{-t} A U_t$ and to study the generator H of the group $\{U_t\}_{t\in\mathbb{R}}$ (instead of the group itself) using the usual self-adjoint formulation of commutators methods.

3 Time changes of horocycle flows

Let Σ be a compact Riemann surface of genus ≥ 2 and let $M := T^1\Sigma$ be the unit tangent bundle of Σ. The compact 3-manifold M carries a probability measure μ_Ω (induced by a canonical volume form Ω) which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ and the geodesic flow $\{F_{2,t}\}_{t\in\mathbb{R}}$. Both flows correspond to right translations on M when M is identified with a homogeneous space $\Gamma \setminus \mathsf{PSL}(2;\mathbb{R})$, for some cocompact lattice Γ in $\mathsf{PSL}(2;\mathbb{R})$ (see [7, Sec. II.3 & Sec. IV.1]). We write $U_j(t)$ ($j = 1, 2$, $t \in \mathbb{R}$) for the operators given by

$$U_j(t)\,\varphi := \varphi \circ F_{j,t}, \quad \varphi \in C(M).$$

One can check that the families $\{U_j(t)\}_{t\in\mathbb{R}}$ define strongly continuous unitary groups in the Hilbert space $\mathcal{H} := \mathsf{L}^2(M, \mu_\Omega)$, and that $U_j(t)C^\infty(M) \subset C^\infty(M)$ for each $t \in \mathbb{R}$. It follows from Nelson's theorem [3, Prop. 5.3] that the generator of the group $\{U_j(t)\}_{t\in\mathbb{R}}$

$$H_j\,\varphi := \text{s-}\lim_{t\to 0} it^{-1}\big(U_j(t) - 1\big)\varphi, \quad \varphi \in \mathcal{D}(H_j) := \Big\{\varphi \in \mathcal{H} \mid \lim_{t\to 0} |t|^{-1}\big\|\big(U_j(t) - 1\big)\varphi\big\| < \infty\Big\},$$

is essentially self-adjoint on $C^\infty(M)$, and one has

$$H_j\,\varphi := -i\mathscr{L}_{X_j}\varphi, \quad \varphi \in C^\infty(M),$$

with X_j the divergence-free vector field associated to $\{F_{j,t}\}_{t\in\mathbb{R}}$ and $\mathscr{L}_{X_j}$ the corresponding Lie derivative.

It is a classical result that the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic [23] and mixing of all orders [43], and that $U_1(t)$ has countable Lebesgue spectrum for each $t \neq 0$ (see [38, Prop. 2.2] and [46]). Moreover, the groups $\{U_1(t)\}_{t\in\mathbb{R}}$ and $\{U_2(t)\}_{t\in\mathbb{R}}$ satisfy the commutation relation

$$U_2(s)\,U_1(t)\,U_2(-s) = U_1(\mathrm{e}^s\, t), \quad s, t \in \mathbb{R}, \tag{3.1}$$

(here we consider the negative horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}} \equiv \{F^-_{1,t}\}_{t\in\mathbb{R}}$, but everything we say can be adapted to the positive horocycle flow by inverting a sign, see [7, Rem. IV.1.2]). By applying the strong derivative $i\,\mathrm{d}/\mathrm{d}t$ at $t = 0$ in (3.1), one gets that $U_2(s)H_1U_2(-s)\varphi = \mathrm{e}^s\, H_1\varphi$ for each $\varphi \in C^\infty(M)$. Since $C^\infty(M)$ is a core for H_1, one infers that H_1 is H_2-homogeneous in the sense of [8]; namely,

$$U_2(s)H_1U_2(-s) = \mathrm{e}^s\, H_1 \quad \text{on} \quad \mathcal{D}(H_1). \tag{3.2}$$

It follows that H_1 is of class $C^\infty(H_2)$ with

$$\big[iH_1, H_2\big] = H_1. \tag{3.3}$$

Now, consider a C^1 vector field with the same orientation and proportional to X_1, that is, a vector field fX_1 with $f \in C^1\big(M;(0,\infty)\big)$. The vector field fX_1 has the same integral curves as X_1, but with reparametrised time coordinate. Indeed, it is known (see [32, Sec. 1]) that the formula

$$t = \int_0^{h(p,t)} \frac{\mathrm{d}s}{f\big(F_{1,s}(p)\big)}\,, \quad t \in \mathbb{R},\ p \in M,$$

defines for each $p \in M$ a strictly increasing function $\mathbb{R} \ni t \mapsto h(p,t) \in \mathbb{R}$ satisfying $h(p,0) = 0$ and $\lim_{t\to\pm\infty} h(p,t) = \pm\infty$. Furthermore, the implicit function theorem implies that the map $t \mapsto h(p,t)$ is C^1 with $\frac{\mathrm{d}}{\mathrm{d}t}h(p,t) = f\big(F_{1,h(p,t)}(p)\big)$. Therefore, the function $\mathbb{R} \ni t \mapsto \widetilde{F}_{1,t}(p) \in M$ given by $\widetilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$ satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{F}_1(p,t) = (fX_1)_{\widetilde{F}_1(p,t)}\,, \quad \widetilde{F}_1(p,0) = p,$$

meaning that $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is the flow of fX_1 (note that $\widetilde{F}_{1,t}(p)$ is of class C^1 in the p-variable and of class C^2 in the t-variable as predicted by the general theory [2, Sec. 2.1]). Since the divergence $\mathrm{div}_{\Omega/f}(fX_1)$ of fX_1 with respect to the volume form Ω/f is zero, the operators

$$\widetilde{U}_1(t)\varphi := \varphi \circ \widetilde{F}_{1,t}\,, \quad \varphi \in C(M),$$

define a strongly continuous unitary group $\{\widetilde{U}_1(t)\}_{t\in\mathbb{R}}$ in the Hilbert space $\widetilde{\mathcal{H}} := \mathsf{L}^2(M, \mu_\Omega/f)$. The generator $\widetilde{H} := -i\mathscr{L}_{fX_1}$ of $\{\widetilde{U}_1(t)\}_{t\in\mathbb{R}}$ is essentially self-adjoint on $C^1(M) \subset \widetilde{\mathcal{H}}$ due to Nelson's theorem.

In the following lemma, we introduce two auxiliary operators which will be useful for the spectral analysis of $\widetilde{H}$.

Lemma 3.1. *Let $f \in C^1\big(M;(0,\infty)\big)$, then*

(a) the operator

$$\mathscr{U} : \mathcal{H} \to \widetilde{\mathcal{H}}, \quad \varphi \mapsto f^{1/2}\varphi,$$

is unitary, with adjoint $\mathscr{U}^* : \widetilde{\mathcal{H}} \to \mathcal{H}$ given by $\mathscr{U}^*\psi = f^{-1/2}\psi$,

(b) the symmetric operator

$$H\varphi := f^{1/2}H_1f^{1/2}\varphi, \quad \varphi \in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$, and the closure of H (which we denote by the same symbol) is unitarily equivalent to $\widetilde{H}$,

(c) for each $z \in \mathbb{C}\setminus\mathbb{R}$, the operator $H_1 + zf^{-1}$ is invertible with bounded inverse, and satisfies

$$(H+z)^{-1} = f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}f^{-1/2}. \tag{3.4}$$

Proof. Point (a) follows from a direct calculation taking into account the boundedness of f from below and from above. For (b), observe that

$$H\varphi = f^{-1/2} f H_1 f^{1/2}\varphi = \mathscr{U}^* \widetilde{H}\mathscr{U}\varphi$$

for each $\varphi \in \mathscr{U}^* C^1(M)$. So, H is essentially self-adjoint on $\mathscr{U}^* C^1(M) \equiv C^1(M)$, and the closure of H is unitarily equivalent to $\widetilde{H}$. To prove (c), take $z \equiv \lambda + i\mu \in \mathbb{C} \setminus \mathbb{R}$, $\varphi \in \mathcal{D}\big(H_1 + zf^{-1}\big) \equiv \mathcal{D}(H_1)$ and $\{\varphi_n\} \subset C^\infty(M)$ such that $\lim_n \|\varphi - \varphi_n\|_{\mathcal{D}(H_1)} = 0$. Then, it follows from (b) that

$$\big\|\big(H_1 + zf^{-1}\big)\varphi\big\|^2 = \lim_n \big\|f^{-1/2}(H+z)f^{-1/2}\varphi_n\big\|^2 \ge \inf_{p\in M} f^{-2}(p)\,\mu^2\big\|\varphi\big\|^2,$$

and thus $H_1 + zf^{-1}$ is invertible with bounded inverse (see [3, Lemma 3.1]). Now, to show (3.4), take $\psi = (H+z)\zeta$ with $\zeta \in C^1(M)$, observe that

$$(H+z)^{-1}\psi - f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1} f^{-1/2}\psi = 0, \tag{3.5}$$

and then use the density of $(H+z)C^1(M)$ in $\mathcal{H}$ to extend the identity (3.5) to all of $\mathcal{H}$. □

The operators H and $\widetilde{H}$ are unitarily equivalent due to Lemma 3.1(b). Therefore, one can either work with H in $\mathcal{H}$ or with $\widetilde{H}$ in $\widetilde{\mathcal{H}}$ to determine the spectral properties associated with the time change fX_1. For convenience, we present our results for the operator H. We start by proving some regularity properties of f and H with respect to H_2. The function

$$g := \frac{1}{2} - \frac{1}{2}\mathscr{L}_{X_2}\big(\ln(f)\big)$$

pops up naturally:

Lemma 3.2. *Let $f \in C^1\big(M;(0,\infty)\big)$, $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Then,*

(a) the multiplication operator f^α satisfies $f^\alpha \in C^1(H_2)$ with $\big[if^\alpha, H_2\big] = -\alpha\, f^\alpha \mathscr{L}_{X_2}\big(\ln(f)\big)$,

(b) $(H+z)^{-1} \in C^1(H_2)$ with $\big[i(H+z)^{-1}, H_2\big] = -(H+z)^{-1}(Hg+gH)(H+z)^{-1}$.

Proof. (a) The chain rule for Lie derivatives and the strict positivity of f imply that

$$\mathscr{L}_{X_2}(f^\alpha) = \alpha\, f^{\alpha-1}\mathscr{L}_{X_2}(f) = \alpha\, f^\alpha \mathscr{L}_{X_2}\big(\ln(f)\big).$$

Thus, one has for each $\varphi \in C^\infty(M)$

$$\big\langle \varphi, if^\alpha H_2\varphi\big\rangle - \big\langle H_2\varphi, if^\alpha\varphi\big\rangle = \big\langle \varphi, \big[if^\alpha, H_2\big]\varphi\big\rangle = \big\langle \varphi, -\alpha\, f^\alpha \mathscr{L}_{X_2}\big(\ln(f)\big)\varphi\big\rangle.$$

Since $f^\alpha \mathscr{L}_{X_2}\big(\ln(f)\big) \in \mathsf{L}^\infty(M)$, it follows by the density of $C^\infty(M)$ in $\mathcal{D}(H_2)$ that $f^\alpha \in C^1(H_2)$ with $\big[if^\alpha, H_2\big] = -\alpha\, f^\alpha \mathscr{L}_{X_2}\big(\ln(f)\big)$.

(b) Let $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}$. Then, one infers from Equations (3.2) and (3.4) that

$$\mathrm{e}^{-itH_2}(H+z)^{-1}\,\mathrm{e}^{itH_2}\,\varphi = \mathrm{e}^{-itH_2}\, f^{-1/2}\,\mathrm{e}^{itH_2}\,\big(\mathrm{e}^t\, H_1 + z\,\mathrm{e}^{-itH_2}\, f^{-1}\,\mathrm{e}^{itH_2}\big)^{-1}\,\mathrm{e}^{-itH_2}\, f^{-1/2}\,\mathrm{e}^{itH_2}\,\varphi.$$

So, one gets from point (a), Equation (3.4) and Lemma 3.1(b) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}(H+z)^{-1}\,\mathrm{e}^{itH_2}\,\varphi\Big|_{t=0}\\
&= \big[if^{-1/2}, H_2\big]\big(H_1 + zf^{-1}\big)^{-1} f^{-1/2}\varphi + f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}\big[if^{-1/2}, H_2\big]\varphi\\
&\qquad - f^{-1/2}\big(H_1 + zf^{-1}\big)^{-1}\big(H_1 + z\big[if^{-1}, H_2\big]\big)\big(H_1 + zf^{-1}\big)^{-1} f^{-1/2}\varphi\\
&= \frac{1}{2}\mathscr{L}_{X_2}\big(\ln(f)\big)(H+z)^{-1}\varphi + \frac{1}{2}(H+z)^{-1}\mathscr{L}_{X_2}\big(\ln(f)\big)\varphi\\
&\qquad - (H+z)^{-1}\big\{H + z\mathscr{L}_{X_2}\big(\ln(f)\big)\big\}(H+z)^{-1}\varphi\\
&= \frac{1}{2}(H+z)^{-1}H\mathscr{L}_{X_2}\big(\ln(f)\big)(H+z)^{-1}\varphi + \frac{1}{2}(H+z)^{-1}\mathscr{L}_{X_2}\big(\ln(f)\big)H(H+z)^{-1}\varphi\\
&\qquad - (H+z)^{-1}H(H+z)^{-1}\varphi\\
&= -(H+z)^{-1}(Hg+gH)(H+z)^{-1}\varphi,
\end{aligned}$$

which implies the claim. $\square$

In [51] we used the operator H_2 as a conjugate operator for H (in fact for H^2). This led us to impose, as A. G. Kushnirenko in [40, Thm. 2], the strict positivity of the function g in order to get at some point a strict Mourre estimate. Here, we will show that this can be avoided if one uses a conjugate operator taking into account the unique ergodicity of the horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$, as presented in Section 2. We start with the definition of the new conjugate operator. We use for $L>0$ the notations g_L and $\widetilde{g_L}$ for the following averages of g along the time-changed flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$:

$$g_L := \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g\circ \widetilde{F}_{1,-t}\big) \qquad \text{and} \qquad \widetilde{g_L} := \frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\,\big(g\circ \widetilde{F}_{1,-s}\big).$$

Lemma 3.3 (Conjugate operator)**.** *Let $f\in C^1\big(M;(0,\infty)\big)$ and $L>0$.*

(a) For each $\varphi\in C^1(M)$, one has the equality

$$\frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH}\,\varphi = -i\big(\mathscr{L}_X+\tfrac{1}{2}\operatorname{div}_\Omega X\big)\varphi,$$

with $X:=X_2+2\widetilde{g_L}fX_1$ and $\operatorname{div}_\Omega X = 2\widetilde{g_L}\mathscr{L}_{X_1}(f)+2(g-g_L)$ the divergence of X relative to the volume form Ω.

(b) If $f\in C^3\big(M;(0,\infty)\big)$, then the operator

$$A_L\varphi := \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH}\,\varphi,\quad \varphi\in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$ (and its closure is denoted by the same symbol).

Proof. (a) We start by collecting some information on the function $\widetilde{g_L}$. For each $s\in\mathbb{R}$, we have

$$\mathrm{e}^{isH}\,g\,\mathrm{e}^{-isH} = \mathrm{e}^{is\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\,g\,\mathrm{e}^{-is\,\mathscr{U}^*\widetilde{H}\mathscr{U}} = \mathscr{U}^*\,\mathrm{e}^{is\widetilde{H}}\,g\,\mathrm{e}^{-is\widetilde{H}}\,\mathscr{U} = g\circ\widetilde{F}_{1,-s}\,.$$

Thus,

$$\frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\ \mathrm{e}^{isH}\,g\,\mathrm{e}^{-isH} = \frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\,\big(g\circ\widetilde{F}_{1,-s}\big) = \widetilde{g_L}$$

and

$$\mathscr{L}_{fX_1}(\widetilde{g_L}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\,\big(g\circ\widetilde{F}_{1,-s}\circ\widetilde{F}_{1,\tau}\big)\Big|_{\tau=0} = \frac{1}{L}\int_0^L \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\tau-t}^{\tau}\mathrm{d}u\,\big(g\circ\widetilde{F}_{1,u}\big)\Big|_{\tau=0} = g-g_L\,. \tag{3.6}$$

This implies that $H\widetilde{g_L}\varphi\in\mathcal{H}$ for each $\varphi\in C^1(M)$ since

$$H\widetilde{g_L}\varphi = \widetilde{g_L}H\varphi+\big[H,\widetilde{g_L}\big]\varphi = \widetilde{g_L}H\varphi - i\mathscr{L}_{fX_1}(\widetilde{g_L})\varphi = \widetilde{g_L}H\varphi - i(g-g_L)\,\varphi. \tag{3.7}$$

Now, take $\varphi\in C^1(M)$ and $\psi\in\mathcal{D}(H)$, and set $H_2(\tau):=(i\tau)^{-1}\big(\mathrm{e}^{i\tau H_2}-1\big)$ for each $\tau\in\mathbb{R}$. Then, one has the equalities

$$\begin{aligned}
&\Big\langle\psi,\Big(\frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2(\tau)\,\mathrm{e}^{-itH}-H_2(\tau)\Big)\varphi\Big\rangle\\
&=\Big\langle(H-i)\,\psi,\frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{isH}(H+i)^{-1}H_2(\tau)(H-i)^{-1}\,\mathrm{e}^{-isH}(H-i)\,\varphi\Big\rangle\\
&=\Big\langle(H-i)\,\psi,\frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\ \mathrm{e}^{isH}(H+i)^{-1}\big[iH,H_2(\tau)\big](H-i)^{-1}\,\mathrm{e}^{-isH}(H-i)\,\varphi\Big\rangle\\
&=\Big\langle(H-i)\,\psi,-\frac{1}{L}\int_0^L \mathrm{d}t\int_0^t \mathrm{d}s\ \mathrm{e}^{isH}(H-i)(H+i)^{-1}\big[i(H-i)^{-1},H_2(\tau)\big]\,\mathrm{e}^{-isH}(H-i)\,\varphi\Big\rangle.
\end{aligned}$$

But, we know from Lemma 3.2(b) that $\text{s-}\lim_{\tau\searrow 0}\big[i(H-i)^{-1},H_2(\tau)\big]=-(H-i)^{-1}(Hg+gH)(H-i)^{-1}$ and we know from (3.7) that $H\widetilde{g_L}\,\varphi\in\mathcal H$. So, one obtains that

$$
\begin{aligned}
&\left\langle\psi,\frac1L\int_0^L\mathrm dt\;\mathrm e^{itH}\,H_2\,\mathrm e^{-itH}\,\varphi-H_2\,\varphi\right\rangle\\
&=\left\langle(H-i)\,\psi,\frac1L\int_0^L\mathrm dt\int_0^t\mathrm ds\;\mathrm e^{isH}(H+i)^{-1}(Hg+gH)(H-i)^{-1}\,\mathrm e^{-isH}(H-i)\,\varphi\right\rangle\\
&=\big\langle(H-i)\,\psi,(H+i)^{-1}\big(H\widetilde{g_L}+\widetilde{g_L}H\big)(H-i)^{-1}(H-i)\,\varphi\big\rangle\\
&=\big\langle\psi,\big(H\widetilde{g_L}+\widetilde{g_L}H\big)\varphi\big\rangle,
\end{aligned}
$$

which implies the equality

$$
\frac1L\int_0^L\mathrm dt\;\mathrm e^{itH}\,H_2\,\mathrm e^{-itH}\,\varphi=H_2\,\varphi+\big(H\widetilde{g_L}+\widetilde{g_L}H\big)\varphi,
$$

due to the density of $\mathcal D(H)$ in $\mathcal H$. Now, the equations $\operatorname{div}_\Omega(X_1)=\operatorname{div}_\Omega(X_2)=0$ and (3.6) imply that

$$
\operatorname{div}_\Omega X=\operatorname{div}_\Omega X_2+\operatorname{div}_\Omega(2\widetilde{g_L}fX_1)=2\widetilde{g_L}\mathscr L_{X_1}(f)+2f\mathscr L_{X_1}(\widetilde{g_L})=2\widetilde{g_L}\mathscr L_{X_1}(f)+2(g-g_L).
$$

So, one infers that

$$
\begin{aligned}
\frac1L\int_0^L\mathrm dt\;\mathrm e^{itH}\,H_2\,\mathrm e^{-itH}\,\varphi&=\big(-i\mathscr L_{X_2}+2\widetilde{g_L}H+\big[H,\widetilde{g_L}\big]\big)\varphi\\
&=\big(-i\mathscr L_{X_2}+2\widetilde{g_L}f^{1/2}H_1f^{1/2}-if\mathscr L_{X_1}(\widetilde{g_L})\big)\varphi\\
&=-i\big(\mathscr L_{X_2}+2\widetilde{g_L}f\mathscr L_{X_1}+\widetilde{g_L}\,\mathscr L_{X_1}(f)+f\mathscr L_{X_1}(\widetilde{g_L})\big)\varphi\\
&=-i\big(\mathscr L_X+\tfrac12\operatorname{div}_\Omega X\big)\varphi,
\end{aligned}
$$

which proves the claim.

(b) If $f\in C^3\big(M;(0,\infty)\big)$, then X is a C^2 vector field on the compact manifold M , and thus X admits a complete flow $\{F_t\}_{t\in\mathbb R}$ with $F_t(p)$ of class C^2 in the p-variable [2, Sec. 2.1]. For each $t\in\mathbb R$, let $\det_\Omega(F_t)\in C^1(M;\mathbb R)$ be the unique function satisfying $F_t^*\Omega=\det_\Omega(F_t)\Omega$ [2, Def. 2.5.18]. Since F_0 is the identity map, we have $\det_\Omega(F_0)=1$ and thus $\det_\Omega(F_t)>0$ for all $t\in\mathbb R$ by continuity of $F_t(p)$ in the t-variable (see [2, Prop. 2.5.19 & 2.5.20(ii)]). In particular, one can define for each $t\in\mathbb R$ the operator

$$
U(t)\,\varphi:=\{\det_\Omega(F_t)\}^{1/2}\,\varphi\circ F_t\,,\quad \varphi\in C(M).
$$

Some routine computations using [2, Prop. 2.5.20] show that $\{U(t)\}_{t\in\mathbb R}$ defines a strongly continuous unitary group in $\mathcal H$ satisfying $U(t)C^1(M)\subset C^1(M)$ for each $t\in\mathbb R$. Thus, it follows from Nelson's theorem that the generator D of the group $\{U(t)\}_{t\in\mathbb R}$ is essentially self-adjoint on $C^1(M)$. Furthermore, standard computations (see [2, Sec. 5.4]) show that

$$
D\,\varphi=-i\big(\mathscr L_X+\tfrac12\operatorname{div}_\Omega X\big)\varphi
$$

for each $\varphi\in C^1(M)$. This, together with point (a), shows that the operators D and A_L coincide on $C^1(M)$, and thus that A_L is essentially self-adjoint on $C^1(M)$. □

Remark 3.4. *We believe it might be possible to prove the essential self-adjointness of the operator A_L for time changes of class C^2, instead of time changes of class C^3 as presented in Lemma 3.3. Doing so, one would extend all the results of this section to time changes of class C^2, since Lemma 3.3 is the only instance where a regularity assumption stronger than C^2 is needed.*

We now prove regularity properties of H and H^2 with respect to A_L.

Lemma 3.5. *Let $f\in C^3\big(M;(0,\infty)\big)$, $L>0$ and $z\in\mathbb C\setminus\mathbb R$. Then,* *(a)* $(H+z)^{-1} \in C^1(A_L)$ *with*

$$\big[i(H+z)^{-1}, A_L\big] = -(H+z)^{-1}\big(Hg_L + g_L H\big)(H+z)^{-1},$$

(b) $(H^2+1)^{-1} \in C^1(A_L)$ *with*

$$\big[i(H^2+1)^{-1}, A_L\big] = -(H^2+1)^{-1}\big(H^2 g_L + 2Hg_L H + g_L H^2\big)(H^2+1)^{-1},$$

(c) the multiplication operator g_L *satisfies* $g_L \in C^1(A_L)$ *with* $\big[ig_L, A_L\big] = -\mathscr{L}_X(g_L)$*,*

(d) $(H^2+1)^{-1} \in C^2(A_L)$*.*

Proof. (a) Let $\varphi \in C^1(M)$. Then, Lemma 3.2(b) implies that

$$\begin{aligned}
&\big\langle \varphi, i(H+z)^{-1}A_L\varphi\big\rangle - \big\langle A_L\varphi, i(H+z)^{-1}\varphi\big\rangle \\
&= -\frac{1}{L}\int_0^L \mathrm{d}t \,\big\langle \mathrm{e}^{-itH}\varphi, (H+z)^{-1}(Hg+gH)(H+z)^{-1}\,\mathrm{e}^{-itH}\varphi\big\rangle \\
&= -\big\langle \varphi, (H+z)^{-1}\big\{H\big(\tfrac{1}{L}\textstyle\int_0^L \mathrm{d}t\, \mathrm{e}^{itH} g\,\mathrm{e}^{-itH}\big) + \big(\tfrac{1}{L}\textstyle\int_0^L \mathrm{d}t\,\mathrm{e}^{itH} g\,\mathrm{e}^{-itH}\big)H\big\}(H+z)^{-1}\,\mathrm{e}^{-itH}\varphi\big\rangle.
\end{aligned}$$

Since $\frac{1}{L}\int_0^L \mathrm{d}t\, \mathrm{e}^{itH} g\,\mathrm{e}^{-itH} = \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g \circ \widetilde{F}_{1,-t}\big) = g_L$, it follows that

$$\big\langle \varphi, i(H+z)^{-1}A_L\varphi\big\rangle - \big\langle A_L\varphi, i(H+z)^{-1}\varphi\big\rangle = -\big\langle \varphi, (H+z)^{-1}\big(Hg_L + g_L H\big)(H+z)^{-1}\varphi\big\rangle,$$

and one concludes using the density of $C^1(M)$ in $\mathcal{D}(A_L)$.

(b) Let $\varphi \in \mathcal{H}$. Then, it follows from point (a) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}(H^2+1)^{-1}\,\mathrm{e}^{itH_2}\varphi\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itH_2}(H+i)^{-1}\,\mathrm{e}^{itH_2}\,\mathrm{e}^{-itH_2}(H-i)^{-1}\,\mathrm{e}^{itH_2}\varphi\Big|_{t=0} \\
&= -(H+i)^{-1}\big(Hg_L+g_LH\big)(H+i)^{-1}(H-i)^{-1}\varphi - (H+i)^{-1}(H-i)^{-1}\big(Hg_L+g_LH\big)(H-i)^{-1}\varphi \\
&= -(H^2+1)^{-1}\big(H^2g_L + 2Hg_LH + g_LH^2\big)(H^2+1)^{-1}\varphi,
\end{aligned}$$

which implies the claim.

(c) Let $\varphi \in C^1(M)$, then we know from Lemma 3.3 that

$$\big\langle \varphi, ig_LA_L\varphi\big\rangle - \big\langle A_L\varphi, ig_L\varphi\big\rangle = \big\langle \varphi, \big[g_L, \mathscr{L}_X + \tfrac{1}{2}\operatorname{div}_\Omega X\big]\varphi\big\rangle = \big\langle \varphi, -\mathscr{L}_X(g_L)\varphi\big\rangle.$$

Since $\mathscr{L}_X(g_L) \in \mathsf{L}^\infty(M)$, it follows by the density of $C^1(M)$ in $\mathcal{D}(A_L)$ that $g_L \in C^1(A_L)$ with $[ig_L, A_L] = -\mathscr{L}_X(g_L)$.

(d) Direct computations using point (b) show that

$$\begin{aligned}
&\big[i(H^2+1)^{-1}, A_L\big] \\
&= -(H^2+1)^{-1}\big\{(H^2+1)g_L + 2(H+i)g_L(H-i) \\
&\qquad\qquad + 2i(H+i)g_L - 2ig_L(H-i) + g_L(H^2+1)\big\}(H^2+1)^{-1} \\
&= -2\,\mathsf{Re}\,\big\{g_L(H^2+1)^{-1} + 2i(H-i)^{-1}g_L(H^2+1)^{-1} + (H-i)^{-1}g_L(H+i)^{-1}\big\}.
\end{aligned}$$

Morevover, we know from points (a)-(c) that the operators g_L, $(H^2+1)^{-1}$, $(H+i)^{-1}$ and $(H-i)^{-1}$ belong to $C^1(A_L)$. So, one infers from standard results on the space $C^1(A_L)$ (see [4, Prop. 5.1.5]) that $\big[i(H^2+1)^{-1}, A_L\big]$ also belongs to $C^1(A_L)$. ☐

In order to apply the theory of Section 2, one has to prove at some point a positive commutator estimate. If the function f were the constant function $f \equiv 1$, then one would have the equalities $H = H_1$,

$$A_L = \frac{1}{L}\int_0^L \mathrm{d}t\; \mathrm{e}^{itH_1}\, H_2\, \mathrm{e}^{-itH_1} = H_2 + \frac{1}{L}\int_0^L \mathrm{d}t\, tH_1 = H_2 + \frac{LH_1}{2}\,,$$

and

$$\big[iH^2, A_L\big] = \left[iH_1^2, H_2 + \frac{LH_1}{2}\right] = 2H_1^2$$

due to (3.3). Therefore, one would immediately obtain a strict Mourre estimate for $H^2 \equiv H_1^2$. This suggests to study the positivity of the commutator $\big[iH^2, A_L\big]$ also in the case $f \not\equiv 1$. A glimpse at Lemma 3.5(b) tells us that $\big[iH^2, A_L\big]$ is equal to the operator $H^2 g_L + 2Hg_LH + g_LH^2$, which does not seem to exhibit any explicit positivity. However, if the function g_L were positive, then all the operators g_L, H^2 and Hg_LH would be positive, and thus the sum $H^2g_L + 2Hg_LH + g_LH^2$ would be more likely to be positive as a whole. In fact, this is exactly what happens and this was the whole point of choosing the conjugate operator A_L as we did. Thanks to the unique ergodicity of the horocycle flow, one has $g_L > 0$ if $L > 0$ is big enough and the operator H^2 satisfies a strict Mourre estimate with respect to A_L:

Lemma 3.6 (Strict Mourre estimate for H^2)**.** *Let $f \in C^3\big(M;(0,\infty)\big)$ and take $L > 0$ big enough. Then, $g_L > 0$ and one has for each bounded Borel set $J \subset (0,\infty)$ that*

$$E^{H^2}(J)\big[iH^2, A_L\big]E^{H^2}(J) \ge aE^{H^2}(J) \quad \textit{with} \quad a := 2\inf(J)\cdot \inf_{p\in M} g_L(p) > 0.$$

Proof. (i) The horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic with respect to the measure μ_Ω [23]. Therefore, we know from the theory of time changes on compact metric spaces [32, Prop. 3] that the flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is also uniquely ergodic with respect to the measure

$$\mathrm{d}\widetilde{\mu}_\Omega := \frac{f^{-1}\mathrm{d}\mu_\Omega}{\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\,.$$

It follows that (see [49, Prop. 1.3.4])

$$\begin{aligned}\lim_{L\to\infty} g_L = \lim_{L\to\infty}\left(\frac{1}{2} - \frac{1}{2L}\int_0^L \mathrm{d}t\, \mathscr{L}_{X_2}\big(\ln(f)\big)\circ \widetilde{F}_{1,-t}\right) &= \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}_\Omega\, \mathscr{L}_{X_2}\big(\ln(f)\big)\\ &= \frac{1}{2} + \frac{1}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\int_M \mathrm{d}\mu_\Omega\, \mathscr{L}_{X_2}\big(f^{-1}\big)\\ &= \frac{1}{2} + \frac{i}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\big\langle 1, H_2 f^{-1}\big\rangle\\ &= \frac{1}{2}\end{aligned}$$

uniformly on M. Thus, $g_L > 0$ if $L > 0$ is big enough.

(ii) We know from Equation (2.2) and Lemma 3.5(b) that

$$E^{H^2}(J)\big[iH^2, A_L\big]E^{H^2}(J) = E^{H^2}(J)\big(H^2g_L + 2Hg_LH + g_LH^2\big)E^{H^2}(J).$$

But, point (i) implies that

$$E^{H^2}(J)2Hg_LHE^{H^2}(J) \ge aE^{H^2}(J) \quad \text{with} \quad a = 2\inf(J)\cdot \inf_{p\in M} g_L(p) > 0.$$

Therefore, it is sufficient to show that $E^{H^2}(J)\big(H^2g_L + g_LH^2\big)E^{H^2}(J) \ge 0$.

So, for any $\varepsilon > 0$ let $H_\varepsilon^2 := H^2\big(\varepsilon^2 H^2 + 1\big)^{-1}$ and $H_\varepsilon^\pm := H(\varepsilon H \pm i)^{-1}$. Then, the inclusion $g_L^{1/2} \in C^1(H)$ (which can be proved as in Lemma 3.5(c)) implies that

$$\operatorname{s-lim}_{\varepsilon\searrow 0}\big[H_\varepsilon^\pm, g_L^{1/2}\big] = \pm \operatorname{s-lim}_{\varepsilon\searrow 0}(\varepsilon H \pm i)^{-1}\big[iH, g_L^{1/2}\big](\varepsilon H \pm i)^{-1} = \pm i\big[g_L^{1/2}, H\big].$$

Therefore, for each $\varphi \in \mathcal{H}$ it follows that

$$\begin{aligned}
&\big\langle \varphi, E^{H^2}(J)\big(H^2 g_L + g_L H^2\big)E^{H^2}(J)\varphi\big\rangle \\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H_\varepsilon^2 g_L^{1/2} g_L^{1/2} + g_L^{1/2} g_L^{1/2} H_\varepsilon^2\big)E^{H^2}(J)\varphi\big\rangle \\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(\big[H_\varepsilon^2, g_L^{1/2}\big]g_L^{1/2} + 2g_L^{1/2}H_\varepsilon^2 g_L^{1/2} + g_L^{1/2}\big[g_L^{1/2}, H_\varepsilon^2\big]\big)E^{H^2}(J)\varphi\big\rangle \\
&\ge \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(\big[H_\varepsilon^2, g_L^{1/2}\big]g_L^{1/2} + g_L^{1/2}\big[g_L^{1/2}, H_\varepsilon^2\big]\big)E^{H^2}(J)\varphi\big\rangle \\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H_\varepsilon^+\big[H_\varepsilon^-, g_L^{1/2}\big]g_L^{1/2} + \big[H_\varepsilon^+, g_L^{1/2}\big]H_\varepsilon^- g_L^{1/2} \\
&\qquad\qquad + g_L^{1/2}\big[g_L^{1/2}, H_\varepsilon^+\big]H_\varepsilon^- + g_L^{1/2}H_\varepsilon^+\big[g_L^{1/2}, H_\varepsilon^-\big]\big)E^{H^2}(J)\varphi\big\rangle \\
&= \lim_{\varepsilon\searrow 0}\big\langle \varphi, E^{H^2}(J)\big(H\big[H, g_L^{1/2}\big]g_L^{1/2} + \big[H_\varepsilon^+, g_L^{1/2}\big]g_L^{1/2}H_\varepsilon^- + \big[H_\varepsilon^+, g_L^{1/2}\big]\big[H_\varepsilon^-, g_L^{1/2}\big] \\
&\qquad\qquad + g_L^{1/2}\big[g_L^{1/2}, H\big]H + H_\varepsilon^+ g_L^{1/2}\big[g_L^{1/2}, H_\varepsilon^-\big] + \big[g_L^{1/2}, H_\varepsilon^+\big]\big[g_L^{1/2}, H_\varepsilon^-\big]\big)E^{H^2}(J)\varphi\big\rangle \\
&= \big\langle \varphi, E^{H^2}(J)\big(H\big[H, g_L^{1/2}\big]g_L^{1/2} + \big[H, g_L^{1/2}\big]g_L^{1/2}H + 2\big[H, g_L^{1/2}\big]^2 + g_L^{1/2}\big[g_L^{1/2}, H\big]H \\
&\qquad\qquad\qquad\qquad + Hg_L^{1/2}\big[g_L^{1/2}, H\big]\big)E^{H^2}(J)\varphi\big\rangle \\
&= \big\langle \varphi, E^{H^2}(J)2\big[H, g_L^{1/2}\big]^2 E^{H^2}(J)\varphi\big\rangle \\
&\ge 0,
\end{aligned}$$

which implies the claim. □

Using the previous results for H^2, one can finally determine the structure of the spectrum of H (and thus that of $\widetilde{H}$):

Theorem 3.7 (Spectral properties of H)**.** *Let $f \in C^3\big(M; (0,\infty)\big)$. Then, H has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbb{C}\cdot f^{-1/2}$. In particular, the self-adjoint operator $\widetilde{H}$ associated to the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbb{C}\cdot 1$.*

Proof. We know from Lemmas 3.5(d) and 3.6 that $(H^2+1)^{-1} \in C^2(A_L)$ and that H^2 satisfies a strict Mourre estimate on each bounded Borel subset of $(0,\infty)$. It follows by Theorem 2.1 that H^2 has purely absolutely continuous spectrum, except at 0, where it may have an eigenvalue. Accordingly, the Hilbert space $\mathcal{H}$ admits the orthogonal decomposition

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\rm ac}(H^2),$$

with $\mathcal{H}_{\rm ac}(H^2)$ the subspace of absolute continuity of H^2.

Now, the function $\lambda \mapsto \lambda^2$ has the Luzin N property on $\mathbb{R}$; namely, if J is a Borel subset of $\mathbb{R}$ with Lebesgue measure zero, then J^2 also has Lebesgue measure zero. It follows that $\mathcal{H}_{\rm ac}(H^2) \subset \mathcal{H}_{\rm ac}(H)$, with $\mathcal{H}_{\rm ac}(H)$ the subspace of absolute continuity of H (see Proposition 29, Section 3.5.4 of [6]). Furthermore, we have that

$$\ker(H^2) = \ker(H) = \mathscr{U}^*\ker(\widetilde{H}) = \mathbb{C}\cdot f^{-1/2}$$

due to the equality $H = \mathscr{U}^*\widetilde{H}\mathscr{U}$ and the ergodicity of the flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$. We thus infer that

$$\mathcal{H} = \ker(H^2) \oplus \mathcal{H}_{\rm ac}(H^2) \subset \ker(H) \oplus \mathcal{H}_{\rm ac}(H).$$

So, one necessarily has $\mathcal{H} = \ker(H) \oplus \mathcal{H}_{\rm ac}(H)$, meaning that H has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\ker(H) \equiv \mathbb{C} \cdot f^{-1/2}$. Since $H = \mathscr{U}^* \widetilde{H} \mathscr{U}$, this implies that $\widetilde{H}$ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbb{C} \cdot 1$. $\square$

Theorem 3.7 establishes the absolute continuity of time changes of horocycle flows on compact surfaces of constant negative curvature for time changes of class C^3. This improves Theorem 6 of [18], where G. Forni and C. Ulcigrai show the same result for time changes in a Sobolev space of order $> 11/2$ (under the same assumption, the authors of [18] also show that the maximal spectral type is equivalent to Lebesgue). This also complements Theorem 4.2 of [51], where the absolute continuity is shown for surfaces of finite volume and time changes of class C^2 under the additional condition of A. G. Kushnirenko.

We note that it would be interesting to see if the technics of this section could be adapted to the case of horocycle flows on surfaces of finite volume or surfaces of non-constant negative curvature. In the first case, one would have to deal with the fact that the horocycle flow is not uniquely ergodic (see [13, 14]), while in the second case one would have to deal with the fact that the horocycle flow is uniquely ergodic, but with the Margulis parametrisation and with respect to the Bowen-Margulis measure (see [12, 42]).

4 Skew products over translations

Let X be a compact metric abelian Banach Lie group with normalised Haar measure μ (such a group is isomorphic to a subgroup of the torus $\mathbb{T}^{\aleph_0}$, see [31, Thm. 8.45]). Take $\{y_t\}_{t\in\mathbb{R}}$ a C^1 one-parameter subgroup of X and let $\{F_t\}_{t\in\mathbb{R}}$ be the corresponding translation flow, *i.e.*,

$$F_t(x) := y_t x, \quad t \in \mathbb{R},\ x \in X.$$

Assume that the translation F_1 is ergodic (so that both flows $\{F_\ell\}_{\ell\in\mathbb{Z}}$ and $\{F_t\}_{t\in\mathbb{R}}$ are uniquely ergodic, see [11, Thm. 4.1.1] and [39, Sec. 1.2.2]) and associate to $\{F_t\}_{t\in\mathbb{R}}$ the operators

$$V_t\varphi := \varphi \circ F_t, \quad t \in \mathbb{R},\ \varphi \in C(X).$$

Due to the continuity of the map $\mathbb{R} \ni t \mapsto y_t \in X$ and the smoothness of the group operation, the family $\{V_t\}_{t\in\mathbb{R}}$ extends to a strongly continuous unitary group in $\mathcal{H} := \mathsf{L}^2(X,\mu)$ satisfying $V_t C^\infty(X) \subset C^\infty(X)$ for each $t \in \mathbb{R}$. It follows from Nelson's theorem [3, Prop. 5.3] that the generator of the group $\{V_t\}_{t\in\mathbb{R}}$

$$H\varphi := \text{s-}\lim_{t\to 0} it^{-1}\big(V_t - 1\big)\varphi, \quad \varphi \in \mathcal{D}(H) := \Big\{\varphi \in \mathcal{H} \mid \lim_{t\to 0} |t|^{-1}\big\|\big(V_t - 1\big)\varphi\big\| < \infty\Big\},$$

is essentially self-adjoint on $C^\infty(X)$, and one has

$$H\varphi := -i\mathscr{L}_Y\varphi, \quad \varphi \in C^\infty(X),$$

with Y the C^0 divergence-free vector field associated to $\{F_t\}_{t\in\mathbb{R}}$ and $\mathscr{L}_Y$ the corresponding Lie derivative. Furthermore, the operator V_1 has pure point spectrum $\sigma(V_1) = \overline{\big\{\gamma(y_1) \mid \gamma \in \widehat{X}\big\}}$, with $\widehat{X}$ the character group of X (see [52, Thm. 3.5]).

Now, let G be a compact metric abelian group with Haar measure ν and character group $\widehat{G}$, and let $\phi : X \to G$ be a measurable function (a cocycle). Then, one can define the skew product $T : X \times G \to X \times G$ given by $T(x,z) := \big(y_1 x, \phi(x) z\big)$ and the corresponding unitary operator

$$W\psi := \psi \circ T, \quad \psi \in \mathsf{L}^2(X \times G, \mu \times \nu). \tag{4.1}$$

It is known [25, Sec. 3.1] that the operator W is reduced by the orthogonal decomposition

$$\mathsf{L}^2(X \times G, \mu \times \nu) = \bigoplus_{\chi\in\widehat{G}} L_\chi, \quad L_\chi := \big\{\varphi \otimes \chi \mid \varphi \in \mathcal{H}\big\},$$

and that the restriction $W|_{L_\chi}$ is unitarily equivalent to the unitary operator

$$U_\chi\varphi := (\chi\circ\phi)V_1\varphi, \quad \varphi\in\mathcal{H}.$$

Furthermore, the operator U_χ satisfies the following purity law: the spectrum of U_χ has uniform multiplicity and is either purely punctual, purely singularly continuous or purely Lebesgue (this follows from Helson's analysis [30]; see [27, Thm. 2], [26, Thm. 4] and [41, p. 8560]).

In the sequel, we treat the case where the cocycle ϕ satisfies the following assumption:

Assumption 4.1 (Cocycle)**.** *The map $\phi : X\to G$ satisfies $\phi=\xi\,\eta$, where*

(i) $\xi : X\to G$ is a continuous group homomorphism,

(ii) $\eta\in C(X;G)$ has a Lie derivative $\mathscr{L}_Y(\chi\circ\eta)$ which satisfies the following Dini-type condition along the flow $\{F_t\}_{t\in\mathbb{R}}$:

$$\int_0^1\frac{\mathrm{d}t}{t}\,\big\|\mathscr{L}_Y(\chi\circ\eta)\circ F_t-\mathscr{L}_Y(\chi\circ\eta)\big\|_{\mathsf{L}^\infty(X)}=\int_0^1\frac{\mathrm{d}t}{t}\,\big\|V_t\mathscr{L}_Y(\chi\circ\eta)V_{-t}-\mathscr{L}_Y(\chi\circ\eta)\big\|_{\mathscr{B}(\mathcal{H})}<\infty.$$

We start the analysis with a first lemma on the regularity of the operators U_χ. We use the fact that the map $\mathbb{R}\ni t\mapsto(\chi\circ\xi)(y_t)\in\mathbb{S}^1$ is a character on $\mathbb{R}$, and thus of class C^∞. We also use the notations

$$\xi_0:=\frac{\mathrm{d}}{\mathrm{d}t}\,(\chi\circ\xi)(y_t)\Big|_{t=0},\qquad g:=|\xi_0|^2-\xi_0\,\frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta}\qquad\text{and}\qquad A:=-i\,\xi_0 H,$$

and observe that $\xi_0\in i\,\mathbb{R}$, that $g:X\to\mathbb{R}$ satisfies the Dini-type condition along $\{F_t\}_{t\in\mathbb{R}}$, that A is self-adjoint with $\mathcal{D}(A)\supset\mathcal{D}(H)$ and that ξ_0, g and A depend on χ (even if we do not specify it in the notation).

Lemma 4.2. *Let ϕ satisfy Assumption 4.1. Then, $U_\chi\in C^{1+0}(A)$ with $\big[A,U_\chi\big]=g\,U_\chi$.*

Proof. Since A and V_1 commute, one has for each $\varphi\in C^\infty(X)$ that

$$\big\langle A\varphi,U_\chi\varphi\big\rangle-\big\langle\varphi,U_\chi A\varphi\big\rangle=\big\langle\varphi,\big[A,\chi\circ\phi\big]V_1\varphi\big\rangle=\big\langle\varphi,-\xi_0\,\mathscr{L}_Y(\chi\circ\phi)\,V_1\varphi\big\rangle.$$

Furthermore, the homomorphism property of χ and ξ and the Leibniz rule for Lie derivatives imply that

$$\mathscr{L}_Y(\chi\circ\phi)=\mathscr{L}_Y(\chi\circ\xi)(\chi\circ\eta)+(\chi\circ\xi)\,\mathscr{L}_Y(\chi\circ\eta)=\left(\xi_0+\frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta}\right)(\chi\circ\phi).$$

It follows that

$$\big\langle A\varphi,U_\chi\varphi\big\rangle-\big\langle\varphi,U_\chi A\varphi\big\rangle=\big\langle\varphi,g\,U_\chi\varphi\big\rangle,$$

with $g\in\mathsf{L}^\infty(X)$. So, one has $U_\chi\in C^1(A)$ with $[A,U_\chi]=g\,U_\chi$ due to the density of $C^\infty(X)$ in $\mathcal{D}(A)$.

To show that $U_\chi\in C^{1+0}(A)$, one has to check that $\int_0^1\frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}[A,U_\chi]\,\mathrm{e}^{itA}-[A,U_\chi]\big\|_{\mathscr{B}(\mathcal{H})}<\infty$. But since $[A,U_\chi]=g\,U_\chi$ with $U_\chi\in C^1(A)$, one is reduced to showing that

$$\int_0^1\frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,g\,\mathrm{e}^{itA}-g\big\|_{\mathscr{B}(\mathcal{H})}<\infty\iff\int_0^{-i\,\xi_0}\frac{\mathrm{d}s}{s}\,\big\|V_s\,gV_{-s}-g\big\|_{\mathscr{B}(\mathcal{H})}<\infty,$$

which is is readily verified due to the Dini-type condition satisfied by g. □

Since $U_\chi\in C^1(A)$, we know from Section 2 that the operator

$$A_n\varphi:=\frac{1}{n}\sum_{\ell=0}^{n-1}U_\chi^{-\ell}\,A\,U_\chi^\ell\varphi=\frac{1}{n}\sum_{\ell=0}^{n-1}U_\chi^{-\ell}\big[A,U_\chi^\ell\big]\varphi+A\varphi,\quad n\in\mathbb{N}^*,\ \varphi\in\mathcal{D}(A_n):=\mathcal{D}(A),$$

is self-adjoint. In the next lemma, we prove regularity properties of the operator U_χ with respect to A_n and the strict Mourre estimate for U_χ. The averages of the function g along the flow $\{F_\ell\}_{\ell\in\mathbb{Z}}$, *i.e.*,

$$g_n := \frac{1}{n}\sum_{\ell=0}^{n-1} g\circ F_{-\ell}\,,\quad n\in\mathbb{N}^*,$$

appear in a natural way.

Lemma 4.3 (Strict Mourre estimate for U_χ)**.** *Let ϕ satisfy Assumption 4.1 with $\chi\circ\xi\not\equiv 1$, and suppose F_1 is ergodic. Then,*

(a) one has $U_\chi\in C^{1+0}(A_n)$ with $[A_n,U_\chi]=g_nU_\chi$,

(b) if n is big enough, one has $g_n>0$ and $(U_\chi)^*\big[A_n,U_\chi\big]\ge a$ with $a:=\inf_{x\in X} g_n(x)>0$.

Proof. (a) We know from Lemma 4.2 that $U_\chi\in C^{1+0}(A)$. So, it follows from the abstract result [17, Lemma 4.1] that $U_\chi\in C^{1+0}(A_n)$ with $[A_n,U_\chi]=\frac{1}{n}\sum_{\ell=0}^{n-1}U_\chi^{-\ell}[A,U_\chi]U_\chi^{\ell}$. Using the equality $[A,U_\chi]=gU_\chi$, one thus obtains that

$$\big[A_n,U_\chi\big]=\left(\frac{1}{n}\sum_{\ell=0}^{n-1}U_\chi^{-\ell}\,g\,U_\chi^{\ell}\right)U_\chi=\left(\frac{1}{n}\sum_{\ell=0}^{n-1}g\circ F_{-\ell}\right)U_\chi=g_nU_\chi,$$

which concludes the proof of the claim.

(b) Due to the unique ergodicity of the discrete flow $\{F_\ell\}_{\ell\in\mathbb{Z}}$, we know that $\lim_{n\to\infty}g_n=\int_X \mathrm{d}\mu\, g$ uniformly on X. Using the fact that $\chi\circ\eta=\mathrm{e}^{if_{\chi,\eta}}$ for some real function $f_{\chi,\eta}\in\mathcal{D}(H)$, we thus deduce that

$$\lim_{n\to\infty}g_n=\int_X\mathrm{d}\mu\,g=|\xi_0|^2-\xi_0\int_X\mathrm{d}\mu\,\frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta}=|\xi_0|^2+\xi_0\big\langle 1,Hf_{\chi,\eta}\big\rangle=|\xi_0|^2$$

uniformly on X. But, since the character $\chi\circ\xi:X\to\mathbb{S}^1$ is nontrivial, we know that $\xi_0\neq 0$ due to the unique ergodicity of the continuous flow $\{F_t\}_{t\in\mathbb{R}}$ (see [11, Thm. 4.1.1']). Therefore, $g_n>0$ if $n>0$ is big enough, and point (a) implies that

$$(U_\chi)^*\big[A_n,U_\chi\big]=(U_\chi)^*g_nU_\chi\ge a,$$

with $a=\inf_{x\in X}g_n(x)>0$. □

Using what precedes, we can determine the spectral properties of the operators U_χ and W (see (4.1) for the definition of W):

Theorem 4.4 (Spectral properties of U_χ and W)**.** *Let ϕ satisfy Assumption 4.1 with $\chi\circ\xi\not\equiv 1$, and suppose that F_1 is ergodic. Then, the operator U_χ has purely Lebesgue spectrum. In particular, the restriction of W to the subspace $\bigoplus_{\chi\in\widehat{G},\,\chi\circ\xi\not\equiv 1}L_\chi\subset\mathsf{L}^2(X\times G,\mu\times\nu)$ has countable Lebesgue spectrum.*

Proof. We know from Lemma 4.3 that $U_\chi\in C^{1+0}(A_n)$ and that U_χ satisfies a strict Mourre estimate on all of $\mathbb{S}^1$. It follows by Theorem 2.2 that U_χ has a purely absolutely continuous spectrum, and thus has a purely Lebesgue spectrum due to the purity law. The claim on W follows from what precedes if one takes into account the separability of the Hilbert space $\mathsf{L}^2(X\times G,\mu\times\nu)$. □

Theorem 4.4 provides a general criterion for the presence of countable Lebesgue spectrum for skew products over translations. In the particular case where $X=\mathbb{T}^d\simeq\mathbb{R}^d/\mathbb{Z}^d$ and $G=\mathbb{T}^{d'}\simeq\mathbb{R}^{d'}/\mathbb{Z}^{d'}$ for some $d,d'\ge 1$, the flow $\{F_t\}_{t\in\mathbb{R}}$ is given (in additive notation) by

$$F_t(x):=ty+x\ (\mathrm{mod}\ \mathbb{Z}^d),\quad t\in\mathbb{R},\ x\in\mathbb{T}^d,$$

for some $y:=(y_1,y_2,\dots,y_d)\in\mathbb{R}^d$. So, one has $\mathscr{L}_Y=y\cdot\nabla$, and F_1 is ergodic if and only if the numbers $y_1,y_2,\dots,y_d,1$ are rationally independent [11, Sec. 3.1]. Furthermore, each group homomorphism $\xi:\mathbb{T}^d\to$

$\mathbb{T}^{d'}$ is given by $\xi(x) := Nx \pmod{\mathbb{Z}^{d'}}$ for some $d' \times d$ matrix N with integer entries, and each character $\chi_m \in \widehat{\mathbb{T}^{d'}}$ is given by $\chi_m(z) := \mathrm{e}^{2\pi i m\cdot z}$ for some $m \in \mathbb{Z}^{d'}$. Therefore,

$$\chi_m \circ \xi \not\equiv 1 \iff \mathrm{e}^{2\pi i m\cdot Nx} \neq 1 \text{ for some } x \in \mathbb{T}^d \iff N^{\mathsf{T}} m \neq 0 \in \mathbb{Z}^d,$$

and we obtain the following corollary of Theorem 4.4.

Corollary 4.5 (The case of tori)**.** *Let $y_1, y_2, \ldots, y_d, 1 \in \mathbb{R}$ be rationally independent, let $\chi_m \in \widehat{\mathbb{T}^{d'}}$ be given by $\chi_m(z) := \mathrm{e}^{2\pi i m\cdot z}$ for some $m \in \mathbb{Z}^{d'}$ and let $\phi : \mathbb{T}^d \to \mathbb{T}^{d'}$ satisfy $\phi = \xi + \eta$, where*

(i) $\xi : \mathbb{T}^d \to \mathbb{T}^{d'}$ is given by $\xi(x) := Nx \pmod{\mathbb{Z}^{d'}}$ for some $d' \times d$ matrix N with integer entries,

(ii) $\eta \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ has a derivative $y \cdot \nabla(m \cdot \eta)$ which satisfies the Dini-type condition

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| y \cdot \nabla(m \cdot \eta) \circ F_t - y \cdot \nabla(m \cdot \eta) \right\|_{\mathsf{L}^\infty(X)} < \infty. \tag{4.2}$$

Assume also that $N^{\mathsf{T}} m \neq 0$. Then, the operator $U_{\chi_m} \equiv \mathrm{e}^{2\pi i m\cdot(\xi+\eta)} V_1$ has purely Lebesgue spectrum. In particular, the restriction of W to the subspace $\bigoplus_{m\in\mathbb{Z}^{d'}, N^{\mathsf{T}} m\neq 0} L_{\chi_m}$ has countable Lebesgue spectrum.

In the case $d = d' = 1$, Corollary 4.5 implies that the restriction of W to $\bigoplus_{m\in\mathbb{Z}\setminus\{0\}} L_{\chi_m}$ has countable Lebesgue spectrum if $\phi(x) = Nx + \eta(x)$, with $N \in \mathbb{Z} \setminus \{0\}$ and with $\eta \in C^1(\mathbb{T}; \mathbb{T})$ such that η' is Dini-continuous. Since the properties of being Dini-continuous and being of bounded variation are are mutually independent [29, Sec. 2], this complements Theorem 1 of [35], where A. Iwanik, M. Lemańzyk and D. Rudolph show the same result under the condition that η is absolutely continuous with η' of bounded variation (see also [33, Sec. 2] for another sufficient condition given in terms of the Fourier coefficients of η). Results prior to [35] along this line can be found in the papers of A. G. Kushnirenko [40] and G. H. Choe [10].

In the case $d, d' \geq 1$, Corollary 4.5 implies that the restriction of W to $\bigoplus_{m\in\mathbb{Z}^{d'}, N^{\mathsf{T}} m\neq 0} L_{\chi_m}$ has countable Lebesgue spectrum if $\phi(x) = Nx + \eta(x)$, with N a $d' \times d$ matrix with integer entries and with $\eta \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ such that $y \cdot \nabla(m \cdot \eta)$ exists and satisfies the Dini-type condition (4.2). This complements Section 3 of [34], where A. Iwanik shows the same result for functions $\eta \in C^1(\mathbb{T}^d; \mathbb{T}^{d'})$ with Fourier coefficients satisfying some decay assumption (see also the works of B. Fayad [16] and K. Frączek [19, 20] for related results on the spectrum of skew products on tori).

We note that it would be interesting to see if the technics of this section could be adapted to the case of cocycles taking values in non-abelian groups, such as the case of $\mathsf{SU}(2)$ considered in [21].

5 Furstenberg transformations

For each integer $n \geq 1$, we denote by μ_n the normalised Haar measure on the torus $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$ and we set $\mathcal{H}_n := \mathsf{L}^2(\mathbb{T}^n, \mu_n)$ for the corresponding Hilbert space. Furstenberg transformations [22, Sec. 2.3] are the invertible measure-preserving maps $T_d : \mathbb{T}^d \to \mathbb{T}^d$ ($d \geq 2$) given by

$$\begin{aligned} &T_d(x_1, x_2, \ldots, x_d) \\ &:= \big(x_1 + y, x_2 + b_{2,1}x_1 + h_1(x_1), \ldots, x_d + b_{d,1}x_1 + \cdots + b_{d,d-1}x_{d-1} + h_{d-1}(x_1, x_2, \ldots, x_{d-1})\big) \pmod{\mathbb{Z}^d}, \end{aligned}$$

where $y \in \mathbb{R} \setminus \mathbb{Q}$, $b_{j,k} \in \mathbb{Z}$, $b_{\ell,\ell-1} \neq 0$ for $\ell \in \{2, \ldots, d\}$ and each $h_j : \mathbb{T}^j \to \mathbb{R}$ satisfies a uniform Lipschitz condition in the variable x_j. The corresponding Koopman operator

$$W_d : \mathcal{H}_d \to \mathcal{H}_d, \quad \varphi \mapsto \varphi \circ T_d, \tag{5.1}$$

is reduced by the orthogonal decompositions

$$\mathcal{H}_d = \mathcal{H}_1 \oplus \bigoplus_{j\in\{2,\ldots,d\}} \big(\mathcal{H}_j \cap \mathcal{H}_{j-1}^\perp\big) = \mathcal{H}_1 \oplus \bigoplus_{j\in\{2,\ldots,d\},\, k\in\mathbb{Z}\setminus\{0\}} \mathcal{H}_{j,k}, \tag{5.2}$$

where the subspaces $\mathcal{H}_{j,k} \subset \mathcal{H}_j$ are defined by $\mathcal{H}_{j,k} := \overline{\mathrm{Span}}\{\eta \otimes \chi_k \mid \eta \in \mathcal{H}_{j-1})\}$, with $\chi_k \in \widehat{\mathbb{T}}$ the character given by $\chi_k(x_j) := \mathrm{e}^{2\pi i k x_j}$ (see [11, Sec. 13.3] for details). Furthermore, the restriction $W_d|_{\mathcal{H}_{j,k}}$ is unitarily equivalent to the unitary operator given by

$$U_{j,k}\,\eta := \mathrm{e}^{2\pi i k\phi_j}\, W_{j-1}\,\eta, \quad \eta \in \mathcal{H}_{j-1}, \tag{5.3}$$

with $\phi_j(x_1, x_2, \ldots, x_{j-1}) := b_{j,1}x_1 + \cdots + b_{j,j-1}x_{j-1} + h_{j-1}(x_1, x_2, \ldots, x_{j-1})$.

The operators $U_{j,k} = \mathrm{e}^{2\pi i k\phi_j}\, W_{j-1}$ are similar to the operators $U_\chi = (\chi \circ \phi)V_1$ studied in Section 4. So, we apply to them the same method. First, we define an operator (vector field) A which commutes with W_{j-1} and has an appropriate commutator with $\mathrm{e}^{2\pi i k\phi_j}$, and then we use as a congugate operator the average $\frac{1}{n}\sum_{\ell=0}^{n-1} U_{j,k}^{-\ell}\, A\, U_{j,k}^{\ell}$ of A along the flow $\{U_{j,k}^{\ell}\}_{\ell\in\mathbb{Z}}$ generated by $U_{j,k}$.

We start with the definition of the operator A and then we prove regularity properties of the operators $U_{j,k}$ with respect to A. For this, we recall that the translation group $\{V_{t,j-1}\}_{t\in\mathbb{R}}$ in $\mathcal{H}_{j-1}$ given by

$$\big(V_{t,j-1}\eta\big)(x_1, x_2, \ldots, x_{j-1}) := \eta\big(x_1, x_2, \ldots, x_{j-1} - t \,(\mathrm{mod}\, \mathbb{Z})\big), \quad t \in \mathbb{R},\ \eta \in C(\mathbb{T}^{j-1}),$$

has self-adjoint generator $P_{j-1} := -i\partial_{j-1}$ which is essentially self-adjoint on $C^\infty(\mathbb{T}^{j-1})$. Also, for $j \in \{2, \ldots, d\}$ and $k \in \mathbb{Z} \setminus \{0\}$, we use the notations

$$g := 1 + (b_{j,j-1})^{-1}\partial_{j-1}h_{j-1} \qquad \text{and} \qquad A := (2\pi k b_{j,j-1})^{-1}P_{j-1},$$

and observe that A is self-adjoint with $\mathcal{D}(A) = \mathcal{D}(P_{j-1})$ and that $g \in \mathsf{L}^\infty(\mathbb{T}^{j-1})$ due to the uniform Lipschitz condition satisfied by h_{j-1} in the variable x_{j-1}. We also note that g and A depend on j and k, even if we do not specify it in the notation.

Lemma 5.1. *Let $j \in \{2, \ldots, d\}$ and $k \in \mathbb{Z} \setminus \{0\}$. Assume that h_{j-1} is of class C^1 in the variable x_{j-1} and that $\partial_{j-1}h_{j-1}$ satisfies the following Dini-type condition in the variable x_{j-1} :*

$$\int_0^1 \frac{\mathrm{d}t}{t}\, \big\|V_{t,j-1}(\partial_{j-1}h_{j-1})V_{-t,j-1} - (\partial_{j-1}h_{j-1})\big\|_{\mathscr{B}(\mathcal{H}_{j-1})} < \infty. \tag{5.4}$$

Then, one has $U_{j,k} \in C^{1+0}(A)$ with $[A, U_{j,k}] = g\,U_{j,k}$.

Proof. Since A and W_{j-1} commute and since h_{j-1} satisfies a uniform Lipschitz condition in the variable x_{j-1}, one has for each $\eta \in C^\infty(\mathbb{T}^{j-1})$ that

$$\big\langle A\,\eta, U_{j,k}\,\eta\big\rangle_{\mathcal{H}_{j-1}} - \big\langle \eta, U_{j,k}\,A\,\eta\big\rangle_{\mathcal{H}_{j-1}} = \big\langle \eta, \big[A, \mathrm{e}^{2\pi i k\phi_j}\big]W_{j-1}\eta\big\rangle_{\mathcal{H}_{j-1}} = \big\langle \eta, g\,U_{j,k}\,\eta\big\rangle_{\mathcal{H}_{j-1}},$$

with $g \in \mathsf{L}^\infty(\mathbb{T}^{j-1})$. So, one has $U_{j,k} \in C^1(A)$ with $[A, U_{j,k}] = g\,U_{j,k}$ due to the density of $C^\infty(\mathbb{T}^{j-1})$ in $\mathcal{D}(A)$.

To show that $U_{j,k} \in C^{1+0}(A)$, one has to check that $\int_0^1 \frac{\mathrm{d}t}{t}\, \big\|\,\mathrm{e}^{-itA}[A, U_{j,k}]\,\mathrm{e}^{itA} - [A, U_{j,k}]\big\|_{\mathscr{B}(\mathcal{H}_{j-1})} < \infty$. But since $[A, U_{j,k}] = g\,U_{j,k}$ with $U_{j,k} \in C^1(A)$, one is reduced to showing that

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t}\, \big\|\,\mathrm{e}^{-itA}\, g\, \mathrm{e}^{itA} - g\big\|_{\mathscr{B}(\mathcal{H}_{j-1})} < \infty \\
\iff &\int_0^{(2\pi k b_{j,j-1})^{-1}} \frac{\mathrm{d}s}{s}\, \big\|V_{s,j-1}(\partial_{j-1}h_{j-1})V_{-s,j-1} - (\partial_{j-1}h_{j-1})\big\|_{\mathscr{B}(\mathcal{H}_{j-1})} < \infty,
\end{aligned}$$

which is is readily verified due to the Dini-type condition satisfied by $\partial_{j-1}h_{j-1}$. □

Since $U_{j,k} \in C^1(A)$, we know from Section 2 that the operator

$$A_n\eta := \frac{1}{n}\sum_{\ell=0}^{n-1} U_{j,k}^{-\ell}\, A\, U_{j,k}^{\ell}\,\eta = \frac{1}{n}\sum_{\ell=0}^{n-1} U_{j,k}^{-\ell}\big[A, U_{j,k}^{\ell}\big]\eta + A\eta, \quad n \in \mathbb{N}^*,\ \eta \in \mathcal{D}(A_n) := \mathcal{D}(A),$$

is self-adjoint. In the next lemma, we prove regularity properties of the operator $U_{j,k}$ with respect to A_n and the strict Mourre estimate for $U_{j,k}$. The averages of the function g along the flow $\{T_{j-1}^{\ell}\}_{\ell\in\mathbb{Z}}$, *i.e.*,

$$g_n := \frac{1}{n}\sum_{\ell=0}^{n-1} g\circ T_{j-1}^{-\ell}, \quad n\in\mathbb{N}^*,$$

appear in a natural way.

Lemma 5.2 (Strict Mourre estimate for $U_{j,k}$)**.** *Let $j\in\{2,\ldots,d\}$ and $k\in\mathbb{Z}\setminus\{0\}$. Assume that h_{j-1} is of class C^1 in the variable x_{j-1} and that $\partial_{j-1}h_{j-1}$ satisfies the Dini-type condition* (5.4)*. Then,*

- *(a) one has $U_{j,k}\in C^{1+0}(A_n)$ with $[A_n,U_{j,k}]=g_n U_{j,k}$,*
- *(b) if n is big enough, one has $g_n>0$ and $(U_{j,k})^*[A_n,U_{j,k}]\ge a$ with $a:=\inf_{x\in\mathbb{T}^{d-1}} g_n(x)>0$.*

Proof. (a) We know from Lemma 5.1 that $U_{j,k}\in C^{1+0}(A)$. So, it follows from the abstract result [17, Lemma 4.1] that $U_{j,k}\in C^{1+0}(A_n)$ with $[A_n,U_{j,k}]=\frac{1}{n}\sum_{\ell=0}^{n-1}U_{j,k}^{-\ell}[A,U_{j,k}]U_{j,k}^{\ell}$. Using the equality $[A,U_{j,k}]=gU_{j,k}$, one thus obtains that

$$[A_n,U_{j,k}]=\left(\frac{1}{n}\sum_{\ell=0}^{n-1}U_{j,k}^{-\ell}gU_{j,k}^{\ell}\right)U_{j,k}=\left(\frac{1}{n}\sum_{\ell=0}^{n-1}g\circ T_{j-1}^{-\ell}\right)U_{j,k}=g_nU_{j,k},$$

which concludes the proof of the claim.

(b) It is known from [22, Thm. 2.1] that the transformation T_{j-1} is uniquely ergodic. So, one has that

$$\lim_{n\to\infty} g_n=\int_{\mathbb{T}^{j-1}}\mathrm{d}\mu_{j-1}\,g=1+i(b_{j,j-1})^{-1}\big\langle 1,P_{j-1}h_{j-1}\big\rangle_{\mathcal{H}_{j-1}}=1$$

uniformly on $\mathbb{T}^{j-1}$. Therefore, $g_n>0$ if n is big enough, and point (a) implies that

$$(U_{j,k})^*[A_n,U_{j,k}]=(U_{j,k})^*g_nU_{j,k}\ge a,$$

with $a=\inf_{x\in\mathbb{T}^{d-1}}g_n(x)>0$. □

Using what precedes, we can determine the spectral properties of the operator W_d (see (5.1) for the definition of W_d):

Theorem 5.3 (Spectral properties of W_d)**.** *For each $j\in\{2,\ldots,d\}$, assume that h_{j-1} is of class C^1 in the variable x_{j-1} and that $\partial_{j-1}h_{j-1}$ satisfies the Dini-type condition* (5.4)*. Then, W_d has countable Lebesgue spectrum in the orthocomplement of $\mathcal{H}_1$.*

Proof. Let $j\in\{2,\ldots,d\}$ and $k\in\mathbb{Z}\setminus\{0\}$. Then, we know from Lemma 5.2 that $U_{j,k}\in C^{1+0}(A_n)$ and that $U_{j,k}$ satisfies a strict Mourre estimate on all of $\mathbb{S}^1$. It follows by Theorem 2.2 that $U_{j,k}$ has a purely absolutely continuous spectrum in $\mathcal{H}_{j-1}$, and thus that W_d has purely purely absolutely continuous spectrum in the orthocomplement of $\mathcal{H}_1$ due to the orthogonal decomposition (5.2). Since W_1 has pure point spectrum and T_d is ergodic, it follows from the standard purity law (see [25, Thm. 8]) that W_d has countable Lebesgue spectrum in the orthocomplement of $\mathcal{H}_1$. □

Theorem 5.3 complements Corollary 3 of [35], where A. Iwanik, M. Lemańzyk and D. Rudolph show that W_d has countable Lebesgue spectrum in the orthocomplement of $\mathcal{H}_1$ under the assumption (independent from the Dini-type condition (5.4)) that $\partial_{j-1}h_{j-1}$ is of bounded variation in the variable x_{j-1}.

As a final comment, we note that it would be interesting to see if the technics of this section could be adapted to variants of Furstenberg transformations (such as the ones studied in [28], [36] or [44]).

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